Appendix E Numerical Implementation of the Cap Model

E.1 Introduction

The numerical algorithm for the cap model was published by Sandler and Rubin in an attempt "to facilitate the general use of the cap model in dynamic computations, as well as in model fitting" (Sandler and Rubin 1979). The cap model algorithm was designed for use in either finite element or finite difference codes and is applicable to both static and dynamic problems (Chen and Baladi 1985). Of notable foresight on the part of the designers was their use of function statements within the model, which allow substantial changes to be made to the cap model's potential functions with little programming effort. This feature has allowed investigators to simulate a wide variety of natural and man-made materials with high degrees of fidelity between model and material response. Despite the many published variations of the cap model, the original cap model algorithm developed by Sandler and Rubin still forms the foundation of most current cap model algorithms.

The cap model algorithm is essentially an implementation of Equation D.7. To march the calculation through time, the user must input the stresses ${}^{t}\sigma_{ij}$ and the location of the cap at time t, which is explicitly defined by the term $l({}^{t}\kappa)$ and implicitly defined by the hardening parameter ${}^{t}\kappa$, and the strain increments from the solution of the field equations for the current time step ${}^{t+\Delta t}d\varepsilon_{ij}$. The cap model returns the new stresses ${}^{t+\Delta t}\sigma_{ij}$ and the updated cap location and hardening parameter $l\,({}^{t+\Delta t}\kappa)$ and ${}^{t+\Delta t}\kappa$ at time $t+\Delta t$. A given strain increment may invoke four different types of stress paths that coincide with four different algorithms within the cap model itself:

(a) an elastic algorithm.

(b) a failure envelope algorithm,

(c) a hardening cap algorithm, or

(d) a tension cutoff algorithm,

In the following text, a description of the four numerical algorithms is provided. The descriptions are based upon previous descriptions by Baladi and Akers (1981), Chen and Baladi (1985), Sandler and Rubin (1979), and Meier (1989). To simplify the presentation, a description of the cap model's response in the tensile regime will be deferred to the later part of this section.

E.2 Elastic algorithm

To start the numerical procedure, it is assumed that the given strain increments produce an entirely elastic stress path. A set of elastic trial stresses are calculated from

$$ {}^{E}J_1 = {}^{t}J_1 + 3K\,{}^{t+\Delta t}d\varepsilon_{kk} \qquad \text{E.1} $$

and

$$ {}^{E}s_{ij} = {}^{t}s_{ij} + 2G\,{}^{t+\Delta t}de_{ij} \qquad \text{E.2} $$

The elastic trial stresses are tested with respect to the tension cutoff, the failure envelope, and then the cap. If these surfaces are not violated by the trial stresses, the actual stress path is an elastic path, and the new stresses are the elastic trial stresses, i.e., ${}^{t+\Delta t}J_1 = {}^{E}J_1$ and ${}^{t+\Delta t}s_{ij} = {}^{E}s_{ij}$.

E.3 Failure envelope algorithm

If the following conditions exist when the elastic trial stresses are tested with respect to the failure envelope,

$$ {}^{E}J_1 \geq L({}^{t}\kappa) $$

$$ h({}^{E}J_1, \sqrt{{}^{E}J_{2D}}) = \sqrt{{}^{E}J_{2D}} - Q({}^{E}J_1) \geq 0 $$

then the elastic trial stresses have violated the failure envelope, and the given strain increment must be a combination of elastic and plastic strains. The trial stresses must be corrected such that (a) the final stress state falls on the failure surface and satisfies the following relation

$$ h({}^{t+\Delta t}J_1, \sqrt{{}^{t+\Delta t}J_{2D}}) = 0 \qquad \text{E.3} $$

and (b) the resulting elastic and plastic strain increments add up to the given strain increments ${}^{t+\Delta t}d\varepsilon_{ij}$.

The mathematical statement that requires the final stresses to lie on the fixed failure surface is given as

$$ dh = \frac{\partial h}{\partial \sigma_{ij}} d\sigma_{ij} = 0 \qquad \text{E.4} $$

Assuming small strain increments, Equation E.4 can be numerically approximated by the following expression

$$ dh = \sqrt{{}^{E}J_{2D}} - \sqrt{{}^{t+\Delta t}J_{2D}} - Q({}^{E}J_1) + Q({}^{t+\Delta t}J_1) \qquad \text{E.5} $$

which reduces to

$$ dh = \sqrt{{}^{E}J_{2D}} - Q({}^{E}J_1) \qquad \text{E.6} $$

since the final stress point must lie on the failure surface, i.e.,

$$\sqrt{{}^{t+\Delta t}J_{2D}} - Q({}^{t+\Delta t}J_1) = 0 \qquad \text{E.7}$$

Equation E.6 may be substituted into Equation E.4 and expanded in the following manner

$$\begin{aligned}\sqrt{{}^{E}J_{2D}} - Q({}^{E}J_1) &= \frac{\partial h}{\partial \sigma_{ij}} d\sigma_{ij} \\ &= \left(\frac{\partial h}{\partial J_1}\frac{\partial J_1}{\partial \sigma_{ij}} + \frac{\partial h}{\partial \sqrt{J_{2D}}}\frac{\partial \sqrt{J_{2D}}}{\partial \sigma_{ij}} \right) d\sigma_{ij} \\ &= \frac{\partial h}{\partial J_1} dJ_1 + \frac{1}{2\sqrt{J_{2D}}}\frac{\partial h}{\partial \sqrt{J_{2D}}} {}^{E}s_{ij}\, ds_{ij}\end{aligned} \qquad \text{E.8}$$

where $dJ_1 = {}^{E}J_1 - {}^{t}J_1$ and $ds_{ij} = {}^{E}s_{ij} - {}^{t}s_{ij}$. From Equations E.1 and E.2, we know that $dJ_1 = 3K\, {}^{t+\Delta t}d\varepsilon_{kk}$ and $ds_{ij} = 2G\, {}^{t+\Delta t}de_{ij}$, and these expressions may be substituted into Equation E.8 to give

$$\begin{aligned}\sqrt{{}^{E}J_{2D}} - Q({}^{E}J_1) &= 3K\, {}^{t+\Delta t}d\varepsilon_{kk} \frac{\partial h}{\partial J_1} \\ &\quad + \frac{G}{\sqrt{J_{2D}}}\frac{\partial h}{\partial \sqrt{J_{2D}}} {}^{E}s_{ij}\, {}^{t+\Delta t}de_{ij}\end{aligned} \qquad \text{E.9}$$

If we substitute Equation E.9 into the numerator of Equation D.5 and recognize that $h = f$, $\frac{\partial h}{\partial J_1} = -\frac{\partial Q}{\partial J_1}$, $\frac{\partial h}{\partial \sqrt{J_{2D}}} = 1$, and $\frac{\partial h}{\partial \kappa} = 0$ for the fixed failure envelope, an expression for $d\lambda$

may be written as

$$d\lambda = \frac{\sqrt{{}^{E}J_{2D}} - Q({}^{E}J_1)}{9K\left(\frac{\partial Q}{\partial J_1}\right)^2 + G} \qquad \text{E.10}$$

Substituting the above expression into Equation 4.20, the final expression for the plastic strain increment is obtained

$$d\varepsilon_{kk}^{p} = -3\left[\frac{\sqrt{{}^{E}J_{2D}} - Q({}^{E}J_1)}{9K\left(\frac{\partial Q}{\partial J_1}\right)^2 + G}\right]\frac{\partial Q}{\partial J_1} \qquad \text{E.11}$$

An expression for ${}^{t+\Delta t}J_1$ may be developed in the following manner

$$\begin{aligned} {}^{t+\Delta t}J_1 &= {}^{t}J_1 + 3K\,d\varepsilon_{kk}^{e} \\ &= {}^{t}J_1 + 3K\,{}^{t+\Delta t}d\varepsilon_{kk} - 3K\,d\varepsilon_{kk}^{p} \\ &= {}^{E}J_1 - 3K\,d\varepsilon_{kk}^{p} \end{aligned} \qquad \text{E.12}$$

where $d\varepsilon_{kk}^{p}$ is defined by Equation E.11. A "tentative" value of ${}^{t+\Delta t}J_1$ may be calculated from Equation E.12; this value is tentative because it must be tested against the current position of the cap, which is defined by the value of $L({}^{t}\kappa)$.

If ${}^{t+\Delta t}J_1 < L({}^{t}\kappa)$, which indicates the stress point has violated the cap, then corner coding is required, i.e., the cap must intersect the failure envelope forming a corner, and the value of ${}^{t+\Delta t}J_1$ must be adjusted. Adhering to the imposed conditions of normality, a stress state lying on the failure envelope produces dilatant plastic volumetric strains. Since cap expansion can only result from compressive plastic volumetric strains, the cap is stationary, and the new stress state can not move beyond the intersection of the cap and the failure envelope. Thus, the final stress state is ${}^{t+\Delta t}J_1 = L({}^t\kappa)$, and the updated hardening parameter is ${}^{t+\Delta t}\kappa = {}^t\kappa$.

If ${}^{t+\Delta t}J_1 > L({}^t\kappa)$, then the final stresses will depend upon the form of the hardening function. Equation 4.7 is the simplest form of the hardening function to use because it only permits plastic volumetric compaction, i.e., the cap is only allowed to expand. As in the above case, a stress state lying on the failure envelope produces dilatant plastic volumetric strains. Since the hardening function defined by Equation 4.7 prescribes no cap movement due to dilatant volumetric strains, the cap is stationary. Thus, the final stress state is ${}^{t+\Delta t}J_1$, i.e., no adjustment is required, and the updated hardening parameter is ${}^{t+\Delta t}\kappa = {}^t\kappa$. If the hardening function takes the form of Equation 4.6, which allows the cap to expand and contract, the cap is adjusted (in this case contracted) to a position prescribed by

$$ {}^{t+\Delta t}l = {}^t l + \left.\frac{\partial l}{\partial \varepsilon_{kk}^p}\right|_{{}^t l} d\varepsilon_{kk}^p \tag{E.13} $$

and a tentative value of ${}^{t+\Delta t}\kappa$ is obtained. The new position of the cap must be compared to the value of ${}^{t+\Delta t}J_1$. If the cap has contracted such that ${}^{t+\Delta t}J_1 < L({}^{t+\Delta t}\kappa) = {}^{t+\Delta t}l$, both ${}^{t+\Delta t}\kappa$ and ${}^{t+\Delta t}J_1$ must be adjusted such that ${}^{t+\Delta t}J_1 = L({}^{t+\Delta t}\kappa) = {}^{t+\Delta t}l$. This is accomplished by starting with the following relation

$$ {}^E J_1 - 3K\,d\varepsilon_{kk}^p = {}^{t+\Delta t}J_1 = L({}^{t+\Delta t}\kappa) = {}^t l + \left.\frac{\partial l}{\partial \varepsilon_{kk}^p}\right|_{{}^t l} d\varepsilon_{kk}^p \tag{E.14} $$

eliminating $d\varepsilon_{kk}^p$, which is the third unknown, by substituting the following

$$d\varepsilon_{kk}^{p} = \frac{{}^{E}J_1 - {}^{t}l}{\left.\frac{\partial l}{\partial \varepsilon_{kk}^{p}}\right]_{{}^{t}l} + 3K} \tag{E.15}$$

from which one can show that

$$\begin{aligned} {}^{t+\Delta t}J_1 &= {}^{E}J_1 - 3K\frac{{}^{E}J_1 - {}^{t}l}{\left.\frac{\partial l}{\partial \varepsilon_{kk}^{p}}\right]_{{}^{t}l} + 3K} \\ &= \frac{{}^{E}J_1\left(\left.\frac{\partial l}{\partial \varepsilon_{kk}^{p}}\right]_{{}^{t}l} + 3K\right) - 3K({}^{E}J_1 - {}^{t}l)}{\left.\frac{\partial l}{\partial \varepsilon_{kk}^{p}}\right]_{{}^{t}l} + 3K} \end{aligned} \tag{E.16}$$

which in turn simplifies to

$${}^{t+\Delta t}l = l({}^{t+\Delta t}\kappa) = {}^{t+\Delta t}J_1 = \frac{\left.\frac{\partial l}{\partial \varepsilon_{kk}^{p}}\right]_{{}^{t}l}{}^{E}J_1 + 3K\,{}^{t}l}{\left.\frac{\partial l}{\partial \varepsilon_{kk}^{p}}\right]_{{}^{t}l} + 3K} \tag{E.17}$$

Having calculated the final value of ${}^{t+\Delta t}J_1$, we must calculate the new components of the deviatoric stress tensor ${}^{t+\Delta t}s_{ij}$. The expressions for ${}^{t+\Delta t}s_{ij}$ are developed below.

Recognizing the path independence of linear elastic constitutive equations, we can write

$${}^{t+\Delta t}s_{ij} = {}^{t}s_{ij} + 2G\,de_{ij}^{e} \tag{E.18}$$

Substituting Equation E.2 into Equation E.18 and performing a simple manipulation one obtains

$$ {}^{t+\Delta t}s_{ij} = {}^{E}s_{ij} - 2\,G\,de^{p}_{ij} \tag{E.19} $$

Recalling Equation 4.22 and recognizing that $\dfrac{\partial f}{\sqrt{J_{2D}}} = 1$, we can write

$$ de^{p}_{ij} = \frac{d\lambda}{2\sqrt{J_{2D}}}\,s_{ij} \tag{E.20} $$

which can be substituted into Equation E.19 giving

$$ {}^{t+\Delta t}s_{ij} = {}^{E}s_{ij} - \frac{d\lambda\,G}{\sqrt{{}^{t+\Delta t}J_{2D}}}\,{}^{t+\Delta t}s_{ij} \tag{E.21} $$

After rearranging the above equation one obtains

$$ {}^{t+\Delta t}s_{ij}\left(1 + \frac{d\lambda\,G}{\sqrt{{}^{t+\Delta t}J_{2D}}}\right) = {}^{E}s_{ij} \tag{E.22} $$

Squaring each side of Equation E.22 and multiplying by ½ produces

$$ {}^{t+\Delta t}J_{2D}\left(1 + \frac{d\lambda\,G}{\sqrt{{}^{t+\Delta t}J_{2D}}}\right)^{2} = {}^{E}J_{2D} \tag{E.23} $$

Taking the square root of each side and rearranging terms, one obtains

$$ \frac{\sqrt{{}^{t+\Delta t}J_{2D}}}{\sqrt{{}^{E}J_{2D}}} = 1 + \frac{d\lambda\,G}{\sqrt{{}^{t+\Delta t}J_{2D}}} \tag{E.24} $$

Replacing the right-hand side of Equation E.24 with the expressions in Equation E.22, one obtains

$$\frac{\sqrt{{}^{t+\Delta t}J_{2D}}}{\sqrt{{}^{E}J_{2D}}} = \frac{{}^{t+\Delta t}s_{ij}}{{}^{E}s_{ij}} \tag{E.25}$$

which may be rewritten for our use as

$${}^{t+\Delta t}s_{ij} = \frac{\sqrt{{}^{t+\Delta t}J_{2D}}}{\sqrt{{}^{E}J_{2D}}}\,{}^{E}s_{ij} \tag{E.26}$$

to calculate the new deviator stress tensor components.

E.4 Cap algorithm

If the failure envelope is not violated by the elastic trial stresses, the trial stresses are checked against the loading function for the cap. If the following conditions exist

$$H({}^{E}J_1, \sqrt{{}^{E}J_{2D}}, {}^{t}\kappa) > 0$$

and

$${}^{E}J_1 < X({}^{t}\kappa)$$

or

$${}^{E}J_1 \leq L({}^{t}\kappa)$$

then the cap algorithm is invoked and the position of the cap is adjusted until

$$H({}^{t+\Delta t}J_1, \sqrt{{}^{t+\Delta t}J_{2D}}, {}^{t+\Delta t}\kappa) = 0$$

An iterative procedure is used in the cap algorithm. To start the procedure, a trial value of $dl^{(i)}$ is assumed in order to calculate a new trial cap position ${}^{t+\Delta t}l^{(i)} = l^{(i)} = {}^{t}l + dl^{(i)}$, where the superscript *i* denotes an iterative value. In addition, trial values of

$\kappa^{(i)}$, $L(\kappa^{(i)})$, $X(\kappa^{(i)})$, and $d\varepsilon_{kk}^{p}$ are computed. Finally, a trial value of J_1 is computed from the following relation

$$J_1^{(i)} = {}^{E}J_1 - 3K\,d\varepsilon_{kk}^{p} \qquad \text{E.27}$$

If $J_1^{(i)} \le X(\kappa^{(i)})$, a smaller value of $dl^{(i)}$ is assumed. If $J_1^{(i)} \ge L(\kappa^{(i)})$, a larger value of $dl^{(i)}$ is assumed. This process is carried on until the condition $L(\kappa^{(i)}) \le J_1^{(i)} < X(\kappa^{(i)})$ is satisfied. The final value of *l* is one which satisfies the following equation to some desired accuracy

$$\sqrt{{}^{t+\Delta t}J_{2D}} + \frac{G\,d\varepsilon_{kk}^{p}}{3\xi} = \sqrt{{}^{E}J_{2D}} \qquad \text{E.28}$$

where

$$\xi = -\left.\frac{\partial F}{\partial J_1}\right|_{J_1^{(i)},\,\kappa^{(i)}} = -\left.\frac{\partial F}{\partial J_1}\right|_{J_1^{(i)},\,l^{(i)}} \qquad \text{E.29}$$

The derivation of Equation E.28 is outlined in the following text.

If we start with Equation 4.22 and substitute for $d\lambda$ using Equation 4.20, one obtains

$$de_{ij}^{p} = s_{ij}\,d\varepsilon_{kk}^{p}\left(\frac{1}{6\sqrt{J_{2D}}}\,\frac{\partial H}{\partial\sqrt{J_{2D}}}\right)\Big/\frac{\partial H}{\partial J_1} \qquad \text{E.30}$$

Recognizing that $f = H$, $\partial H/\partial\sqrt{J_{2D}} = 1$ and $\partial H/\partial J_1 = -\partial F/\partial J_1 = \xi$, we can rewrite Equation E.30 as

$$de_{ij}^{p} = s_{ij}\,\frac{d\varepsilon_{kk}^{p}}{6\,\xi\sqrt{J_{2D}}} \qquad \text{E.31}$$

Substituting the above into Equation E.19 and rearranging gives

$$s_{ij}\left(1+\frac{G\,d\varepsilon_{kk}^{p}}{3\,\xi\sqrt{J_{2D}}}\right)={}^{E}s_{ij} \tag{E.32}$$

Performing the same operations on the above equation as was used on Equations E.22-E.24, one obtains Equation E.28.

The solution of Equation E.28 is obtained through the use of an iterative convergence routine known as the modified regula falsi method (Sandler and Rubin 1979). A dimensionless function $P(l)$ is defined as

$$P(l)=\begin{cases}\dfrac{l(\kappa)-J_1^{(i)}}{l(\kappa)-X(\kappa)} & \text{if } J_1^{(i)}\le X(\kappa)\\[2ex] \dfrac{\sqrt{{}^{E}J_{2D}}-\sqrt{{}^{t+\Delta t}J_{2D}^{(i)}}-\dfrac{G\,d\varepsilon_{kk}^{p}}{3\,\xi}}{\sqrt{{}^{E}J_{2D}}+\sqrt{{}^{t+\Delta t}J_{2D}^{(i)}}+\dfrac{G\,d\varepsilon_{kk}^{p}}{3\,\xi}} & \text{if } X(\kappa)<J_1^{(i)}<L(\kappa)\\[2ex] \dfrac{X(\kappa)-J_1^{(i)}}{L(\kappa)-X(\kappa)} & \text{if } J_1^{(i)}\ge L(\kappa)\end{cases} \tag{E.33}$$

where the solution $P(l)=0$ is also the solution of Equation E.28. If we can show that ${}^{E}J_1<{}^{t+\Delta t}l^{(i)}<{}^{t}l$, then $P(l)$ is bounded and monotonic in the strict sense, and the solution $P(l)=0$ is unique and can be found to any desired degree of accuracy (Sandler and Rubin 1979). An expression for the degree of accuracy or tolerated error is given by

$$\left|\sqrt{{}^{E}J_{2D}}-\sqrt{{}^{t+\Delta t}J_{2D}}-\frac{G\,d\varepsilon_{kk}^{p}}{3\,\xi}\right|<N\,Q[\,X(\kappa)\,] \tag{E.34}$$

where a tolerance of $N/h(\infty,\sqrt{J_{2D}})=10^{-3}$ (in dimensionless format) is typically used.

To show that ${}^{E}J_1 < {}^{t+\Delta t}l^{\,(i)} < {}^{t}l$, we must recognize that $J_1^{(i)} < L\,({}^{t}\kappa)$ must be true, since it is a condition for invoking the cap algorithm. In addition, since

$$J_1^{(i)} = {}^{E}J_1 - 3K\,d\varepsilon_{kk}^{p} \qquad \text{E.35}$$

we know that $J_1^{(i)} > {}^{E}J_1$, because the plastic volumetric strain increment is negative during volumetric compaction, i.e., when the cap expands. This means that the final value of ${}^{t+\Delta t}J_1$ must lie in the range

$${}^{E}J_1 < {}^{t+\Delta t}J_1 < L\,({}^{t}\kappa)$$

Now let us determine the lower limit of ${}^{t+\Delta t}J_1^{(i)}$, which will lead us to the lower limit of ${}^{t+\Delta t}l^{\,(i)}$. The cap exhibits its furthest expansion when ${}^{E}J_1$ is at the intersection of the cap and the failure envelope. When the cap is in this position, ${}^{t+\Delta t}J_1 = L\,({}^{t+\Delta t}\kappa) = {}^{E}J_1$, which implies that the lower bound of ${}^{t+\Delta t}l^{\,(i)}$ is

$${}^{t+\Delta t}l = l\,({}^{t+\Delta t}\kappa) = L\,({}^{t+\Delta t}\kappa) = {}^{E}J_1$$

The upper bound of ${}^{t+\Delta t}l^{\,(i)}$ is simply the value at time *t*, i.e., ${}^{t+\Delta t}l = {}^{t}l$. Combining these expressions, the range of ${}^{t+\Delta t}l^{\,(i)}$ must be

$${}^{E}J_1 < {}^{t+\Delta t}l^{\,(i)} < {}^{t}l$$

With the above conditions satisfied, the solution to Equation E.28 may be obtained. This concludes the description of the cap algorithm. A description of the cap model's response in the tensile regime follows.

E.5 Tensile algorithm

Sandler and Rubin (1979) recognized that soil tensile data is seldom obtained in the laboratory and therefore dealt with tensile behavior in a simple manner. They also cautioned potential users of the simplistic nature of the cap model in the tension regime. A tension failure response is invoked if ${}^{E}J_1 > T$, where *T* is the tension cutoff or limit. Sandler and Rubin recommended that the final stresses be defined as ${}^{t+\Delta t}J_1 = T$ and ${}^{t+\Delta t}s_{ij} = 0$ when the tension cutoff is exceeded. For materials using the definition of the hardening parameter defined by Equation 4.6, the plastic volumetric strain is defined by

$$d\varepsilon^p_{kk} = d\varepsilon_{kk} + \frac{{}^EJ_1 - {}^tJ_1}{3K} \qquad \text{E.36}$$

and an updated hardening parameter is determined. If the tension cutoff is not exceeded but ${}^EJ_1 > 0$, i.e., the elastic trial stress still lies in the tension regime, the stress state must be checked against both the failure envelope and the von Mises transition using the following inequality

$$\sqrt{{}^EJ_{2D}} \geq \min\left\{ Q\left({}^EJ_1\right),\ F\left[L\left({}^t\kappa\right), {}^t\kappa \right] \right\}$$

Stress states violating the von Mises transition must be returned to that surface using the same logic implemented for the failure envelope. Stress states lying on the von Mises transition will produce no plastic volumetric strains due to the imposed normality conditions. In addition, the von Mises transition is fixed because the cap hardening surface does not expand.