The Minimax Control Chart for Multivariate Quality Control

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(ABSTRACT)

A new multicharacteristic control chart designed to detect shifts in the mean of a multivariate process is proposed. It is assumed that the correlation matrix is known and that the distribution of the data is multivariate normal. The new chart is based on the minimum standardized sample mean ($Z_{[1]}$) and the maximum standardized sample mean ($Z_{[p]}$) of p correlated quality variables or characteristics. For this reason the chart has been named the Minimax control chart. A method for calculating probabilities for the joint distribution of $Z_{[1]}$ and $Z_{[p]}$ is developed. This method is used to determine the position of the four control limits of the chart; the upper and lower control limits of $Z_{[1]}$, and the upper and lower control limits of $Z_{[p]}$. The control limits of the chart are determined such that the chart has a fixed probability of Type I error.

The chart's performance is compared to that of the Chi-squared control chart in terms of the average run length for several combinations of the parameters of the chart. Among these parameters are the sample size, the number of variables, the probability of

Type I error, the correlation matrix, and the direction and magnitude of the shift in the mean. The proposed chart outperforms the Chi-squared chart in all the cases studied where the covariance matrix has non-negative elements.

The new chart provides an easy way for diagnosing the system when a signal occurs. That is to say, the chart provides a means to identify the source of the problem when a shift in the mean occurs. The criteria established for diagnosing the system is based on the positions of $Z_{[1]}$ and $Z_{[p]}$ in the Minimax chart. Thus, to diagnose the signals no further analysis is needed. The diagnosing criteria are shown to be particularly effective when the shifts in the mean are either axial or diagonal.

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Chapter 1

Introduction

A control chart is a statistical instrument used to monitor the stability of a process over time. A process for which the output quality is determined by one measurable characteristic is said to be in statistical control if its mean and its standard deviation are not statistically different from their respective expected values. Thus, to decide whether a process is in statistical control, a control chart is used to make sequential tests of hypotheses on the mean and standard deviation of the variable being monitored.

The quality of many products is determined by more than one correlated variables. In these so called multivariate processes, the process is assumed to be in control if the mean vector and the covariance matrix remain stable over time. To test for control in multivariate processes the traditional univariate approach to quality control (i.e., monitoring each variable with a separate control chart) should not be applied. Monitoring multivariate processes using multiple univariate control charts has two disadvantages; namely, raising the overall probability of Type I error, and ignoring the dependence between the variables. Thus, multivariate control charts capable of monitoring all variables in the same test of hypothesis should be used in multivariate problems.

The advantage of monitoring several variables at a fixed probability of Type I error makes the multivariate control charts attractive to practitioners. However, there is a price to pay; multivariate control charts are not usually as easy to use and interpret as univariate control charts. In univariate control charts, a signal is instantaneously interpreted as a shift in the variable that is being measured. Unfortunately, this is not the case in multivariate control charts where a signal is usually interpreted as a shift in an unknown set of the variables being monitored. The identification of the set of variables that have experienced a shift in the mean is usually done after the chart signals. In other words, a signal does not provide information on the identification of the set of variables which have changed. Thus, the interpretation of signals is usually not straightforward in a multivariate problem.

Another practical disadvantage of the available methods to monitor multivariate processes is that the computational work needed to use them usually turns out to be far from trivial. For example, in most multivariate control charts, matrix operations such as transposition, multiplication, and inversion are common. These complexities tend to reduce the use of multivariate control charts in environments where automatization of the data collection and analysis is not economically acceptable.

To address the previously mentioned disadvantages and to meet the formerly described properties of multivariate control charts, the charts should be designed to meet the following criteria:

a) The probability of Type I error must remain fixed for any number of variables.

b) Some type of diagnosing criteria must be available to interpret the signals in the control chart. In other words, when a signal occurs, the source of the problem should be identified after applying and interpreting the preestablished criteria designed for this purpose.

c) The computational work needed to use the chart should be modest enough to allow a quick analysis of each sample in non-automated processes.

In this dissertation a new multivariate control chart that meets these three design criteria is proposed. The new chart has been named the *Minimax control chart* because it is based on monitoring the *maximum standardized sample mean* ($Z_{[p]}$) and the *minimum standardized sample mean* ($Z_{[1]}$) of samples taken from a multivariate process. It is assumed that the data are multivariate normally distributed and that the correlation matrix is known and that it remains constant over time.

The objective of this dissertation is to state how to construct a Minimax control chart such that the three design criteria mentioned above are met, and to evaluate its properties compared to a well known control chart: The Chi-Squared control chart. The charts are compared in terms of the average run lengths and the ability to identify the source of a problem when a signal occurs.

One of the most significant contributions of this dissertation is the development of a method for calculating probabilities for the joint distribution of $Z_{[1]}$ and $Z_{[p]}$. This method is used to determine the position of the four control limits of the chart (the upper and lower control limits $Z_{[1]}$, and the upper and lower control limits of $Z_{[p]}$) such that the chart has a fixed probability of Type I error. An algorithm to calculate the control limits is also provided.

The Minimax chart is designed to detect shifts in the mean of a multivariate process and to diagnose them. The new chart provides an easy way for diagnosing the system when a signal occurs. That is, the chart helps identify the source of the problem when a shift in the mean occurs. A set of criteria is designed such that no further analysis is needed to diagnose the shifts in the mean. Plotting $Z_{[1]}$ and $Z_{[p]}$ in the Minimax control chart is sufficient. The diagnosing criteria are shown to be particularly effective when the shifts in the mean of the process are either in one variable at a time or in all variables at the same time.

The ability of the Minimax chart to detect and diagnose shifts in the mean is studied under different combinations of the following parameters: the sample size, the number of variables, the probability of Type I error of the chart, the magnitude of the shift in the mean, the type of shift in the mean, and the correlation structure of the variables. In addition, the performance of the Minimax control chart is compared to that of the Chi-squared control chart in terms of the out-of-control average run length for several combinations of the parameters of the chart.

The results show that the difference in the out-of-control ARL between the Minimax and the Chi-squared charts tends to decrease as the magnitude of the shift increases. In most of the cases studied the difference in ARL is less than one for shifts in the mean of magnitude greater than 1.5. In addition, the Minimax chart is found to be faster than the Chi-squared chart in detecting shifts in the mean when the variables are non-negatively correlated.

One further advantage of the Minimax chart over the Chi-squared is that $Z_{[1]}$ and $Z_{[p]}$ are plotted on a time axis. Thus, it is possible to make inferences about the process based on time. Another advantage is that the diagnosis of the shifts is made right at the moment of plotting $Z_{[1]}$ and $Z_{[p]}$ on the chart without the need of any other analysis. These two advantages plus a faster detection of shifts in the mean in non-negatively correlated problems make the Minimax chart more attractive than the Chi-squared chart at least for the cases studied in this dissertation.

In conclusion, the Minimax control chart is shown to meet all the requirements of a good multivariate control chart. Specifically, it can be designed to have a fixed probability of Type I error and to diagnose the shifts in the mean while the computational complexity required to use the chart does not go beyond arithmetic. Furthermore, the Minimax control chart outperforms the Chi-squared chart in terms of the out-if-control ARL in the majority of the cases studied.

This dissertation is organized in seven chapters of which this introduction is the first. Chapter 2 contains a summary of the pertinent literature. The formal statement of the problem and its solution are presented in chapters 3 and 4, respectively. Afterwards, Chapter 5 presents an evaluation of the performance of the Minimax chart in terms of the out-of-control ARL, and a comparison against the Chi-squared chart, for several combinations of parameters. Chapters 6 is a summary of the conclusions and findings of this research. The possible future research is presented in Chapter 7.

Chapter 2

Literature Review

This chapter is divided in three sections. In Section 2.1 a summary of the historical development of univariate control charts is provided. The objective of this section is to identify the major contributions made in the field of univariate control charts rather than to give a detailed explanation of the papers cited. Section 2.2 is dedicated to multivariate quality control. Since this is the topic studied in this dissertation, the papers cited in that section are discussed in more detail. In particular, there are two papers that describe approaches to multivariate quality problems that are similar to the approach presented in this dissertation. Thus, these two papers are given the most attention.

The evaluation of multivariate normal probabilities is needed to design the Minimax control chart. For this reason Section 2.3 is dedicated to the multivarite normal distribution. The objective of this section is to provide some references that present different approaches to evaluating these multivariate probabilities. Comments on the precision attained and the highest dimensions that can be evaluated with each method are also provided.

2.1 Control Charts for one Variable Processes

The control chart was originally proposed by Shewhart (1931) and has been widely accepted and applied. Shewhart control charts are mostly used to monitor the mean and the variance of a process. Examples of Shewhart control charts to monitor the mean and the variance are the $\overline{X}$ and the R charts. One disadvantage of Shewhart control charts is that they use only the last sample from the process to decide whether the mean has changed or not. In situations where the shifts in the process are gradual, the last few samples may contain information regarding the process quality. To use this information, Page (1955) introduces warning lines to the Shewhart charts. His idea is that r out of N sample points between the warning lines and the control limits might be evidence of a shift in the process. In fact, he shows that the incorporation of warning lines and another stopping rule (run rule) makes the control chart more efficient. The evaluation of the case where several run rules are used simultaneously is described by Champ and Woodall (1987).

Although Shewhart charts are effective, easy to use, and statistically correct, they perform poorly in the detection of small shifts in the process mean. The CUSUM control chart, due to Page (1954), and the EWMA chart due to Roberts (1959), have proved to be more efficient in the detection of small shifts in the mean and to have similar properties, as described by Lucas and Saccucci (1990). Champ and Woodall (1987) compare the performance of the CUSUM chart with that of the Shewhart chart with run rules. They find that the Shewhart chart is more sensitive to small shifts in the mean when used with run rules but not as good as the CUSUM chart.

In general, the traditional control chart is based on the assumption that the samples are taken in fixed sampling intervals (FSI) of time. Reynolds, Amin, Arnold, and Nachlas (1988) introduce the idea of a variable sampling interval (VSI) applied to Shewhart $\bar{x}$ charts. They show that the VSI $\bar{x}$ chart is more efficient than the FSI counterpart. The underlying intuitive justification of the VSI is that when there is evidence of a possible change in the mean, a sample should be taken sooner than in the case where the sample does not show any evidence of change in the mean. Further studies on VSI charts have been made by Reynolds and Arnold (1989), and Reynolds (1989).

2.2 Multivariate Control Charts

Hotelling (1947, 1951) initiates the study of multivariate quality control by providing a solution to the problem of maintaining a specified probability of Type I error when dealing with multiple variables by means of the T^2 statistic. For a detailed description of Hotelling's T^2 statistic see Appendix B. Ghare and Torgerson (1968) develop a control chart for simultaneously monitoring the mean of two variables when the covariance matrix is known. In particular, they describe how to construct elliptical control regions for bivariate normal random variables using the Q statistic, which is the same as the χ^2 statistic described in Appendix B. Their control chart is different from the Chi-squared control chart in the sense that they can only monitor two variables in a two dimensional plot while the Chi-squared chart can be used to monitor more than two variables. However, the statistic used by both methods is the same, the χ^2 statistic. The difference between the methods is based on the use of the statistic. In the Chi-squared chart the χ^2 statistic is plotted against time. In the method of Ghare and Torgerson the Q (or χ^2) statistic is used to solve for the values of the two variables (x and y) that result in a specified percentile of the χ^2 statistic. The resulting control chart is an ellipse in the x y plane. Note that the Chi-squared control chart permits the observation of the process as a function of time while Ghare and Torgerson's method does not. However, the elliptical shape of Ghare and Torgerson's control chart allows the user to observe both variables in the plot while the Chi-squared chart does not.

Multivariate control charts are, in many cases, extensions of univariate control charts. For example, multivariate versions of the Shewhart, CUSUM and EWMA charts have already been developed. Woodall and Ncube (1985) introduce the idea of a multivariate CUSUM (MCUSUM). They show that for the bivariate case the MCUSUM control chart is preferable to Hotelling's T^2 control chart. They compare the MCUSUM chart to that of Hotelling in terms of the out-of-control ARLs and give a detailed explanation of how to calculate the ARLs. They also conclude that the meaning of an out of control signal is clear if the variables have physical interpretation. Among the authors that have worked in the development of the MCUSUM theory are Alwan (1986), Crosier (1988), and Pignatiello and Runger (1990).

The multivariate EWMA (MEWMA) is introduced by Lowry, Woodall, Champ, and Rigdon (1992). They show that the multivariate EWMA has similar ARL characteristics to the MCUSUM control chart to detect a shift in the mean vector of a multivariate normal distribution. They also conclude that the T^2 control chart should be used in conjunction with the MCUSUM and the MEWMA if not only gradual, but also abrupt shifts in the mean can occur in the process.

Hawkins (1991) proposes Shwehart and CUSUM charts based on a vector of scaled residuals from the regression of each variable on all others. This method is particularly useful in cascade-type processes. The cascade effect refers to the situation in which a product is manufactured in a sequence of processes. Thus, if process j departs from control, the remaining processes (j+1, j+2, ...) could be affected by this shift, but not the previous ones. In this paper he states that the assumption of a known covariance matrix is of little harm if a sufficiently large sample is used to estimate the matrix. Although no ARL comparisons are made by Hawkins, his method provides a good separation of location from scale when a signal occurs and immediate association of signals with particular variables. Separation of location from scale is a characteristic that many other multivariate control charts do not have since these charts are usually developed to detect shifts in the mean and not in the covariance structure of the process.

One common approach to multivariate problems is to transform the correlated data to uncorrelated data using the eigenvalues and eigenvectors of the data matrix. This approach is usually referred as the principal components method. The method, as presented by Jackson and Mudholkar (1979) and by Jackson (1980), assumes that the data is multivariate normal and that the correlation matrix is known. Their method uses the Q statistic (Q = sum of squares of the residuals) to test whether the method can be applied, and the T^2 statistic (not Hotelling's T^2) to test for control. If the method can be applied and the process happens to be out of control, then the individual principal components are tested to determine the nature of the disturbance on the process. In this way the signals are diagnosed. The typical criticism of this method is that the linear combinations of variables used to normalize the data, though independent from each other, are not easy to interpret. Thus, the diagnosis sometimes has no real meaning in terms of what should be done to correct the problem.

In some quality control applications the quality characteristic cannot be measured directly because of cost limitations, destructive inspection, or because the quality characteristic is impossible to measure. In these situations the typical approach is to measure some other characteristics of the process that are correlated to the variable of interest. This process of measuring indirect variables instead of the characteristic of interest is called screening. Tang (1988) and Tang and Tang (1989) use linear regression models to predict the behavior of the quality characteristic by using the screening variables as the independent variables. They set the process to be out of control when the cost of the shift in the mean exceeds a fixed value c. Latter, Drezner and Wesolowsky (1995) expand their work by relaxing the assumptions of equal cost for different directions of the shifts in the mean of the process.

Tracy, Young, and Mason (1992) provide an exact method based on the beta distribution for calculating the probabilities needed to find the limits of a multivariate control chart for individual observations (n=1). This method is particularly useful when a process is in its initial stage and only individual observations are available. Recently, Mason, Tracy, and Young (1995) have presented another approach to identify the source of signals when a T^2 control chart is used and the sample size is one. Their approach consists of decomposing the T^2 statistic into independent components, each of them resulting in a statistic that is similar to an individual T^2 variate. They show that Hawkins' (1993) regression adjustment method, as well as the ranking technique of Doganaksoy, Faltin, and Tucker (1991), are subsets of their procedure.

Another approach to solve the problem of identification in multivariate problems is that proposed by Hayter and Tsui (1994). They use simultaneous confidence intervals for the means being tested that supposedly identify the variables responsible for the shift in the process, while a specified α value is maintained. Their procedure triggers an alarm when

$$M = \max_{1 \le i \le p} \frac{|x_i - \mu_i^0|}{\sigma_i} > C_{R,\alpha},$$

where $C_{R,\alpha}$ is a (1-α)% critical point which should be determined using multivariate normal probabilities. The content of this paper is similar to the research presented in this dissertation and is therefore examined in greater detail. The method they present is limited to less than five variables since they do not provide a practical method to evaluate the multivariate normal distribution. This dissertation (see Section 2.3) presents a better method for evaluating multivariate normal probabilities. Another limitation of their control chart is that they do not provide any method to evaluate their intuitive rule for identification of signals. Surprisingly, they assume that a signal will always identify the source of the shift in the process. Section 4.5 in this dissertation shows that this assumption is not true. Another limitation found in their paper is that they cannot compare the out-of-control ARL of their control chart to that of the Chi-squared because they would need the distribution of $|Z_{[p]}|$ when the mean vector is different from zero and they do not have it. The distribution of $|Z_{[p]}|$ can be evaluated for any shift in the mean and any correlation structure using the joint distribution of $Z_{[1]}$ and $Z_{[p]}$ developed in this dissertation. Finally, they only use one control limit in their chart ignoring that placing upper and lower control limits on their statistic could result in smaller out-of-control ARL's.

Timm (1996) presents another approach to the multivariate problem. The method is called Finite Intersection Tests (FIT). This procedure is designed to signal when the process is out of control, but not to diagnose the system. He defines the process to be out of control if

$$\max_{1 \le i \le p} |T_i| = \frac{|x_i - \mu_{oi}|}{\sigma_{ii}^{1/2}} > T^2_{p,\alpha},$$

where $T^2_{p,\alpha}$ is the upper α critical value of the multivariate t distribution with df = ∞. This approach is similar to the one presented in this dissertation because it defines a signal in terms of the maximum of the absolute value of the standardized sample means. However, Timm also shows that the method presented by Hayter and Tsui in 1994 is an FIT. Accordingly, as in Hayter and Tsui, he does not provide an evaluation of out-of-control ARLs to compare his method to other existing multivariate control charts, because he lacks the probability distribution of $|Z_{[p]}|$ when the mean vector is different from zero.

The literature studied here shows that no approach similar to the one presented in this dissertation has yet been developed. Thus, the Minimax control chart is a new approach to the multivariate quality control problem and its study is worthwhile.

2.3 The Multivariate Normal Distribution

Multivariate control charts such as that proposed by Hayter and Tsui (1994) depend on the evaluation of multivariate normal probabilities. This is usually a limitation of these charts because the numerical integration of multivariate normal probabilities, even using Monte Carlo methods, is usually limited to 5 or 6 variables as described by Deák (1978), and by Hayter and Tsui (1994). Deák (1980) develops a method to rapidly evaluate multivariate normal probabilities of higher dimensions. Using his approach Deák can evaluate three significant digit probabilities of ten dimensions in five seconds, up to twenty dimensions in one minute, and up to fifty dimensions in ten minutes.

Even though many methods have been developed to calculate multivariate normal probabilities, (e.g. Milton (1972), Dutt (1973), and Deák (1976, 1980, 1986)), none of these can assure a fast and precise calculation of the probabilities. Ducrocq and Colleau (1986) analyze the precision of the methods described by Dutt (1973) and Deák (1976, 1980, 1986). They conclude that Dutt's method is remarkably precise for dimensions 1 to 5, except when there are high-absolute-valued correlations, and that Deák's method is less precise though better suited for higher dimensions (6 to 20). However, there is no assurance that any of the methods will always provide more than three digits of precision. In addition, these methods have proved to loose precision when the truncation points of the integral are not in the set (-3, 3).

Y. L. Tong (1990) describes a method to reduce the normal multiple integral to one integral given that the correlation matrix has a particular structure. His method is summarized as follows:

Let the correlation matrix ρ of the vector $\mathbf{x}$ be such that $\rho_{ij} = \lambda_i \lambda_j$ for $\lambda_i \in [-1,1]$ and for $i \neq j$, $i=1, 2, \ldots, p$. Also let $\mathbf{x} \sim N_p(\mu, \Sigma)$ i.e., $\mathbf{x}$ is multivariate normal with mean μ and covariance matrix Σ, and let $Z_1, Z_2, \ldots, Z_p$ be i.i.d. N(0,1). Finally, define a p dimensional rectangle A as:

$$A=\{\mathbf{x}: \mathbf{x} \in \mathbb{R}^n, b_i \leq x_i \leq a_i, i=1, 2, \ldots, p\},$$

where $-\infty \leq b_i < a_i \leq \infty$, $i=1, 2, \ldots, p$. Now, the probability integral can be expressed as:

$$P[\mathbf{x} \in A] = \int_{-\infty}^{\infty} \prod_{i=1}^{p} \left[\Phi\left(\frac{(a_i - \mu_i)/\sigma_i + \lambda_i z}{\sqrt{1-\lambda_i^2}} \right) - \Phi\left(\frac{(b_i - \mu_i)/\sigma_i + \lambda_i z}{\sqrt{1-\lambda_i^2}} \right) \right] \phi(z)\,dz,$$

where Φ and ϕ are the N(0,1) distribution function and density function, respectively.

This method is used in this dissertation and it is found that the time saved by making the reduction of dimension is acceptable for small dimensional integrals (20 or fewer variables). However, for large dimensions, the integrand becomes too large. This leads to two problems; namely, the need for large amounts of computer memory to solve the integrals, and an increase in the computer time needed to evaluate the integral. Even with these limitations, this method of evaluating multivariate normal probabilities is faster and more precise than that recommended by Hayter and Tsui (1994). Information on the implementation and availability of software to compute multidimensional integrals can be found in Kahaner (1991).

Chapter 3

Problem Statement

3.1 Notation, Assumptions, and Definitions

A new multicharacteristic control chart to monitor the mean of p measurable quality characteristics (or variables) represented by $\mathbf{X} = (X_1, X_2, \ldots, X_p)^T$ is proposed[1]. Assume that the distribution of $\mathbf{X}$ is multivariate normal with known mean vector $\mu = (\mu_1, \mu_2, \ldots \mu_p)^T$ and covariance matrix Σ with elements σ_{ij}^2, where σ_{ii}^2 represents the variance of the i^{th} variable. Also assume that only changes in μ are possible and that Σ remains constant over time.

Samples of size n are taken every t units of time. Each sample consists of measuring p characteristics of each of the n items sampled. Note that the assumption of a constant Σ implies that the k^{th} observation (or sample) is independent of the $(k+1)^{st}$ even when all variables are mutually correlated. This assumption of independent successive observations might not always be realistic. However, many well known control charts (e.g. the $\overline{X}$ chart) rely on the same assumption which is usually supported by the fact that

[1] All vectors in this dissertation are boldface and are defined as column vectors.

samples are separated in time. In summary, correlation is permitted within observations but not between observations, where an observation consists of the inspection of one sampled product or item.

As an example, consider a particular product whose quality is determined by p=3 quality variables: height, width, and length. A sample of size n=5 items is taken every hour and measurements of the three characteristics are recorded for each sample item. The nature of the production process causes the three variables to be correlated. However, the process operates such that successive items are uncorrelated. That is, the quality of the $i^{\underline{th}}$ item is independent of that of the $(i+1)^{\underline{st}}$. An example of the data matrix for a similar case is shown in Figure 3.1. In this matrix, the variable X_{ij} represents the measurement of the variable i of item j.

$$
\begin{array}{r|cccc|c}
 & \underline{\mathit{Item}\ 1} & \underline{\mathit{Item}\ 2} & \cdots & \underline{\mathit{Item}\ n} & \\
\hline
X_1 = \mathit{Length} & X_{11} & X_{12} & \cdots & X_{1n} & \overline{X}_1 \\
X_2 = \mathit{Width} & X_{21} & X_{22} & \cdots & X_{2n} & \overline{X}_2 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
X_p = \mathit{Height} & X_{p1} & X_{p2} & \cdots & X_{pn} & \overline{X}_p \\
\hline
\end{array}
$$

Figure 3.1 - An example of a Data Matrix

The process is assumed to have sudden shifts in the mean such that the new mean μ' is given by

$$\mu' = \mu + \Delta \odot \sigma, \tag{3.1}$$

where σ is the standard deviation vector and the vector Δ represents the magnitude of the shift from the mean vector measured in standard deviations.[2] That is, the components of the vector Δ are of the form

$$\Delta_i = \frac{(\mu'_i - \mu_i)}{\sigma_{ii}}. \tag{3.2}$$

For example, for p = 3, a 1.5 standard deviation change in μ_1 is represented by $\Delta = (1.5, 0, 0)^T$.

Whenever $\mu' = \mu$ (i.e. $\Delta = \mathbf{0}$) the process is said to be in statistical control. To decide whether the process is in statistical control or not, the *Minimax control chart* is proposed as a method to test the following hypothesis in each sample:

$$H_0\colon \Delta = \mathbf{0}$$

$$H_1\colon \Delta \neq \mathbf{0}. \tag{3.3}$$

A detailed description of how to use and interpret the Minimax chart is given in the next section.

---

[2] In equation (3.1) the operator $\odot$ is known as the Hadamard product of two vectors and signifies that the two vectors are multiplied element-wise. In general, if $A = [a_{ij}]$ and $B = [b_{ij}]$ are two matrices of equal dimension, then $A \odot B = [a_{ij}\, b_{ij}]$.

3.2 The Minimax Control Chart

3.2.1 Standardization of the Mean

The principal idea behind the Minimax control chart is to standardize all p means and to monitor the maximum and the minimum of those standardized sample means. To do this, the sample average vector $\overline{\mathbf{X}} = \left(\overline{X}_1,\ \overline{X}_2,\ \dots\ \overline{X}_p\right)^T$ is calculated and its elements are standardized using equation (3.4):

$$Z_i = \frac{\overline{X}_i - \mu_i}{\sigma_{ii} / \sqrt{n}}. \tag{3.4}$$

The vector $\mathbf{Z} = [Z_i]$ is now defined as the *standardized sample mean vector*.

Next, define the *maximum standardized sample mean*[3] ($Z_{[p]}$) as the maximum of the elements of the vector $\mathbf{Z}$, or

$$Z_{[p]} = \max_i (Z_i),\ i = 1, 2, \dots, p. \tag{3.5}$$

Similarly, define the *minimum standardized sample mean* ($Z_{[1]}$) as the minimum of the elements of the vector $\mathbf{Z}$, or

$$Z_{[1]} = \min_i (Z_i),\ i = 1, 2, \dots, p. \tag{3.6}$$

---

[3] Subscripts in Squared brackets are used to represent order statistics.

It should be clear from (3.4), (3.5), and (3.6) that $Z_{[p]}$ and $Z_{[1]}$ represent the variables in the sample with respective maximum and minimum standardized deviations from their respective mean. Note that $Z_{[1]}$ does not necessarily represent the standardized value of X_1 because $Z_{[i]}$ is an order statistic. To keep track of the correspondence between X_j and $Z_{[i]}$, define Ψ_i as a pointer that points from $Z_{[i]}$ to the variable that it represents (X_j). This pointer is formally expressed as:

$$\Psi_i = \{j : Z_{[i]} = Z_j;\ i, j = 1,\ldots,p\}. \tag{3.7}$$

Now, Ψ_1 and Ψ_p respectively point to the variables that exhibit the smallest and largest standardized sample means.

For example, consider again the case where X_1=length, X_2=width, and X_3=height of a particular item. Let the mean vector μ = (10, 15, 5) and variance vector σ^2=(0.02^2, 0.10^2, 0.01^2). A sample of size 5 (n=5) is taken and analyzed. Table 3.1 shows the resulting data.

Table 3.1- Simulated Sample Data for p=3

variable	μ	σ	observations					$\overline{X}$	Z
X_1	10	0.02	10.013	9.981	9.985	10.004	9.998	9.996	-0.425
X_2	15	0.10	15.014	14.981	14.991	15.077	14.962	15.005	0.112
X_3	5	0.01	5.009	5.007	4.997	5.004	5.005	5.004	0.984

The resulting average vector is $\overline{\mathbf{X}} = (9.996,\ 15.005,\ 5.004)^T$. Now, the components of the standardized sample mean vector **Z** are calculated using equation (3.4). For example: $Z_1 = (9.996 - 10) / (0.02 / 5^{1/2}) = -0.425$. The resulting vector is $\mathbf{Z} = (-0.425, 0.112, 0.984)^T$. Then, applying (3.5) and (3.6), or simply ordering the elements of **Z**, results in $Z_{[1]} = -0.425$, $Z_{[2]} = 0.112$, and $Z_{[p]} = Z_{[3]} = 0.984$. Note that the first and third pointers will point to the variables with respective minimum and maximum standardized sample means. Applying equation (3.7) gives: $\Psi_2 = 2$, $\Psi_1 = 1$, and $\Psi_3 = 3$. It is fairly reasonable to say that Ψ_1 or Ψ_3 point to the variable(s) whose means have the highest probability of having changed. However, more information is needed to decide whether the mean of any of these variables has indeed changed. The next section gives more details on the construction of the Minimax chart.

3.2.2 Detection of Shifts in the Mean

The Minimax chart consists of monitoring $Z_{[1]}$ and $Z_{[p]}$ by placing upper and lower control limits on both of these statistics as shown in Figure 3.2. Denote the upper control limit on $Z_{[p]}$ by $UCL_{[p]}$ and the lower control limit on $Z_{[p]}$ by $LCL_{[p]}$. Similarly, denote the respective upper and lower control limits on $Z_{[1]}$ by $UCL_{[1]}$ and $LCL_{[1]}$. These control limits have to be calculated such that the chart has a predefined probability of Type I error (α). In other words, the parameters α_1, α_2, α_3, and α_4 shown in Figure 3.2 have to be set such that the chart has a particular probability of Type I error (this will be discussed in Chapter 4). These four parameters are formally defined as:

$$\alpha_1 = P[Z_{[1]} \in (-\infty, LCL_{[1]}) \mid \mathbf{\Delta} = \mathbf{0}] \tag{3.8}$$

$$\alpha_2 = P[Z_{[1]} \in (UCL_{[1]}, \infty) \mid \mathbf{\Delta} = \mathbf{0}] \tag{3.9}$$

$$\alpha_3 = P[Z_{[p]} \in (-\infty, LCL_{[p]}) \mid \mathbf{\Delta} = \mathbf{0}] \tag{3.10}$$

$$\alpha_4 = P[Z_{[p]} \in (UCL_{[p]}, \infty) \mid \mathbf{\Delta} = \mathbf{0}], \tag{3.11}$$

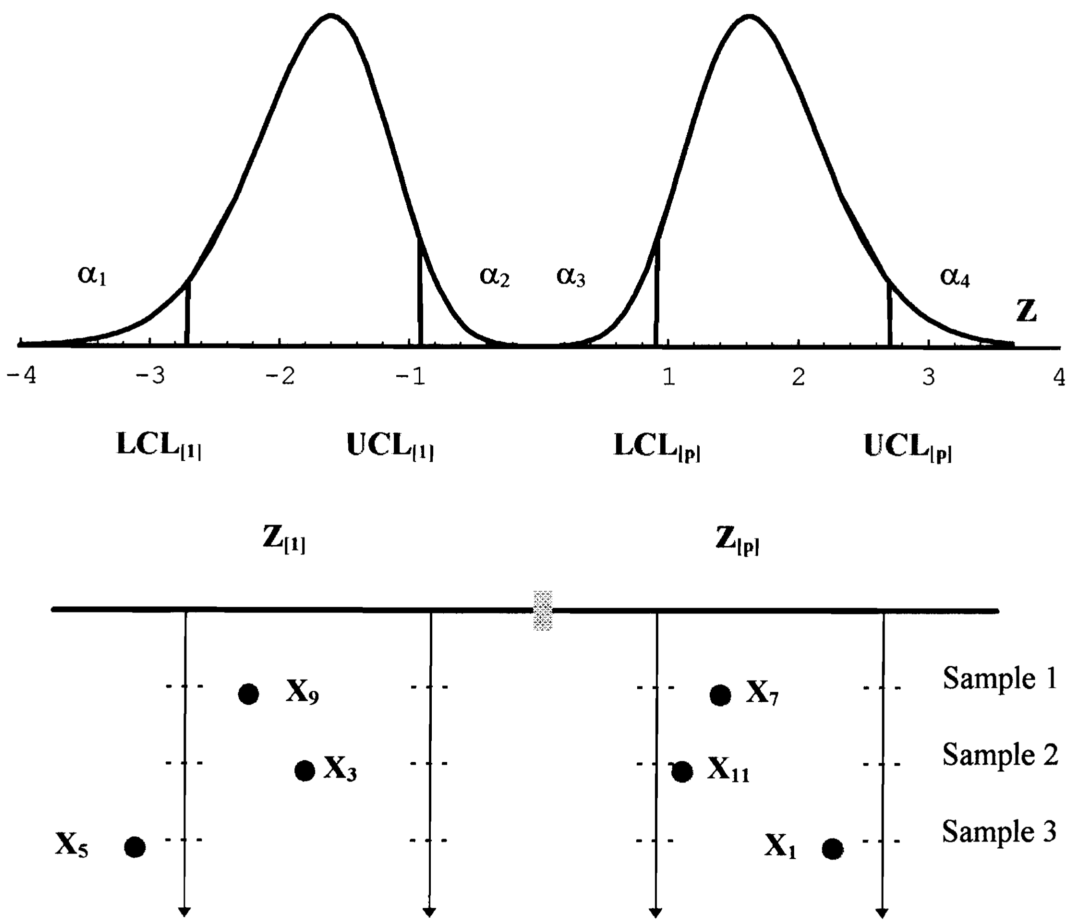

Figure 3.2 - The Minimax Control Chart

The graphical representation of the Minimax chart shown in Figure 3.2 is based on density functions of $Z_{[1]}$ and $Z_{[p]}$ for p=15 variables. In this case, the chart has non-overlapping control limits. However, for small values of p, the control limits on $Z_{[1]}$ and $Z_{[p]}$ might overlap. In that case, the two charts should be graphed on different axes. In any case, the control chart consists of plotting $Z_{[1]}$ and $Z_{[p]}$ and deciding whether they lie inside their respective control limits.

Suppose that a sequence of samples is taken from a process and that the resulting plots of $Z_{[1]}$ and $Z_{[p]}$ in the Minimax chart are as shown in Figure 3.2. In this example samples 1 and 2 lie inside the control limits, while sample 3 lies outside. In particular $Z_{[1]}$ lies below $LCL_{[1]}$. This *signal* should be interpreted as an out-of-control situation. In this case, it should be concluded that the mean of $X_{\Psi_1} = X_5$ has decreased. Similar to this type of signal, points in other regions of the control chart have their respective intuitive interpretations. A general discussion of the ways the different signals should be interpreted is given in the next section.

3.2.3 Diagnosis of Signals

As mentioned in Chapter 1, a good multicharacteristic control chart must not only identify shifts in the mean, but should also give some information about the shift that has occurred when the chart signals. In this way a diagnosis can be made for the system. In other words, after the chart signals it is known that something is wrong in the process, but the cause of the signal is still unknown. Then, it is necessary to make a *diagnosis*; that is, to predict which of the p variables have experienced a shift in the mean.

The Minimax chart is designed to signal under the presence of any type of shift in the mean. However, to simplify the process of diagnosing the system when a signal occurs, it is assumed that only three types of shifts in the mean occur in the process. These types of shifts are introduced in the next paragraph.

The criteria used to diagnose a signal of the Minimax chart are dependent upon the types of shifts that can occur in the process being monitored. The most common changes in direction of the mean studied in the literature are axial and diagonal shifts. The term *axial shift* suggests that only one variable's mean changes; for example, a change in mean in the i^{th} variable which is represented as $\Delta = (0, 0, \ldots, \Delta_i, \ldots, 0)^T$. *Diagonal shifts* are those where all means change by the same magnitude. These shifts are represented as $\Delta = (\Delta_1, \Delta_2, \ldots, \Delta_p)^T$, $|\Delta_i| = |\Delta_j| \ \forall \ (i, j)$. A special case of diagonal shifts is when $\Delta_i = \Delta_j$, and Δ_i

$> 0 \ \forall$ (i, j). These are called *positive diagonal shifts* in the mean because all means experience the same positive shift. Similarly, *negative diagonal shifts* in the mean are those where $\Delta_i = \Delta_j$, and $\Delta_i < 0 \ \forall$ (i, j).

In general, when any type of shift in the mean occurs in a process, the variable that causes the signal (X_{Ψ_p} or X_{Ψ_1}) is expected to be out-of-control. However, there will be situations in which several variables, but not all, experience a shift in the mean. In these situations, the Minimax chart is expected to identify, at most, two of these variables; those pointed by X_{Ψ_1} and X_{Ψ_p} when a signal occurs. However, if it is assumed that the only shifts that can occur in the process mean are axial, positive diagonal, or negative diagonal, then the Minimax chart is a dependable tool for diagnosing the signals in the process. This will be discussed during this dissertation.

Again, note that this assumption on the types of shifts in the mean is made just to evaluate the ability of the chart to diagnose the system after a signal has occurred and thus, this assumption has no effect on the way that the Minimax is designed. In other words, the Minimax chart is going to be designed to signal under the presence of any type of shift in the mean of the process.

The following definitions are used to explain how the Minimax chart can be used to diagnose signals when the process experiences axial, positive diagonal, or negative diagonal shifts in the mean.

There are only nine events that can occur in a Minimax chart when $Z_{[1]}$ and $Z_{[p]}$ are plotted. These events are the combinations of the three possible outcomes for $Z_{[1]}$ and $Z_{[p]}$. Based on the various positions, with respect to the control limits, in which $Z_{[1]}$ can lie, the following outcomes are possible:

a - <u>a</u>bove its upper control limit,

b - <u>b</u>elow its lower control limit,

c - inside its <u>c</u>ontrol limits,

and for any position of $Z_{[1]}$, $Z_{[p]}$ has the same three possible outcomes (a, b, or c). Thus, there are $3\times3 = 9$ possible events. Denote by $E_{u,v}$ the **E**vent where $Z_{[1]}$ lies in position u and $Z_{[p]}$ lies in position v, where u, v $\in$ {a, b, c}. For example, $E_{a,c}$ is the event where $Z_{[1]}$ lies <u>a</u>bove its upper control limit and $Z_{[p]}$ is in <u>c</u>ontrol. Now, define nine mutually exclusive and collectively exhaustive events based on the position of $Z_{[1]}$ and $Z_{[p]}$ in the Minimax chart as:

$$E_{b,b} = \left\{Z_{[1]} \in \left(-\infty,\ LCL_{[1]}\right)\right\} \cap \left\{Z_{[p]} \in \left(-\infty,\ LCL_{[p]}\right)\right\} \tag{3.12}$$

$$E_{b,c} = \left\{Z_{[1]} \in \left(-\infty,\ LCL_{[1]}\right)\right\} \cap \left\{Z_{[p]} \in \left(LCL_{[p]},\ UCL_{[p]}\right)\right\} \tag{3.13}$$

$$E_{b,a} = \left\{Z_{[1]} \in \left(-\infty,\ LCL_{[1]}\right)\right\} \cap \left\{Z_{[p]} \in \left(UCL_{[p]},\ \infty\right)\right\} \tag{3.14}$$

$$E_{c,b} = \left\{Z_{[1]} \in \left(LCL_{[1]},\ UCL_{[1]}\right)\right\} \cap \left\{Z_{[p]} \in \left(-\infty,\ LCL_{[p]}\right)\right\} \tag{3.15}$$

$$E_{c,c} = \left\{Z_{[1]} \in \left(LCL_{[1]},\ UCL_{[1]}\right)\right\} \cap \left\{Z_{[p]} \in \left(LCL_{[p]},\ UCL_{[p]}\right)\right\} \tag{3.16}$$

$$E_{c,a} = \left\{Z_{[1]} \in \left(LCL_{[1]},\ UCL_{[1]}\right)\right\} \cap \left\{Z_{[p]} \in \left(UCL_{[p]},\ \infty\right)\right\} \tag{3.17}$$

$$E_{a,b} = \left\{Z_{[p]} \in \left(-\infty,\ LCL_{[p]}\right)\right\} \cap \left\{Z_{[1]} \in \left(UCL_{[1]},\ \infty\right)\right\} \tag{3.18}$$

$$E_{a,c} = \left\{Z_{[p]} \in \left(LCL_{[p]},\ UCL_{[p]}\right)\right\} \cap \left\{Z_{[1]} \in \left(UCL_{[1]},\ \infty\right)\right\} \tag{3.19}$$

$$E_{a,a} = \left\{Z_{[p]} \in \left(UCL_{[p]},\ \infty\right)\right\} \cap \left\{Z_{[1]} \in \left(UCL_{[1]},\ \infty\right)\right\} \tag{3.20}$$

Note that, excepting $E_{c,c}$, all the previous events represent a signal in the Minimax chart. Thus, there are eight different possible types of signals. From these signals, there are two events that seem to be less probable under the assumption of axial, positive diagonal, or negative diagonal shifts in the mean. These events are $E_{b,a}$ and $E_{a,b}$. Note that these two events represent situations where at least two variables have changed in different directions. Thus, the probability of occurrence of $E_{b,a}$ and $E_{a,b}$ are expected to be small compared to the other six events. The other six are expected to have a higher probability of occurrence and have a reasonably intuitive interpretation which is discussed in detail in the remainder of this section. A formal probabilistic evaluation of the interpretation of these signals is presented in Section 4.5.

Since only axial, positive diagonal, or negative diagonal shifts in the mean are considered for the interpretation of signals, these types of shifts are now going to be matched to the events that appear to have the highest probability of representing them when a signal occurs.

Axial Shifts:

A positive shift in the mean of one variable along its axis should result in a large $Z_{[p]}$ caused by the variable that experienced the shift. Thus, it is expected that the case $E_{c,a}$ corresponds to a positive axial shift in the mean of the variable that gives the signal (X_{Ψ_p}). Similarly, $E_{b,c}$ is expected to correspond to a negative axial shift in the mean of the variable that gives the signal (X_{Ψ_1}).

Diagonal Shifts:

A positive shift in the means of all variables should result in a large $Z_{[1]}$ caused by any of the variables that experienced the shift. In other words, since the means of all the variables have increased, the minimum of all observations should be large compared to the minimum of the in-control case. Thus, it is expected that $E_{a,c}$ corresponds to a positive diagonal shift in the mean. Note that this shift is **not** exclusively caused by the variable that gives the signal (X_{Ψ_1}) but by all variables because the shift is diagonal. Another case that can be interpreted as a positive diagonal shift in the mean is $E_{a,a}$. In this case, the diagonal shift caused both a large minimum and a large maximum. Similarly, $E_{c,b}$ and $E_{b,b}$ should be interpreted as possible consequences of a negative diagonal shift in the mean.

If these interpretations are correct, then the Minimax control chart is a simple solution to two multivariate quality problems: detecting the absence of control and diagnosing the apparent quality problem (i.e., identifying the set of variables responsible for the shift when a signal occurs). The formal probabilistic evaluation of the correctness of these interpretations of the signals of the Minimax control chart is discussed in Section 4.5. Afterwards, some numerical examples are evaluated in Chapter 5.

The operation of the Minimax chart and the interpretation of its different types of signals have been discussed in order to provide the reader with a good idea of what the remainder of this dissertation is all about. The next section provides a more general summary of the scope of this dissertation.

3.3 Scope of the Research

Chapter 4 includes the statistical properties of the Minimax chart which are summarized as:

- A general algorithm to determine the position of the control limits that result in a specified probability of Type I error of the chart.

- The mathematical development that leads to the determination of the probability of Type II error for the chart.
- The expressions needed to determine the probability of correctly diagnosing the system when a signal occurs.

Afterwards, in Chapter 5, the ability of the Minimax chart to detect shifts in the mean and to correctly diagnose the system when a signal occurs are evaluated under several experimental conditions. After all the experimental charts are evaluated and the data gathered are analyzed, the following questions are answered:

1. For how many variables can we provide a fast calculation of the control limits of the Minimax control chart?

2. How should α_1, α_2, α_3, and α_4 be assigned to enhance the chart's performance depending on the magnitude and direction of the shift to be detected and on the correlation matrix ρ?

3. How does the performance of the Minimax chart compare to that of the Chi-Squared control chart? See Appendix B for an introduction to the Chi-Squared chart. The

comparison should be done in terms of the out-of-control average run length (or the expected time to signal given that a shift has occurred).

4. How should n be selected such that the chart has a particular probability of Type II error for a given shift in the mean?

5. Can the Minimax chart be used to identify the source of a problem after a signal occurs? To what extent or under which circumstances can the identification of the problem be assured?

Chapter 4

Statistical Properties of the Minimax Control Chart

In this Chapter the statistical properties of the Minimax chart are mathematically developed. After reading this chapter, the reader should understand the construction, use, and evaluation of the performance of the Minimax control chart.

4.1 Expected Value, Variance, and Covariance of the Standardized Sample Mean

The properties of the statistic Z_i are very important for the development of the Minimax control chart. Thus, expressions for its expected value, variance, and covariance are now developed. First note that

$$E[Z_i] = E\left[\frac{\bar{x}_i - \mu_i}{\sigma_{ii}/\sqrt{n}}\right] = \frac{E[\bar{x}_i - \mu_i]}{\sigma_{ii}/\sqrt{n}} = \frac{E[\bar{x}_i] - E[\mu_i]}{\sigma_{ii}/\sqrt{n}} = \frac{\mu_i' - \mu_i}{\sigma_{ii}/\sqrt{n}} = \sqrt{n}\Delta_i \tag{4.1}$$

and thus, when the process is in control,

$$E[\mathbf{Z}] = 0. \tag{4.2}$$

Secondly, the variance of Z_i is given by

$$\mathrm{Var}[Z_i] = \mathrm{Var}\left[\frac{\bar{X}_i - \mu_i}{\sigma_{ii}/\sqrt{n}}\right] = \mathrm{Var}\left[\frac{\bar{X}_i}{\sigma_{ii}/\sqrt{n}}\right] = \frac{\mathrm{Var}[\bar{X}_i]}{(\sigma_{ii}/\sqrt{n})^2} = \frac{\sigma_{ii}^2}{n\sigma_{ii}^2/n} = 1, \tag{4.3}$$

and finally, the covariance between Z_i and Z_j is

$$
\begin{aligned}
\mathrm{Cov}[Z_i, Z_j] &= E[Z_i Z_j] - E[Z_i]E[Z_j] \\
&= E[Z_i Z_j] = E\left[\left(\frac{\overline{X}_i - \mu_i}{\sigma_{ii}/\sqrt{n}}\right)\left(\frac{\overline{X}_j - \mu_j}{\sigma_{jj}/\sqrt{n}}\right)\right] \\
&= \frac{E\left[(\overline{X}_i - \mu_i)(\overline{X}_j - \mu_j)\right]}{\sigma_{ii}/\sqrt{n}\,\sigma_{jj}/\sqrt{n}} = \frac{E\left[\overline{X}_i\,\overline{X}_j - \overline{X}_i\,\mu_j - \mu_i\,\overline{X}_j + \mu_j\,\mu_i\right]}{\sigma_{ii}\,\sigma_{jj}/n} \\
&= \frac{E\left[\overline{X}_i\,\overline{X}_j\right] - E\left[\overline{X}_i\right]\mu_j - \mu_i\,E\left[\overline{X}_j\right] + \mu_j\,\mu_i}{\sigma_{ii}\,\sigma_{jj}/n} = \frac{E\left[\overline{X}_i\,\overline{X}_j\right] - \mu_j\,\mu_i}{\sigma_{ii}\,\sigma_{jj}/n} \\
&= \frac{\mathrm{Cov}\left[\overline{X}_i\,\overline{X}_j\right]}{\sigma_{ii}\,\sigma_{jj}/n} = \frac{\mathrm{Cov}\left[X_i\,X_j\right]}{n\,\sigma_{ii}\,\sigma_{jj}/n} \\
&= \rho_{ij}
\end{aligned}
\tag{4.4}
$$

where ρ_{ij} is the correlation coefficient for X_i and X_j. In developing the expression for $\mathrm{Cov}[Z_i, Z_j]$, the following fact is used:

If $\mathbf{X}_1 = (X_{11}, X_{12}, \ldots, X_{1n})^T$ and $\mathbf{X}_2 = (X_{21}, X_{22}, \ldots, X_{2n})^T$ respectively represent samples of size n for the variables X_1 and X_2, and X_{1i} and X_{2j} are dependent for i = j and independent otherwise, then

$$
\mathrm{Cov}\left[\overline{X}_1, \overline{X}_2\right] = \frac{\mathrm{Cov}\left[X_1, X_2\right]}{n}, \tag{4.5}
$$

where $\overline{X}_k = \sum_{l=1}^{n} \frac{X_{kl}}{n}$.

The proof of this result is shown in Appendix A.

4.2 The distributions of $Z_{[1]}$ and $Z_{[p]}$

Another important point in the development of the Minimax control chart is the marginal distributions of $Z_{[1]}$ and $Z_{[p]}$, because these distributions are needed to determine the control limits of the chart. In this section, these marginal distributions are defined for two cases: p independent variables and p dependent variables.

Note that, in general, since $Z_{[p]} = \max_i (Z_i)$, i = 1, 2, ..., p, then $Z_i \leq Z_{[p]}\ \forall$ i. Thus, the cumulative probabilities of $Z_{[p]}$, where $Z_{[p]}$ is simply the maximum of p observations, can be calculated using the following expression:

$$P[Z_{[p]} \leq \gamma] = P[Z_1 \leq \gamma, Z_2 \leq \gamma, \ldots, Z_p \leq \gamma] \tag{4.6}$$

where Z_1, Z_2, ..., Z_p are p standard normal variables with any particular correlation structure.

In the particular case of independent variables (i.e., when $\rho = \mathbf{I}_p$) the cumulative distribution of $Z_{[p]}$ is given by

$$P[Z_{[p]} \leq \gamma] = P^p [Z \leq \gamma]. \tag{4.7}$$

For the general case, note that since each element of the vector $\mathbf{Z}$ is a standard normal variate, the distribution of $\mathbf{Z}$ is multivariate normal with expectation and covariance matrix as respectively described in equations (4.1) and (4.4). Thus, by virtue of equation (4.6), the cumulative distribution of $Z_{[p]}$ is in general given by

$$F_{Z_{[p]}}(\gamma) = P\left[Z_{[p]} < \gamma\right] = \int_{-\infty}^{\gamma} \cdots \int_{-\infty}^{\gamma} f(\mathbf{Z}) \prod_{i=1}^{p} dz_i \tag{4.8}$$

where $f(\mathbf{Z})$ is the multivariate normal density function given by

$$f(\mathbf{Z}) = (2\pi)^{-p/2} |\rho|^{-1/2} \exp\left\{-\frac{1}{2}(\mathbf{Z} - E[\mathbf{Z}])' \rho^{-1} (\mathbf{Z} - E[\mathbf{Z}])\right\}. \tag{4.9}$$

Recall that it was shown that under the stated assumptions, ρ (the correlation matrix of **X**) is equal to the covariance matrix of **Z**. Thus, from now on, ρ represents the covariance matrix of **Z**.

Similarly, the cumulative probabilities of $Z_{[1]}$, where $Z_{[1]}$ is simply the minimum of p observations, can in general be calculated using the following expression:

$$P[Z_{[1]} \leq \gamma] = 1 - P[Z_{[1]} \geq \gamma] = 1 - P[Z_1 > \gamma, Z_2 > \gamma, \ldots, Z_p > \gamma]. \tag{4.10}$$

Equation (4.10) is based on the fact that if $Z_{[1]} = \min_i(Z_i)$, i = 1, 2, ..., p, then $Z_i > Z_{[1]}$ $\forall$ i. Thus, equation (4.10) holds for any correlation structure on the vector **Z**.

In the particular case of independent variables the evaluation of equation (4.10) results in

$$P[Z_{[1]} \leq \gamma] = 1 - P^p[Z > \gamma]. \tag{4.11}$$

For the general case, by virtue of equation (4.10), the cumulative distribution of $Z_{[1]}$ is given by

$$F_{Z_{[1]}}(\gamma) = P\left[Z_{[1]} < \gamma\right] = 1 - \int_{\gamma}^{\infty} \cdots \int_{\gamma}^{\infty} f(\mathbf{z}) \prod_{i=1}^{p} dz_i \, , \tag{4.12}$$

where $f(\mathbf{Z})$ is the multivariate normal density function given in equation (4.9).

Now, the control limits of the Minimax chart can be calculated using the previously described probability distributions. However, before defining the control limits it is helpful to define a way to find the probability of Type I and Type II errors in the Minimax chart. This is the purpose of the next section.

4.3 Probability of Type I and II Errors for the Minimax Control Chart

4.3.1 Assumptions and Definitions

As is customary, define α and β_Δ as the respective probabilities of Type I and II errors for the Minimax control chart. Since the Minimax chart consists of two charts, that of $Z_{[1]}$ and that of $Z_{[p]}$, define $\alpha_{[1]}$ and $\alpha_{[p]}$ as the respective probabilities of Type I error for the $Z_{[1]}$ chart and for the $Z_{[p]}$ chart. Note that based on the definitions of α_1 and α_2 given in equations (3.8) and (3.9),

$$\alpha_{[1]} = \alpha_1 + \alpha_2 \tag{4.13}$$

$$\alpha_{[p]} = \alpha_3 + \alpha_4. \tag{4.14}$$

Figure 3.2 can help the reader interpreting the former two equations.

Assuming that positive shifts in the mean are as important to detect as negative ones, and using the fact that the distribution of $Z_{[1]}$ is a mirror image of that of $Z_{[p]}$ for a given μ and Σ, the following relation is also assumed:

$$\alpha_{[1]} = \alpha_{[p]} \tag{4.15}$$

Note that this assumption is not indispensable for the development of the Minimax chart; however, it simplifies its construction.

4.3.2 A Special Case: Independent Variables

Thinking of the Minimax chart as two hypotheses being tested with a fixed α value, it can be concluded that there are three situations in which a false alarm can occur. These are:

1. $Z_{[1]}$ and $Z_{[p]}$ give a false alarm,
2. $Z_{[1]}$ gives a false alarm but not $Z_{[p]}$
3. $Z_{[p]}$ gives a false alarm but not $Z_{[1]}$.

Thus, when all variables are independent, α can be found using the following expression:

$$\alpha = \alpha_{[1]} \times \alpha_{[p]} + \alpha_{[1]} \times \left[1 - \alpha_{[p]}\right] + \left[1 - \alpha_{[1]}\right] \times \alpha_{[p]}, \tag{4.16}$$

which under the assumption expressed in equation (4.15) simplifies to

$$\begin{aligned} \alpha &= \alpha_{[1]} \times \alpha_{[1]} + \alpha_{[1]} \times \left[1 - \alpha_{[1]}\right] + \left[1 - \alpha_{[1]}\right] \times \alpha_{[1]} \\ &= \alpha_{[1]}^2 + 2\alpha_{[1]} - 2\alpha_{[1]}^2 \\ &= 2\alpha_{[1]} - \alpha_{[1]}^2. \end{aligned} \tag{4.17}$$

Again, to find β, think of the Minimax chart as two hypotheses; one to test whether $Z_{[1]}$ is in control and another to test whether $Z_{[p]}$ is in control. From these two tests, only four outcomes are possible if defined in terms of the probability of Type II error in either test. These are:

1. A Type II error occurs in $Z_{[1]}$ and $Z_{[p]}$
2. A Type II error occurs in $Z_{[1]}$ but not in $Z_{[p]}$
3. A Type II error occurs in $Z_{[p]}$ but not in $Z_{[1]}$
4. A Type II error does not occur neither in $Z_{[p]}$ nor in $Z_{[1]}$

From these, the first three represent situations in which a Type II error occurs. Thus, when all variables are independent, β_Δ can be found using the following expression:

$$\beta_\Delta = \left(1 - \beta_{Z_{[p]}}\right) \times \left(1 - \beta_{Z_{[1]}}\right) + \left(1 - \beta_{Z_{[p]}}\right) \times \beta_{Z_{[1]}} + \beta_{Z_{[p]}} \times \left(1 - \beta_{Z_{[1]}}\right) \tag{4.18}$$

where $\beta_{Z_{[1]}}$ is the probability that the $Z_{[1]}$ chart does not detect a given shift, or

$$\beta_{Z_{[1]}} = P\left[Z_{[1]} \in \left(LCL_{[1]},\ UCL_{[1]}\right) \mid \Delta \neq 0\right], \tag{4.19}$$

and $\beta_{Z_{[p]}}$ is the probability that the $Z_{[p]}$ chart does not detect a given shift, or

$$\beta_{Z_{[p]}} = P\left[Z_{[p]} \in \left(LCL_{[p]},\ UCL_{[p]}\right) \mid \Delta \neq \mathbf{0}\right]. \tag{4.20}$$

Now, to find $\beta_{Z_{[p]}}$ and $\beta_{Z_{[1]}}$, use the probability distributions of $Z_{[p]}$ and $Z_{[1]}$ given in equations (4.7) and (4.11), respectively.

At this point, expressions for the calculation of α and β_Δ for the independent variables case have been developed. In the next two sections, two different general solutions for the calculation of α and β_Δ are discussed.

4.3.3 First General Solution for α and β_Δ

Note that in the general case $\alpha \neq \alpha_{[1]} + \alpha_{[p]} - \alpha_{[1]}\,\alpha_{[p]}$, since the occurrence of a Type I error in the $Z_{[1]}$ chart and the occurrence of a Type I error in the $Z_{[p]}$ chart are not independent events. Thus, α must be calculated using the following expression:

$$\alpha = 1 - P\left[\ \left\{Z_{[1]} \in \left(LCL_{[1]},\ UCL_{[1]}\right) \mid \Delta = \mathbf{0}\right\} \cap \left\{Z_{[p]} \in \left(LCL_{[p]},\ UCL_{[p]}\right) \mid \Delta = \mathbf{0}\right\}\ \right]. \tag{4.21}$$

Equation (4.21) is based on the complement of the event where no false alarm occurs. This equation can equivalently be represented, for $\Delta = \mathbf{0}$, as

$$\alpha = 1 - \int_{LCL_{[P]}}^{UCL_{[P]}} \int_{LCL_{[1]}}^{UCL_{[1]}} f\left(z_{[1]}, z_{[p]}\right) dz_{[1]}\, dz_{[p]}\,, \tag{4.22}$$

where $f(z_{[1]}, z_{[p]})$ is the joint distribution of $Z_{[1]}$ and $Z_{[p]}$. This development is appropriate but relies on the knowledge of the joint distribution of $Z_{[1]}$ and $Z_{[p]}$. Unfortunately this distribution is still unknown in the general case. For this reason, another approach to finding α is required. However, this approach is presented because it is going to provide improvements in the speed of the calculation of the control limits whenever the joint distribution of $Z_{[1]}$ and $Z_{[p]}$ is known. In the next section, a general solvable approach to finding the probability of Type I and Type II errors for the Minimax chart is discussed.

4.3.4 Second General Solution for α and β_Δ

As stated in the previous section, the probability of a Type I error (α) must be calculated from the following expression:

$$\alpha = 1 - P\left[\left\{ \left\{ Z_{[1]} \in \left(LCL_{[1]},\ UCL_{[1]} \right) \right\} \cap \left\{ Z_{[p]} \in \left(LCL_{[p]},\ UCL_{[p]} \right) \right\} \right\} \mid \Delta = \mathbf{0} \right]. \qquad (4.23)$$

Similarly, the probability of a Type II error (β), which will be needed in the evaluation of the performance of the chart, is defined as

$$\beta_\Delta = P\left[\left\{ \left\{ Z_{[1]} \in \left(LCL_{[1]},\ UCL_{[1]} \right) \right\} \cap \left\{ Z_{[p]} \in \left(LCL_{[p]},\ UCL_{[p]} \right) \right\} \right\} \mid \Delta \right],\ \Delta \neq \mathbf{0}. \qquad (4.24)$$

As previously stated, the joint distribution of $Z_{[1]}$ and $Z_{[p]}$ is unknown for the general case. Fortunately, there is an alternate method of determining α and it is developed next. First, recall that only nine events can occur when $Z_{[1]}$ and $Z_{[p]}$ are plotted in the Minimax chart [see equations (3.12) through (3.20)]. Figure 4.1 is a graphical representation of these events. Note that in Figure 4.1 the marginal distributions of $Z_{[1]}$ and $Z_{[p]}$ are used instead of the joint distribution and thus, the shaded areas below the curves represent, but are not equal to, the probabilities of the events.

Note that when $\Delta = \mathbf{0}$, event $E_{c,c}$ is the only event where a Type I error has not occurred in either of the two charts. Thus, by complementary events, when $\Delta = \mathbf{0}$ equation (4.21) can be expressed as

$$\alpha = 1 - \mathrm{P}\left[\mathrm{E}_{c,c} \mid \Delta = \mathbf{0}\right]. \tag{4.25}$$

Similarly, when $\Delta \neq \mathbf{0}$ the probability of Type II error (β) is given by

$$\beta_\Delta = \mathrm{P}[\mathrm{E}_{c,c} \mid \Delta],\ \Delta \neq \mathbf{0}. \tag{4.26}$$

Thus, finding $\mathrm{P}[\mathrm{E}_{c,c}]$ for any Δ is sufficient to determine α and β_Δ. To determine $\mathrm{P}[\mathrm{E}_{c,c}]$ define the following tail probabilities for $Z_{[1]}$ and $Z_{[p]}$:

$$t_1 = \mathrm{P}[Z_{[1]} < \mathrm{LCL}_{[1]}] = 1 - \int_{\mathrm{LCL}_{[1]}}^{\infty} \cdots \int_{\mathrm{LCL}_{[1]}}^{\infty} f(\mathbf{z}) \prod_{i=1}^{p} dz_i \tag{4.27}$$

$$t_2 = \mathrm{P}[Z_{[1]} > \mathrm{UCL}_{[1]}] = \int_{\mathrm{UCL}_{[1]}}^{\infty} \cdots \int_{\mathrm{UCL}_{[1]}}^{\infty} f(\mathbf{z}) \prod_{i=1}^{p} dz_i \tag{4.28}$$

$$t_3 = \mathrm{P}[Z_{[p]} < \mathrm{LCL}_{[p]}] = \int_{-\infty}^{\mathrm{LCL}_{[p]}} \cdots \int_{-\infty}^{\mathrm{LCL}_{[p]}} f(\mathbf{z}) \prod_{i=1}^{p} dz_i \tag{4.29}$$

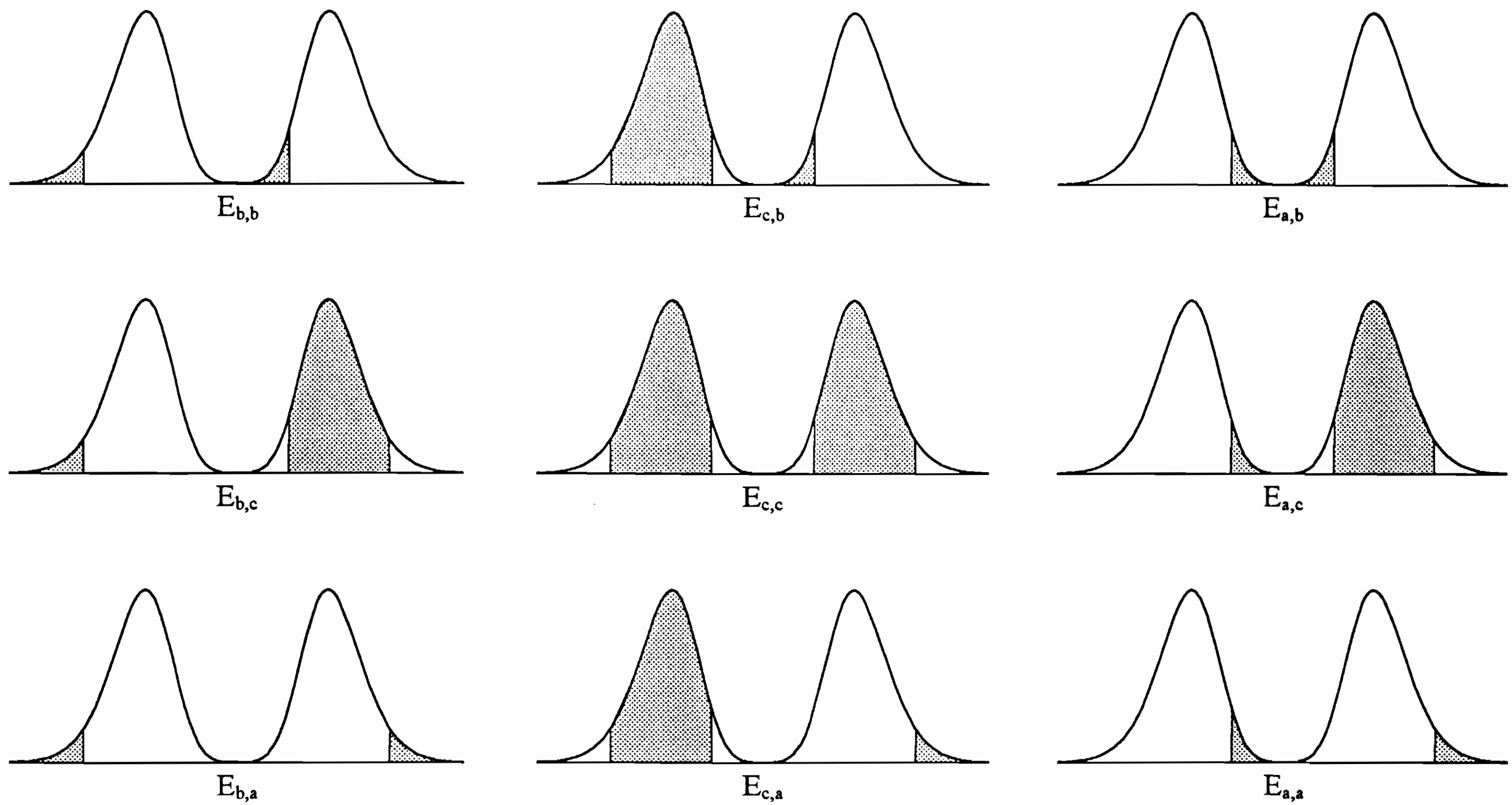

Figure 4.1: Graphical Representation of the Possible Events in a Minimax Chart

$$t_4 = \mathrm{P}[\mathrm{Z}_{[p]} > \mathrm{UCL}_{[p]}] = 1 - \int_{-\infty}^{\mathrm{UCL}_{[p]}} \cdots \int_{-\infty}^{\mathrm{UCL}_{[p]}} f(\mathbf{z}) \prod_{i=1}^{p} dz_i \qquad (4.30)$$

Note that when $\mathbf{\Delta} = \mathbf{0}$ these tail probabilities are simply the chart parameters ($t_1 = \alpha_1$, $t_2 = \alpha_2$, $t_3 = \alpha_3$, and $t_4 = \alpha_4$). Now, since all the events of the $E_{u,v}$ type are mutually exclusive,

$$\mathrm{P}\left[\mathrm{E}_{b,b} \cup \mathrm{E}_{b,c} \cup \mathrm{E}_{b,a}\right] = \mathrm{P}\left[\mathrm{E}_{b,b}\right] + \mathrm{P}\left[\mathrm{E}_{b,c}\right] + \mathrm{P}\left[\mathrm{E}_{b,a}\right]. \qquad (4.31)$$

Also note that

$$\mathrm{P}\left[\mathrm{E}_{b,b} \cup \mathrm{E}_{b,c} \cup \mathrm{E}_{b,a}\right] = \mathrm{P}\left[\mathrm{Z}_{[1]} \in \left(-\infty,\ \mathrm{LCL}_{[1]}\right) \mid \Delta\right]; \qquad (4.32)$$

thus,

$$\mathrm{P}[\mathrm{E}_{b,b}] + \mathrm{P}[\mathrm{E}_{b,c}] + \mathrm{P}[\mathrm{E}_{b,a}] = \mathrm{P}\left[\mathrm{Z}_{[1]} \in (-\infty,\ \mathrm{LCL}_{[1]}) \mid \Delta\right] = t_1. \qquad (4.33)$$

Similarly, the following equations can be defined because the events of the type $E_{u,v}$ are mutually exclusive:

$$\mathrm{P}[\mathrm{E}_{c,b}] + \mathrm{P}[\mathrm{E}_{c,c}] + \mathrm{P}[\mathrm{E}_{c,a}] = \mathrm{P}\left[\mathrm{Z}_{[1]} \in (\mathrm{LCL}_{[1]},\ \mathrm{UCL}_{[1]}) \mid \Delta\right] = 1 - (t_1 + t_2) \qquad (4.34)$$

$$\mathrm{P}[\mathrm{E}_{a,b}] + \mathrm{P}[\mathrm{E}_{a,c}] + \mathrm{P}[\mathrm{E}_{a,a}] = \mathrm{P}\left[\mathrm{Z}_{[1]} \in (\mathrm{UCL}_{[1]},\ \infty) \mid \Delta\right] = t_2 \qquad (4.35)$$

$$\mathrm{P}[\mathrm{E}_{b,b}] + \mathrm{P}[\mathrm{E}_{c,b}] + \mathrm{P}[\mathrm{E}_{a,b}] = \mathrm{P}\left[\mathrm{Z}_{[p]} \in \left(-\infty,\ \mathrm{LCL}_{[p]}\right) \mid \Delta\right] = t_3 \qquad (4.36)$$

$$\mathrm{P}[\mathrm{E}_{b,c}] + \mathrm{P}[\mathrm{E}_{c,c}] + \mathrm{P}[\mathrm{E}_{a,c}] = \mathrm{P}\left[\mathrm{Z}_{[p]} \in \left(\mathrm{LCL}_{[p]},\ \mathrm{UCL}_{[p]}\right) \mid \Delta\right] = 1 - (t_3 + t_4) \qquad (4.37)$$

It can be easily verified that equations (4.33), (4.34), (4.35), (4.36), and (4.37) are linearly independent. Thus, these are five equations in nine unknowns. Fortunately, four unknowns ($P[E_{b,b}]$, $P[E_{b,a}]$, $P[E_{a,b}]$, $P[E_{a,a}]$) can be calculated using the fact that

$$\{(\gamma_1 < Z_i < \gamma_2),\ i = 1, 2, \ldots, p\} \Leftrightarrow \{Z_{[1]} > \gamma_1 \text{ and } Z_{[p]} < \gamma_2\} \tag{4.38}$$

where $\gamma_1 < \gamma_2$, and $\gamma_1, \gamma_2 \in \Re$. Applying the relationship in (4.38) in the context of the Minimax chart results in the following equation:

$$P\left[\left\{Z_{[1]} > \gamma_1\right\} \cap \left\{Z_{[p]} < \gamma_2\right\}\right] = \int_{\gamma_1}^{\gamma_2} \cdots \int_{\gamma_1}^{\gamma_2} f(\mathbf{z}) \prod_{i=1}^{p} dz_i \,. \tag{4.39}$$

Now $P[E_{a,b}]$ can be calculated using equation (4.39), as shown here:

$$\begin{aligned} P[E_{a,b}] &= P\left[\left\{\left\{Z_{[1]} \in \left(UCL_{[1]},\ \infty\right)\right\} \cap \left\{Z_{[p]} \in \left(-\infty,\ LCL_{[p]}\right)\right\}\right\} \mid \Delta\right] \\ &= \int_{UCL_{[1]}}^{LCL_{[p]}} \cdots \int_{UCL_{[1]}}^{LCL_{[p]}} f(\mathbf{z}) \prod_{i=1}^{p} dz_i \end{aligned} \tag{4.40}$$

Note that if $LCL_{[p]} < UCL_{[1]}$ in equation (4.40), then $E_{a,b}$ is an impossible event and thus, $P[E_{a,b}] = 0$. Consequently, $P[E_{a,b}]$ is given by

$$P\left[E_{a,b}\right] = \begin{cases} \int_{UCL_{[1]}}^{LCL_{[p]}} \cdots \int_{UCL_{[1]}}^{LCL_{[p]}} f(\mathbf{z}) \prod_{i=1}^{p} dz_i & LCL_{[p]} > UCL_{[1]} \\ 0 & \text{otherwise} \end{cases} \tag{4.41}$$

$P[E_{b,a}]$ can also be calculated using equation (4.39). First, note that in general, for any two events e_1 and e_2,

$$P\left[\overline{e_1} \cap \overline{e_2}\right] = 1 - P\left[e_1 \cup e_2\right]$$
$$= 1 - \left(P\left[e_1\right] + P\left[e_2\right] - P\left[e_1 \cap e_2\right]\right) \tag{4.42}$$
$$= 1 - P\left[e_1\right] - P\left[e_2\right] + P\left[e_1 \cap e_2\right]$$

thus,

$$P\left[e_1 \cap e_2\right] = -1 + P\left[e_1\right] + P\left[e_2\right] + P\left[\overline{e_1} \cap \overline{e_2}\right]. \tag{4.43}$$

Now use equation (4.43) to determine $P[E_{b,a}]$ as follows:

$$\begin{aligned}
P\left[E_{b,a}\right] &= P\left[\left\{\left\{Z_{[1]} \in \left(-\infty,\ LCL_{[1]}\right)\right\} \cap \left\{Z_{[p]} \in \left(UCL_{[p]},\ \infty\right)\right\}\right\} \mid \Delta\right] \\
&= -1 + P\left[Z_{[1]} \in \left(-\infty,\ LCL_{[1]}\right) \mid \Delta\right] + P\left[Z_{[p]} \in \left(UCL_{[p]},\ \infty\right) \mid \Delta\right] + \\
&\quad P\left[\overline{\left\{Z_{[1]} \in \left(-\infty,\ LCL_{[1]}\right) \mid \Delta\right\}} \cap \overline{\left\{Z_{[p]} \in \left(UCL_{[p]},\ \infty\right) \mid \Delta\right\}}\right] \\
&= -1 + t_1 + t_4 + P\left[\left\{\left\{Z_{[1]} \in \left(LCL_{[1]},\ \infty\right)\right\} \cap \left\{Z_{[p]} \in \left(-\infty,\ UCL_{[p]}\right)\right\}\right\} \mid \Delta\right] \\
&= -1 + t_1 + t_4 + \int_{LCL_{[1]}}^{UCL_{[p]}} \cdots \int_{LCL_{[1]}}^{UCL_{[p]}} f(\mathbf{z}) \prod_{i=1}^{p} dz_i
\end{aligned} \tag{4.44}$$

To find the other two unknowns ($P[E_{a,a}]$ and $P[E_{b,b}]$) use the fact that for any two events e_1 and e_2

$$P\left[e_1 \cap e_2\right] = P\left[e_1\right] - P\left[e_1 \cap \overline{e_2}\right] \tag{4.45}$$

or,

$$P\left[e_1 \cap e_2\right] = P\left[e_2\right] - P\left[\overline{e_1} \cap e_2\right]. \tag{4.46}$$

Now, use equation (4.45) to find $P[E_{a,a}]$ as follows:

$$
\begin{aligned}
P[E_{a,a}] &= P\left\{\left\{Z_{[1]} \in \left(UCL_{[1]},\ \infty\right)\right\} \cap \left\{Z_{[p]} \in \left(UCL_{[p]},\ \infty\right)\right\}\right\} \mid \Delta \\
&= P\left[Z_{[1]} \in \left(UCL_{[1]},\ \infty\right) \mid \Delta\right] - P\left[\left\{\begin{matrix}\left\{Z_{[1]} \in \left(UCL_{[1]},\ \infty\right)\right\} \cap \\ \overline{\left\{Z_{[p]} \in \left(UCL_{[p]},\ \infty\right)\right\}}\end{matrix}\right\} \mid \Delta\right] \\
&= t_2 - P\left\{\left\{Z_{[1]} \in \left(UCL_{[1]},\ \infty\right)\right\} \cap \left\{Z_{[p]} \in \left(-\infty,\ UCL_{[p]}\right)\right\}\right\} \mid \Delta \\
&= t_2 - \int_{UCL_{[1]}}^{UCL_{[p]}} \cdots \int_{UCL_{[1]}}^{UCL_{[p]}} f(\mathbf{z}) \prod_{i=1}^{p} dz_i
\end{aligned}
\tag{4.47}
$$

Similarly, use equation (4.46) to find $P[E_{b,b}]$ as shown here:

$$
\begin{aligned}
P[E_{b,b}] &= P\left\{\left\{Z_{[1]} \in \left(-\infty,\ LCL_{[1]}\right)\right\} \cap \left\{Z_{[p]} \in \left(-\infty,\ LCL_{[p]}\right)\right\}\right\} \mid \Delta \\
&= P\left[Z_{[p]} \in \left(-\infty,\ LCL_{[p]}\right) \mid \Delta\right] - P\left[\left\{\begin{matrix}\overline{\left\{Z_{[1]} \in \left(-\infty,\ LCL_{[1]}\right)\right\}} \cap \\ \left\{Z_{[p]} \in \left(-\infty,\ LCL_{[p]}\right)\right\}\end{matrix}\right\} \mid \Delta\right] \\
&= t_3 - P\left\{\left\{Z_{[1]} \in \left(LCL_{[1]},\ \infty\right)\right\} \cap \left\{Z_{[p]} \in \left(-\infty,\ LCL_{[p]}\right)\right\}\right\} \mid \Delta \\
&= t_3 - \int_{LCL_{[1]}}^{LCL_{[p]}} \cdots \int_{LCI_{[1]}}^{LCL_{[p]}} f(\mathbf{z}) \prod_{i=1}^{p} dz_i
\end{aligned}
\tag{4.48}
$$

Recall that when the process is in control (i.e. Δ=**0**), in equations (4.44), (4.47) and (4.48) $t_1=\alpha_1$, $t_2=\alpha_2$, $t_3=\alpha_3$, and $t_4=\alpha_4$.

As can be easily verified, equations (4.33), (4.34), (4.35), (4.36), and (4.37) remain independent after $P[E_{b,b}]$, $P[E_{b,a}]$, $P[E_{a,b}]$, and $P[E_{a,a}]$ are determined. Thus, these form a system of five equations in five unknowns. After solving the system of equations the exact value of α and β_Δ for a given set of α_1, α_2, α_3, and α_4 can be found using equations (4.25) and (4.26) respectively.

It is important to mention that the joint distribution of $Z_{[1]}$ and $Z_{[p]}$ is not formally known. However, all probabilities that could have been calculated with that joint distribution were calculated using some simple probability rules. Note that expressions for $P[E_{b,b}]$, $P[E_{b,a}]$, $P[E_{a,b}]$, and $P[E_{a,a}]$ are now available. By examining the structure of these events in terms of the inequalities that they represent, the reader should realize that any joint probability of the random variables $Z_{[1]}$ and $Z_{[p]}$ can be calculated using the approach described in this section.

In this section, a general method to find α and β_Δ for the Minimax control chart has been developed. In the next section, the formulas provided in Section 4.3.4 for the determination of α and β_Δ are used in the development of a method to determine the control limits for the Minimax chart.

4.4 Determining the Control Limits of the Minimax Control Chart

The objective of this section is to provide an algorithm to determine the set of control limits (or the set of values of α_1, α_2, α_3, and α_4) that meet the required α for a given *scenario*. A *scenario* is the situation described by the set of parameters $\{p, \rho\}$. The task of setting the control limits when the variables are independent is easier than when they are correlated. For that reason, a simple method to define the control limits when $\rho = \mathbf{I}_p$ is discussed first. Afterwards, a general method is also provided.

4.4.1 A Special Case: Independent Variables

For simplicity, assume that α_1, α_2, α_3, and α_4 are known at this point. Recall that it had been shown in Section 4.2 that for independent variables

$$P[Z_{[p]} \leq \gamma] = P^p [Z < \gamma]. \tag{4.49}$$

Now, the definition of α_4 given in equation (3.11) can be used in conjunction with equation (4.49) to determine $UCL_{[p]}$. That is, to find $UCL_{[p]}$, substitute $UCL_{[p]}$ for γ in equation (4.49) and let the equation be equal to 1 - α_4,

$$P[Z_{[p]} \leq UCL_{[p]}] = P^p [Z < UCL_{[p]}] = 1-\alpha_4; \tag{4.50}$$

thus,

$$P[Z < UCL_{[p]}] = [1- \alpha_4]^{1/p}. \tag{4.51}$$

Since Z is a standard normal variable, solving for $UCL_{[p]}$ in (4.51) results in the following expression:

$$UCL_{[p]} = \{\tau\text{: } \Phi(\tau) = [1 - \alpha_4]^{1/p}\}, \tag{4.52}$$

where $\Phi(.)$ is the cumulative standard normal distribution. Since a closed form for the cumulative normal distribution does not exist, equation (4.52) must be solved numerically.

In a similar way, to find $LCL_{[p]}$, substitute $LCL_{[p]}$ for γ in the cumulative distribution of $Z_{[p]}$ given in (4.49) and make the equation equal to α_3,

$$P[Z_{[p]} \leq LCL_{[p]}] = P^p [Z < LCL_{[p]}] = \alpha_3. \tag{4.53}$$

After solving for $LCL_{[p]}$, equation (4.53) leads to

$$LCL_{[p]} = \{\tau\text{: } \Phi(\tau) = [\alpha_3]^{1/p}\}. \tag{4.54}$$

The control limits for $Z_{[1]}$ can be defined using the same approach as that used for the control limits of $Z_{[p]}$. In particular, recall that in the case of independent variables

$$P[Z_{[1]} \leq \gamma] = 1 - P^p [Z \geq \gamma]. \tag{4.55}$$

Thus, using the same approach as that used to generate (4.52) and (4.54), the control limits of $Z_{[1]}$ and $Z_{[p]}$ can be found by solving for $UCL_{[1]}$ and $LCL_{[1]}$ in the following equations:

$$P[Z_{[1]} \leq UCL_{[1]}] = 1 - P^p [Z > UCL_{[1]}] = 1 - \alpha_2 \tag{4.56}$$

$$P[Z_{[1]} \leq LCL_{[1]}] = 1 - P^{p}[Z > LCL_{[1]}] = \alpha_1 \tag{4.57}$$

Some algebraic manipulations in equations (4.56) and (4.57) result in following expressions:

$$UCL_{[1]} = \{\tau: \Phi(\tau) = 1 - [\alpha_2]^{1/p}\} \tag{4.58}$$

$$LCL_{[1]} = \{\tau: \Phi(\tau) = 1 - [1 - \alpha_1]^{1/p}\} \tag{4.59}$$

Note that $LCL_{[1]} = - UCL_{[p]}$ and $UCL_{[1]} = - LCL_{[p]}$. This is not surprising since the normal distribution is symmetric; and thus, the distribution of $Z_{[p]}$ is a mirror image of that of $Z_{[1]}$. The assumption that $\alpha_1 = \alpha_4$ and $\alpha_2 = \alpha_3$ is responsible for this symmetry of the control limits.

Up to this point the control limits for the chart have been defined based on the assumption that α_1, α_2, α_3, and α_4 are known. However, the values of these parameters that will make the chart have a particular value of α are still unknown.

To determine the values of α_1, α_2, α_3, and α_4 that result in a particular value of α, recall that it has been shown in Section 4.3.2 that when the variables are independent

$$\alpha = 2\alpha_{[1]} - \alpha_{[1]}^2 . \tag{4.60}$$

Now, using the quadratic equation to solve for $\alpha_{[1]}$ in (4.60) yields:

$$\alpha_{[1]} = \frac{2 \pm \sqrt{2^2 - 4(-1)\alpha}}{2} \qquad (4.61)$$
$$= 1 \pm \sqrt{1-\alpha}$$

Note that $\alpha_{[1]}$ is a probability; thus, based on the possible values that $\alpha_{[1]}$ can assume, the correct solution is

$$\alpha_{[1]} = 1 - \sqrt{1-\alpha}\,. \qquad (4.62)$$

Now, since

$$\alpha_{[1]} = \alpha_1 + \alpha_2 \qquad (4.63)$$

and α is known, assigning a value to α_1 gives the value of α_2 that meets the desired α. The result is:

$$\alpha_2 = 1 - \alpha_1 - \sqrt{1-\alpha} \qquad (4.64)$$

Note that the assignment of α_1 must be such that $\alpha_1 \in (0, \alpha/2)$. For example: Let $\alpha_1 = 6/10\ \alpha/2$. Now, since it has been assumed that

$$\alpha_1 = \alpha_4 \text{ and } \alpha_2 = \alpha_3, \qquad (4.65)$$

knowing α_1 and α_2 is sufficient to determine all the control limits for the Minimax chart by using equations (4.52), (4.54), (4.58), and (4.59).

In this way the control limits for the control chart are defined when the variables are uncorrelated. Note that the value of α_1 has been assigned arbitrarily. However, there must be a value of α_1 that maximizes the performance of the chart for a particular scenario. This "optimal" value of α_1 (or α_4) is determined for different scenarios in Chapter 5.

4.4.2 A General Method to Set the Control Limits

Recall that it is necessary to define α as a chart parameter such that the chart has a specified in-control ARL. That is to say, the values of α_1, α_2, α_3, and α_4 should be assigned such that a specified value of α is met. An initial estimate for α_1, α_2, α_3, and α_4 for a given μ and Σ can be found by assuming that the variables are independent. For this purpose, use the method described in Section 4.4.1. To determine the exact control limits use the cumulative distributions of $Z_{[p]}$ and $Z_{[1]}$ given in (4.8) and (4.12). By combining these equations with the definitions of α_1, α_2, α_3, and α_4, given in equations (3.8) through (3.11), the control limits for $Z_{[1]}$ and $Z_{[p]}$ for any matrix ρ, ($\rho = \mathbf{I}_p$ or $\rho \neq \mathbf{I}_p$) result in the following equations:

$$UCL_{[p]} = \left\{ \tau : \int_{-\infty}^{\tau} \cdots \int_{-\infty}^{\tau} f(\mathbf{z}) d\mathbf{z} = 1 - \alpha_4 \right\} \tag{4.66}$$

$$LCL_{[p]} = \left\{ \tau : \int_{-\infty}^{\tau} \cdots \int_{-\infty}^{\tau} f(\mathbf{z}) d\mathbf{z} = \alpha_3 \right\} \tag{4.67}$$

$$UCL_{[1]} = \left\{ \tau : 1 - \int_{\tau}^{\infty} \cdots \int_{\tau}^{\infty} f(\mathbf{z}) d\mathbf{z} = 1 - \alpha_2 \right\} \tag{4.68}$$

$$LCL_{[1]} = \left\{ \tau : 1 - \int_{\tau}^{\infty} \cdots \int_{\tau}^{\infty} f(\mathbf{z}) d\mathbf{z} = \alpha_1 \right\}. \tag{4.69}$$

Note that these definitions of the control limits cannot be solved in closed form because the multivariate normal distribution does not exist in closed form. Consequently, the determination of the control limits in the general case has to be performed using numerical integration. In particular, a numerical line search must be performed to determine the value of τ that makes equations (4.66) to (4.69) hold for a given scenario. To perform this search the following algorithm is used:

1. Assign values to α, α_1 and δ (the acceptable difference between the desired in-control ARL and the in-control ARL; e.g. δ=0.5).
2. Search for the value of τ that makes (4.69) hold and determine $LCL_{[1]}$.
3. Set $UCL_{[p]} = -LCL_{[1]}$.
4. Set α_2 assuming $\rho = \mathbf{I}_p$ (i.e. $\alpha_2 = \alpha_{[1]} - \alpha_1$)
5. Determine $UCL_{[1]}$ using equation (4.68)
6. Determine $P[E_{c,c} \mid \Delta=\mathbf{0}]$
7. If $\dfrac{1}{1-P\left[E_{c,c} \mid \Delta = 0\right]} \notin \left(\dfrac{1}{\alpha} \pm \delta\right)$ Reset the value of α_2 and Goto Step 5
8. If $\dfrac{1}{1-P\left[E_{c,c} \mid \Delta = 0\right]} \in \left(\dfrac{1}{\alpha} \pm \delta\right)$ Set $LCL_{[p]} = -UCL_{[1]}$.
9. End.

To conclude this section, note that the control limits for any covariance structure can be determined using the former algorithm. In the next section, the equations needed to determine the probability of correctly diagnosing a given shift in the mean are mathematically developed.

4.5 Diagnosing the System when a Signal Occurs

The objective of this section is to develop a method of calculating the probability of making a correct diagnosis for the system given that a signal occurs. The criteria for signal diagnosis described in Section 3.2 are used for this purpose. For diagnosing the system, assume that only positive diagonal and positive axial shifts in the mean occur in the process (if positive shifts can be diagnosed correctly, so can negative shifts). Under this assumption a correct diagnosis is attained when:

1. The right type of shift in the mean is diagnosed (either axial or diagonal).
2. The variable that has experienced the shift in the mean is correctly identified when the shift is axial.

Note that when the shift is diagonal, all variables experience the same shift in the mean; thus, no particular variable is responsible for the shift, but all of them are. After the diagnosis is made, the magnitude of the shift is estimated from the means corresponding to the set of variables responsible for the shift. To diagnose the type of shift, the diagnosis criteria described in Section 3.2 are used. Using the assumption of positive shifts in the mean, those criteria condense to the following two statements:

Diagnosis Criterion # 1:

- Axial Shifts - It is expected that $E_{c,a}$ corresponds to a positive axial shift in the mean of the variable that gives the signal (X_{Ψ_p}).

Diagnosis Criterion # 2:

- Diagonal Shifts - It is expected that $E_{a,c}$ and $E_{a,a}$ correspond to a positive diagonal shift in the means of all variables.

Recall that equations (3.12) through (3.20) describe all possible events when $Z_{[1]}$ and $Z_{[p]}$ are plotted in the Minimax chart. From these events, only $E_{c,c}$ does not represent an out-of-control situation. The remaining eight events represent all the different possible signals in the Minimax chart.

In order to define the probability of correctly diagnosing a shift in the mean, let the following definitions hold for a given shift in the mean Δ:

1. P[CD] = P[<u>C</u>orrectly <u>D</u>iagnosing a given shift in the mean]
2. P[Signal & CD] = P[A **Signal** occurs <u>**AND**</u> Correct **D**iagnosis is achieved]

3. P[CD | Signal] = P[**C**orrectly **D**iagnosing a shift **given** that a **Signal** occurs]

The methods of determining the probability of a correct diagnosis for axial and diagonal shifts in the mean using the former three definitions are developed next.

4.5.1 Diagnosis of Diagonal Shifts

Diagonal shifts are the easiest to diagnose since all variables are directly associated with this type of shift. Thus, in diagonal shifts P[CD] is independent of the set of variables that give the signal. Now, based on Diagnosis Criterion # 2, when this type of shift has occurred, the probability of a correct diagnosis is equal to the probability of the occurrence of $E_{a,a}$ or $E_{a,c}$. Since $E_{a,a}$ and $E_{a,c}$ are mutually exclusive, then, given a diagonal shift,

$$P[\text{Signal \& CD}] = P[E_{a,a}] + P[E_{a,c}]. \tag{4.70}$$

Thus, for positive diagonal shifts in the mean,

$$\begin{aligned} P[\text{CD} \mid \text{Signal}] &= \frac{P[\text{Signal \& CD}]}{P[\text{Signal}]} \\ &= \frac{P[E_{a,a}] + P[E_{a,c}]}{1 - P[E_{c,c}]} \end{aligned} \tag{4.71}$$

Now, find $P[E_{a,a}]$, $P[E_{a,c}]$, and $P[E_{c,c}]$ by solving the set of equations described in Section 4.3.4. In this way, equation (4.71) can be used to determine the probability of correctly diagnosing the system when a diagonal shift in the mean occurs.

It should be easy to realize that for negative diagonal shifts in the mean equation (4.72) shown below gives the probability of making the correct diagnose when a signal occurs.

$$\begin{aligned} P[CD \mid Signal] &= \frac{P[Signal \ \& \ CD]}{P[Signal]} \\ &= \frac{P[E_{b,b}] + P[E_{c,b}]}{1 - P[E_{c,c}]} \end{aligned} \tag{4.72}$$

Again, to solve equation (4.72) find $P[E_{a,a}]$, $P[E_{a,c}]$, and $P[E_{c,c}]$ by solving the set of equations described in Section 4.3.4.

4.5.2 Diagnosis of Axial Shifts

The diagnosis of axial shifts is not as easy as that of the diagonal shifts since now the diagnosis involves the identification of the variable that is out of control. An example of this type of diagnosis is done first; afterwards, the general equations to find P[CD | Shift] are developed.

Assume that p=3 and Δ=(1, 0, 0). It is necessary to find the probability of correctly diagnosing this shift given that the chart has signaled, this is P[CD | Signal]. Note that the Diagnosis Criteria # 1 states that a positive shift in the mean results in the occurrence of $E_{c,a}$. Now recall that $E_{c,a}$ is defined as:

$$E_{c,a} = \left\{Z_{[1]} \in \left(LCL_{[1]},\ UCL_{[1]}\right)\right\} \cap \left\{Z_{[p]} \in \left(UCL_{[p]},\ \infty\right)\right\}. \tag{4.73}$$

Next, use the definition of conditional probability to express P[CD | Signal] as shown in equation (4.74):

$$P[CD \mid Signal] = \frac{P[Signal \ \& \ CD]}{P[Signal]}. \tag{4.74}$$

Notice that in this particular example, the fact that $E_{c,a}$ is a subset of all the possible signals, as well as that a correct diagnosis will only be achieved if $E_{c,a}$ occurs, justify the following relationship:

$$P[Signal \ \& \ CD] = P[Signal \ \& \ E_{c,a} \ \& \ CD] = P[E_{c,a} \ \& \ CD]. \tag{4.75}$$

Thus,

$$P[CD \mid Signal] = \frac{P[E_{c,a} \ \& \ CD]}{P[Signal]}. \tag{4.76}$$

Now, since Δ=(1, 0, 0), a correct diagnosis is attained if $X_{\Psi_p} = X_1$ and $E_{c,a}$ occurs. This is expressed formally in the following equation:

$$P[E_{c,a} \ \& \ CD] = P[E_{c,a} \ \& \ X_{\Psi_p} = X_1]. \tag{4.77}$$

A detailed examination of equation (4.77) leads to the following equation:

$$\begin{aligned}
P\left[E_{c,a} \ \& \ CD\right] &= P\left[\left\{\begin{matrix}\left(Z_1 > UCL_{[p]}\right) \text{ AND } \left(LCL_{[1]} < Z_2 < Z_1\right) \text{ AND} \\ \left(LCL_{[1]} < Z_3 < Z_1\right)\end{matrix}\right\} \mid \Delta\right] \\
&\quad - P\left[\left\{\begin{matrix}\left(Z_1 > UCL_{[p]}\right) \text{ AND } \left(UCL_{[1]} < Z_2 < \infty\right) \text{ AND} \\ \left(UCL_{[1]} < Z_3 < \infty\right)\end{matrix}\right\} \mid \Delta\right] \\
&= P\left[\left\{\begin{matrix}\left\{Z_1 \in \left(UCL_{[p]}, \infty\right)\right\} \cap \left\{Z_2 \in \left(LCL_{[1]}, Z_1\right) \mid \Delta\right\} \cap \\ \left\{Z_3 \in \left(LCL_{[1]}, Z_1\right)\right\}\end{matrix}\right\} \mid \Delta\right] \\
&\quad - P\left[\left\{\begin{matrix}\left\{Z_1 \in \left(UCL_{[p]}, \infty\right)\right\} \cap \left\{Z_2 \in \left(UCL_{[1]}, \infty\right) \mid \Delta\right\} \cap \\ \left\{Z_3 \in \left(UCL_{[1]}, \infty\right)\right\}\end{matrix}\right\} \mid \Delta\right]
\end{aligned} \tag{4.78}$$

Note that equation (4.78) can be evaluated by integrating the multivariate normal density function as shown in equation (4.79):

$$\begin{aligned}
P[E_{c,a} \ \& \ CD] &= \int_{UCL_{[p]}}^{\infty} \int_{LCL_{[1]}}^{Z_1} \int_{LCL_{[1]}}^{Z_1} f(\mathbf{Z}|\Delta) \ dZ_2 \ dZ_3 \ dZ_1 \\
&\quad - \int_{UCL_{[p]}}^{\infty} \int_{UCL_{[1]}}^{\infty} \int_{UCL_{[1]}}^{\infty} f(\mathbf{Z}|\Delta) \ dZ_2 \ dZ_3 \ dZ_1
\end{aligned} \tag{4.79}$$

Thus, for this particular example, P[CD | Signal] can be found by substituting equation (4.79) into equation (4.76) and using the fact that

$$P[\text{Signal}] = 1 - P[E_{c,c} \mid \Delta = (1, 0, 0)]. \tag{4.80}$$

However, a general solution to equation (4.76) is still needed.

To find the general solution assume that a shift in variable k has occurred. In this case, the general solution to equitation (4.76) can be found by substituting the following expression for its numerator:

$$P[E_{c,a} \ \& \ CD] = \int_{UCL_{[p]}}^{\infty}\left(\int_{LCL_{[1]}}^{Z_k}\cdots\int_{LCL_{[1]}}^{Z_k} f(\mathbf{Z}|\Delta) \prod_{\substack{i=1 \\ i\neq k}}^{p} dZ_i\right) dZ_k \\ -\int_{UCL_{[p]}}^{\infty}\left(\int_{UCL_{[1]}}^{\infty}\cdots\int_{UCL_{[1]}}^{\infty} f(\mathbf{Z}|\Delta) \prod_{\substack{i=1 \\ i\neq k}}^{p} dZ_i\right) dZ_k \tag{4.81}$$

and equation (4.80) for a general Δ for its denominator. The resulting equation is shown below:

$$P[CD \mid Signal] = \frac{1}{1 - P[E_{c,c}|\Delta]}\left(\begin{array}{l}\int_{UCL_{[p]}}^{\infty}\left(\int_{LCL_{[1]}}^{Z_k}\cdots\int_{LCL_{[1]}}^{Z_k} f(\mathbf{Z}|\Delta) \prod_{\substack{i=1 \\ i\neq k}}^{p} dZ_i\right) dZ_k \\ -\int_{UCL_{[p]}}^{\infty}\left(\int_{UCL_{[1]}}^{\infty}\cdots\int_{UCL_{[1]}}^{\infty} f(\mathbf{Z}|\Delta) \prod_{\substack{i=1 \\ i\neq k}}^{p} dZ_i\right) dZ_k\end{array}\right) \tag{4.82}$$

Finally, equation (4.82) gives the probability of a correct diagnosis when a positive axial shift in the mean occurs. For negative axial shifts, assume that the mean of variable k has shifted. It can be easily shown that a general solution is found by replacing the numerator in (4.76) with $P[E_{b,c}$ & CD] and using equation (4.83) below to solve for $P[E_{b,c}$ & CD]:

$$P[E_{b,c} \;\&\; CD] = \int_{-\infty}^{LCL_{[1]}} \left(\int_{Z_k}^{UCL_{[p]}} \cdots \int_{Z_k}^{UCL_{[p]}} f(\mathbf{Z}|\Delta) \prod_{\substack{i=1 \\ i \neq k}}^{p} dZ_i \right) dZ_k - \int_{-\infty}^{LCL_{[1]}} \left(\int_{-\infty}^{LCL_{[p]}} \cdots \int_{-\infty}^{LCL_{[p]}} f(\mathbf{Z}|\Delta) \prod_{\substack{i=1 \\ i \neq k}}^{p} dZ_i \right) dZ_k \quad (4.83)$$

The resulting equation to determine the probability of a correct diagnosis when a negative diagonal shift in the mean occurs is:

$$P[CD \mid Signal] = \frac{1}{1 - P[E_{c,c}|\Delta]} \left(\begin{array}{l} \int_{-\infty}^{LCL_{[1]}} \left(\int_{Z_k}^{UCL_{[p]}} \cdots \int_{Z_k}^{UCL_{[p]}} f(\mathbf{Z}|\Delta) \prod_{\substack{i=1 \\ i \neq k}}^{p} dZ_i \right) dZ_k \\ - \int_{-\infty}^{LCL_{[1]}} \left(\int_{-\infty}^{LCL_{[p]}} \cdots \int_{-\infty}^{LCL_{[p]}} f(\mathbf{Z}|\Delta) \prod_{\substack{i=1 \\ i \neq k}}^{p} dZ_i \right) dZ_k \end{array} \right). \quad (4.84)$$

Thus, general expressions to determine the probability of a correct diagnosis for axial and diagonal shifts in the mean are now available. At this point all the needed theoretical development to set and use a Minimax control chart has been completed. The next step is an evaluation of the performance of the chart under different scenarios and a comparison of it to other multivariate control charts. This analysis is the purpose of the next chapter.

Chapter 5

Experimental Performance Measurements for The Minimax Control Chart

5.1 Selection of the Experimental Conditions

The Minimax chart has been developed in a general way. In other words, a different Minimax chart exists for each possible combination of its parameters. With the intention of studying the effect of these parameters on the performance of the chart, these parameters are going to be divided in four categories. First, recall that a *scenario* has been defined as the set of parameters $\{p, \rho\}$. For each scenario, a completely different Minimax chart is defined by a set of *tuning parameters*: $\{n, \alpha_4\}$. These are *tuned-up* by the designer so that the chart has the desired performance properties in terms of Type I and Type II errors.

The performance properties of the chart are defined by α and β, the *performance parameters*. Now, define $\lambda(\Delta)$ as the magnitude of the shift in the vector of means for which β_Δ holds, and S as the direction or type of shift in the mean for which β_Δ holds. Since $\lambda(\Delta)$ and S are uncontrollable in a real situation, they are called the *uncontrollable* *parameters*. More details on the parameters S and $\lambda(\Delta)$ are going to be given latter in this section.

Note that α_3 is not in the *tuning* set of parameters since it is completely determined by α and α_4 (see the algorithm to calculate the control limits of the Minimax chart in Section 4.4.2). Similarly, α_1 and α_2 are not in the tuning set because $\alpha_1 = \alpha_4$ and $\alpha_2 = \alpha_3$. Although α_4 is dependent on α, [$\alpha_4 \in (0, \alpha/2)$], it is included in the tuning set because it is adjustable within its range of possible values.

The following list summarizes the previously discussed nomenclature:

- The set of parameters that represent a *scenario* is $\{p, \rho\}$.
- The set of *tuning parameters* is $\{n, \alpha_4\}$.
- The set of *performance parameters* is $\{\alpha, \beta\}$.
- The set of *uncontrollable parameters* is $\{\lambda(\Delta), S\}$.

The former eight parameters are going to be used to run a deterministic simulation (which from now on will be referred to as an experiment) to evaluate the performance of the chart under different situations. The objective of the experiment is to determine the position of the *tuning parameters* that minimizes the average run length (ARL, see Appendix C) given a set of *performance* and *uncontrollable parameters* for several *scenarios*. Afterwards, it is also desired to predict, within the experimental region, the values of the *tuning* parameters that optimize the performance of the chart in terms of the out-of-control ARL and the probability of correctly diagnosing the signals. Finally, a comparison between the Minimax and the Chi-squared chart's ARLs is made under the same experimental conditions (see Appendix B). For these purposes, an experiment is run using all of the parameters (p, ρ, n, α_4, α, β, $\lambda(\Delta)$, and S) as the experimental factors and the associated ARL and probability of correctly diagnosing a signal as the responses. In the following section, the selection of the experimental levels for all the parameters is discussed.

5.1.1 Scenarios

The selected levels of the number of variables are p=2, p=3, and p=4. The assignment of the levels of the correlation structures are based on the fact that for any set of variables there is no loss of generality in scaling all p variances such that $\sigma_{ii} = 1 \; \forall \; i$. Assuming that such scaling is done, the covariance matrix of X becomes equal to the correlation matrix of X since $\rho_{ij} = \dfrac{\sigma_{ij}^2}{\sigma_{ii}\sigma_{jj}} = \sigma_{ij}^2$. Thus, there is no loss of generality in studying the behavior of the chart for different correlation matrices instead of covariance matrices. In order to delimit the scenarios to be studied, the types of perturbations on the correlation matrix have been chosen to meet the following criteria:

$$\rho = [\rho_{i,j}],\ \rho_{i,j} = \begin{cases} 1 & for\ i = j \\ r & o.w. \end{cases},\ \text{where } r \in [-1,1] \tag{5.1}$$

Note that equation (5.1) implies that only covariance matrices with equal off-diagonal elements are going to be studied. The final selection of the levels of ρ is to set r in equation (5.1) as follows: r = {-3/10, 0, 3/10}. This is done with the objective of studying independent variables, negatively correlated variables, and positively correlated variables. The purpose of studying changes in covariance is **not** to assess the ability of the chart to detect changes in covariance, but to evaluate its robustness to different covariance structures. Recall that the distributions of $Z_{[1]}$ and $Z_{[p]}$ are a function of ρ, and thus, so are the control limits and the performance of the Minimax control chart.

5.1.2 Tuning Parameters

The chosen levels for the sample size are n=1 and n=5. The parameter α_4 was assigned nine levels. For simplicity of notation, the vector of levels of α_4 is expressed as ξ, where ξ_k (the k[th] element of ξ) is as expressed in equation (5.2)

$$\xi_k = \frac{k}{10} \times \frac{\alpha}{2},\ k = 1, \ldots, 9. \tag{5.2}$$

The objective of this assignment is to select from these nine values the one that minimizes the ARL, i.e., the one that optimizes the performance of the chart. Details on how to perform this selection are given in Section 5.2.

5.1.3 Performance Parameters

Because of the associated ARLs, the chosen levels of α are α = 1/200 and α = 1/125. The parameter β_Δ is not assigned levels in the experiment since it is directly associated with the experimental response (ARL). That is, the response is a function of β_Δ.

5.1.4 Uncontrollable Parameters

To assign the levels of the magnitude of the shifts in mean the multivariate χ^2 noncentrality parameter $\lambda^2(\mu')$ is used. This represents the *squared distance* between the in-control mean μ and the out-of-control mean μ'. The quantity $\lambda^2(\mu')$ is given by

$$\lambda^2(\mu') = (\mu' - \mu)^T \Sigma^{-1} (\mu' - \mu). \tag{5.3}$$

Since there is no loss of generality in scaling all means such that $\mu = \mathbf{0}$ and all variances such that $\Sigma = \rho$, equation (5.3) can be expressed as

$$\lambda^2(\Delta) = \Delta^T \rho^{-1} \Delta, \tag{5.4}$$

where the components of the vector Δ are as given in (3.2). The square root of the non centrality parameter (i.e., $\lambda(\mu')$) is referred as the *distance* between μ' and μ. Note that when $\rho = \mathbf{I}_p$, $\lambda(\mu')$ represents the Euclidean or straight line distance from μ' to μ, that is, the norm of μ'. From now on we will reserve the word *distance* to represent the *magnitude* of a particular shift in mean, i.e. $\lambda(\mu')$, or in a simpler form, $\lambda(\Delta)$.

Finally, the experimental levels of the magnitude of the shifts in mean have been chosen so that they result in distances going from 0 to 3 in steps of 0.5 units. That is, for a particular ρ matrix the vector Δ must be assigned such that $\lambda(\Delta) = 0, 0.5, 1, \ldots, 3$. This assignment has to be done using equation (5.4). For example: to find the value of Δ that makes $\lambda(\Delta) = 0.5$, search for the value of Δ for which

$$\lambda^2(\Delta) = 0.5^2 = \Delta^T \rho^{-1} \Delta. \tag{5.5}$$

Note that the number of solutions to equation (5.5) depends on the values of p and ρ. What is certain is that for any scenario there is more than one vector Δ that satisfies the equation. To delimit that number of solutions, the uncontrollable parameter S is introduced in the experiment. S refers to the direction of the shift in the mean. The levels for the type of shift in the mean (S) have been chosen to be: S = Axial shift and S = Diagonal shift.

5.1.5 Summary of the Experimental Conditions

So far all the experimental levels of each of the parameters have been assigned. A summary of the experimental conditions is shown in Table 5.1. Recall that n, p, r, and α

represent a completely distinct Minimax chart because changes in S, α_4, and λ take place within each set of the values of n, p, r, and α. Thus, setting the conditions shown in Table 5.1 results in: (3 levels of p) $\times$ (3 levels of r) $\times$ (2 levels of n) $\times$ (2 levels of α) = 36 different Minimax charts. For each of these charts, the out-of-control ARL is calculated for 7 levels of $\lambda(\Delta)$, 2 levels of S, and 9 levels of α_4. Note that the levels of $\lambda(\Delta)$ and S result in 2×7=14 shifts in the mean.

Table 5.1 - Summary of the Experimental Conditions

Factor Name	Factor	Levels
Number of variables	p	{2, 3, 4}
Correlation coefficient	r	{-3/10, 0, 3/10}
Sample Size	n	{1, 5}
$P[Z_{[p]} > UCL_{[p]} \mid \Delta=0] = \alpha_4$	ξ	$\left\{\frac{1}{10}\times\frac{\alpha}{2}, \frac{2}{10}\times\frac{\alpha}{2}, \ldots, \frac{9}{10}\times\frac{\alpha}{2}\right\}$
Probability of Type I error	α	{1/125, 1/200},
Direction of the shift	S	{Axial, Diagonal},
Magnitude of the shift	$\lambda(\Delta)$	{0, 1/2, ..., 3}

The objective of the nine levels of α_4 is to choose the value of α_4 that minimizes, for each control chart, the average of the out-of-control ARLs which is calculated from the 2×7=14 shifts in the mean. That is to say, the objective of the nine levels of α_4 is to choose the optimal value of α_4 for each control chart, assuming that Axial and Diagonal shifts are equally likely. Finally, $36 \times 7 \times 2 \times 9 = 4536$ experimental conditions are set and the performance of the chart in terms of the out-of-control ARL is measured in each of them.

After the optimal value of α_4 is chosen, two more responses are measured for each of the 14 shifts in the mean in each of the 36 charts. These responses include: (1) the probability that a correct diagnosis is made in the system when a signal occurs (or $P[CD \mid \Delta]$ as described in Section 4.5), and (2) the out-of-control ARL for the Chi-Squared chart.

The experiment is run with a computer program that appropriately evaluates each of the experimental conditions using the statistical properties of the Minimax chart developed in Chapter 4. The program is coded in Mathematica Version 2.2 (1994) and the code is provided in Appendix G. After the data is collected they are analyzed as described in the next section.

5.2 Data Analysis:

After the experiment is completed and the data are gathered, the data are analyzed. The analysis consists of:

1. Selecting, for each of the 36 charts, the value of α_4 that results in the minimum out-of-control ARL, and plotting the out-of-control ARL and P[CD] for axial and diagonal shifts in the mean against the 7 levels of $\lambda(\Delta)$.

2. Fitting regression models to predict the behavior of the chart as a function of its parameters within the experimental region, and comparing the performance of the Minimax control chart to that of the Chi-squared chart.

The purpose of the next section is to explain in detail how this analysis is done, and to present the results obtained. The objectives of the regression models are discussed in detail in Section 5.2.2.

5.2.1 Selection of the Best Charts

Section 5.2.1.1 explains the general method used to select the value of α_4 that results in the best control chart for each experimental condition. The general results of this selection are also provided in graphical form in Section 5.2.1.2. That is to say, in that section some examples of the out-of-control ARLs, as functions of the seven shifts in the mean, are plotted for the optimal value of α_4 for some particular charts.

5.2.1.1 Selection of α_4

The first analysis is to select for each of the 36 charts the value of α_4 that minimizes the out-of-control ARL. To do this let $\mathrm{ARL}^{\mathrm{M}}_{i,j,k}$ represent the ARL for the

Minimax chart when it has experienced an S_i type of shift in the mean (i = 1, 2) of magnitude $\lambda(\Delta)_j$ (j = 2, ..., 7) and $\alpha_4 = \xi_k$ (k = 1, ..., 9). Now, the average ARL for the Minimax chart for a particular value of ξ_k is thus given by:

$$\overline{\mathrm{ARL}}_k^{\mathrm{M}} = \frac{1}{2} \times \frac{1}{6} \times \sum_{i=1}^{2} \sum_{j=2}^{7} ARL_{i,j,k}^{M} . \tag{5.6}$$

Similarly, the average ARL for the Chi-squared chart is expressed as:

$$\overline{\mathrm{ARL}}^{\mathrm{C}} = \frac{1}{6} \times \sum_{j=2}^{7} \mathrm{ARL}_j^{\mathrm{C}} . \tag{5.7}$$

Recall that the Chi-Squared chart is not sensitive to the direction of the shift but to its magnitude $\lambda(\Delta)$, thus, the ARL for the Chi-squared chart is the same for either axial or diagonal shifts in the mean. Note that $\overline{\mathrm{ARL}}^{\mathrm{C}}$ is not used for the selection of α_4. It is introduced here for simplicity and it will be used to compare the performance of both charts.

Also let Ω_k represent the expected difference in ARL between the Minimax and the Chi-squared chart when $\alpha_4 = \xi_k$. The expression for Ω_k is given by

$$\Omega_{\mathrm{k}} = \mathrm{E}\left[\mathrm{ARL}^{\mathrm{C}}\right] - \mathrm{E}\left[\mathrm{ARL}_k^{\mathrm{M}}\right], \ k = 1, 2, \ldots, 9 . \tag{5.8}$$

Since the expected values in (5.8) must be estimated using experimental data, Ω_k is approximated as

$$\Omega_k \approx \overline{\mathrm{ARL}}^{\mathrm{C}} - \overline{\mathrm{ARL}}_k^{\mathrm{M}} . \tag{5.9}$$

Note that a positive Ω_k means that the Minimax chart is expected to be faster than the Chi-Squared chart in detecting shifts from the set of uncontrollable parameters ($\lambda(\Delta)$, S). Thus, the optimal value of α_4 is selected as that with the associated maximum Ω_k, or defining

$$\xi^* = \left\{ \xi_m : \Omega_m = \max_{k=1}^{9}\{\Omega_k\} \right\} \tag{5.10}$$

the optimal value of α_4 is as given by

$$\alpha_4 = \xi^* . \tag{5.11}$$

In developing the selection criteria for the optimal α_4, it is assumed that axial, diagonal, and any other shifts in the mean are equally likely, as well as all elements of the vector $\lambda(\Delta)$. The value of α_4 that minimizes the out-of-control ARL for a given scenario and values of n and α is therefore selected using equation (5.11).

5.2.1.2 Graphical Representation of the Results

Figure 5.1 shows a comparison between the optimal ARL of the Minimax chart and the ARL of the Chi-squared chart. The probabilities of correctly diagnosing the system are also shown in the figure. These results correspond to the experimental condition n=1, p=4, r=3/10, and α=1/200 (plots for all experimental conditions are provided in Appendix D). As can be read in top of the figure, the value of α_4 that optimizes the performance of the chart for this experimental condition is: $a_4 = \frac{9}{10} \times \frac{\alpha}{2}$.

Figure 5.1 also shows that the Minimax chart is, on average, 1.68 samples faster than the Chi-squared chart in detecting shifts in the mean when these are described by the experimental levels of the uncontrollable parameters S and $\lambda(\Delta)$. Note that in this case, the Minimax chart is much faster than the Chi-squared chart in detecting diagonal shifts in the mean, but it is slower in the detection of axial shifts. This behavior is shown in 32 of the 36 charts studied. The four cases where p=4 and r=-3/10 (see Figures D.17, D.18, D.35, and D.36) show that the Chi-squared chart is faster detecting diagonal shifts. Thus, excepting these four cases, it seems that the Minimax chart is more powerful (in a probabilistic sense) than the Chi-squared chart in the detection of diagonal shifts in the mean. Also note that based on the statistic Ω, all plots in Appendix D show that the Minimax chart is better than the Chi-squared chart when r is either 0 or 3/10, because $\Omega>0$ in these cases. However, when r=-3/10 the Chi-squared chart tends to perform better in terms of the out-of-control ARL.

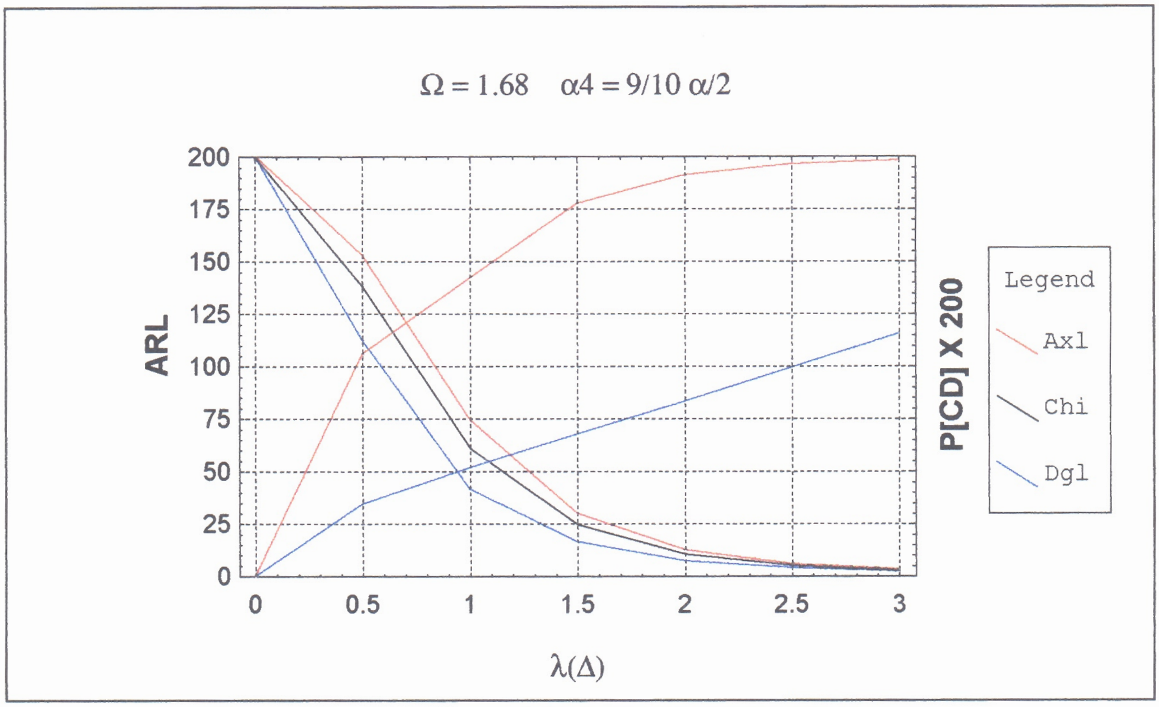

Figure 5.1 - Experimental Case n=1, p=4, r=3/10, α=1/200

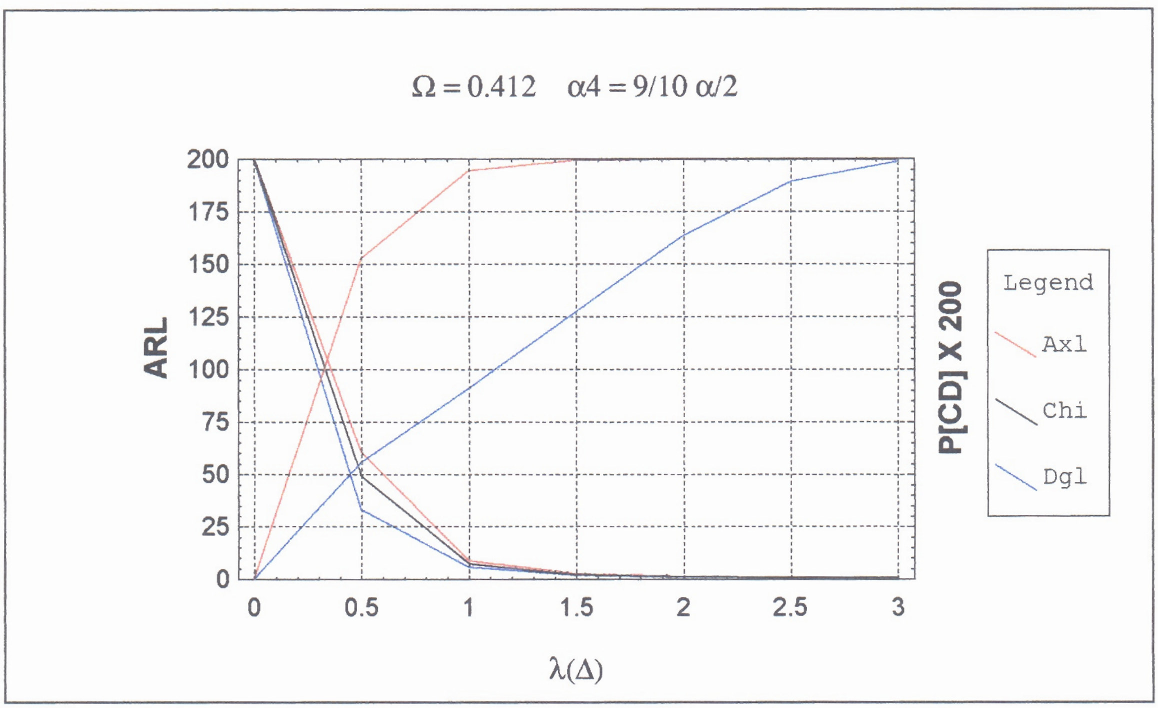

Figure 5.2 - Experimental Case n=5, p=4, r=3/10, α=1/200

Note that in Figure 5.1 the diagonal P[CD] is always smaller than the axial P[CD]. This pattern is present in the majority of the cases studied where $r=0$ or $r=3/10$. Another pattern of interest is that the axial P[CD] is approximately equal to 1 whenever $n=5$ and $\lambda(\Delta)>1.5$. Thus, the criteria established to diagnose axial shifts seems to be excellent for relatively large shifts in the mean and moderate sample sizes.

Figure 5.2 shows the same experimental condition as that shown in Figure 5.1 but for $n=5$. In this case, because of the increase in n, the ARLs of both, the Chi-squared and the Minimax charts, have decreased substantially. However, the average difference in ARL is 0.415, which is less than the previous case. Thus, in this particular case, the Chi-squared chart takes more advantage of the increase in sample size than does the Minimax chart.

Note that the increase in sample size also increases the probabilities of correctly diagnosing the system. Thus, increasing the sample size has two positive effects on the Minimax chart: decreasing the out-of-control ARL, and increasing the probability of correctly diagnosing the system when a signal occurs.

Since trying to analyze all 36 charts in detail is too extensive a task, to generalize the comparison between these charts in the experimental region, six regression models have been set and analyzed. The resulting regression equations and their interpretations are provided in Section 5.2.2.

5.2.2 Regression Models

In this section, six regression models are set with the data generated using the set of parameters described in Table 5.1. In the development of the models, a backward type of regression analysis was performed. A model that includes all the main factors and the two-factor interactions was used, where the factors are: p, n, α, α_4, and $\lambda(\Delta)$. For these factors the responses to be analyzed include: $\overline{ARL}^{M}$, Ω, α_3, ξ^*, Axial P[CD], and Diagonal P[CD]. All of these responses can be either read or calculated from the data shown in Appendix E. This appendix shows all the results obtained when each of the experimental conditions was evaluated. From this appendix, the appropriate sets of dependent and independent variables were selected to build the regression models.

The resulting models are based on a p-value to leave the regression model of 0.10. That is, variables with p-values larger than 0.10 were considered to have no significant effect on the response. A complete printout of the results is provided in Appendix F. The results obtained on each of the models are shown in the next sections.

5.2.2.1 Model for $\overline{\text{ARL}}^{M}$

Objective:

To predict the performance (ARL) of the Minimax chart as a function of the parameters included in the experiment. In other words, to estimate the effect of the experimental parameters on the ARL and to identify the direction in which the out-of-control ARL decreases.

Fitted Equation: (Adjusted $R^2 = 0.67$)

$$\overline{\text{ARL}}^{M} = 40.56091 - 68.23509\ \lambda - 9.90579\ n - 8.48683\ \alpha + 15.42119\ \alpha\ \lambda + 7.83961\ \lambda\ n$$

Conclusions:

- The factor p does not have a significant effect on $\overline{\text{ARL}}^{M}$. This means that, in the experimental region, the performance of the Minimax chart is not negatively affected by the increase in the number of variables.
- $\overline{\text{ARL}}^{M}$ is mainly determined by λ, n and α. Note that the factors r and α_4, do not appear to have a significant effect on $\overline{\text{ARL}}^{M}$.
- The factor λ has the largest effect on $\overline{\text{ARL}}^{M}$. An Increase in λ implies a decrease in $\overline{\text{ARL}}^{M}$. However, as can be inferred by the $\alpha\ \lambda$ interaction, the negative effect of λ

on $\overline{\text{ARL}}^{\text{M}}$ is reduced if both λ and α are both at their high levels. This means that when small shifts in the mean need to be detected, α should be set to larger values than when large shifts in the mean need to be detected. Similarly, the negative effect of λ on $\overline{\text{ARL}}^{\text{M}}$ is reduced if λ and n are both set to their high levels.

- As expected, an increase in n implies a decrease in ARL. However, the interaction λ n suggests that in order to decrease $\overline{\text{ARL}}^{\text{M}}$, n should be large for small shifts in the mean, and small if the shifts in the mean are large. Nonetheless, the effect of n alone is larger in magnitude than that of the interaction λ n, thus, the interpretation of the interaction is subject to the effect of n. In other words, an increase in n will always tend to reduce $\overline{\text{ARL}}^{\text{M}}$ regardless of the level of λ.

5.2.2.2 Model for Ω

Objective:

To predict the performance of the Minimax chart compared to that of the Chi-squared as a function of the parameters included in the experiment and to estimate the direction in which the Minimax outperforms the Chi-squared chart.

Fitted Equation: (Adjusted $R^2 = 0.58$)

$$\Omega = 3.74675 + 2.79367\ r - 2.16862\ p + 1.19173\ \alpha + 0.58088\ n - 9.41086\ \alpha\ \alpha_4 + 9.11265\ \alpha_4\ p + 5.84694\ \alpha\ r - 4.35445\ \alpha_4\ n - 3.94281\ \alpha_4\ r - 3.19444\ \alpha\ p - 2.47360\ p\ r + 1.83169\ \alpha\ n + 1.65021\ n\ r + 0.80860\ \lambda\ n$$

Conclusions:

- Since the effect of most of the interactions is much larger than the effect of the main factors, the interpretation of the model is complicated. However, some general conclusions can still be made.
- The magnitude of the shift in the mean (λ) does not have a strong effect on the difference in ARL. Thus, on average, both charts are equally effective in detecting small, moderate, and large shifts in the mean.
- Increase in r implies an increase in Ω. Thus, the Minimax chart performs better when the correlations are positive. However, when negative correlations are present the Chi-squared performs better. Note that the interaction α r has a large positive effect on Ω. Thus, when the variables are negatively correlated, the best way to improve the performance of the Minimax chart relative to the Chi-squared chart, is to make α small.
- There are four interactions that include α_4, and the effects of these interactions is, in all cases, larger than the effect of any of the main factors. Thus, the position of α_4

strongly determines the difference in performance between the Chi-squared and the Minimax chart.

- By taking the partial derivatives of the function of Ω with respect of each of the experimental factors, the maximal Ω is found to be at:

  $p = 0.5513$, $n = 0.0$, $\lambda = -0.0498077$, $\alpha = 0.124233$, $\alpha_4 = 0.546908$, $r = 0.977646$

  Note that the above results are in a (-1, 1) scale, thus, the maximum is approximately found at the following conditions:

  $p = 3$ or 4, $n = 3$, $\lambda = 3/2$, $\alpha = 1/165$, $\alpha_4 = 7/10\ \alpha/2$, $r = 3/10$

  These results suggest that a local maximum exists in the experimented region and thus, there is not such a thing as a direction in which Ω always increases.

5.2.2.3 Model for α_3

Objective:

To assess the dependency of α_3 on the correlation structure of a particular scenario for a given value of α_4. If the assumption of independence does not affect drastically the value of α_3, then the calculation of the control limits could be sped up by assuming that the variables are independent.

Fitted Equation: (Adjusted R^2 =0.99)

$$\alpha_3 = 0.00136 - 0.00168\,\alpha_4 + 0.00077\,\alpha + 0.00006\,r - 0.00004\,p + 0.00006\,\alpha_4\,r - 0.00002\,p\,r$$

Conclusion:

- Because of the effect of r and its interaction with p and α_4 it can be concluded that it is not a good idea to assume independence between the variables to find the control limits.

5.2.2.4 Model for ξ^*

Objective:

To predict the value of α_4 that results in the minimum ARL for the Minimax chart for a given set of parameters. In other words, the objective is to give a reasonable estimate of the value of α_4 that makes the Minimax chart minimize the out-of-control ARL for a given experimental condition inside the region studied.

Fitted Equation: (Adjusted R^2 = 0.96)

$$\xi^* = 6.30556 + 2.79167\,r - 0.33333\,p + 0.30556\,n + 0.68750\,p\,r - 0.20833\,n\,r$$

Conclusions:

- The optimal value of α_4 is mainly a function of r, p and n, of these, r has the largest effect on the optimal value of α_4.
- In the experimental region, α_4 should be large when r is positive or 0, and small otherwise.

5.2.2.5 Model for Diagonal P[CD]

Objective:

To evaluate the effectiveness of the diagnosis criteria described in Section 4.5 and its dependence on the parameters of the chart.

Fitted Equation: (Adjusted R^2 =0.77)

$$P[CD_{Diag}] = 0.60666 + 0.36815\ \lambda - 0.12812\ r + 0.06435\ n + 0.07191\ \alpha_4\ n + 0.05634\ \lambda\ n - 0.03871\ p\ r$$

Conclusions:

- The factor λ has the largest effect on $P[CD_{Diag}]$. The larger the λ the better the diagnosis of diagonal shifts.

- The factor r has the second largest effect on the $P[CD_{Diag}]$, the larger the r the smaller the $P[CD_{Diag}]$. Thus, the diagnosis of diagonal shifts in the mean is better for negatively correlated variables than for positively correlated ones.

5.2.2.6 Model for Axial P[CD]

Objective:

To evaluate the effectiveness of the diagnosis criteria described in Section 4.5 and its dependence on the parameters of the chart.

Fitted Equation: (Adjusted R^2 =0.75)

$$P[CD_{Axial}] = 0.79269 + 0.41101\,\lambda + 0.15089\,r + 0.08967\,n - 0.05529\,p + 0.02662\,\alpha - 0.18581\,\alpha_4\,r + 0.12809\,\alpha\,r + 0.11929\,\alpha_4\,p - 0.08224\,\alpha_4\,n - 0.04456\,\alpha\,p + 0.03072\,\alpha\,n$$

Conclusions:

- Again, λ plays an important role in the probability of correctly diagnosing the system. The larger the λ the larger the probability of making a correct diagnose.
- Same as in the $P[CD_{Diag}]$ case, r plays the second most important role in diagnosing the system when axial shifts in the mean occur. However, now r has a positive effect on $P[CD_{Axial}]$. Thus, for axial shifts in the mean, the chart diagnoses better the system when variables are positively correlated than when the variables are negatively

correlated. Note that α_4 r and α r are the two strongest interactions and both of them contain r.

This concludes this chapter. The next chapter presents conclusions concerning the use and evaluation of the Minimax chart as an alternative multivariate control chart.

Chapter 6

Conclusions

In general, it has been shown that the Minimax chart is a valuable resource to use in multivariate quality control problems. It has been demonstrated that the Minimax chart is a competitive multivariate control chart because it is reasonably fast in detecting shifts in the mean, it allows the user to set an exact probability of Type I error in the multivariate problem, it provides a simple solution to the problem of diagnosing signals, and, once it is designed, is perhaps the most user friendly multivariate control chart available.

Calculation of control limits:

The speed of calculation of the control limits on the Minimax chart is mainly dependent on the correlation matrix and on the dimension of the problem. If the variables are independent then the calculation of the control limits is basically instantaneous. If the correlation matrix ρ is such that $\rho_{ij} = \lambda_i \lambda_j$ for $\lambda_i \in [-1,1]$, $i \neq j$, then the control limits can be found for many variables since the computation of multivariate normal probabilities can be reduced to evaluating a one dimensional integral. When the structure of the correlation matrix cannot be matched to that previously described, then the control limits can be calculated for four variables using numerical integration. The results reported in this paragraph are based on the experience of the author using an IBM PC compatible with a

Pentium processor running at 66 MHz with 16 MB of RAM. Note that the methods provided by Deák (1980) or Dutt (1973) to calculate multivariate normal probabilities should not be used to estimate the control limits of the Minimax chart because these cannot assure more than three digits of precision. By inspecting the resulting values of α_3 for each experimental condition, the reader should realize that at least four digits of precision are needed to set the control limits.

Sample size determination:

No analytical solution in closed form was found for the problem of determining the value of n that results in a particular probability of Type II error for a given shift in the mean. This was the case because to do this, a closed form solution to the multivariate normal probability integrals would be necessary and this problem is still unsolved. Thus, to solve this problem the user needs to specify the exact shift in all variables means and then, using a numerical search, determine the position of the control limits that will result in the specified probability of Type II error.

Diagnosing the signals:

In terms of the diagnosis of the system, it was found that the heuristic criteria described in Section 3.2.3 results in an increasing probability of a correct diagnosis as the shift in the mean increases in magnitude. This behavior was found in all the cases studied. It was also found that in all cases where r was non negative, the probability of a correct diagnosis of an axial shift in the mean was greater than 90% for $\lambda(\Delta) > 1$ and n=5. For negatively correlated variables, the criteria used to diagnose diagonal shifts seems to be better than that designed to diagnose axial shifts, especially when the sample size is equal to one.

Optimal value of α_4:

As expected, the value of α_4 has a significant effect on the performance of the chart. In the cases studied, the value of α_4 that optimized the performance of the chart turned out to be a function of n, p, and r, as described in Section 5.2.2.4. Thus, the optimal value of α_4 cannot be generalized but must be determined every time that a Minimax chart is to be designed.

Comparing the Minimax to the Chi-squared chart:

In terms of the out-of-control ARL, the Minimax chart has proved to be an excellent tool to monitor multivariate processes. Based on the cases studied and the assumptions made in this dissertation, the Minimax chart is usually faster than the Chi-squared control chart in detecting shifts in the process mean. In particular, for non-negatively correlated variables, the Minimax chart was always faster than the Chi-squared in detecting shifts in the mean. It was also found that the difference in the out-of-control ARL between the Minimax and the Chi-squared charts tends to decrease as the magnitude of the shift increases. In most cases the difference in ARL was less than one for shifts in the mean of magnitude greater than 1.5.

One further advantage of the Minimax chart over the Chi-squared is that $Z_{[1]}$ and $Z_{[p]}$ are plotted in a time axis, thus, it is possible to make inferences on the process based on time. For example, the process could be said to be out of control if $Z_{[p]} > k$ for c consecutive times, where k and c are two predetermined constants. Another advantage is that the diagnosis of the system is made right at the moment of plotting $Z_{[1]}$ and $Z_{[p]}$ on the chart without the need of any other analysis. These two advantages, plus a faster detection of shifts in the mean in non-negatively correlated problems, make the Minimax chart more attractive than the Chi-squared chart at least on the cases studied in this dissertation.

Chapter 7

Future Research

Incorporation of Run Rules:

One possible enhancement to the Minimax chart is to develop run rules to detect, or diagnose, shifts in the mean. In this way, the probability of detecting shifts in the mean, or the probability of correctly diagnosing the system, can be increased to higher levels. Some examples of these rules are:

- Signal if the same variable is the maximum (minimum) for k consecutive times, and conclude that the mean of that same variable has increased (decreased).
- Signal if the minimum (maximum) is larger than a cut off value γ, and conclude that a positive (negative) diagonal shift in the mean has occurred.

Sequential Tests in the Same Sample:

If the assumption of either axial or diagonal shifts in the mean is relaxed, then the interpretation of the signal changes. In this situation the fact that only one variable is identified as responsible for the shift does not exclude the possibility that other variables have also experienced a shift in the mean. For example in a 15 variables case it is possible that three variables experience a shift at the same time. With the diagnosing criteria established in this dissertation only one of these variables can be identified as responsible for the shift. This problem can be addressed with sequential Minimax control charts. The basic idea is to plot a sequential Minimax chart for each variable other than the ones that have signaled until no signal is detected. In the example given, four sequential tests in the same sample would be necessary to detect the shift in the three variables. From these four tests, the first three should signal and the last one should not. The first test is done in 15 variables, the second in 14, the third in 13, and the fourth test in 12 variables.

Separation of Scale from Location in the Shifts:

Another possible extension to this dissertation is to study the sensitivity of the chart to changes in covariance. To diagnose the signals, it sounds reasonable to say that if more than two points lie outside the control limits in the same sample, then the covariance structure has changed. In this case, the Minimax chart will be capable of differentiating location shifts (shifts in the mean) from scale shifts (shifts in the covariance structure) in the process. It is also important to investigate the Minimax chart when the covariance matrix is assumed to be unknown. That is, compare the Minimax chart to Hotelling's T^2 control chart. One more addition to this dissertation is to compare the axial and diagonal P[CD] of the Minimax chart with that of the Chi-squared.

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Appendices

Appendix A

The Covariance of Two Averages

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)^T$ respectively represent a sample of size n for the variables X and Y. Assume that x_i and y_j are dependent for i=j and that they are independent otherwise. Under this conditions $\mathrm{Cov}[\bar{x}\,\bar{y}] = \dfrac{\mathrm{Cov}[x,y]}{n}$.

Proof:

First note that by definition

$$\mathrm{Cov}[x, y] = \mathrm{E}[(x-\mu_x)\,(y-\mu_y)] = \mathrm{E}[xy] - \mu_x\,\mu_y. \tag{A.1}$$

Also note that

$$\begin{aligned}\mathrm{E}[\bar{x}\,\bar{y}] &= \frac{1}{n^2}\mathrm{E}\left[\sum_{i=1}^{n} x_i \sum_{j=1}^{n} y_j\right] = \frac{1}{n^2}\mathrm{E}\left[\sum_{i=1}^{n}\sum_{j=1}^{n} x_i y_j\right] \\ &= \frac{1}{n^2}\mathrm{E}\left[\sum_{k=1}^{n} x_k y_k + \sum_{i=1}^{n}\sum_{\substack{j=1 \\ i\neq j}}^{n} x_i y_j\right]\end{aligned} \tag{A.2}$$

Note that since x_i and y_j are independent for $i\neq j$ then,

$$\frac{1}{\mathrm{n}^2}\mathrm{E}\left[\sum_{i=1}^{n}\sum_{\substack{j=1 \\ i\neq j}}^{n} x_i y_j\right] = \frac{n(n-1)}{n^2}\mathrm{E}[x_i]\ \mathrm{E}[y_j] = \frac{(n-1)}{n}\mu_x\mu_y. \tag{A.3}$$

Since the expectation of the sum is equal to the sum of the expectations then,

$$\frac{1}{n^2}\mathrm{E}\left[\sum_{k=1}^{n} x_k y_k\right] = \frac{1}{n^2}\sum_{k=1}^{n}\mathrm{E}[x_k y_k] = \frac{1}{n}\mathrm{E}[x_k y_k] \tag{A.4}$$

Now, from equation (A.1)

$$\mathrm{E}[x_k y_k] = Cov[x,y] + \mu_x\mu_y \tag{A.5}$$

thus, combining (A.4) and (A.5)

$$\frac{1}{n^2}\mathrm{E}\left[\sum_{k=1}^{n} x_k y_k\right] = \frac{1}{n}\left(Cov[x_k, y_k] + \mu_{x_k}\mu_{y_k}\right). \tag{A.6}$$

Note that in (A.6) $Cov[x_k, y_k] = Cov[x, y]$ since x_i and y_j are dependent for i=j and are independent otherwise. Thus all the correlation between x_i and y_j is due to the case where i=j. Also note that in (A.6) $\mu_{x_k}\mu_{y_k} = \mu_x\mu_y$. Thus,

$$\frac{1}{n^2}\mathrm{E}\left[\sum_{k=1}^{n} x_k y_k\right] = \frac{1}{n}\left(Cov[x, y] + \mu_x\mu_y\right) \tag{A.7}$$

and, thus combining equations (A.2) and (A.7)

$$\mathrm{E}\left[\bar{x}\,\bar{y}\right] = \frac{1}{n}\left[Cov[x, y] + \mu_x\mu_y\right] + \frac{(n-1)}{n}\mu_x\mu_y = \frac{Cov[x, y]}{n} + \mu_x\mu_y. \tag{A.8}$$

Finally, combining equations (A.1) and (A.8)

$$Cov[\bar{x}, \bar{y}] = E[\bar{x}, \bar{y}] - E[\bar{x}]\,E[\bar{y}] = \frac{Cov[x, y]}{n} + \mu_x\mu_y - E[\bar{x}]\,E[\bar{y}] = \frac{Cov[x, y]}{n} \tag{A.9}$$

and this completes the proof.

Appendix B

The Chi-Squared Control Chart

In this appendix the basic concepts of multivariate hypothesis testing techniques needed for the development of this dissertation are discussed. Hypothesis testing in a multivariate context is a widely used technique in the statistics field. Thus, the basic concepts of multivariate hypothesis testing can be found in any elementary multivariate analysis text as that of Renchester (1995).

Multivariate test for H_0: $\mu=\mu_0$, H_1: $\mu\neq\mu_0$, with Σ and μ known.

Under the assumption of a multivariate vector of p elements the test statistic Z^2 in (B.1)

$$Z^2 = n\left(\overline{\mathbf{X}}-\mu\right)^T \Sigma^{-1}\left(\overline{\mathbf{X}}-\mu\right) \tag{B.1}$$

is distributed as $\chi^2_{\alpha,p}$ if H_0 is true, where the parameter p represents the degrees of freedom associated with the statistic. In this case H_0 is accepted if $Z^2 < \chi^2_{\alpha,p}$ and rejected otherwise.

When H_1 is true, $Z^2 \sim$ Non Central Chi-Squared distribution with p degrees of freedom and non-centrality parameter $n\lambda^2(\Delta)$. Thus, H_0 is rejected if $Z^2 > \chi^2_{\alpha,p}$ and accepted otherwise.

Multivariate test for H_0: $\mu=\mu_0$, H_1: $\mu\neq\mu_0$, with Σ unknown and μ known.

Under the assumption of a multivariate vector of p elements the test statistic T^2 in (B.2)

$$T^2 = n\left(\overline{\mathbf{X}} - \mu\right)^T S^{-1}\left(\overline{\mathbf{X}} - \mu\right), \tag{B.2}$$

where S is the is the covariance matrix estimated from the sample, is distributed as $T^2_{\alpha,\,p,\,n-1}$. The distribution of T^2 is indexed by two parameters: the dimension p and the degrees of freedom n-1. In this case we reject H_0 if $T^2 > T^2_{\alpha,\,p,\,n-1}$. This is the well known Hotelling's T^2 statistic, which application to multivariate quality control is well described by Alt (1985).

In this dissertation the χ^2 statistic is used instead of the T^2 to test whether the mean vector has changed or not because it is assumed that the covariance matrix is known. The so called Chi-squared control chart is simply based on the sequential use of the χ^2 statistic to test for control in multivariate quality problems.

Appendix C

Calculation of the Average Run Length (ARL)

One commonly used measure of performance in control charts is the average run length (ARL). Based on the assumption that the samples are sufficiently separated in time such that they are independent from each other, the in-control run length, defined as the number of samples taken until the first false alarm occurs, can be modeled using the geometric distribution. Thus, the in-control ARL is the expected value of a geometric random variable with parameter α, where α is the probability of a false alarm. In the Minimax chart α is as given in (4.23) and thus the ARL is calculated using equation C.1 below.

$$\mathrm{ARL} = \frac{1}{\mathrm{P[point\ out\ of\ the\ control\ limits} \mid \Delta = 0]} = \frac{1}{\alpha} \tag{C.1}$$

The out-of-control run length, defined as the number of samples taken until the control chart signals given that the process is out of control, can also be modeled using the geometric distribution. In the Minimax chart it is assumed that subsequent samples are independent of each other, thus the distribution of the out-of-control run length is geometric with parameter β_Δ, where β_Δ is as defined in equation (4.24). Thus, the out-of-control ARL for the Minimax chart is as given in equation (C.2) below.

$$\mathrm{ARL} = \frac{1}{1-\beta_\Delta} \tag{C.2}$$

Appendix D

Plots of the ARL for the Experimental Conditions

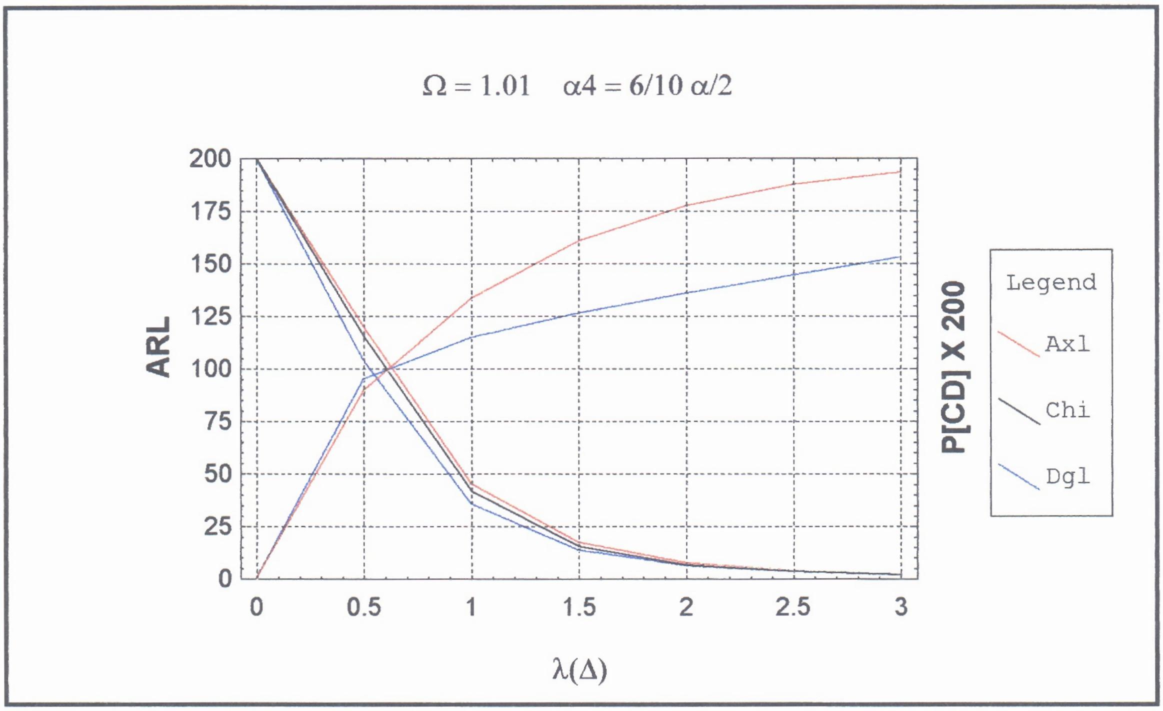

Figure D.1- Experimental Case n=1, p=2, r=0/10, α=1/200

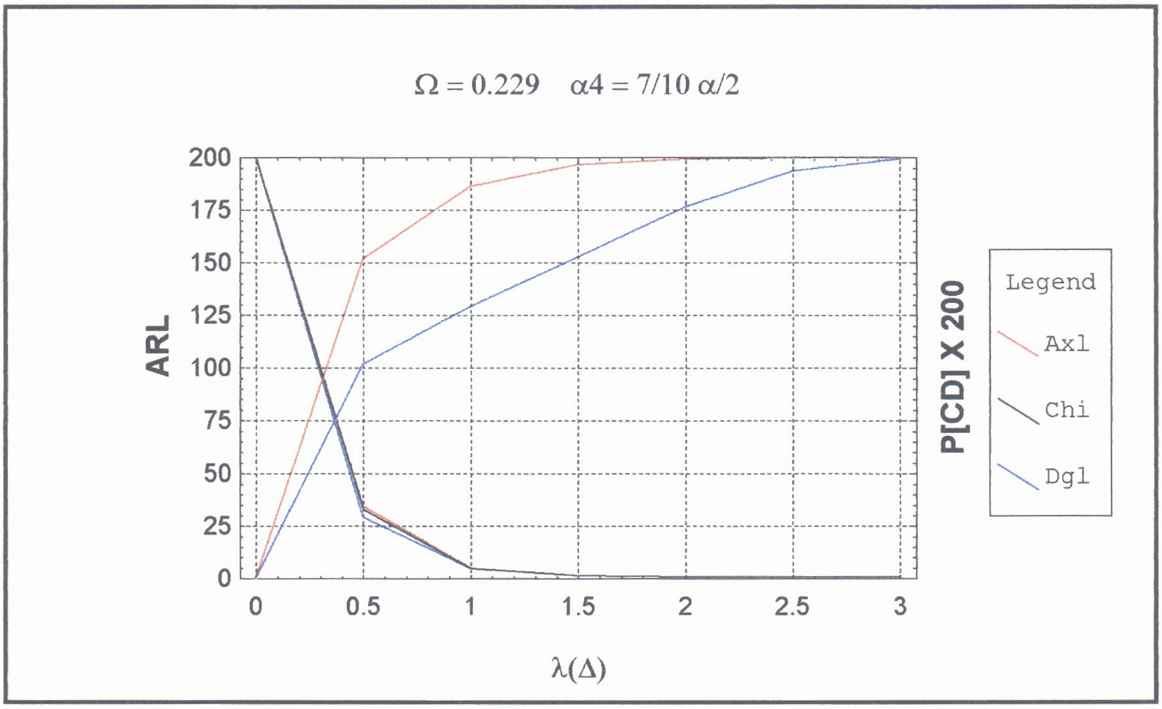

Figure D.2 - Experimental Case n=5, p=2, r=0/10, α=1/200

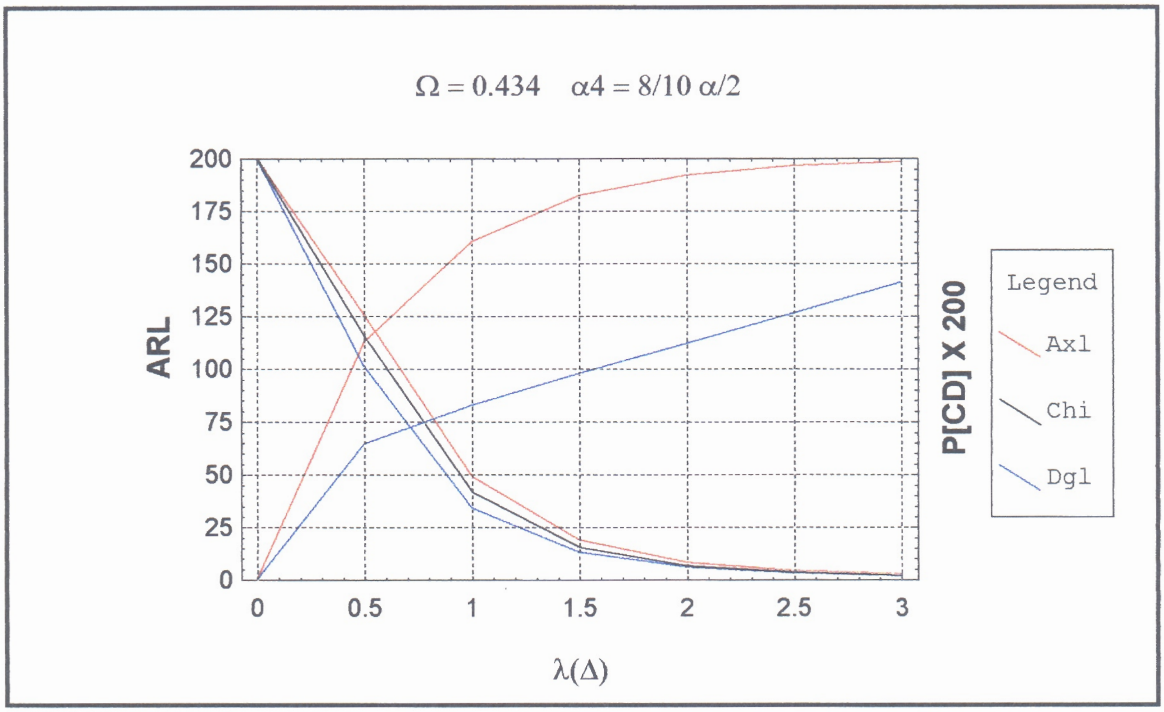

Figure D.3- Experimental Case n=1, p=2, r=3/10, α=1/200

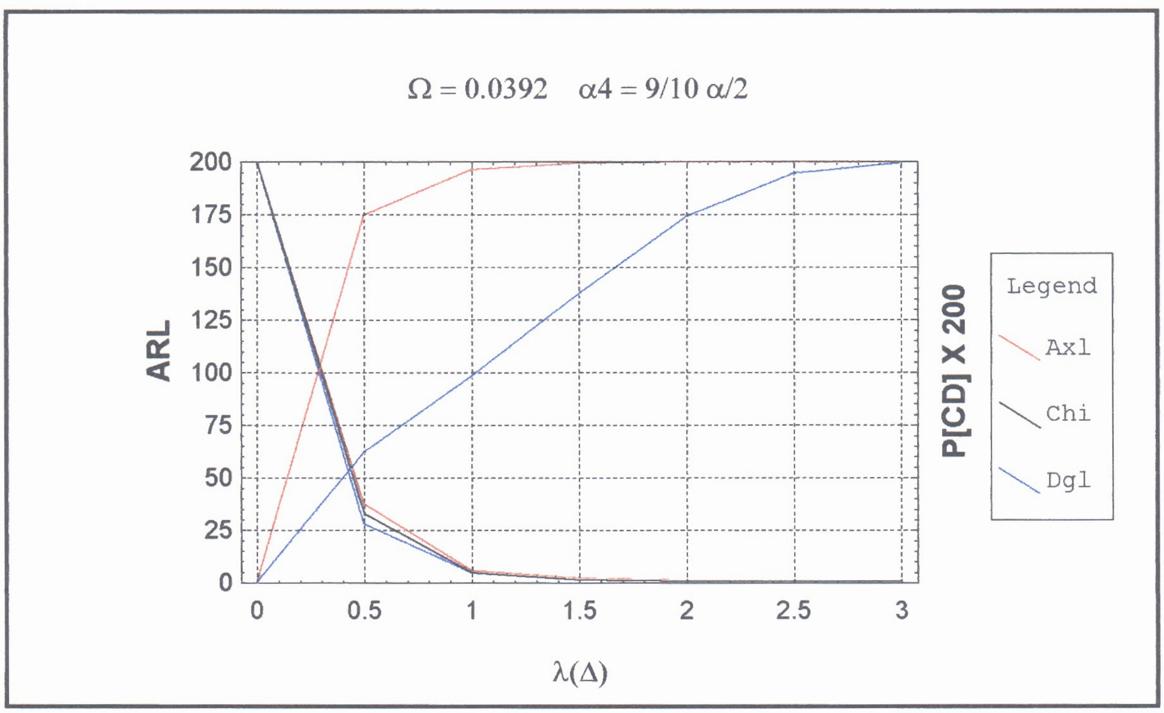

Figure D.4 - Experimental Case n=5, p=2, r=3/10, α=1/200

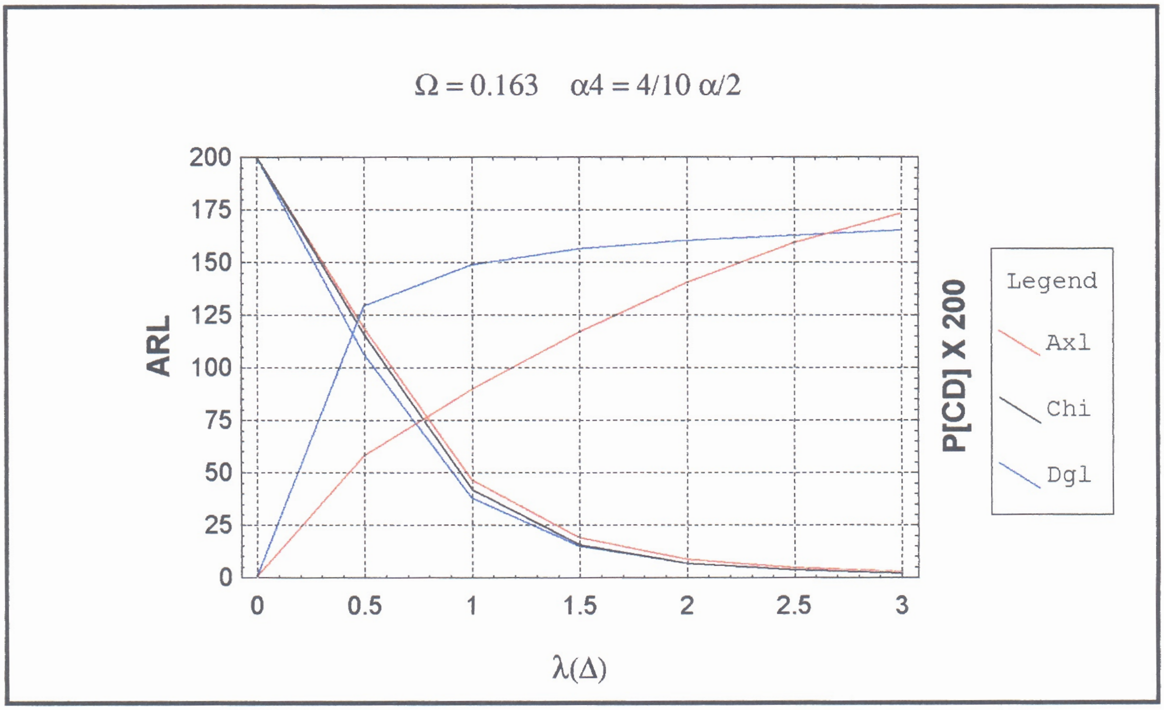

Figure D.5- Experimental Case n=1, p=2, r=-3/10, α=1/200

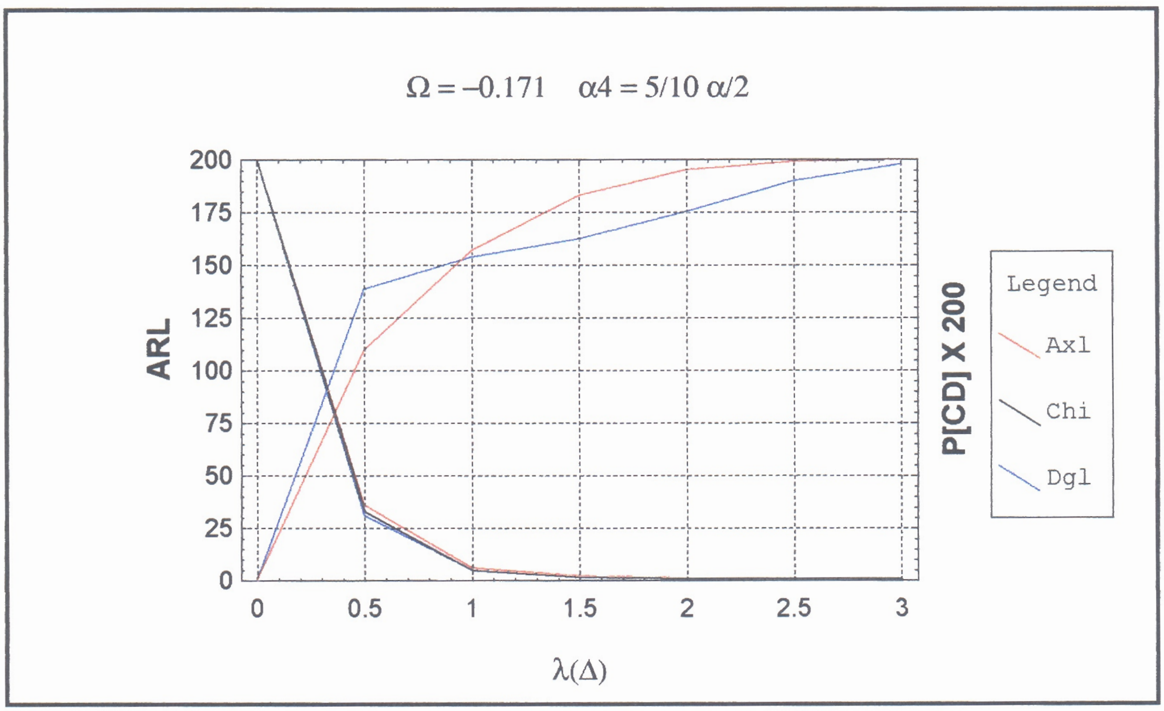

Figure D.6 - Experimental Case n=5, p=2, r=-3/10, α=1/200

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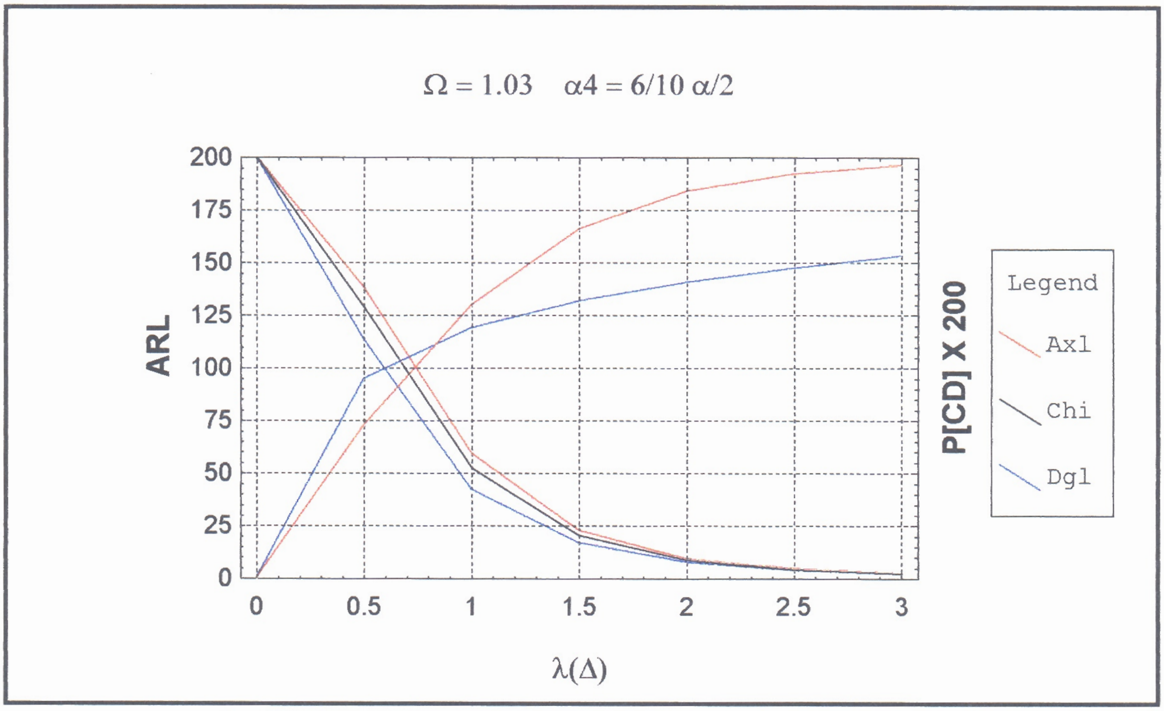

Figure D.7- Experimental Case n=1, p=3, r=0/10, α=1/200

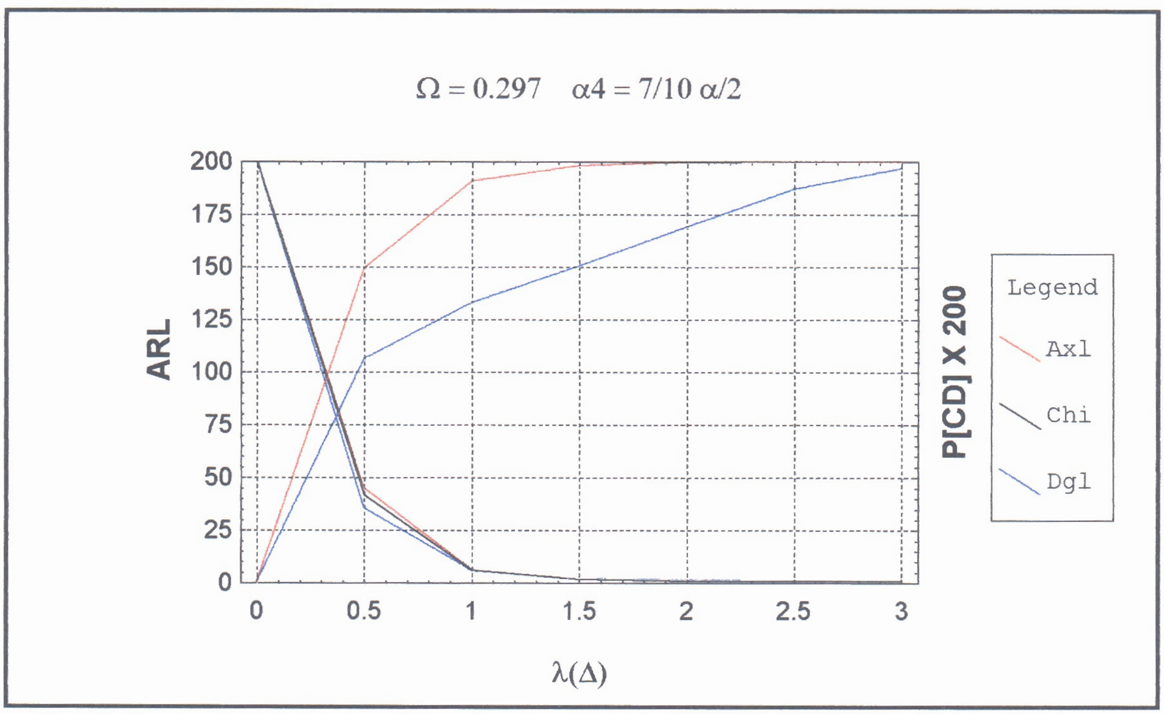

Figure D.8 - Experimental Case n=5, p=3, r=0/10, α=1/200

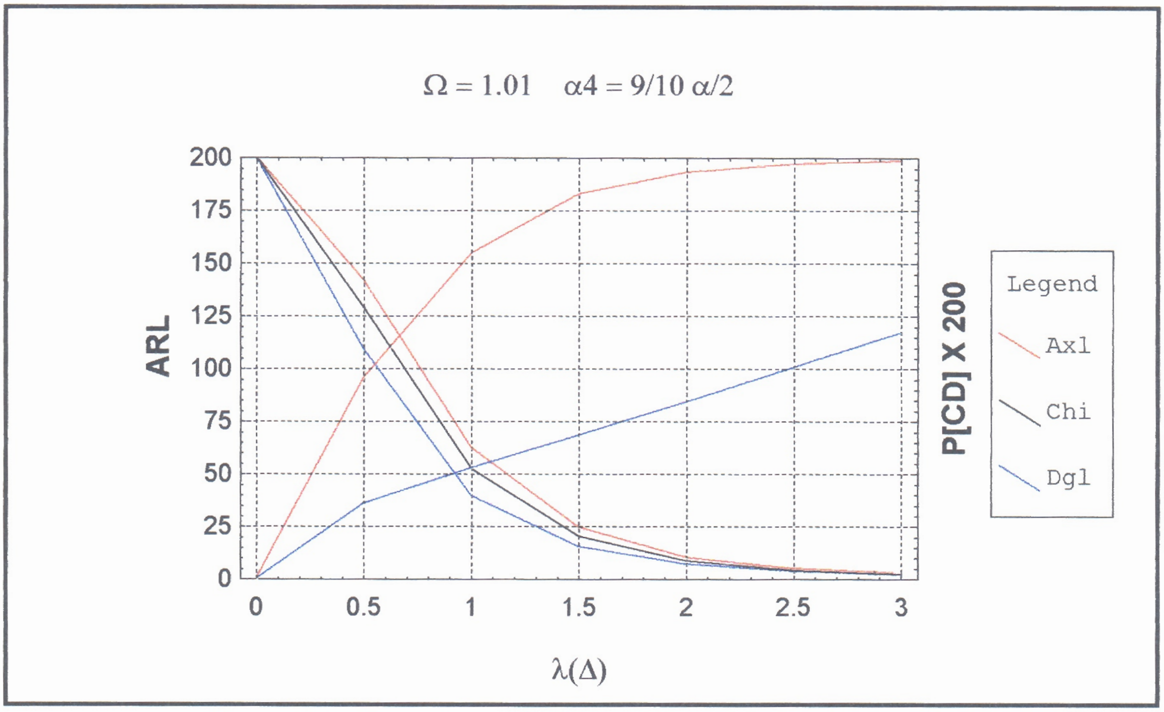

Figure D.9- Experimental Case n=1, p=3, r=3/10, α=1/200

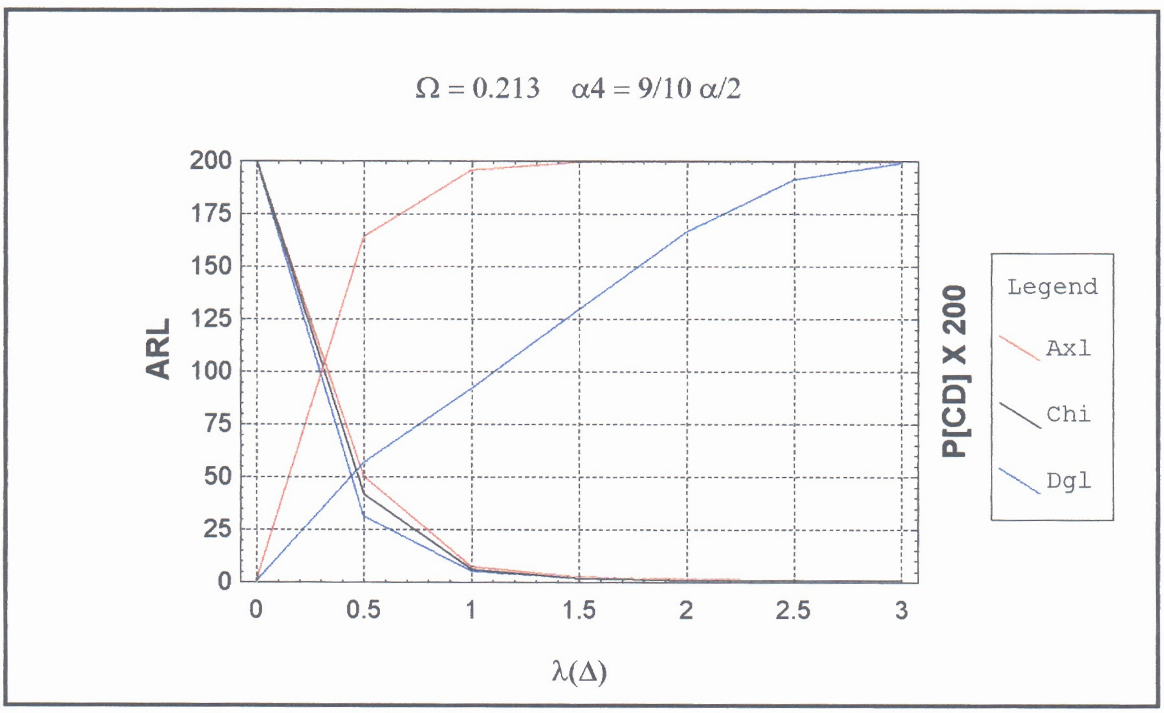

Figure D.10 - Experimental Case n=5, p=3, r=3/10, α=1/200

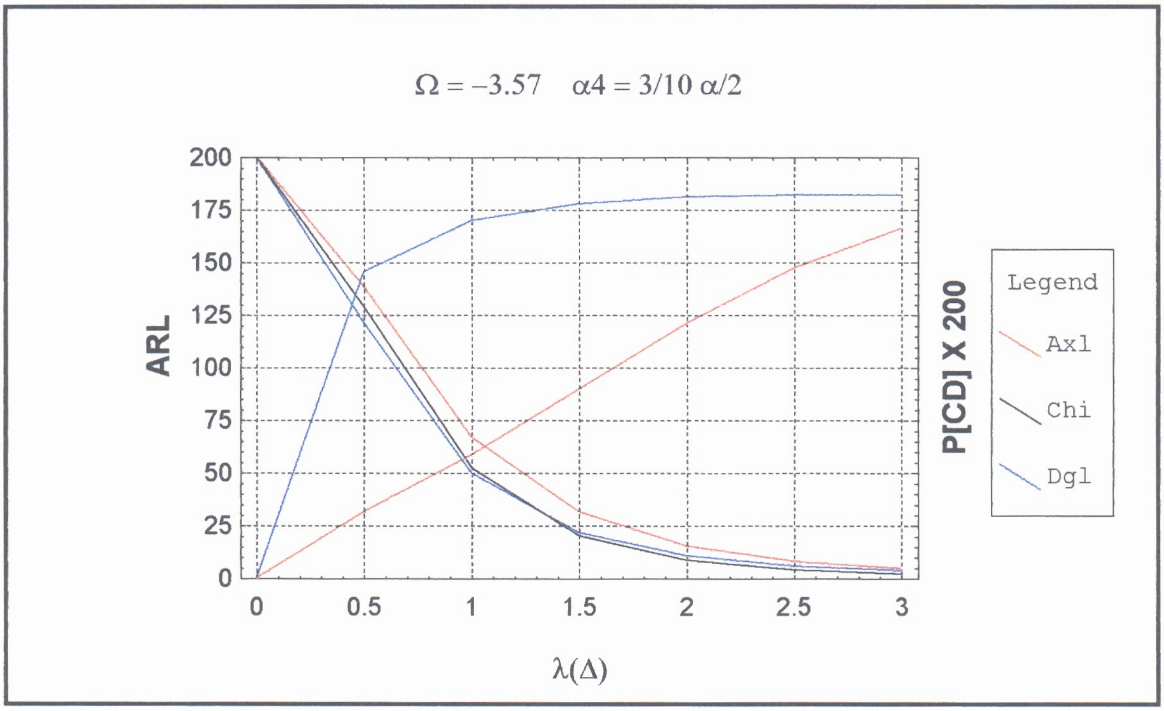

Figure D.11- Experimental Case n=1, p=3, r=-3/10, α=1/200

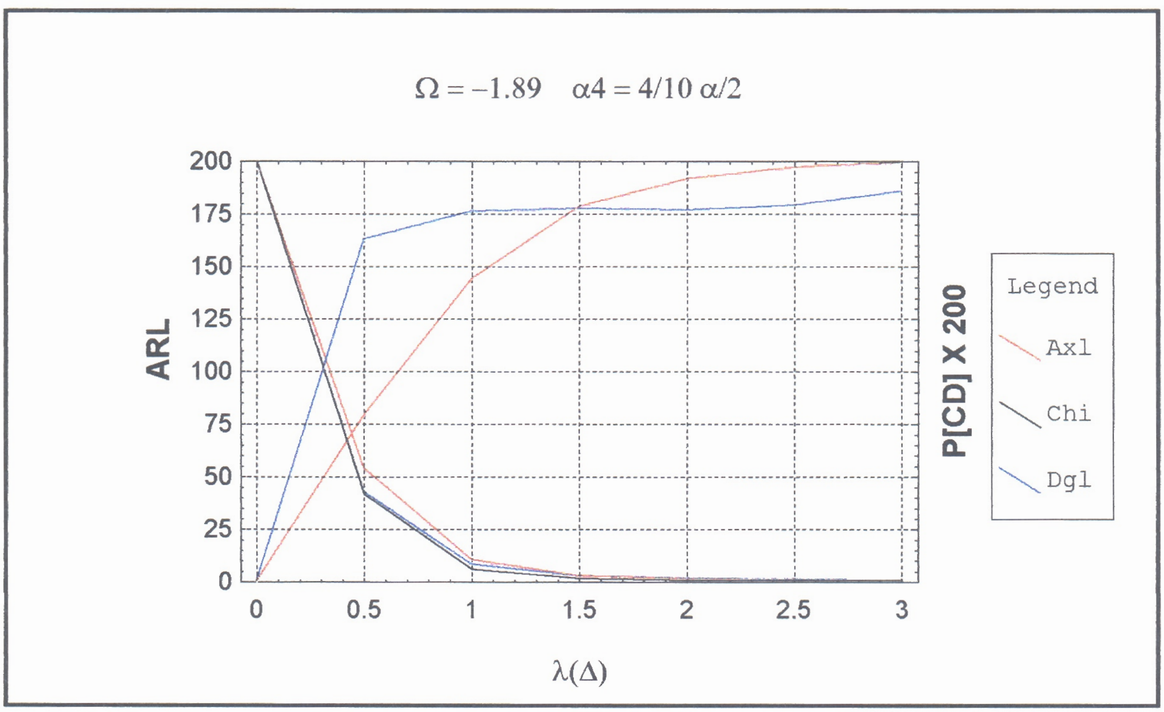

Figure D.12 - Experimental Case n=5, p=3, r=-3/10, α=1/200

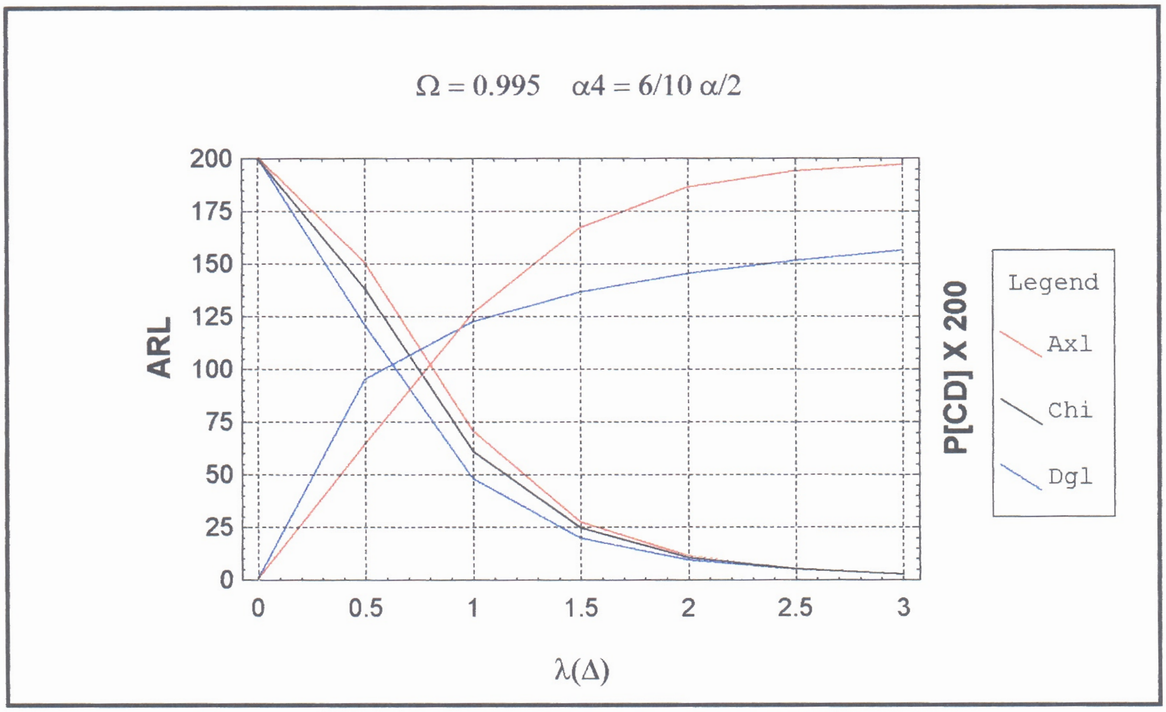

Figure D.13- Experimental Case n=1, p=4, r=0/10, α=1/200

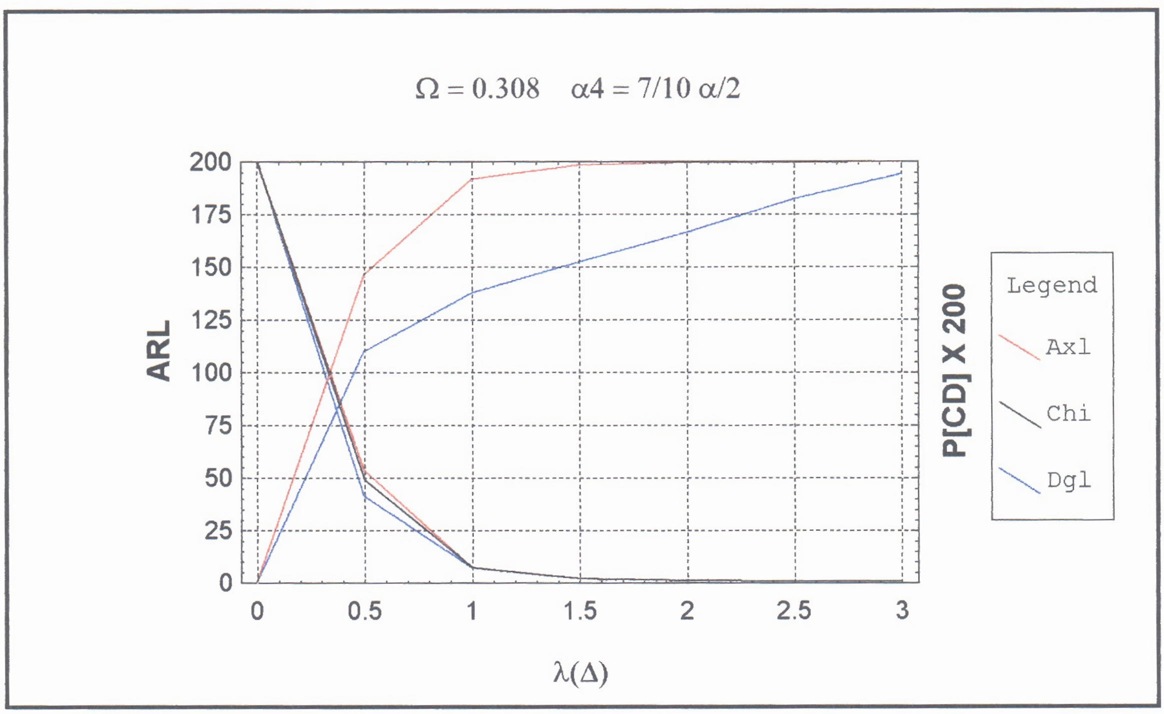

Figure D.14 - Experimental Case n=5, p=4, r=0/10, α=1/200

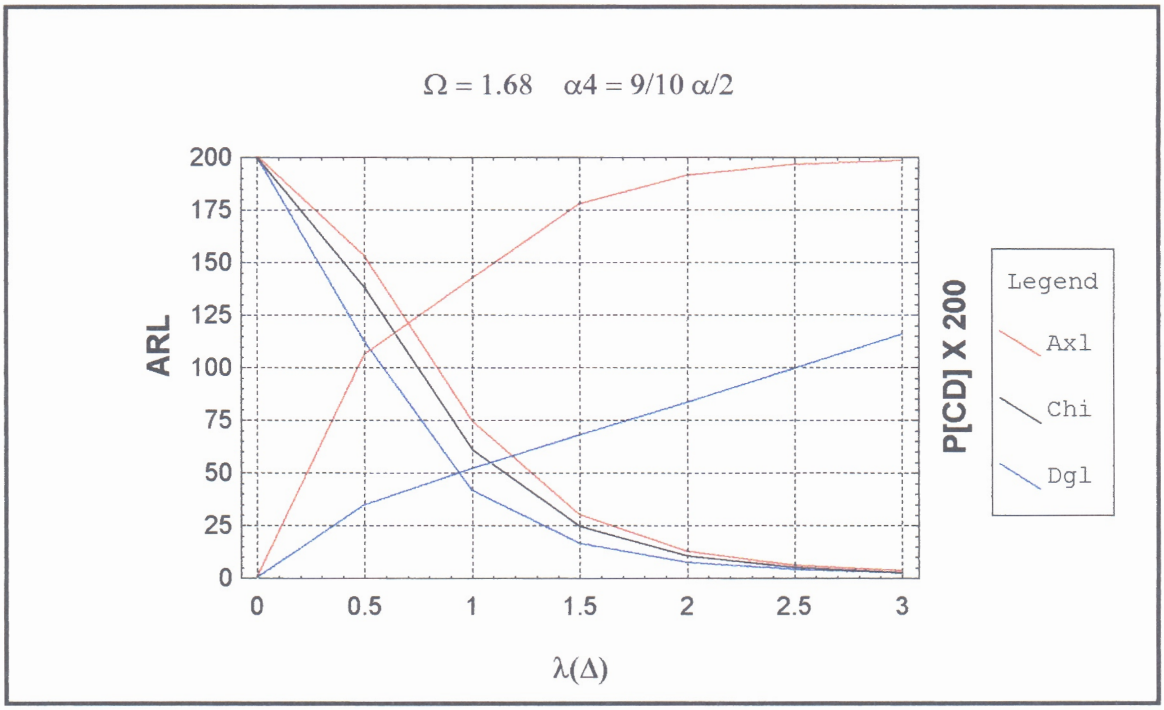

Figure D.15- Experimental Case n=1, p=4, r=3/10, α=1/200

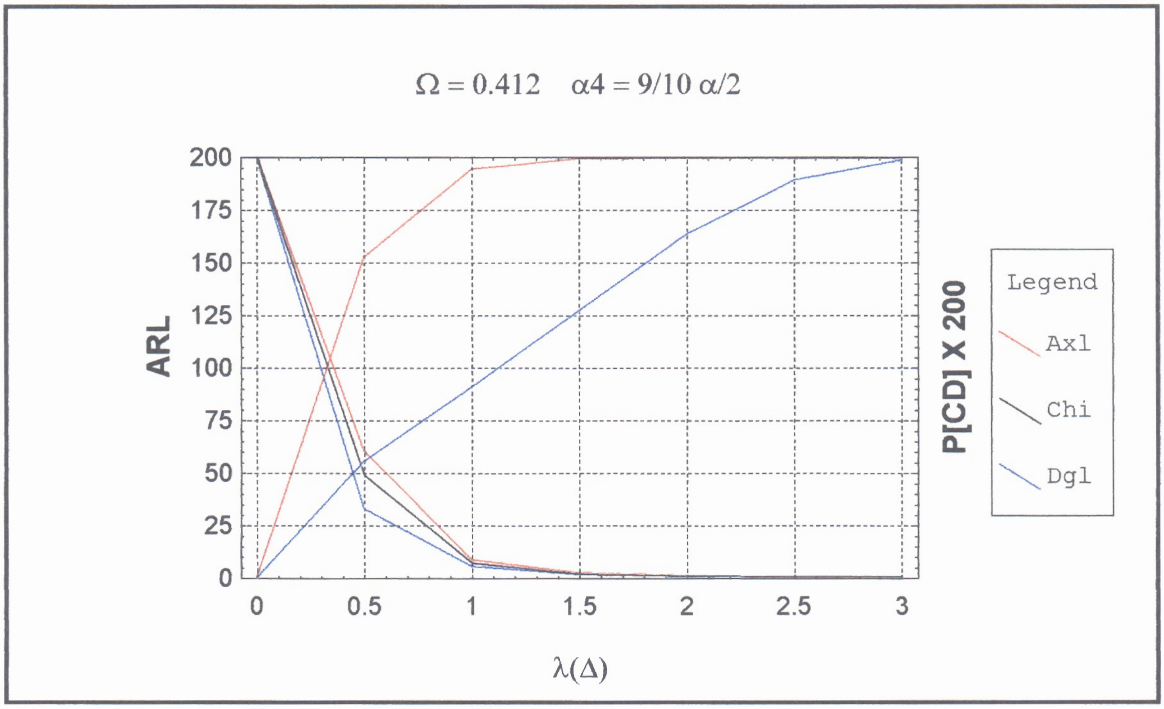

Figure D.16 - Experimental Case n=5, p=4, r=3/10, α=1/200

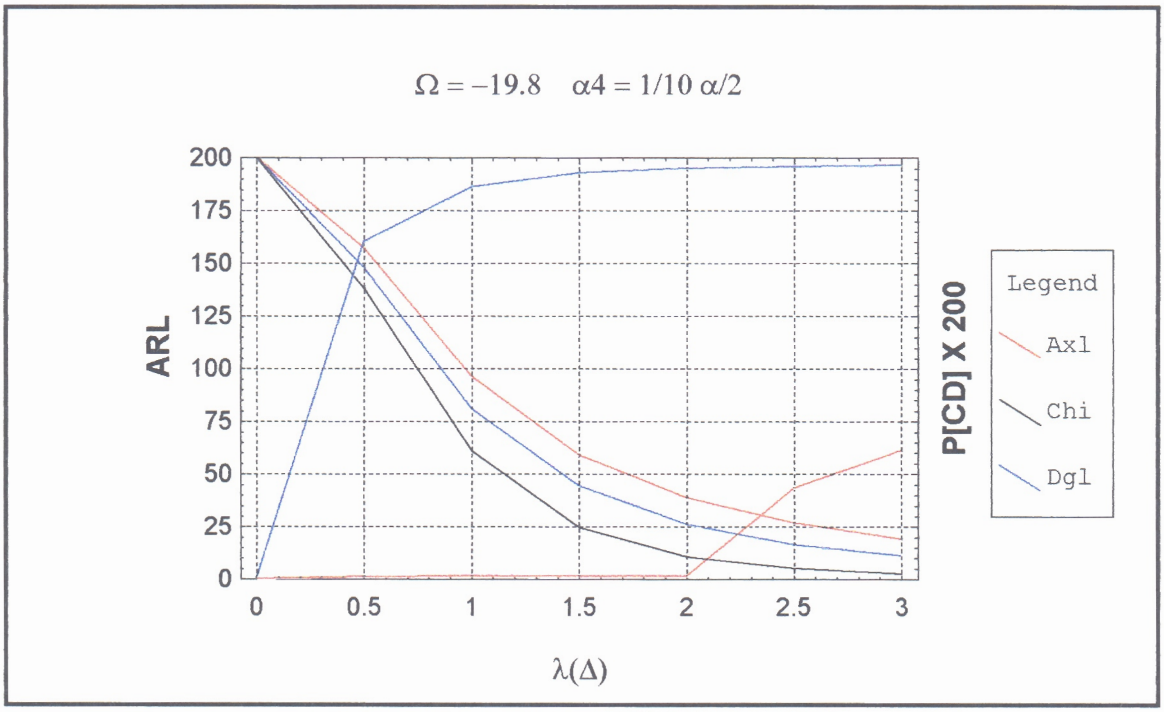

Figure D.17- Experimental Case n=1, p=4, r=-3/10, α=1/200

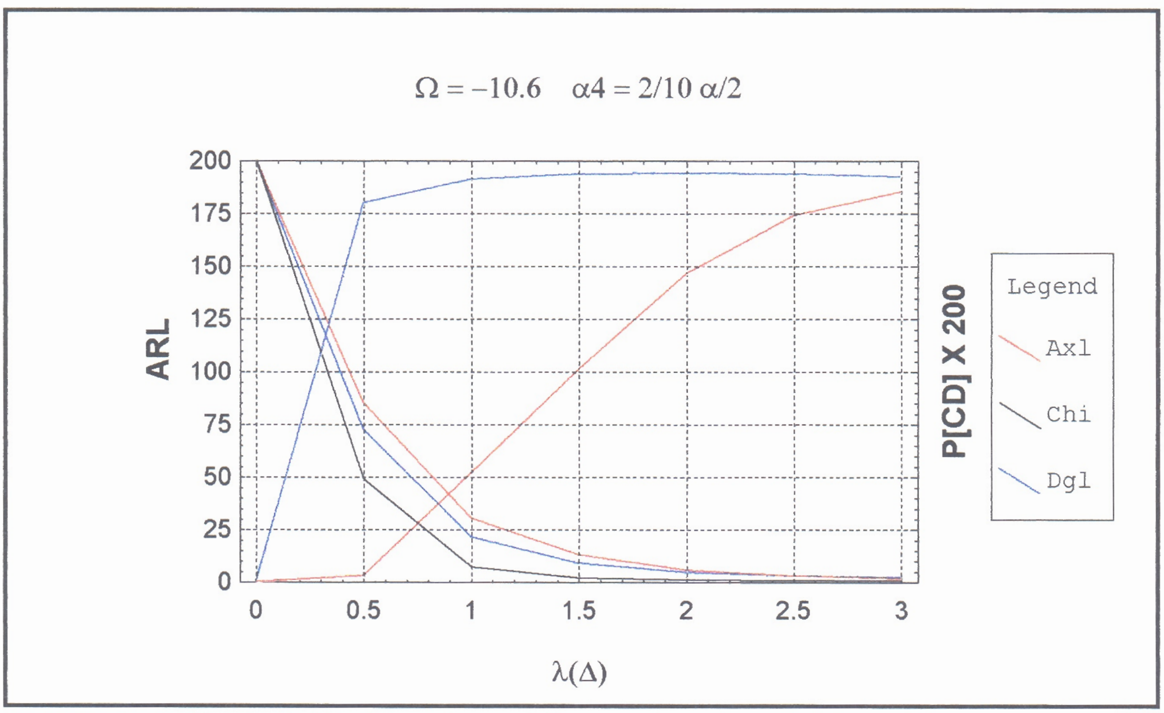

Figure D.18 - Experimental Case n=5, p=4, r=-3/10, α=1/200

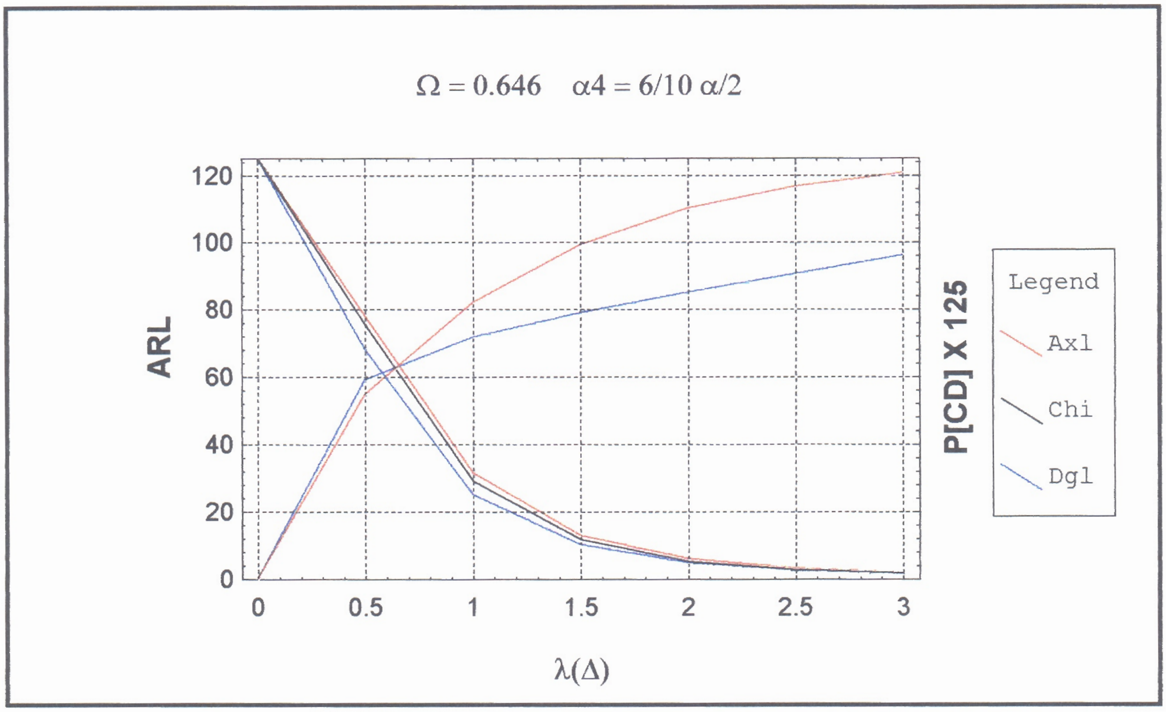

Figure D.19- Experimental Case n=1, p=2, r=0/10, α=1/125

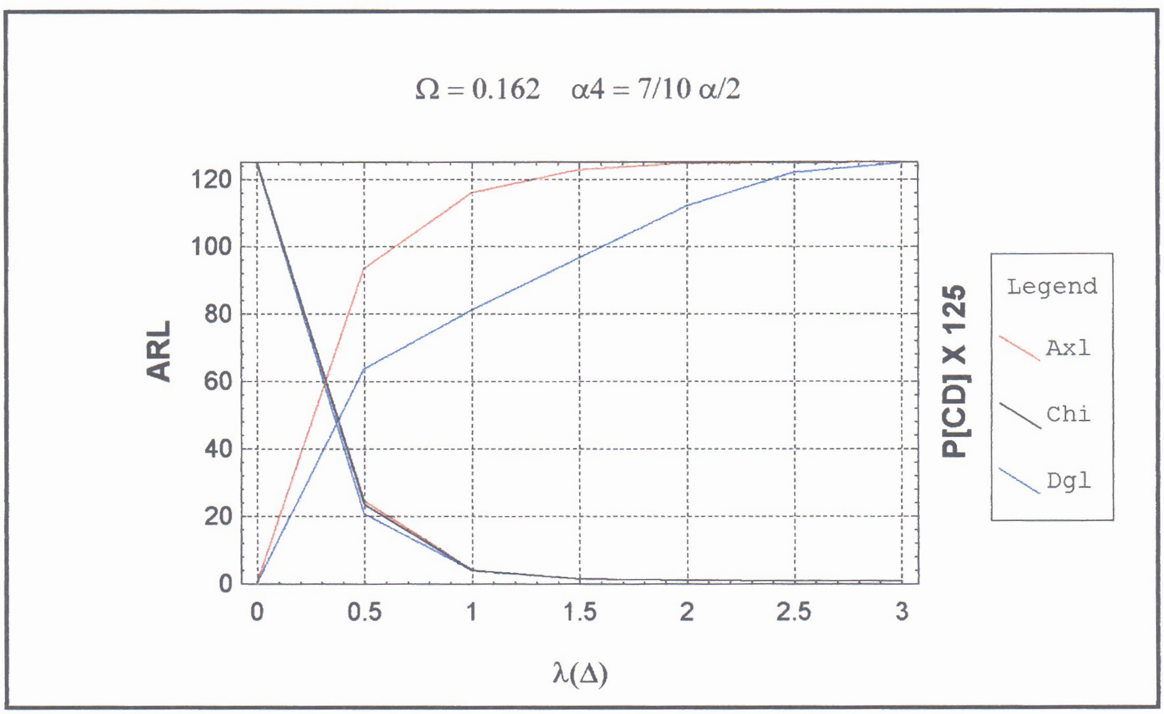

Figure D.20 - Experimental Case n=5, p=2, r=0/10, α=1/125

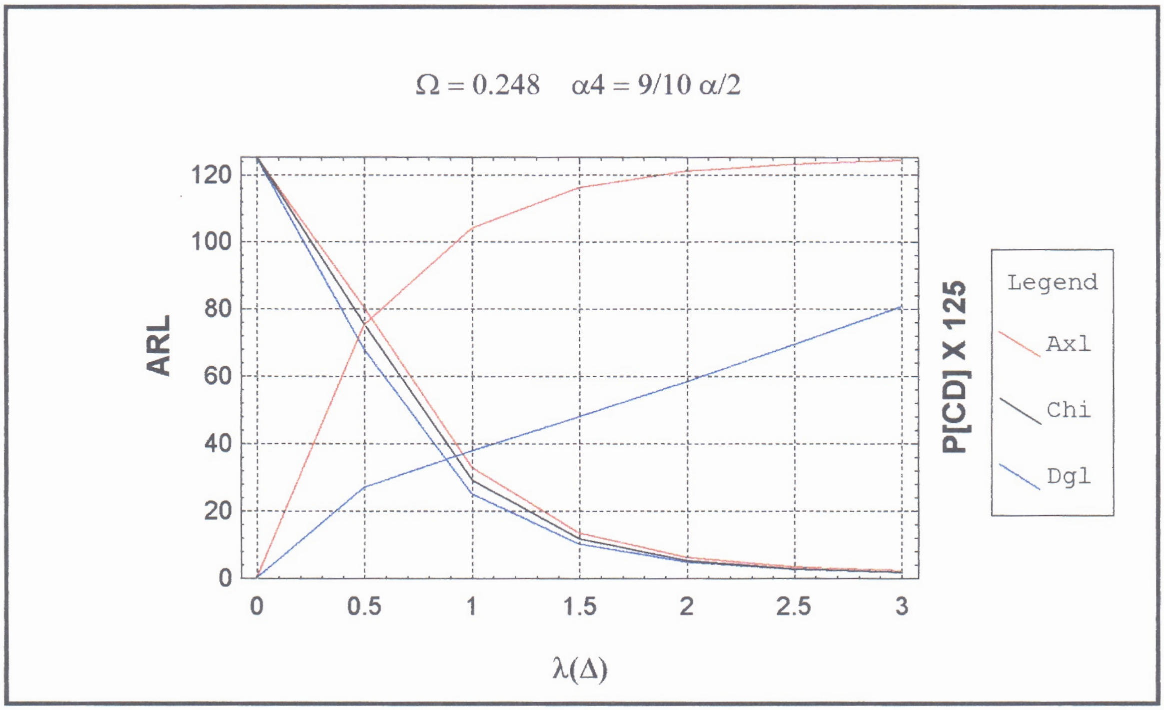

Figure D.21- Experimental Case n=1, p=2, r=3/10, α=1/125

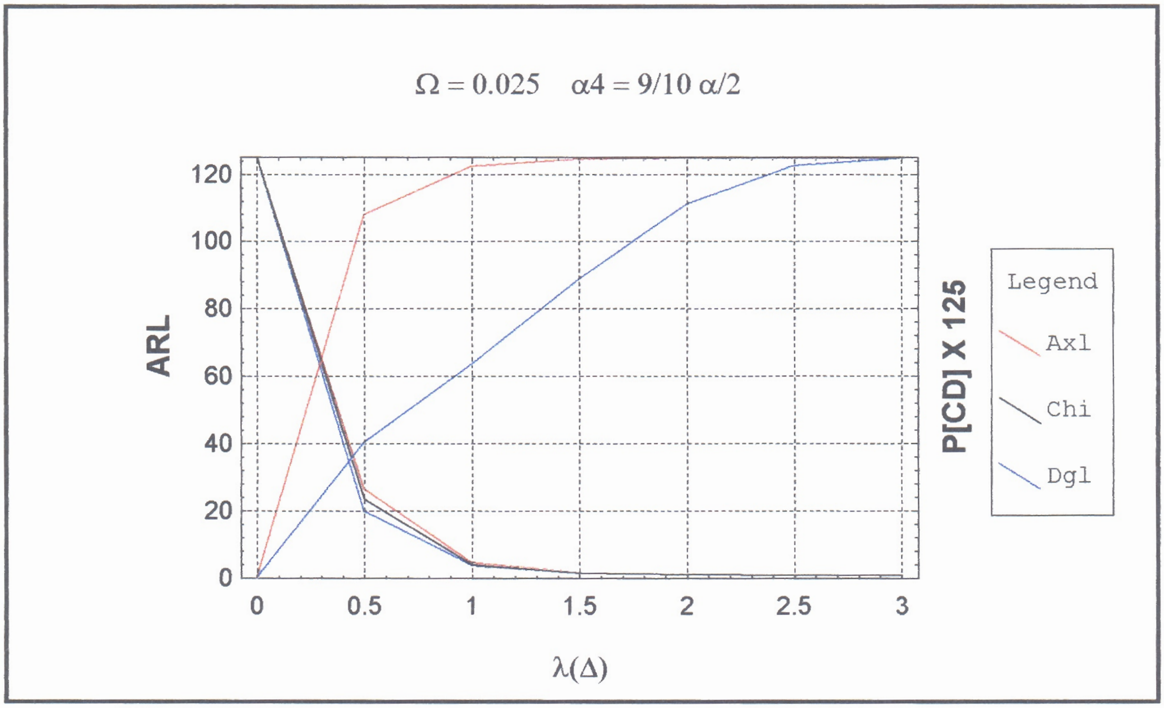

Figure D.22 - Experimental Case n=5, p=2, r=3/10, α=1/125

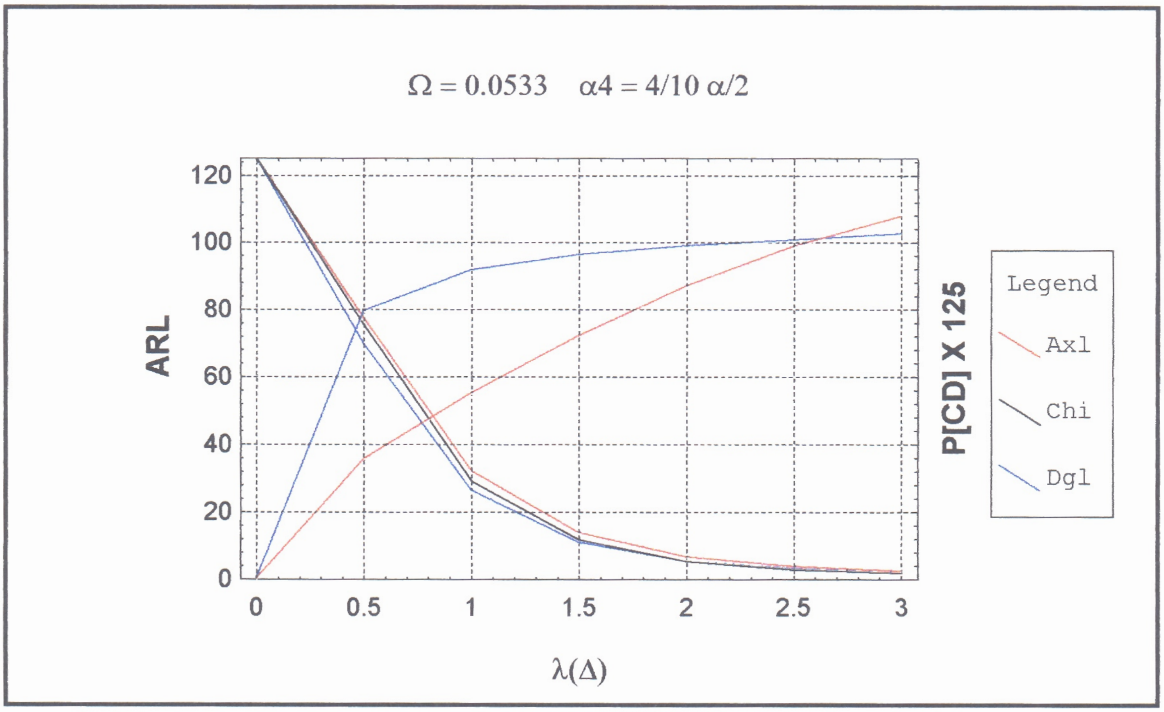

Figure D.23- Experimental Case n=1, p=2, r=-3/10, α=1/125

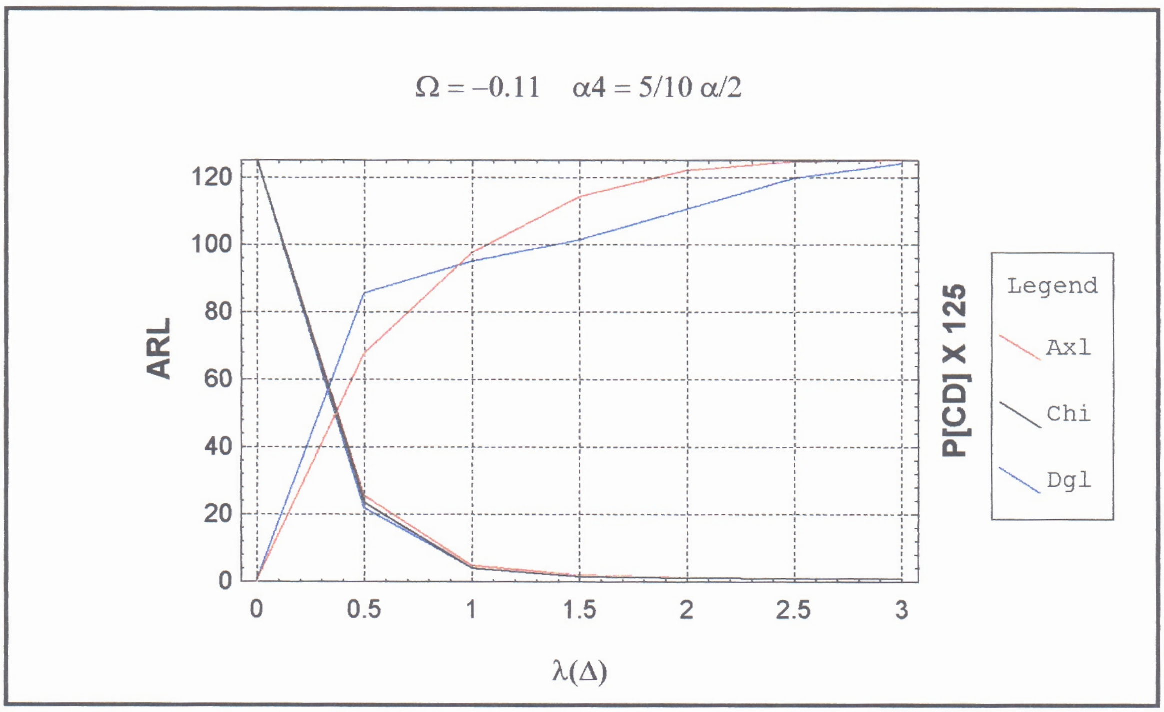

Figure D.24 - Experimental Case n=5, p=2, r=-3/10, α=1/125

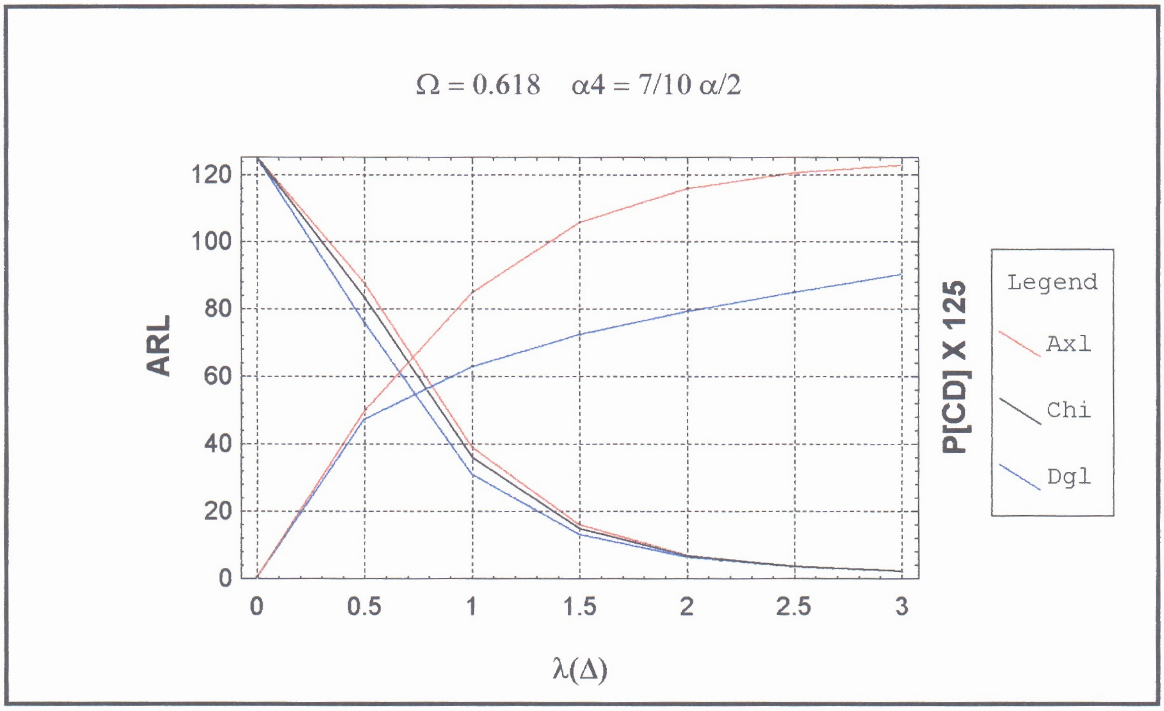

Figure D.25- Experimental Case n=1, p=3, r=0/10, α=1/125

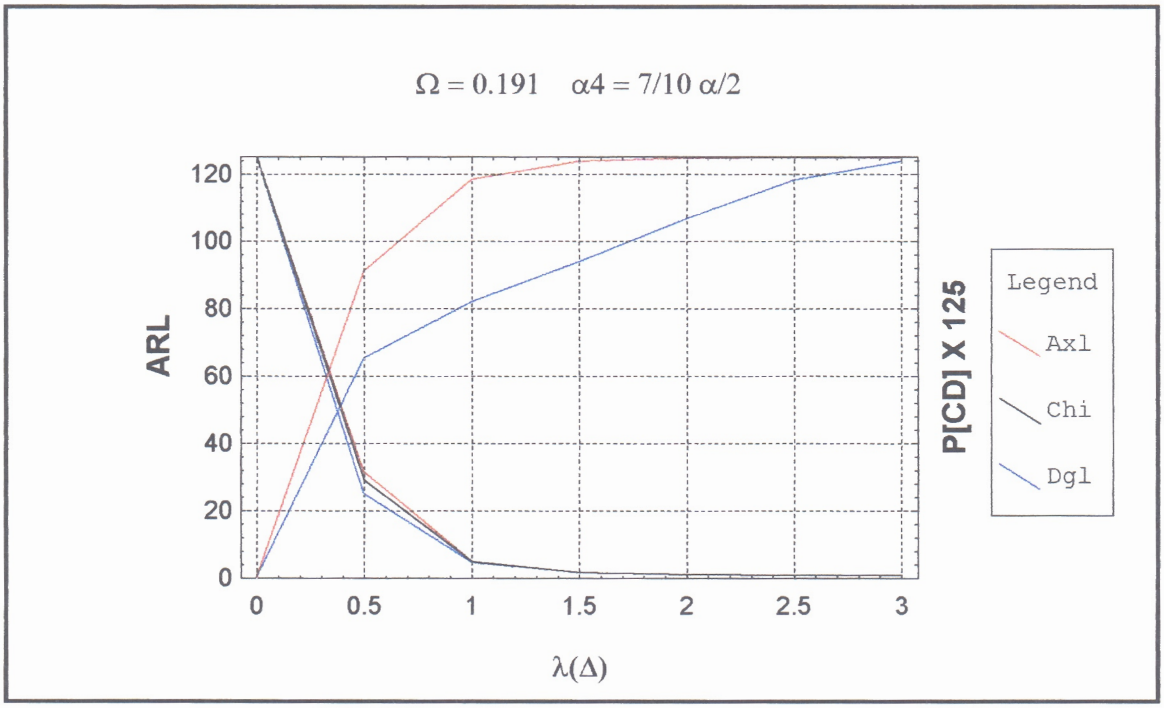

Figure D.26 - Experimental Case n=5, p=3, r=0/10, α=1/125

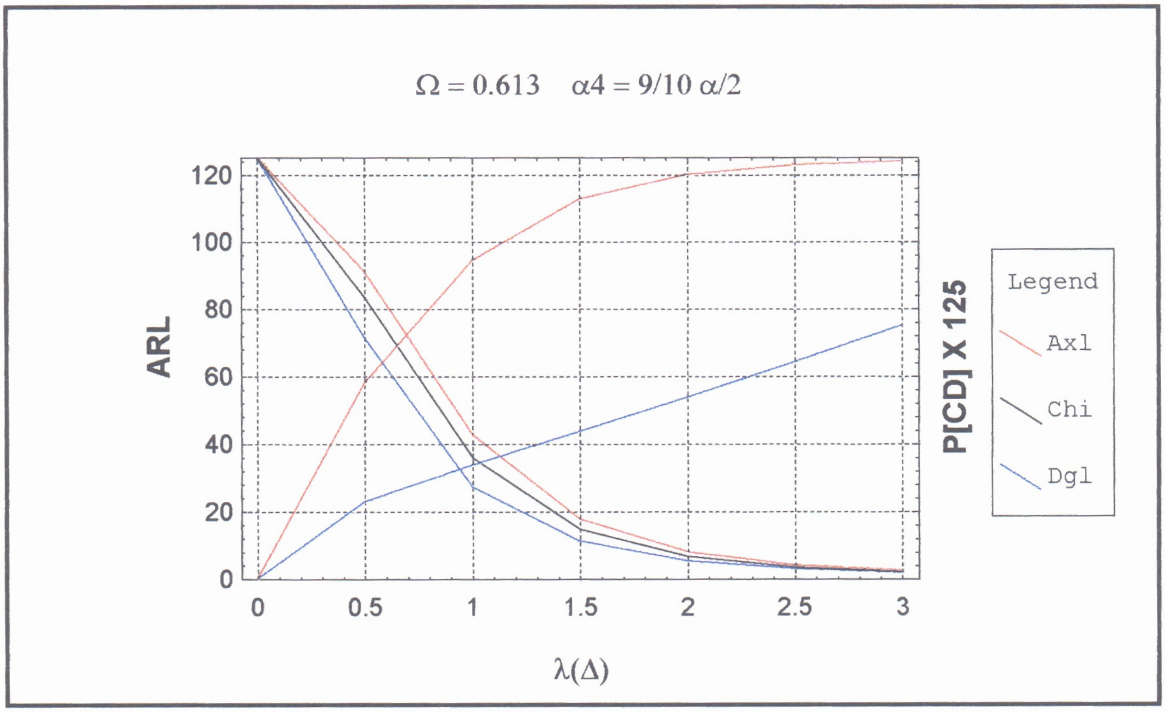

Figure D.27- Experimental Case n=1, p=3, r=3/10, α=1/125

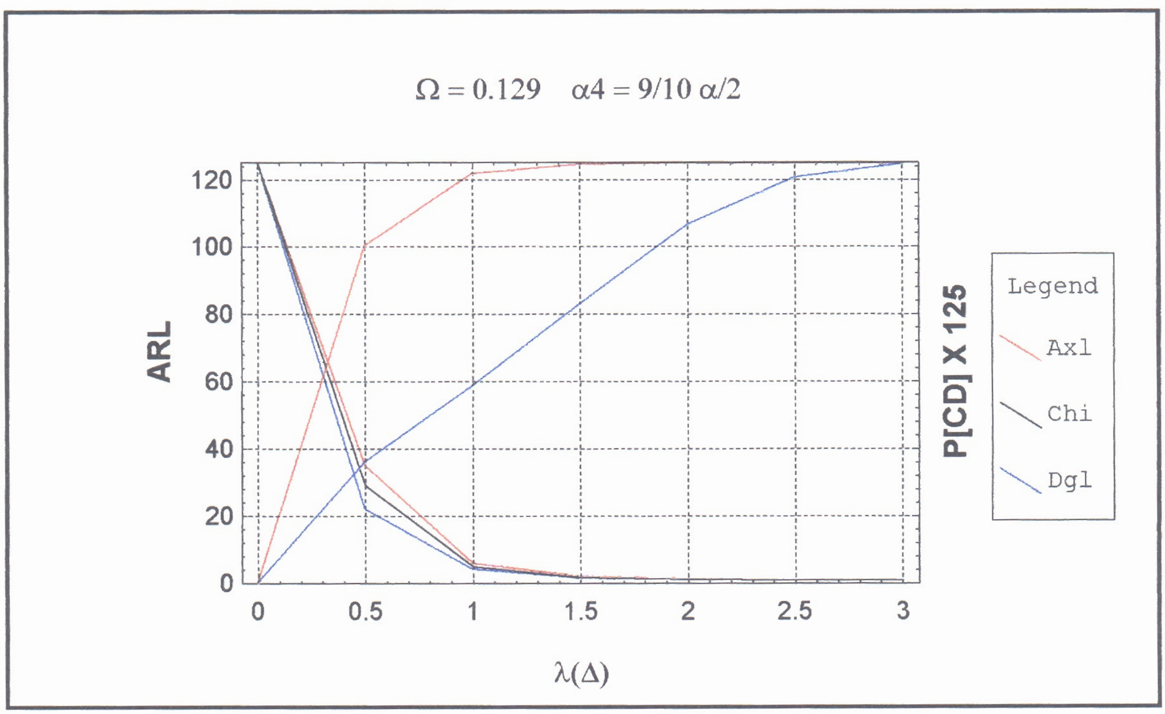

Figure D.28 - Experimental Case n=5, p=3, r=3/10, α=1/125

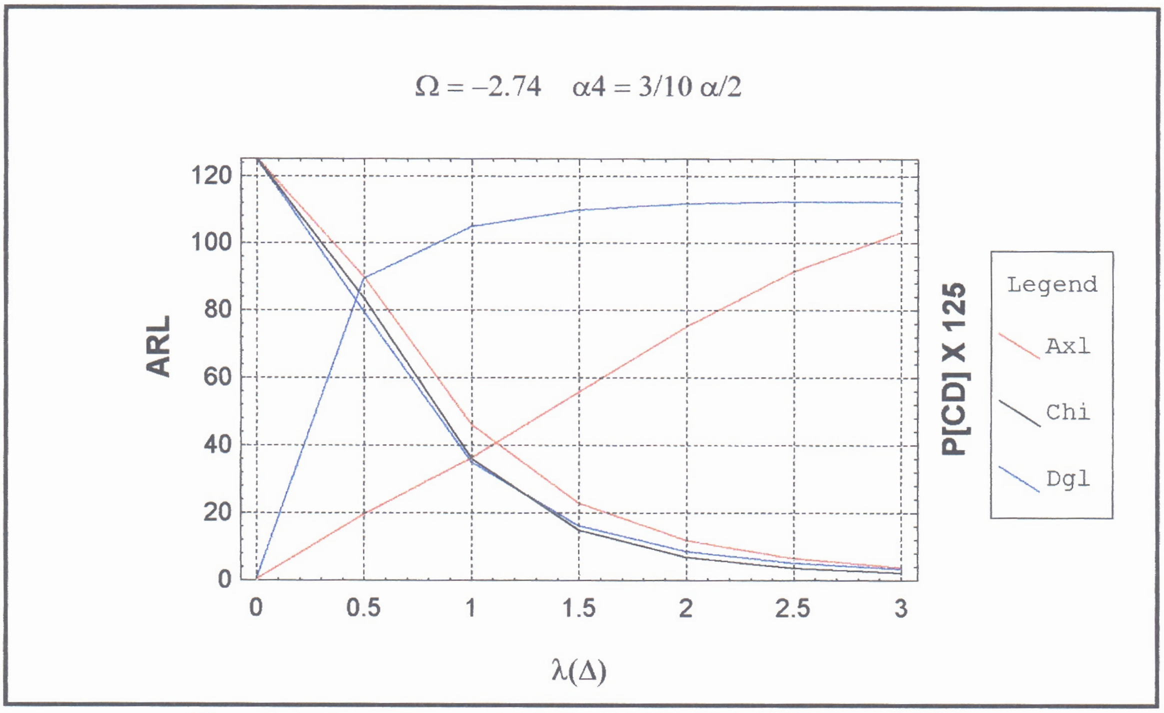

Figure D.29- Experimental Case n=1, p=3, r=-3/10, α=1/125

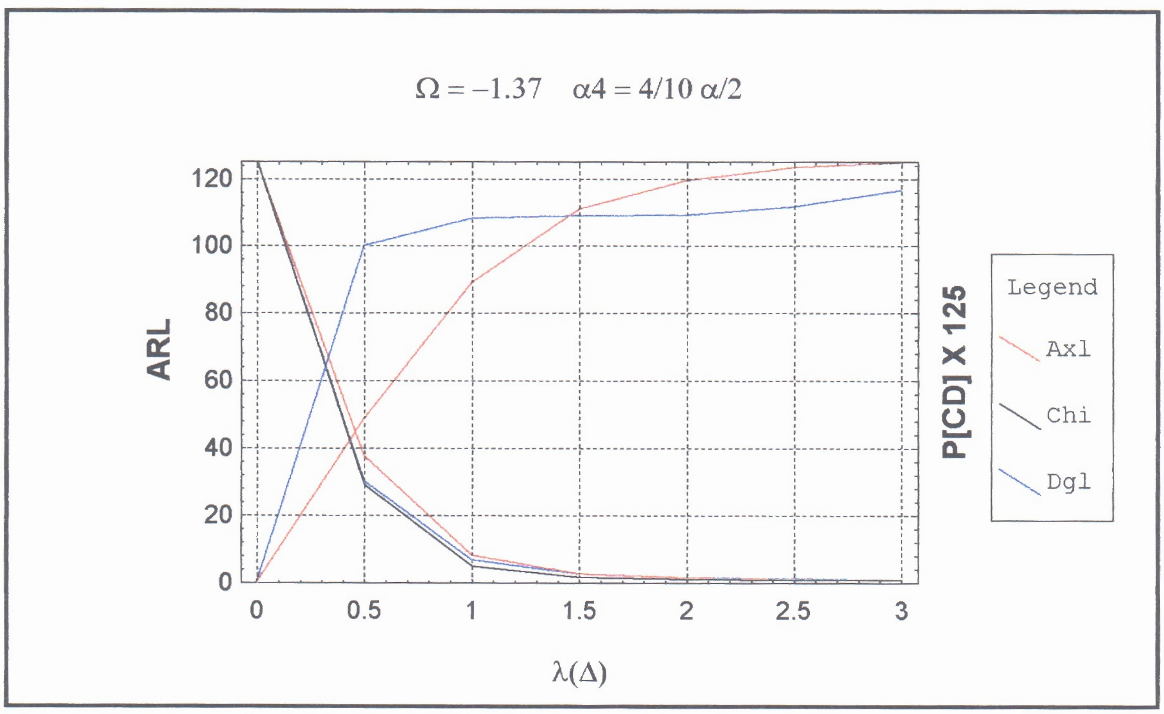

Figure D.30 - Experimental Case n=5, p=3, r=-3/10, α=1/125

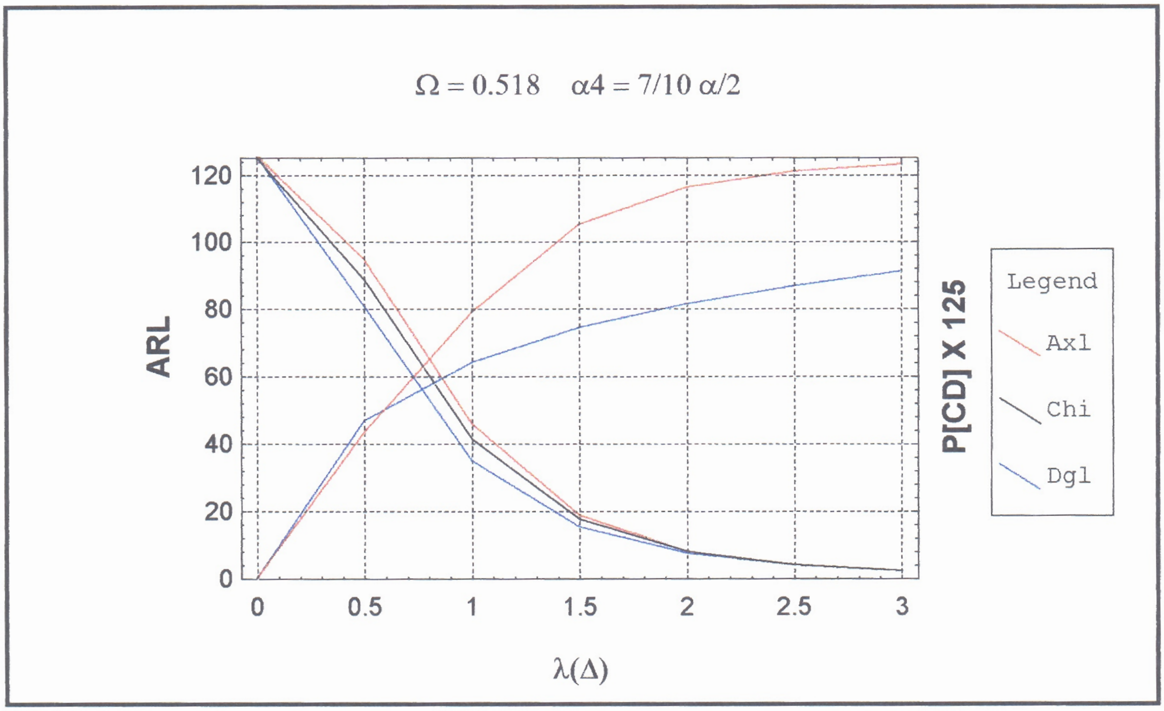

Figure D.31- Experimental Case n=1, p=4, r=0/10, α=1/125

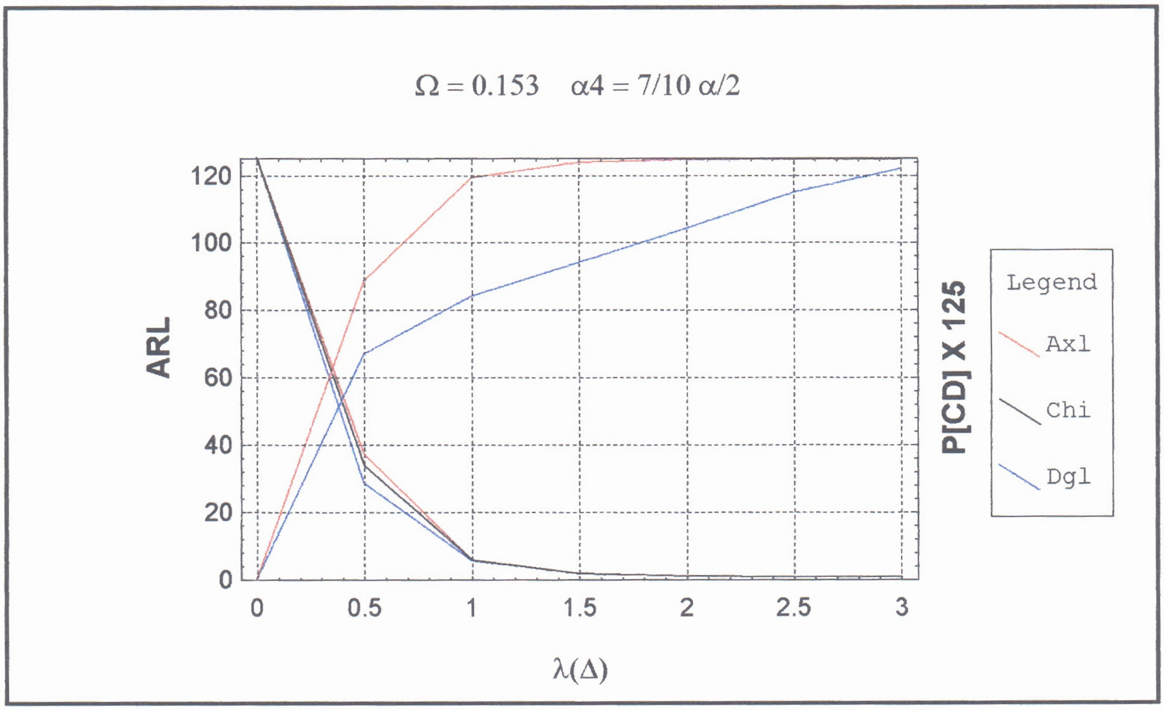

Figure D.32 - Experimental Case n=5, p=4, r=0/10, α=1/125

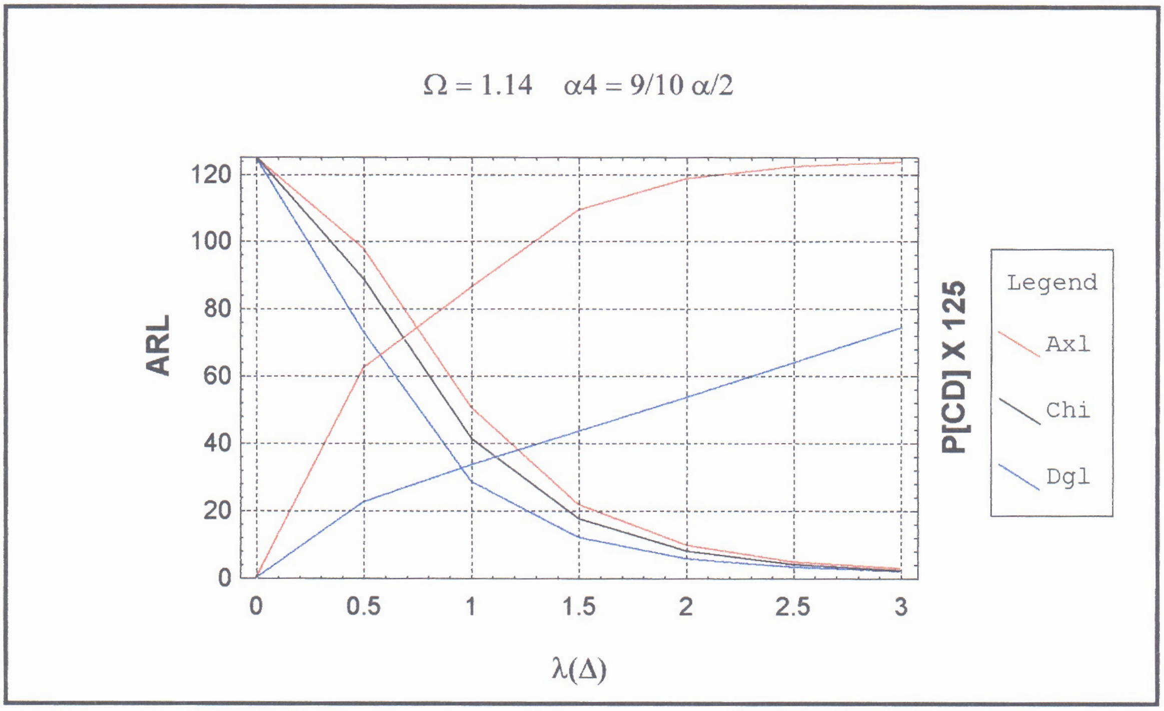

Figure D.33- Experimental Case n=1, p=4, r=3/10, α=1/125

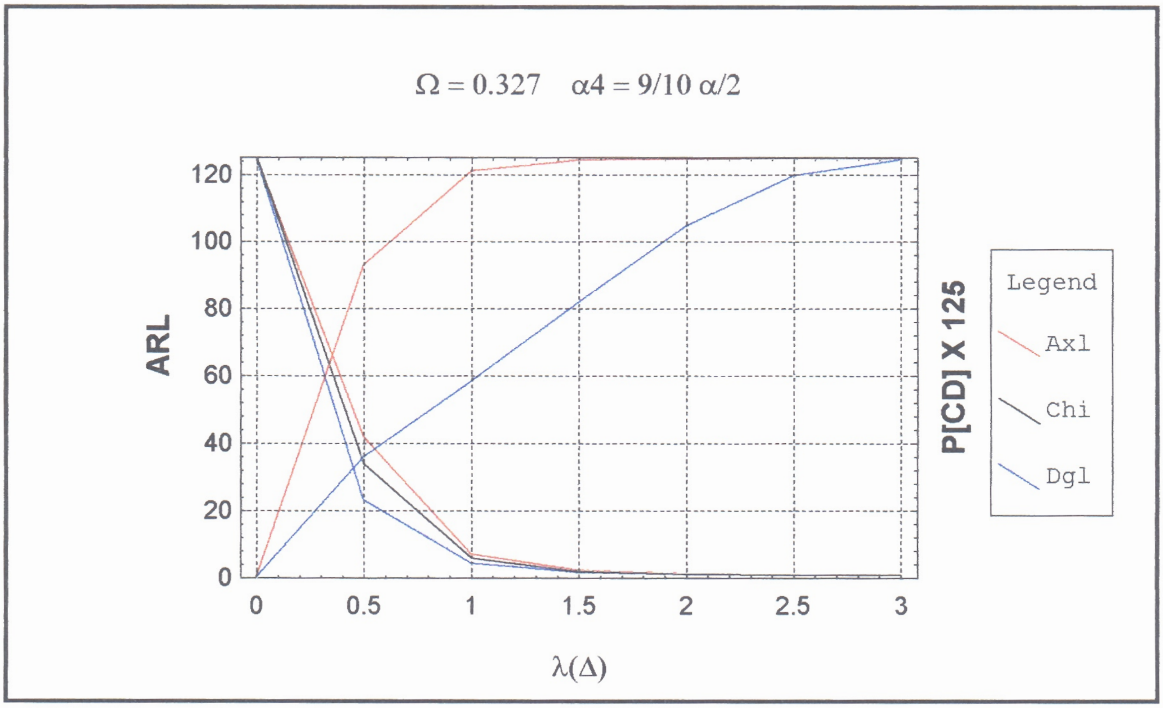

Figure D.34 - Experimental Case n=5, p=4, r=3/10, α=1/125

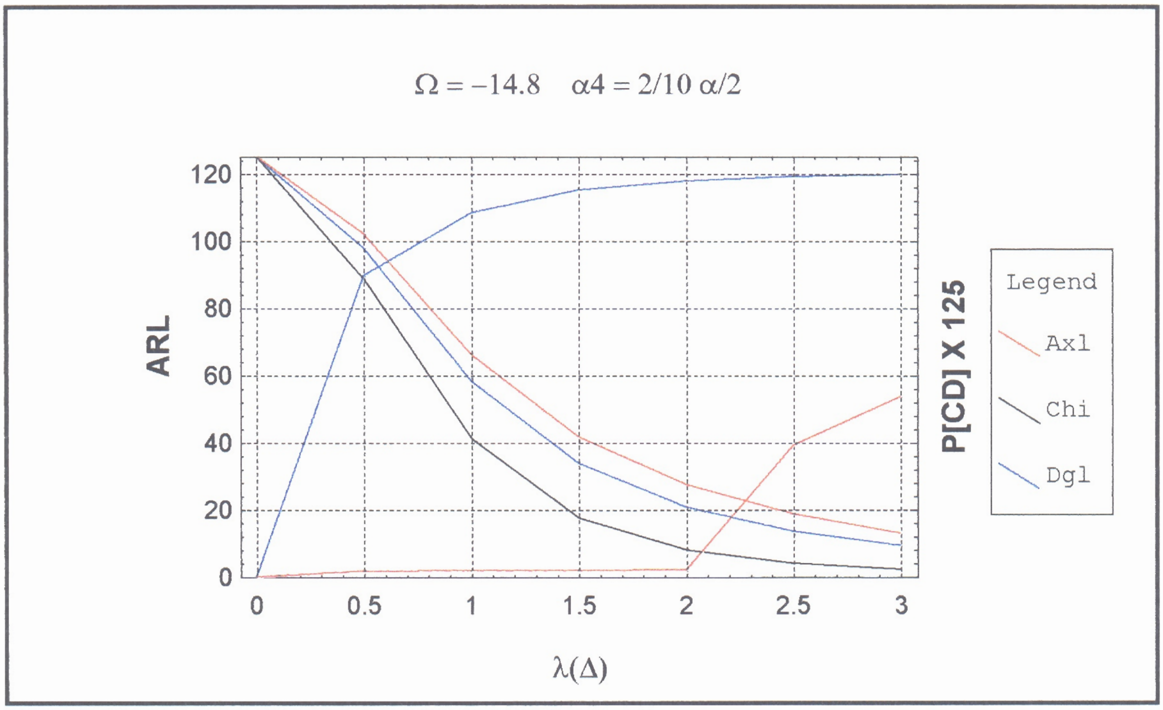

Figure D.35- Experimental Case n=1, p=4, r=-3/10, α=1/125

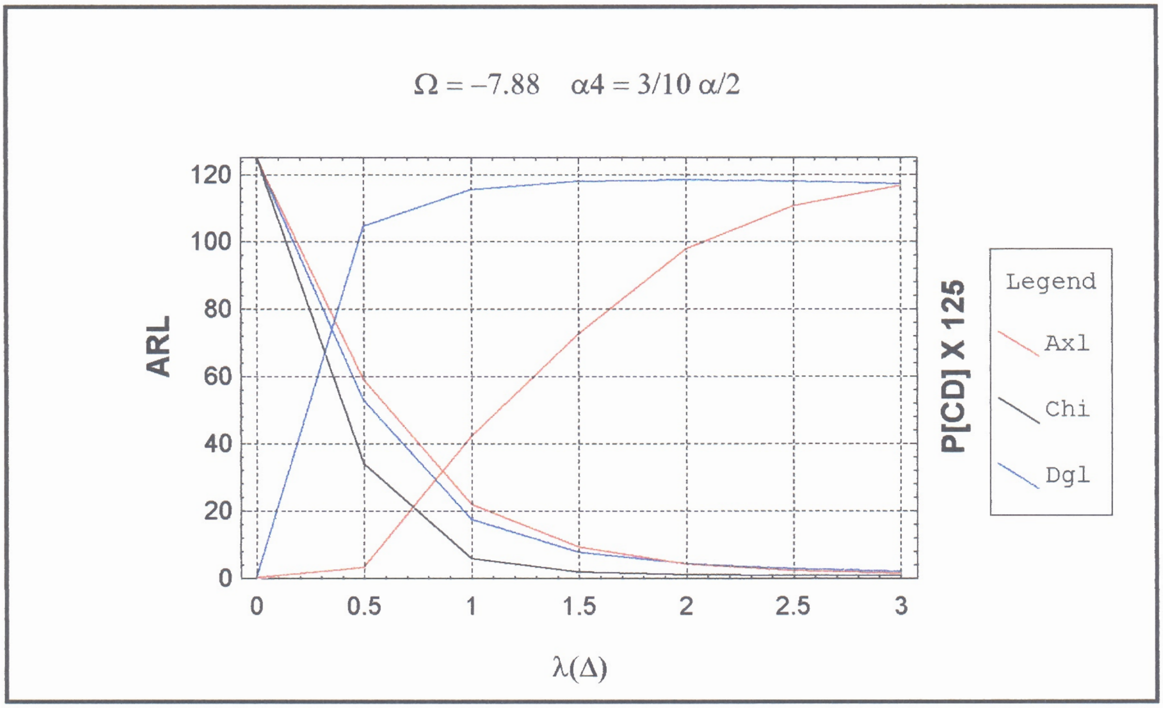

Figure D.36 - Experimental Case n=5, p=4, r=-3/10, α=1/125

Appendix E

Experimental Data

Table A3
Experimental Data

n	r	α	α_4	α_3	UCLp	LCLp	ϕ	$\lambda(\Delta)$	ARLAx	ARLDi	ARLChi	PCDAx	PCDDi
1	0	0.005	0.0015	0.00105	3.17457	-1.84687	6	0.0	199.99	199.99	200.00	0.00000	0.00000
1	0	0.005	0.0015	0.00105	3.17457	-1.84687	6	0.5	119.79	103.65	115.53	0.44766	0.47474
1	0	0.005	0.0015	0.00105	3.17457	-1.84687	6	1.0	45.04	35.48	41.92	0.66721	0.57399
1	0	0.005	0.0015	0.00105	3.17457	-1.84687	6	1.5	17.10	13.57	15.78	0.80285	0.63234
1	0	0.005	0.0015	0.00105	3.17457	-1.84687	6	2.0	7.39	6.14	6.88	0.88633	0.67907
1	0	0.005	0.0015	0.00105	3.17457	-1.84687	6	2.5	3.75	3.29	3.55	0.93638	0.72137
1	0	0.005	0.0015	0.00105	3.17457	-1.84687	6	3.0	2.24	2.06	2.16	0.96562	0.76357
5	0	0.005	0.00175	0.00080	3.12955	-1.90666	7	0.0	199.97	199.97	200.00	0.00000	0.00000
5	0	0.005	0.00175	0.00080	3.12955	-1.90666	7	0.5	34.27	29.11	32.94	0.75758	0.50866
5	0	0.005	0.00175	0.00080	3.12955	-1.90666	7	1.0	5.02	4.65	4.92	0.93085	0.64516
5	0	0.005	0.00175	0.00080	3.12955	-1.90666	7	1.5	1.67	1.65	1.67	0.98234	0.76208
5	0	0.005	0.00175	0.00080	3.12955	-1.90666	7	2.0	1.10	1.10	1.10	0.99636	0.88190
5	0	0.005	0.00175	0.00080	3.12955	-1.90666	7	2.5	1.01	1.01	1.01	0.99891	0.96736
5	0	0.005	0.00175	0.00080	3.12955	-1.90666	7	3.0	1.00	1.00	1.00	0.99912	0.99568
1	0.3	0.005	0.002	0.00067	3.08801	-2.28214	8	0.0	200.07	200.07	200.00	0.00000	0.00000
1	0.3	0.005	0.002	0.00067	3.08801	-2.28214	8	0.5	125.41	100.99	115.53	0.56338	0.32175
1	0.3	0.005	0.002	0.00067	3.08801	-2.28214	8	1.0	48.90	33.90	41.92	0.80052	0.41367
1	0.3	0.005	0.002	0.00067	3.08801	-2.28214	8	1.5	18.70	12.88	15.78	0.91003	0.48787
1	0.3	0.005	0.002	0.00067	3.08801	-2.28214	8	2.0	8.07	5.83	6.88	0.95937	0.55980
1	0.3	0.005	0.002	0.00067	3.08801	-2.28214	8	2.5	4.08	3.13	3.55	0.98171	0.63164
1	0.3	0.005	0.002	0.00067	3.08801	-2.28214	8	3.0	2.42	1.98	2.16	0.99172	0.70410
5	0.3	0.005	0.00225	0.00039	3.05267	-2.40869	9	0.0	200.02	200.02	200.00	0.00000	0.00000
5	0.3	0.005	0.00225	0.00039	3.05267	-2.40869	9	0.5	37.24	27.91	32.94	0.87310	0.30912
5	0.3	0.005	0.00225	0.00039	3.05267	-2.40869	9	1.0	5.49	4.49	4.92	0.98042	0.49099
5	0.3	0.005	0.00225	0.00039	3.05267	-2.40869	9	1.5	1.78	1.62	1.67	0.99620	0.68692
5	0.3	0.005	0.00225	0.00039	3.05267	-2.40869	9	2.0	1.13	1.09	1.10	0.99858	0.86873
5	0.3	0.005	0.00225	0.00039	3.05267	-2.40869	9	2.5	1.01	1.01	1.01	0.99885	0.97209
5	0.3	0.005	0.00225	0.00039	3.05267	-2.40869	9	3.0	1.00	1.00	1.00	0.99887	0.99750

Table A3 (Continued)
Experimental Data

p	n	r	α	α_4	α_3	UCLp	LCLp	ϕ	$\lambda(\Delta)$	ARLAx	ARLDi	ARLChi	PCDAx	PCDDi
2	1	-0.3	0.005	0.001	0.00151	3.29053	-1.42181	4	0.0	200.00	200.00	200.00	0.00000	0.00000
2	1	-0.3	0.005	0.001	0.00151	3.29053	-1.42181	4	0.5	118.72	106.24	115.53	0.28882	0.64538
2	1	-0.3	0.005	0.001	0.00151	3.29053	-1.42181	4	1.0	46.15	37.41	41.92	0.44691	0.74349
2	1	-0.3	0.005	0.001	0.00151	3.29053	-1.42181	4	1.5	18.59	14.62	15.78	0.58316	0.78059
2	1	-0.3	0.005	0.001	0.00151	3.29053	-1.42181	4	2.0	8.42	6.71	6.88	0.70062	0.80061
2	1	-0.3	0.005	0.001	0.00151	3.29053	-1.42181	4	2.5	4.36	3.62	3.55	0.79433	0.81363
2	1	-0.3	0.005	0.001	0.00151	3.29053	-1.42181	4	3.0	2.59	2.26	2.16	0.86407	0.82554
2	5	-0.3	0.005	0.00125	0.00126	3.22723	-1.45678	5	0.0	200.00	200.00	200.00	0.00000	0.00000
2	5	-0.3	0.005	0.00125	0.00126	3.22723	-1.45678	5	0.5	35.84	30.73	32.94	0.54766	0.69317
2	5	-0.3	0.005	0.00125	0.00126	3.22723	-1.45678	5	1.0	5.74	5.06	4.92	0.78464	0.76894
2	5	-0.3	0.005	0.00125	0.00126	3.22723	-1.45678	5	1.5	1.87	1.78	1.67	0.91405	0.81167
2	5	-0.3	0.005	0.00125	0.00126	3.22723	-1.45678	5	2.0	1.15	1.14	1.10	0.97399	0.87541
2	5	-0.3	0.005	0.00125	0.00126	3.22723	-1.45678	5	2.5	1.01	1.01	1.01	0.99527	0.94896
2	5	-0.3	0.005	0.00125	0.00126	3.22723	-1.45678	5	3.0	1.00	1.00	1.00	0.99911	0.98856
3	1	0	0.005	0.0015	0.00102	3.29039	-1.27856	6	0.0	200.00	200.00	200.00	0.00000	0.00000
3	1	0	0.005	0.0015	0.00102	3.29039	-1.27856	6	0.5	138.24	113.45	129.19	0.36334	0.47447
3	1	0	0.005	0.0015	0.00102	3.29039	-1.27856	6	1.0	59.19	42.22	52.41	0.65013	0.59526
3	1	0	0.005	0.0015	0.00102	3.29039	-1.27856	6	1.5	22.64	16.79	20.41	0.82977	0.65959
3	1	0	0.005	0.0015	0.00102	3.29039	-1.27856	6	2.0	9.35	7.68	8.80	0.91952	0.70309
3	1	0	0.005	0.0015	0.00102	3.29039	-1.27856	6	2.5	4.48	4.07	4.38	0.96145	0.73645
3	1	0	0.005	0.0015	0.00102	3.29039	-1.27856	6	3.0	2.55	2.49	2.55	0.98110	0.76610
3	5	0	0.005	0.00175	0.00077	3.24675	-1.33167	7	0.0	200.00	200.00	200.00	0.00000	0.00000
3	5	0	0.005	0.00175	0.00077	3.24675	-1.33167	7	0.5	44.86	35.53	41.76	0.74511	0.53095
3	5	0	0.005	0.00175	0.00077	3.24675	-1.33167	7	1.0	6.12	5.87	6.21	0.95308	0.66501
3	5	0	0.005	0.00175	0.00077	3.24675	-1.33167	7	1.5	1.83	1.95	1.90	0.99010	0.75199
3	5	0	0.005	0.00175	0.00077	3.24675	-1.33167	7	2.0	1.12	1.18	1.15	0.99745	0.84465
3	5	0	0.005	0.00175	0.00077	3.24675	-1.33167	7	2.5	1.01	1.02	1.01	0.99872	0.93531
3	5	0	0.005	0.00175	0.00077	3.24675	-1.33167	7	3.0	1.00	1.00	1.00	0.99883	0.98465

Table A3 (Continued)
Experimental Data

p	n	r	α	α_4	α_3	UCLp	LCLp	ϕ	$\lambda(\Delta)$	ARLAx	ARLDi	ARLChi	PCDAx	PCDDi
3	1	0.3	0.005	0.00225	0.00033	3.17102	-2.02138	9	0.0	200.06	200.06	200.00	0.00000	0.00000
3	1	0.3	0.005	0.00225	0.00033	3.17102	-2.02138	9	0.5	141.85	109.20	129.19	0.47782	0.17946
3	1	0.3	0.005	0.00225	0.00033	3.17102	-2.02138	9	1.0	62.37	39.26	52.41	0.77304	0.26348
3	1	0.3	0.005	0.00225	0.00033	3.17102	-2.02138	9	1.5	24.33	15.33	20.41	0.91299	0.34178
3	1	0.3	0.005	0.00225	0.00033	3.17102	-2.02138	9	2.0	10.26	6.93	8.80	0.96601	0.42143
3	1	0.3	0.005	0.00225	0.00033	3.17102	-2.02138	9	2.5	5.00	3.66	4.38	0.98569	0.50292
3	1	0.3	0.005	0.00225	0.00033	3.17102	-2.02138	9	3.0	2.84	2.25	2.55	0.99327	0.58657
3	5	0.3	0.005	0.00225	0.00033	3.17102	-2.02138	9	0.0	200.06	200.06	200.00	0.00000	0.00000
3	5	0.3	0.005	0.00225	0.00033	3.17102	-2.02138	9	0.5	49.96	31.06	41.76	0.81832	0.28194
3	5	0.3	0.005	0.00225	0.00033	3.17102	-2.02138	9	1.0	7.16	5.03	6.21	0.97767	0.45966
3	5	0.3	0.005	0.00225	0.00033	3.17102	-2.02138	9	1.5	2.09	1.74	1.90	0.99568	0.64703
3	5	0.3	0.005	0.00225	0.00033	3.17102	-2.02138	9	2.0	1.19	1.12	1.15	0.99811	0.83188
3	5	0.3	0.005	0.00225	0.00033	3.17102	-2.02138	9	2.5	1.02	1.01	1.01	0.99845	0.95534
3	5	0.3	0.005	0.00225	0.00033	3.17102	-2.02138	9	3.0	1.00	1.00	1.00	0.99849	0.99446
3	1	-0.3	0.005	0.00075	0.00176	3.48076	-0.60524	3	0.0	199.99	199.99	200.00	0.00000	0.00000
3	1	-0.3	0.005	0.00075	0.00176	3.48076	-0.60524	3	0.5	138.53	121.46	129.19	0.15709	0.72701
3	1	-0.3	0.005	0.00075	0.00176	3.48076	-0.60524	3	1.0	66.89	49.91	52.41	0.29308	0.84998
3	1	-0.3	0.005	0.00075	0.00176	3.48076	-0.60524	3	1.5	31.43	21.71	20.41	0.44860	0.88917
3	1	-0.3	0.005	0.00075	0.00176	3.48076	-0.60524	3	2.0	15.40	10.72	8.80	0.60577	0.90524
3	1	-0.3	0.005	0.00075	0.00176	3.48076	-0.60524	3	2.5	8.00	5.99	4.38	0.73746	0.91109
3	1	-0.3	0.005	0.00075	0.00176	3.48076	-0.60524	3	3.0	4.49	3.75	2.55	0.83107	0.91126
3	5	-0.3	0.005	0.001	0.00151	3.40294	-0.62612	4	0.0	200.07	200.07	200.00	0.00000	0.00000
3	5	-0.3	0.005	0.001	0.00151	3.40294	-0.62612	4	0.5	54.21	42.61	41.76	0.39592	0.81448
3	5	-0.3	0.005	0.001	0.00151	3.40294	-0.62612	4	1.0	10.32	8.33	6.21	0.71992	0.88117
3	5	-0.3	0.005	0.001	0.00151	3.40294	-0.62612	4	1.5	2.94	2.91	1.90	0.89289	0.88764
3	5	-0.3	0.005	0.001	0.00151	3.40294	-0.62612	4	2.0	1.42	1.58	1.15	0.95783	0.88421
3	5	-0.3	0.005	0.001	0.00151	3.40294	-0.62612	4	2.5	1.07	1.16	1.01	0.98579	0.89594
3	5	-0.3	0.005	0.001	0.00151	3.40294	-0.62612	4	3.0	1.01	1.03	1.00	0.99689	0.92867

Table A3 (Continued)
Experimental Data

p	n	r	α	α_4	α_3	UCLp	LCLp	ϕ	$\lambda(\Delta)$	ARLAx	ARLDi	ARLChi	PCDAx	PCDDi
4	1	0	0.005	0.0015	0.00100	3.37046	-0.92314	6	0.0	200.49	200.49	200.00	0.00000	0.00000
4	1	0	0.005	0.0015	0.00100	3.37046	-0.92314	6	0.5	150.32	120.79	138.15	0.32113	0.47510
4	1	0	0.005	0.0015	0.00100	3.37046	-0.92314	6	1.0	70.45	47.90	60.96	0.63121	0.61125
4	1	0	0.005	0.0015	0.00100	3.37046	-0.92314	6	1.5	27.18	19.71	24.62	0.83449	0.68188
4	1	0	0.005	0.0015	0.00100	3.37046	-0.92314	6	2.0	10.91	9.16	10.63	0.93023	0.72609
4	1	0	0.005	0.0015	0.00100	3.37046	-0.92314	6	2.5	5.05	4.86	5.19	0.96799	0.75663
4	1	0	0.005	0.0015	0.00100	3.37046	-0.92314	6	3.0	2.77	2.94	2.93	0.98446	0.78065
4	5	0	0.005	0.00175	0.00075	3.32773	-0.97146	7	0.0	200.48	200.48	200.00	0.00000	0.00000
4	5	0	0.005	0.00175	0.00075	3.32773	-0.97146	7	0.5	53.47	41.06	49.19	0.73050	0.54879
4	5	0	0.005	0.00175	0.00075	3.32773	-0.97146	7	1.0	6.98	7.08	7.45	0.95859	0.68787
4	5	0	0.005	0.00175	0.00075	3.32773	-0.97146	7	1.5	1.95	2.28	2.13	0.99167	0.76064
4	5	0	0.005	0.00175	0.00075	3.32773	-0.97146	7	2.0	1.14	1.28	1.19	0.99756	0.83069
4	5	0	0.005	0.00175	0.00075	3.32773	-0.97146	7	2.5	1.01	1.05	1.02	0.99838	0.91049
4	5	0	0.005	0.00175	0.00075	3.32773	-0.97146	7	3.0	1.00	1.00	1.00	0.99936	0.96987
4	1	0.3	0.005	0.00225	0.00031	3.25280	-1.77062	9	0.0	200.38	200.38	200.00	0.00000	0.00000
4	1	0.3	0.005	0.00225	0.00031	3.25280	-1.77062	9	0.5	153.25	112.65	138.15	0.52985	0.17308
4	1	0.3	0.005	0.00225	0.00031	3.25280	-1.77062	9	1.0	74.52	41.62	60.96	0.71139	0.25880
4	1	0.3	0.005	0.00225	0.00031	3.25280	-1.77062	9	1.5	30.03	16.44	24.62	0.88851	0.33805
4	1	0.3	0.005	0.00225	0.00031	3.25280	-1.77062	9	2.0	12.55	7.45	10.63	0.95579	0.41736
4	1	0.3	0.005	0.00225	0.00031	3.25280	-1.77062	9	2.5	5.95	3.91	5.19	0.98273	0.49719
4	1	0.3	0.005	0.00225	0.00031	3.25280	-1.77062	9	3.0	3.28	2.38	2.93	0.99101	0.57816
4	5	0.3	0.005	0.00225	0.00031	3.25280	-1.77062	9	0.0	200.38	200.38	200.00	0.00000	0.00000
4	5	0.3	0.005	0.00225	0.00031	3.25280	-1.77062	9	0.5	60.51	33.07	49.19	0.76218	0.27760
4	5	0.3	0.005	0.00225	0.00031	3.25280	-1.77062	9	1.0	8.67	5.40	7.45	0.97209	0.45496
4	5	0.3	0.005	0.00225	0.00031	3.25280	-1.77062	9	1.5	2.35	1.82	2.13	0.99540	0.63641
4	5	0.3	0.005	0.00225	0.00031	3.25280	-1.77062	9	2.0	1.26	1.14	1.19	0.99766	0.81690
4	5	0.3	0.005	0.00225	0.00031	3.25280	-1.77062	9	2.5	1.03	1.01	1.02	0.99810	0.94588
4	5	0.3	0.005	0.00225	0.00031	3.25280	-1.77062	9	3.0	1.00	1.00	1.00	0.99830	0.99213

Table A3 (Continued)
Experimental Data

p	n	r	α	α_4	α_3	UCLp	LCLp	ϕ	$\lambda(\Delta)$	ARLAx	ARLDi	ARLChi	PCDAx	PCDDi
4	1	-0.3	0.005	0.00025	0.00225	3.83375	-0.01331	1	0.0	200.00	200.00	200.00	0.00000	0.00000
4	1	-0.3	0.005	0.00025	0.00225	3.83375	-0.01331	1	0.5	157.11	147.92	138.15	0.00570	0.80128
4	1	-0.3	0.005	0.00025	0.00225	3.83375	-0.01331	1	1.0	96.18	81.02	60.96	0.00679	0.93049
4	1	-0.3	0.005	0.00025	0.00225	3.83375	-0.01331	1	1.5	58.88	44.24	24.62	0.00669	0.96445
4	1	-0.3	0.005	0.00025	0.00225	3.83375	-0.01331	1	2.0	38.45	25.92	10.63	0.00662	0.97540
4	1	-0.3	0.005	0.00025	0.00225	3.83375	-0.01331	1	2.5	26.55	16.34	5.19	0.21483	0.98011
4	1	-0.3	0.005	0.00025	0.00225	3.83375	-0.01331	1	3.0	18.86	11.00	2.93	0.30464	0.98246
4	5	-0.3	0.005	0.0005	0.00200	3.66221	-0.02312	2	0.0	200.00	200.00	200.00	0.00000	0.00000
4	5	-0.3	0.005	0.0005	0.00200	3.66221	-0.02312	2	0.5	85.16	72.53	49.19	0.01423	0.89960
4	5	-0.3	0.005	0.0005	0.00200	3.66221	-0.02312	2	1.0	30.18	21.45	7.45	0.26006	0.95728
4	5	-0.3	0.005	0.0005	0.00200	3.66221	-0.02312	2	1.5	12.76	8.84	2.13	0.50557	0.96885
4	5	-0.3	0.005	0.0005	0.00200	3.66221	-0.02312	2	2.0	5.61	4.68	1.19	0.73236	0.97066
4	5	-0.3	0.005	0.0005	0.00200	3.66221	-0.02312	2	2.5	2.74	2.96	1.02	0.87028	0.96805
4	5	-0.3	0.005	0.0005	0.00200	3.66221	-0.02312	2	3.0	1.64	2.12	1.00	0.92604	0.96297
2	1	0	0.008	0.0024	0.00170	3.03549	-1.73668	6	0.0	124.99	124.99	125.00	0.00000	0.00000
2	1	0	0.008	0.0024	0.00170	3.03549	-1.73668	6	0.5	78.09	68.16	75.44	0.43767	0.47309
2	1	0	0.008	0.0024	0.00170	3.03549	-1.73668	6	1.0	31.45	24.94	29.31	0.65619	0.57329
2	1	0	0.008	0.0024	0.00170	3.03549	-1.73668	6	1.5	12.76	10.15	11.77	0.79391	0.63177
2	1	0	0.008	0.0024	0.00170	3.03549	-1.73668	6	2.0	5.87	4.87	5.46	0.88007	0.67969
2	1	0	0.008	0.0024	0.00170	3.03549	-1.73668	6	2.5	3.15	2.76	2.98	0.93251	0.72436
2	1	0	0.008	0.0024	0.00170	3.03549	-1.73668	6	3.0	1.99	1.82	1.91	0.96348	0.76971
2	5	0	0.008	0.0028	0.00130	2.98867	-1.79802	7	0.0	124.97	124.97	125.00	0.00000	0.00000
2	5	0	0.008	0.0028	0.00130	2.98867	-1.79802	7	0.5	24.37	20.69	23.40	0.74650	0.50899
2	5	0	0.008	0.0028	0.00130	2.98867	-1.79802	7	1.0	4.11	3.78	4.02	0.92619	0.64766
2	5	0	0.008	0.0028	0.00130	2.98867	-1.79802	7	1.5	1.53	1.50	1.52	0.98107	0.77165
2	5	0	0.008	0.0028	0.00130	2.98867	-1.79802	7	2.0	1.07	1.07	1.07	0.99589	0.89483
2	5	0	0.008	0.0028	0.00130	2.98867	-1.79802	7	2.5	1.00	1.01	1.00	0.99842	0.97392
2	5	0	0.008	0.0028	0.00130	2.98867	-1.79802	7	3.0	1.00	1.00	1.00	0.99859	0.99690

Table A3 (Continued)
Experimental Data

p	n	r	α	α_4	α_3	UCLp	LCLp	ϕ	$\lambda(\Delta)$	ARLAx	ARLDi	ARLChi	PCDAx	PCDDi
2	1	0.3	0.008	0.0036	0.00066	2.90795	-2.28394	9	0.0	125.02	125.02	125.00	0.00000	0.00000
2	1	0.3	0.008	0.0036	0.00066	2.90795	-2.28394	9	0.5	80.41	68.06	75.44	0.60140	0.21544
2	1	0.3	0.008	0.0036	0.00066	2.90795	-2.28394	9	1.0	32.91	24.95	29.31	0.83075	0.30279
2	1	0.3	0.008	0.0036	0.00066	2.90795	-2.28394	9	1.5	13.31	10.17	11.77	0.92709	0.38349
2	1	0.3	0.008	0.0036	0.00066	2.90795	-2.28394	9	2.0	6.11	4.88	5.46	0.96724	0.46743
2	1	0.3	0.008	0.0036	0.00066	2.90795	-2.28394	9	2.5	3.28	2.76	2.98	0.98468	0.55499
2	1	0.3	0.008	0.0036	0.00066	2.90795	-2.28394	9	3.0	2.06	1.82	1.91	0.99242	0.64515
2	5	0.3	0.008	0.0036	0.00066	2.90795	-2.28394	9	0.0	125.02	125.02	125.00	0.00000	0.00000
2	5	0.3	0.008	0.0036	0.00066	2.90795	-2.28394	9	0.5	26.39	19.91	23.40	0.86177	0.32171
2	5	0.3	0.008	0.0036	0.00066	2.90795	-2.28394	9	1.0	4.46	3.66	4.02	0.97724	0.50836
2	5	0.3	0.008	0.0036	0.00066	2.90795	-2.28394	9	1.5	1.62	1.47	1.52	0.99511	0.70918
2	5	0.3	0.008	0.0036	0.00066	2.90795	-2.28394	9	2.0	1.09	1.07	1.07	0.99785	0.88751
2	5	0.3	0.008	0.0036	0.00066	2.90795	-2.28394	9	2.5	1.01	1.00	1.00	0.99816	0.97878
2	5	0.3	0.008	0.0036	0.00066	2.90795	-2.28394	9	3.0	1.00	1.00	1.00	0.99818	0.99833
2	1	-0.3	0.008	0.0016	0.00242	3.15591	-1.32633	4	0.0	125.00	125.00	125.00	0.00000	0.00000
2	1	-0.3	0.008	0.0016	0.00242	3.15591	-1.32633	4	0.5	77.48	69.85	75.44	0.28349	0.63605
2	1	-0.3	0.008	0.0016	0.00242	3.15591	-1.32633	4	1.0	32.11	26.30	29.31	0.44130	0.73432
2	1	-0.3	0.008	0.0016	0.00242	3.15591	-1.32633	4	1.5	13.72	10.92	11.77	0.57749	0.77066
2	1	-0.3	0.008	0.0016	0.00242	3.15591	-1.32633	4	2.0	6.58	5.31	5.46	0.69529	0.79090
2	1	-0.3	0.008	0.0016	0.00242	3.15591	-1.32633	4	2.5	3.59	3.01	2.98	0.79009	0.80556
2	1	-0.3	0.008	0.0016	0.00242	3.15591	-1.32633	4	3.0	2.25	1.98	1.91	0.86144	0.82056
2	5	-0.3	0.008	0.002	0.00203	3.09024	-1.36269	5	0.0	125.00	125.00	125.00	0.00000	0.00000
2	5	-0.3	0.008	0.002	0.00203	3.09024	-1.36269	5	0.5	25.31	21.82	23.40	0.54051	0.68322
2	5	-0.3	0.008	0.002	0.00203	3.09024	-1.36269	5	1.0	4.62	4.08	4.02	0.77947	0.75928
2	5	-0.3	0.008	0.002	0.00203	3.09024	-1.36269	5	1.5	1.68	1.60	1.52	0.91274	0.80981
2	5	-0.3	0.008	0.002	0.00203	3.09024	-1.36269	5	2.0	1.11	1.10	1.07	0.97484	0.88296
2	5	-0.3	0.008	0.002	0.00203	3.09024	-1.36269	5	2.5	1.01	1.01	1.00	0.99553	C.95661
2	5	-0.3	0.008	0.002	0.00203	3.09024	-1.36269	5	3.0	1.00	1.00	1.00	0.99881	0.99114

Table A3 (Continued)
Experimental Data

p	n	r	α	α_4	α_3	UCLp	LCLp	ϕ	$\lambda(\Delta)$	ARLAx	ARLDi	ARLChi	PCDAx	PCDDi
3	1	0	0.008	0.0028	0.00123	3.11039	-1.24112	7	0.0	125.00	125.00	125.00	0.00000	0.00000
3	1	0	0.008	0.0028	0.00123	3.11039	-1.24112	7	0.5	87.77	76.05	83.55	0.39582	0.37652
3	1	0	0.008	0.0028	0.00123	3.11039	-1.24112	7	1.0	39.03	30.81	36.07	0.67778	0.50151
3	1	0	0.008	0.0028	0.00123	3.11039	-1.24112	7	1.5	15.77	13.05	14.96	0.84375	0.57776
3	1	0	0.008	0.0028	0.00123	3.11039	-1.24112	7	2.0	6.95	6.28	6.85	0.92467	0.63366
3	1	0	0.008	0.0028	0.00123	3.11039	-1.24112	7	2.5	3.56	3.48	3.62	0.96279	0.67929
3	1	0	0.008	0.0028	0.00123	3.11039	-1.24112	7	3.0	2.16	2.21	2.22	0.98107	0.72110
3	5	0	0.008	0.0028	0.00123	3.11039	-1.24112	7	0.0	125.00	125.00	125.00	0.00000	0.00000
3	5	0	0.008	0.0028	0.00123	3.11039	-1.24112	7	0.5	31.48	24.92	29.18	0.72732	0.52223
3	5	0	0.008	0.0028	0.00123	3.11039	-1.24112	7	1.0	4.97	4.67	4.97	0.94621	0.65605
3	5	0	0.008	0.0028	0.00123	3.11039	-1.24112	7	1.5	1.66	1.73	1.71	0.98806	0.75092
3	5	0	0.008	0.0028	0.00123	3.11039	-1.24112	7	2.0	1.09	1.13	1.11	0.99660	0.85298
3	5	0	0.008	0.0028	0.00123	3.11039	-1.24112	7	2.5	1.01	1.01	1.01	0.99801	0.94414
3	5	0	0.008	0.0028	0.00123	3.11039	-1.24112	7	3.0	1.00	1.00	1.00	0.99813	0.98793
3	1	0.3	0.008	0.0036	0.00054	3.03069	-1.91095	9	0.0	125.07	125.07	125.00	0.00000	0.00000
3	1	0.3	0.008	0.0036	0.00054	3.03069	-1.91095	9	0.5	91.24	71.49	83.55	0.46363	0.18384
3	1	0.3	0.008	0.0036	0.00054	3.03069	-1.91095	9	1.0	42.78	27.36	36.07	0.75451	0.26976
3	1	0.3	0.008	0.0036	0.00054	3.03069	-1.91095	9	1.5	17.85	11.33	14.96	0.90077	0.34922
3	1	0.3	0.008	0.0036	0.00054	3.03069	-1.91095	9	2.0	8.00	5.43	6.85	0.95933	0.43057
3	1	0.3	0.008	0.0036	0.00054	3.03069	-1.91095	9	2.5	4.12	3.03	3.62	0.98205	0.51465
3	1	0.3	0.008	0.0036	0.00054	3.03069	-1.91095	9	3.0	2.46	1.96	2.22	0.99111	0.60153
3	5	0.3	0.008	0.0036	0.00054	3.03069	-1.91095	9	0.0	125.07	125.07	125.00	0.00000	0.00000
3	5	0.3	0.008	0.0036	0.00054	3.03069	-1.91095	9	0.5	34.82	21.96	29.18	0.80093	0.28850
3	5	0.3	0.008	0.0036	0.00054	3.03069	-1.91095	9	1.0	5.74	4.05	4.97	0.97268	0.46991
3	5	0.3	0.008	0.0036	0.00054	3.03069	-1.91095	9	1.5	1.87	1.56	1.71	0.99407	0.66425
3	5	0.3	0.008	0.0036	0.00054	3.03069	-1.91095	9	2.0	1.15	1.09	1.11	0.99711	0.85076
3	5	0.3	0.008	0.0036	0.00054	3.03069	-1.91095	9	2.5	1.02	1.01	1.01	0.99754	0.96435
3	5	0.3	0.008	0.0036	0.00054	3.03069	-1.91095	9	3.0	1.00	1.00	1.00	0.99758	0.99604

Table A3 (Continued)
Experimental Data

p	n	r	α	α_4	α_3	UCLp	LCLp	ϕ	$\lambda(\Delta)$	ARLAx	ARLDi	ARLChi	PCDAx	PCDDi
3	1	-0.3	0.008	0.0012	0.00281	3.35349	-0.53827	3	0.0	125.01	125.01	125.00	0.00000	0.00000
3	1	-0.3	0.008	0.0012	0.00281	3.35349	-0.53827	3	0.5	89.77	79.60	83.55	0.15387	0.71358
3	1	-0.3	0.008	0.0012	0.00281	3.35349	-0.53827	3	1.0	45.90	34.99	36.07	0.28928	0.83775
3	1	-0.3	0.008	0.0012	0.00281	3.35349	-0.53827	3	1.5	22.71	16.14	14.96	0.44396	0.87682
3	1	-0.3	0.008	0.0012	0.00281	3.35349	-0.53827	3	2.0	11.68	8.39	6.85	0.59959	0.89216
3	1	-0.3	0.008	0.0012	0.00281	3.35349	-0.53827	3	2.5	6.35	4.91	3.62	0.73010	0.89732
3	1	-0.3	0.008	0.0012	0.00281	3.35349	-0.53827	3	3.0	3.73	3.19	2.22	0.82367	0.89713
3	5	-0.3	0.008	0.0016	0.00242	3.27315	-0.55996	4	0.0	125.00	125.00	125.00	0.00000	0.00000
3	5	-0.3	0.008	0.0016	0.00242	3.27315	-0.55996	4	0.5	37.62	30.11	29.18	0.38955	0.79993
3	5	-0.3	0.008	0.0016	0.00242	3.27315	-0.55996	4	1.0	8.02	6.62	4.97	0.71183	0.86484
3	5	-0.3	0.008	0.0016	0.00242	3.27315	-0.55996	4	1.5	2.53	2.52	1.71	0.88636	0.87130
3	5	-0.3	0.008	0.0016	0.00242	3.27315	-0.55996	4	2.0	1.33	1.46	1.11	0.95536	0.87244
3	5	-0.3	0.008	0.0016	0.00242	3.27315	-0.55996	4	2.5	1.05	1.12	1.01	0.98580	0.89271
3	5	-0.3	0.008	0.0016	0.00242	3.27315	-0.55996	4	3.0	1.00	1.02	1.00	0.99686	0.93232
4	1	0	0.008	0.0028	0.00121	3.19435	-0.89112	7	0.0	125.41	125.41	125.00	0.00000	0.00000
4	1	0	0.008	0.0028	0.00121	3.19435	-0.89112	7	0.5	94.78	80.91	88.83	0.34600	0.37384
4	1	0	0.008	0.0028	0.00121	3.19435	-0.89112	7	1.0	45.79	35.00	41.49	0.63079	0.51250
4	1	0	0.008	0.0028	0.00121	3.19435	-0.89112	7	1.5	18.68	15.33	17.81	0.83983	0.59528
4	1	0	0.008	0.0028	0.00121	3.19435	-0.89112	7	2.0	8.02	7.48	8.16	0.92907	0.65168
4	1	0	0.008	0.0028	0.00121	3.19435	-0.89112	7	2.5	3.98	4.13	4.22	0.96660	0.69367
4	1	0	0.008	0.0028	0.00121	3.19435	-0.89112	7	3.0	2.33	2.58	2.51	0.98319	0.72870
4	5	0	0.008	0.0028	0.00121	3.19435	-0.89112	7	0.0	125.41	125.41	125.00	0.00000	0.00000
4	5	0	0.008	0.0028	0.00121	3.19435	-0.89112	7	0.5	37.23	28.60	33.97	0.70808	0.53546
4	5	0	0.008	0.0028	0.00121	3.19435	-0.89112	7	1.0	5.64	5.56	5.88	0.95358	0.67279
4	5	0	0.008	0.0028	0.00121	3.19435	-0.89112	7	1.5	1.76	1.99	1.89	0.98935	0.75228
4	5	0	0.008	0.0028	0.00121	3.19435	-0.89112	7	2.0	1.11	1.21	1.15	0.99656	0.83299
4	5	0	0.008	0.0028	0.00121	3.19435	-0.89112	7	2.5	1.01	1.03	1.01	0.99771	0.91848
4	5	0	0.008	0.0028	0.00121	3.19435	-0.89112	7	3.0	1.00	1.00	1.00	0.99758	0.97491

Table A3 (Continued)
Experimental Data

p	n	r	α	α_4	α_3	UCLp	LCLp	ϕ	$\lambda(\Delta)$	ARLAx	ARLDi	ARLChi	PCDAx	PCDDi
4	1	0.3	0.008	0.0036	0.00053	3.11561	-1.65968	9	0.0	124.92	124.92	125.00	0.00000	0.00000
4	1	0.3	0.008	0.0036	0.00053	3.11561	-1.65968	9	0.5	97.75	73.27	88.83	0.49815	0.18059
4	1	0.3	0.008	0.0036	0.00053	3.11561	-1.65968	9	1.0	50.51	28.74	41.49	0.69052	0.26840
4	1	0.3	0.008	0.0036	0.00053	3.11561	-1.65968	9	1.5	21.80	12.02	17.81	0.87348	0.34847
4	1	0.3	0.008	0.0036	0.00053	3.11561	-1.65968	9	2.0	9.69	5.77	8.16	0.94874	0.42891
4	1	0.3	0.008	0.0036	0.00053	3.11561	-1.65968	9	2.5	4.86	3.21	4.22	0.97754	0.51071
4	1	0.3	0.008	0.0036	0.00053	3.11561	-1.65968	9	3.0	2.81	2.05	2.51	0.98891	0.59441
4	5	0.3	0.008	0.0036	0.00053	3.11561	-1.65968	9	0.0	124.92	124.92	125.00	0.00000	0.00000
4	5	0.3	0.008	0.0036	0.00053	3.11561	-1.65968	9	0.5	41.68	23.15	33.97	0.74309	0.28743
4	5	0.3	0.008	0.0036	0.00053	3.11561	-1.65968	9	1.0	6.88	4.29	5.88	0.96675	0.46732
4	5	0.3	0.008	0.0036	0.00053	3.11561	-1.65968	9	1.5	2.08	1.62	1.89	0.99269	0.65470
4	5	0.3	0.008	0.0036	0.00053	3.11561	-1.65968	9	2.0	1.20	1.10	1.15	0.99649	0.83721
4	5	0.3	0.008	0.0036	0.00053	3.11561	-1.65968	9	2.5	1.02	1.01	1.01	0.99729	0.95664
4	5	0.3	0.008	0.0036	0.00053	3.11561	-1.65968	9	3.0	1.00	1.00	1.00	0.99715	0.99436
4	1	-0.3	0.008	0.0008	0.00321	3.54000	0.01698	2	0.0	125.00	125.00	125.00	0.00000	0.00000
4	1	-0.3	0.008	0.0008	0.00321	3.54000	0.01698	2	0.5	102.20	98.00	88.83	0.01359	0.71784
4	1	-0.3	0.008	0.0008	0.00321	3.54000	0.01698	2	1.0	66.13	58.39	41.49	0.01618	0.86768
4	1	-0.3	0.008	0.0008	0.00321	3.54000	0.01698	2	1.5	41.75	33.92	17.81	0.01602	0.92088
4	1	-0.3	0.008	0.0008	0.00321	3.54000	0.01698	2	2.0	27.44	20.76	8.16	0.01644	0.94240
4	1	-0.3	0.008	0.0008	0.00321	3.54000	0.01698	2	2.5	18.72	13.54	4.22	0.31452	0.95283
4	1	-0.3	0.008	0.0008	0.00321	3.54000	0.01698	2	3.0	13.02	9.36	2.51	0.43018	0.95832
4	5	-0.3	0.008	0.0012	0.00282	3.43149	0.00585	3	0.0	125.00	125.00	125.00	0.00000	0.00000
4	5	-0.3	0.008	0.0012	0.00282	3.43149	0.00585	3	0.5	58.90	52.85	33.97	0.02479	0.83524
4	5	-0.3	0.008	0.0012	0.00282	3.43149	0.00585	3	1.0	21.72	17.44	5.88	0.33547	0.92220
4	5	-0.3	0.008	0.0012	0.00282	3.43149	0.00585	3	1.5	9.13	7.63	1.89	0.57941	0.94232
4	5	-0.3	0.008	0.0012	0.00282	3.43149	0.00585	3	2.0	4.15	4.19	1.15	0.78113	0.94599
4	5	-0.3	0.008	0.0012	0.00282	3.43149	0.00585	3	2.5	2.19	2.71	1.01	0.88265	0.94270
4	5	-0.3	0.008	0.0012	0.00282	3.43149	0.00585	3	3.0	1.43	1.97	1.00	0.93183	0.93654

Appendix F

Tabulated Results of the Regression Analyses

Regression Analysis for α_3:

{ParameterTable ->

	Estimate	SE	TStat	PValue
1	0.00136414	$4.62537\ 10^{-6}$	294.925	0
p	-0.0000378047	$4.11666\ 10^{-6}$	-9.18335	0
p r	-0.0000198674	$5.51303\ 10^{-6}$	-3.60372	0.00115938
r	0.0000602052	0.0000123133	4.88946	0.0000344889
α_4 r	0.0000636744	$8.54735\ 10^{-6}$	7.44961	0
α	0.000765395	$7.39148\ 10^{-6}$	103.551	0
α_4	-0.00167653	0.0000217316	-77.1472	0

,

RSquared -> 0.999548

AdjustedRSquared -> 0.999455,

EstimatedVariance -> $3.49008\ 10^{-10}$,

ANOVATable ->

	DoF	SoS	MeanSS	FRatio	PValue}
Model	6	0.0000223829	$3.73049\ 10^{-6}$	10688.8	0
Error	29	$1.01212\ 10^{-8}$	$3.49008\ 10^{-10}$		
Total	35	0.0000223931			

Regression Analysis for ξ:

{ParameterTable ->

	Estimate	SE	TStat	PValue
1	6.30556	0.083453	75.5582	0
p	-0.333333	0.102209	-3.2613	0.00283598
p r	0.6875	0.125179	5.49211	0
n	0.305556	0.083453	3.66141	0.000994679
n r	-0.208333	0.102209	-2.03831	0.0507294
r	2.79167	0.102209	27.3134	0
α	0.138889	0.083453	1.66428	0.106827

,

RSquared -> 0.965317, AdjustedRSquared -> 0.958142,

EstimatedVariance -> 0.250718,

ANOVATable ->

	DoF	SoS	MeanSS	FRatio	PValue}
Model	6	202.368	33.728	134.525	0
Error	29	7.27083	0.250718		
Total	35	209.639			

Regression Analysis for Ω:

{ParameterTable ->

	Estimate	SE	TStat	PValue
1	3.74675	0.736222	5.08915	0
p	-2.16862	0.298126	-7.27418	0
n	0.580879	0.227433	2.55407	0.0112741
r	2.79367	0.332776	8.39506	0
α	1.19173	0.261997	4.54864	0
p r	-2.4736	1.02455	-2.41432	0.0165235
α p	-3.19444	0.674657	-4.7349	0
α_4 p	9.11265	1.85363	4.9161	0
n r	1.65021	0.747401	2.20793	0.0282072
α n	1.83169	0.482217	3.79849	0.000185064
α_4 n	-4.35445	1.31347	-3.31523	0.00105929
λ n	0.808601	0.300951	2.68682	0.0077244
α r	5.84694	1.08877	5.37025	0
α_4 r	-3.94281	0.839834	-4.69475	0
α α_4	-9.41086	1.87605	-5.01632	0

,

RSquared -> 0.605424, AdjustedRSquared -> 0.582115,

EstimatedVariance -> 10.144,

ANOVATable ->

	DoF	SoS	MeanSS	FRatio	PValue}
Model	14	3688.81	263.487	25.9746	0
Error	237	2404.13	10.144		
Total	251	6092.95			

Regression Analysis for $\overline{ARL}^{M}$:

{ParameterTable ->

	Estimate	SE	TStat	PValue	,
1	40.5609	2.13942	18.9589	0	
λ	-68.2351	3.20912	-21.2628	0	
n	-9.90579	2.13942	-4.63014	0	
α	-8.48683	2.13942	-3.96689	0.0000955547	
λ n	7.83961	3.20912	2.44291	0.0152735	
α λ	15.4212	3.20912	4.80542	0	

RSquared -> 0.678155, AdjustedRSquared -> 0.671613,

EstimatedVariance -> 1153.43,

ANOVATable ->

	DoF	SoS	MeanSS	FRatio	PValue}
Model	5	597872.	119574.	103.668	0
Error	246	283744.	1153.43		
Total	251	881615.			

Regression Analysis for Axial P[CD]:

{ParameterTable ->

	Estimate	SE	TStat	PValue
1	0.792686	0.0398976	19.868	0
λ	0.411006	0.0177231	23.1904	0
p	-0.0552939	0.0154298	-3.58357	0.000410717
n	0.0896737	0.0121224	7.39737	0
r	0.150888	0.0191656	7.87283	0
α	0.0266185	0.0151487	1.75715	0.0801727
α p	-0.04456	0.0189298	-2.35396	0.0193848
α_4 p	0.119292	0.0332275	3.59016	0.000400991
α n	0.0307196	0.015788	1.94576	0.0528559
α_4 n	-0.0822375	0.0259155	-3.1733	0.00170487
α r	0.128093	0.0600979	2.1314	0.0340768
α_4 r	-0.185815	0.0473086	-3.92771	0.000112264
α α_4	-0.141971	0.0998709	-1.42155	0.156461

,

RSquared -> 0.761453, AdjustedRSquared -> 0.749476,

EstimatedVariance -> 0.0351801,

ANOVATable ->

	DoF	SoS	MeanSS	FRatio	PValue}
Model	12	26.8389	2.23658	63.575	0
Error	239	8.40805	0.0351801		
Total	251	35.247			

Regression Analysis for Diagonal P[CD]:

{ParameterTable ->

	Estimate	SE	TStat	PValue
1	0.606663	0.0097843	62.0038	0
λ	0.36815	0.0146052	25.2067	0
n	0.0643471	0.00983608	6.54194	0
r	-0.128122	0.0119476	-10.7236	0
p r	-0.0387148	0.0146061	-2.65059	0.00855866
α_4 n	0.0719138	0.0173302	4.14963	0.000045969
λ n	0.0563356	0.0146052	3.85722	0.000146661

,

RSquared -> 0.775765, AdjustedRSquared -> 0.770274,

EstimatedVariance -> 0.023891,

ANOVATable ->

	DoF	SoS	MeanSS	FRatio	PValue}
Model	6	20.2501	3.37502	141.267	0
Error	245	5.85329	0.023891		
Total	251	26.1034			

Appendix G

Mathematica Code for the Minimax Control Chart

Main program "Evalcase.ma" : Calls the package Minimax.m and uses it to determine the control limits and ARL of the case specified by the user.

Evalcase.ma

```
(*************************************************************************************
The following lines are the initialization of packages needed for the
program to run properly.  Execute them before starting any session.
*************************************************************************************)
SetDirectory["C:\\1files\\1tesisfl"];
Needs["Statistics`ContinuousDistributions`"]
Needs["Minimax`"]

(*************************************************************************************
Please input the following parameters for the Minimax control chart
*************************************************************************************)
p = 5; (* p is the number of variables *)

n=5; (* n is the sample size *)

DesignAlpha = 1/125; (*probability of Type I error of the minimax chart*)

Rho = 0/10; (* Rho is the correlation coeficient for all {x[i], x[j]}*)

Lambda=0; (* Lambda is the magnitude of the shift *)

AxialShift=True; (* AxialShift=True implies an Axial shift in the mean
                    AxialShift=False implies a diagonal shift in the mean*)

FindControlLimits=True; (*This means that the control limits (CL) must be calculated.
        FindControlLimits=False means that the CL are known or had been found previously *)
Phi = 5/10; (* 1/10 < Phi < 9/10, Phi is used to assign Alpha4*)

(*************************************************************************************
Use the following instructions to evaluate the scenario specified
        by the parameters entered in the last statements.  You don't need to
        change these lines!!!  The program will recognize if the input data
        is logical or not.
*************************************************************************************)
Alpha4=DesignAlpha/2 * Phi;

Sigma = Rho+DiagonalMatrix[Table[1-Rho,{p}]];  (*Sigma is the correlation matrix of X*)

Approved=True; (*Approved is used to determine if the information needed to evaluate a case is correctly
        provided by the user*)
```

```
If[Not[FindControlLimits],
	If[Lambda == 0,
		Print["Set Lambda > 0 before calculating Out of Control ARL"];
		Approved=False;
	,
		ControlLimits={-LCLp,-UCLp,UCLp,LCLp};
	];
,
	If[Lambda!=0,
		Print["Set Lambda=0 before trying to find the control limits!"];
		Approved=False;
	];
];

If[Approved,
	{Alpha3,UCLp,LCLp,ARLMinMax}=EvaluateCase[Sigma,Lambda,p,AxialShift,n,
		DesignAlpha,Alpha4,FindControlLimits,ControlLimits];
	Print["Alpha3=",Alpha3," UCLp=",UCLp," LCLp=",LCLp,"  ARL=",ARLMinMax]
,
	Print["Some information is missing or input parameters are not logical.
	\n Please make the appropriate corrections."]
];
```

Package “Minimax.m” : Called by the main program “Evalcase.ma” to determine the control limits and ARL of the case specified by the user.

Minimax.m

```
(*Minimax Control Chart Statistics*)
BeginPackage[
        "Minimax`",
        "Statistics`InverseStatisticalFunctions`",
        "Statistics`Common`DistributionsCommon`",
        "Statistics`DescriptiveStatistics`",
        "Statistics`NormalDistribution`",
        "Statistics`ContinuousDistributions`",
]


CumF::usage = " Definition of the cummulative multivariate  normal function"

EvaluateCase::usage = "EvaluateCase[Sigma_,Lambda_,p_,AxialShift_,n_,
        DesignAlpha_,Alpha4_,FindControlLimits_,ControlLimits_]
        Returns Alpha3, UCLp, LCLp, and ARLMinMax for the input parameters
        Sigma, Lambda, p, AxialShift (True if an axial shift is to be
        evaluated. When False a Diagonal shift is evaluated), n, DesignAlpha,
        Alpha4, and FindControlLimits (True if the control limits are unknown)"

CalcAlpha::usage = "CalcAlpha[UCLp, LCLp, UCL1, LCL1, F, Alpha3, Alpha4]
        Calculates THE PROBABILITY OF TYPE I ERROR for the specified control
        limits UCLp, LCLp, UCL1, and LCL1. F is the joint density of the
        vector Z which must be evaluated from a to b in all the limits of
        the integral."

CalcBeta::usage = "CalcBeta[UCLp,LCLp,UCL1,LCL1,F] Calculates Beta
        for the specified control limits UCLp, LCLp, UCL1, and LCL1. F is the
        joint density of the vector Z which must be evaluated from a to b in
        all the limits of the integral."

CalcUCLChiSq::usage = "CalcUCLChiSq[DesignAlpha_,p_] Calculates the UCL
        of the Chi-Squared control chart for the specified probability of
        Type I error=DesignAlpha and number of variables=p."

CalcARLChiSq::usage = "CalcARLChiSq[p_,n_,Lambda_,UCLChiSq_]
        Calculates the ARL of the Chi-Squared Control Chart for number of
        variables=p, sample size=n, distance ot the shift in the mean=
        Lambda, and the Upper Control Limit of the Chart=UCLChiSq"

CalcLCLp::usage = "CalcLCLp[DesignAlpha,Alpha4,UCLp_,LCL1_,F_]
        Calculates LCLp for the given values of DesignAlpha, Alpha4, and UCLp"
```

```
CalcUCLp::usage = "CalcUCLp[Alpha4_,F_] Calculates UCLp for the given Alpha4 and F"
Begin["`Private`"]

CalcUCLChiSq[DesignAlpha_,p_] :=
        Replace[ULCS,FindRoot[CDF[ChiSquareDistribution[p],ULCS]==
                (1-DesignAlpha), {ULCS, 1, 100}]]

CalcARLChiSq[p_,n_,Lambda_,UCLChiSq_]:=
1/(1-CDF[NoncentralChiSquareDistribution[p,(n*
                Lambda^2)], UCLChiSq])

CalcAlpha[UCLp_,LCLp_,UCL1_,LCL1_,F_,Alpha3_,Alpha4_] :=
Module[{t1, t2, t3, t4, PA, PB, PC, PD, PE, PG, PH, PI, Answer},
        t1=Alpha4;
        t2=Alpha3;
        t3=Alpha3;
        t4=Alpha4;
        If[LCLp > UCL1, PG=F[UCL1, LCLp], PG=0];
        PC=-1+t1+t4+F[LCL1, UCLp];
        PI=t2-F[UCL1, UCLp];
        PA=PI;
        PH:=t2-PI-PG;
        (*PD:=t3-PG-PA;*)
        PB:=t1-PA-PC;
        PE:=Re[1-t3-t4-PH-PB];
        Answer=1-PE
]

CalcBeta[UCLp_,LCLp_,UCL1_,LCL1_,F_] :=
Module[{t1, t2, t3, t4, PA, PB, PC, PD, PE, PG, PH, PI, Answer},
        t1=1-F[LCL1, Infinity];
        t2=F[UCL1, Infinity];
        t3=F[-Infinity, LCLp];
        t4=1-F[-Infinity, UCLp];
        If[LCLp > UCL1, PG=F[UCL1, LCLp], PG=0];
        PC=-1+t1+t4+F[LCL1, UCLp];
        PI=t2-F[UCL1, UCLp];
        PA=t3-F[LCL1, LCLp];
        PH:=t2-PI-PG;
        (*PD:=t3-PG-PA;*)
        PB:=t1-PA-PC;
        PE:=Re[1-t3-t4-PH-PB];
        Answer=PE
]
```

```
CalcLCLp[DesignAlpha_,Alpha4_,UCLp_,LCL1_,F_,p_] :=
Module[{Err,NearLarge,NearSmall,Shift,Counter,NearestSmallARL,
        NearestLargeARL,UCL1List,Alpha3List,ARLList,Alpha3},
        (* Assigning LCLp assuming independent variables *)
        StdPhi[z_] := CDF[NormalDistribution[0,1],z];
        AlphaMin=1-Sqrt[1-DesignAlpha];
        Alpha3=AlphaMin-Alpha4;
        LCLp=Replace[t,FindRoot[StdPhi[t]==(Alpha3)^(1/p),{t,0}]];
        (*Searching for LCLp to meet the required DesignAlpha*)
        Err=100;
        NearLarge= NearSmall = False;
        Shift= 0.5;
        Counter=0;
        NearestSmallARL=0;
        NearestLargeARL=100/DesignAlpha;
        While[Err > .5,
                Counter++;
                UCL1=-LCLp;
                Alpha3=F[-Infinity, LCLp];
                Alpha = CalcAlpha[UCLp,LCLp,UCL1,LCL1,F,Alpha3,Alpha4];
                ARL = 1/Alpha;
                Err = Abs[ARL - 1/DesignAlpha];
                If [(ARL > NearestSmallARL)&&(ARL < 1/DesignAlpha),
                        NearestSmallARL=ARL;
                        NearestSmallLCLp=LCLp;
                        NearSmall=True;
                ];
                If [(ARL < NearestLargeARL)&&(ARL > 1/DesignAlpha),
                        NearestLargeARL=ARL;
                        NearestLargeLCLp=LCLp;
                        NearLarge=True;
                ];
                If[NearLarge && NearSmall,
                        ForcastLCLp = Interpolation[{{NearestLargeARL,NearestLargeLCLp},
                                {NearestSmallARL,NearestSmallLCLp}},InterpolationOrder->1];
                        SaveLCLp=LCLp;
                        LCLp = ForcastLCLp[1/DesignAlpha];
                ,
                        If[Counter>1,
                                Fw=Fit[{{SaveARL,SaveLCLp}, {ARL,LCLp}}, {1,PredARL},
                                        {PredARL}];
                                PredARL=1/DesignAlpha;
                                SaveLCLp=LCLp;
                                LCLp=Fw;
                        ,
                                If[1/Err < 0.001, Shift = (1/Err)/.001 * .5];
                                SaveLCLp=LCLp;
                                If [ARL < 1/DesignAlpha,
                                        LCLp=LCLp-Shift;
```

```
                                        ,
                                                LCLp=LCLp+Shift;
                                        ]; (*End If*)
                                ]; (*End If*)
                        ]; (*End If*)
                        SaveARL=ARL;
                ]; (*End While*)
                LCLp=SaveLCLp;
                Answer={LCLp,Alpha3,ARL}
] (*End Module*)

CalcUCLp[Alpha4_,F_,p_]:=
Module[{Err,NearLarge,NearSmall,Shift,Counter,NearestSmallAlpha4,
        NearestLargeAlpha4},
        (*Searching for UCLp to meet the required DesignAlpha*)
        Err=1;
        NearLarge = NearSmall = False;
        Shift= 0.5;
        Counter=0;
        (* Assigning UCLp assuming independent variables *)
        StdPhi[z_]:=CDF[NormalDistribution[0,1],z];
        UCLp=Replace[t,FindRoot[StdPhi[t]==(1-Alpha4)^(1/p),{t,0}]];
        NearestSmallTail4=0;
        NearestLargeTail4=1;

        While[Err > 0.00001,
                Counter++;
                Tail4=1-F[-Infinity, UCLp];
                Err = Abs[Tail4-Alpha4];
                If [Tail4 > NearestSmallTail4 && (Tail4 < Alpha4),
                        NearestSmallTail4=Tail4;
                        NearestSmallUCLp=UCLp;
                        NearSmall=True;
                ];
                If [Tail4 < NearestLargeTail4 && (Tail4 > Alpha4),
                        NearestLargeTail4=Tail4;
                        NearestLargeUCLp=UCLp;
                        NearLarge=True; (*Tail4 > ALpha4 but close to it*)
                ];
                If[NearLarge && NearSmall,
                        ForcastUCLp = Interpolation[{{NearestLargeTail4,NearestLargeUCLp},
                                {NearestSmallTail4,NearestSmallUCLp}},InterpolationOrder->1];
                        SaveUCLp=UCLp;
                        UCLp = ForcastUCLp[Alpha4];
                ,
                        If[Counter>1,

        Fw=Fit[{{SaveAlpha4,SaveUCLp},{Tail4,UCLp}},{1,PredAlpha4},{PredAlpha4}];
                                PredAlpha4=Alpha4;
```

```
                                        SaveUCLp=UCLp;
                                        UCLp=Fw;
                                ,
                                        If[Err < 0.001, Shift = Err/.001 * .5];
                                        SaveUCLp=UCLp;
                                        If [Tail4 < Alpha4,
                                                UCLp=UCLp-Shift;
                                        ,
                                                UCLp=UCLp+Shift;
                                        ]; (*End If*)
                                ]; (*End If*)
                        ]; (*End If*)
                        SaveAlpha4=Tail4;
                ]; (*End While*)
                UCLp=SaveUCLp;
                Answer=UCLp
] (*End Module*)

CumF[MuZ_,Theta_,p_,a_,b_]:=CompoundExpression[
StdPhi[z_] := CDF[NormalDistribution[0,1],z];
F1:=Re[N[Integrate[ Simplify[ 1/Sqrt[2 Pi] Exp[-(Yy^2)/2]*
        Simplify[Product[
                StdPhi[ ((b-MuZ[[i]]) + Theta[[i]]*Yy)/Sqrt[ 1-Theta[[i]]^2 ] ] -
                StdPhi[ ((a-MuZ[[i]]) + Theta[[i]]*Yy)/Sqrt[ 1-Theta[[i]]^2 ] ],
        {i,1,p}]]],
{Yy,-Infinity,Infinity}]]];

Return[F1]

](*End of CompoundExpression*)


EvaluateCase[Sigma_,Lambda_,p_,AxialShift_,n_,DesignAlpha_,Alpha4_,
        FindControlLimits_,ControlLimits_]:=CompoundExpression[
        (* Defining MuZ and search of Delta (dt) for a given Lambda. Delta(i)
        is the shift on mean of x(i) measured in standard deviations of x(i) *)
        If[AxialShift,
                Delta = Replace[dt,
                        FindRoot[Sqrt [Table[If[i==1,dt,0],{i, p}] . Inverse[Sigma] .
                        Table[If[i==1,dt,0],{i, p}]] == Lambda, {dt, 0,5}]];
                MuZ = Table[If[i==1,Delta*Sqrt[n],0],{i, p}];
                MuX = Table[If[i==1,Delta,0],{i, p}];
        ,(* Diagonal Shifts *)
                Delta = Replace[dt,
                        FindRoot[Sqrt [Table[dt,{i, p}] . Inverse[Sigma] .
                        Table[dt,{i, p}]] == Lambda, {dt, 0,5}]];
                MuZ = Table[Delta*Sqrt[n],{i, p}];
                MuX = Table[Delta,{i, p}];
        ];
```

```
If[FindControlLimits, (*Set the control limits for a specified DesignAlpha*)
        Theta = Table[Sqrt[Sigma[[1,2]]],{i,p}];
        (*Theta is Lambda in Y.L. Tong 8.1.5, Sigma[[1,2]=ro]*)
        F[a_,b_]=CumF[MuZ,Theta,p,a,b];
        (*Searching for UCLp*)
        UCLp=CalcUCLp[Alpha4,F,p];
        LCL1=-UCLp;

        (*Searching for LCLp to meet the required DesignAlpha*)
        {LCLp,Alpha3,ARLMinMax}=CalcLCLp[DesignAlpha,Alpha4,UCLp,LCL1,F,p];
        UCL1=-LCLp;
];
If[Lambda>0,(*Calculate ARL for a given shift in the mean. *)
        {UCL1,LCL1,UCLp,LCLp}=ControlLimits;
        Theta = Table[Sqrt[Sigma[[1,2]]],{i,p}];
        (*Theta is Lambda in Y.L. Tong 8.1.5, Sigma[[1,2]=ro]*)
        F[a_,b_]=CumF[MuZ,Theta,p,a,b];
        MyBeta = CalcBeta[UCLp,LCLp,UCL1,LCL1,F];
        ARLMinMax = 1/(1-MyBeta);
];
Return[{Alpha3,UCLp,LCLp,ARLMinMax}]
](*End of Compound Expression for EvaluateCase*)


End[ ] (*End of the Private Context*)

EndPackage[ ]
```

Vita

Ariel Sepúlveda has a bachelor degree in Industrial Engineering and a master in Industrial Engineering from the University of Puerto Rico at Mayagüez where he has served as an instructor for two years.

Ariel Sepúlveda