Appendix 3b Proofs for Part 3b

Proof of Lemma 3.b.1: Observe that

$$\alpha = \frac{\sum ((\hat{f}^{(i)} - \hat{g}^{(i)}) + (\hat{f} - \hat{g}))((\hat{f}^{(i)} - \hat{g}^{(i)}) - (\hat{f} - \hat{g}))}{n}$$

$$= \frac{\sum ((\hat{f}^{(i)} + \hat{f}) - (\hat{g}^{(i)} + \hat{g}))((\hat{f}^{(i)} - \hat{f}) - (\hat{g}^{(i)} - \hat{g}))}{n}$$

$$\leq \left| \frac{\sum ((\hat{f}^{(i)} + \hat{f}) - (\hat{g}^{(i)} + \hat{g}))((\hat{f}^{(i)} - \hat{f}) - (\hat{g}^{(i)} - \hat{g}))}{n} \right|$$

$$\leq \left\| ((\hat{f}^{(i)} + \hat{f}) - (\hat{g}^{(i)} + \hat{g})) \right\| \left\| ((\hat{f}^{(i)} - \hat{f}) - (\hat{g}^{(i)} - \hat{g})) \right\|$$

$$\leq \left\| ((\hat{f} - \hat{g}) + (\hat{f} - \hat{g}) + (\hat{g} - \hat{g}^{(i)}) + (\hat{f}^{(i)} - \hat{f})) \right\| \left\| ((\hat{f}^{(i)} - \hat{f}) - (\hat{g}^{(i)} - \hat{g})) \right\|$$

$$\leq 4 \left\| \hat{f} - \hat{g} \right\| \sqrt{\left\| ((\hat{f}^{(i)} - \hat{f}) - (\hat{g}^{(i)} - \hat{g})) \right\|^2}$$

$$\leq 4 \left\| \hat{f} - \hat{g} \right\| \sqrt{(\left\| (\hat{f}^{(i)} - \hat{f}) \right\|^2 + 2 \left\| (\hat{f}^{(i)} - \hat{f}) \right\| \left\| (\hat{g}^{(i)} - \hat{g}) \right\| + \left\| (\hat{g}^{(i)} - \hat{g}) \right\|^2)}$$

$$= 4 \left\| \hat{f} - \hat{g} \right\| \sqrt{(O_P(n^{-2}) + O_P(n^{-1} g_n^2) + O_P(g_n^4)}$$

$$= 4 \left\| \hat{f} - \hat{g} \right\| (O_P(n^{-1}) + O_P(n^{-.5} g_n) + O_P(g^{2}{}_{n})) \qquad \text{(A3.B.1)}$$

So $\alpha = \begin{cases} O_P(g_n{}^2), & if \ \lim_{n \to \infty} \boldsymbol{d}_n \neq 0 \\ O_P(g_n{}^3), & if \ \boldsymbol{d}_n = 0 \end{cases}$,

by assumption A5, and Lemma 3.a.1, as desired.//.

Proof of Corollary 3.b.1: For two non-negative Real numbers a and b,

$$\sqrt{|a^2 - b^2|} = \sqrt{|a-b|\,|a+b|}$$

$$\geq |a-b|.$$

Letting a = $\left\| \hat{f}^{(i)} - \hat{g}^{(i)} \right\|$, and b = $\left\| \hat{f} - \hat{g} \right\|$, we have

$$\left\| \hat{f}^{(i)} - \hat{g}^{(i)} \right\| - \left\| \hat{f} - \hat{g} \right\| = \begin{cases} O_P(g_n), & \text{if } \lim_{n\to\infty} d_n \neq 0 \\ O_P(g_n^{\;1.5}), & \text{if } d_n = 0 \end{cases}$$

by Lemma 3.b.1, as desired.//.

Proof of Lemma 3.b.2: First observe that the expression can be rewritten

$$\frac{\sum (y_i - \hat{g}^{(i)})(\hat{f}^{(i)} - \hat{g}^{(i)})}{n(\left\| \hat{f} - \hat{g} \right\|^2)} = \frac{\sum (y_i - \hat{f}^{(i)} + \hat{f}^{(i)} - \hat{g}^{(i)})(\hat{f}^{(i)} - \hat{g}^{(i)})}{n(\left\| \hat{f} - \hat{g} \right\|^2)}$$

$$= \frac{\sum (q_i + e_i - \hat{f}^{(i)})(\hat{f}^{(i)} - \hat{g}^{(i)})}{n(\left\| \hat{f} - \hat{g} \right\|^2)} + \frac{\left\| \hat{f}^{(i)} - \hat{g}^{(i)} \right\|^2}{\left\| \hat{f} - \hat{g} \right\|^2}$$

$$= \frac{\sum (q_i + e_i - \hat{f}^{(i)})(\hat{f}^{(i)} - \hat{g}^{(i)})}{n(\left\| \hat{f} - \hat{g} \right\|^2)} + \frac{\left\| \hat{f}^{(i)} - \hat{g}^{(i)} \right\|^2}{\left\| \hat{f} - \hat{g} \right\|^2}$$

$$= \frac{\sum (q_i - \hat{f} + \hat{f} - \hat{f}^{(i)})(\hat{f}^{(i)} - \hat{g}^{(i)})}{n(\left\| \hat{f} - \hat{g} \right\|^2)} + \frac{\sum e_i(\hat{f}^{(i)} - \hat{g}^{(i)})}{n(\left\| \hat{f} - \hat{g} \right\|^2)} + \frac{\left\| \hat{f}^{(i)} - \hat{g}^{(i)} \right\|^2}{\left\| \hat{f} - \hat{g} \right\|^2}$$

= T1 + T2 + T3 (say).

$$\text{T1} = \frac{\sum (q_i - \hat{f})(\hat{f}^{(i)} - \hat{g}^{(i)})}{n(\left\| \hat{f} - \hat{g} \right\|^2)} + \frac{\sum (\hat{f} - \hat{f}^{(i)})(\hat{f}^{(i)} - \hat{g}^{(i)})}{n(\left\| \hat{f} - \hat{g} \right\|^2)}$$

$$\leq \frac{\left\| q_i - \hat{f} \right\| \left\| \hat{f}^{(i)} - \hat{g}^{(i)} \right\|}{\left\| \hat{f} - \hat{g} \right\|^2} + \frac{\left\| \hat{f} - \hat{f}^{(i)} \right\| \left\| \hat{f}^{(i)} - \hat{g}^{(i)} \right\|}{\left\| \hat{f} - \hat{g} \right\|^2}$$

(by the Cauchy-Schwartz Inequality)

$$\leq \frac{\left\| q_i - \hat{f} \right\| (1 + O_P(g_n))}{\left\| \hat{f} - \hat{g} \right\|} + \frac{\left\| \hat{f} - \hat{f}^{(i)} \right\| (1 + O_P(g_n))}{\left\| \hat{f} - \hat{g} \right\|}$$

(by Corollary 3.b.1). So asymptotically

$$\text{T1} = \frac{\left\| q_i - \hat{f} \right\|}{\left\| \hat{f} - \hat{g} \right\|} + \frac{\left\| \hat{f} - \hat{f}^{(i)} \right\|}{\left\| \hat{f} - \hat{g} \right\|}$$

$$\leq \frac{\left\| q_i - f(\boldsymbol{b}^*) \right\|}{\left\| \hat{f} - \hat{g} \right\|} + \frac{\left\| f(\boldsymbol{b}^*) - \hat{f} \right\|}{\left\| \hat{f} - \hat{g} \right\|} + \frac{\left\| \hat{f} - \hat{f}^{(i)} \right\|}{\left\| \hat{f} - \hat{g} \right\|}$$

$$= \frac{d_n + O_P(n^{-.5}) + O_P(n^{-1})}{\left\| \hat{f} - \hat{g} \right\|}$$

(by Burman and Chaudhuri (1992) equation 6.20). Next,

$$\text{T2} = \frac{\sum e_i (\hat{f}^{(i)} - \hat{g}^{(i)})}{n(\left\| \hat{f} - \hat{g} \right\|^2)} = \frac{O_P(d_n n^{-.5}) + O_P(g_n n^{-.5})}{\left\| \hat{f} - \hat{g} \right\|^2}$$

(by Burman and Chaudhuri (1992) equation 6.19). Finally,

$$\text{T3} = \frac{\left\| \hat{f}^{(i)} - \hat{g}^{(i)} \right\|^2}{\left\| \hat{f} - \hat{g} \right\|^2}$$

$$= \frac{\left\| \hat{f} - \hat{g} \right\|^2 + (\left\| \hat{f}^{(i)} - \hat{g}^{(i)} \right\|^2 - \left\| \hat{f} - \hat{g} \right\|^2)}{\left\| \hat{f} - \hat{g} \right\|^2}$$

$$= 1 + \frac{\boldsymbol{a}}{\left\| \hat{f} - \hat{g} \right\|^2} = O_P(1)$$

(by Lemma 3.b.1). So that $\frac{\sum (y_i - \hat{g}^{(i)})(\hat{f}^{(i)} - \hat{g}^{(i)})}{n(\left\| \hat{f} - \hat{g} \right\|^2)}$ = T1 + T2 + T3

$$= \frac{d_n + O_P(n^{-.5}) + O_P(n^{-1})}{\left\| \hat{f} - \hat{g} \right\|} + \frac{O_P(d_n n^{-.5}) + O_P(g_n n^{-.5})}{\left\| \hat{f} - \hat{g} \right\|^2} + O_P(1) = O_P(1)$$

by Assumptions A2 and A5, Lemmas 3.a.1 and 3.a.4, and the fact that d_n is finite, as desired.//.

Proof of Lemma 3.b.3: Observe that

$$R1 = \frac{\sum (y_i - \hat{g}^{(i)})(\hat{f}^{(i)} - \hat{g}^{(i)})}{n(\|\hat{f} - \hat{g}\|^2)} \left(\frac{\boldsymbol{a}}{\|\hat{f} - \hat{g}\|^2 + \boldsymbol{a}} \right)$$

$$= O_P(1) \left(\frac{\boldsymbol{a}}{\|\hat{f} - \hat{g}\|^2 + \boldsymbol{a}} \right) = \left(\frac{\boldsymbol{a}}{\|\hat{f} - \hat{g}\|^2 + \boldsymbol{a}} \right) \qquad \text{(A3.B.2)}$$

by Lemma 3.b.2. Finally, we have

$$\text{R1} = \begin{cases} O_P(g_n{}^2), & \text{if } \boldsymbol{d}_n \neq 0 \\ O_P(g_n), & \text{if } \boldsymbol{d}_n = 0 \end{cases}$$

by the combination of Lemmas 3.a.1 and 3.b.1, as desired.//.

Proof of Lemma 3.b.4: The proof for both parts follows from Lemmas 3.a.1, 3.a.3 and 3.b.3, along with equation 3.B.4, as desired.//.

Proof of Lemma 3.b.5: The proof of part a) follows from Lemma 3.a.4 along with equation A3.B.1. The proof of b) follows from Lemma 3.a.4 along with equations A3.B.1 and A3.B.2. The proof of c) follows from Lemma 3.a.4 along with equations A3.B.1, A3.B.2, and 3.B.4, as desired.//.

Proof of Theorem 3.B.2: The proof for both parts follows from Lemmas 3.a.1, 3.b.4, equation A3.A.1 (with subsequent comments and replacing $\hat{\boldsymbol{l}}^{*C}$ with $\hat{\boldsymbol{l}}^{P}$) and Theorem 3.A.1, as desired.//.

Proof of Theorem 3.B.4: The proof for both parts follows from Lemmas 3.a.4, 3.b.5, equation A3.A.1 (with subsequent comments and replacing $\hat{\boldsymbol{l}}^{*C}$ with $\hat{\boldsymbol{l}}^{P}$) and Theorem 3.A.3, as desired.//.