Appendix 3c Proofs for Part 3c

Proof of Lemma 3.c.1: Note that

$$\text{T1} = (\sum\sum_{i,j}((\hat{f}^{(i)} - \hat{g}^{(i)})(\hat{f}^{(j)} - \hat{g}^{(j)})) / n^2) - (\sum((\hat{f}^{(i)} - \hat{g}^{(i)})^2) / n^2)$$

$$= (\sum\sum_{i,j}((\hat{f}^{(i)} - \hat{g}^{(i)})(\hat{f}^{(j)} - \hat{g}^{(j)})) / n^2) - O_P(n^{-1})$$

$$= (\sum(\hat{f}^{(i)} - \hat{g}^{(i)}) / n)(\sum(\hat{f}^{(j)} - \hat{g}^{(j)}) / n) - O_P(n^{-1}).$$

Next observe that $\sum(\hat{f}^{(i)} - \hat{g}^{(i)}) / n$

$$= (\sum(\hat{f}^{(i)} - \hat{f}) / n) + (\sum(\hat{f} - \hat{g}) / n) + (\sum(\hat{g} - \hat{g}^{(i)}) / n)$$

$$\leq O_P(n^{-1}) + (\sum(\hat{f} - \boldsymbol{q}) / n) + (\sum(\boldsymbol{q} - \hat{g}) / n) + O_P(\boldsymbol{g}_n^2)$$

(by Burman and Chaudhuri (1992) results 6.20 and 6.21)

$$\leq O_P(n^{-1}) + O_P(n^{-.5}) + O_P(\boldsymbol{g}_n) + O_P(\boldsymbol{g}_n^2) \qquad \text{(A3.C.1)}$$

(from assumption A2 and Burman and Chaudhuri (1992) equation 6.1).

$$= O_P(\boldsymbol{g}_n).$$

Note that this is consistent with Corollary 3.b.1. So that T1 is $O_P(\boldsymbol{g}_n^2)$ as desired.//.

Proof of Lemma 3.c.2: Bishop, Fienberg, and Holland (1975) state that "you are only as big as your standard deviation." We will proceed along these lines.

$$V\left(\frac{\sum \boldsymbol{e}_i}{n}\right) = E\left(\left(\frac{\sum \boldsymbol{e}_i}{n}\right)^2\right)$$

$$= n^{-2}(E\left(\sum \boldsymbol{e}_i^2\right) + E\left(\sum\sum_{i \neq j} \boldsymbol{e}_i \boldsymbol{e}_j\right))$$

$$= n^{-2}(n\boldsymbol{s}^2 + 0)$$

$$= n^{-1}\boldsymbol{s}^2.$$

So that $\dfrac{\sum \boldsymbol{e}_i}{n} = O_P(n^{-.5})$ as desired.//.

Proof of Lemma 3.c.3:

$$\frac{(\sum_{i=1}^{n}((Y_i-\hat{g}^{(i)})(\hat{f}^{(i)}-\hat{g}^{(i)}))/n)}{(\sum_{i=1}^{n}(\hat{f}^{(i)}-\hat{g}^{(i)})^2/n)} \le \frac{\left\|q-\hat{g}^{(i)}\right\|\left\|\hat{f}^{(i)}-\hat{g}^{(i)}\right\|+n^{-1}\sum e_i(\hat{f}^{(i)}-\hat{g}^{(i)})}{\left\|\hat{f}^{(i)}-\hat{g}^{(i)}\right\|^2}$$

$$\le \frac{\left\|q-\hat{g}\right\|+\left\|\hat{g}-\hat{g}^{(i)}\right\|}{\left\|\hat{f}^{(i)}-\hat{g}^{(i)}\right\|}+\frac{O_P(n^{-.5})(O_P(g_n)+d_n)}{\left\|\hat{f}^{(i)}-\hat{g}^{(i)}\right\|^2}$$

(by the triangle inequality, and Burman and Chaudhuri (1992) equation 6.19)

$$= \begin{cases} \dfrac{O_P(g_n)+O_P(g_n^2)}{O_P(1)}+O_P(n^{-.5}), \, if \ \lim_{n\to\infty} d_n \ne 0 \\ O_P(1)+O_P(g_n^{-1}n^{-.5}) \qquad\qquad , if \ d_n = 0 \end{cases}$$

(by Lemmas 3.a.1, and 3.b.1, Corollary 3.b.1, and assumption A2). So that

$$\frac{(\sum_{i=1}^{n}((Y_i-\hat{g}^{(i)})(\hat{f}^{(i)}-\hat{g}^{(i)}))/n)}{(\sum_{i=1}^{n}(\hat{f}^{(i)}-\hat{g}^{(i)})^2/n)} = \begin{cases} O_P(g_n), \, if \ \lim_{n\to\infty} d_n \ne 0 \\ O_P(1), \quad if \ d_n = 0 \end{cases}$$

as desired.//.

Proof of Lemma 3.c.4: Again note that

$$\frac{\sum\sum_{i\ne j}((Y_i-\hat{g}^{(i)})(\hat{f}^{(j)}-\hat{g}^{(j)}))}{n^2} = \frac{\sum\sum_{i,j}((Y_i-\hat{g}^{(i)})(\hat{f}^{(j)}-\hat{g}^{(j)}))}{n^2} - O_P(n^{-1})$$

$$= \frac{\sum\sum_{i,j}((q_i+e_i-\hat{g}^{(i)})(\hat{f}^{(j)}-\hat{g}^{(j)}))}{n^2} - O_P(n^{-1})$$

$$= \frac{\sum(q_i-\hat{g}^{(i)})\sum(\hat{f}^{(i)}-\hat{g}^{(i)})}{n^2} + \frac{\sum(e_i)\sum(\hat{f}^{(i)}-\hat{g}^{(i)})}{n^2} - O_P(n^{-1})$$

$$\le (O_P(g_n)+O_P(g_n^2))O_P(g_n)+O_P(n^{-.5})O_P(g_n)-O_P(n^{-1}) = O_P(g_n^2)$$

by Lemmas 3.a.1, 3.c.1 (particularly equation A3.C.1) and 3.c.2, Corollary 3.b.1, and assumption A2, as desired.//.

Proof of Lemma 3.c.5: The proof follows by combining Lemmas 3.c.1 and 3.c.4, as desired.//.