Appendix 4a Proofs for Part 4a

Proof of Lemma 4.a.1: Note that if d_n stays away from zero, $\|q - f(b^{**})\|$ = $O_P(1)$. But

$$\|q - f(b^{**})\| = \|\hat{g} - (\hat{g} - (q - f(b^{**}))\|$$

$$\leq \|\hat{g}\| + \|(\hat{g} - (q - f(b^{**}))\|$$

$$= \|\hat{g}\| + O_P(g_n)$$

by assumption A2, so that

$$\|\hat{g}\| = O_P(1) .$$

If d_n = 0 then by assumption A2 $\|\hat{g}\|$

$$= \|\hat{g} - (q - f(b^{**}))\| = O_P(g_n)$$

as desired.//.

Proof of Lemma 4.a.2: First observe that l^* (given by 4.A.1) becomes

$$\frac{<\hat{g}, q - f> + <\hat{g}, f - \hat{f}>}{\|\hat{g}\|^2}$$

$$\leq \frac{\|\hat{g}\|\,\|q - f\| + \|\hat{g}\|\,\|f - \hat{f}\|}{\|\hat{g}\|^2}$$

(by the Cauchy-Schwarz Inequality)

$$= \frac{\|q - f\| + \|f - \hat{f}\|}{\|\hat{g}\|} \qquad \text{(A4.A.1)}$$

(We will often use f to designate $f(b^{**})$). If d_n = 0, then by Lemma 4.a.1, and 4.1, A4.A.1

$$= \frac{0 + O_P(n^{-.5})}{O_P(g_n)} = O_P(n^{-.5} g_n^{-1}) .$$

If d_n stays away from zero we want to see how quickly $l*$ approaches 1. To that end 1- $l*$

$$= \frac{\|\hat{g}\|^2 - <\hat{g}, \boldsymbol{q} - \hat{f}>}{\|\hat{g}\|^2}$$

$$= \frac{<\hat{g},\hat{g}> - <\hat{g}, \boldsymbol{q} - \hat{f}>}{\|\hat{g}\|^2}$$

$$= \frac{<\hat{g},\hat{g} - \boldsymbol{q} + \hat{f}>}{\|\hat{g}\|^2}$$

$$\leq \frac{\|\hat{g}\|\,\|\hat{g} - (\boldsymbol{q} - f)\| + \|\hat{g}\|\,\|(\hat{f} - f)\|}{\|\hat{g}\|^2}$$

(by the Triangle and Cauchy-Schwarz Inequalities)

$$= \frac{\|\hat{g} - (\boldsymbol{q} - f)\| + \|(\hat{f} - f)\|}{\|\hat{g}\|} \qquad \text{(A4.A.2)}$$

So by Lemma 4.a.1, equation 4.1, and assumption A2, A4.A.2

$$= \frac{O_P(\boldsymbol{g}_n) + O_P(n^{-.5})}{O_P(1)}$$

$$= O_P(\boldsymbol{g}_n),$$

as desired.//.

Proof of Lemma 4.a.3: Consider $\hat{l}* - l*$

$$= \frac{<\hat{g}, Y - \hat{f}>}{\|\hat{g}\|^2} - \frac{<\hat{g}, \boldsymbol{q} - \hat{f}>}{\|\hat{g}\|^2} = \frac{<\hat{g}, \boldsymbol{e}>}{\|\hat{g}\|^2}$$

Note that $<\hat{g}, \boldsymbol{e}> = n^{-1}\sum_{i=1}^{n}\hat{g}(x_i)\boldsymbol{e}_i = n^{-1}\sum_{i=1}^{n}\frac{\sum_{j=1}^{n}W_2(x_i,x_j)(Y_j - \hat{f}_j)}{n}\boldsymbol{e}_i$

$$= n^{-2}\sum_{i=1}^{n}\boldsymbol{e}_i\sum_{j=1}^{n}W_2(x_i,x_j)(\boldsymbol{q}_j + \boldsymbol{e}_j - \hat{f}_j)$$

=

$$n^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n}\boldsymbol{e}_i W_2(x_i,x_j)(\boldsymbol{q}_j - f_j) + n^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n} W_2(x_i,x_j)\boldsymbol{e}_i\boldsymbol{e}_j - n^{-2}\sum_{i=1}^{n}\boldsymbol{e}_i\sum_{j=1}^{n} W_2(x_i,x_j)(\hat{f}_j - f_j)$$

= T1 + T2 - T3 (say).

Note that E(T1) = 0. Hence, we turn our attention to V(T1) = E(T1^2)

$$\leq n^{-4}\sum_{i=1}^{n} c\left(\sum_{j=1}^{n} W_2\ (x_i,x_j)(\boldsymbol{q}_j - f_j)\right)^2 .$$

(for some real positive constant c, by Whittle's Inequality (Whittle (1960)))

$$\leq c_2 n^{-2}\boldsymbol{d}_n^2\boldsymbol{t}_n$$

for some positive constant c_2, by Assumption A4, the definition of $\boldsymbol{d}_n$, and the Cauchy-Schwarz Inequality. So V(T1)

$$= O_P(\boldsymbol{g}_n^2 n^{-1}\boldsymbol{d}_n^2)$$

by Burman and Chaudhuri (1992) results 6.14, and 6.16. So that T1

$$= O_P(n^{-.5}\boldsymbol{g}_n\boldsymbol{d}_n)$$

Next observe that

$$\text{T2} = n^{-2}\sum\sum_{i\neq j} W_2(x_i,x_j)\boldsymbol{e}_i\boldsymbol{e}_j + n^{-2}\sum_{i=1}^{n} W_2(x_i,x_i)\boldsymbol{e}_i^2$$

so that E(T2) approaches 0 asymptotically. Note that

$$V(\text{T2}) = O_P(n^{-1}\boldsymbol{g}_n^2)$$

by Whittle's Inequality (Whittle (1960)), Assumptions A3 and A4, and the definition of $\boldsymbol{e}_i$. Then

$$\text{T2} = O_P\ (n^{-.5}\boldsymbol{g}_n)$$

(note also that this result is confirmed by Burman and Chaudhuri (1992) result 6.14).

T3 = $n^{-2}\sum_{i=1}^{n} e_i \sum_{j=1}^{n} W_2(x_i, x_j)(\hat{f}_j - f_j)$

$$= n^{-2}\sum_{i=1}^{n} e_i \sum_{j=1}^{n} W_2(x_i, x_j)(n^{-1}\sum_{k=1}^{n} W_1(x_j, x_k) e_k + O_P(n^{-1}))$$

(by Assumption A1)

$$= n^{-2}\sum_{i=1}^{n} e_i (\sum_{j=1}^{n} W_2(x_i, x_j)(n^{-1}\sum_{k=1}^{n} W_1(x_j, x_k) e_k) + \sum_{j=1}^{n} W_2(x_i, x_j)(O_P(n^{-1})))$$

$$= n^{-3}\sum_{i=1}^{n} e_i (\sum_{j=1}^{n}\sum_{k=1}^{n} W_2(x_i, x_j) W_1(x_j, x_k) e_k) + n^{-2}\sum_{i=1}^{n} e_i \sum_{j=1}^{n} W_2(x_i, x_j)(O_P(n^{-1}))$$

$$= \text{T31} + O_P(n^{-.5} g_n^2) \text{ (say)}$$

by noting the convergence rate of T1. Next rewrite T31 as

$$n^{-2}\sum_{i=1}^{n}\sum_{k=1}^{n} e_k e_i (n^{-1}\sum_{j=1}^{n} W_2(x_i, x_j) W_1(x_j, x_k))$$

and observe that by Whittle's Inequality (Whittle (1960)), V(T31)

$$\leq n^{-4}\sum_{i=1}^{n}\sum_{k=1}^{n} c(n^{-1}\sum_{j=1}^{n} W_2(x_i, x_j) W_1(x_j, x_k))^2$$

(for some real positive constant c)

$$\leq n^{-2} c_2 t_n$$

(for some positive real constant c_2) by assumptions A1 and A4. So that T31 (and T3)

$$= O_P(n^{-.5} g_n)$$

by Burman and Chaudhuri (1992) results 6.14 and 6.16.

Then T1 + T2 - T3 =

$$< \hat{g}, e > = O_P(n^{-.5} g_n)$$

so that

$$\hat{l}* - l* = \frac{O_P(n^{-.5} g_n)}{\|\hat{g}\|^2} \qquad \text{(A4.A.3)}$$

$$= \begin{cases} O_P(n^{-.5} g_n), & if \lim_{n\to\infty} \boldsymbol{d}_n \neq 0 \\ O_P(n^{-.5} g_n^{-1}), & if\ \boldsymbol{d}_n = 0 \end{cases}$$

by Lemma 4.a.1, as desired.//.

Proof of Lemma 4.a.4: We will prove parts a, b and c in order. First note that

$$\|\hat{g}\| = \|\hat{g} + (\boldsymbol{q} - f) - (\boldsymbol{q} - f)\|$$

$$\leq \|\hat{g} - (\boldsymbol{q} - f)\| + \|(\boldsymbol{q} - f)\|$$

$$= O_P(g_n) + \boldsymbol{d}_n$$

by the Triangle Inequality and Assumption A2. Hence, $\|\hat{g}\|$ takes the larger of $O_P(g_n)$ or $O_P(\boldsymbol{d}_n)$

as n approaches ∞. Next observe that $\boldsymbol{l}^* = \frac{<\hat{g}, \boldsymbol{q} - \hat{f}>}{\|\hat{g}\|^2}$

$$= \frac{<\hat{g}, \boldsymbol{q} - f>}{\|\hat{g}\|^2} + \frac{<\hat{g}, f - \hat{f}>}{\|\hat{g}\|^2}$$

$$\leq \frac{\|\boldsymbol{q} - f\|}{\|\hat{g}\|} + \frac{\|f - \hat{f}\|}{\|\hat{g}\|}$$

$$= \frac{\boldsymbol{d}_n}{\|\hat{g}\|} + \frac{O_P(n^{-.5})}{\|\hat{g}\|} \qquad \text{(A4.A.4)}$$

by the Cauchy-Schwartz inequality, Assumption A1, and 4.1. By simply observing the form of

$\boldsymbol{l}^*$ above it is clear from Assumption A2 that $\boldsymbol{l}^*$ approaches 1 at the rate of $O_P(g_n)$. Using

Lemma 4.a.4 (part a), we see that term A4.A.4

$$= \begin{cases} O_P(\boldsymbol{d}_n g_n^{-1}), if\ \frac{n^{-.5}}{g_n} < \frac{\boldsymbol{d}_n}{g_n} < 1 \\ O_P(n^{-.5} g_n^{-1}), if\ \frac{\boldsymbol{d}_n}{g_n} < \frac{n^{-.5}}{g_n} < 1 \end{cases}$$

Finally, combining A4.A.3 with Lemma 4.a.4 (part a) it is clear that $\hat{l}*-l*$ =

$$\begin{cases} O_P(n^{-.5}g_n d_n^{-2}), \text{ if } \frac{d_n}{g_n} > 1 \\ O_P(n^{-.5}g_n^{-1}), \text{ if } \frac{d_n}{g_n} < 1 \end{cases}$$

as desired.//.

Proof of Lemma 4.a.5: Note that Lemmas 4.a.1, 4.a.3 and 4.a.4 together imply that

$\left|\hat{l}*-l*\right|\|\hat{g}\| = O_P(n^{-.5})$ in not only either of the cases in which $d_n = 0, or \lim_{n\to\infty} d_n \neq 0$ but in

any of the situations in which $\lim_{n\to\infty} d_n = 0$ as well.//.

Proof of Theorem 4.A.1: Observe that $\left\| l*\hat{g} + \hat{f} - q \right\|$

$$= \left\| l*\hat{g} - (q - f) + (\hat{f} - f) \right\|$$

$$= \left\| l*(\hat{g} - (q - f)) + (1 - l*)(q - f) + (\hat{f} - f) \right\|$$

$$\leq |l*| \left\| (\hat{g} - (q - f)) \right\| + |1 - l*| \left\| (q - f) \right\| + \left\| (\hat{f} - f) \right\| \qquad \text{(A4.A.5)}$$

by the Triangle inequality. We may easily assert by Assumptions A1 and A2, Lemma 4.a.2 and equation 4.1, that A4.A.5

$$= O_P(1)O_P(g_n) + O_P(g_n)O_P(1) + O_P(n^{-.5})$$

$$= O_P(g_n)$$

if $\lim_{n\to\infty} d_n \neq 0$, and A4.A.5

$$= O_P(n^{-.5}g_n^{-1})O_P(g_n) + 0 + O_P(n^{-.5})$$

$$= O_P(n^{-.5}),$$

if $d_n = 0$, as desired.//.

Proof of Theorem 4.A.2: As in Part 1a, observe that $\left\| \hat{\boldsymbol{l}} * \hat{g} + \hat{f} - \boldsymbol{q} \right\|^2 - \left\| \boldsymbol{l} * \hat{g} + \hat{f} - \boldsymbol{q} \right\|^2$

$$= n^{-1} \sum_{i=1}^{n} ((\hat{\boldsymbol{l}} * \hat{g} + \hat{f} - \boldsymbol{q})^2 - (\boldsymbol{l} * \hat{g} + \hat{f} - \boldsymbol{q})^2)$$

$$= n^{-1} \sum_{i=1}^{n} ((\hat{\boldsymbol{l}} *^2 - \boldsymbol{l} *^2) \hat{g}^2 + 2(\hat{\boldsymbol{l}} * - \boldsymbol{l} *)(\hat{g}\hat{f} - \hat{g}\boldsymbol{q}))$$

$$= n^{-1} \sum_{i=1}^{n} ((\hat{\boldsymbol{l}} * - \boldsymbol{l} *) \hat{g} [(\hat{\boldsymbol{l}} * + \boldsymbol{l} *) \hat{g} + 2(\hat{f} - \boldsymbol{q})]$$

$$= n^{-1} \sum_{i=1}^{n} ((\hat{\boldsymbol{l}} * - \boldsymbol{l} *) \hat{g} [(\hat{\boldsymbol{l}} * \hat{g} + \hat{f} - \boldsymbol{q}) + (\boldsymbol{l} * \hat{g} + \hat{f} - \boldsymbol{q})]$$

$$\leq \left| \hat{\boldsymbol{l}} * - \boldsymbol{l} * \right| \left\| \hat{g} \right\| \left\| \hat{\boldsymbol{l}} * \hat{g} + \hat{f} - \boldsymbol{q} \right\| + \left| \hat{\boldsymbol{l}} * - \boldsymbol{l} * \right| \left\| \hat{g} \right\| \left\| \boldsymbol{l} * \hat{g} + \hat{f} - \boldsymbol{q} \right\| \qquad \text{(A4.A.6)}$$

by the Cauchy-Schwartz inequality. Next rewrite A4.A.6 as

$$T3^2 - T5^2 \leq T4 * T3 + T4 * T5 \text{ (say).}$$

Then

$$T3^2 \leq T4 * T3 + T4 * T5 + T5^2 \qquad \text{(A4.A.7)}$$

Lemma 4.a.5 indicates that $T4 = \left| \hat{\boldsymbol{l}} * - \boldsymbol{l} * \right| \left\| \hat{g} \right\|$ converges at a rate of $O_P(n^{-.5})$, which is at least as fast as $\left\| \boldsymbol{l} * \hat{g} + \hat{f} - \boldsymbol{q} \right\|$ in the two cases considered here. We will show that $T5 = \left\| \boldsymbol{l} * \hat{g} + \hat{f} - \boldsymbol{q} \right\|$ and $T3 = \left\| \hat{\boldsymbol{l}} * \hat{g} + \hat{f} - \boldsymbol{q} \right\|$ converge at exactly the same rate.

First, suppose that $T3$ converges at a slower rate than $T5$. By Lemma 4.a.5, A4.A.7 becomes

$$T3^2 \leq k * T4 * T3 + k * T5^2$$

(where $k \in \aleph$). Then

$$T3 \leq k^{.5} * (T4 + T5)$$

which implies that $T3$ converges at a rate at least as fast as $T5$. This is a contradiction (#).

Next, suppose that $T5$ converges at a slower rate than $T3$. This is a somewhat absurd assumption, nevertheless we will proceed as planned. Notice that we may rewrite A4.A.6 as

$$T5^2 - T3^2 \leq T4 * T3 + T4 * T5$$

so that

$$T5^2 \leq T4 * T3 + T4 * T5 + T3^2$$

and

$$T5 \leq T4^{.5} * (T3^{.5} + T5^{.5}) + T3$$

$$\leq T4^{.5} * (k * T5^{.5}) + T3$$

again for some $k \in \aleph$, by our assumption. Then $T5$ converges at a rate at least as fast as $(T4 * T5)^{.5}$ (which is a #, based on Lemma 4.a.5, assumption A5, and Theorem 4.A.1.a), or $T5$ converges at a rate at least as fast as $T3$ (which is a # of the assumption of this part). #. So $T5 = \left\| \boldsymbol{l} * \hat{g} + \hat{f} - \boldsymbol{q} \right\|$ and $T3 = \left\| \hat{\boldsymbol{l}} * \hat{g} + \hat{f} - \boldsymbol{q} \right\|$ converge at exactly the same rate for the cases $\boldsymbol{d}_n = 0, or \lim_{n \to \infty} \boldsymbol{d}_n \neq 0$, as desired.//.

Proof of Theorem 4.A.3: Recall that A4.A.5 had that $\left\| \boldsymbol{l} * \hat{g} + \hat{f} - \boldsymbol{q} \right\|$

$$\leq |\boldsymbol{l} *| \left\| (\hat{g} - (\boldsymbol{q} - f)) \right\| + |1 - \boldsymbol{l} *| \left\| (\boldsymbol{q} - f) \right\| + \left\| (\hat{f} - f) \right\|.$$

The result follows as a consequence of combining this result with Lemma 4.a.4, as desired.//.

Proof of Theorem 4.A.4: Note that in the proof of Theorem 4.A.2 we proved that asymptotically $\left\| \hat{\boldsymbol{l}} * \hat{g} + \hat{f} - \boldsymbol{q} \right\| = \left\| \boldsymbol{l} * \hat{g} + \hat{f} - \boldsymbol{q} \right\|$. The proof of that theorem was dependent upon the fact that $\left| \hat{\boldsymbol{l}} * - \boldsymbol{l} * \right| \left\| \hat{g} \right\|$ converges at a rate of $O_P(n^{-.5})$; at least as fast as $\left\| \boldsymbol{l} * \hat{g} + \hat{f} - \boldsymbol{q} \right\|$. Observe that the same holds true when $\lim_{n \to \infty} \boldsymbol{d}_n = 0$, by Theorem 4.A.3.

Then Theorem 4.A.4 is proven identically with the exception that the second false supposition will use Lemma 4.a.5, assumption A5 and Theorem 4.A.3.a) for the first #. So, again

$$\left\| \hat{\boldsymbol{l}} * \hat{g} + \hat{f} - \boldsymbol{q} \right\| = \left\| \boldsymbol{l} * \hat{g} + \hat{f} - \boldsymbol{q} \right\|$$

asymptotically when $\lim_{n \to \infty} \boldsymbol{d}_n = 0$, as desired.//.