Appendix 4b Proofs for Part 4b

Proof of Lemma 4.b.1: Note that α can be written as

$$\frac{\sum(\hat{g}^{(i)} - \hat{g})(\hat{g}^{(i)} + \hat{g})}{n}$$

$$\leq \left\|\hat{g}^{(i)} - \hat{g}\right\| \left\|\hat{g}^{(i)} + \hat{g}\right\|$$

$$= \left\|\hat{g}^{(i)} - \hat{g}\right\| \left\|\hat{g} + \hat{g} + \hat{g}^{(i)} - \hat{g}\right\|$$

$$\leq \left\|\hat{g}^{(i)} - \hat{g}\right\| \left(\left\|2\hat{g}\right\| + \left\|\hat{g}^{(i)} - \hat{g}\right\|\right)$$

(by the Cauchy-Schwarz and Triangle Inequalities)

$$= \begin{cases} O_P(g_n{}^2), & if \lim_{n\to\infty} \boldsymbol{d}_n \neq 0 \\ O_P(g_n{}^3), & if\ \boldsymbol{d}_n = 0 \end{cases}$$

by Burman and Chaudhuri (1992) result 6.20 and Lemma 4.a.1, as desired.//.

Proof of Corollary 4.b.1: The proof follows the same reasoning as that of Corollary 3.b.1, except that we let a = $\left\|\hat{g}^{(i)}\right\|$, and b = $\left\|\hat{g}\right\|$. The result follows from Lemma 4.b.1, as desired.//.

Proof of Lemma 4.b.2: Observe that $\dfrac{\sum(\hat{g}^{(i)})(Y_i - \hat{f}^{(i)})}{n(\left\|\hat{g}\right\|^2)}$

$$= \frac{\sum(\hat{g}^{(i)})(\boldsymbol{q}_i + \boldsymbol{e}_i + f(\boldsymbol{b}^{**}) - f(\boldsymbol{b}^{**}) - \hat{f}^{(i)})}{n(\left\|\hat{g}\right\|^2)}$$

$$= \frac{\sum(\hat{g}^{(i)})(\boldsymbol{q}_i - f(\boldsymbol{b}^{**}))}{n(\left\|\hat{g}\right\|^2)} + \frac{\sum(\hat{g}^{(i)})(\boldsymbol{e}_i)}{n(\left\|\hat{g}\right\|^2)} + \frac{\sum(\hat{g}^{(i)})(f(\boldsymbol{b}^{**}) - \hat{f}^{(i)})}{n(\left\|\hat{g}\right\|^2)}$$

= T1 + T2 + T3 (say).

$$\text{T1} = \frac{\sum(\hat{g}^{(i)})(\boldsymbol{q}_i - f(\boldsymbol{b}^{**}))}{n(\left\|\hat{g}\right\|^2)}$$

$$= \frac{\sum (\hat{g}^{(i)} + \hat{g} - \hat{g})(\boldsymbol{q}_i - f(\boldsymbol{b}^{**}))}{n(\|\hat{g}\|^2)}$$

$$\leq \frac{\|\hat{g}^{(i)} - \hat{g}\|\boldsymbol{d}_n + \|\hat{g}\|\boldsymbol{d}_n}{\|\hat{g}\|^2}$$

(by the Cauchy-Schwarz and the Triangle Inequalities, and the definition in section 4a preceding equation 4.1)

$$= \frac{O_P(\boldsymbol{g}_n^2 \boldsymbol{d}_n) + \|\hat{g}\|\boldsymbol{d}_n}{\|\hat{g}\|^2}$$

by Burman and Chaudhuri (1992) result 6.20. Recall that T2

$$= \frac{\sum (\hat{g}^{(i)})(\boldsymbol{e}_i)}{n(\|\hat{g}\|^2)} = \frac{O_P(\boldsymbol{g}_n n^{-.5})}{\|\hat{g}\|^2}$$

(by Burman and Chaudhuri (1992) results 6.14 and 6.16).

$$\text{T3} = \frac{\sum (\hat{g}^{(i)})(f(\boldsymbol{b}^{**}) - \hat{f}^{(i)})}{n(\|\hat{g}\|^2)}$$

$$= \frac{\sum (\hat{g}^{(i)} - \hat{g} + \hat{g})(f(\boldsymbol{b}^{**}) - \hat{f}^{(i)})}{n(\|\hat{g}\|^2)}$$

$$\leq \frac{\|\hat{g}^{(i)} - \hat{g}\| \, \|f(\boldsymbol{b}^{**}) - \hat{f}^{(i)}\| + \|\hat{g}\| \, \|f(\boldsymbol{b}^{**}) - \hat{f}^{(i)}\|}{\|\hat{g}\|^2}$$

(by the Cauchy-Schwarz and Triangle Inequalities)

$$= \frac{\|\hat{g}^{(i)} - \hat{g}\| \, \|f(\boldsymbol{b}^{**}) - \hat{f} + \hat{f} - \hat{f}^{(i)}\| + \|\hat{g}\| \, \|f(\boldsymbol{b}^{**}) - \hat{f} + \hat{f} - \hat{f}^{(i)}\|}{\|\hat{g}\|^2}$$

$$\leq \frac{\|\hat{g}^{(i)} - \hat{g}\| \, \|f(\boldsymbol{b}^{**}) - \hat{f}\| + \|\hat{g}^{(i)} - \hat{g}\| \, \|\hat{f} - \hat{f}^{(i)}\| + \|\hat{g}\| \, \|f(\boldsymbol{b}^{**}) - \hat{f}\| + \|\hat{g}\| \, \|\hat{f} - \hat{f}^{(i)}\|}{\|\hat{g}\|^2}$$

$$= \frac{O_P(\boldsymbol{g}_n^2 n^{-.5}) + O_P(\boldsymbol{g}_n^2 n^{-1}) + \|\hat{g}\| \, O_P(n^{-.5}) + \|\hat{g}\| \, O_P(n^{-1})}{\|\hat{g}\|^2}$$

by Burman and Chaudhuri (1992) results 6.20 and 6.21, and equation 4.1. Then T1 + T2 + T3

$$= \frac{O_P(g_n^2 \boldsymbol{d}_n)}{\|\hat{g}\|^2} + \frac{\boldsymbol{d}_n}{\|\hat{g}\|} + \frac{O_P(g_n^2 n^{-.5})}{\|\hat{g}\|^2} + \frac{O_P(n^{-.5})}{\|\hat{g}\|} \qquad \text{(A4.B.1)}$$

$$= \begin{cases} O_P(1), & \boldsymbol{d}_n \neq 0 \\ O_P(g_n^{-1} n^{-.5}), & \boldsymbol{d}_n = 0 \end{cases}$$

by Lemma 4.A.1, as desired.//.

Proof of Lemma 4.b.3: Recall that $R1$ = $\dfrac{\sum(\hat{g}^{(i)})(Y_i - \hat{f}^{(i)})}{n(\|\hat{g}\|^2)}\left(\dfrac{\boldsymbol{a}}{\|\hat{g}\|^2 + \boldsymbol{a}}\right)$. Then the result follows by Lemmas 4.a.1, 4.b.1, and 4.b.2, as desired.//.

Proof of Lemma 4.b.4: The proof for both parts follows from Lemmas 4.a.1, and 4.b.3, along with equation 4.B.3, as desired.//.

Proof of Lemma 4.b.5: The proof is in stages. The proof of part a) follows from the proof of Lemmas 4.a.4, 4.b.1, combined with Burman and Chaudhuri (1992) result 6.20. The proof of part b) follows from the result of part a), Lemma 4.a.4, equation A4.B.1 and the fact that $R1$ = $\dfrac{\sum(\hat{g}^{(i)})(Y_i - \hat{f}^{(i)})}{n(\|\hat{g}\|^2)}\left(\dfrac{\boldsymbol{a}}{\|\hat{g}\|^2 + \boldsymbol{a}}\right)$. The proof of part c) follows from the result of part b), and Lemma 4.a.4, along with equation 4.B.3, as desired.//.

Proof of Theorem 4.B.2: The proofs for both parts follow from Lemmas 4.a.1 and 4.b.4, and Theorem 4.A.1, along with equation 4.B.4, as desired.//.

Proof of Theorem 4.B.4: The proof follows from Lemmas 4.a.4 and 4.b.5, and Theorem 4.A.3, along with equation 4.B.4, as desired.//.