Appendix 5b Proofs for Part 5b

Proof of Lemma 5.b.1: Observe that

\[
\left\| \hat{f} - \hat{g}_{LL} \right\| = \left\| (\hat{f} - f) + (\Theta - \hat{g}_{LL}) + (f - \Theta) \right\|
\leq \left\| \hat{f} - f \right\| + \left\| \Theta - \hat{g}_{LL} \right\| + \left\| f - \Theta \right\|
= O_p(n^{-5}) + O_p(\tau_n^5 n^{-5}) + \left\| f - \Theta \right\|
= \begin{cases} 
O_p(1), & \text{if } \lim_{n \to \infty} \delta_n \neq 0 \\
O_p(\tau_n^5 n^{-5}), & \text{if } \delta_n = 0
\end{cases}
\]

as desired.//.

Proof of Lemma 5.b.2: Note that

\[
|1 - \lambda^{*T}| = \left| \frac{\langle \hat{f} - \hat{g}_{LL}, \hat{f} - \hat{g}_{LL} - (\Theta - \hat{g}_{LL}) \rangle}{\left\| \hat{f} - \hat{g}_{LL} \right\|^2} \right|
= \left| \frac{\langle \hat{f} - \hat{g}_{LL}, \hat{f} - \Theta \rangle}{\left\| \hat{f} - \hat{g}_{LL} \right\|^2} \right|
\]

Then using the Burman and Chaudhuri (1992) proof of Lemma 5.2, and replacing \( \gamma_i \) by \( \tau_i^5 n^{-5} \),
the result is proved.//.

Proof of Lemma 5.b.3: Note that

\[
(\lambda^{*T} - \lambda^{*T}) = \frac{\langle \hat{f} - \hat{g}_{LL}, Y - \hat{g}_{LL} \rangle - \langle \hat{f} - \hat{g}_{LL}, \Theta - \hat{g}_{LL} \rangle}{\left\| \hat{f} - \hat{g}_{LL} \right\|^2}
= \frac{\langle (\hat{f} - \hat{g}_{LL}), (Y - \Theta) \rangle}{\left\| \hat{f} - \hat{g}_{LL} \right\|^2}
= \sum_{i=1}^{n} \frac{(\hat{f}_i - \hat{g}_{LL})\epsilon_i}{n\left\| \hat{f} - \hat{g}_{LL} \right\|^2}.
\]
Next observe that

\[
\frac{\sum_{i=1}^{n} (\hat{f}_i - \hat{g}_{\text{LLI}}) \epsilon_i}{n} = \frac{\sum_{i=1}^{n} (\hat{f}_i - f_i) \epsilon_i}{n} + \frac{\sum_{i=1}^{n} (f_i - \Theta) \epsilon_i}{n} + \frac{\sum_{i=1}^{n} (\Theta - \hat{g}_{\text{LLI}}) \epsilon_i}{n} = T_1 + T_2 + T_3 \ (\text{say}).
\]

(A5.B.1)

We will give asymptotic results for each of the terms T2, T3 and T1 (in that order).

Observe that \( E(T2) = 0 \), and that \( E(T2^2) = V(T2^2) \)

\[
= n^{-2} E\left( \left( \sum_{i=1}^{n} (f_i - \Theta) \epsilon_i \right)^2 \right)
\leq cn^{-2} \left( \sum_{i=1}^{n} (f_i - \Theta)^2 \right)
\]

(for some constant \( c \in \mathbb{R} \), by Whittle’s Inequality (Whittle (1960)))

\[
= \delta_x^2 O_p(n^{-1}).
\]

So that

\[
T_2 = \delta_x O_p(n^{-5}).
\]

Recall that T3 =

\[
\frac{\sum_{i=1}^{n} (\Theta - \hat{g}_{\text{LLI}}) \epsilon_i}{n}
\leq \frac{\sum_{i=1}^{n} \left| (\Theta - \hat{g}_{\text{LLI}}) \right| \epsilon_i}{n}
\leq \frac{\sum_{i=1}^{n} c(\tau_n^5 n^{-5}) \epsilon_i}{n}
\]

(A5.B.2)

for some constant \( c \in \mathbb{R}^+ \), with probability approaching 1 (by choice of \( c \)), by the proof of Lemma 5.a.1, and the definition of convergence in distribution.
Next notice that the terms in A5.B.2 are stochastically independent (since the bandwidth
is a function of \( n \) which is independent of \( \varepsilon_i \) for all \( i \)) so that we may employ Whittle’s Inequality
(Whittle (1960)). Since \( E(T3) = \)
\[
O_p(\tau_n^{-5}n^{-5}),
\]
we also have that asymptotically \( V(T3) = E(T3^2) \)
\[
\leq E \left( \left( \frac{\sum_{i=1}^{n} c(\tau_n^{5}n^{-5})|\varepsilon_i|}{n} \right)^2 \right)
\]
(for some constant \( c \in \mathbb{R}^+ \), with probability approaching 1 (by choice of \( c \)), by inequality A5.B.2)
\[
= c^2 (\tau_n n^{-1}) E \left( \left( \frac{\sum_{i=1}^{n} |\varepsilon_i|}{n} \right)^2 \right)
\]
\[
\leq c_2 (\tau_n n^{-2})
\]
(for some constant \( c_2 \in \mathbb{R}^+ \), by Whittle’s Inequality (Whittle (1960)). Then by definition of
convergence in distribution, it follows that
\[
T3 = O_p(\tau_n^{-5} n^{-1}) .
\]
Finally, observe that by assumption A1, \( T1 = \)
\[
\frac{\sum_{i=1}^{n} ((\hat{f}_i - f_i) \varepsilon_i)}{n}
\]
\[
= \frac{\sum_{i=1}^{n} (n^{-1} \sum_{j=1}^{n} W_i(x_i, x_j) \varepsilon_j) + O_p(n^{-1}) \varepsilon_i}{n}
\]
\[
= \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} W_i(x_i, x_j) \varepsilon_i \varepsilon_j}{n^2} + O_p(n^{-1.5})
\]
\[
= T11 + O_p(n^{-1.5}) \text{ (say)}.
\]
We also have that \( E(T_{11}) \)
\[
\leq \sum_{i=1}^{n} \frac{25W_i(x_i, x_i)}{n^2}
= O_p(n^{-1}).
\]
Then, as before, asymptotically \( V(T_{11}) = E(T_{11}^2) \)
\[
\leq \frac{c \sum_{i=1}^{n} \sum_{j=1}^{n} W_i^2(x_i, x_j)}{n^2}
\leq \frac{c_2}{n^3}
\]
(for some constant \( c \in \mathbb{R} \), by Whittle’s Inequality (Whittle (1960))
\[
= O_p(n^{-3}).
\]
(for some constant \( c_2 \in \mathbb{R} \), by condition A1)
\[
= O_p(n^{-3}).
\]
So that
\[
T_{11} = O_p(n^{-1.5}),
\]
and consequently
\[
T_1 = O_p(n^{-1.5}).
\]
Thus A5.B.1 becomes
\[
O_p(n^{-1.5}) + \delta_n O_p(n^{-5}) + O_p(\tau_n^{-5} n^{-1}). \quad \text{(A5.B.3)}
\]
Finally, combining A5.B.3 with Lemma 5.b.1 we have \( \hat{\Lambda}^{*L} - \lambda^{*L} \)
\[
= \begin{cases} 
O_p(\tau_n^5 n^{-1}) + O_p(n^{-5}), & \text{if } \lim_{n \to \infty} \delta_n \neq 0 \\
O_p(\tau_n^{-5}), & \text{if } \delta_n = 0
\end{cases}
\]
as desired. //.
Proof of Theorem 5.B.1: Observe that 

\[
\| \lambda^* L \hat{f} + (1 - \lambda^* L) \hat{g}_{LL} - \Theta \|
\]

\[
= \| \lambda^* L \hat{f} - \lambda^* L \Theta + (1 - \lambda^* L) \hat{g}_{LL} - (1 - \lambda^* L) \Theta \|
\]

\[
= \| \lambda^* L (\hat{f} - \Theta) + (1 - \lambda^* L) (\hat{g}_{LL} - \Theta) \|
\]

\[
\leq \| \lambda^* L (\hat{f} - \Theta) \| + \| (1 - \lambda^* L) (\hat{g}_{LL} - \Theta) \|
\]

(by the Triangle Inequality)

\[
\leq |\lambda^* L| \| (\hat{f} - f) \| + \| (f - \Theta) \| + |(1 - \lambda^* L) \| \| (\hat{g}_{LL} - \Theta) \|
\]

(by the Cauchy-Schwarz Inequality)

\[
\leq |\lambda^* L| \| (\hat{f} - f) \| + \| (f - \Theta) \| + |1 - \lambda^* L| \| (\hat{g}_{LL} - \Theta) \|
\]

(by the Triangle Inequality)

\[
= |\lambda^* L| \left( O_p(n^{-5}) + \delta_n \right) + |1 - \lambda^* L| \left( O_p(\tau_n^5 n^{-5}) \right)
\]

(A5.B.4)

by Lemmas 5.a.1, 5.a.3 and by definition of $\delta_n$. Then using Lemma 5.b.2 we have that A5.B.4

\[
= \begin{cases} 
O_p(\tau_n^5 n^{-5}), & \text{if } \lim_{n \to \infty} \delta_n \neq 0 \\
O_p(n^{-5}), & \text{if } \delta_n = 0 
\end{cases}
\]

and the theorem is proved. //.

Proof of Theorem 5.B.2: Observe that

\[
\| \lambda^* L \hat{f} + (1 - \lambda^* L) \hat{g}_{LL} - \Theta \|^2 - \| \lambda^* L \hat{f} + (1 - \lambda^* L) \hat{g}_{LL} - \Theta \|^2
\]

\[
= \sum_{i=1}^n (\lambda^* L \hat{f} + (1 - \lambda^* L) \hat{g}_{LL} - \Theta)^2 - \sum_{i=1}^n (\lambda^* L \hat{f} + (1 - \lambda^* L) \hat{g}_{LL} - \Theta)^2
\]

\[
= \frac{\sum_{i=1}^n (t_i^2 - \Theta)}{n} - \frac{\sum_{i=1}^n (t_i^2 - \Theta)}{n} \text{ (say)}
\]

\[
= \frac{\sum_{i=1}^n ((t_i^2 - t_i^2) - 2\Theta(t_i^2 - t_i))}{n}
\]

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\[
\sum_{i=1}^{n} \frac{((t_1 - t_2)(t_1 + t_2 - 2\Theta))}{n}
\]

\[
= \sum_{i=1}^{n} \frac{((t_1 - t_2)(t_1 - t_2 + 2t_2 - 2\Theta))}{n}
\]

\[
= \sum_{i=1}^{n}(t_1 - t_2)^2 \frac{n}{n} + \sum_{i=1}^{n}(t_1 - t_2)2(t_2 - \Theta)
\]

\[
= \sum_{i=1}^{n} \frac{(\hat{\lambda}^L - \lambda^*)^2}{n} (\hat{f} - \hat{g}_{LL})^2 + 2\sum_{i=1}^{n} (\hat{\lambda}^L - \lambda^*)(\hat{f} - \hat{g})(\lambda^*) \left( \hat{f} + (1 - \lambda^*)\hat{g}_{LL} - \Theta \right)
\]

\[
\leq (\hat{\lambda}^L - \lambda^*)^2 \|\hat{f} - \hat{g}_{LL}\|^2 + 2|\hat{\lambda}^L - \lambda^*| \|\hat{f} - \hat{g}_{LL}\|\|\lambda^* \hat{f} + (1 - \lambda^*)\hat{g}_{LL} - \Theta\|
\]

by the Cauchy-Schwarz Inequality. Then

\[
\left\|\hat{\lambda}^L \hat{f} + (1 - \hat{\lambda}^L)\hat{g}_{LL} - \Theta\right\|^2
\]

\[
\leq (\hat{\lambda}^L - \lambda^*)^2 \|\hat{f} - \hat{g}_{LL}\|^2 + 2|\hat{\lambda}^L - \lambda^*| \|\hat{f} - \hat{g}_{LL}\|\|\lambda^* \hat{f} + (1 - \lambda^*)\hat{g}_{LL} - \Theta\| + \left\|\lambda^* \hat{f} + (1 - \lambda^*)\hat{g}_{LL} - \Theta\right\|^2
\]

so that

\[
\left\|\hat{\lambda}^L \hat{f} + (1 - \hat{\lambda}^L)\hat{g}_{LL} - \Theta\right\|
\]

\[
\leq (\hat{\lambda}^L - \lambda^*) \|\hat{f} - \hat{g}_{LL}\| + (2|\hat{\lambda}^L - \lambda^*| \|\hat{f} - \hat{g}_{LL}\|\|\lambda^* \hat{f} + (1 - \lambda^*)\hat{g}_{LL} - \Theta\|)^5
\]

\[
+ \left\|\lambda^* \hat{f} + (1 - \lambda^*)\hat{g}_{LL} - \Theta\right\|.
\]

(A5.B.5)

Using Lemmas 5.b.1, 5.b.3 and Theorem 5.B.1 along with A5.B.5, we have that

\[
\left\|\hat{\lambda}^L \hat{f} + (1 - \hat{\lambda}^L)\hat{g}_{LL} - \Theta\right\| = \begin{cases} O_p(\tau^5 n^{-5}), & \text{if } \lim_{n \to \infty} \delta_n \neq 0 \\ O_p(n^{-5}), & \text{if } \delta_n = 0 \end{cases}
\]

as desired.//.