Appendix 5b Proofs for Part 5b

Proof of Lemma 5.b.1: Observe that

$$\left\| \hat{f} - \hat{g}_{LL} \right\| = \left\| (\hat{f} - f) + (\boldsymbol{q} - \hat{g}_{LL}) + (f - \boldsymbol{q}) \right\|$$

$$\leq \left\| \hat{f} - f \right\| + \left\| \boldsymbol{q} - \hat{g}_{LL} \right\| + \left\| f - \boldsymbol{q} \right\|$$

$$= O_P(n^{-.5}) + O_P(\boldsymbol{t}_n^{.5} n^{-.5}) + \left\| f - \boldsymbol{q} \right\|$$

$$= \begin{cases} O_P(1), & if \ \lim_{n\to\infty} \boldsymbol{d}_n \neq 0 \\ O_P(\boldsymbol{t}_n^{.5} n^{-.5}), & if \ \boldsymbol{d}_n = 0 \end{cases}$$

as desired.//.

Proof of Lemma 5.b.2: Note that

$$\left| 1 - \boldsymbol{l} *^L \right| = \left| \frac{< \hat{f} - \hat{g}_{LL}, \hat{f} - \hat{g}_{LL} - (\boldsymbol{q} - \hat{g}_{LL}) >}{\left\| \hat{f} - \hat{g}_{LL} \right\|^2} \right|$$

$$= \left| \frac{< \hat{f} - \hat{g}_{LL}, \hat{f} - \boldsymbol{q} >}{\left\| \hat{f} - \hat{g}_{LL} \right\|^2} \right|$$

Then using the Burman and Chaudhuri (1992) proof of Lemma 5.2, and replacing $\boldsymbol{g}_n$ by $\boldsymbol{t}_n^{.5} n^{-.5}$, the result is proved.//.

Proof of Lemma 5.b.3: Note that

$$(\hat{\boldsymbol{l}} *^L - \boldsymbol{l} *^L) = \frac{< \hat{f} - \hat{g}_{LL}, \boldsymbol{Y} - \hat{g}_{LL} > - < \hat{f} - \hat{g}_{LL}, \mathbf{q} - \hat{g}_{LL} >}{\left\| \hat{f} - \hat{g}_{LL} \right\|^2}$$

$$= \frac{< (\hat{f} - \hat{g}_{LL}), (\boldsymbol{Y} - \mathbf{q}) >}{\left\| \hat{f} - \hat{g}_{LL} \right\|^2}$$

$$= \frac{\sum_{i=1}^n (\hat{f}_i - \hat{g}_{LLi}) \boldsymbol{e}_i}{n \left\| \hat{f} - \hat{g}_{LL} \right\|^2}.$$

Next observe that

$$\frac{\sum_{i=1}^{n}(\hat{f}_i - \hat{g}_{LLi})e_i}{n}$$

$$= \frac{\sum_{i=1}^{n}(\hat{f}_i - f_i)e_i}{n} + \frac{\sum_{i=1}^{n}(f_i - q_i)e_i}{n} + \frac{\sum_{i=1}^{n}(q_i - \hat{g}_{LLi})e_i}{n} \qquad \text{(A5.B.1)}$$

= T1 + T2 + T3 (say).

We will give asymptotic results for each of the terms T2, T3 and T1 (in that order).

Observe that E(T2) = 0, and that $E(\mathrm{T2}^2) = V(\mathrm{T2}^2)$

$$= n^{-2}E((\sum_{i=1}^{n}(f_i - q_i)e_i)^2)$$

$$\leq cn^{-2}(\sum_{i=1}^{n}(f_i - q_i)^2)$$

(for some constant $c \in \Re$, by Whittle's Inequality (Whittle (1960)))

$$= d_n^2\, O_P(n^{-1}).$$

So that

$$\mathrm{T2} = d_n\, O_P(n^{-.5}).$$

Recall that T3 =

$$\frac{\sum_{i=1}^{n}(q_i - \hat{g}_{LLi})e_i}{n}$$

$$\leq \frac{\sum_{i=1}^{n}\left|(q_i - \hat{g}_{LLi})\right|\left|e_i\right|}{n}$$

$$\leq \frac{\sum_{i=1}^{n}c(t_n^{.5}n^{-.5})\left|e_i\right|}{n} \qquad \text{(A5.B.2)}$$

for some constant $c \in \Re^+$, with probability approaching 1 (by choice of c), by the proof of Lemma 5.a.1, and the definition of convergence in distribution.

Next notice that the terms in A5.B.2 are stochastically independent (since the bandwidth is a function of n which is independent of $\boldsymbol{e}_i$ for all i) so that we may employ Whittle's Inequality (Whittle (1960)). Since E(T3) =

$$O_P(\boldsymbol{t}_n^{.5} n^{-.5}),$$

we also have that asymptotically V(T3) = E(T3^2)

$$\leq E\left(\left(\frac{\sum_{i=1}^{n} c(\boldsymbol{t}_n^{.5} n^{-.5})\left|\boldsymbol{e}_i\right|}{n}\right)^2\right)$$

(for some constant $c \in \Re^+$, with probability approaching 1 (by choice of c), by inequality A5.B.2)

$$= c^2(\boldsymbol{t}_n n^{-1})E\left(\left(\frac{\sum_{i=1}^{n}\left|\boldsymbol{e}_i\right|}{n}\right)^2\right)$$

$$\leq c_2(\boldsymbol{t}_n n^{-2})$$

(for some constant $c_2 \in \Re^+$, by Whittle's Inequality (Whittle (1960)). Then by definition of convergence in distribution, it follows that

$$\text{T3} = O_P(\boldsymbol{t}_n^{.5} n^{-1}).$$

Finally, observe that by assumption A1, T1 = $\dfrac{\sum_{i=1}^{n}(\hat{f}_i - f_i)\boldsymbol{e}_i}{n}$

$$= \frac{\sum_{i=1}^{n}((n^{-1}\sum_{j=1}^{n} W_1(x_i, x_j)\boldsymbol{e}_j) + O_P(n^{-1}))\boldsymbol{e}_i}{n}$$

$$= \frac{\sum_{i=1}^{n}\sum_{j=1}^{n} W_1(x_i, x_j)\boldsymbol{e}_i\boldsymbol{e}_j}{n^2} + O_P(n^{-1.5})$$

$$= \text{T11} + O_P(n^{-1.5}) \text{ (say).}$$

We also have that E(T11)

$$\leq \frac{\sum_{i=1}^{n} .25 W_1(x_i, x_i)}{n^2}$$

$$= O_P(n^{-1}).$$

Then, as before, asymptotically V(T11) = E(T11^2)

$$\leq \frac{c}{n^2} \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} W_1^2(x_i, x_j)}{n^2}$$

(for some constant $c \in \Re$, by Whittle's Inequality (Whittle (1960))

$$\leq \frac{c_2}{n^3}$$

(for some constant $c_2 \in \Re$, by condition A1)

$$= O_P(\ n^{-3}).$$

So that

$$\text{T11} = O_P(n^{-1.5}).$$

and consequently

$$\text{T1} = O_P(n^{-1.5}).$$

Thus A5.B.1 becomes

$$O_P(n^{-1.5}) + \boldsymbol{d}_n O_P(n^{-.5}) + O_P(\boldsymbol{t}_n^{.5}\ n^{-1}). \qquad \text{(A5.B.3)}$$

Finally, combining A5.B.3 with Lemma 5.b.1 we have $(\hat{\boldsymbol{l}}*^L - \boldsymbol{l}*^L)$

$$= \begin{cases} O_P(\boldsymbol{t}_n^{.5} n^{-1}) + O_P(n^{-.5}), & if \ \lim_{n\to\infty} \boldsymbol{d}_n \neq 0 \\ O_P(\boldsymbol{t}_n^{-.5}), & if \ \boldsymbol{d}_n = 0 \end{cases}$$

as desired.//.

Proof of Theorem 5.B.1: Observe that $\left\| \lambda^{*L} \hat{f} + (1-\lambda^{*L})\hat{g}_{LL} - \theta \right\|$

$$= \left\| \lambda^{*L} \hat{f} - \lambda^{*L}\theta + (1-\lambda^{*L})\hat{g}_{LL} - (1-\lambda^{*L})\theta \right\|$$

$$= \left\| \lambda^{*L} (\hat{f} - \theta) + (1-\lambda^{*L})(\hat{g}_{LL} - \theta) \right\|$$

$$\leq \left\| \lambda^{*L} (\hat{f} - \theta) \right\| + \left\| (1-\lambda^{*L})(\hat{g}_{LL} - \theta) \right\|$$

(by the Triangle Inequality)

$$\leq \left| \lambda^{*L} \right| \left\| (\hat{f} - \theta) \right\| + \left| (1-\lambda^{*L}) \right| \left\| (\hat{g}_{LL} - \theta) \right\|$$

(by the Cauchy-Schwarz Inequality)

$$\leq \left| \lambda^{*L} \right| \left(\left\| (\hat{f} - f) \right\| + \left\| (f - \theta) \right\| \right) + \left| (1-\lambda^{*L}) \right| \left\| (\hat{g}_{LL} - \theta) \right\|$$

(by the Triangle Inequality)

$$= \left| \lambda^{*L} \right| (O_P(n^{-.5}) + \delta_n) + \left| (1-\lambda^{*L}) \right| O_P(\tau_n^{.5} n^{-.5}) \qquad \text{(A5.B.4)}$$

by Lemmas 5.a.1, 5.a.3 and by definition of δ_n. Then using Lemma 5.b.2 we have that A5.B.4

$$= \begin{cases} O_P(\tau_n^{.5} n^{-.5}), & \text{if } \lim_{n\to\infty} \delta_n \neq 0 \\ O_P(n^{-.5}), & \text{if } \delta_n = 0 \end{cases}$$

and the theorem is proved.//.

Proof of Theorem 5.B.2: Observe that

$$\left\| \hat{\lambda}^{*L} \hat{f} + (1-\hat{\lambda}^{*L})\hat{g}_{LL} - \theta \right\|^2 - \left\| \lambda^{*L} \hat{f} + (1-\lambda^{*L})\hat{g}_{LL} - \theta \right\|^2$$

$$= \frac{\sum_{i=1}^{n} (\hat{\lambda}^{*L} \hat{f} + (1-\hat{\lambda}^{*L})\hat{g}_{LL} - \theta)^2}{n} - \frac{\sum_{i=1}^{n} (\lambda^{*L} \hat{f} + (1-\lambda^{*L})\hat{g}_{LL} - \theta)^2}{n}$$

$$= \frac{\sum_{i=1}^{n} (t_1 - \theta)^2}{n} - \frac{\sum_{i=1}^{n} (t_2 - \theta)^2}{n} \text{ (say)}$$

$$= \frac{\sum_{i=1}^{n} ((t_1^2 - t_2^2) - 2\theta(t_1 - t_2))}{n}$$

$$= \frac{\sum_{i=1}^{n}((t_1 - t_2)(t_1 + t_2 - 2\boldsymbol{q}))}{n}$$

$$= \frac{\sum_{i=1}^{n}((t_1 - t_2)(t_1 - t_2 + 2t_2 - 2\boldsymbol{q}))}{n}$$

$$= \frac{\sum_{i=1}^{n}(t_1 - t_2)^2}{n} + \frac{\sum_{i=1}^{n}(t_1 - t_2)2(t_2 - \boldsymbol{q})}{n}$$

$$= \frac{\sum_{i=1}^{n}(\hat{\boldsymbol{l}}*^L - \boldsymbol{l}*^L)^2(\hat{f} - \hat{g}_{LL})^2}{n} + \frac{2\sum_{i=1}^{n}(\hat{\boldsymbol{l}}*^L - \boldsymbol{l}*^L)(\hat{f} - \hat{g})(\boldsymbol{l}*^L\ \hat{f} + (1 - \boldsymbol{l}*^L)\hat{g}_{Ll} - \boldsymbol{q})}{n}$$

$$\leq (\hat{\boldsymbol{l}}*^L - \boldsymbol{l}*^L)^2\left\|\hat{f} - \hat{g}_{LL}\right\|^2 + 2\left|(\hat{\boldsymbol{l}}*^L - \boldsymbol{l}*^L)\right|\left\|\hat{f} - \hat{g}_{LL}\right\|\left\|(\boldsymbol{l}*^L\ \hat{f} + (1 - \boldsymbol{l}*^L)\hat{g}_{LL} - \boldsymbol{q})\right\|.$$

by the Cauchy-Schwarz Inequality. Then

$$\left\|\hat{\boldsymbol{l}}*^L\ \hat{f} + (1 - \hat{\boldsymbol{l}}*^L)\hat{g}_{LL} - \boldsymbol{q}\right\|^2$$

$$\leq (\hat{\boldsymbol{l}}*^L - \boldsymbol{l}*^L)^2\left\|\hat{f} - \hat{g}_{LL}\right\|^2 + 2\left|\hat{\boldsymbol{l}}*^L - \boldsymbol{l}*^L\right|\left\|\hat{f} - \hat{g}_{LL}\right\|\left\|\boldsymbol{l}*^L\ \hat{f} + (1 - \boldsymbol{l}*^L)\hat{g}_{LL} - \boldsymbol{q}\right\| + \left\|\boldsymbol{l}*^L\ \hat{f} + (1 - \boldsymbol{l}*^L)\hat{g}_{LL} - \boldsymbol{q}\right\|^2$$

so that

$$\left\|\hat{\boldsymbol{l}}*^L\ \hat{f} + (1 - \hat{\boldsymbol{l}}*^L)\hat{g}_{LL} - \boldsymbol{q}\right\|$$

$$\leq (\hat{\boldsymbol{l}}*^L - \boldsymbol{l}*^L)\left\|\hat{f} - \hat{g}_{LL}\right\| + (2\left|\hat{\boldsymbol{l}}*^L - \boldsymbol{l}*^L\right|\left\|\hat{f} - \hat{g}_{LL}\right\|\left\|\boldsymbol{l}*^L\ \hat{f} + (1 - \boldsymbol{l}*^L)\hat{g}_{LL} - \boldsymbol{q}\right\|)^{.5}$$

$$+ \left\|\boldsymbol{l}*^L\ \hat{f} + (1 - \boldsymbol{l}*^L)\hat{g}_{LL} - \boldsymbol{q}\right\|. \qquad \text{(A5.B.5)}$$

Using Lemmas 5.b.1, 5.b.3 and Theorem 5.B.1 along with A5.B.5, we have that

$$\left\|\hat{\boldsymbol{l}}*^L\ \hat{f} + (1 - \hat{\boldsymbol{l}}*^L)\hat{g}_{LL} - \boldsymbol{q}\right\| = \begin{cases} O_P(\boldsymbol{t}_n^{.5} n^{-.5}), & if \lim_{n\to\infty} \boldsymbol{d}_n \neq 0 \\ O_P(n^{-.5}), & if\ \boldsymbol{d}_n = 0 \end{cases}$$

as desired.//.