Appendix B

Derivation of Basic Equations

B.1 Continuity Equation

Following White [36], inlet and outlet mass fluxes for a two-phase control volume can be expressed as in Fig. B.1. For a steady state analysis mass in must equal mass out. Therefore, mass out subtracted from mass in yields:

$$\frac{\partial}{\partial x}(\alpha\rho u)dxdydz + \frac{\partial}{\partial y}(\alpha\rho v)dxdydz + \frac{\partial}{\partial z}(\alpha\rho w)dxdydz = \Gamma dxdydz \qquad (B.1)$$

Dividing out the *dxdydz* volume term, and converting from cartesian to cylindrical coordinates yields,

$$\frac{1}{r}\frac{\partial}{\partial r}(r\alpha\rho v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\alpha\rho v_\theta) + \frac{\partial}{\partial z}(\alpha\rho v_z) = \Gamma \qquad (B.2)$$

For axisymmetric coordinates, we eliminate the θ component and slightly change nomenclature by defining $x=z$, $u=v_z$, and $u=v_r$:

$$\frac{1}{r}\frac{\partial}{\partial r}(r\alpha\rho v) + \frac{\partial}{\partial x}(\alpha\rho u) = \Gamma \qquad (B.3)$$

Multiplying by r gives the final equation for use in non-dimensionalization:

$$\frac{\partial}{\partial r}(r\alpha\rho v) + \frac{\partial}{\partial x}(r\alpha\rho u) = r\Gamma \qquad (B.4)$$

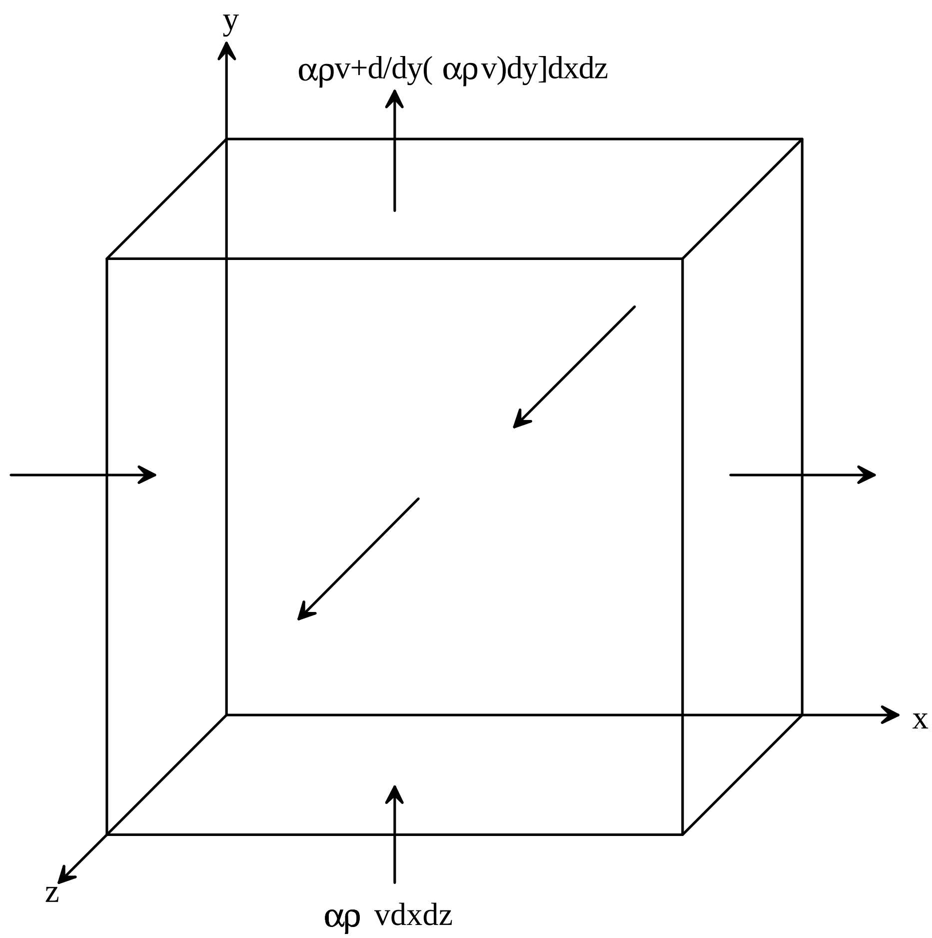

Figure B.1 - Control Volume for Derivation of Basic Equations

B.2 Momentum Equation

Again, following the procedure outlined in White [36], we write the basic momentum expression $\sum F = M_{out} - M_{in}$ for an infinitesimal control volume with two phases present. The sum of the forces consists of surface stresses, pressure, and interfacial drag (gravity and other body forces are being ignored). The momentum flux terms consist of momentum carried into and out of the control volume, as in single phase flow, but also an interfacial momentum transfer due to mass transfer. Assembling these terms in cartesian coordinates yields:

$$
\begin{aligned}
&\hat{i}\left[\frac{\partial}{\partial x}(\alpha\sigma_{xx}) + \frac{\partial}{\partial y}(\alpha\sigma_{yx}) + \frac{\partial}{\partial z}(\alpha\sigma_{zx})\right]dxdydz \\
&+\hat{j}\left[\frac{\partial}{\partial x}(\alpha\sigma_{xy}) + \frac{\partial}{\partial y}(\alpha\sigma_{yy}) + \frac{\partial}{\partial z}(\alpha\sigma_{zy})\right]dxdydz \\
&+\hat{k}\left[\frac{\partial}{\partial x}(\alpha\sigma_{xz}) + \frac{\partial}{\partial y}(\alpha\sigma_{yz}) + \frac{\partial}{\partial z}(\alpha\sigma_{zz})\right]dxdydz \\
&+ P\nabla\alpha dxdydz + \vec{F}_{int}dxdydz \\
&= \left[\frac{\partial}{\partial x}\left(\alpha\rho u\vec{V}\right) + \frac{\partial}{\partial y}\left(\alpha\rho v\vec{V}\right) + \frac{\partial}{\partial z}\left(\alpha\rho w\vec{V}\right)\right]dxdydz \\
&+ \left\{MAX[\Gamma, 0]\vec{V}_{other} - MAX[-\Gamma, 0]\vec{V}\right\}dxdydz \qquad (B.5)
\end{aligned}
$$

If the surface forces shown above are broken up into viscous and pressure forces, as follows,

$$
\begin{aligned}
&\sigma_{xx} = \tau_{xx} - P \qquad \sigma_{xy} = \tau_{xy} \qquad \sigma_{xz} = \tau_{xz} \\
&\sigma_{yx} = \tau_{yx} \qquad \sigma_{yy} = \tau_{yy} - P \qquad \sigma_{yz} = \tau_{yz} \\
&\sigma_{zx} = \tau_{zx} \qquad \sigma_{zy} = \tau_{zy} \qquad \sigma_{zz} = \tau_{zz} - P \qquad (B.6)
\end{aligned}
$$

then the momentum equation can be expressed as:

$$
\begin{aligned}
&\hat{i}\left[\frac{\partial}{\partial x}(\alpha\sigma_{xx})+\frac{\partial}{\partial y}(\alpha\sigma_{yx})+\frac{\partial}{\partial z}(\alpha\sigma_{zx})\right]\\
&+\hat{j}\left[\frac{\partial}{\partial x}(\alpha\sigma_{xy})+\frac{\partial}{\partial y}(\alpha\sigma_{yy})+\frac{\partial}{\partial z}(\alpha\sigma_{zy})\right]\\
&+\hat{k}\left[\frac{\partial}{\partial x}(\alpha\sigma_{xz})+\frac{\partial}{\partial y}(\alpha\sigma_{yz})+\frac{\partial}{\partial z}(\alpha\sigma_{zz})\right]\\
&-\hat{i}\,\frac{\partial}{\partial x}(\alpha P)-\hat{j}\,\frac{\partial}{\partial y}(\alpha P)-\hat{k}\,\frac{\partial}{\partial z}(\alpha P)+P\nabla\alpha+F_{int}\\
&=\nabla\bullet\left(\alpha\rho\vec{V}\vec{V}\right)-\left\{MAX[\Gamma,0]\vec{V}_{other}-MAX[-\Gamma,0]\vec{V}\right\} \qquad (B.7)
\end{aligned}
$$

Simplifying the above expression to the general tensor form yields:

$$
\nabla\bullet\left(\alpha\rho\vec{V}\vec{V}\right)=-\alpha\nabla P+\nabla\bullet\left(\alpha\tau_{ij}\right)+F_{int}+\left\{MAX[\Gamma,0]\vec{V}_{other}-MAX[-\Gamma,0]\vec{V}\right\} \qquad (B.8)
$$

Now this equation can be expanded into any coordinate system. For the z, or x, direction in cylindrical coordinates it becomes:

$$
\begin{aligned}
&\frac{1}{r}\frac{\partial}{\partial r}(r\alpha\rho v_r v_z)+\frac{1}{r}\frac{\partial}{\partial\theta}(\alpha\rho v_\theta v_z)+\frac{\partial}{\partial z}\left(\alpha\rho v_z^2\right)\\
&=-\alpha\frac{\partial P}{\partial z}+\frac{1}{r}\frac{\partial}{\partial r}(\alpha r\tau_{rz})+\frac{1}{r}\frac{\partial}{\partial\theta}(\alpha\tau_{\theta z})+\frac{\partial}{\partial z}(\alpha\tau_{zz})\\
&+F_{int,z}+MAX[\Gamma,0]v_{z,other}-MAX[-\Gamma,0]v_z \qquad (B.9)
\end{aligned}
$$

Using the Stokes deformation law, the shear stresses can be expressed as functions of viscosity and velocity gradient. Note that in the following equations the term containing the second coefficient of viscosity does not appear, as it is neglected:

$$
\begin{aligned}
&\tau_{rr}=2\mu\frac{\partial v_r}{\partial r} \qquad \tau_{\theta\theta}=2\mu\left(\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{v_r}{r}\right) \qquad \tau_{zz}=2\mu\frac{\partial v_z}{\partial z}\\
&\tau_{r\theta}=\tau_{\theta r}=\mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)+\frac{1}{r}\frac{\partial v_r}{\partial\theta}\right] \qquad \tau_{\theta z}=\tau_{z\theta}=\mu\left(\frac{\partial v_\theta}{\partial z}+\frac{1}{r}\frac{\partial v_z}{\partial\theta}\right)\\
&\tau_{zr}=\tau_{rz}=\mu\left(\frac{\partial v_z}{\partial r}+\frac{\partial v_r}{\partial z}\right) \qquad (B.10)
\end{aligned}
$$

Making this substitution for the z-direction and neglecting θ component terms yields the following expression: