$$\frac{1}{r}\frac{\partial}{\partial r}(r\alpha\rho v_r v_z) + \frac{\partial}{\partial z}\left(\alpha\rho v_z^2\right) = -\alpha\frac{\partial P}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}\left[\alpha r\mu\left(\frac{\partial v_z}{\partial r} + \overbrace{\frac{\partial v_r}{\partial z}}^{negl.}\right)\right] + \overbrace{\frac{\partial}{\partial z}\left(2\mu\alpha\frac{\partial v_z}{\partial z}\right)}^{negl.} + F_{int,z}$$
$$+ MAX[\Gamma, 0]v_{z,other} - MAX[-\Gamma, 0]v_z \qquad (B.11)$$

The indicated terms are neglected per the parabolic approximation. The final equation results after a nomenclature change (x=z, u=v_z, and u=v_r):

$$\frac{1}{r}\frac{\partial}{\partial r}(r\alpha\rho v u) + \frac{\partial}{\partial x}\left(\alpha\rho u^2\right) = -\alpha\frac{\partial P}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}\left[\alpha r\mu\frac{\partial u}{\partial r}\right] + F_{int,x}$$
$$+ MAX[\Gamma, 0]v_{z,other} - MAX[-\Gamma, 0]v_z \qquad (B.12)$$

For the radial momentum equation:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\alpha\rho v_r^2\right) + \frac{1}{r}\frac{\partial}{\partial\theta}(\alpha\rho v_\theta v_r) + \frac{\partial}{\partial z}(\alpha\rho v_z v_r)$$
$$= -\alpha\frac{\partial P}{\partial r} + \frac{1}{r}\frac{\partial}{\partial r}(\alpha r\tau_{rr}) + \frac{1}{r}\frac{\partial}{\partial\theta}(\alpha\tau_{\theta r}) + \frac{\partial}{\partial z}(\alpha\tau_{zr}) - \frac{\alpha\tau_{\theta\theta}}{r}$$
$$F_{int,r} + MAX[\Gamma, 0]v_{r,other} - MAX[-\Gamma, 0]v_r \qquad (B.13)$$

Following the procedure outlined above for the x-momentum equation, the r-momentum equation eventually reduces to:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\alpha\rho v^2\right) + \frac{\partial}{\partial x}(\alpha\rho u v) = -\alpha\frac{\partial P}{\partial r} + \frac{1}{r}\frac{\partial}{\partial r}\left(2\alpha r\mu\frac{\partial v}{\partial r}\right) + \frac{\partial}{\partial x}\left(\alpha\mu\frac{\partial u}{\partial r}\right) - \frac{2\mu\alpha}{r^2}v$$
$$+ F_{int,r} + [\Gamma, 0]v_{other} - [-\Gamma, 0]v \qquad (B.14)$$

B.3 Energy Equation

Following White [36], again, the First Law of Thermodynamics applied to an infinitesimal control volume can be expressed as:

$$\dot{Q} - \dot{W} = \left[\frac{\partial}{\partial x}(\alpha\rho u\varepsilon) + \frac{\partial}{\partial y}(\alpha\rho v\varepsilon) + \frac{\partial}{\partial z}(\alpha\rho w\varepsilon)\right]dxdydz + E_{int} \qquad (B.15)$$

where ε is the total convected energy (internal+kinetic+flow work), and E_{int} is the interfacial energy transfer due to mass transfer. The following expressions detail these terms specifically. The heat transfer term amounts to:

$$\dot{Q} = [\nabla \bullet (\alpha k \nabla T) + q_{int}]dxdydz \qquad (B.16)$$

where the first term is the standard conduction heat transfer occuring within the phase and the second term is the interfacial heat transfer. For work transfer the following expression is applicable:

$$\dot{W} = \left[-\vec{V} \bullet (\nabla \bullet \alpha\tau_{ij}) - \Phi - \frac{1}{2}\left(\vec{V} + \vec{V}_{other} \right) \bullet \vec{F}_{int} \right] dxdydz \qquad (B.17)$$

where the first term is work done by the shear forces, the second term is viscous dissipation (neglected) and the third term is work done by the interfacial drag forces. Since each phase has a different velocity, it is assumed that an average velocity is appropriate for this work term. The interfacial energy transfer due to mass transfer is expressed as:

$$E_{int} = \left[-MAX[\Gamma, 0]\frac{V^2_{other}}{2} + MAX[-\Gamma, 0]\frac{V^2}{2} - \Gamma h \right] dxdydz \qquad (B.18)$$

where the first two terms are kinetic energy transfer. These terms reflect that whatever loss of kinetic energy that occurs in one phase is a gain for the other phase. The second term is the enthalpy transferred between phases. Substituting the above expressions into the First Law expression yields:

$$\nabla \bullet (\alpha k \nabla T) + \vec{V} \bullet \left(\nabla \bullet \alpha\tau_{ij}\right) + MAX[\Gamma, 0]\frac{V_{other}^2}{2} - MAX[-\Gamma, 0]\frac{V^2}{2} + \Gamma h + q_{int}$$
$$+ \frac{1}{2}\left(\vec{V} + \vec{V}_{other}\right) \bullet \vec{F}_{int} = \frac{\partial}{\partial x}(\alpha\rho u\varepsilon) + \frac{\partial}{\partial y}(\alpha\rho v\varepsilon) + \frac{\partial}{\partial z}(\alpha\rho w\varepsilon) \qquad (B.19)$$

Using the following definition:

$$\varepsilon = u + \frac{1}{2}V^2 + \frac{P}{\rho} = h + \frac{1}{2}V^2 = h + \frac{1}{2}\left(u^2 + v^2 + w^2\right) \qquad (B.20)$$

where u is internal energy, and substituting gives:

$$\nabla \bullet (\alpha k \nabla T) + \vec{V} \bullet \left(\nabla \bullet \alpha\tau_{ij}\right) + MAX[\Gamma, 0]\frac{V_{other}^2}{2} - MAX[-\Gamma, 0]\frac{V^2}{2} + \Gamma h + q_{int}$$
$$+ \frac{1}{2}\left(\vec{V} + \vec{V}_{other}\right) \bullet \vec{F}_{int} = \frac{\partial}{\partial x}(\alpha\rho uh) + \frac{\partial}{\partial y}(\alpha\rho vh) + \frac{\partial}{\partial z}(\alpha\rho wh)$$
$$\frac{\partial}{\partial x}\left\{\alpha\rho u\left[\frac{1}{2}\left(u^2 + v^2 + w^2\right)\right]\right\} + \frac{\partial}{\partial y}\left\{\alpha\rho v\left[\frac{1}{2}\left(u^2 + v^2 + w^2\right)\right]\right\}$$
$$+ \frac{\partial}{\partial z}\left\{\alpha\rho w\left[\frac{1}{2}\left(u^2 + v^2 + w^2\right)\right]\right\} \qquad (B.21)$$

Next, the mechanical energy equation is subtracted from this expression to give the thermal energy equation. The mechanical energy equation is formed by taking the dot product of velocity with the momentum equation (Eqn. B.8), which gives:

$$\vec{V} \bullet \nabla \bullet \left(\alpha\rho \vec{V}\vec{V}\right) = -\vec{V} \bullet \alpha\nabla P + \vec{V} \bullet \left(\nabla \bullet \alpha\tau_{ij}\right) + \vec{V} \bullet \vec{F}_{int}$$
$$+ MAX[\Gamma, 0]\,\vec{V} \bullet \vec{V}_{other} - MAX[-\Gamma, 0]\,\vec{V} \bullet \vec{V} \qquad (B.22)$$

The details of this subtraction are not shown here since they only amount to a large amount of tedious algebra.

The result is:

$$\nabla \bullet (\alpha\rho \vec{V} h) = \nabla \bullet (\alpha k \nabla T) + \vec{V} \bullet (\alpha \nabla P) + \Gamma h + q_{int}$$
$$\underbrace{+MAX[\Gamma, 0]\left(\frac{1}{2}V_{other}^2 + \frac{1}{2}V^2 - \vec{V} \bullet \vec{V}_{other}\right) + \frac{1}{2}\left(\vec{V}_{other} - \vec{V}\right) \bullet \vec{F}_{int}}_{\text{viscous dissipation - always positive}} \qquad (B.23)$$

Note that the last two terms represent viscous dissipation, since all mechanical energy transferred was cancelled out with the subraction of the mechanical energy equation. The next step is to convert enthalpy to temperature in the above equation. This is done using the property relationship (Bejan, [37]):

$$\nabla h = c_p \nabla T + \frac{1 - \beta T}{\rho} \nabla P \qquad (B.24)$$

where β is the coefficient of thermal expansion. Taking the dot product of velocity with this relationship multiplied by $(\alpha\rho)$ gives:

$$\alpha\rho \vec{V} \bullet \nabla h = \alpha\rho c_p \vec{V} \bullet \nabla T + \alpha(1 - \beta T) \vec{V} \bullet \nabla P \qquad (B.25)$$

By the chain rule of calculus this equation can be expressed as follows:

$$\nabla \bullet \left(\alpha\rho \vec{V} h\right) - h \underbrace{\nabla \bullet \left(\alpha\rho \vec{V}\right)}_{=\Gamma\,(\text{continuity})}$$
$$= c_p \nabla \bullet \left(\alpha\rho \vec{V} T\right) - c_p T \underbrace{\nabla \bullet \left(\alpha\rho \vec{V}\right)}_{=\Gamma\,(\text{continuity})} + \alpha(1 - \beta T) \vec{V} \bullet \nabla P \qquad (B.26)$$

Substituting this result into the energy equation to remove h and dividing through by c_p yields:

$$\nabla \bullet \left(\alpha\rho \vec{V} T\right) = \frac{1}{c_p}\nabla \bullet (\alpha k \nabla T) + \frac{\beta T}{c_p}\vec{V} \bullet \alpha \nabla P + T\Gamma + \frac{q_{int}}{c_p}$$
$$+ \frac{1}{c_p}MAX[\Gamma, 0]\left(\frac{1}{2}V_{other}^2 + \frac{1}{2}V^2 - \vec{V} \bullet \vec{V}_{other}\right) + \frac{1}{2c_p}\left(\vec{V}_{other} - \vec{V}\right) \bullet \vec{F}_{int} \qquad (B.27)$$

This expression is in general vector notation. Expanding in axisymmetric coordinates yields:

$$\frac{1}{r}\frac{\partial}{\partial r}(r\alpha\rho v T) + \frac{\partial}{\partial x}(\alpha\rho u T) = \frac{1}{c_p}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\alpha k\frac{\partial T}{\partial r}\right) + \overbrace{\frac{\partial}{\partial x}\left(\alpha k\frac{\partial T}{\partial x}\right)}^{negl.}\right]$$
$$+ \frac{\beta T}{c_p}\left(\overbrace{v\alpha\frac{\partial P}{\partial r}}^{negl.} + u\alpha\frac{\partial P}{\partial x}\right) + T\Gamma + \frac{q_{int}}{c_p}$$
$$+ \frac{1}{c_p}MAX[\Gamma, 0]\left[\frac{1}{2}\left(\overbrace{v_{other}^2}^{negl.} + u_{other}^2 + \overbrace{v^2}^{negl.} + u^2\right) - \overbrace{v v_{other}}^{negl.} - u u_{other}\right]$$
$$+ \frac{1}{2c_p}\left[\overbrace{(v_{other} - v)F_{int,r}}^{negl.} + (u_{other} - u)F_{int,x}\right] \qquad (B.28)$$

where the noted terms are neglected due to the parabolic approximation. The final form of the energy equation is thus:

$$\frac{1}{r}\frac{\partial}{\partial r}(r\alpha\rho v T) + \frac{\partial}{\partial x}(\alpha\rho u T) = \frac{1}{c_p}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\alpha k\frac{\partial T}{\partial r}\right)\right] + \frac{\beta T}{c_p}u\alpha\frac{\partial P}{\partial x} + T\Gamma + \frac{q_{int}}{c_p}$$
$$\frac{1}{c_p}MAX[\Gamma, 0]\left[\frac{1}{2}(u_{other}^2 + u^2) - u u_{other}\right] + \frac{1}{2c_p}(u_{other} - u)F_{int,x} \qquad (B.29)$$