CHAPTER 6
OPTIMAL STABLE SAND-SILT CHANNEL

6.1. BACKGROUND

Many rivers have the ability to transport sediment in the form of bedload, as well as in suspension. The additional processes involved make it more difficult to represent these rivers, compared with those whose sediment is coarse enough to ensure that only bedload transport occurs. The following is a brief background of the processes involved, models that have been developed, as well as a discussion of how these can possibly be modified into a more consistent and realistic model.

Parker (1978) developed a model based on the idea that suspended sediment tends to deposit on the banks of the channel at the same rate that sediment is eroded from the banks; part of it going back into suspension, the other part becoming bed load. He used two governing relations; the conservation of suspended sediment mass, and the conservation of total sediment mass. The second relation specifies that the amount of sediment laterally diffused and deposited on the banks, is equal to the amount eroded from the bank and returned to the bed region. This ensures that the channel will be in a state of dynamic equilibrium. He again used singular perturbation techniques in order to solve the problem.

Ikeda and Izumi (1991) developed a more rigorous model; based on Parker's analysis and also utilizing singular perturbation techniques. They pointed out that Parker's model had the inconsistency of allowing suspended sediment to be laterally diffused to the banks even though the depth-averaged suspended sediment concentration was uniform everywhere. In developing their equation for sediment mass-balance, Ikeda and Izumi used the following expression for the vertically integrated lateral volumetric transport of suspended sediment F_L:

$$F_L = -\varepsilon_y \int_0^D \frac{\partial c}{\partial y} dz = -\varepsilon_y \left(\partial \zeta / \partial y\right) + \varepsilon_y c|_{z=D} \frac{dD}{dy} \qquad (58)$$

where

ε_y = lateral sediment diffusivity induced by fluid turbulence

c = temporally averaged sediment concentration

$c|_{z=D}$ = near-bottom suspended sediment concentration

D = local depth

y = lateral coordinate taken positive toward the channel center from the left water margin

z = vertical coordinate taken positive downward from the water surface

$\zeta = \int_0^D c dz$ = vertically integrated suspended sediment concentration

Parker neglected the second term on the right-hand side of Equation 58. Ikeda and Izumi stated that this term plays an important role in defining the geometry of the channel, and showed that its inclusion in the formulation caused significant differences between their results and Parker's.

The problem with these two models is that they do not take momentum-diffusion into account. Ikeda and Izumi justified that momentum-diffusion could be neglected by assuming that the lateral slope at any point along the channel boundary is small, and varies slowly in the the lateral direction. This assumption, while valid along the flat-bed region, is not reasonable along the bank region, where curvature of the boundary is appreciable. In the bank region, the cross-channel velocity gradients are steepest, and therefore, lateral momentum flux is highest. It could be said that for very wide channels, the assumption can still be considered acceptable since the effects of the bank region for such channels become limited to the vicinity of the junction point. However, for channels whose banks make up a considerable part of the entire boundary, this is not so. Recalling the threshold channel case, where consideration of momentum-diffusion resulted in a significantly different bank shape, it is expected that momentum-diffusion will have a significant role to play in this case as well.

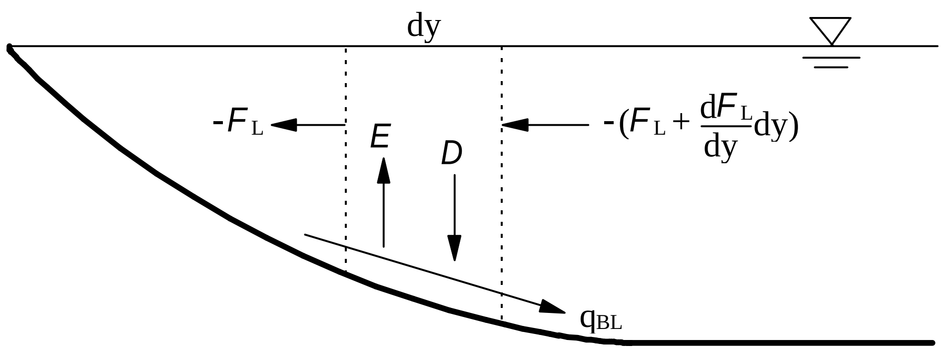

Figure 6.1. Definition diagram; conservation of suspended sediment mass

The sand-silt channel shall be assumed to support both bedload and suspended sediment transport, and that capacity conditions in terms of suspended sediment exist. In the central region of the channel, the higher shear stresses entrain higher amounts of sediment in suspension. The resulting lateral concentration gradient causes mass transport of suspended sediment from the center of the channel toward its banks, where it eventually settles. At the same time, erosion of the channel bank occurs as bank material either becomes part of lateral bedload due to gravity, or is reentrained in suspension. To represent the behavior of the suspended sediment, the conservation of suspended sediment mass is used in this model. Figure 6.1 shows this to be described by the following equation:

$$-dF_L/dy = E - D \tag{59}$$

where

E = rate of erosion from bottom

D = deposition rate

However, this equation alone does not ensure that the channel will be in a state of dynamic equilibrium. The condition is guaranteed when the amount of sediment that is deposited on the bank is equal to the amount of sediment that is eroded from it. This will be represented by a conservation of total sediment mass equation.

$$F_L + q_{BL} = 0 \tag{60}$$

where
q_{BL} = lateral bedload rate per unit longitudinal length

Equations 59 and 60, together with the momentum-diffusion equation (Equation 2), can be used as the bases for a suspended sediment transport model.

6.2. CONSERVATION OF SUSPENDED SEDIMENT MASS

The vertically integrated lateral volumetric transport of suspended sediment F_L is given by equation 61:

$$F_L = -\varepsilon_y \int_0^D \frac{\partial c}{\partial y} dz = -\varepsilon_y (\partial \zeta / \partial y) + \varepsilon_y c|_{z=D} \frac{dD}{dy} \tag{61}$$

The lateral sediment diffusivity ε_y, and the vertical sediment diffusivity ε_z, are assumed to have constant values of $0.2u_{*c}D_c$ and $0.1u_{*c}D_c$ respectively, where $u_{*c} = \sqrt{\tau_c/\rho}$ = the friction velocity corresponding to the shear stress at the center of the channel τ_c (Ikeda and Izumi, 1991).

The bed concentration $c|_{z=D}$ can be deduced from experimental or field data approximating zero-flux flow. In this study, it will be related to flow conditions in the form proposed by Ikeda, Izumi, and Ito (1989):

$$c|_{z=D} = 0.0025(u_{*G} / v_s)^2 \tag{62}$$

where
v_s = settling velocity of the suspended sediment
$u_{*G} = \sqrt{\tau_G/\rho}$ = friction velocity corresponding to grain shear stress τ_G

The expression for vertically integrated concentration of suspended sediment, $z = \int_0^D c dz$, can be derived by using the following equation for suspended sediment concentration (Ikeda and Izumi, 1991):

$$c = c|_{z=D} \exp\left(-\frac{v_s}{e_z}(D-z)\right) \tag{59}$$

where
ε_z = vertical sediment diffusivity

Since ε_z is assumed to be constant in Equation 63, the vertically integrated concentration then becomes

$$z = \frac{e_z}{v_s}\left\{1 - \exp\left[-\left(\frac{v_s}{e_z}\right)D\right]\right\}c|_{z=D} \tag{64}$$

However, $v_s D / e_z$ >>1 for natural sand rivers (Ikeda and Izumi, 1991). Thus Equation 64 can be approximated by

$$z = \frac{e_z}{v_s} c|_{z=D} \tag{65}$$

The deposition rate D is correlated with the bottom suspended sediment concentration as follows:

$$D = v_s c|_{z=D} \tag{66}$$

Rearranging Equation 65 into an expression for $c|_{z=D}$,and plugging this into Equation 66,

$$D = \left(v_s^2 / \varepsilon_z\right) z \tag{67}$$

The erosion rate E is given by the following expression (Izumi and Ikeda, 1991):

$$E = 0.0025(u_{*G} / v_s)^2 v_s \tag{68}$$

E can be expressed as a function of dimensionless total shear stress τ^* provided u_{*G}, which depends on grain shear stress τ_G, can be expressed as a function of τ^*. Engelund and Hansen (1967) proposed the following formula relating τ^* and dimensionless grain shear stress τ^*_G:

$$\tau^*_G - \tau^*_{cr} = 0.4\tau^{*2} \tag{69}$$

where

$\tau^*_G = \tau_G/\rho R_s g d_{50}$

$\tau^*_{cr} = \tau_{cr}/\rho R_s g d_{50}$ = dimensionless critical shear stress for bedload motion

$\tau^* = \tau/\rho R_s g d_{50}$

Using Equation 69, and noting that $\tau^*_G >> \tau^*_{cr}$ for beds with active suspended load transport, Equation 68 becomes

$$E = 0.001\tau^{*2} v_s / v_s^{*2} \tag{70}$$

where

$$v_s^* = v_s \Big/ \sqrt{R_s g d_{50}}$$

The equation for the conservation of suspended sediment mass (equation 59) can now be expressed as follows:

$$-\varepsilon_y \frac{\partial^2 z}{\partial y^2} + 2\left(\frac{v_s}{\varepsilon_z}\right)\left(\frac{dD}{dy}\frac{dz}{dy} + z\,\frac{d^2 D}{dy^2}\right) = 0.001\left(\frac{\tau^*}{v_s^*}\right)^2 v_s - z\left(\frac{v_s^2}{\varepsilon_z}\right) \tag{71}$$

6.3. CONSERVATION OF TOTAL SEDIMENT MASS

The lateral bed load rate per longitudinal length q_{BL} can be related to the longitudinal bed load rate per unit width q_B using the following equation (Johannesson and Parker, 1989):

$$\frac{q_{BL}}{q_B} = \frac{1+\beta\mu}{f^*\mu}\left(\frac{\tau^*_{cr}}{\tau^*_G}\right)\frac{dD}{dy} \tag{72}$$

where f^* is a correction factor. For β = 0.85 and μ = 0.43, f^* = 1.19 (Johannesson and Parker, 1989).

The Meyer-Peter and Muller (1948) formula is used to predict q_B:

$$q_B = 8\left(\tau^*_G - \tau^*_{cr}\right)^{1.5}\sqrt{R_s g d_{50}^3} \tag{73}$$

Using Equations 69 and 73, Equation 72 becomes

$$q_{BL} = 1.85\tau^{*2}\sqrt{R_s g d_{50}^3}\,\frac{dD}{dy} \tag{74}$$

The equation for conservation of total sediment mass (Equation 60) can now be expressed as follows:

$$-e_y \frac{\P^2 z}{\P y^2} + 2\left(\frac{v_s}{e_z}\right)\left(\frac{dD}{dy}\frac{dz}{dy} + z\,\frac{d^2 D}{dy^2}\right) + 1.85\sqrt{R_s g d_{50}^3}\left(2t^*\frac{dt^*}{dy}\frac{dD}{dy} + t^{*2}\frac{d^2 D}{dy^2}\right) = 0 \qquad (75)$$

6.4. BOUNDARY SHEAR STRESS

The boundary shear stress τ^* can be expressed as a dimensionless stress-depth δ^* as follows:

$$\tau^* = \frac{\delta^* \mathrm{S}}{\mathrm{R_s d_{50}^*}} \qquad (76)$$

where

d_{50}^* = d_{50}/D_c = dimensionless grain size such that 50% of the sediment is finer.

Recall that δ^* is given by Equation 2, which accounts for the effect of momentum-diffusion in the formulation.

6.5. REDUCTION AND SOLUTION

Subtracting Equation 75 from Equation 71, and putting all terms on the left-hand side, results in the following relation:

$$-1.85\sqrt{R_s g d^3}\left(2t^*\frac{dt^*}{dy}\frac{dD}{dy} + t^{*2}\frac{d^2 D}{dy^2}\right) - 0.001\left(\frac{t^*}{v_s^*}\right)^2 v_s + z\left(\frac{v_s^2}{e_z}\right) = 0 \qquad (77)$$

Equation 77 accounts for both the conservation of suspended sediment mass, and the conservation of total sediment mass.

By using Equation 76 to express τ^* in terms of δ^*, and non-dimensionalizing the other terms, Equation 77 is reduced to the following:

$$k_1\left(2d^*\frac{dd^*}{dy^*}\frac{dD^*}{dy^*} + d^{*2}\frac{d^2 D^*}{dy^{*2}}\right) + k_2 d^{*2} + k_3 z(d^*) = 0 \qquad (78)$$

where

$k_1 = -1.85\sqrt{\dfrac{g}{R_s^3 d_{50}}}\, S^2 D_c$

$k_2 = -0.001\sqrt{\dfrac{g}{R_s^3 d_{50}^3}}\, \dfrac{(SD_c)^2}{v_s^*}$

$k_3 = \dfrac{R_s g d_{50} v_s^{*2}}{e_z}$

Equations 78 and 2 (the momentum-diffusion equation) form a system that is in terms of D^* and δ^*. They will be solved in a manner similar to that of the optimal stable gravel channel case. Firstly, the bank profile is determined. Secondly, the width of the flat-bed is determined; making sure that the bed and bank solutions are continuous at the junction point. Lastly, the bed and bank solutions are combined to give the complete solution.

The following are the boundary conditions for this case:

Bank Solution:

At the junction between the bed and bank regions, $y^* = B_f^* / 2$:

1. dimensionless depth, $D^* = 1$. (79)
2. lateral slope, $dD^*/dy^* = 0$. (80)

At the water margin, $y^* = \left(B_f^* + B_s^*\right)/ 2$:

3. $D^* = 0$. (81)

Bed Solution:

At the channel's center, $y^* = 0$:

1. stress depth gradient, $d\delta^*/dy^* = 0$. (82)

At the junction of the bed and bank regions, $y^* = B_f^* / 2$:

2. dimensionless stress depth, $\delta^* = \delta_{cr}^*$. (83)
3. lateral momentum flux, $\psi D^{*2}\{(dD^*/dy^*)^2+1\}(d\delta^*/dy^*)$ is continuous. (84)

Once the bed and bank solutions have been put together, the complete channel profile will have been obtained. The distribution of shear stresses over the entire channel boundary will have been determined as well. To verify this shear stress distribution, Equation 2 is solved over the entire channel half-width subject to the following conditions:

At the center of the channel, $y^* = 0$:

1. $d\delta^*/dy^* = 0$. (85)

At the water margin, $y^* = \left(B_f^* + B_s^*\right)/ 2$:

2. $\delta^* = 0$. (86)

As in the gravel channel case, the Runge-Kutta-Merson technique will be used to solve the governing equations for this case.