Appendix B

Error Probability for Spectrum-Sliced FSK Systems

Here we derive a simplified expression for the error probability of spectrum-sliced FSK systems, assuming a chi-squared approximation for the signal and noise statistics. The analysis follows that developed by Jacobs [60].

Equation (4.28) is stated here as a ready reference.

$$P_e = \frac{1}{\left[(2m-1)!\right]^2} \int_0^{\infty} v^{2m-1} \exp(-v)\,dv \int_0^{v\beta} u^{2m-1} \exp(-u)\,du \qquad (4.28)$$

Considering the second integral (I_2) on the RHS we have

$$I_2 = \int_0^{v\beta} u^{m-1} \exp(-u)\,du = (m-1)\int_0^{v\beta} u^{m-2} \exp(-u)\,du - (v\beta)^{m-1} \exp(-v\beta) \qquad (B.1)$$

which leads after further simplification to

$$I_2 = (m-1)! - \exp(-v\beta)\left[\begin{array}{l}(v\beta)^{m-1} + (m-1)(v\beta)^{m-2} + (m-1)(m-2)(v\beta)^{m-3} \\ +(m-1)(m-2)(m-3)(v\beta)^{m-4} + \ldots\ldots\ldots\ldots\ldots(m-1)!\end{array}\right]. \qquad (B.2)$$

Hence the main expression in Eq. (4.28) can now be expressed as

$$P_e = \frac{1}{\left[(m-1)!\right]^2}\left[\int_0^{\infty} v^{m-1} \exp(-v)\,dv \left\{I_2\right\}\right]. \qquad (B.3)$$

Simplifying Eq. (B.3) by using the properties of the gamma function

$$P_e = \frac{(m-1)!}{\left[(m-1)!\right]^2}\left[\int_0^{\infty} v^{m-1} \exp(-v)\,dv\right] = \frac{1}{(m-1)!}\Gamma(m) = 1 \qquad (B.4)$$

and simplifying the other terms of Eq. (B.3), it can be shown that [60]

$$P_e = 1 - \frac{1}{(1+\beta)^m} \sum_{i=0}^{m-1} \left({}^{m-1+i}C_i \right) \left(\frac{\beta}{1+\beta} \right)^i \tag{B.5}$$

By using the relation

$$\sum_{i=0}^{\infty} \left({}^{m-1+i}C_i \right) \left(\frac{\beta}{1+\beta} \right)^i = \left(1 - \frac{\beta}{1+\beta} \right)^{-m} \tag{B.6}$$

we can rewrite the above equation as

$$\begin{aligned} P_e &= \frac{1}{(1+\beta)^m} \sum_{i=m}^{\infty} \left({}^{m-1+i}C_i \right) \left(\frac{\beta}{1+\beta} \right)^i \\ &= \frac{\beta^m}{(1+\beta)^{2m}} \left({}^{2m-1}C_m \right) \sum_{k=0}^{\infty} a_k \left(\frac{\beta}{1+\beta} \right)^k \end{aligned} \tag{B.7}$$

where

$$a_k = \left(\frac{2m-1+k}{m+k} \right) a_{k-1} \; ; \; a_0 = 1. \tag{B.8}$$

Now we know that

$$1 < \frac{a_k}{a_k - 1} \leq \frac{2}{1+1/\mathrm{m}} \tag{B.9}$$

is bounded away from above and below by geometric series, so that

$$\left({}^{2m-1}C_m \right) \frac{\beta^m}{(1+\beta)^{2m-1}} < P_e < \left({}^{2m-1}C_m \right) \left[\frac{1+1/m}{(1-\beta)+(1/m)(1+\beta)} \right] \frac{\beta^m}{(1+\beta)^{2m-1}} . \tag{B.10}$$

For the practical case in fiber systems where $m >> 1$, Stirling's approximation may be used to show that

$$\left({}^{2m-1}C_m \right) \approx \frac{4^m}{\sqrt{4\pi m}} \tag{B.11}$$

which leads to

$$P_e \approx \frac{1}{\sqrt{4\pi m}} \frac{(4\beta)^m}{(1+\beta)^{2m-1}} . \tag{B.12}$$