OPTIMAL BOUNDARY AND DISTRIBUTED CONTROLS FOR THE VELOCITY TRACKING PROBLEM FOR NAVIER-STOKES FLOWS

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Abstract

The velocity tracking problem is motivated by the desire to match a desired target flow with a flow which can be controlled through time dependent distributed forces or time dependent boundary conditions. The flow model is the Navier-Stokes equations for a viscous incompressible fluid and different kinds of controls are studied. Optimal distributed and boundary controls minimizing a quadratic functional and an optimal bounded distributed control are investigated. The distributed optimal and the bounded control are compared with a linear feedback control. Here, a unified mathematical formulation, covering several specific classes of meaningful control problems in bounded domains, is presented with a complete and detailed analysis of all these time dependent optimal control velocity tracking problems. We concentrate not only on questions such as existence and necessary first order conditions but also on discretization and computational aspects. The first order necessary conditions are derived in the continuous, in the semidiscrete time approximation and in the fully finite element discrete case. This derivation is needed to obtain an accurate meaningful numerical algorithm with a satisfactory convergence rate. The gradient algorithm is used and several numerical computations are performed to compare and understand the limits imposed by the theory. Some computational aspects are discussed without which problems of any realistic size would remain intractable.

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Chapter 1

Introduction

1.1 Motivation

Optimal control theory of distributed parameter systems has several applications in the engineering sciences. The subject has been under rapid development since the early works of Fattorini [1], Friedman [2] and Butkovskii [3]. Other early major contributors include Lions [4],[5],[6], Ahmed [7] and Barbu [8]. The impact of this development to continuum mechanics has been essentially in the branch of solid mechanics. Fursikov [9],[10],[11] established several theorems relating the Navier-Stokes equations and the minimization of certain functionals, thus starting the control of distributed systems in fluid mechanics. Turbulence control was the theme of these early works in control. The occurrence of turbulence can have a devastating effect on performance in many fluid mechanical systems and this is why there was considerable technical interest in control flow problems. Now the purpose of control ranges from reducing the drag on bodies, to increasing the lift coefficient or improving heat and mass transfer characteristics. By a small perturbation of the thin boundary layer it is sometimes possible to obtain quite dramatic effects on important flow parameters. In the paper of Bushnell and McGinley [12] we find a concise review of various techniques for turbulence control of wall flow. Profile design, continuous suction and blowing, additives, and compliant walls are examples of strategies to affect the boundary layer by passive actions. More recent (see Metcalfe [13] or Thomas [14]) is the development of active means of controlling the boundary layer, usually by anti-phase modal superposition in transition flows, or by suppression of coherent structures in turbulent flows. Another developing area in which active control strategies are of interest is combustion and chemical reacting flows. A review of these techniques can be found in McManus [15]. Some researchers have attempted to apply optimal control ideas to flow control problems. An example, concerning strategies to control dam water gates for the prevention of flood flows, can be found in the work of Kawahara [16]. The objective is to suppress waves generated by sudden operations of dam gates in order to avoid the damage that the reflected waves cause on the upstream area. The hydraulic model is based on the shallow-water equations, and the water elevation is controlled by minimizing a quadratic objective function using a gradient or conjugate gradient method.

Another important class of problems in which optimal control ideas has been used is design problems. Numerical approaches to optimal-shape problems involving flow equations are discussed by Jameson [17] and Glowinski [18], [19]. Of course, these articles are concerned with the optimal design of airfoils.

The use of nonlinear avection and viscous diffusion models in control problem can be found in literature. For example see [20],[20],[22] .

The use of the Navier-Stokes equations for optimal control problem has been developed by Abergel [23], Alekseyev [24], Fattorini [25], Glowinski [26], Gunzburger, Hou and Svobodny [27],[28],[29] and Sritharan [30]. These papers address questions such as the formulation of feasible problems, existence of optimal controls, first order necessary conditions for optimality and discretizations issues.

The velocity tracking problem is motivated by the desire to match a desired target flow with a flow which can be controlled through distributed forces or boundary conditions. The possibility of driving a fluid in a particular configuration opens a wide range of possible applications. Some papers on this subject have been written by Gunzburger, Hou and Svobodny [28],[31] for the stationary tracking problem and by Hou and Yan [32],[33] on the dynamics of the nonsteady tracking problem with distributed controls.

Here, a unified mathematical formulation, covering several specific classes of meaningful control problems in a bounded domain, is presented. In this work a complete and detailed analysis of the time dependent optimal control tracking velocity problem is made with distributed or boundary controls. We concentrate not only on questions such as existence and necessary first order conditions but also on discretization and computational aspects. Several numerical computations are performed to compare and understand the limits imposed by the theory. The computation by itself is an outstanding problems that can dominate all other aspects of the optimal control theory. For this reason, the application of the frontal method and storage tape techniques are needed in order to treat systems of realistic size in time and space.

This work is organized as follows. Notations, used in the following chapters, and the variational form of the Navier-Stokes equations are introduced in Chapter 1. Chapter 2 treats the distributed optimal control tracking velocity problem with a quadratic functional. This means that the minimization of this quadratic functional is the criterion to judge if the solution of the Navier-Stokes equations matches the target flow. In Chapter 3 a bounded limit is imposed on the distributed optimal control. Chapter 4 shows that a linear feedback can be used not only to improve the algorithm of the optimal control problem but it can be used by itself as a good control, avoiding the heavy task of solving the optimization algorithm. If our goal is only to match the desired flow by the designated flow, without minimizing the energy, then the linear feedback algorithm is a very good solution of the problem. Finally, in Chapter 5, boundary controls are treated. This type of controls has a wide applicability in the fluid mechanics setting as it can be easily implemented in real practical applications.

1.2 Notation

The analysis of partial differential equations naturally involves function spaces which are defined in term of functions and their derivatives. Thus, the natural spaces to study these equations are the Sobolev spaces and the isotropic Sobolev spaces if time is involved. The Navier-Stokes equations stem from the conservation of mass and momentum and their study needs also some properties related with the divergence and curl operators. In the following subsections we cover the basic definition and properties of Sobolev spaces (see [34], [35], [36] and [37]). Then, we introduce the notation for the fiber spaces in time and focus on the divergence and curl operators (see [36] and [38]).

1.2.1 Domain

In our approach to the problem the flow domain is always a bounded connected open set Ω in $\mathbb{R}^2$. The properties of the spaces involved are connected with the boundary and a precise definition of the boundary is needed to understand these spaces. We shall not discuss various geometric conditions such as the segment property or the cone property of the boundary. When domains, which are not smooth, are used, the boundary of the domain is a Lipschitz boundary.

We say that $\Omega \subset \mathbb{R}^2$ is Lipschitz if every point on the boundary Γ has a neighborhood I such that, after an affine change of coordinates, $\Gamma \cap I$ is described by the equation $z = \phi(s)$ where ϕ is uniformly Lipschitz continuous.

The hypothesis, which requires $\Omega \cap I$ to be on one side of $\Gamma \cap I$, is not necessary as the domain is connected. We note that a Lipschitz-continuous boundary has almost everywhere a normal vector $\vec{n}$ and this definition includes domains with corners, which are standard in all applications. Unfortunately these domains do not have all the properties that are required for a rigorous mathematical treatment of the problem. In this case, we can always approximate the domain with a more smooth domain and apply the rigorous mathematical treatment to that approximate domain.

We say that Ω is of class C^k, $k \geq 1$, if every point on Γ has a neighborhood I so that $\Gamma \cap I$ is a C^k-surface.

In general, for a good behavior of the tangent and normal vector to the boundary, we shall assume $\Gamma \in C^2$.

1.2.2 Distribution space $\mathcal{D}'(\Omega)$

We shall denote by $\mathcal{D}(\Omega)$ the set of all $C^\infty(\Omega)$ functions with compact support contained in Ω and $\mathcal{D}(\bar{\Omega})$ the set of all $C^\infty(\Omega)$ functions with compact support contained in the closure of Ω. $\mathcal{D}(\bar{\Omega})$ can be also defined as the linear space of restrictions to $\bar{\Omega}$ of the functions in $C_0^\infty(\mathbb{R})$. Of course $\mathcal{D}(\Omega)$ is a vector space and we can provide this space with a topology of a topological vector space but this topology is difficult to manipulate. For almost all practical purposes, the notion of convergence of sequences is sufficient. *Let $\{f_n\}_{n=1}^\infty$ be a sequence in $\mathcal{D}(\Omega)$ with supports K_n. The sequence converges to $f \in \mathcal{D}(\Omega)$ if there exists a fixed compact set $K \in \Omega$ such that $K_n \subseteq K$ for all n and all the sequences of the $|\alpha| - th$ derivatives of $\{f_n\}_{n=1}^\infty$ converge uniformly on K to the $|\alpha| - th$ derivative of f.* We have that the space $\mathcal{D}(\Omega)$ is dense in $C^k(\Omega)$ for all $k \geq 0$ and in $L^p(\Omega)$ for all $1 \leq p < \infty$. Every sequence that converges in $\mathcal{D}(\Omega)$ converges in each of these spaces.

A continuous linear form on $\mathcal{D}(\Omega)$ is called a distribution on Ω. We denote by $\mathcal{D}'(\Omega)$ the set of distributions on Ω. If $T \in \mathcal{D}'(\Omega)$, we denote by $< T, \phi >$ its value on the function $\phi \in \mathcal{D}(\Omega)$. *To say that T is continuous on $\mathcal{D}(\Omega)$ means that for every sequence $\{\phi_n\}_{n=1}^\infty \in \mathcal{D}(\Omega)$ such that $\phi_n \to \phi$ we have $< T, \phi_n > \to < T, \phi >$.* Then, it is easily to check that $\mathcal{D}'(\Omega)$ is a vector space which is the topological dual of $\mathcal{D}(\Omega)$. *A sequence of distributions $\{T_n\}_{n=1}^\infty$ converges to the distribution T if for every $\phi \in \mathcal{D}(\Omega)$ we have $< T_n - T, \phi > \to 0$ for $n \to \infty$.* We note that if $\{T_n\}_{n=1}^\infty$ is a sequence of distributions such that for all $\phi \in \mathcal{D}(\Omega)$, $< T_n, \phi >$ converges when $n \to \infty$, then the linear form converges to an element of $\mathcal{D}'(\Omega)$. If $T \in \mathcal{D}'(\Omega)$, the derivative $D_i T$, which coincides with the usual differentiation of continuously differentiable functions, is defined by $< D_i T, \phi > = - < T, D_i \phi >$. From this definition and the notion of convergence in $\mathcal{D}'(\Omega)$ we can state that *if $\{T_n\}_{n=1}^\infty$ is a sequence of distributions which converges to T in $\mathcal{D}'(\Omega)$ when $n \to \infty$, then we have that for all α, $D^\alpha T_n \to D^\alpha T$ in $\mathcal{D}'(\Omega)$ when $n \to \infty$.*
The distribution derivative is compatible with the classical derivative and has some common properties. *Let Ω connected, and let $u \in \mathcal{D}'(\Omega)$ be such that $\nabla u = 0$ then, u is a constant.* In general we can not write a definite integral of a generalized function but it is possible to show that a primitive exists. *Let $I = (a, b)$ be an open interval in $\mathbb{R}$ and let $f' \in \mathcal{D}'(I)$. Then there exists $u' \in \mathcal{D}'(\Omega)$ such that $u' = f$. The primitive u is unique up to a constant.*

1.2.3 Sobolev spaces

Definition

We recall that the space $L^2(\Omega)$ is a separable Hilbert space if equipped with the inner product

$$(u, v) = \int_\Omega u \cdot v d\vec{x} \tag{1.1}$$

and the norm

$$\|u\| = \sqrt{(u,u)}.$$

A definition of non-negative integer Sobolev space can be the following:
let k be a non-negative integer. Then we define $H^k(\Omega)$ to be the set of all distribution $u \in L^2(\Omega)$ such that $D^\alpha u \in L^2(\Omega)$ for $|\alpha| \leq k$. In $H^k(\Omega)$, we define a norm by

$$\|u\|_k^2 = \sum_{|\alpha|\leq k} \|D^\alpha u\|^2 \tag{1.2}$$

and the inner product by

$$(u,v)_k = \sum_{|\alpha|\leq k} \int_\Omega D^\alpha u D^\alpha v d\vec{x}. \tag{1.3}$$

The Sobolev space $H^k(\Omega)$ is a separable Hilbert space and it can be identified with the completion of the linear space $C^k(\bar{\Omega})$ with the norm $\|\cdot\|_k^2$. It is interesting to extend the notion of Sobolev spaces with nonintegral value of $k \geq 0$.
Let Ω be an open subset of $\mathbb{R}^2$, $m \geq 0$ be an integer and $s = m + \sigma$ where $0 < \sigma < 1$. We denote $H^s(\Omega)$ the space of all distributions $u \in H^m(\Omega)$ defined on Ω such that

$$\int_\Omega \int_\Omega \frac{|\partial^\alpha u(\vec{x}) - \partial^\alpha v(\vec{y})|^2}{\|\vec{x}-\vec{y}\|^{2(1+\sigma)}} d\vec{x}d\vec{y} < +\infty \quad \forall |\alpha| = m. \tag{1.4}$$

Again it can be shown that $H^s(\Omega)$ is a Hilbert space if equipped with the norm

$$\|u\|_s^2 = \|u\|_m^2 + \sum_{|\alpha|=m} \int_\Omega \int_\Omega \frac{|\partial^\alpha u(\vec{x}) - \partial^\alpha v(\vec{y})\|^2}{\|\vec{x}-\vec{y}\|^{2(1+\sigma)}} d\vec{x}d\vec{y} < +\infty \quad \forall |\alpha| = m \tag{1.5}$$

and the relative scalar product.
If $p < \infty$ then $\mathcal{D}(\Omega)$ is dense in $L^P(\Omega)$. This in general not the case for the Sobolev space and this motivates the following definition. *We shall denote $H_0^s(\Omega)$ the closure of $\mathcal{D}(\Omega)$ in $H^k(\Omega)$.* $H_0^k(\Omega)$ is in general a proper subspace of $H^k(\Omega)$ when Ω is different from $\mathbb{R}$.
Since test functions are generally not dense in $H^k(\Omega)$, linear functionals are not necessarily distributions. There are some nonlinear functionals which vanish on all test functions.
We denote $H^{-k}(\Omega)$ the set of all linear functionals on $H_0^k(\Omega)$ and $(H^k)^*$ the set of all linear functionals on $H^k(\Omega)$.
From this definition if $f \in H^{-k}(\Omega)$ with k integer then, there exists functions $g_\alpha \in L^2(\Omega)$ such that

$$f = \sum_{|\alpha|\leq k} D^\alpha g_\alpha.$$

In general $(H^k)^*$ is a proper subspace of $H_0^k(\Omega)$. When Ω is the whole $\mathbb{R}^2$ or is a compact manifold of class C^l, $l > k$ then $H^{-k}(\Omega)$ denotes also the dual of $H^k(\Omega)$.

It is possible to generalize the Sobolev space to $L^p(\Omega)$ spaces. For example, for each integer $m \geq 0$ and real p with $1 \leq p < \infty$, we define the Sobolev space:

$$W^{m,p}(\Omega) = \{v \in L^p(\Omega); \partial^\alpha v \in L^p(\Omega) \quad \forall |\alpha| \leq m\},$$

which is a Banach space with the norm

$$\|u\|_{m,p}^2 = \left(\sum_{|\alpha| \leq m} \int_\Omega \|D^\alpha u\|^2 d\vec{x} \right)^{1/p}.$$

For $p = 2$ we recover the $H^m(\Omega)$ spaces.

General and interpolation properties

Now we recall some properties of integral and fractional order Sobolev spaces. Normally, Sobolev spaces of integral orders are used but we need also fractional order spaces to treat the boundary conditions.

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^n$ be an open set and $H^m(\Omega)$ be an integral or fractional order Sobolev space.*
i) $H^m(\Omega)$ endowed with the appropriate scalar product is a Hilbert space.
ii) If $m \geq m'$, then, $H^m(\Omega)$ is contained with continuous injection, in $H^{m'}(\Omega)$.
iii) If $u \in H^m(\Omega)$ then, $D_i u \in H^{m-1}(\Omega)$.
iv) $H^m(\mathbb{R}^n)$ $(m \geq 0)$ coincides with the space of tempered distributions u such that $(1 + |w|^2)^{m/2}\hat{u} \in L^2(\mathbb{R}^n)$ where $\hat{u}$ is the Fourier transform of u in w variable. The norm $\|u\|_m$ is equivalent to $\left(\int_{\mathbb{R}^n} (1 + |w|^2)^{m/2} |\hat{u}|^2 dw\right)^{1/2}$.

From iii) we see that for $m > 0$ there is an alternative Fourier definition of the Sobolev spaces using Fourier. When Ω is an open, we can define

$$H^m(\Omega) = \{v \in L^2(\Omega); \exists \hat{v} \in H^m(\mathbb{R}^n) \text{ with } \hat{v}|_\Omega = v\}$$

with the norm

$$\|v\|_s = \inf_{\hat{v} \in H^s(\mathbb{R}^n), \hat{v}|_\Omega = v} \|\hat{v}\|_{s,\mathbb{R}^n}.$$

In addition we have the following Interpolation Theorem [36]

Theorem 1.2 *Let Ω be a bounded Lipschitz-continuous open subset of $\mathbb{R}^n$. Let $\theta \in [0, 1]$ ant let s_i and t_i be two pairs of real numbers with $0 \leq t_i \leq s_i$ for $i = 1, 2$ and let $\mathcal{L}_i$ and*

$\mathcal{L}_\theta$ denote respectively $\mathcal{L}(H^{s_i}; H^{t_i})$ and $\mathcal{L}(H^{(1-\theta)s_1+\theta s_2}; H^{(1-\theta)t_1+\theta t_2})$. Let π an operator in $\mathcal{L}_1 \cap \mathcal{L}_2$; then π also belongs to $\mathcal{L}_\theta$ and there exists a constant C such that

$$\|\pi\|_{\mathcal{L}_\theta} \leq C\|\pi\|_{\mathcal{L}_1}^{1-\theta} \cdot \|\pi\|_{\mathcal{L}_2}^{\theta}$$

and a multiplication theorem [36]

Theorem 1.3 *Assume that Ω is a bounded Lipschitz-continuous open subset of $\mathbb{R}^n$. Let m_1, m_2 and m be three nonnegative integral or non negative fractional Sobolev space orders such that $m_1 \geq m, m_2 \geq m$ and either*

$$\begin{aligned} m_1 + m_2 - m \geq n/2 \qquad & if \quad m_i > m \\ & \qquad or \\ m_1 + m_2 - m > n/2. \qquad & if \quad m_i \geq m \end{aligned}$$

Then, the mapping $u, v \rightarrow u \cdot v$ is a continuous bilinear map from $H^{m_1}(\Omega) \times H^{m_2}(\Omega)$ into $H^m(\Omega)$.

Extension and density properties

The extension and density theorem can be stated in the following way [38].

Theorem 1.4 *Let Ω be an open Lipschitz-continuous subset of $\mathbb{R}^n$ and $H^m(\Omega)$ be a positive integral or fractional order Sobolev space.*
i) Let u in $H^m(\Omega)$ and let $\tilde{u}$ denote its extension by zero outside Ω. If $\tilde{u} \in H^m(\mathbb{R}^n)$ then $u \in H_0^m(\Omega)$.
ii) If in addition Γ is bounded and $m \geq 1$, there exists a continuous linear extension operator P from $H^m(\Omega)$ to $H^m(\mathbb{R}^n)$:

$$Pu|_\Omega = u \quad \forall u \in H^m(\Omega).$$

Theorem 1.5 *Let Ω be an open bounded Lipschitz-continuous subset of $\mathbb{R}^n$ and $H^m(\Omega)$ be a positive integral or fractional order Sobolev space.*
i) The space $\mathcal{D}(\bar{\Omega})$ is dense in $H^m(\Omega)$
ii) The space $\mathcal{D}(\Omega)$ is not dense in $H^m(\Omega)$ with $m \geq 1$ but is dense in $H_0^m(\Omega)$ and in $H^0(\Omega)$.

The density theorem is fundamental for the treatment of these spaces. One can do everything for the functions in the dense set and then argue the general case by taking the limit.

Imbedding and compactness theorem

We now come to the fundamental Sobolev Imbedding theorem which essentially relates different Sobolev spaces and spaces of smooth functions (see [36]).

Theorem 1.6 *Let Ω be an open Lipschitz continuous subset of $\mathbb{R}^n$ and m_1 and m_2 positive integral or fractional indices of a Sobolev space. The following imbeddings hold:*

$$H^{m_1+1}(\Omega) \subset H^{m_1}(\Omega) \tag{1.6}$$

$$H_2^m(\Omega) \subset C^{m_1}(\Omega) \quad m_1 > m_2 + n/2. \tag{1.7}$$

Moreover, if Ω is bounded, the last inclusion holds in $C^{m_1}(\bar{\Omega})$ and the imbedding of $H^{m_1+1}(\Omega)$ into $H^{m_1}(\Omega)$ is compact. In addition these compact imbeddings are also valid for negative m_1 and for all nonnegative integer m_1 the injection of $H_0^{m_1+1}(\Omega)$ into $H_0^{m_1}(\Omega)$ is compact.

This theorem will be used constantly. For instance, in the next paragraphs we shall use the fact that $L^2(\Omega)$ is compactly imbedded in $H^{-1}(\Omega)$ and that $H^1(\Omega)$ and $H_0^1(\Omega)$ are compactly imbedded into $L^2(\Omega)$.

The trace theorem

Finally in this subsection, we address the question if the functions in Sobolev spaces can be restricted to the boundary of the domain. It is also interesting to define precisely how smooth boundary data have to be so that a function in $H^m(\Omega)$ can assume those boundary data (see [38]).

Theorem 1.7 *Let m, l be positive integers such that $m > l$. Assume that Ω is a bounded open set of class C^k.*
i) There exists a bounded trace operator $\gamma_0 : H^m(\Omega) \to H^{m-1/2}(\Gamma)$. Moreover, γ_0 has a bounded right inverse.
ii) There exists a continuous trace operator

$$\gamma_l : H^m(\Omega) \to \Pi_{j=0}^{l} H^{k-j-1/2}(\Gamma)$$

with the property that

$$\gamma_l \phi = \left(\phi, \frac{\partial \phi}{\partial n}, \dots \frac{\partial^l \phi}{\partial n^l}\right)$$

for every smooth ϕ. The operator γ_l has a bounded right inverse.
iii) $H_0^m(\Omega)$ is the set of all those functions in $u \in H^m(\Omega)$ for which

$$u = \frac{\partial u}{\partial n} = \dots.. = \frac{\partial^{m-1} u}{\partial n^{m-1}} = 0$$

on γ_{m-1}.

1.2.4 Gradient, divergence and curl operators

Definition

Let Ω be an open subset of $\mathbb{R}^n$ (in our case $n = 2$). *For every* $v \in \mathcal{D}'(\Omega)$ *we put*

$$gradv = \left(\frac{\partial v}{\partial x_1}, \frac{\partial v}{\partial x_2}, ..., \frac{\partial v}{\partial x_n} \right), \tag{1.8}$$

which defines the linear differential operator denoted by $grad$ (or $\vec{\nabla}$) from $\mathcal{D}'(\Omega)$ to $\mathcal{D}'(\Omega)^n$. *We define the linear differential operator denoted by div (or* $\nabla \cdot$*) from* $\mathcal{D}'(\Omega)^n$ *to* $\mathcal{D}'(\Omega)$ *by*

$$\nabla \cdot \vec{v} = \sum_{i=1}^{n} \frac{\partial v_i}{\partial x_i} \quad \forall \vec{v} \in \mathcal{D}'(\Omega)^n. \tag{1.9}$$

In the case where $n = 3$, for all $\vec{v} = (v_1, v_2, v_3)$ we put

$$curl(\vec{v}) = \left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}, \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}, \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \right) \tag{1.10}$$

and thus define a linear differential operator from $\mathcal{D}'(\Omega)^3$ into $\mathcal{D}'(\Omega)^3$. In our case $n = 2$ and we write

$$curl(\vec{v}) = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}, \quad \forall \vec{v} = (v_1, v_2) \in \mathcal{D}'(\Omega)^2. \tag{1.11}$$

We also need to introduce, in the case $n = 2$, the linear differential operator, denoted by $Curl$, from $\mathcal{D}'(\Omega)$ into $\mathcal{D}'(\Omega)^2$ defined by

$$Curlv = \left(\frac{\partial v}{\partial x_2}, -\frac{\partial v}{\partial x_1} \right) \quad \forall v \in \mathcal{D}'(\Omega). \tag{1.12}$$

In the case n=2 we remark the diagram [38]:

$$\mathcal{D}'(\Omega) \overset{grad}{\rightarrow} \mathcal{D}'(\Omega)^2 \overset{curl}{\rightarrow} \mathcal{D}'(\Omega)$$
$$\mathcal{D}'(\Omega) \overset{Curl}{\rightarrow} \mathcal{D}'(\Omega)^2 \overset{div}{\rightarrow} \mathcal{D}'(\Omega)$$

or

$$\mathcal{C}^{k+2}(\Omega) \overset{grad}{\rightarrow} \mathcal{C}^{k+1}(\Omega)^2 \overset{curl}{\rightarrow} \mathcal{C}^{k}(\Omega)$$
$$\mathcal{C}^{k+2}(\Omega) \overset{Curl}{\rightarrow} \mathcal{C}^{k+1}(\Omega)^2 \overset{div}{\rightarrow} \mathcal{C}^{k}(\Omega)$$

with

$$curl(gradv) = 0 \quad \forall v \in \mathcal{D}'(\Omega)$$
$$div(Curlv) = 0 \quad \forall v \in \mathcal{D}'(\Omega)$$

thus:

$$Ran(grad) \subset Ker(curl)$$
$$Ran(Curl) \subset Ker(div).$$

<u>The space $H(grad,\Omega)$, $H(div,\Omega)$ and $H(curl,\Omega)$</u>

Let Ω be an open set in $\mathbb{R}^2$. We always shall use the same notation for a space $X(\Omega)$ and for a vector space $X(\Omega)^n$, i.e., we shall denote X^n as X. Let us introduce these spaces:

$$\begin{aligned} H(grad,\Omega) &= \{v \in L^2(\Omega), gradv \in L^2(\Omega)\} \\ H(div,\Omega) &= \{\vec{v} \in L^2(\Omega), \nabla\cdot\vec{v} \in L^2(\Omega)\} \\ H(curl,\Omega) &= \{\vec{v} \in L^2(\Omega), curl(\vec{v}) \in L^2(\Omega)\}. \end{aligned} \tag{1.13}$$

The first space is $H^1(\Omega)$ and the other spaces are Hilbert spaces when equipped with the *div* and *curl* norm respectively (i.e. $\|\vec{u}\|^2_{H(div,\Omega)} = \|\vec{u}\|^2 + \|div(\vec{u})\|^2$ and $\|\vec{u}\|^2_{H(curl,\Omega)} = \|\vec{u}\|^2 + \|curl(\vec{u})\|^2$). We denote the closure of $\mathcal{D}(\Omega)$ in these spaces by $H^1_0(\Omega)$, $H_0(div,\Omega)$ and $H_0(curl,\Omega)$. If the boundary is Lipschitz continuous, the space $\mathcal{D}(\bar{\Omega})$ is dense in these spaces and there exists [38]

$$\begin{aligned} &\gamma_0 : H^1(\Omega) \to H^{1/2}(\Omega) \\ &\gamma_0 v = v|_\Gamma \\ &\gamma_n : H(div,\Omega) \to H^{-1/2}(\Omega) \\ &\gamma_n v = \vec{v}\cdot\vec{n}|_\Gamma \\ &\gamma_\tau : H(curl,\Omega) \to H^{-1/2}(\Omega) \\ &\gamma_\tau = \vec{v}\times\vec{n}|_\Gamma \end{aligned}$$

continuous linear mappings whose kernels are $H^1_0(\Omega)$, $H_0(div,\Omega)$ and $H_0(curl,\Omega)$ respectively.

If Ω is a connected bounded open set the image of the divergence operator is

$$\begin{aligned} &div(H^1(\Omega)) = L^2(\Omega) \\ &div(H^1_0(\Omega)) = \{u \in L^2(\Omega) : \int_\Omega u dx = 0\}. \end{aligned}$$

We shall denote the kernel of the divergence operator by [38]

$$\begin{aligned} &\mathcal{V}(\Omega) = \{\vec{u} \in \mathcal{D}(\Omega), \nabla\cdot\vec{u} = 0\}. \\ &V(\Omega) = \{\vec{u} \in H^1_0(\Omega), \nabla\cdot\vec{u} = 0\} \\ &H_0(div0,\Omega) = \{u \in L^2(\Omega), \quad \nabla\cdot\vec{u} = 0, \quad \vec{u}\cdot\vec{n} = 0\} \end{aligned}$$

and denote $W(\Omega)$ the closure of $\mathcal{V}(\Omega)$ in $L^2(\Omega)$. The sets $V(\Omega)$, $H_0(div0, \Omega)$ and $W(\Omega)$ are closed subspaces of $L^2(\Omega)$ and $\mathcal{V}(\Omega)$ is a dense set in all these subspaces. In other words, the elements of $V(\Omega)$ and those of $W(\Omega)$ can be approximated by regular functions in $\mathcal{D}(\Omega)$ with zero divergence which implies that the dual space of $V(\Omega)$ denoted by $V(\Omega)^*$ can be considered as a space of distributions with zero divergence.
The kernel of the *grad* operator (from $H^1(\Omega)$ into $L^2(\Omega)$), being comprised of constant functions, has dimension 1 for bounded connected Ω. We define

$$\mathcal{H}^1 = \{u \in H^1(\Omega), \nabla^2 u = 0 \quad in \quad \Omega\}$$

and then the space $L^2(\Omega)$ admits the following orthogonal decomposition:

$$\begin{aligned} L^2(\Omega) &= gradH^1(\Omega) \oplus H_0(div0, \Omega) \\ L^2(\Omega) &= gradH_0^1(\Omega) \oplus H(div0, \Omega) \\ L^2(\Omega) &= gradH_0^1(\Omega) \oplus grad\mathcal{H}^1 \oplus H_0(div0, \Omega). \end{aligned}$$

This says that a necessary and sufficient condition for an element $u \in L^2(\Omega)$ to be of the form $u = grad(p)$ with $p \in H^1(\Omega)$ is that u shall be orthogonal to the space $H_0(div0, \Omega)$ or to $\mathcal{V}(\Omega)$.
Let Ω be a bounded and connected open set; then, the kernel of the *curl* in $L^2(\Omega)^n$ is denoted by $H(curl0, \Omega)$ and it is the sum of two orthogonal spaces $grad(H^1)$ and another vector space of finite dimension N (equal to the number of cuts needed to render the open domain simply connected) defined by $\{u \in L^2(\Omega) : curl(u) = 0, div(u) = 0, u \cdot n|_\Gamma\}$. The image of the *curl* (or *Curl* if $n = 2$) is the space

$$curl(H^1) = \{\vec{u} \in L^2(\Omega)^n, \nabla \cdot \vec{u} = 0 \quad \int_\Gamma \vec{u} \cdot \vec{n} d\vec{x} = 0\}$$

which is a closed subspace of the kernel of the divergence

1.2.5 Vector-valued distributions

Let X be a Banach space and (a, b) an open set of $\mathbb{R}$. *We denote by* $L^p((a, b); X)$ *(*$1 \le p < \infty$*) the space of functions*

$$\begin{aligned} & f(t) : (a, b) \to X \\ & \textit{such that } f \textit{ is measurable and} \\ & \|f\|_{L^p((a,b);X)} = \left(\int_a^b \|f(t)\|_X^p \right)^{1/p} \qquad (1.14) \end{aligned}$$

$$(1.15)$$

is finite. We also denote by $L^\infty((a,b);X)$ the space of function f from (a,b) into X such that f is measurable and is bounded almost everywhere over (a,b) and we set

$$\|f\|_{L^\infty((a,b);X)} = \inf_{\|f(t)\| \leq M a.e.} (M).$$

These vector spaces are Banach spaces. If $u \in L^p((a,b);X)$ and $f \in X^*$ then

$$< f, \int_a^b u dt >= \int_a^b < f, u dt > .$$

If $u \in L^1((a,b);X)$ and if X is a space of functions of the variable x (for example $X = L^p(\Omega)$), then u is identified with a function $u(x,t)$ such that $u(t)$ denotes the function $x \to u(x,t)$ for almost all t. The distributional derivative du/dt is identified with the derivative $\frac{\partial u}{\partial t}$ of u in $\mathcal{D}(\Omega \times (a,b))$.
Let (X,Y) be a pair Banach spaces with $X \to Y$ a continuous injection. Then, we have a continuous injection from $\mathcal{D}((a,b);X)$ into $\mathcal{D}((a,b);Y)$ and $L^p((a,b);X)$ is a continuous injection into $L^p((a,b);Y)$.

We consider two real, separable Hilbert spaces V, H. We suppose that V is dense in H so that, by identifying H and its dual H^*, we have continuous injections from $V \to H \to V^*$, where each space being dense in the following.
Let $a, b \in \mathbb{R}$. We denote by $\mathcal{H}^1((a,b);V)$ the space:

$$\mathcal{H}^1((a,b);V) = \{u : u \in L^2((a,b);V), u_t \in L^2((a,b);V^*)\}$$

This space equipped with the norm

$$\|u\|_{\mathcal{H}^1} = (\|u\|^2_{L^2((a,b);V)} + \|u_t\|^2_{L^2((a,b);V^*)})^{1/2}$$

is a Hilbert space. The space $\mathcal{D}([a,b];V)$ is dense in $\mathcal{H}^1((a,b);V)$ and every $u \in \mathcal{H}^1((a,b);V)$ is almost everywhere equal to a continuous function over $[a,b]$ in H. Further the imbedding into $C^0([a,b];H)$ is continuous where the space $C^0([a,b];H)$ is equipped with the norm of uniform convergence.

Finally, we define the anisotropic Sobolev spaces.
Let r and $s \geq 0$ and $Q = (a,b) \times \Omega$. We define

$$H^{r,s}(Q) = L^2((a,b);H^r(\Omega)) \cap H^s((a,b);L^2(\Omega)) \tag{1.16}$$

with the norm

$$\|u\|_{H^{r,s}} = (\|u\|^2_{L^2((a,b);H^r)} + \|u\|^2_{H^s((a,b);L^2)})^{1/2}.$$

1.3 The Navier-Stokes equations

The classical formulation

The standard derivation of the classical fluid dynamics equations in Eulerian form and details can be found in L.Landau and E. Lifschitz [39] and in G.K Batchelor [40]. The Navier-Stokes equations for Newtonian fluids can be written as

$$\begin{aligned}
&\frac{\partial \varrho}{\partial t} + \nabla \cdot (\varrho \vec{u}) = 0 \\
&\frac{\partial(\varrho u_i)}{\partial t} + \nabla \cdot (\varrho \vec{u} u_i) - \sum_{j=1}^{N} \frac{\partial}{\partial x_j} \mu (\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}) - \frac{\partial(\lambda \nabla \cdot \vec{u})}{\partial x_i} + \frac{\partial p}{\partial x_i} = \varrho f_i \\
&\frac{\partial(\varrho e)}{\partial t} + \nabla \cdot (\varrho \vec{u} e) + p \nabla \cdot \vec{u} - \sum_{j=1}^{N} \frac{\partial}{\partial x_j} (k \frac{\partial T}{\partial x_j}) = \frac{\mu}{2} \sum_{j=1}^{N} (\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j})^2 + \lambda (\nabla \cdot \vec{u})^2 \\
&\text{for } 1 \le i \le N
\end{aligned} \tag{1.17}$$

where $\varrho, \vec{u}, e, p, T$ are the density, the velocity of the fluid, the internal energy, the pressure and the temperature. The coefficients μ, λ and k, which are in general function of T and p, are called the first and second viscosity and conduction coefficient, respectively. In order to close the system, it remains to describe two variables as function of the others. The dependent variables must obey some given state equations (generally $p = p(\varrho, T)$, $e = e(\varrho, T)$).

We use the Navier-Stokes equations in the homogeneous, incompressible case. The system can be deduced from eq(1.17) by setting ϱ as a positive constant and by introducing the kinematic viscosity $\nu = \mu(\varrho)/\varrho$ and a reduced pressure field p/ϱ. The classical problem for an incompressible flow problem can be formulated in the following manner. Let Ω be an open and bounded domain in $\mathbb{R}^n, n = 2, 3$ with a Lipschitz-continuous boundary Γ and $\vec{f} \in H^{-1}(\Omega)$ be the given body force per unit mass. We want to find $\vec{u}$ and p, which are the velocity field and the pressure, solving the incompressible Navier-Stokes equations

$$\begin{aligned}
&\frac{\partial \vec{u}}{\partial t} - \nu \nabla^2 \vec{u} + (\vec{u}.\vec{\nabla})\vec{u} + \vec{\nabla} p = \vec{f} \qquad in \quad \Omega \\
&\vec{\nabla}.\vec{u} = 0 \qquad in \quad \Omega
\end{aligned} \tag{1.18}$$

with boundary conditions on Γ and an initial condition $\vec{u}_0$. In the following chapters we shall consider only Dirichlet boundary conditions.

The weak formulation

In order to obtain a weak formulation we can multiply the equation, which represents momentum conservation , by a function $\vec{v} \in H_0^1(\Omega)$ and the conservation of mass equation by a function $q \in L^2_0(\Omega)$. We recall that

$$L^2_0(\Omega) = \{v \in L^2(\Omega) : \int_\Omega v d\vec{x} = 0\}.$$

We now define the bilinear forms

$$a(\vec{u}, \vec{v}) = 2\nu \sum_{i,j=1}^{n} (D_{ij}(\vec{u}), D_{ij}(\vec{v})) \qquad \forall \vec{u}, \vec{v} \in H^1(\Omega) \tag{1.19}$$

$$b(\vec{v}, q) = -\int_\Omega q \vec{\nabla} \cdot \vec{v} d\vec{x} \quad \forall q \in L^2_0 \quad \forall \vec{v} \in H^1(\Omega) \tag{1.20}$$

and the trilinear form

$$c(\vec{w}; \vec{u}, \vec{v}) = \sum_{i,j=1}^{n} \int_\Omega w_j \left(\frac{\partial u_i}{\partial x_j} \right) v_i d\vec{x}. \qquad \forall \vec{w}, \vec{u}, \vec{v} \in H^1(\Omega) \tag{1.21}$$

The Navier-Stokes problem in weak form can be formulated as :
given $\vec{f} \in H^{-1}(\Omega)^n$, *find* $(\vec{u}, p) \in H^1(\Omega) \times L^2_0(\Omega)$ *satisfying*

$$< \frac{d\vec{u}}{dt}, \vec{v} > + a(\vec{u}, \vec{v}) + c(\vec{u}; \vec{u}, \vec{v}) + b(\vec{u}, p) = < f, v > \quad \forall v \in H^1_0(\Omega) \tag{1.22}$$

$$b(\vec{u}, q) = 0 \quad \forall q \in L^2_0(\Omega) \tag{1.23}$$

with boundary conditions and initial velocity field $\vec{u}_0$. $< \cdot, \cdot >$ denotes the duality pairing between $H^1_0(\Omega)$ and $H^{-1}(\Omega)$. If the boundary conditions are homogeneous Dirichlet boundary conditions we can take the test function in $V(\Omega)$ and the linear form $b(\vec{v}, p)$ vanishes. We have only the equation

$$< \frac{d\vec{u}}{dt}, \vec{v} > + a(\vec{u}, \vec{v}) + c(\vec{u}; \vec{u}, \vec{v}) = < f, v > \quad \forall v \in V(\Omega). \tag{1.24}$$

The equivalence with the classical problem is now clear. Every function in $\mathcal{D}'(\Omega)$ orthogonal to $\mathcal{V}(\Omega)$, which is dense in $V(\Omega)$, can be written as a gradient of some p in $\mathcal{D}'(\Omega)$. Specific boundary conditions and properties of the linear forms will be introduced in the next chapters.

For details concerning notation employed one may consult [36], [41] and [35]. For the study of the weak solutions we can refer to the celebrated results of J.Leray [42] and O.A. Ladyzhenskaya [43]. Some recent regularity results can be found in [35], in [44] and in [45].

Chapter 2

Velocity tracking problem with quadratic functional control

2.1 Introduction

In this chapter, and in the following chapters, we present systematic approaches to the mathematical formulation and numerical resolution of the time dependent control problem of tracking the velocity for Navier-Stokes flow in a bounded two-dimensional domain.

The control acts as a distributed force, the state of the system $(\vec{u}, p)$ is the solution of the Navier-Stokes system of partial differential equations modelling the flow evolution and the cost function is some quadratic functional involving the state and the control variables. The minimum of this functional corresponds to the minimum possible distance, measured in the L^2 norm, between a given target velocity $\vec{U}$ and the state velocity $\vec{u}$, within the bounds of the control variable. The norm of the distance and the control norm should be scaled judiciously in order to yield a good velocity tracking. We consider two-dimensional flows because the Navier-Stokes equations may not be well posed and the corresponding Frechet map may not be differentiable in three-dimensions.

In this chapter, we will formulate the problem in a convenient mathematical way, then we will prove the existence of an optimal control, and characterize such an optimal control by deriving the first-order necessary optimal conditions associated with the problem. Finally, once the optimality conditions are available, we will write down a gradient algorithm and numerically determine a solution through suitable approximations. The most arduous difficulties for this quadratic functional control problem stem from the fact that the mathematical model does not fit extremely well to the physical idea of tracking the velocity. Clearly, when one minimizes this functional, one is no longer purely minimizing the difference between the control velocity and the desired velocity. However, the minimization of this quadratic functional with respect to the state and control variables can still drive the state velocity $\vec{u}$ to the target velocity $\vec{U}$ over a period of time and effectively limit the size of the control as well.

In section 2.2, we will treat the continuous optimal control problem. In section 2.3 and 2.4, we will analyze the semidiscrete approximation and the fully discrete space-time approximation, respectively. Finally, in section 2.5, some numerical experiments are performed and compared.

2.2 Distributed control problem

2.2.1 Formulation of the optimal control problem

In this section, we describe the problem of time distributed control for the Navier-Stokes equations that models the velocity tracking problem through a quadratic functional. This problem reflects the desire to steer, over time, a candidate velocity field $\vec{u}$ to a target velocity field $\vec{U}$ by appropriately controlling the body force. The choice of a quadratic functional reflects the desire to analyze the problem from the simplest mathematical point of view.

Classical formulation

We consider a two-dimensional flow over the physical domain Ω with boundary Γ. The equations considered here are the nondimensional incompressible Navier-Stokes equations on the interval of time $(0, T)$

$$\begin{cases} \vec{u}_t(t,\vec{x}) + (\vec{u}\cdot\vec{\nabla})\vec{u}(t,\vec{x}) - \nu\nabla^2\vec{u}(t,\vec{x}) + \vec{\nabla}p(t,\vec{x}) = \vec{f}(t,\vec{x}) - \vec{F}(t,\vec{x}) \\ \vec{\nabla}\cdot\vec{u}(t,\vec{x}) = 0 \end{cases} \tag{2.1}$$

with initial velocity $\vec{u}(0,\vec{x}) = \vec{u}_0(\vec{x})$ for $\vec{x} \in \Omega$ and boundary condition $\vec{u}(t,\vec{x}) = 0$ for $\vec{x} \in \Gamma$ and $t \in [0, T]$. The vector $\vec{u} = (u_1, u_2)$ is the velocity vector, p is the pressure and ν is the kinematics viscosity. We note that the Reynolds number is equal to $1/\nu$. The function $\vec{u}_0$ must be divergence free and must satisfy the boundary conditions. The control is a distributed control: the right-hand side of the momentum equation in (2.1), i.e. the volume force $\vec{f}(t,\vec{x})$. The optimal control problem is formulated as follows:

Find a control $\vec{f}$ minimizing the cost function

$$\begin{aligned} L(\vec{f}) = \quad & \frac{\alpha}{2}\int_0^T\int_\Omega(\vec{u}-\vec{U})^2 d\vec{x}dt + \frac{\beta}{2}\int_0^T\int_\Omega(\vec{f}-\vec{F})^2 d\vec{x}dt + \\ & \frac{\gamma}{2}\int_\Omega(\vec{u}(T,\vec{x})-\vec{U}(T,\vec{x}))^2 d\vec{x} \end{aligned} \tag{2.2}$$

where $\vec{u}$ is solution of eq(2.1).

This means that we act upon the system by changing the body forces around the function $\vec{F}(t,\vec{x})$, which is a fixed body force. The term $\vec{F}(t,\vec{x})$ has been introduced for completeness and we usually set $\vec{F}(t,\vec{x}) = 0$ or define $\vec{g} = \vec{f}(t,\vec{x}) - \vec{F}(t,\vec{x})$ as the total external body force.

The minimization of the $\int_0^T \int_\Omega (\vec{u} - \vec{U})^2 d\vec{x}dt$ term is the real goal of the velocity tracking problem; the $\int_0^T \int_\Omega (\vec{f} - \vec{F})^2 d\vec{x}dt$ term is introduced in order to bound the control function and to prove the existence of an optimal control. The $\int_\Omega (\vec{u}(T,\vec{x}) - \vec{U}(T,\vec{x}))^2 d\vec{x}$ term is necessary in order to keep the solution $\vec{u}$ close to $\vec{U}$ at the time T.

Weak formulation of the LQF control problem

We consider an open bounded set $\Omega \subset \mathbb{R}^2$ with a Lipschitz-continuous boundary Γ. We begin by defining the particular target field $\vec{U}$ for which we made some assumptions.

$\vec{U}$ is said to be in the set of admissible target velocities U_{ad} if

$$\begin{cases} \vec{U} = \vec{U}(t,\vec{x}) \in C([0,T]; H^2(\Omega)) \\ \vec{\nabla}_{\vec{x}} \cdot \vec{U}(t,\vec{x}) = 0 \quad \forall \vec{x} \in \Omega \\ \vec{U}(t,\vec{x}) = 0 \quad \forall \vec{x} \in \Gamma \\ \vec{F}_{\vec{U}}(t,\vec{x}) = \vec{U}_t(t,\vec{x}) - \nu \nabla^2 \vec{U}(t,\vec{x}) + (\vec{U}(t,\vec{x}) \cdot \vec{\nabla}) \vec{U}(t,\vec{x}) \end{cases} \tag{2.3}$$

where $\vec{F}_{\vec{U}} \in L^\infty((0,T); L^2(\Omega))$.

Let $\vec{u} \in L^2((0,T); H_0^1(\Omega))$ and $p \in L^2((0,T); L_0^2(\Omega))$ denote the state variables, i.e. the velocity and pressure fields, respectively, and let $\vec{g} = \vec{f} - \vec{F} \in L^2((0,T); L^2(\Omega))$ denote the distributed control. The state variables are constrained to satisfy the weak form of the Navier-Stokes equations for a.e. t in $[0,T]$, i.e

$$\begin{cases} (\vec{u}_t, \vec{v}) + \nu a(\vec{u}, \vec{v}) + c(\vec{u}; \vec{u}, \vec{v}) + b(\vec{v}, p) = (\vec{g}, \vec{v}) \qquad \forall v \in H_0^1(\Omega) \\ b(\vec{u}, q) = 0 \quad \forall q \in L_0^2(\Omega) \\ \vec{u} = 0 \qquad \vec{x} \in \Gamma \\ \vec{u}(0,\vec{x}) = \vec{u}_0(\vec{x}) \qquad \vec{\nabla} \cdot \vec{u}_0 = 0 \qquad \vec{u}_0 \in L^2(\Omega) \end{cases} \tag{2.4}$$

More precisely, *given $\vec{g} \in L^2((0,T); H^{-1}(\Omega))$ and $\vec{u}_0, \in V(\Omega)$ then $(\vec{u}, p) \in (L^2((0,T); H_0^1(\Omega)) \times L^2((0,T); L_0^2(\Omega)))$ is called a weak solution for the Navier-Stokes equations if it satisfies the equations (2.4) with initial velocity $\vec{u}_0$.*

If $\vec{u}$ is a solution of the eq(2.1), then it is also solution of the weak formulation eq(2.4). If $\vec{u}$ is solution of eq(2.4), then it satisfies eq(2.1) in the distribution sense on $(0,T)$. If $\vec{g}, \vec{u}_0$ are given as above, then we can recall that there exists a unique admissible weak solution $(\vec{u}, p)$ of (2.4), such that $\vec{u} \in C([0,T]; H_0^1(\Omega)) \cap L^2((0,T); H_0^1(\Omega))$ and $\vec{u}_t \in L^2((0,T); H^{-1}(\Omega))$.

We note that if the control $\vec{g} \in L^2((0,T); H^{-1}(\Omega))$, then $\vec{u} \in L^2((0,T); H_0^1(\Omega))$ or, equivalently, $\vec{u}$ in $\mathcal{H}((0,T)\times\Omega)$ (see [35]).

Since we need a certain regularity for the solutions, we choose the control function $\vec{g}$ to belong to $L^2((0,T); L^2(\Omega))$. An admissible solution for our optimal control problem can be defined as follows.

Given T , $\vec{g} \in L^2((0,T); L^2(\Omega))$, $\vec{u}_0$ a divergence free vector in $H_0^1(\Omega)$ and $\vec{U} \in U_{ad}$, then a weak solution of the eq(2.4) $(\vec{u}, p, \vec{g})$ is called an admissible solution for our optimal control problem if $u \in \mathcal{H}((0,T)\times\Omega)$, $p \in L^2((0,T); L_0^2(\Omega))$, and the functional $L(\vec{g})$ is bounded.

The set of all admissible solutions is defined as A_d.

P_L form of the weak formulation

The optimal control problem, in the "P_L form", can be formulated as follows:

Let $\vec{g} \in L^2((0,T); L^2(\Omega))$ and $\vec{u}_0 \in V(\Omega)$. Given $\vec{U} \in U_{ad}$, find $(\vec{u}, p, \vec{g}) \in A_d$ such that the control $\vec{g}$ minimizes the cost function

$$\begin{aligned} L(\vec{g}) = \quad & \int_0^T \int_\Omega [\frac{\alpha}{2}(\vec{u} - \vec{U})^2 + \frac{\beta}{2}(\vec{g})^2] d\vec{x} dt + \\ & \frac{\gamma}{2}\int_\Omega (\vec{u}(T) - \vec{U}(T))^2 d\vec{x} \end{aligned} \tag{2.5}$$

The weak formulation of the Navier-Stokes system under the operator form is probably the best formulation for studying our problem. Let us define the operators A, B and C in the usual way (see [36]):

$$(A\vec{u}, \vec{v}) = a(\vec{u}, \vec{v}) \qquad \forall \vec{u} \in H^1(\Omega), \vec{v} \in H_0^1(\Omega) \tag{2.6}$$
$$(C(\vec{w})\vec{u}, \vec{v}) = c(\vec{w}; \vec{u}, \vec{v}) \qquad \forall \vec{w}, \vec{u} \in H^1(\Omega), \vec{v} \in H_0^1(\Omega) \tag{2.7}$$
$$(B\vec{v}, q) = b(\vec{v}, q) \qquad \forall \vec{v} \in H^1(\Omega) \qquad \forall q \in L_0^2(\Omega). \tag{2.8}$$

With this notation the formulation of the problem P_L becomes:

given $\vec{g} \in L^2((0,T); L^2(\Omega))$, $\vec{u}_0$ in $V(\Omega)$ and $\vec{U} \in U_{ad}$, find $(\vec{u}, p, \vec{g})$ in $L^2((0,T); H_0^1(\Omega)) \times L^2((0,T); L_0^2(\Omega)) \times L^2((0,T); L^2(\Omega))$ such as that $(\vec{u}, p)$ is the solution of

$$P_L \begin{cases} \vec{u}_t + \nu A\vec{u} + C(\vec{u})\vec{u} + B^* p = \vec{g} \\ B\vec{u} = 0 \\ \vec{u} = 0 \qquad \vec{x} \in \Gamma \\ \vec{u}(0, \vec{x}) = \vec{u}_0(\vec{x}) \quad \vec{x} \in \Omega \end{cases} \tag{2.9}$$

and such that $\vec{g}$ minimizes the cost function in eq(2.5).

<u>Q_L form of the weak formulation</u>

Finally, it is very useful to associate with the optimal control problem P_L the optimal control problem Q_L formulated in the space $V(\Omega)$. To this purpose, we set $V(\Omega) = Ker(B)$ and note that this definition is consistent with the previous definition of the space $V(\Omega)$. We set also the usual projections πA and πC of the operators A and C, respectively, on the space $V(\Omega)$. The auxiliary optimal control problem Q_L can be formulated as follows: *given $\vec{g} \in L^2((0,T); L^2(\Omega))$, $\vec{u}_0 \in V(\Omega)$ and $\vec{U} \in U_{ad}$, find $(\vec{u}, \vec{g})$ in $L^2((0,T); V(\Omega)) \times L^2((0,T); W(\Omega))$ such as that $\vec{u}$ is the solution of*

$$Q_L \begin{cases} \vec{u}_t + \nu(\pi A)\vec{u} + (\pi C)(\vec{u})\vec{u} + = \pi\vec{g} \\ \vec{u}(0, \vec{x}) = \vec{u}_0(\vec{x}) \in V(\Omega) \end{cases} \tag{2.10}$$

and $\vec{g}$ minimizes the cost function in eq(2.5).
We can recall that if $\Gamma \in C^2$ and also that $\vec{u}_0 \in V(\Omega)$ and $\vec{g} \in L^2((0,T); W(\Omega))$, then the solution of the equation in (2.10) is in $C([0,T]; V(\Omega)) \cap L^2((0,T); H^2 \cap H_0^1)$ and $\vec{u}_t \in L^2((0,T); W(\Omega))$ [35].
Of course, a solution of the optimal control problem Q_L is also a solution for the optimal control problem P_L. The converse can be also established. We remark that the weak formulation in eq(2.10) is equivalent to the classical formulation in the distribution sense on $(0,T)$.

2.2.2 Existence of the optimal control solution

We prove the existence for the problem Q_L which implies the existence of the optimal control problem in P_L. Here Ω is an open bounded domain with a Lipschitz-continuous boundary Γ.

Theorem 2.1 *Given $\vec{u}_0 \in W(\Omega)$, then there exists a solution $\vec{g} \in L^2((0,T); L^2(\Omega))$ and $\vec{u} \in C([0,T];, W(\Omega)) \cap L^2((0,T); V(\Omega))$ of the problem Q_L defined in (2.10).*

Proof: As the set A_d is bounded and not empty ($(\vec{u}, p, 0) \in A_d$), let $\vec{g}_n$ be a minimizing sequence for the problem (2.10) and set $\vec{u}_n = \vec{u}(\vec{g}_n)$. The sequence $\{\vec{g}_n\}$ is bounded in $L^2((0,T); L^2(\Omega))$ and the corresponding solution $\vec{u}_n$ is bounded in $C([0,T]; W(\Omega)) \cap L^2((0,T); V(\Omega))$. This follows from the well-known existence theorems for solutions of unsteady N.S. equations [35]. As the Hilbert space is a reflexive space, every ball is weakly compact and thus there is a pair $(\vec{u}, \vec{g})$ and a subsequence of $(\vec{u}_n, \vec{g}_n)$ that converges weakly to $(\vec{u}, \vec{g})$. We abuse the notation and write again that

$$\begin{array}{llll} \vec{g}_n \rightarrow \vec{g} & in & L^2((0,T); L^2(\Omega)) & weakly \\ \vec{u}_n \rightarrow \vec{u} & in & L^2((0,T); V(\Omega)) & weakly \\ \vec{u}_n \rightarrow \vec{u} & in & L^\infty((0,T); W(\Omega)) & * -weakly \end{array}$$

Now the pair $(\vec{u}, \vec{g})$ satisfies the Navier-Stokes equations (2.9) and minimizes the functional. In fact, by the lower semicontinuity of the functional in (2.5) we have

$$\int_0^T \int_\Omega \vec{g}^2 d\vec{x}dt \leq \liminf_{n\to\infty} \int_0^T \int_\Omega \vec{g}_n^2 d\vec{x}dt$$

$$\int_0^T \int_\Omega \vec{u}^2 d\vec{x}dt \leq \liminf_{n\to\infty} \int_0^T \int_\Omega \vec{u}_n^2 d\vec{x}dt$$

$$\int_\Omega \vec{u}^2(T) d\vec{x} \leq \liminf_{n\to\infty} \int_\Omega \vec{u}_n^2(T) d\vec{x}$$

which implies

$$L(\vec{g}) \leq \liminf_{n\to\infty} L(\vec{g}_n).$$

Besides, a priori estimate (see [35] or [44]) for $\vec{u}$ in a fractional time order Sobolev space yields in our case that $\vec{u}_n$ converges strongly to $\vec{u} \in L^2((0,T); V(\Omega))$. Now we consider the weak Navier-Stokes system in eq(2.10) with state $\vec{u}_n$ and control $\vec{g}_n$. Let $\vec{w}$ be in $\mathcal{V}(\Omega)$ and $\psi(t)$ be a continuously differentiable function on $[0,T]$ with $\psi(T) = 0$. We multiply eq(2.10) by $\psi(\tau)\vec{w}$ and then integrate by parts in τ

$$-\int_0^T (\vec{u}_n, \psi'(\tau)\vec{w})d\tau + \nu \int_0^T a(\vec{u}_n, \psi(\tau)\vec{w})d\tau + \int_0^T c(\vec{u}_n; \vec{u}_n, \psi(\tau)\vec{w})d\tau =$$
$$(\vec{u}_0, \psi(0)\vec{w}) + \int_0^T (\vec{g}_n, \psi(\tau)\vec{w})d\tau.$$

We can pass to the limit inside the linear and the nonlinear terms. In fact if $\vec{u}_n$ converges to $\vec{u}$ in $L^2((0,T); V(\Omega))$ weakly and $L^2((0,T); W(\Omega))$ strongly, then for any $\vec{z} \in C^1((0,T); \mathcal{D}(\Omega))$ we have

$$\lim_{n\to\infty} \int_0^T c(\vec{u}_n; \vec{u}_n, \vec{z}(\tau))d\tau = \int_0^T c(\vec{u}; \vec{u}, \vec{z}(\tau))d\tau \tag{2.11}$$

(see [35]). If $\psi = \phi \in \mathcal{D}(0,T)$ the pair $(\vec{u}, \vec{g})$ satisfies the Navier-Stokes equations (2.10) in the distribution sense. Since $\mathcal{V}(\Omega)$ is dense in $V(\Omega)$, then is still true for any $\vec{w}$ in $V(\Omega)$ by a continuity argument. $\sqcup$

2.2.3 First-order necessary condition

In this section we proceed to derive the first-order necessary condition associated with the problem (2.10). We will show that the optimal solution must satisfy the first-order necessary condition. By studying the case in which the Gateaux derivative vanishes we can get a possible candidate for the optimal control solution. Before stating the equation for this possible optimal solution we need some auxiliary results.

Lemma 2.1 *Let $\vec{w}$ be the solution of this system of equations in weak form*

$$\begin{cases} \vec{w}_t + \nu(\pi A)\vec{w} + \delta[(\pi C)(\vec{w})\vec{u} + (\pi C)(\vec{u})\vec{w}] + \sigma(\pi C)(\vec{w})\vec{w} = \vec{g} \\ \vec{w} \in V(\Omega) \end{cases} \tag{2.12}$$

with initial value $\vec{w}(0,\vec{x}) = 0$ and homogeneous boundary condition. The solution $\vec{w}$ for positive real values of δ and σ has the following properties:
i) if Ω is an open bounded set with Lipschitz-continuous boundary Γ, $\vec{g} \in L^2((0,T);V^*(\Omega))$ and $\vec{u} \in L^\infty((0,T);W(\Omega)) \cap L^2((0,T);V(\Omega))$, then the solution $\vec{w}$ of (2.12), belongs to $L^\infty((0,T);W(\Omega)) \cap L^2((0,T);V(\Omega))$ and

$$\int_0^T \|\vec{w}\|_1^2 dt \leq C_1 \int_0^T \|\vec{g}\|_{V^*}^2 dt$$

where C_1 is a constant depending on Ω and ν.
ii) If we assume also that $\Omega \in C^2$, $\vec{u} \in L^\infty((0,T);V(\Omega)) \cap L^2((0,T);H^2(\Omega))$ and $\vec{g} \in L^2((,T);W(\Omega))$ then $\vec{w} \in L^\infty((0,T);V(\Omega)) \cap L^2((0,T);H^2(\Omega))$ and

$$\|\vec{w}\|_1^2(t) \leq C_2 \int_0^T \|\vec{g}\|^2 d\tau \quad \forall t \in (0,T)$$

where C_2 is a constant depending on Ω and ν.

Proof: i) The proof can be achieved using the standard techniques based on the weak formulation of eq(2.12). We take the scalar product with $\vec{w}$ and using the orthogonality propriety of the trilinear form

$$c(\vec{w};\vec{u},\vec{v}) = -c(\vec{w};\vec{v},\vec{u}) \quad c(\vec{w};\vec{u},\vec{u}) = 0 \quad \forall \vec{u},\vec{w},\vec{v} \in H^1(\Omega) \tag{2.13}$$

we obtain

$$\frac{d}{dt}\|\vec{w}\|^2 + 2\nu\|\nabla\vec{w}\|^2 + 2\delta c(\vec{w};\vec{u},\vec{w}) = 2(\vec{g},\vec{w}).$$

The trilinear form and the right-hand side can be bounded using a particular two dimensional estimate and the Poincare's and Young's inequalities respectively as

$$|c(\vec{w};\vec{u},\vec{w})| \leq \sqrt{2}\|\nabla\vec{u}\| \cdot \|\vec{w}\| \cdot \|\nabla\vec{w}\| \quad \forall \vec{u},\vec{w} \in H_0^1(\Omega)$$

$$|(\vec{g},\vec{w})| \leq \|\vec{w}\|_V \|\vec{g}\|_{V^*} \leq \lambda\|\nabla\vec{w}\| \cdot \|\vec{g}\|_{V^*} \leq \frac{\nu}{2}\|\nabla\vec{w}\|^2 + \frac{\lambda}{2\nu}\|\vec{g}\|_{V^*}^2.$$

Thus we have

$$\frac{d}{dt}\|\vec{w}\|^2 + \nu\|\nabla\vec{w}\|^2 \leq \frac{\lambda}{\nu}\|\vec{g}\|_{V^*}^2 + 2\delta\sqrt{2}\|\nabla\vec{u}\| \cdot \|\vec{w}\| \cdot \|\nabla\vec{w}\|$$

where λ can be written in terms of Poincare's constant. Applying again the Young's inequality to the last right-hand side term we have

$$\frac{d}{dt}\|\vec{w}\|^2 + \frac{\nu}{2}\|\nabla\vec{w}\|^2 \leq \frac{\lambda}{\nu}\|\vec{g}\|^2_{V^*} + \phi(t)\|\vec{w}\|^2 \tag{2.14}$$

or

$$\frac{d}{dt}[\|\vec{w}\|^2 \exp\left(-\int_0^t \phi(\tau)d\tau\right)] + \frac{\nu}{2}\|\nabla\vec{w}\|^2 \leq \frac{\lambda}{\nu}\|\vec{g}\|^2_{V^*}$$

where $\phi(t) = \frac{\sqrt{2}\delta}{\nu}\|\nabla\vec{u}\|^2$ is an integrable function ($\vec{u} \in L^2((0,T);V(\Omega))$). After integration over the interval $(0,t)$ we have

$$\|\vec{w}\|^2(t)e^{(-\int_0^t \phi(\tau)d\tau)} + \frac{\nu}{2}\int_0^t \|\nabla\vec{w}\|^2(\tau)d\tau \leq \frac{\lambda}{\nu}\int_0^T \|\vec{g}\|^2_{V^*}(\tau)d\tau \tag{2.15}$$

as $\vec{w} = 0$ at $t = 0$. From eq(2.15) it follows that $\|\vec{w}\|^2$ is bounded and so $\vec{w} \in L^\infty((0,T);W(\Omega))$. For $t = T$ also it follows that

$$\int_0^T \|\nabla\vec{w}\|^2 dt \leq C' \int_0^T \|\vec{g}\|^2_{V^*} dt.$$

As

$$\int_0^T \|\vec{w}\|_1^2 dt \leq C'(1 + C_p)\int_0^T \|\vec{g}\|^2_{V^*} dt \leq C_1 \int_0^T \|\vec{g}\|^2_{V^*} dt$$

then $\vec{w} \in L^2((0,T);V(\Omega))$.

ii) In the same manner we proceed to evaluate an estimate for $\|\nabla\vec{w}\|$. We need to estimate only the term $\|\nabla\vec{w}\|^2$. We recall that (πA) is an operator from $L^2(\Omega) \to W(\Omega)$ and $\|(\pi A)\vec{w}\|$ is equivalent to the $H^2(\Omega)$ norm on $H^2(\Omega) \cap V(\Omega)$ (see [44] for more details). We take the scalar product of the eq(2.12) with $(\pi A)\vec{w}$ to obtain

$$\begin{aligned}&\frac{d}{dt}\|\nabla\vec{w}\|^2 + 2\nu\|(\pi A)\vec{w}\|^2 = \\ &-2\delta[c(\vec{w};\vec{u},(\pi A)\vec{w}) + c(\vec{u};\vec{w},(\pi A)\vec{w})] - 2\sigma c(\vec{w};\vec{w},(\pi A)\vec{w}) + 2(\vec{g},(\pi A)\vec{w}).\end{aligned}$$

Now we use [35]

$$|c(\vec{a};\vec{b},\vec{d})| \leq K_1\|\vec{a}\|^{\frac{1}{2}}\|\nabla\vec{a}\|^{\frac{1}{2}}\|\nabla\vec{b}\|^{\frac{1}{2}}\|(\pi A)\vec{b}\|^{\frac{1}{2}}\|\vec{d}\| \tag{2.16}$$

$\forall(\vec{a};\vec{b},\vec{d}) \in V(\Omega) \times (V(\Omega) \cap H^2(\Omega)) \times W(\Omega)$ in order to obtain

$$\begin{aligned}&\frac{d}{dt}\|\nabla\vec{w}\|^2 + 2\nu\|(\pi A)\vec{w}\|^2 \leq \|\vec{g}\| \cdot \|(\pi A)\vec{w}\| + \\ &2\delta K_1[\|\nabla\vec{w}\|^{\frac{1}{2}}\|\vec{w}\|^{\frac{1}{2}}\|\nabla\vec{u}\|^{\frac{1}{2}}\|(\pi A)\vec{u}\|^{\frac{1}{2}}\|(\pi A)\vec{w}\| + \\ &\|\nabla\vec{u}\|^{\frac{1}{2}}\|\vec{u}\|^{\frac{1}{2}}\|\nabla\vec{w}\|^{\frac{1}{2}}\|(\pi A)\vec{w}\|^{\frac{3}{2}}] + 2\sigma K_1\|\nabla\vec{w}\| \cdot \|\vec{w}\|^{\frac{1}{2}}\|(\pi A)\vec{w}\|^{\frac{3}{2}}\end{aligned}$$

and once again Poicare's inequality in order to obtain

$$\frac{d}{dt}\|\nabla\vec{w}\|^2 + 2\nu\|(\pi A)\vec{w}\|^2 \leq \|\vec{g}\|\cdot\|(\pi A)\vec{w}\| +$$
$$2\delta[K_3\|\nabla\vec{w}\|\cdot\|\nabla\vec{u}\|^{\frac{1}{2}}\|(\pi A)\vec{u}\|^{\frac{1}{2}}\|(\pi A)\vec{w}\| + K_2\|\nabla\vec{u}\|\cdot\|\nabla\vec{w}\|^{\frac{1}{2}}\|(\pi A)\vec{w}\|^{\frac{3}{2}}] +$$
$$2\sigma K_1\|\nabla\vec{w}\|\cdot\|\vec{w}\|^{\frac{1}{2}}\|(\pi A)\vec{w}\|^{\frac{3}{2}}$$

We apply Young's inequality in the generalized form

$$ab \leq \frac{c}{p}a^p + \frac{1}{p' c^{\frac{p'}{p}}}b^{p'}$$

where $a, b > 0, 1 < p < +\infty, c > 0, p' = p/(p-1)$, to all four terms on the right-hand side in order to get the common term $2\nu/5\|(\pi A)\vec{w}\|^2$. The final result is

$$\frac{d}{dt}\|\nabla\vec{w}\|^2 + \frac{2\nu}{5}\|(\pi A)\vec{w}\|^2 \leq K_4\|\vec{g}\|^2 +$$
$$K_4(\|\nabla\vec{w}\|^2\|\nabla\vec{u}\|\cdot\|(\pi A)\vec{u}\| + \|\nabla\vec{u}\|^4\|\nabla\vec{w}\|^2 + \|\nabla\vec{w}\|^4\|\vec{w}\|^2)$$

where K_4 is the largest constant that appears in the right-hand side terms. If we set $\phi(t) = K_4(\|\nabla\vec{u}\|\cdot\|(\pi A)\vec{u}\| + \|\nabla\vec{u}\|^4 + \|\nabla\vec{w}\|^2\|\vec{w}\|^2)$, we obtain

$$\frac{d}{dt}\|\nabla\vec{w}\|^2 + \frac{2\nu}{5}\|(\pi A)\vec{w}\|^2 \leq K_4\|\vec{g}\|^2 + \phi(t)\|\nabla\vec{w}\|^2 \tag{2.17}$$

or

$$\frac{d}{dt}[\|\nabla\vec{w}\|^2 \exp(-\int_0^t \phi(\tau)d\tau)] + \frac{2\nu}{5}\|(\pi A)\vec{w}\|^2 \leq K_4\|\vec{g}\|^2.$$

From part i) we have that $\vec{u} \in L^2((0,T); H^2(\Omega)) \cap L^\infty((0,T); V(\Omega))$ and $\vec{w}$ in $L^2((0,T); V)$ $\cap\, L^\infty((0,T); W)$, The function $\phi(t)$ is integrable and the integral exponential function is well defined. In fact

$$\int_0^T (\|\nabla\vec{u}\|\cdot\|(\pi A)\vec{u}\| + \|\nabla\vec{u}\|^4 + \|\nabla\vec{w}\|^2\|\vec{w}\|^2)dt \leq T\|\vec{u}\|^4_{L^\infty((0,T);V)} +$$
$$\|\vec{u}\|_{L^\infty((0,T);V)}\int_0^T \|(\pi A)\vec{u}\|dt + \|\vec{w}\|^2_{L^\infty((0,T);W)}\int_0^T \|\nabla\vec{w}\|^2 dt \leq \infty.$$

After integration over the interval $(0,t)$ eq(2.2.3) can be written as

$$\|\nabla\vec{w}\|^2(t)e^{-\int_0^t \phi(\tau)d\tau} + \frac{2\nu}{5}\int_0^t \|(\pi A)\vec{w}\|^2(\tau)d\tau \leq K_4\int_0^T \|\vec{g}\|^2 d\tau. \tag{2.18}$$

From eq(2.18) follows that $\|\nabla\vec{w}\|^2$ is bounded and $\vec{w} \in L^\infty((0,T); V(\Omega))$. Since

$$\|\vec{w}\|_1^2 = \|\vec{w}\|^2 + \|\nabla\vec{w}\|^2 \leq (1 + C_p)\|\nabla\vec{w}\|^2$$

we have

$$\|\vec{w}\|_1^2(t) \leq C(t)\int_0^T \|\vec{g}\|^2 d\tau \leq C_2 \int_0^T \|\vec{g}\|^2 d\tau \quad \forall t \in (0,T)$$

as $C(t)$ is a bounded function. For $t = T$ from eq(2.18) it follows that

$$\int_0^T \|(\pi A)\vec{w}\|^2 dt \leq \frac{5K_4}{2\nu}\int_0^T \|\vec{g}\|^2 dt.$$

If we recall that the norm $\|(\pi A)\vec{w}\|$ is equivalent to the $H^2(\Omega)$ norm on $H^2(\Omega) \cap V(\Omega)$, then $\vec{w} \in L^2((0,T); H^2(\Omega) \cap V(\Omega))$.⊔
We can remark that the existence of the solution of the eq(2.12) can be proved with the standard technique used for the Navier-Stokes equations. Now it is very useful to state and prove here some results concerning the differentiability of on the operator πC

Lemma 2.2 *We have:*
i) Given $\vec{a} \in V(\Omega)$, $\pi C(\vec{a})\vec{a}$ is a differentiable form from $V(\Omega)$ into $V^*(\Omega)$ and

$$(\pi C)'(\vec{a})\vec{b} = (\pi C)(\vec{a})\vec{b} + (\pi C)(\vec{b})\vec{a} \quad \forall \vec{b} \in V(\Omega) \tag{2.19}$$

ii) the adjoint $(\pi C)'^*$ of $(\pi C)'$ defined by $((\pi C)'(\vec{a})\vec{b}, \vec{c}) = (\vec{b}, (\pi C)'^*(\vec{a})\vec{c})$, in the case of the Navier-Stokes system, takes the form

$$((\pi C)'^*(\vec{a}) \cdot \vec{b}, \vec{c}) = \int_\Omega \sum_{i,j=1}^2 c_j \left(\frac{\partial a_i}{\partial x_j} b_i - a_i \frac{\partial b_j}{\partial x_i} \right) d\vec{x} \tag{2.20}$$

Proof: i) We need to show that

$$\sup_{\vec{v} \in V - \{0\}} \left(\frac{|((\pi C)(\vec{b})\vec{b} - (\pi C)(\vec{a})\vec{a} - (\pi C)'(\vec{a})(\vec{b} - \vec{a}), \vec{v})|}{\|\vec{b} - \vec{a}\| \cdot \|\vec{v}\|} \right)$$

vanishes when $\|\vec{b} - \vec{a}\|$ tends to zero. We have from the definition

$$\begin{aligned} &((\pi C)(\vec{b})\vec{b}, \vec{v}) - (\pi C)(\vec{a})\vec{a}, \vec{v}) - ((\pi C)'(\vec{a})(\vec{b} - \vec{a}), \vec{v}) = \\ &c(\vec{b}, \vec{b}; \vec{v}) - c(\vec{a}, \vec{a}; \vec{v}) - c(\vec{a}, \vec{b} - \vec{a}; \vec{v}) - c(\vec{b} - \vec{a}, \vec{a}; \vec{v}) = \\ &c(\vec{b} - \vec{a}, \vec{b} - \vec{a}; \vec{v}). \end{aligned} \tag{2.21}$$

From the continuity property of the trilinear form $c(\vec{u}; \vec{v}, \vec{w})$ we have

$$\sup_{\vec{v} \in V - \{0\}} \left(\frac{|((\pi C)(\vec{b})\vec{a} - (\pi C)(\vec{a})\vec{b} - (\pi C)'(\vec{a})(\vec{b} - \vec{a}), \vec{v})|}{\|\vec{b} - \vec{a}\| \cdot \|\vec{v}\|} \right) \leq K\|\vec{b} - \vec{a}\|$$

which gives the result as $\|\vec{b}-\vec{a}\|$ tends to zero.
ii) From the definition of the trilinear form in this particular case we use integration by parts to obtain:

$$(\vec{c},(\pi C)'(\vec{a})\vec{b}) = \int_\Omega \sum_{i,j=1}^{2} c_j \left(\frac{\partial a_j}{\partial x_i} b_i - a_j \frac{\partial b_j}{\partial x_j}\right) d\vec{x} =$$

$$\int_\Omega \sum_{i,j=1}^{2} \left(\frac{\partial a_i}{\partial x_j} b_i - a_i \frac{\partial b_j}{\partial x_i}\right) c_j d\vec{x} = ((\pi C)'(\vec{a})^*\vec{b},\vec{c})$$

because the vector vanishes on Γ. $\sqcup$

Using the previous lemmas we are now ready to state and prove the existence of the Gateaux derivative. It is useful to remark that the Gateaux derivatives makes sense whenever one is able to prove the uniqueness of the solution of the Navier-Stokes system. In the time dependent two dimensional case this is possible.

Theorem 2.2 *Given $\Omega \in C^2$ and $\vec{u}_0 \in V(\Omega)$. The mapping $\vec{u} = \vec{u}(\vec{g})$ from $L^2((0,T);L^2(\Omega))$ to $\mathcal{H}((0,T)\times\Omega)$ has a Gateaux derivatives $\frac{D\vec{u}}{D\vec{g}}\cdot\vec{h}$ in every direction $\vec{h}$ in $L^2((0,T);L^2(\Omega))$. Furthermore, $\tilde{w}(h) = \frac{D\vec{u}}{D\vec{g}}\cdot\vec{h}$ is the solution of the problem*

$$\begin{cases} \tilde{w}_t + \nu(\pi A)\tilde{w} + (\pi C)'(\vec{u}(\vec{g}))\tilde{w} = \vec{h} \\ \tilde{w} \in V(\Omega) \\ \tilde{w}(t,\vec{x}) = 0 \quad \vec{x} \in \Gamma \\ \tilde{w}(0,\vec{x}) = 0 \quad \vec{x} \in \Omega. \end{cases} \tag{2.22}$$

where $\tilde{w} \in L^\infty((0,T);V(\Omega)) \cap L^2((0,T);H^2(\Omega))$.

Proof: Let $\vec{g}$ and $\vec{h}$ be given in $L^2((0,T);W(\Omega))$. We need to prove the following result:

$$\lim_{s\to 0}\left(\frac{\|\vec{u}_{\vec{g}+s\vec{h}} - \vec{u}_{\vec{g}}) - s\tilde{w}(\vec{h})\|_{L^2((0,T);V)}}{|s|}\right) = 0$$

We set $\tilde{u} = (\vec{u}_{\vec{g}+s\vec{h}} - \vec{u}_{\vec{g}}) - s\tilde{w}(\vec{h})$ so that $\tilde{u}$ is the solution of the evolution equation

$$\begin{cases} \frac{d\tilde{u}}{dt} + \nu(\pi A)\tilde{u} + (\pi C)(\vec{u}_{\vec{g}+s\vec{h}})\vec{u}_{\vec{g}+s\vec{h}} - (\pi C)(\vec{u}_{\vec{g}})\vec{u}_{\vec{g}} - \\ (\pi C)'(\vec{u}_{\vec{g}})s\tilde{w} = 0 \\ \tilde{u} \in V(\Omega) \\ \tilde{u}(t,\vec{x}) = 0 \quad \vec{x} \in \Gamma \\ \tilde{u}(0,\vec{x}) = 0 \quad \vec{x} \in \Omega. \end{cases} \tag{2.23}$$

If we define the function $\vec{k} \in H^{-1}(\Omega)$ as follows

$$\vec{k} = (\pi C)(\vec{u}_{\vec{g}+s\vec{h}})\vec{u}_{\vec{g}+s\vec{h}} - (\pi C)(\vec{u}_{\vec{g}})\vec{u}_{\vec{g}} - (\pi C)'(\vec{u}_{\vec{g}})(\vec{u}_{\vec{g}+s\vec{h}} - \vec{u}_{\vec{g}})$$

then, the problem in eq(2.23) becomes

$$\begin{cases} \frac{d\tilde{u}}{dt} + \nu(\pi A)\tilde{u} + (\pi C)'(\vec{u}_{\vec{g}})\tilde{u} = \vec{k} \\ \tilde{u} \in V(\Omega) \\ \tilde{u}(t,\vec{x}) = 0 \quad \vec{x} \in \Gamma \\ \tilde{u}(0,\vec{x}) = 0 \quad \vec{x} \in \Omega. \end{cases} \tag{2.24}$$

In order to estimate $\int_0^T \|\tilde{u}\|_1^2 dt$ we use the property in (2.19)

$$(\pi C)'(\vec{u}) \cdot \tilde{u} = (\pi C)(\vec{u})\tilde{u} + (\pi C)(\tilde{u})\vec{u}$$

and the lemma 2.1 part i) with $\sigma = 0, \delta = 1, \vec{g} = \vec{k}$ to obtain

$$\int_0^T \|\tilde{u}\|_1^2 dt \leq C_1 \int_0^T \|\vec{k}\|_{V^*}^2 dt$$

Now we need to evaluate the right-hand side term above. Taking the scalar product of the equation defining $\vec{k}$ with the vector $\vec{v} \in H_0^1(\Omega)$ we have

$$\begin{aligned} &\| < \vec{k}, \vec{v} > | = |((\pi C)(\vec{u}_{\vec{g}+s\vec{h}})\vec{u}_{\vec{g}+s\vec{h}}, \vec{v}) - ((\pi C)(\vec{u}_{\vec{g}})\vec{u}_{\vec{g}}, \vec{v}) - ((\pi C)'(\vec{u}_{\vec{g}})\hat{u}, \vec{v})| = \\ &|c(\vec{u}_{\vec{g}+s\vec{h}}, \vec{u}_{\vec{g}+s\vec{h}}, \vec{v}) - c(\vec{u}_{\vec{g}}, \vec{u}_{\vec{g}}, \vec{v}) - c(\vec{u}_{\vec{g}}, \hat{u}, \vec{v}) - c(\hat{u}, \vec{u}_{\vec{g}}, \vec{v})| = \\ &|c(\hat{u}, \vec{u}_{\vec{g}+s\vec{h}}, \vec{v}) - c(\hat{u}, \vec{u}_{\vec{g}}, \vec{v})| = |c(\hat{u}, \hat{u}, \vec{v})| \leq K\|\hat{u}\|_1^2 \|\vec{v}\|_1 \end{aligned} \tag{2.25}$$

where $\hat{u} = \vec{u}_{\vec{g}+s\vec{h}} - \vec{u}_{\vec{g}}$. Hence as a result of this estimate we obtain

$$\int_0^T \|\tilde{u}(t)\|_1^2 dt \leq C \int_0^T \|\vec{u}_{\vec{g}+s\vec{h}} - \vec{u}_{\vec{g}}\|_1^4 dt.$$

To estimate the norm of $\hat{u}$ in $L^\infty((0,T); V(\Omega))$ we set again $\hat{u} = \vec{u}_{\vec{g}+s\vec{h}} - \vec{u}_{\vec{g}}$. We can bound its norm in the latter space by a constant independent of s. We can use the lemma 2.1 with $\sigma = 1$ and $\delta = 1$ to get that

$$\|\hat{u}\|_1^2 = \|\vec{u}_{\vec{g}+s\vec{h}} - \vec{u}_{\vec{g}}\|_1^2 \leq C_1 |s|^2 \int_0^T \|\vec{h}\|^2 d\tau$$

as $\hat{u}$ is the solution of

$$\begin{cases} \frac{d\hat{u}}{dt} + \nu(\pi A)\hat{u} + (\pi C)(\vec{u}_{\vec{g}})\hat{u} + (\pi C)(\hat{u})\vec{u}_{\vec{g}} + (\pi C)(\hat{u})\hat{u} = s\vec{h} \\ \hat{u} \in V(\Omega) \\ \hat{u}(t,\vec{x}) = 0 \quad \vec{x} \in \Gamma \\ \hat{u}(0,\vec{x}) = 0 \quad \vec{x} \in \Omega. \end{cases} \tag{2.26}$$

Thus

$$\sup_{0\leq t\leq T}(\|\hat{u}(t)\|_1) \leq C_3|s|$$

Our claim will follow from the estimate

$$\int_0^T \|\tilde{u}(t)\|_1^2 dt \leq C_4|s|^4. \tag{2.27}$$

From the regularity of $\vec{h}$ it follows that $\tilde{w} \in L^\infty((0,T);V(\Omega)) \cap L^2((0,T);H^2(\Omega))$. $\sqcup$

For a variation $\delta\vec{h}_2 \in L^2((0,T);L^2(\Omega))$ the Gateaux derivative of the Navier-Stokes system can be written in the following form

$$\begin{cases} \tilde{w}_t + \nu\nabla^2\tilde{w} + (\tilde{w}\cdot\nabla)\vec{u} + (\vec{u}\cdot\nabla)\tilde{w} + \nabla\tilde{q} = \delta\vec{h}_2 \\ \tilde{w} \in V(\Omega) \\ \tilde{w}(t,\vec{x}) = 0 \quad \vec{x}\in\Gamma \\ \tilde{w}(0,\vec{x}) = 0 \quad \vec{x}\in\Omega. \end{cases} \tag{2.28}$$

The Gateuax derivative gives useful information about the sensitivity of the system around a particular point but complete information requires one to solve the problem for every direction $\delta\vec{g}$. In order to minimize the functional we need only an integral over all these directions. The linear adjoint equation is the information that we need as the final result of this integration. In order to show this, we have to show this preliminary result.

Lemma 2.3 *Given $\Omega \in C^2$ and $\vec{u}_0 \in V(\Omega)$. Let $\vec{h}_1$ be given in $L^2((0,T);L^2(\Omega))$ and let $\tilde{w}(\vec{h}_1)$ be defined in eq(2.22). For every $\vec{h}_2$ in $L^2((0,T);L^2(\Omega))$, we have*

$$\int_\Omega [\vec{w}\tilde{w}]_0^T d\vec{x} + \int_0^T\int_\Omega \vec{h}_2\tilde{w}(\vec{h}_1)d\vec{x}dt = \int_0^T\int_\Omega \vec{w}(\vec{h}_2)\vec{h}_1 d\vec{x}dt$$

where $\vec{w}$ is the solution of the adjoint problem

$$\begin{cases} -\vec{w}_t + \nu(\pi A)\vec{w} + (\pi C)'^*(\vec{u}(\vec{g}))\vec{w} = \vec{h}_2 \\ \vec{w}\in V(\Omega) \\ \vec{w} = 0 \quad \forall\vec{w}\in\Gamma \\ \vec{w}(T,\vec{x}) = \gamma(\vec{u}(T) - \vec{U}(T)). \end{cases} \tag{2.29}$$

with final value $\vec{w}(T,\vec{x}) = \gamma(\vec{u}(T) - \vec{U}(T))$ for all $\vec{x}\in\Omega$ and homogeneous boundary condition on Γ.

Proof: The integral on the right-hand side contains $\vec{h}_1$ for which we can use eq(2.22); then we can proceed by integration by parts. We note that the integration by parts is fully justified by the regularity proprieties of the quantities involved. In the following mathematical passages we use the fact that the operator A is selfadjoint (see [44]) and the adjoint operator of $(\pi C)'(\vec{u}_f)$ is $(\pi C)'^*(\vec{u}_f)$ so that

$$\begin{aligned}
&\int_0^T \int_\Omega \vec{w}(\vec{h}_2)\vec{h}_1 d\vec{x}dt = \\
&\int_0^T \int_\Omega \vec{w}\left(\tilde{w}_t + \nu(\pi A)\tilde{w} + (\pi C)'(\vec{u}_{\vec{g}})\tilde{w}\right) d\vec{x}dt = \\
&\int_0^T \int_\Omega \tilde{w}\left(-\vec{w}_t + \nu(\pi A)\vec{w} + (\pi C)'^*(\vec{u}_{\vec{g}})\vec{w}\right) d\vec{x}dt + \int_\Omega [\vec{w}\tilde{w}]_0^T d\vec{x} = \\
&\int_0^T \int_\Omega \vec{h}_2\tilde{w}(\vec{h}_1) d\vec{x}dt + \int_\Omega [\vec{w}\tilde{w}]_0^T d\vec{x}. \quad \sqcup
\end{aligned}$$

It is easy to show that the Gateaux derivative should be zero where the optimal control problem attains its solution.

Theorem 2.3 *Let $\vec{u}_0$ be in $V(\Omega)$. If $(\vec{u}, \vec{g})$ is an optimal pair for the problem in eq(2.9), then the Gateaux derivative of $L(\vec{g})$ vanishes at $(\vec{u}, \vec{g})$.*

Proof: If $(\vec{u}, \vec{g})$ is an optimal pair, then for every $\vec{h} \in L^2((0,T); L^2(\Omega))$ and for every $\lambda \in \mathbb{R}$ we have from the definition of an optimal solution

$$L(\vec{g} + \lambda\vec{h}) \geq L(\vec{g}).$$

The above inequality implies

$$\frac{L(\vec{g} + \lambda\vec{h}) - L(\vec{g})}{\lambda} \geq 0 \quad if \quad \lambda \geq 0$$

and

$$\frac{L(\vec{g} + \lambda\vec{h}) - L(\vec{g})}{\lambda} \leq 0 \quad if \quad \lambda \leq 0,$$

that is

$$\frac{L(\vec{g} + \lambda\vec{h}) - L(\vec{g})}{\lambda} = 0 \quad if \quad \lambda = 0,$$

and thus the Gateaux derivative must vanish. $\sqcup$

In the next theorem we will show that if the Gateaux derivative vanishes, then $\vec{g}$ must be proportional to the solution of the adjoint system.

Theorem 2.4 *Given $\Omega \in C^2$ and $\vec{u}_0 \in V(\Omega)$. If $(\vec{u}, \vec{g})$ is an optimal pair for the problem in eq(2.9), then we have $\vec{g} = -\frac{1}{\beta}\vec{w}$ where $\vec{w}$ is the solution of this adjoint problem*

$$\begin{cases}
-\vec{w}_t + \nu\nabla^2\vec{w} + (\nabla\vec{u})^T\vec{w} - (\vec{u}\cdot\nabla)\vec{w} + \nabla q = \alpha(\vec{u} - \vec{U}) \\
\vec{w} \in V(\Omega) \\
\vec{w}(t, \vec{x}) = 0 \quad \vec{x} \in \Gamma \\
\vec{w}(T, \vec{x}) = \gamma(\vec{u}(T) - \vec{U}(T)) \quad \vec{x} \in \Omega.
\end{cases} \tag{2.30}$$

and $\vec{g} \in L^\infty((0,T);V(\Omega)) \cap L^2((0,T);H^2(\Omega))$.

Proof: Let $(\vec{u},\vec{g})$ be an optimal pair solution of the problem defined in eq (2.9). We compute the Gateaux derivative of the functional $L(\vec{g})$ in the direction of $\vec{h}$, then the lemma 2.3 completes the proof. We have

$$\frac{DL(\vec{g})}{D\vec{g}} \cdot \vec{h} = \int_0^T \int_\Omega [\alpha(\vec{u}-\vec{U})\left(\frac{D\vec{u}}{D\vec{g}} \cdot \vec{h}\right) + \beta\vec{g}\cdot\vec{h}]d\vec{x}dt +$$
$$\int_\Omega \gamma(\vec{u}(T,\vec{x}) - \vec{U}(T,\vec{x}))\left(\frac{D\vec{u}}{D\vec{g}} \cdot \vec{h}\right)(T,\vec{x})d\vec{x} =$$
$$\int_0^T \int_\Omega [\alpha(\vec{u}-\vec{U})\tilde{w} + \beta\vec{g}\cdot\vec{h}]d\vec{x}dt + \gamma\int_\Omega(\vec{u}(T) - \vec{U}(T))\tilde{w}(T)d\vec{x}$$

Now using lemma 2.3, we can integrate by parts to obtain

$$\frac{DL(\vec{g})}{D\vec{g}} \cdot \vec{h} =$$
$$\int_0^T \int_\Omega [\alpha(\vec{u}-\vec{U})\tilde{w} + \beta\vec{g}\cdot\vec{h}]d\vec{x}dt + \int_\Omega \gamma(\vec{u}(T,\vec{x}) - \vec{U}(T,\vec{x}))\tilde{w}(T,\vec{x})d\vec{x} =$$
$$\int_0^T \int_\Omega [\vec{w} + \beta\vec{g}]\cdot\vec{h}d\vec{x}dt$$

where $\vec{w}$ is the solution of the system in eq(2.29). Now from the theorem 2.3, if $(\vec{u},\vec{g})$ is a solution of the optimal problem the Gateaux derivative must be zero. The scalar product with a generic vector of $L^2(\Omega)$ is zero so that by the completeness of the Hilbert space, $\vec{g} = -\frac{1}{\beta}\vec{w}$. The regularity of $\vec{g}$ follows from the regularity proprieties showed for $\vec{w}$ $\sqcup$.

Now in order to obtain the solution of our optimal control problem we have to solve the Navier-Stokes system and the adjoint system

$$\begin{cases} (\vec{u}_t,\vec{v}) + \nu a(\vec{u},\vec{v}) + c(\vec{u};\vec{u},\vec{v}) + b(\vec{v},p) = (\vec{g},\vec{v}) \qquad \forall v \in H_0^1(\Omega) \\ -(\vec{g}_t,\vec{v}) + \nu a(\vec{g},\vec{v}) + c(\vec{g};\vec{u},\vec{v}) + c(\vec{u};\vec{g},\vec{v}) = -\frac{\alpha}{\beta}(\vec{u}-\vec{U}) \\ b(\vec{u},q) = 0 \quad \forall q \in L_0^2(\Omega) \\ b(\vec{g},q) = 0 \end{cases} \tag{2.31}$$

with initial velocity $\vec{u}(0,\vec{x}) = \vec{u}_0(\vec{x})$, final condition $\vec{g}(T,\vec{x}) = -\frac{\gamma}{\beta}(\vec{u}(T) - \vec{U}(T))$ and homogenous boundary conditions on Γ. The above system of equations is the weak formulation of the following system

$$\begin{cases} \vec{u}_t(t,\vec{x}) + (\vec{u}.\vec{\nabla})\vec{u} - \nu\nabla^2\vec{u} + \vec{\nabla}p(t,\vec{x}) = \vec{g} \\ \vec{\nabla}\cdot\vec{u} = 0 \\ -\vec{g}_t(t,\vec{x}) + \nu\nabla^2\vec{g} + (\nabla\vec{u})^T\vec{g} - (\vec{u}\cdot\nabla)\vec{g} + \vec{\nabla}\sigma = -\frac{\alpha}{\beta}(\vec{u}-\vec{U}) \\ \vec{\nabla}\cdot\vec{g} = 0 \end{cases} \tag{2.32}$$

with the same initial, final and boundary condition. We can remark that $\vec{g}$ and $\vec{u}$ are in $L^\infty((0,T);V(\Omega))\cap L^2((0,T);H^2\cap H^1_0(\Omega))$ for all finite values of α,β and γ.

2.3 Semidiscrete time approximation

2.3.1 Formulation of the semidiscrete time approximation optimal control

In order to compute the solution discussed in the previous section we need to discretize this problem in time and in space. Here we give some considerations about the semidiscrete time approximation.

Let $\sigma_N=\{t_n\}_{n=0}^N$ be a partition of $[0,T]$ into equal intervals $\Delta t=T/N$ with $t_0=0$ and $t_N=T$. For each fixed Δt (or N) and for every quantity $q(t,\vec{x})$ we associate the corresponding set $\{q^{(n)}(\vec{x})\}_{n=0}^N$ and a continuous piecewise linear function $q^N=q^N(t,\vec{x})$ such that $q^N(t_n,\vec{x})=q^{(n)}(\vec{x})$ for all $n=0,1,..N$. We will denote with bold letters $\mathbf{q}$ the vector $(q^{(1)},q^{(2)},.....q^{(N)})$ of the discrete time components. Also the space X^N will be denoted as $\mathbf{X}$. On this partition we define the discrete target velocity as $\vec{U}^{(n)}(\vec{x})=\vec{U}(t_n,\vec{x})$ for $n=0,1...N$ when $\vec{U}\in U_{ad}$. The state variables $\vec{u}^{(n)}\in H^1_0(\Omega)$ and $p^{(n)}\in L^2_0(\Omega)$ are constrained to satisfy the semidiscrete Navier-Stokes equations

$$\begin{cases} \frac{1}{\Delta t}(\vec{u}^{(n)}-\vec{u}^{(n-1)})+\nu A\vec{u}^{(n)}+C(\vec{u}^{(n)})\vec{u}^{(n)}+B^*p^{(n)}=\vec{g}^{(n)} \\ B\vec{u}^{(n)}=0 \\ \vec{u}^{(n)}(\vec{x})=0 \qquad \vec{x}\in\Gamma \end{cases} \tag{2.33}$$

for $n=1,2,..N$ with initial velocity $\vec{u}^{(0)}=\vec{u}_0(\vec{x})\in V(\Omega)$. This represents a backward Euler time discretization. The optimization is achieved by mean of the minimization of the discrete functional

$$L^N=\frac{\alpha\Delta t}{2}\sum_{n=1}^N\|\vec{u}^{(n)}-\vec{U}^{(n)}\|^2+\frac{\beta\Delta t}{2}\sum_{n=1}^N\|\vec{g}^{(n)}\|^2+\frac{\gamma}{2}\|\vec{u}^{(N)}-\vec{U}^{(N)}\|^2. \tag{2.34}$$

This functional corresponds to the right-point discretization rule of the functional 2.5. Of course, if Δt tends to zero, this functional tends to the corresponding continuous functional.

The formulation of the problem P_L in the semidiscrete approximation becomes:
given $\Delta t=T/N$, $\vec{u}_0\in V(\Omega)$ *and* $\vec{U}\in U_{ad}$, *find* $(\vec{\mathbf{u}},\mathbf{p},\vec{\mathbf{g}})$ *in* $\mathbf{H^1_0(\Omega)}\times\ \mathbf{L^2_0(\Omega)}\times\ \mathbf{L^2(\Omega)}$ *such*

that $(\vec{\mathbf{u}}, \mathbf{p})$ *is the solution of eq(2.33) and* $\vec{\mathbf{g}}$ *minimizes the cost function in eq(2.34).*
We note that in this formulation the value of $\vec{g}^{(0)}$ is not defined from the solution of the Navier-Stokes problem and can be arbitrary chosen as an extension of the corresponding continuous linear function $\vec{g}^N(t,\vec{x})$ in $C((0,T);L^2(\Omega))$. More complicated discretizations of the functional could also involve $\vec{g}^{(0)}$.

The auxiliary optimal control problem Q_L can be formulated as follows:
given $\Delta t = T/N$, $\vec{u}_0 \in V(\Omega)$ *and* $\vec{U} \in U_{ad}$, *find* $(\vec{\mathbf{u}}, \vec{\mathbf{g}})$ *in* $(\mathbf{V(\Omega)} \times \mathbf{L^2(\Omega)})$ *such that* $\vec{u}^{(n)}$ *is the solution of*

$$\frac{1}{\Delta t}(\vec{u}^{(n)} - \vec{u}^{(n-1)}) + \nu(\pi A)\vec{u}^{(n)} + (\pi C)(\vec{u}^{(n)})\vec{u}^{(n)} = \pi\vec{g}^{(n)} \tag{2.35}$$

for n=1,2..N with initial value $\vec{u}^{(0)} = \vec{u}_0 \in V(\Omega)$ *and* $\vec{\mathbf{g}}$ *minimizes the cost functional in eq(2.34).*

For the semidiscrete Navier-Stokes homogeneous boundary problem defined in eq(2.35) we recall this useful result [35]:

Theorem 2.5 *Given* $\Delta t = T/N$, *if* $\vec{\mathbf{g}} \in \mathbf{L^2(\Omega)}$ *then there exists at least one* $\vec{\mathbf{u}}$ *satisfying the eq(2.33) with the following estimates*

$$\|\vec{u}^{(n)}\|^2 \le d \quad n = 1,2..,N \tag{2.36}$$

$$\Delta t \sum_{n=1}^{N} \|\nabla \vec{u}^{(n)}\|^2 \le \frac{d}{\nu} \tag{2.37}$$

$$\sum_{n=1}^{N} \|\vec{u}^{(n)} - \vec{u}^{(n-1)}\|^2_{V^*} \le d \cdot K \tag{2.38}$$

where $d = \|\vec{u}^{(0)}\|^2 + \sum_{n=1}^{N} \|\vec{g}^{(n)}\|^2 \Delta t$ *and* K *a constant independent of* Δt.

Hence $\vec{u}^N \in L^\infty((0,T);V(\Omega))$ and $\vec{u}'^N \in L^2((0,T);V^*(\Omega))$. Also if $\vec{g}^N \to \vec{g} \in L^2((0,T);L^2(\Omega))$, then $\vec{u}^N \to \vec{u}$ where $\vec{u}$ is the solution of the continuous Navier-Stokes system of equations (see [35]).

2.3.2 Existence and consistency of the semidiscrete optimal control problem

If $\vec{\mathbf{g}}$ is in $\mathbf{L^2(\Omega)}$, then the existence of the optimal control problem can be proved. This fact is an easy consequence of the definition of the optimal control problem and the boundness of the functional.

Lemma 2.4 *Let $\Delta t = T/N$, $\vec{u}_0 \in W(\Omega)$ and $\vec{U} \in U_{ad}$. If $(\vec{\mathbf{u}}, \vec{\mathbf{g}})$ is the solution of the semidiscrete optimal control problem then there exists two constants C_1 and C_2 independent of Δt such that*

$$\sum_{n=1}^{N} \|\vec{g}^{(n)}\|^2 \Delta t \leq C_1 \tag{2.39}$$

$$\sum_{n=1}^{N} \|\vec{u}^{(n)}\|^2 \Delta t \leq C_2. \tag{2.40}$$

Hence we have $\vec{g}^N \in L^2((0,T); L^2(\Omega))$ and $\vec{u}^N \in L^2((0,T); W(\Omega))$ for all N.

Proof: Let $\vec{\mathbf{g}}$ be zero and $\vec{\mathbf{u}}$ be the solution of eq(2.33). Using the result in eq(2.37) the functional yields

$$\begin{aligned} L^N(0) \quad = \quad & \frac{\alpha \Delta t}{2} \sum_{n=1}^{N} \|\vec{u}^{(n)} - \vec{U}^{(n)}\|^2 + \frac{\gamma}{2} \|\vec{u}^{(N)} - \vec{U}^{(N)}\|^2 \leq \\ & \frac{\alpha T + \gamma}{2} \|\vec{U}\|^2_{L^\infty((0,T);V)} + \frac{\alpha \Delta t}{2} \sum_{n=1}^{N} \|\vec{u}^{(n)}\|^2 + \frac{\gamma}{2} \|\vec{u}^{(N)}\|^2 \leq \\ & \frac{T\alpha + \gamma}{2} \left(\|\vec{U}\|^2_{L^\infty((0,T);V)} + \|\vec{u}_0\|^2 \right) = C_1. \end{aligned}$$

where C_1 is independent of Δt. Now if $(\vec{\mathbf{u}}, \vec{\mathbf{g}})$ is a solution of our optimal control problem then $L(\vec{\mathbf{g}}) \leq L(0)$. From this inequality we have

$$\frac{\alpha \Delta t}{2} \sum_{n=1}^{N} \|\vec{u}^{(n)} - \vec{U}^{(n)}\|^2 + \frac{\beta \Delta t}{2} \sum_{n=1}^{N} \|\vec{g}^{(n)}\|^2 + \frac{\gamma}{2} \|\vec{u}^{(N)} - \vec{U}^N\|^2 \leq L(0) \leq C_1.$$

From the above inequality follows eq(2.39) and $\Delta t \sum_{n=1}^{N} \|\vec{u}^{(n)} - \vec{U}^{(n)}\|^2 \leq C'$ from which eq(2.40) follows. $\sqcup$

We remark that if Δt is fixed the control $\vec{g}^{(n)}$ is bounded by $C_1/\Delta t$. In this case $\vec{g}^N$ is in $L^\infty((0,T); L^2(\Omega))$. Now we can state and prove the existence for the optimal control problem in a open bounded domain Ω with Lipschitz-continuous boundary Γ.

Theorem 2.6 *Given $\Delta t = T/N$, $\vec{u}_0 \in V(\Omega)$ and $\vec{U} \in U_{ad}$, there exists a pair $(\vec{\mathbf{u}}, \vec{\mathbf{g}})$ in $(\mathbf{V}(\boldsymbol{\Omega}) \times \mathbf{L^2}(\boldsymbol{\Omega}))$ such that $\vec{\mathbf{u}}$ is the solution of eq(2.35) and $\vec{\mathbf{g}}$ minimizes the cost functional.*

Proof: Given N let $\{\vec{\mathbf{g}}_k\}_{k=1}^{\infty}$ be a minimizing sequence in $\mathbf{L^2}(\boldsymbol{\Omega})$. Using the results in eq(2.39) and eq(2.36) we find that the corresponding sequence $\vec{\mathbf{u}}_k$ is uniformly bounded in $\mathbf{V}(\mathbf{\Omega})$. Now we can proceed with a weakly convergent subsequence and show that this subsequence converges to the solution of the optimal control problem in the semidiscrete approximation. We can write

$$\begin{array}{llll} \vec{g}_k^{(n)} \to \vec{g}^{(n)} & in & L^2(\Omega) & weakly \\ \vec{u}_k^{(n)} \to \vec{u}^{(n)} & in & V(\Omega) & weakly \end{array}$$

for $n = 1, 2, ...N$. By using the fact that the injection of $V(\Omega)$ into $L^2(\Omega)$ is compact, the subsequence converges strongly. The lower semicontinuity of the functional in eq(2.34) allows the pair $(\vec{\mathbf{u}}, \vec{\mathbf{g}})$ to minimize the functional. Since we can pass to the limit in the linear and the nonlinear term, the pair also satisfies the Navier-Stokes eq(2.35). In fact, since $\vec{\mathbf{u}}_k$ converges to $\vec{\mathbf{u}}$ strongly in $\mathbf{L^2}(\mathbf{\Omega})$, then for any $\vec{\mathbf{z}} \in \mathcal{V}(\mathbf{\Omega})$ we have

$$\lim_{k\to\infty} c(\vec{\mathbf{u}}_k; \vec{\mathbf{u}}_k, \vec{\mathbf{z}}) = c(\vec{\mathbf{u}}; \vec{\mathbf{u}}, \vec{\mathbf{z}}).$$

Since $\mathcal{V}(\mathbf{\Omega})$ is dense in $\mathbf{V}(\mathbf{\Omega})$, this is still true for any $\vec{w}$ in $V(\Omega)$ by a continuity argument. This allows us to pass to the limit in the semidiscrete equation and complete the proof. $\sqcup$

In order to prove the consistency of our approximation we need to recall a compactness theorem. Let $m, n > 1$ be two integers and X_0, X_1 be two Banach spaces . We define

$$\mathcal{Y}((0,T); n, m; X_0, X_1) = \{v \in L^n((0,T); X_0) : \quad V^* \in L^m((0,T); X_1)\}$$

with the norm

$$\|v\|^2_{\mathcal{Y}} = \|v\|^2_{L^n((0,T);X_0)} + \|v\|^2_{L^m((0,T);X_1)}$$

We recall this compactness theorem in the framework of Banach spaces.

Theorem 2.7 *Let $X_0 \subset X \subset X_1$ be three Banach spaces. X_0 and X_1 are reflexive and the the injections are continuous. If the injection $X_0 \to X_1$ is compact, then the injection from $\mathcal{Y}((0,T); n, m, X_0, X_1)$ into $L^2((0,T); X)$ is compact.*

A proof of this theorem can be found in [35]. Since $V(\Omega) \subset W(\Omega) \subset V^*(\Omega)$ where V and V^* are reflexive and the injections are continuous and $V(\Omega) \to H(\Omega)$ is compact from the Sobolev imbedding theorem, then the injection from $\mathcal{Y}((0,T); 2, 2, V, V^*)$ into $L^2((0,T); W)$ is compact. Hence if a sequence $\vec{v}_k$ converges weakly in $L^2((0,T); V(\Omega))$ and $\vec{v}_k'$ in $L^2((0,T); V^*(\Omega))$ then $\vec{v}_k$ converges strongly in $L^2((0,T); W(\Omega))$. Now we can prove the consistency of our semidiscrete optimal control problem.

Theorem 2.8 *Given $\Delta t = T/N$, $\vec{U} \in U_{ad}$ and $\vec{u}_0 \in V(\Omega)$. For $\Delta T \to 0$ ($N \to \infty$) the solution $\{(\vec{u}^{(n)}, \vec{g}^{(n)})\}_{n=1}^N$ of the problem described in eq(2.35) tends to the optimal control pair $(\vec{u}, \vec{g})$ solution of the corresponding continuous optimal control problem.*

Proof: Let $\Delta t = T/N$, $\vec{u}'^{(n)} = (\vec{u}^{(n)} - \vec{u}^{(n-1)})/\Delta t$, and $\vec{u}'^N$ the corresponding linear function. The sequences $\{\vec{u}^N\}_{N=1}^{\infty}$, $\{\vec{g}^N\}_{N=1}^{\infty}$ and $\{\vec{u}'^N\}_{N=1}^{\infty}$ are uniformly bounded in $L^2((0,T);V(\Omega)) \cap L^{\infty}((0,T);W(\Omega))$, $L^2((0,T);L^2(\Omega))$ and $L^2((0,T);V^*(\Omega))$ respectively. This follows from Lemma 2.4 and from the well-known theorems (eq(2.36)-eq(2.38)) for unsteady Navier-Stokes solutions. Hence from these sequences we can extract some subsequences such that

$$\begin{cases} \vec{u}^K \to \vec{u} \quad L^2(0,T,V(\Omega)) \text{ weakly} \\ \vec{u}^K \to \vec{u} \quad L^{\infty}(0,T,H(\Omega)) \text{ *-weakly} \\ \vec{g}^K \to \vec{g} \quad L^2(0,T,L^2(\Omega)) \text{ weakly} \\ \frac{d\vec{u}^{(k)}}{dt} \to \vec{u}' \quad L^2(0,T,V^*(\Omega)) \text{ weakly.} \end{cases} \tag{2.41}$$

As a consequence of the compactness theorem the convergence of the sequence $\{\vec{u}^N\}_{N=1}^{\infty}$ is strong in $L^2((0,T);W(\Omega))$. Now we can pass to the limit in the system of equations and in the functional. The linear terms do not give problems. Using the fact that the sequence converges weakly in $L^2((0,T);V)$ and strongly in $L^2((0,T);W)$ we can pass to the limit in the nonlinear term (see eq(4.14)). Thus the semidiscrete optimal control problem for $N \to \infty$ is consistent with the continuous one. $\sqcup$

The fact that the semidiscrete approximation approaches the continuous case for $\Delta t \to 0$ allows us to state that $\{\vec{u}^N\}_{N=1}^{\infty}$ is a bounded sequence in $L^{\infty}((0,T);V(\Omega))$. In fact $\vec{u}^N$ is in $L^{\infty}((0,T);V(\Omega))$ for all N and converges to $\vec{u}$ in $L^{\infty}((0,T);V(\Omega))$.

2.3.3 First-order necessary condition

In this section we proceed to derive the first-order necessary condition in term of Gateaux derivative. We will show that the optimal solution must satisfy the first-order necessary condition where this is available which is for small Δt. Following the same procedure as in the continuous case, we can state this analogous theorem.

Theorem 2.9 *Given $\Delta t = T/N$ and $\vec{u}_0 \in V(\Omega)$. If $(\vec{\mathbf{u}}, \vec{\mathbf{g}})$ in $(\mathbf{H_0^1(\Omega)}, \mathbf{L^2(\Omega)})$ is a solution of the semidiscrete optimal control problem then the Gateaux derivative of $L(\vec{\mathbf{g}})$ vanishes at $(\vec{\mathbf{u}}, \vec{\mathbf{g}})$.*

Hence, if the Gateaux derivative exists, it must vanish. Now we should define the range of applicability of this theorem. We need some auxiliary results similar to the continuous case.

Lemma 2.5 *Let $\Delta t = T/N$, and $\vec{w}^{(n)}$ be the solution of this system of equations*

$$\begin{cases} \frac{1}{\Delta t}(\vec{w}^{(n)} - \vec{w}^{(n-1)}) + \nu(\pi A)\vec{w}^{(n)} + \sigma(\pi C)(\vec{w}^{(n)})\vec{w}^{(n)} + \\ \delta[(\pi C)(\vec{w}^{(n)})\vec{u}^{(n)} + (\pi C)(\vec{u}^{(n)})\vec{w}^{(n)}] = \vec{g}^{(n)} \\ \vec{w}^{(n)} \in V(\Omega) \end{cases} \tag{2.42}$$

for n=1,2,...N with initial value $\vec{w}^{(0)}(\vec{x}) = 0$*, homogeneous boundary condition and* $\delta, \sigma \geq 0$*.*
i) Given $\{\vec{g}^N\}_{N=1}^{\infty}$ *and* $\{\vec{u}^N\}_{N=1}^{\infty}$ *convergent sequences in* $L^\infty((0,T);V(\Omega))$ *and* $L^2((0,T);V^*(\Omega))$ *respectively, then there exists a* N_1 *such that if* $N \geq N_1$ *the linear function* $\vec{w}^N$ *is in* $L^\infty((0,T);W(\Omega)) \cap L^2((0,T);V(\Omega))$ *and*

$$\sum_{n=1}^{N} \|\vec{w}^{(n)}\|_1^2 \Delta t \leq C_1 \sum_{n=1}^{N} \|\vec{g}^{(n)}\|_{V^*}^2 \Delta t \tag{2.43}$$

where C_1*, is independent of* Δt*.*
ii) Given $\{\vec{g}^N\}_{N=1}^{\infty}$ *in and* $\{\vec{u}^N\}_{N=1}^{\infty}$ *convergent sequences in* $L^\infty((0,T);V(\Omega))$ *and* $L^\infty((0,T);L^2(\Omega))$ *respectively, then there exists a* N_2 *such that if* $N \geq N_2$ *the solution* $\vec{w}^N$ *is in* $L^\infty((0,T);V(\Omega)) \cap L^2((0,T);H^2(\Omega))$ *and*

$$\|\vec{w}^{(n)}\|_1^2 \leq C_1 \sum_{k=1}^{N} \|\vec{g}^{(k)}\|^2 \Delta t \tag{2.44}$$

for n=1,2,...N.

Proof: The proof can be achieved using the same techniques used in the continuous case. Let $\Delta t = T/N$. Following the same mathematical passages we arrive at

$$\frac{1}{\Delta t}(\vec{w}^{(n)} - \vec{w}^{(n-1)}, \vec{w}^{(n)}) + \frac{\nu}{2}\|\nabla \vec{w}^{(n)}\|^2 \leq \frac{\lambda}{\nu}\|\vec{g}^{(n)}\|_{V^*}^2 + \frac{C}{4}\|\nabla \vec{u}^{(n)}\|^2 \|\vec{w}^{(n)}\|^2$$

where $C = \frac{4\sqrt{2}\delta}{\nu}$. Now

$$2(\vec{a} - \vec{b}, \vec{a}) = \|\vec{a}\|^2 - \|\vec{b}\|^2 + \|\vec{a} - \vec{b}\|^2 \quad \forall \vec{a}, \vec{b} \tag{2.45}$$

so that

$$\|\vec{w}^{(n)}\|^2 - \|\vec{w}^{(n-1)}\|^2 + \|\vec{w}^{(n)} - \vec{w}^{(n-1)}\|^2 + \nu \Delta t \|\nabla \vec{w}^{(n)}\|^2 \leq$$
$$\frac{2\lambda \Delta t}{\nu}\|\vec{g}^{(n)}\|_{V^*}^2 + \frac{C\Delta t \|\nabla \vec{u}^{(n)}\|^2}{2}\|\vec{w}^{(n)}\|^2.$$

Since $\{\vec{u}^N\}_{N=1}^{\infty}$ is bounded in $L^\infty((0,T);V(\Omega))$, there exists a $(\Delta t)_1$ such that

$$1 - \frac{C\Delta t \|\nabla \vec{u}^{(n)}\|^2}{2} \leq e^{-(C\Delta t \|\nabla \vec{u}^{(n)}\|^2)}$$

for all $\Delta t \leq (\Delta t)_1$. As

$$\|\nabla \vec{u}^{(n)}\|^2 = \sum_{m=1}^{n} \|\nabla \vec{u}^{(m)}\|^2 - \sum_{m=1}^{n-1} \|\nabla \vec{u}^{(m)}\|^2$$

we have

$$\|\vec{w}^{(n)}\|^2 e^{-C\Delta t\left(\sum_{m=1}^{n}\|\nabla\vec{u}^{(m)}\|^2-\sum_{m=1}^{n-1}\|\nabla\vec{u}^{(m)}\|^2\right)}+\|\vec{w}^{(n)}-\vec{w}^{(n-1)}\|^2+\nu\Delta t\|\nabla\vec{w}^{(n)}\|^2 \leq$$
$$\|\vec{w}^{(n-1)}\|^2+\frac{2\lambda\Delta t}{\nu}\|\vec{g}^{(n)}\|^2_{V^*}$$

or

$$\|\vec{w}^{(n)}\|^2 e^{-\left(C\Delta t\sum_{m=1}^{n}\|\nabla\vec{u}^{(m)}\|^2\right)}+e^{-\left(C\Delta t\sum_{m=1}^{n-1}\|\nabla\vec{u}^{(m)}\|^2\right)}\left(\|\vec{w}^{(n)}-\vec{w}^{(n-1)}\|^2+\nu\Delta t\|\nabla\vec{w}^{(n)}\|^2\right)$$
$$\leq e^{-\left(C\Delta t\sum_{m=1}^{n-1}\|\nabla\vec{u}^{(m)}\|^2\right)}\|\vec{w}^{(n-1)}\|^2+\frac{2\lambda\Delta t}{\nu}\|\vec{g}^{(n)}\|^2_{V^*}$$

Dropping the term $\|\vec{w}^{(n)}-\vec{w}^{(n-1)}\|^2$ and summing the inequalities for $n=1,...N$ we have

$$\|\vec{w}^{(N)}\|^2 e^{-\left(C\Delta t\sum_{m=1}^{N}\|\nabla\vec{u}^{(m)}\|^2\right)}+ \tag{2.46}$$
$$\nu\Delta t\sum_{n=1}^{N} e^{-\left(C\Delta t\sum_{m=1}^{n-1}\|\nabla\vec{u}^{(m)}\|^2\right)}\|\nabla\vec{w}^{(n)}\|^2 \leq \frac{2\lambda\Delta t}{\nu}\sum_{n=1}^{N}\|\vec{g}^{(n)}\|^2_{V^*}$$

where the exponential functions are well defined as $\vec{u}^N$ is in $L^2((0,T);V(\Omega))$. Summing the inequalities for $n=1,...k$ and rearranging some terms we get

$$\|\vec{w}^{(k)}\|^2 \leq \frac{2\lambda\Delta t}{\nu} e^{\left(C\Delta t\sum_{m=1}^{N}\|\nabla\vec{u}^{(m)}\|^2\right)}\sum_{n=1}^{N}\|\vec{g}^{(n)}\|^2_{V^*} \tag{2.47}$$

From eq(2.47) it follows that $\|\vec{w}^{(k)}\|^2$ is bounded for all k and so the associated linear function $\vec{w}^N$ is in $L^\infty((0,T);W(\Omega))$. Dropping the term $\vec{w}^{(N)}$ in eq(2.46) we have that $\vec{w}^N$ is in $L^2((0,T);V(\Omega))$. Also

$$\sum_{n=1}^{N}\|\vec{w}^{(n)}\|_1^2\Delta t \leq (1+C_p)\sum_{n=1}^{N}\|\nabla\vec{w}^{(n)}\|^2\Delta t \leq C_1\sum_{n=1}^{N}\|\vec{g}^{(n)}\|^2_{V^*}\Delta t.$$

ii) If $\vec{g}^N \in L^\infty((0,T);L^2(\Omega))$ and $N \geq N_1$, from eq(2.43) follows eq(2.44). The fact that $\vec{w}^{(n)}$ is in H^2 can be proved following the continuous case. In fact we get

$$\|\nabla\vec{w}^{(n)}\|^2-\|\nabla\vec{w}^{(n-1)}\|^2+\|\nabla\vec{w}^{(n)}-\nabla\vec{w}^{(n-1)}\|^2+ \tag{2.48}$$
$$\frac{4\Delta t\nu}{5}\|(\pi A)\vec{w}^{(n)}\|^2 \leq 2K_4\Delta t\|\vec{g}^{(n)}\|^2+\frac{\phi^{(n)}\Delta t}{2}\|\nabla\vec{w}^{(n)}\|^2$$

where $\phi^{(n)}=4K_4(\|\nabla\vec{u}^{(n)}\|\cdot\|(\pi A)\vec{u}^{(n)}\|+\|\nabla\vec{u}^{(n)}\|^4+\|\nabla\vec{w}^{(n)}\|^2\|\vec{w}^{(n)}\|^2)$ is a bounded function for all n=1,..N. Thus there exists a Δt_2 such that

$$1-\frac{\phi^{(n)}\Delta t}{2} \leq e^{-(\phi^{(n)}\Delta t)}$$

for all $\Delta t \leq \Delta t_2$. As in the previous estimate we can write $\phi^{(n)} = \sum_{m=1}^{n} \phi^{(m)} - \sum_{m=1}^{n-1} \phi^{(m)}$ and hence

$$\|\nabla \vec{w}^{(n)}\|^2 e^{-\left(\Delta t \sum_{m=1}^{n} \phi^{(m)}\right)} + e^{-\left(\Delta t \sum_{m=1}^{n-1} \phi^{(m)}\right)} \times$$
$$\left(-\|\nabla \vec{w}^{(n-1)}\|^2 + \|\nabla \vec{w}^{(n)} - \nabla \vec{w}^{(n-1)}\|^2 + \frac{4\nu\Delta t}{5}\|(\pi A)\vec{w}^{(n)}\|^2\right) \leq$$
$$2K_4\Delta t\|\vec{g}^{(n)}\|^2_{V^*} e^{-\left(\Delta t \sum_{m=1}^{n-1} \phi^{(m)}\right)} \leq 2K_4\Delta t\|\vec{g}^{(n)}\|^2$$

From this inequality it follows eq(2.44). Summing the inequality for $n = 1, 2...k$ and rearranging some terms we get

$$\frac{4\Delta t\nu}{5}\sum_{n=1}^{N} e^{\left(\Delta t \sum_{m=1}^{n-1} \phi^{(n)}\right)}\|(\pi A)\vec{w}^{(n)}\|^2 \leq 2K_4\Delta t\sum_{n=1}^{N}\|\vec{g}^{(n)}\|^2 \tag{2.49}$$

which implies

$$\|(\pi A)\vec{w}^{(n)}\|^2 \leq \infty.$$

Thus $\vec{w}^{(n)}$ is in $H^2(\Omega)$ since the $\|(\pi A)\vec{w}^{(n)}\|$ is equivalent to the H^2 norm. ⊔

In the next theorem we use the vector notation for the sequences in the space $\mathbf{X} = X^N$ (i.e. $\vec{\mathbf{g}} = \{\vec{g}^{(n)}\}_{n=1}^{N}$ and $\vec{\mathbf{h}} = \{\vec{h}^{(n)}\}_{n=1}^{N}$) to state the existence of the Gateaux derivative for small Δt.

Theorem 2.10 *Let $\Omega \in C^2$, $\Delta t = T/N$, $\vec{\mathbf{h}}$ be in $\mathbf{L^2(\Omega)}$ for all N and $\vec{u}_0 \in V(\Omega)$. With these hypotheses there exists a N_1 such that the mapping $bf\vec{u} = \vec{\mathbf{u}}(\vec{\mathbf{g}})$ from $\mathbf{L^2(\Omega)}$ to $\mathbf{H^1_0(\Omega)}$ has Gateaux derivative $\frac{D\vec{\mathbf{u}}}{D\vec{\mathbf{g}}} \cdot \vec{\mathbf{h}}$ for all $N \geq N_1$. Furthermore $\tilde{w}^{(n)}(\vec{\mathbf{h}}) = \frac{D\vec{u}^{(n)}}{D\vec{\mathbf{g}}} \cdot \vec{\mathbf{h}}$ is solution of this system of equations*

$$\begin{cases} \frac{1}{\Delta t}(\tilde{w}^{(n)} - \tilde{w}^{(n-1)}) + \nu(\pi A)\tilde{w}^{(n)} + (\pi C)'(\vec{u}^{(n)}(\vec{g})) \cdot \tilde{w}^{(n)} = \vec{h}^{(n)} \\ \tilde{w}^{(n)} \in V(\Omega) \end{cases} \tag{2.50}$$

for $n = 1, 2, ...N$ with initial value $\tilde{w}^{(0)}(\vec{x}) = 0$ and homogeneous boundary condition. The associated continuous linear function $\tilde{w}^N$ is in $L^\infty((0,T); W(\Omega))$.

Proof: We need to prove that there exists a Δt_1 such that

$$\lim_{s\to 0}\left(\frac{\sum_{n=1}^{N}\Delta t\|\vec{u}^{(n)}(\vec{\mathbf{g}} + s\vec{\mathbf{h}}) - \vec{u}^{(n)}(\vec{\mathbf{g}})) - s\tilde{w}^{(n)}(\vec{\mathbf{h}})\|_1^2}{|s|}\right) = 0$$

for all $\Delta t \leq \Delta t_1$. We set, as in the continuous case, $\tilde{u}^{(n)} = \vec{u}^{(n)}(\vec{\mathbf{g}} + s\vec{\mathbf{h}}) - \vec{u}^{(n)}(\vec{\mathbf{g}})) - s\tilde{w}^{(n)}(\vec{\mathbf{h}})$ and $\tilde{u}^{(n)}$ now is the solution of the evolution system of equations

$$\begin{cases} \frac{1}{\Delta t}(\tilde{u}^{(n)} - \tilde{u}^{(n-1)}) + \nu(\pi A)\tilde{u}^{(n)} + (\pi C)(\vec{u}^{(n)}_{\vec{g}+s\vec{h}})\vec{u}^{(n)}_{\vec{g}+s\vec{h}} - \\ (\pi C)(\vec{u}^{(n)}_{\vec{g}})\vec{u}^{(n)}_{\vec{g}} - (\pi C)'(\vec{u}^{(n)}_{\vec{g}})s\tilde{w} = 0 \\ \tilde{u}^{(n)} \in V(\Omega) \\ \tilde{u}^{(n)}(t,\vec{x}) = 0 \quad \vec{x} \in \Gamma \\ \tilde{u}^{(n)}(0,\vec{x}) = 0 \quad \vec{x} \in \Omega. \end{cases} \tag{2.51}$$

Again we define

$$\vec{k}^{(n)} = (\pi C)(\vec{u}^{(n)}_{\vec{g}+s\vec{h}})\vec{u}^{(n)}_{\vec{g}+s\vec{h}} - (\pi C)(\vec{u}^{(n)}_{\vec{g}})\vec{u}^{(n)}_{\vec{g}} - (\pi C)'(\vec{u}^{(n)}_{\vec{g}})(\vec{u}^{(n)}_{\vec{g}+s\vec{h}} - \vec{u}^{(n)}_{\vec{g}})$$

where as in the continuous case

$$\|\vec{k}^{(n)}\|^2_{V^*} \leq K\|\hat{u}^{(n)}\|^2_1. \tag{2.52}$$

The eq(2.51) becomes

$$\begin{cases} \frac{1}{\Delta t}(\tilde{u}^{(n)} - \tilde{u}^{(n-1)}) + \nu(\pi A)\tilde{u}^{(n)} + (\pi C)'(\vec{u}^{(n)}_{\vec{g}}) \cdot \tilde{u}^{(n)} = \vec{k}^{(n)} \\ \tilde{u}^{(n)} \in V(\Omega) \\ \tilde{u}^{(n)}(t,\vec{x}) = 0 \quad \vec{x} \in \Gamma \\ \tilde{u}^{(n)}(0,\vec{x}) = 0 \quad \vec{x} \in \Omega. \end{cases} \tag{2.53}$$

The fact that $\vec{u}^N$ is in $L^\infty((0,T);V(\Omega))$ for all N and converges to $\vec{u} \in L^\infty((0,T);V(\Omega))$ implies that the sequence is bounded in $L^\infty((0,T);V(\Omega))$. Using this propriety

$$(\pi C)'(\vec{u}) \cdot \tilde{u} = (\pi C)(\vec{u})\tilde{u} + (\pi C)(\tilde{u})\vec{u}$$

we can apply the previous lemma with $\sigma = 0, \delta = 1, \vec{g} = \vec{k}$. Thus there is an integer N_2 such that for $N \geq N_2$ we have

$$\sum_{n=1}^{N} \|\tilde{u}^{(n)}\|^2_1 \Delta t \leq \sum_{n=1}^{N} \|\vec{k}^{(n)}\|^2_{V^*}\Delta t$$

Now the right-hand term above can be evaluated with the eq(2.52) and gives

$$\sum_{n=1}^{N} \|\tilde{u}^{(n)}\|^2_1 \Delta t \leq C \sum_{n=1}^{N} \|\vec{u}^{(n)}(\vec{\mathbf{g}} + s\vec{\mathbf{h}}) - \vec{u}^{(n)}(\vec{\mathbf{g}})\|^4_1 \Delta t.$$

The functions $\hat{u}^{(n)} = \vec{u}^{(n)}(\vec{\mathbf{g}} + s\vec{\mathbf{h}}) - \vec{u}^{(n)}(\vec{\mathbf{g}})$ are the solutions of

$$\begin{cases} \frac{1}{\Delta t}(\hat{u}^{(n)} - \hat{u}^{(n-1)}) + \nu(\pi A)\hat{u} + (\pi C)(\vec{u}^{(n)}_{\vec{g}})\hat{u}^{(n)} + \\ (\pi C)(\hat{u}^{(n)})\vec{u}^{(n)}_{\vec{g}} + (\pi C)(\hat{u}^{(n)})\hat{u}^{(n)} = s\vec{h}^{(n)} \\ \tilde{u}^{(n)} \in V(\Omega) \\ \hat{u}^{(n)}(t,\vec{x}) = 0 \quad \vec{x} \in \Gamma \\ \hat{u}^{(n)}(0,\vec{x}) = 0 \quad \vec{x} \in \Omega. \end{cases} \tag{2.54}$$

Again we can use the previous lemma with $\sigma = 1$ and $\delta = 1$. There is a Δt_3 such that if $\Delta t_1 \leq min\{\Delta t_2, \Delta t_3\}$ we have

$$\|\hat{u}^{(n)}\|^2_1 = \|\vec{u}^{(n)}(\vec{\mathbf{g}} + s\vec{\mathbf{h}}) - \vec{u}^{(n)}(\vec{\mathbf{g}})\|^2_1 \leq C_1|s|^2 \sum_{n=1}^{N} \|\vec{h}^{(n)}\|^2 \Delta t \leq C_2|s|^2 \tag{2.55}$$

as $\vec{\mathbf{h}}$ is in $\mathbf{L^2(\Omega)}$. Thus we have

$$\frac{\sum_{n=1}^{N} \|\tilde{u}^{(n)}\|_1^2 \Delta t}{|s|} \leq C'|s|^3 \tag{2.56}$$

that tends to zero when $s \to 0$ and so the Gateaux derivative has a precise sense. The fact that $\tilde{w}^N \in L^\infty((0,T); V(\Omega))$ follows from the regularity of $\vec{\mathbf{h}}$ and $\hat{\mathbf{u}}$. $\sqcup$

The existence of the Gateaux derivative has been proved only for small Δt and for $\vec{\mathbf{h}}$ in $\mathbf{L^2(\Omega)}$. Small Δt is not a limitation as our discretisation should be sufficient small to resemble the continuous case. In order to estimate the optimal control $\vec{\mathbf{g}}$ we need to evaluate an integral involving the Gateaux derivative.

Lemma 2.6 *Let $\Delta t = T/N$ and $\vec{u}_0 \in V(\Omega)$. Let $\vec{\mathbf{h}}_1$ and $\vec{\mathbf{h}}_2$ be in $\mathbf{L^2(\Omega)}$ and $\tilde{w}^{(n)}(\vec{\mathbf{h}}_1)$ be defined in eq(2.50). For every $\vec{\mathbf{h}}_2$ we have*

$$\sum_{n=1}^{N} \int_\Omega \vec{h}_2^{(n)} \tilde{w}^{(n)}(\vec{\mathbf{h}}_1) d\vec{x} = \sum_{n=1}^{N} \int_\Omega \vec{w}^{(n)}(\vec{\mathbf{h}}_2) \vec{h}_1^{(n)} d\vec{x} - \int_\Omega \vec{w}^{(N+1)} \tilde{w}^{(N)} d\vec{x}$$

where $\vec{w}^{(n)}$ is the solution of the adjoint problem

$$\begin{cases} -\frac{1}{\Delta t}(\vec{w}^{(n)} - \vec{w}^{(n+1)}) + \nu(\pi A)\vec{w}^{(n)} + (\pi C)'^{*}(\vec{u}_{\vec{g}}^{(n)})\vec{w}^{(n)} = \vec{h}_2^{(n)} \\ \vec{w}^{(n)} \in V(\Omega) \end{cases} \tag{2.57}$$

for n=1,2,....N with final condition $\vec{w}^{(N+1)}(\vec{x}) = \gamma(\vec{u}^{(N)} - \vec{U}^{(N)})$ and homogeneous boundary condition.

Proof: We substitute $\vec{h}_1^{(n)}$ from eq(2.50) and then we proceed by changing index in the summation. We have

$$\Delta t \sum_{n=1}^{N} \int_\Omega \vec{w}^{(n)} \vec{h}_1^{(n)} d\vec{x} =$$
$$\Delta t \sum_{n=1}^{N} \int_\Omega \vec{w}^{(n)} \left(\frac{1}{\Delta t}(\tilde{w}^{(n)} - \tilde{w}^{(n+1)}) + \nu(\pi A)\tilde{w}^{(n)} + (\pi C)'\tilde{w}^{(n)} \right) d\vec{x} =$$
$$\Delta t \sum_{n=1}^{N} \int_\Omega \tilde{w}^{(n)} \left(\frac{1}{\Delta t}\vec{w}^{(n)} + \nu(\pi A)\vec{w}^{(n)} + (\pi C)'^{*}\vec{w}^{(n)} \right) d\vec{x} -$$
$$\sum_{n=1}^{N-1} \int_\Omega \tilde{w}^{(n+1)} \vec{w}^{(n)} d\vec{x} =$$
$$\Delta t \sum_{n=1}^{N} \int_\Omega \tilde{w}^{(n)} \left(-\frac{1}{\Delta t}(\vec{w}^{(n+1)} - \vec{w}^{(n)}) + \nu(\pi A)\vec{w} + (\pi C)'^{*}\vec{w}^{(n)} \right) d\vec{x} +$$
$$\int_\Omega \vec{w}^{N+1} \tilde{w}^N d\vec{x} = \Delta t \sum_{n=1}^{N} \int_\Omega \vec{h}_2 \tilde{w}(\vec{h}_1) d\vec{x} + \int_\Omega \vec{w}^{N+1} \tilde{w}^N d\vec{x}.$$

where we used the fact that the operator A is selfadjoint (see [44]) and the adjoint operator of $(\pi C)'(\vec{u}_g^{(n)})$ is $(\pi C)'^*(\vec{u}_g^{(n)})$. $\sqcup$

Using the standard techniques it is easy to show the existence and the regularity of the solution of the eq(2.57). The regularity of $\vec{\mathbf{w}}$ is a direct consequence of the regularity of $\vec{\mathbf{h}}_2$ and $\vec{\mathbf{u}}$.

Finally we will show that the control must be proportional to the solution of the adjoint system of equations as in the continuous case.

Theorem 2.11 *Given $\Omega \in C^2$ and $\vec{u}_0 \in V(\Omega)$. If $(\vec{\mathbf{u}}, \vec{\mathbf{g}})$ is an optimal pair for the problem in eq(2.35) then there exists a $(\Delta t)^*$ such that for each $\Delta t \leq (\Delta t)^*$ we have $\vec{g}^{(n)} = -\frac{1}{\beta}\vec{w}^{(n)}$ for n=1,2...N where $\vec{w}^{(n)}$ is the solution of this adjoint problem*

$$\left\{ \begin{array}{l} -\frac{1}{\Delta t}(\vec{w}^{(n+1)} - \vec{w}^{(n)}) + \nu\nabla^2\vec{w}^{(n)} + (\nabla\vec{u}^{(n)})^T\vec{w}^{(n)} - \\ (\vec{u}^{(n)} \cdot \nabla)\vec{w}^{(n)} + \nabla q = \alpha(\vec{u}^{(n)} - \vec{U}^{(n)}) \end{array} \right. \tag{2.58}$$

with final condition $\vec{w}^{(N+1)}(\vec{x}) = \gamma(\vec{u}^N - \vec{U}^N)$ and homogeneous boundary condition. Also we have that $\vec{w}^N$ is in $L^\infty((0,T); L^2(\Omega))$.

Proof: Let $(\vec{\mathbf{u}}, \vec{\mathbf{g}})$ be an optimal pair solution of the problem defined in eq (2.35). By the theorem 2.10 there exists a $(\Delta t)^*$ such that for $\Delta t \leq (\Delta t)^*$ we can compute the Gateux derivative of the functional $L(\vec{\mathbf{g}})$ in the direction of $\vec{\mathbf{h}} \in \mathbf{L^2(\Omega)}$. We have

$$\frac{DL^N}{D\vec{\mathbf{g}}} \cdot \vec{\mathbf{h}} = \sum_{n=1}^{N} \int_\Omega [\alpha(\vec{u}^{(n)} - \vec{U}^{(n)}) \left(\frac{D\vec{u}^{(n)}}{D\vec{\mathbf{g}}} \cdot \vec{\mathbf{h}} \right) + \beta\vec{g}^{(n)} \cdot \vec{h}^{(n)}] d\vec{x} +$$
$$\int_\Omega \gamma(\vec{u}^{(N)} - \vec{U}^{(N)}) \left(\frac{D\vec{u}^{(N)}}{D\vec{\mathbf{g}}} \cdot \vec{\mathbf{h}} \right) d\vec{x} =$$
$$\sum_{n=1}^{N} \int_\Omega [\alpha(\vec{u}^{(n)} - \vec{U}^{(n)})\tilde{w}^{(n)} + \beta\vec{g}^{(n)} \cdot \vec{h}^{(n)}] d\vec{x} +$$
$$\int_\Omega \gamma(\vec{u}^{(N)} - \vec{U}^{(N)})\vec{w}^{(N)} d\vec{x}$$

We note that $\alpha(\vec{u}^N - \vec{U}^N)$ is in $L^\infty((0,T); L^2(\Omega))$ for all N and converges to $\alpha(\vec{u} - \vec{U}) \in L^\infty((0,T); L^2(\Omega))$ for $N \to \infty$. Hence $\vec{w}^N \in L^\infty((0,T); L^2(\Omega))$ for all N. Now using the lemma 2.6 we can complete the proof. We have

$$\frac{DL^N}{D\vec{\mathbf{g}}} \cdot \vec{\mathbf{h}} =$$
$$\sum_{n=1}^{N} \int_\Omega [\alpha(\vec{u}^{(n)} - \vec{U}^{(n)})\tilde{w}^{(n)} + \beta\vec{g}^{(n)} \cdot \vec{h}^{(n)}] d\vec{x} +$$
$$\int_\Omega \gamma(\vec{u}^{(N)} - \vec{U}^{(N)})\tilde{w}^{(N)} d\vec{x} = \sum_{n=1}^{N} \int_\Omega [\vec{w}^{(n)} + \beta\vec{g}^{(n)}] \cdot \vec{h}^{(n)} d\vec{x}$$

where $\vec{w}^{(n)}$ is the solution of the system in eq(2.58). From the theorem 2.9 if $(\vec{\mathbf{u}}, \vec{\mathbf{g}})$ is a solution of the optimal problem the Gateaux derivative must be zero. Thus it follows that $\vec{g}^{(n)} = -\frac{1}{\beta}\vec{w}^{(n)}$ for all n. $\sqcup$
Now in order to get the solution of our optimal control problem we have to solve the Navier-Stokes system and the adjoint system

$$\begin{cases} \frac{1}{\Delta t}(\vec{u}^{(n)} - \vec{u}^{(n-1)}, \vec{v}) + \nu a(\vec{u}^{(n)}, \vec{v}) + c(\vec{u}^{(n)}; \vec{u}^{(n)}, \vec{v}) + \\ b(\vec{v}, p^{(n)}) = (\vec{g}^{(n)}, \vec{v}) \\ -\frac{1}{\Delta t}(\vec{g}^{(n+1)} - \vec{g}^{(n)}, \vec{v}) + \nu a(\vec{g}^{(n)}, \vec{v}) + c(\vec{u}^{(n)}; \vec{v}, \vec{g}^{(n)}) + c(\vec{v}; \vec{u}^{(n)}, \vec{g}^{(n)}) + \\ b(\vec{v}, \sigma^{(n)}) = -\frac{\alpha}{\beta}(\vec{u}^{(n)} - \vec{U}^{(n)}, \vec{v}) \quad \vec{v} \in H_0^1(\Omega) \\ b(\vec{u}^{(n)}, q) = 0 \\ b(\vec{g}^{(n)}, q) = 0 \quad \forall q \in L_0^2(\Omega) \end{cases}$$

for $n = 1, 2...N$ with final condition $\vec{g}^{(N+1)}(\vec{x}) = -\frac{\gamma}{\beta}(\vec{u}^{(N)} - \vec{U}^{(N)})$ and initial velocity $\vec{u}^{(0)}(\vec{x}) = \vec{u}_0(\vec{x})$. The above system of equations is the weak formulation of the following system

$$\begin{cases} \frac{1}{\Delta t}(\vec{u}^{(n)} - \vec{u}^{(n-1)}) + (\vec{u}^{(n)}.\vec{\nabla})\vec{u}^{(n)} - \nu\nabla^2\vec{u}^{(n)} + \vec{\nabla}p^{(n)} = \vec{g}^{(n)} \\ \vec{\nabla} \cdot \vec{u} = 0 \\ -\frac{1}{\Delta t}(\vec{g}^{(n+1)} - \vec{g}^{(n)}) + \nu\nabla^2\vec{g}^{(n)} + (\nabla\vec{u})^T\vec{g} - (\vec{u}^{(n)} \cdot \nabla)\vec{g}^{(n)} + \\ \vec{\nabla}\sigma^{(n)} = -\frac{\alpha}{\beta}(\vec{u}^{(n)} - \vec{U}^{(n)}) \\ \vec{\nabla} \cdot \vec{g}^{(n)} = 0 \end{cases} \tag{2.59}$$

for $n = 1, 2...N$ with the same initial, final and homogeneous boundary condition.

2.4 Fully discrete time-space approximation

2.4.1 Assumptions on the finite element spaces

We consider only conforming finite element approximations. Let $X^h \subset H_0^1(\Omega)$ and $S^h \subset L^2(\Omega)$ be two families of finite dimensional subspaces parameterized by h that tends to zero. We also denote $S_0^h = S^h \cap L_0^2(\Omega)$ We make the following assumptions on X^h and S^h :
a) the approximation hypotheses:
there exists an integer l and a constant C, independent of $h, \vec{u}$ and p, such that for $1 \le k \le l$

we have

$$\inf_{\vec{u}_h \in X^h} \|\vec{u}_h - \vec{u}\|_1 \leq Ch^k \|\vec{u}\|_{k+1} \quad \forall \vec{u} \in H^{k+1}(\Omega) \cap H_0^1(\Omega) \tag{2.60}$$

$$\inf_{p_h \in S^h} \|p - p_h\| \leq Ch^k \|p\|_k \qquad \forall p \in H^k(\Omega) \cap L_0^2(\Omega). \tag{2.61}$$

b) the inf-sup condition or L-B-B condition:
there exists a constant C' , independent of h such that

$$\inf_{0 \neq q_h \in S^h} \sup_{0 \neq \vec{u}_h \in X^h} \frac{\int_\Omega q_h div \vec{u}_h}{\|\vec{u}_h\|_1 \|q_h\|} \geq C' > 0. \tag{2.62}$$

This condition assures the stability of the Navier-Stokes discrete solutions.

To preserve the antisymmetry of the trilinear form $c(\vec{u}; \vec{v}, \vec{w})$ on the finite element spaces we introduce the modified trilinear form (see [35])

$$\tilde{c}(\vec{u}; \vec{v}, \vec{w}) = \frac{1}{2}\{c(\vec{u}; \vec{v}, \vec{w}) - c(\vec{u}; \vec{w}, \vec{v})\} \qquad \forall \vec{u}, \vec{v}, \vec{w} \in H_0^1(\Omega)$$

We can recall some useful formulas and inequalities in a two-dimensional domain Ω :

$$c(\vec{u}; \vec{v}, \vec{w}) = \tilde{c}(\vec{u}; \vec{v}, \vec{w}) \qquad \forall \vec{u} \in H_0^1(\Omega) \cap W(\Omega) \quad \forall \vec{v}, \vec{w} \in H_0^1(\Omega)$$

$$\begin{cases} \tilde{c}(\vec{u}; \vec{v}, \vec{w}) = -\tilde{c}(\vec{u}; \vec{w}, \vec{v}) \\ \tilde{c}(\vec{u}; \vec{v}, \vec{v}) = 0 \end{cases} \tag{2.63}$$

and [35]

$$\begin{cases} |\tilde{c}(\vec{u}; \vec{v}, \vec{w})| \leq K_1 \|\nabla \vec{u}\| \cdot \|\vec{v}\|_{L^4(\Omega)} \|\nabla \vec{w}^{(n)}\| \\ |\tilde{c}(\vec{u}; \vec{v}, \vec{w})| \leq K_2 \|\vec{u}\|^{\frac{1}{2}} \|\nabla \vec{u}\|^{\frac{1}{2}} \|\nabla \vec{v}\| \cdot \|\vec{w}\|^{\frac{1}{2}} \|\nabla \vec{w}\|^{\frac{1}{2}} \end{cases} \tag{2.64}$$

for all $\vec{u}, \vec{v}, \vec{w} \in H_0^1(\Omega)$. We remark that the inequality in eq(2.64) is true in the framework of the conforming finite element approximation and only in the two-dimensional case (see [35]).

2.4.2 Formulation of the fully discrete optimal control approximation

Let $\sigma_N = \{t_n\}_{n=0}^N$ be a partition of $[0, T]$ in equal intervals $\Delta t = T/N$ with $t_0 = 0$ and $t_N = T$. For each fixed Δt (or N) and for every quantity $q(t, \vec{x})$, we associate the corresponding set $\{q_h^{(n)}\}_{n=1}^N$. We will denote the vector $(q_h^{(1)}, q_h^{(2)}, ...q_h^{(N)})$ with bold letter $\mathbf{q}_h$ and the space

Y^N as $\mathbf{Y}$. The continuous linear function $\vec{q}_h^N(t,\vec{x})$ is defined by $\vec{q}_h^N(t_n,\vec{x}) = q_h(t_n,\vec{x})$ for all $n = 0, 1, 2...N$.

Given $\Delta t = T/N$, $\vec{\mathbf{g}} \in \mathbf{L^2(\Omega)}$ *and* $\vec{u}_0 \in V(\Omega)$, *then* $(\vec{\mathbf{u}}_h, \mathbf{p}_h)$ *is called a generalized solution for the Navier-Stokes fully discrete time-space approximation if* $u_h^{(n)} \in X^h$, $p_h^{(n)} \in S_0^h$ *and* $(u_h^{(n)}, p_h^{(n)})$ *satisfies the following system of equations*

$$\begin{cases} \frac{1}{\Delta t}(\vec{u}_h^{(n)} - \vec{u}_h^{(n-1)}, \vec{v}_h) + \nu a(\vec{u}_h^{(n)}, \vec{v}_h) + \tilde{c}(\vec{u}_h^{(n)}; \vec{u}_h^{(n)}, \vec{v}_h) + \\ b(\vec{v}_h, p^{(n)}) = (\vec{g}^{(n)}, \vec{v}_h) \quad \forall \vec{v}_h \in X^h(\Omega) \\ b(\vec{v}_h^{(n)}, q_h) = 0 \quad \forall q_h \in S_0^h(\Omega) \end{cases} \tag{2.65}$$

for n=1,2,...N with initial velocity $\vec{u}_h^{(0)} = \pi^h \vec{u}_0(\vec{x})$ *and homogeneous boundary condition.* The optimal control is achieved with the functional

$$L_h^N = \frac{\alpha \Delta t}{2} \sum_{n=1}^{N} \|\vec{u}_h^{(n)} - \vec{U}^{(n)}\|^2 + \frac{\beta \Delta t}{2} \sum_{n=1}^{N} \|\vec{g}^{(n)}\|^2 + \frac{\gamma}{2}\|\vec{u}_h^{(N)} - \vec{U}^N\|^2. \tag{2.66}$$

The formulation of the problem P_L in the fully discrete approximation becomes: *given* $\Delta t = T/N$, $\vec{u}_0 \in V(\Omega)$ *and* $\vec{U} \in U_{ad}$, *find* $(\vec{\mathbf{u}}_h, \mathbf{p}_h, \vec{\mathbf{g}})$ *in* $\mathbf{X^h(\Omega)} \times \mathbf{S_0^h(\Omega)} \times \mathbf{L^2(\Omega)})$ *such that* $(\vec{u}_h^{(n)}, p_h^{(n)})$ *is the solution of eq(2.65) and* $\vec{\mathbf{g}}$ *minimizes the cost function in eq(2.66).*

In analogy with the semidiscrete case for the problem defined in eq(2.65) we have this useful theorem [35].

Theorem 2.12 *Given* $\Delta t = T/N$, *if* $\vec{\mathbf{g}} \in \mathbf{L^2(\Omega)}$ *then there exists at least one* $\vec{\mathbf{u}}_h$ *satisfying the eq(2.65) with the following estimates*

$$\|\vec{u}_h^{(n)}\|^2 \leq d \quad n = 1, 2, ...N \tag{2.67}$$

$$\Delta t \sum_{n=1}^{N} \|\nabla \vec{u}_h^{(n)}\|^2 \leq \frac{d}{\nu} \tag{2.68}$$

$$\sum_{n=1}^{N} \|\vec{u}_h^{(n)} - \vec{u}_h^{(n-1)}\|^2 \leq d \cdot K \tag{2.69}$$

where $d = \|\vec{u}_h^{(0)}\|^2 + \sum_{n=1}^{N} \|\vec{g}^{(n)}\|^2 \Delta t$ *and* K *a constant independent of* Δ *and* h.
Hence $\vec{u}_h^N \in L^\infty((0,T); X^h)$ *and* $\vec{u}_h^N \in L^2((0,T); L^2(\Omega))$.

Also if $\vec{g}^N \to \vec{g} \in L^2((0,T); L^2(\Omega))$ then $\vec{u}_h^N \to \vec{u}$ where $\vec{u}$ is the solution of the continuous Navier-Stokes system of equations.

2.4.3 Existence and consistency of the fully discrete optimal control solution

In order to prove the existence and consistency of the fully discrete optimal control problem we need to show that $\vec{g}^N$ belongs to $L^2((0,T); L^2(\Omega))$.

Lemma 2.7 *Let $\vec{u}_0 \in V(\Omega)$ and $\vec{\mathbf{U}} \in U_{ad}^N$. If $(\vec{\mathbf{u}}_h, \vec{\mathbf{g}})$ is the solution of the fully discrete optimal control problem then there exists two constants C_1 and C_2 independent of Δt and h (for small h) such that*

$$\sum_{n=1}^{N} \|\vec{g}^{(n)}\|^2 \Delta t \leq C_1 \tag{2.70}$$

$$\sum_{n=1}^{N} \|\vec{u}_h^{(n)}\|^2 \Delta t \leq C_2. \tag{2.71}$$

Hence we have $\vec{g}^N \in L^2((0,T); L^2(\Omega))$ and $\vec{u}_h^N \in L^2((0,T); L^2(\Omega))$ for all N.

Proof: The proof is substantially the same as in the semidiscrete case. If $\vec{\mathbf{g}} = 0$, then the functional can be bounded as

$$\begin{aligned} L(\vec{\mathbf{g}}) \leq L(\mathbf{0}) = \frac{T\alpha + \gamma}{2} \left(\|\vec{U}\|^2_{L^\infty((0,T);W)} + \|\pi^h \vec{u}_0\|^2 \right) \leq \\ \frac{T\alpha + \gamma}{2} \left(\|\vec{U}\|^2_{L^\infty((0,T);V)} + \|\vec{u}_0\|^2 \right) = C_1 \end{aligned} \tag{2.72}$$

which implies eq(2.70) and eq(2.71). In the last inequality, we used the approximation hypothesis in eq(2.60). ⊔

Now the existence and the consistency theorems of the optimal control problem in the fully discrete approximation are particular cases of the semidiscrete ones. We state the first and sketch the proof for the latter one

Theorem 2.13 *Given $\Delta t = T/N$, $\vec{u}_0 \in V(\Omega)$ and $\vec{U} \in U_{ad}$, there exists a sequence $(\vec{\mathbf{u}}_h, \vec{\mathbf{g}})$ in $\mathbf{X^h} \times \mathbf{L^2(\Omega)}$ such that $\vec{\mathbf{u}}_h$ is the solution of eq(2.65) and $\vec{\mathbf{g}}$ minimizes the cost function in eq(2.66).*

Theorem 2.14 *Let $\Delta t = T/N$ and $\vec{u}_0^{(n)}$ belong to $V(\Omega)\}_{n=1}^N$. The solution $(\vec{\mathbf{u}}_\mathbf{h}, \vec{\mathbf{g}})$ for the fully discrete optimal control problem tends to the optimal control solution $(\vec{u}, \vec{g})$ of the continuous problem as $\Delta t \to 0$ ($N \to \infty$) and $h \to 0$.*

Proof: Let $X^h \subset H_0^1(\Omega)$ and $S^h \subset L^2(\Omega)$ be some families of finite dimensional subspaces parameterized by h that tends to zero. Let also $\Delta t = T/N$, and $\vec{u}_h^{\prime(n)}$ be defined by $(\vec{u}_h^{(n)} - \vec{u}_h^{(n-1)})/\Delta t$. The sequences $\{\vec{u}_h^N\}_{N=1}^\infty$, $\{\vec{g}^N\}_{N=1}^\infty$ and $\{\vec{u}_h^{\prime N}\}_{N=1}^\infty$ are uniformly bounded in $L^2((0,T); H_0^1(\Omega)) \cap L^\infty((0,T); L^2(\Omega))$, $L^2((0,T); L^2(\Omega))$ and $L^2((0,T); H^{-1}(\Omega))$ respectively. Hence, we can extract from these sequences some subsequences such that

$$\begin{cases} \vec{u}^K \to \vec{u} & L^2(0,T,H_0^1(\Omega)) \text{ weakly} \\ \vec{u}^K \to \vec{u} & L^\infty(0,T,L^2(\Omega)) \text{ *-weakly} \\ \vec{g}^K \to \vec{g} & L^2(0,T,L^2(\Omega)) \text{ weakly} \\ \vec{u}'^K \to \vec{u}' & L^2(0,T,H^{-1}(\Omega)) \text{ weakly} \end{cases} \tag{2.73}$$

where the index K is comprehensive of N and h. As a consequence of the compactness theorem, the convergence of the sequence $\vec{u}^K$ is a strong convergence in $L^2((0,T);L^2(\Omega))$. Now we can pass to the limit in the system of equations and in the functional. Using the fact that the sequence converges weakly in $L^2((0,T);H_0^1)$ and strongly in $L^2((0,T);L^2(\Omega))$ we can pass to the limit in the nonlinear term. The fully discrete time-space optimal control approximation for $N\to\infty$ and $h\to 0$ is consistent with the continuous optimal control problem. $\sqcup$

2.4.4 First-order necessary condition

In this section we proceed to derive the first-order necessary condition in terms of Gateaux derivatives. Before stating the equation for this possible optimal solution, we need some auxiliary results similar to the continuous and semi-discrete case.

Lemma 2.8 *Let $\Delta t = T/N$ and $\lambda,\sigma \geq 0$. Let $\vec{w}_h^{(n)}$ be the solution of the system of equations*

$$\begin{cases} \frac{1}{\Delta t}(\vec{w}_h^{(n)} - \vec{w}_h^{(n-1)}, \vec{v}_h) + \nu a(\vec{w}_h^{(n)}, \vec{v}_h) + \lambda[\tilde{c}(\vec{w}_h^{(n)}; \vec{u}_h^{(n)}, \vec{v}_h) + \\ \tilde{c}(\vec{u}_h^{(n)}; \vec{w}_h^{(n)}, \vec{v}_h)] + \sigma\tilde{c}(\vec{w}_h^{(n)}; \vec{w}_h^{(n)}, \vec{v}_h) + b(\vec{v}_h, p^{(n)}) = (\vec{g}^{(n)}, \vec{v}_h) \\ b(\vec{w}_h^{(n)}, q_h) = 0 \quad \forall q_h \in S^h(\Omega) \quad \forall \vec{v}_h \in X^h(\Omega) \end{cases} \tag{2.74}$$

for n=1,2...N with initial value $\vec{w}_h^{(0)}(\vec{x}) = 0$ and homogeneous boundary condition. Given $\{\vec{u}_h^N\}_{N=1}^\infty$ and $\{\vec{g}^N\}_{N=1}^\infty$ sequences in $L^\infty((0,T);X^h(\Omega))$ and in $L^\infty((0,T);L^2(\Omega))$ respectively, there exists a N_1 such that if $N \geq N_1$ then $\vec{w}_h^N$ is in $L^\infty((0,T);X^h(\Omega))$ and

$$\sum_{n=1}^N \|\vec{w}_h^{(n)}\|_1^2 \Delta t \leq C_1 \sum_{n=1}^N \|\vec{g}^{(n)}\|^2 \Delta t \tag{2.75}$$

or

$$\|\vec{w}_h^{(n)}\|_1^2 \leq C_1 \sum_{n=1}^N \|\vec{g}^{(n)}\|^2 \quad n = 1...N. \tag{2.76}$$

where C_1, is independent of Δt.

Proof: We take the scalar product with $\vec{w}$ and using the orthogonality propriety of the trilinear form in eq(2.63) we obtain

$$\frac{1}{\Delta t}(\vec{w}_h^{(n)} - \vec{w}_h^{(n-1)}, \vec{w}_h^{(n)}) + \nu\|\nabla \vec{w}_h^{(n)}\|^2 + \delta\tilde{c}(\vec{w}_h^{(n)}; \vec{u}_h^{(n)}, \vec{w}_h^{(n)}) = (\vec{g}^{(n)}, \vec{w}_h^{(n)}).$$

The trilinear form can be bounded using this particular two dimensional estimate in eq(2.64)

$$|\tilde{c}(\vec{w};\vec{u},\vec{w})| \leq K_1\|\nabla\vec{u}\|\cdot\|\vec{w}\|\cdot\|\nabla\vec{w}\| \quad \forall \vec{u},\vec{w} \in X^h(\Omega).$$

Using Poincare's and Young's inequality we can bound the right-hand side as

$$|(\vec{g}^{(n)}, \vec{w}_h^{(n)})| \leq \|\vec{w}_h^{(n)}\| \|\vec{g}^{(n)}\| \leq \lambda \|\nabla \vec{w}_h^{(n)}\| \cdot \|\vec{g}^{(n)}\| \leq \frac{\nu}{2}\|\nabla \vec{w}_h^{(n)}\| + \frac{\lambda}{2\nu}\|\vec{g}^{(n)}\|^2.$$

and thus we have

$$\frac{1}{\Delta t}(\vec{w}_h^{(n)} - \vec{w}^{(n-1)}, \vec{w}_h^{(n)}) + \frac{\nu}{2}\|\nabla \vec{w}_h^{(n)}\|^2 \leq \frac{\lambda}{2\nu}\|\vec{g}^{(n)}\|^2 + \\ \delta K_1 \|\nabla \vec{u}_h^{(n)}\| \cdot \|\vec{w}_h^{(n)}\| \cdot \|\nabla \vec{w}_h^{(n)}\|$$

where λ can be written in terms of Poincare's constant. Applying again the Young's inequality to the last right-hand side term we have

$$\frac{1}{\Delta t}(\vec{w}_h^{(n)} - \vec{w}^{(n-1)}, \vec{w}_h^{(n)}) + \frac{\nu}{4}\|\nabla \vec{w}_h^{(n)}\|^2 \leq \frac{\lambda}{2\nu}\|\vec{g}^{(n)}\|^2 + \frac{\phi_h^{(n)}}{4}\|\vec{w}_h^{(n)}\|^2$$

where $\phi_h^{(n)} = \frac{4K_1\delta}{\nu}\|\nabla \vec{u}_h^{(n)}\|^2$. Now

$$2(\vec{a} - \vec{b}, \vec{a}) = \|\vec{a}\|^2 - \|\vec{b}\|^2 + \|\vec{a} - \vec{b}\|^2 \quad \forall \vec{a}, \vec{b} \tag{2.77}$$

so that

$$\|\vec{w}_h^{(n)}\|^2 - \|\vec{w}^{(n-1)}\|^2 + \|\vec{w}_h^{(n)} - \vec{w}^{(n-1)}\|^2 + \frac{\nu \Delta t}{2}\|\nabla \vec{w}_h^{(n)}\|^2 \leq \\ \frac{2\lambda \Delta t}{\nu}\|\vec{g}^{(n)}\|^2 + \frac{\phi_h^{(n)}\Delta t}{2}\|\vec{w}_h^{(n)}\|^2.$$

Since $\vec{u}_h^N$ belongs to $L^\infty((0,T); X^h)$ then there exists a $(\Delta t)_1$ such that

$$1 - \frac{\phi_h^{(n)}\Delta t}{2} \leq e^{-(\phi_h^{(n)}\Delta t)}$$

for all $\Delta t \leq (\Delta t)_1$. As

$$\phi_h^{(n)} = \sum_{m=1}^{n} \phi_h^{(m)} - \sum_{m=1}^{n-1} \phi_h^{(m)}$$

we have

$$\|\vec{w}_h^{(n)}\|^2 e^{-\phi_h^{(n)}\Delta t} - \|\vec{w}^{(n-1)}\|^2 + \|\vec{w}_h^{(n)} - \vec{w}_h^{(n-1)}\|^2 + \\ \frac{\nu \Delta t}{2}\|\nabla \vec{w}_h^{(n)}\|^2 \leq \frac{2\lambda \Delta t}{\nu}\|\vec{g}^{(n)}\|^2$$

or

$$\|\vec{w}_h^{(n)}\|^2 e^{-\left(\Delta t \sum_{m=1}^{n} \phi_h^{(m)}\right)} + e^{-\left(C\Delta t \sum_{m=1}^{n-1} \phi_h^{(m)}\right)} \times \\ \left(\|\vec{w}_h^{(n)} - \vec{w}_h^{(n-1)}\|^2 - \|\vec{w}_h^{(n-1)}\|^2 + \frac{\nu \Delta t}{2}\|\nabla \vec{w}_h^{(n)}\|^2\right) \leq \frac{2\lambda \Delta t}{\nu}\|\vec{g}^{(n)}\|^2$$

Summing the inequalities for $n = 1, ...k$ and rearranging some terms $\|\vec{u}^{(k)}\|^2$, we get

$$\|\vec{w}_h^{(k)}\|^2 \leq \frac{2\lambda\Delta t}{\nu} e^{\left(\Delta t \sum_{m=1}^{N} \phi_h^{(m)}\right)} \sum_{n=1}^{N} \|\vec{g}^{(n)}\|^2 \tag{2.78}$$

which implies

$$\sum_{n=1}^{N} \|\vec{w}_h^{(n)}\|_1^2 \Delta t \leq (1 + C_p) \sum_{n=1}^{N} \|\nabla \vec{w}_h^{(n)}\|^2 \Delta t \leq C_1 \sum_{n=1}^{N} \|\vec{g}^{(n)}\|^2 \Delta t$$

where C_p is the Poisson constant. $\sqcup$

The existence of the Gateaux derivative has been proved in the semidiscrete case for small Δt. The fully discrete version of the theorem can be proved with same procedure. In the proof, eq(2.75- 2.76) substitute eq(2.43- 2.44). In this case we are in $X^h \subset H_0^1$ and so the estimate is not so good. The eq(2.55) becomes

$$\|\hat{u}_h^{(n)}\|_1^2 = \|\vec{u}_h^{(n)}(\vec{\mathbf{g}} + s\delta\vec{\mathbf{g}}) - \vec{u}_h^{(n)}(\vec{\mathbf{g}})\|_1^2 \leq C_1 |s|^2 \sum_{n=1}^{N} \|\vec{h}^{(n)}\|^2. \tag{2.79}$$

N is a fixed number and the proof follows. For N that tends to ∞ the differentiability is assured by the consistency theorem 2.14.

Theorem 2.15 *Let $\Delta t = T/N$, $\vec{u}_0 \in V(\Omega)$ and $\delta\vec{\mathbf{g}}_1$ be in $\mathbf{X^h}$.*
i) There exists a small Δt_1 such that the mapping $\vec{\mathbf{u}}_h = \vec{\mathbf{u}}_h(\vec{\mathbf{g}}_1)$ from $\mathbf{L^2(\Omega)}$ to $\mathbf{X^h}$ has a Gateaux derivative $\frac{D\vec{\mathbf{u}}_h}{D\vec{\mathbf{g}}_1} \cdot \delta\vec{\mathbf{g}}_1$ for all $\Delta t \leq \Delta t_1$. The function $\tilde{w}_h^{(n)}(\delta\vec{\mathbf{g}}_1) = \frac{D\vec{u}_h^{(n)}}{D\vec{\mathbf{g}}_1} \cdot \delta\vec{\mathbf{g}}_1$ is a solution of this system of equations

$$\begin{cases} \frac{1}{\Delta t}(\tilde{w}_h^{(n)} - \tilde{w}_h^{(n-1)}, \vec{v}_h) + \nu a(\tilde{w}_h^{(n)}, \vec{v}_h) + \tilde{c}(\tilde{w}_h^{(n)}; \vec{u}_h^{(n)}, \vec{v}_h) + \\ \tilde{c}(\vec{u}_h^{(n)}; \tilde{w}_h^{(n)}, \vec{v}_h) + b(\vec{v}_h, \sigma_h^{(n)}) = (\delta\vec{g}_1^{(n)}, \vec{v}_h) \quad \forall \vec{v}_h \in X^h(\Omega) \\ b(\tilde{w}_h^{(n)}, q_h) = 0 \quad \forall q_h \in S_0^h(\Omega) \end{cases} \tag{2.80}$$

for n=1,2,...N with initial value $\tilde{w}_h^{(0)}(\vec{x}) = 0$ and homogeneous boundary condition.
ii) Also if $\delta\vec{\mathbf{g}}_2$ is in $\mathbf{L^2(\Omega)}$ then $\tilde{w}^N$ is in $L^\infty((0,T); X^h)$ and

$$\sum_{n=1}^{N} \int_\Omega \delta\vec{g}_2^{(n)} \tilde{w}_h^{(n)}(\delta\vec{\mathbf{g}}_1) d\vec{x} = \sum_{n=1}^{N} \int_\Omega \vec{w}_h^{(n)}(\delta\vec{\mathbf{g}}_2) \delta\vec{g}_1^{(n)} d\vec{x} - \int_\Omega \vec{w}_h^{(N+1)} \tilde{w}_h^{(n)} d\vec{x} \tag{2.81}$$

where $\vec{w}_h^{(n)}$ is the solution of the adjoint problem

$$\begin{cases} \frac{1}{\Delta t}(\vec{w}_h^{(n)} - \vec{w}_h^{(n+1)}, \vec{v}_h) + \nu a(\vec{w}_h^{(n)}, \vec{v}_h) + \tilde{c}(\vec{u}_h^{(n)}; \vec{v}_h, \vec{w}_h^{(n)}) + \\ \tilde{c}(\vec{v}_h; \vec{u}_h^{(n)}, \vec{w}_h^{(n)}) + b(\vec{v}_h, \sigma_h^{(n)}) = (\delta\vec{g}_2^{(n)}, \vec{v}_h) \quad \vec{v}_h \in X^h(\Omega) \\ b(\vec{w}_h^{(n)}, q_h) = 0 \quad q_h \in S_0^h(\Omega) \end{cases} \tag{2.82}$$

for n=1,2,....N with final value $\vec{w}^{(N+1)}(\vec{x}) = \gamma(\vec{u}^N - \vec{U}^N)$ and homogeneous boundary condition.

Finally we can state and prove the theorem that give the control as the solution of the adjoint equation

Theorem 2.16 *Let $\vec{u}_0 \in V(\Omega)$, $\vec{U} \in U_{ad}$ and $\Delta t = T/N$. If $(\vec{\mathbf{u}}_h, \vec{\mathbf{g}})$ is solution of the fully discrete optimal control problem then there exists a $(\Delta t)^*$ such that for each $\Delta t \leq (\Delta t)^*$ and for all n=1,2,...N we have $\vec{g}^{(n)} = -\frac{1}{\beta}\vec{w}_h^{(n)}$. The function $\vec{w}_h^{(n)}$ is the solution of the adjoint problem in eq(2.82)*

Proof: Let $(\vec{\mathbf{u}}_h, \vec{\mathbf{g}})$ be an optimal pair solution of the fully discrete optimal control problem. By using the theorem 2.15 we can compute the Gateux derivative of the functional $L(\vec{\mathbf{g}})$ in the direction of $\delta\vec{\mathbf{g}}$. We have for every $\delta\vec{\mathbf{g}} \in \mathbf{L^2(\Omega)}$

$$\frac{DL_h^N}{D\vec{\mathbf{g}}} \cdot \delta\vec{\mathbf{g}} = \sum_{n=1}^{N} \int_\Omega [\alpha(\vec{u}_h^{(n)} - \vec{U}^{(n)}) \left(\frac{D\vec{u}_h^{(n)}}{D\vec{\mathbf{g}}} \cdot \delta\vec{\mathbf{g}}\right) + \beta\vec{g}^{(n)} \cdot \delta\vec{g}^{(n)}] d\vec{x} +$$
$$\int_\Omega \gamma(\vec{u}_h^{(N)} - \vec{U}^{(N)}) \left(\frac{D\vec{u}_h^{(N)}}{D\vec{\mathbf{g}}} \cdot \delta\vec{\mathbf{g}} d\vec{x}\right) =$$
$$\sum_{n=1}^{N} \int_\Omega [\alpha(\vec{u}_h^{(n)} - \vec{U}^{(n)})\tilde{w}_h^{(n)} + \beta\vec{g}^{(n)} \cdot \delta\vec{g}^{(n)}] d\vec{x} +$$
$$\int_\Omega \gamma(\vec{u}_h^{(N)} - \vec{U}^{(N)})\vec{w}_h^{(N)} d\vec{x}.$$

We can use the lemma 2.8 to complete the proof as $\vec{u}_h^N - \vec{U}^N$ is in $L^2((0,T);L^2(\Omega))$. We have

$$\frac{DL_h^N}{D\vec{\mathbf{g}}} \cdot \delta\vec{\mathbf{g}} = \sum_{n=1}^{N} \int_\Omega [\alpha(\vec{u}_h^{(n)} - \vec{U}_h^{(n)})\tilde{w}_h^{(n)} + \beta\vec{g}^{(n)} \cdot \delta\vec{g}^{(n)}] d\vec{x} +$$
$$\int_\Omega \gamma(\vec{u}_h^{(N)} - \vec{U}^{(N)})\tilde{w}_h^{(N)} d\vec{x} = \sum_{n=1}^{N} \int_\Omega [\vec{w}_h^{(n)} + \beta\vec{g}^{(n)}] \cdot \delta\vec{g}^{(n)} d\vec{x}$$

where $\vec{w}_h^{(n)}$ is the solution of the system in eq(2.82). If $(\vec{\mathbf{u}}_h, \vec{\mathbf{g}})$ is a solution of the optimal problem, the Gateaux derivative must be zero. Since $\delta\vec{\mathbf{g}}, \vec{\mathbf{w}}_h$ and $\vec{\mathbf{g}}$ are in $\mathbf{L^2(\Omega)}$, we can set $\delta\vec{g}^{(n)} = \vec{w}_h^{(n)} + \beta\vec{g}^{(n)}$. Hence it follows that $\vec{g}^{(n)} = -\frac{1}{\beta}\vec{w}_h^{(n)}$ for n=1,2...N $\sqcup$.

2.5 Numerical results

2.5.1 Introduction

In order to determine the optimal control solution we have to solve the following system of equations in the state and control variables $(\vec{u}, p, \vec{g}, \sigma)$

$$\begin{cases} \vec{u}_t + (\vec{u}.\vec{\nabla})\vec{u} - \nu\nabla^2\vec{u} + \vec{\nabla}p = \vec{g} \\ \vec{\nabla}\cdot\vec{u} = 0 \\ -\vec{g}_t + \nu\nabla^2\vec{g} + (\nabla\vec{u})^T\vec{g} - (\vec{u}\cdot\nabla)\vec{g} + \vec{\nabla}\sigma = -\frac{\alpha}{\beta}(\vec{u} - \vec{U}) \\ \vec{\nabla}\cdot\vec{g} = 0 \end{cases} \tag{2.83}$$

with final condition $\vec{g}(T,\vec{x}) = -\frac{\gamma}{\beta}(\vec{u}(T,\vec{x}) - \vec{U}(T,\vec{x}))$, initial velocity $\vec{u}^{(0)}(\vec{x}) = \vec{u}_0(\vec{x})$ and homogeneous boundary conditions.

Let $\sigma_N = \{t_n\}_{n=0}^N$ be a partition of $[0,T]$ into equal intervals $\Delta t = T/N$ with $t_0 = 0$ and $t_N = T$. For a fixed Δt (or N) let $X^h \subset H_0^1(\Omega)$ and $S_0^h \subset L^2(\Omega)$ be two families of finite dimensional subspaces parameterized by h that tends to zero. The eq(2.83) becomes

$$\begin{cases} \frac{1}{\Delta t}(\vec{u}_h^{(n)} - \vec{u}_h^{(n-1)}, \vec{v}_h) + \nu a(\vec{u}_h^{(n)}, \vec{v}_h) + \tilde{c}(\vec{u}_h^{(n)}; \vec{u}_h^{(n)}, \vec{v}_h) + \\ b(\vec{v}_h, p_h^{(n)}) = (\vec{g}_h^{(n)}, \vec{v}_h) \\ -\frac{1}{\Delta t}(\vec{g}_h^{(n+1)} - \vec{g}_h^{(n)}, \vec{v}_h) + \nu a(\vec{g}_h^{(n)}, \vec{v}_h) + \tilde{c}(\vec{u}_h^{(n)}; \vec{v}_h, \vec{g}_h^{(n)}) + \\ \tilde{c}(\vec{v}_h; \vec{u}_h^{(n)}, \vec{g}_h^{(n)}) + b(\vec{v}_h, \sigma_h^{(n)}) = -\frac{\alpha}{\beta}(\vec{u}_h^{(n)} - \vec{U}^{(n)}, \vec{v}_h) \\ b(\vec{u}_h^{(n)}, q_h) = 0 \\ b(\vec{g}_h^{(n)}, q_h) = 0 \qquad \forall q_h \in S_0^h(\Omega) \quad \vec{v}_h \in X^h(\Omega) \end{cases} \tag{2.84}$$

for $n = 1, 2, ...N$ final condition $\vec{g}_h^{(N+1)}(\vec{x}) = -\frac{\gamma}{\beta}(\vec{u}_h^{(N)} - \vec{U}^{(N)})$, initial velocity $\vec{u}_h^{(0)}(\vec{x}) = \pi^h\vec{u}_0(\vec{x})$ and homogeneous boundary conditions.

2.5.2 Numerical algorithm

Let us consider the gradient method for the optimal control problem. We have to split the system in two parts in order to apply the algorithm. Now the fully discrete system consists of:
a) Navier-Stokes equations

$$\begin{cases} \frac{1}{\Delta t}(\vec{u}_h^{(n)} - \vec{u}_h^{(n-1)}, \vec{v}_h) + \nu a(\vec{u}_h^{(n)}, \vec{v}_h) + c(\vec{u}_h^{(n)}; \vec{u}_h^{(n)}, \vec{v}_h) + \\ b(\vec{v}_h, p_h^{(n)}) = (\vec{g}_h^{(n)}, \vec{v}_h) \\ b(\vec{u}_h^{(n)}, q_h) = 0 \qquad \forall q_h \in S_0^h(\Omega) \quad \vec{v}_h \in X^h(\Omega) \end{cases} \tag{2.85}$$

for $n = 1, 2, ...N$ with initial velocity $\vec{u}^{(0)}(\vec{x})_h = \pi^h\vec{u}_0(\vec{x})$ and homogeneous boundary condition;

b) adjoint equation :

$$\begin{cases} -\frac{1}{\Delta t}(\vec{w}_h^{(n+1)} - \vec{w}_h^{(n)}, \vec{v}_h) + \nu a(\vec{w}_h^{(n)}, \vec{v}_h) + c(\vec{u}_h^{(n)}; \vec{v}_h, \vec{w}_h^{(n)}) + \\ c(\vec{v}_h; \vec{u}_h^{(n)}; \vec{w}_h^{(n)}) + b(\vec{v}_h, \sigma_h^{(n)}) = -\alpha(\vec{u}_h^{(n)} - \vec{U}^{(n)}, \vec{v}_h) \\ b(\vec{w}_h^{(n)}, q_h) = 0 \qquad \forall q_h \in S_0^h(\Omega) \quad \vec{v}_h \in X^h(\Omega) \end{cases} \tag{2.86}$$

for $n = 1, 2...N$ with final condition $\vec{w}_h^{(N+1)}(\vec{x}) = -\gamma(\vec{u}_h^{(N)} - \vec{U}^{(N)})$ and homogeneous boundary condition.
The optimal control variable $\vec{g}$ is related to $\vec{w}$ by the equation $\vec{g} = \frac{1}{\beta}\vec{w}$. In the gradient algorithm we satisfy this relation only when convergence is reached. Let $L^{(k)} = L(\vec{\mathbf{g}}_h(k))$ and τ be the tolerance required for the convergence of the functional.

The gradient algorithm proceeds as follows:

a) initial configuration:
i) given $\vec{\mathbf{g}}_h(0), \tau$ and $\epsilon = 1$;
ii) solve for $\vec{\mathbf{u}}_h(0)$ in eq(2.85) with $\vec{\mathbf{g}}_h(0)$;
iii) evaluate $L^{(0)}$;

b) main loop :
iv) solve for $\vec{\mathbf{w}}_h(k)$ in eq(2.86) with $\vec{\mathbf{u}}_h(k-1)$;

c) optimization loop:
v) with $\vec{g}_h^{(n)}(k) = \vec{g}_h^{(n)}(k-1) - \epsilon\left(\beta\vec{g}_h^{(n)}(k-1) - \vec{w}_h^{(n)}(k)\right)$ solve for $\vec{\mathbf{u}}_h(k)$ in eq(2.85);
vi) check if $L^{(k)}$ is less than $L^{(k-1)}$; if $L^{(k)} \leq L^{(k-1)}$ then $\epsilon = 1.5\epsilon$ and go to b); if $L^{(k)} > L^{(k-1)}$ then $\epsilon = .5\epsilon$ and go to c).

The algorithm stops when $|L^{(k)} - L^{(k-1)}|/L^{(k)} \leq \tau$. The idea stems from the expansion in Taylor's series of the functional and the convergence of the algorithm is a direct consequence of the following lemma.

Lemma 2.9 *Let J be a real-valued functional on a Hilbert space X with norm $\|\cdot\|$ and scalar product $<>$. We make the following assumptions:*
i) J is of class C^2 and has a local minimum at a point $\tilde{x}$;
ii) there exists two real numbers r and s and a ball B of X centered at $\tilde{x}$ such that for all $x, y \in X$ and for all u in B we have

$$\begin{cases} J"(u)(x,y) \leq s\|x\| \cdot \|y\| \\ \|x\|^2 r \leq J"(u)(x,x) \end{cases} \tag{2.87}$$

where $J"(u)(x,y)$ is the bilinear form associated with the second derivative. Then, the gradient algorithm converges with initial value $x_0 \in B$ to $\tilde{x}$.

The proof can be found in [56]. Using this lemma we can show that the gradient algorithm converges to the solution.

Theorem 2.17 *Let $(\vec{\mathbf{u}}_h(k), \vec{\mathbf{w}}_h(k), \mathbf{p}_h(k), \sigma_h(k), \vec{\mathbf{g}}_h(k))$ be the k-th step solution of the gradient algorithm and $(\vec{\mathbf{u}}_h, \vec{\mathbf{w}}_h, \mathbf{p}_h, \sigma_h, \vec{\mathbf{g}}_h)$ be the solution of the eq(2.84). Then, there exists a Δt_1 such that if $\Delta t \leq \Delta t_1$, the solution of the gradient algorithm converges to $(\vec{\mathbf{u}}_h, \vec{\mathbf{w}}_h, \mathbf{p}_h, \sigma_h, \vec{\mathbf{g}}_h)$ for any initial guess $\vec{\mathbf{g}}_h(0)$ when $k \to \infty$.*

Proof: In order to prove this theorem we have to satisfy the hypotheses of the lemma 2.9. Let Δt be equal to T/N then for each $\vec{\mathbf{g}}_h$ in $L^2((0,T);X^h)$ the second Frechet derivative $\frac{D^2 L_h^N(\vec{\mathbf{g}}_h)}{D\vec{\mathbf{g}}_{1h} D\vec{\mathbf{g}}_{2h}} \cdot \delta\vec{\mathbf{g}}_{1h}\delta\vec{\mathbf{g}}_{2h}$ can be computed by

$$\frac{D^2 L_h^N(\vec{\mathbf{g}}_h)}{D\vec{\mathbf{g}}_{1h} D\vec{\mathbf{g}}_{2h}} \cdot \delta\vec{\mathbf{g}}_{1h} \cdot \delta\vec{\mathbf{g}}_{2h} = \alpha \sum_{n=1}^{N} \int_\Omega \left(\frac{D\vec{u}_h^{(n)}}{D\vec{\mathbf{g}}_{1h}} \cdot \delta\vec{\mathbf{g}}_{1h} \right) \left(\frac{D\vec{u}_h^{(n)}}{D\vec{\mathbf{g}}_{1h}} \cdot \delta\vec{\mathbf{g}}_{2h} \right) d\vec{x}\Delta t +$$
$$\beta \sum_{n=1}^{N} \int_\Omega \delta\vec{g}_{1h}^{(n)} \delta\vec{g}_{2h}^{(n)} d\vec{x}\Delta t + \gamma \int_\Omega \left(\frac{D\vec{u}_h^{(N)}}{D\vec{\mathbf{g}}_h} \cdot \delta\vec{\mathbf{g}}_{1h} \right) \left(\frac{D\vec{u}_h^{(N)}}{D\vec{\mathbf{g}}_h} \cdot \delta\vec{\mathbf{g}}_{2h} \right) d\vec{x}$$

where $\tilde{w}_{h1}^{(n)} = \frac{D\vec{u}_h^{(n)}}{D\vec{\mathbf{g}}_h} \cdot \delta\vec{\mathbf{g}}_{1h}$ and $\tilde{w}_{h2}^{(n)} = \frac{D\vec{u}_h^{(n)}}{D\vec{\mathbf{g}}_h} \cdot \delta\vec{\mathbf{g}}_{2h}$ are the Gateaux derivatives. Now from lemma 2.8 we have an estimate. There exists a Δt_1 such that if $\Delta t \leq \Delta t_1$ we have

$$\sum_{n=1}^{N} \|\tilde{w}_{1h}^{(n)}\|^2 \Delta t \leq C_1 \sum_{n=1}^{N} \|\delta\vec{g}_{1h}^{(n)}\|_1^2 \Delta t$$
$$\sum_{n=1}^{N} \|\tilde{w}_{2h}^{(n)}\|^2 \Delta t \leq C_1 \sum_{n=1}^{N} \|\delta\vec{g}_{2h}^{(n)}\|_1^2 \Delta t \tag{2.88}$$

where C_1 is a constant. It follows that there exists a constant r such that

$$\frac{D^2 L_h^N(\vec{\mathbf{g}}_h)}{D\vec{\mathbf{g}}_{1h} D\vec{\mathbf{g}}_{2h}} \cdot \delta\vec{\mathbf{g}}_{1h} \cdot \delta\vec{\mathbf{g}}_{2h} \leq r \sum_{n=1}^{N} \|\delta\vec{g}_{1h}^{(n)}\|_1^2 \Delta t \sum_{n=1}^{N} \|\delta\vec{g}_{2h}^{(n)}\|_1^2 \Delta t.$$

Also there exists a constant s such that

$$\frac{D^2 L_h^N(\vec{\mathbf{g}}_h)}{D\vec{\mathbf{g}}_{1h} D\vec{\mathbf{g}}_{2h}} \delta\vec{\mathbf{g}}_{1h}\delta\vec{\mathbf{g}}_{1h} = \alpha \sum_{n=1}^{N} \|\tilde{w}_{h1}^{(n)}\|^2 \Delta t +$$
$$\beta \sum_{n=1}^{N} \|\delta\vec{g}_{1h}^{(n)}\|^2 \Delta t + \gamma \|\tilde{w}_{h1}^{(N)}\|^2 \geq s \sum_{n=1}^{N} \|\delta\vec{\mathbf{g}}_{h2}\|_{L^2((0,T);X^h)}^2 \Delta t.$$

The fact that no limitations are imposed assures that for every initial guess the gradient algorithm converges for small Δt. $\sqcup$

2.5.3 Test 1

We consider a unit square domain $(0,1) \times (0,1) \subset \mathbb{R}^2$ We assume that the time interval [0,1] is divided in equal intervals of time $\Delta t = 1/N$. The finite element spaces are chosen to be piecewise quadratic for the velocity and linear on the pressure. The Taylor-Hood finite elements are used in this calculation on a rectangular mesh. The mesh size is h and calculations with varying mesh sizes have been performed. In this first test we are interest in the convergence history for all the parameters involved and so a simple stationary target velocity $\vec{U} = (U, V)$ is chosen. The target velocity for this test is defined by

$$\begin{aligned}
\phi(t,z) &= (1 - cos(2\pi t z)) \times (1-z)^2 \\
U(x,y) &= 10\frac{d}{dy}\left(\phi(.4,x)\phi(.4,y)\right) \\
V(x,y) &= -10\frac{d}{dx}\left(\phi(.4,x)\phi(.4,y)\right).
\end{aligned}$$

Of course the vector $\vec{U}$ is divergence free. All the pictures are normalized by the maximum values.

<u>Velocity tracking evolution</u>

We can see a first example of control for the initial velocity is

$$\begin{aligned}
u_0(x,y) &= -10U(x,y) \\
v_0(x,y) &= -10V(x,y).
\end{aligned}$$

The evolution is in Fig.2.1 - Fig.2.6. The controlled fluid is on the left and the desired flow is on the right. We start with a high energy flow that rotates in opposite direction with respect to the target flow. As we can see we have distinct phases:
i) Decreasing in magnitude: at the beginning the control does not act on the shape of the flow (Fig.2.1 -Fig.2.3) but on the magnitude.
ii) Changing in shape: in the Fig.2.4 we have a quick change in shape. The change in shape is so quick that is difficult to track with small time steps.
iii) Matching the target flow: in this phase the control changes shape and magnitude in order to match the target flow (Fig.2.5). At $t = .25$ we reach a perfect match.
iv) Tracking velocity: in this phase the control keeps tracking of the target flow (Fig.2.6). The control in this steady phase is excellent and improves near $t = T$.
Fig.2.7 shows the error $\|\vec{u} - \vec{U}\|$ between the controlled flow $\vec{u}$ and the target flow $\vec{U}$. As we can see the error rapidly goes to zero. For this calculation $\Delta t = 0.0125$ and $h = 1/16$. For the same flow Fig.2.8 shows the correspondent value of the norm of the control $\vec{g}$ as a function of time. The control works hard at the beginning in order to steer the controlled

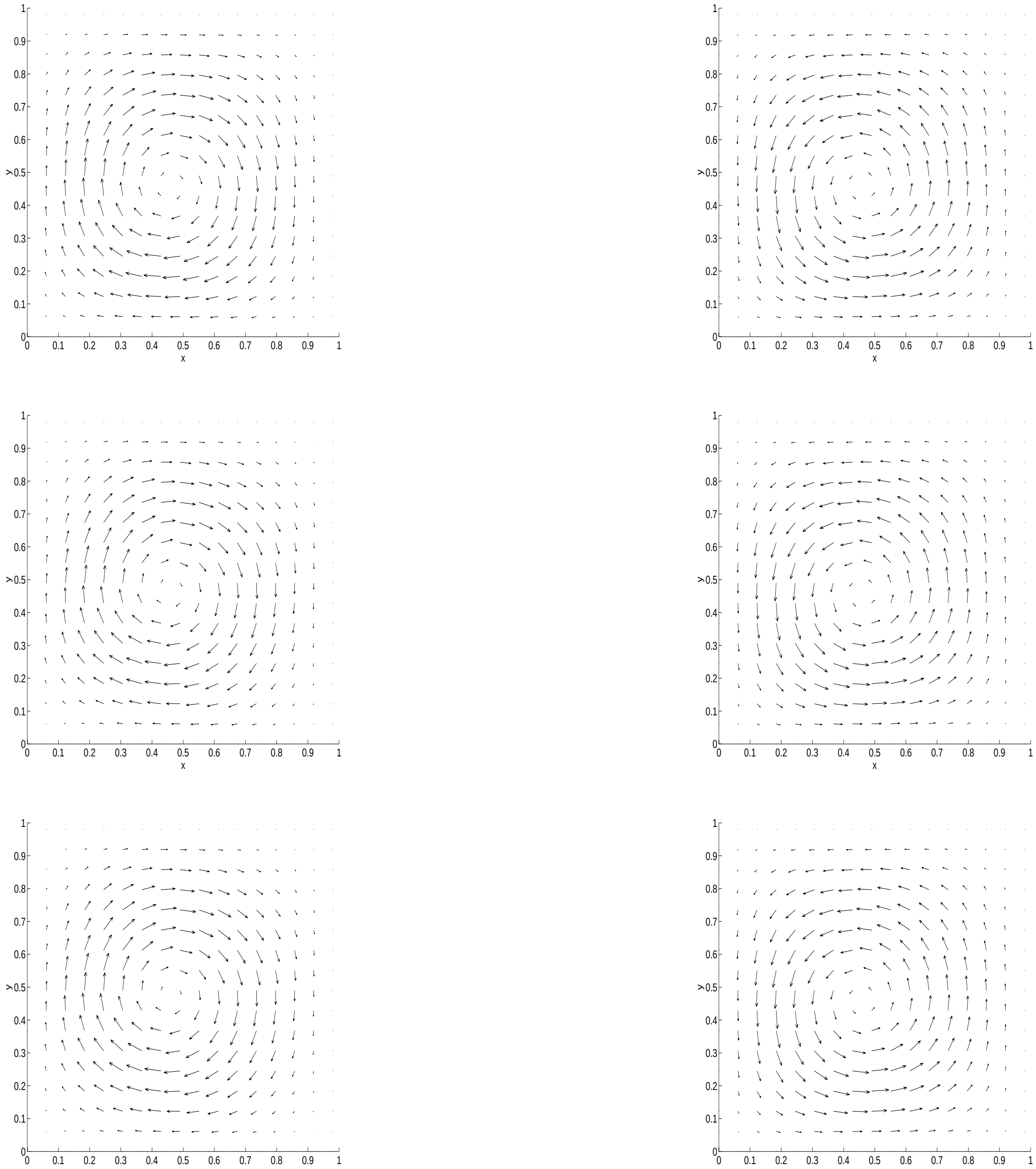

Figure 2.1: Test 1. Controlled(right) and desired(left) flow at $t = 0$ (top), $t = .05$ (middle) and $t = .1$ (bottom)

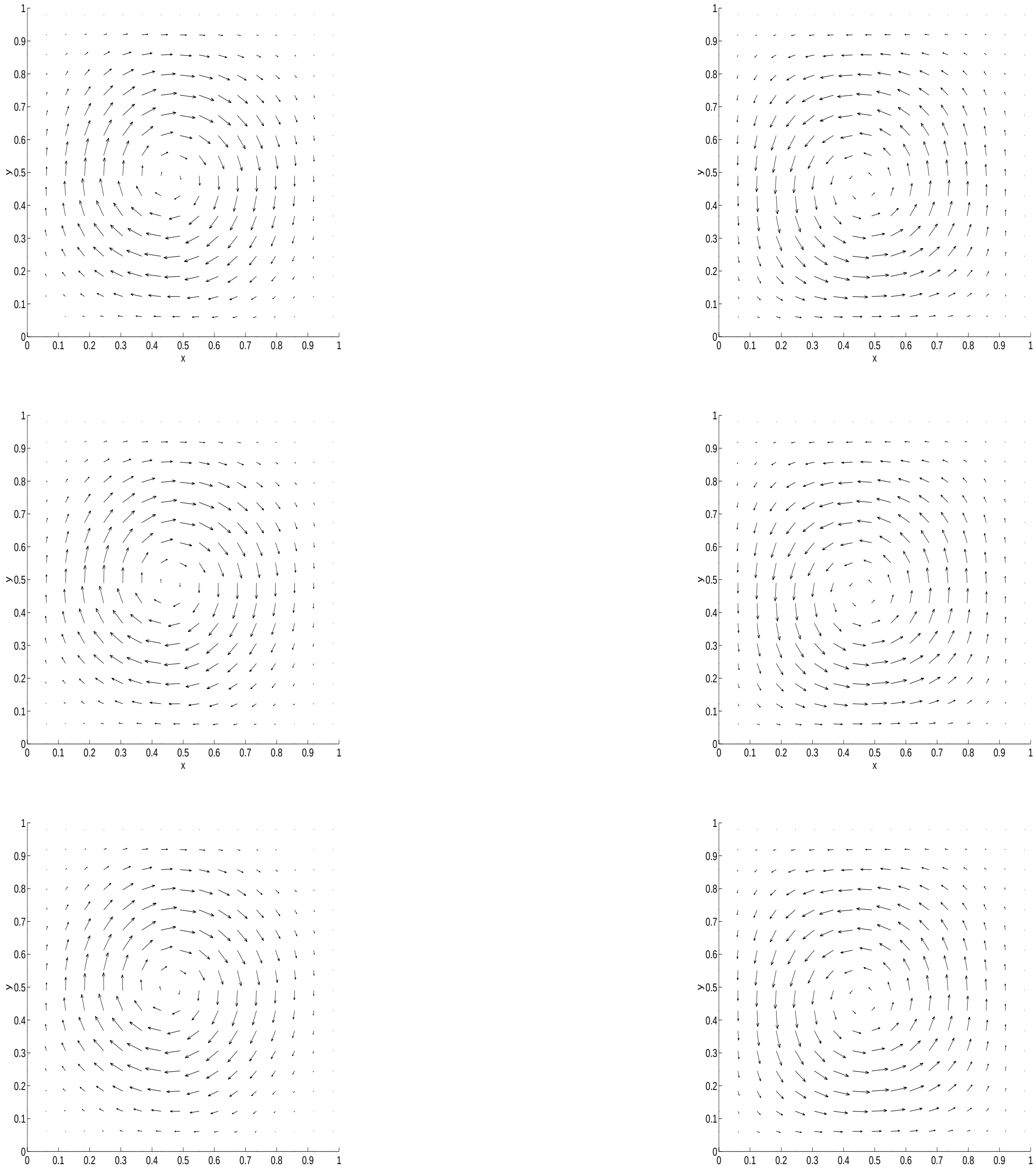

Figure 2.2: Test 1. Controlled(left) and desired(right) flow at $t = .125$ (top), $t = .15$ (middle) and $t = .163$ (bottom)

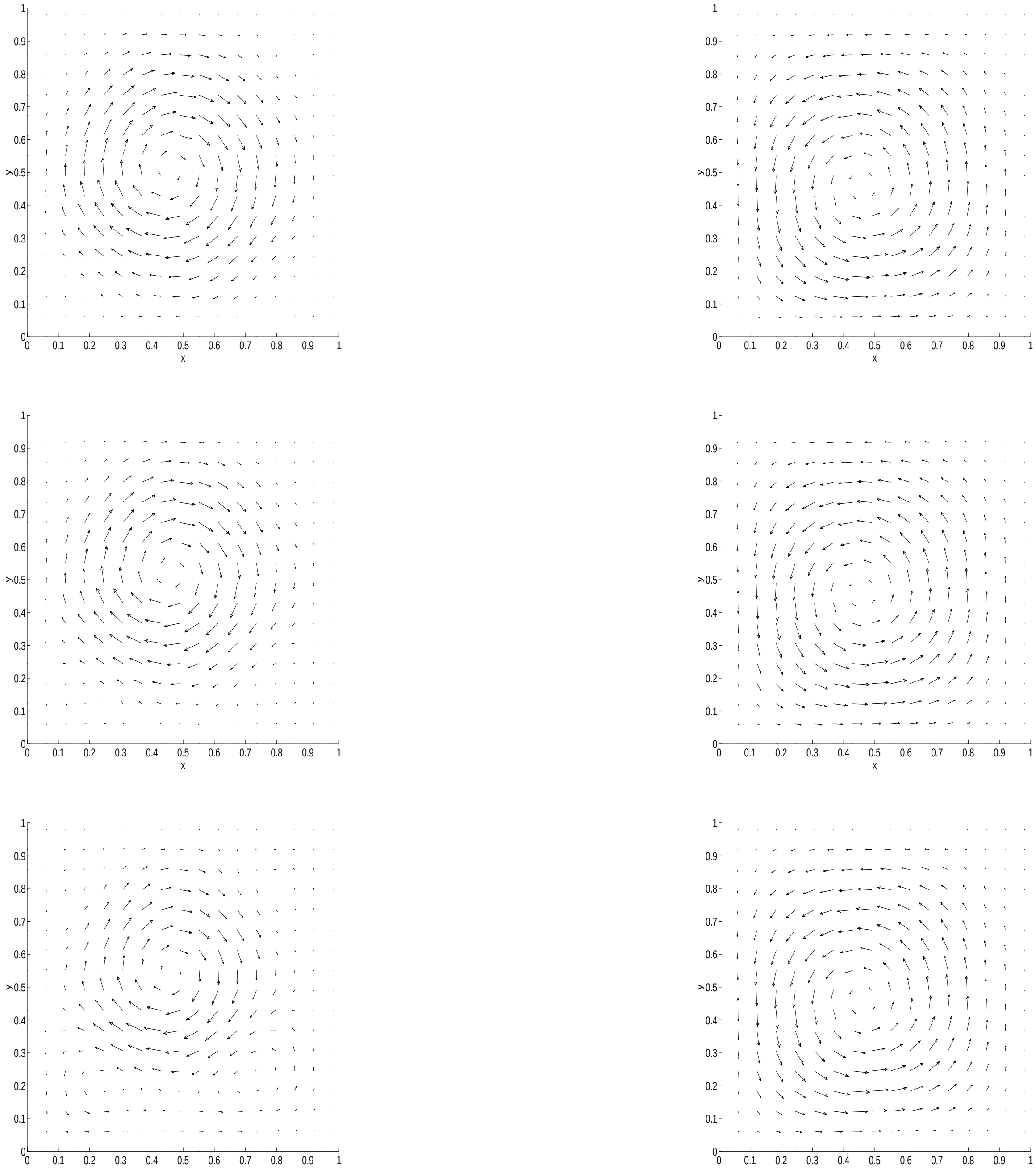

Figure 2.3: Test 1. Controlled(left) and desired(right) flow at $t = .169$ (top), $t = .175$ (middle) and $t = .181$ (bottom)

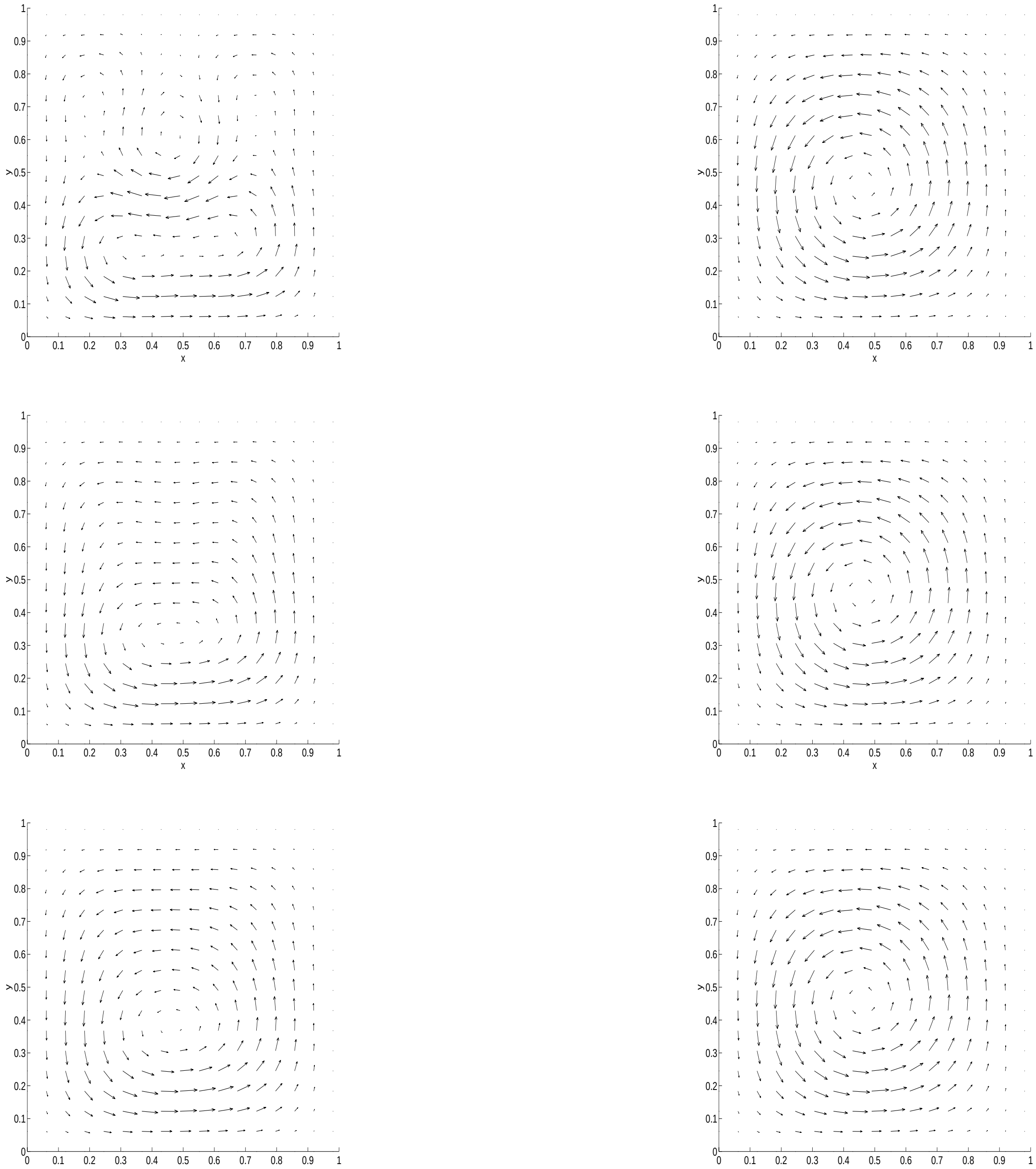

Figure 2.4: Test 1. Controlled(left) and desired(right) flow at $t = .187$ (top), $t = .193$ (middle) and $t = .2$ (bottom)

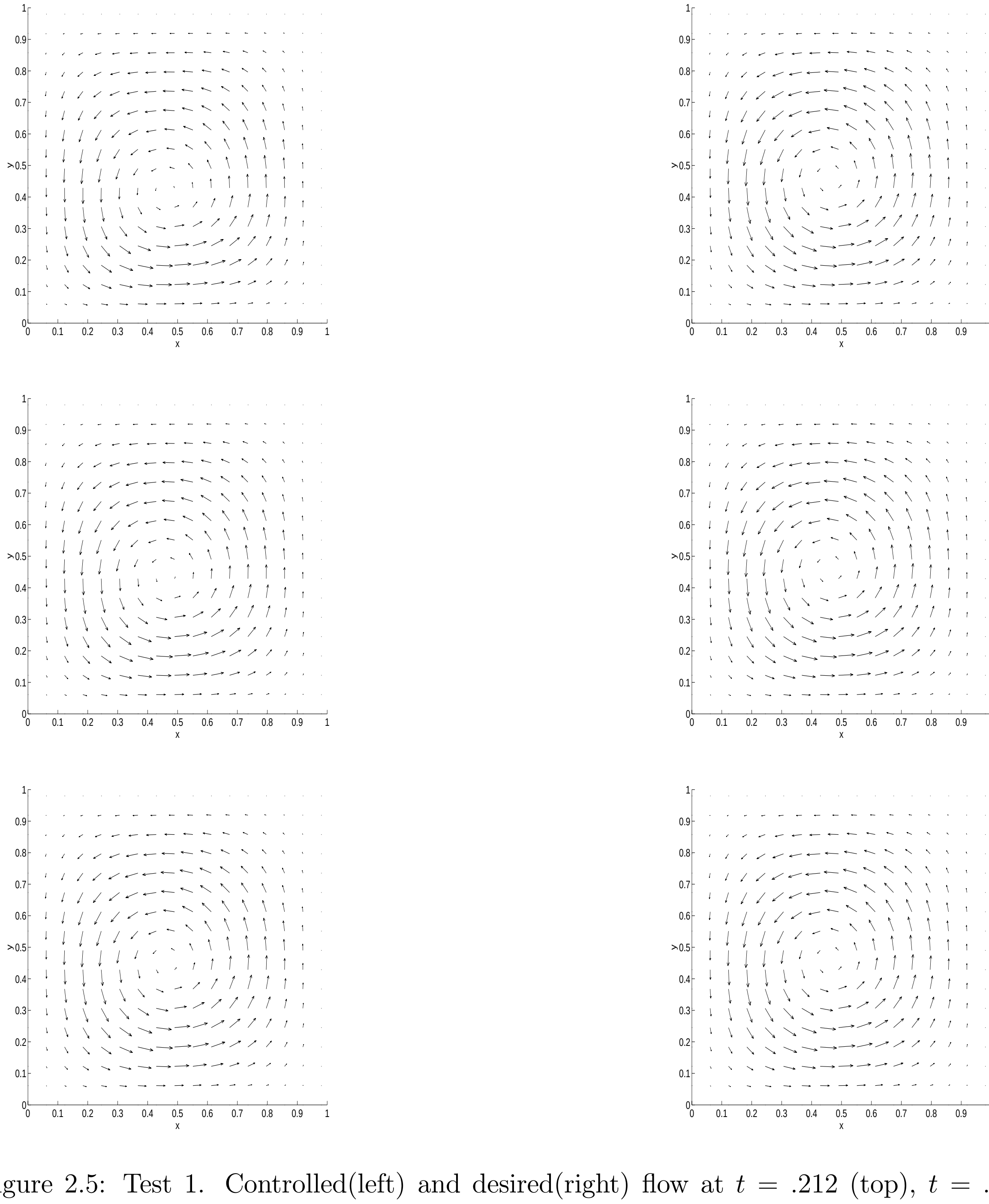

Figure 2.5: Test 1. Controlled(left) and desired(right) flow at $t = .212$ (top), $t = .225$ (middle) and $t = .25$ (bottom)

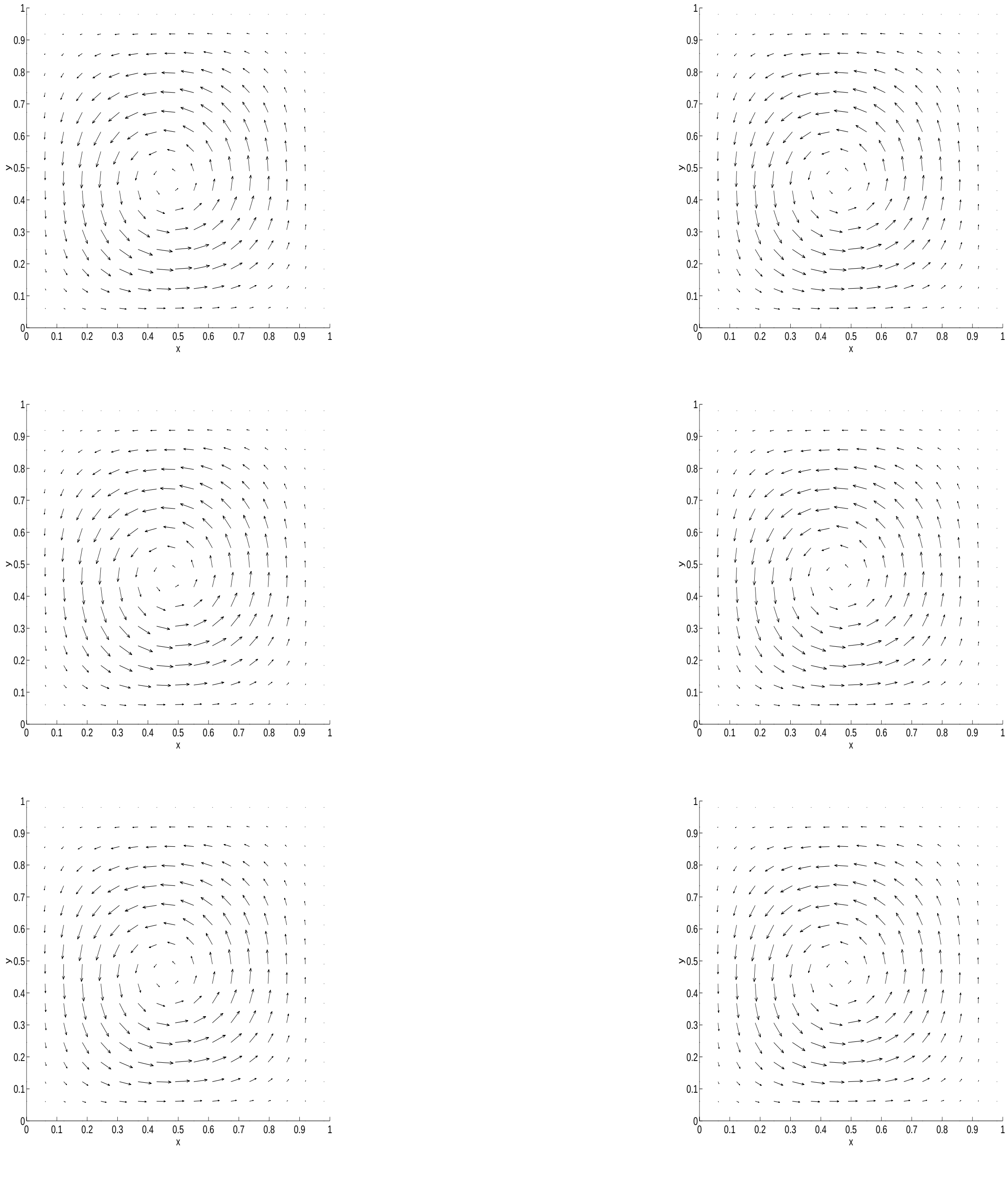

Figure 2.6: Test 1. Controlled(left) and desired(right) flow at $t = .3$(top), $t = .5$ (middle) and $t = 1.$ (bottom)

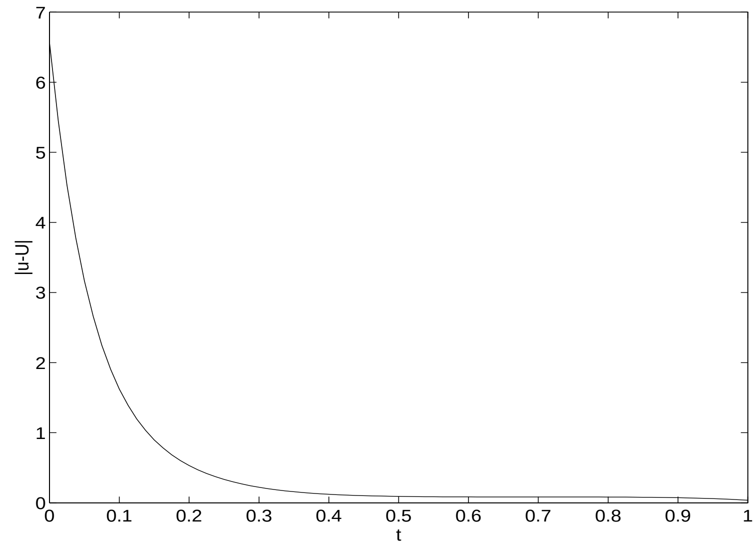

Figure 2.7: Test 1. Error $\|\vec{u} - \vec{U}\|$

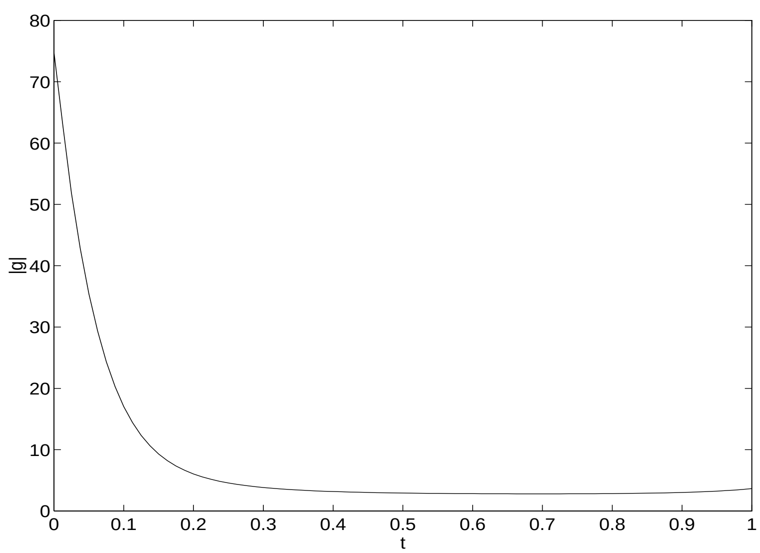

Figure 2.8: Test 1. Control norm $\|g\|$

flow to the desired one and then remains flat. Near $t = T$ the control strength increases in order to minimize the last term in the functional. In this test α has been set to 1, β to .0001 and γ to .5.

Velocity tracking with different values of β and γ

We want to analyse what happens if we change the form of the functional by changing the parameters β and γ. The initial velocity is set to be zero.

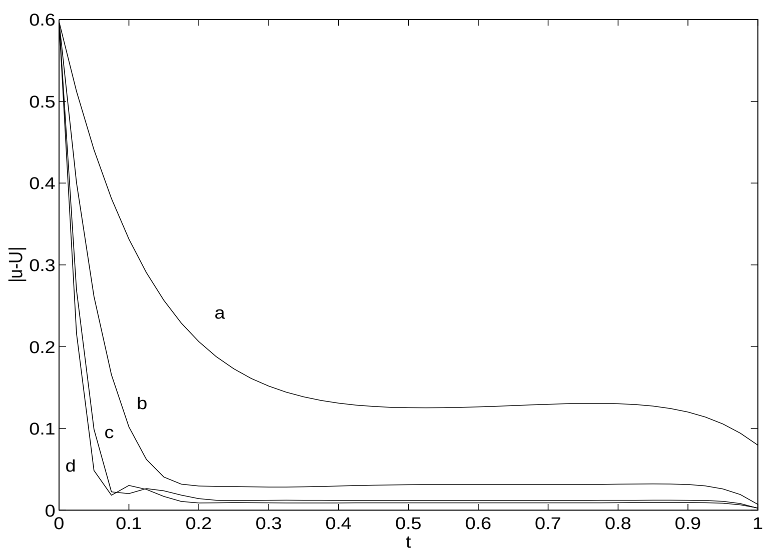

Figure 2.9: Test 1. Error for different β

In Fig.2.9 we have the error $\|\vec{u} - \vec{U}\|$ between the controlled flow $\vec{u}$ and the target flow $\vec{U}$ for different values of β. Starting from the top we have β equals .01 (a), .001 (b), .0001 (c) and .00001 (d). The value of γ in this calculation is hold constant at .5. The time step Δt is again 0.0125 and $h = 1/16$. We can note that the control flow matches very well for values of β less than .001. The reduction of the error in the tracking part is accompanied by a reduction of the matching part near $t = 0$ when β increases.

The norm of the control agrees with the intuitive behaviour of the error. For low values of β the control resembles a delta function plus the body force generated by the target velocity $\vec{U}$. The norm of $\vec{g}$ is shown in Fig.2.10. For different values of γ we see the importance of the last term in the functional. In Fig.2.12 we see the error $\|\vec{u} - \vec{U}\|$ between the controlled flow $\vec{u}$ and the target flow $\vec{U}$ for γ equal to .01 (a), .1 (b) and .5 (c). For small values

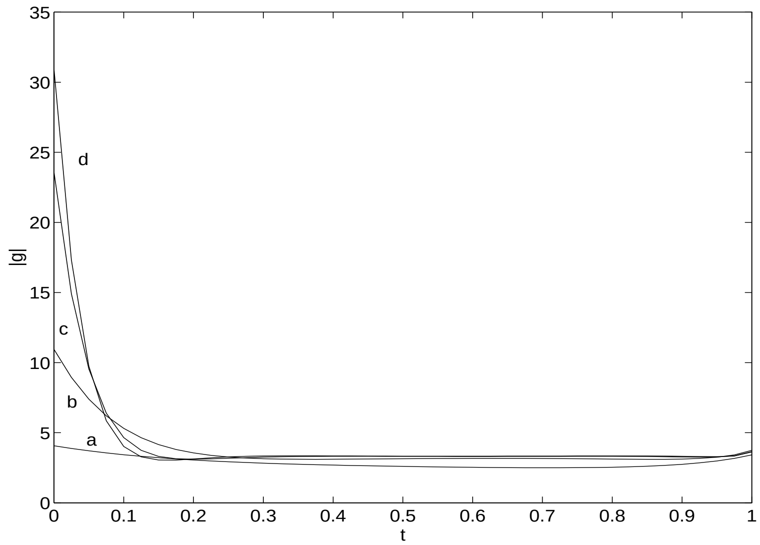

Figure 2.10: Test 1. Control norm $\|g\|$ for different β

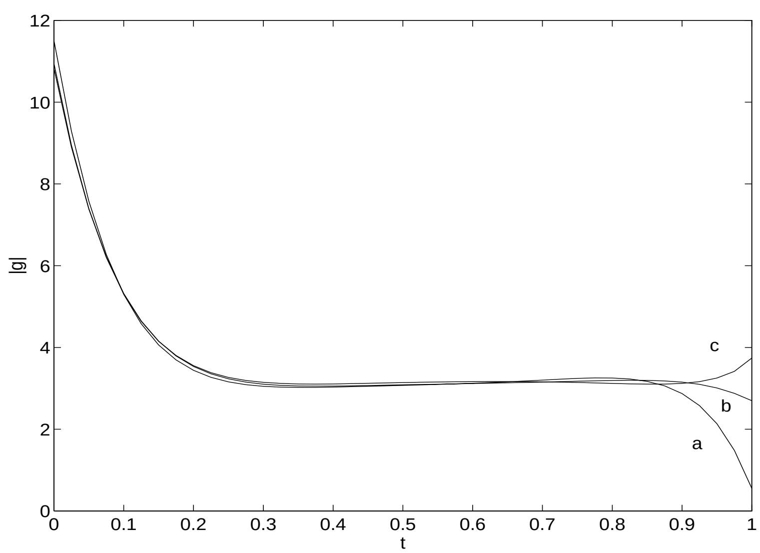

Figure 2.11: Test 1. Control norm $\|g\|$ for different γ

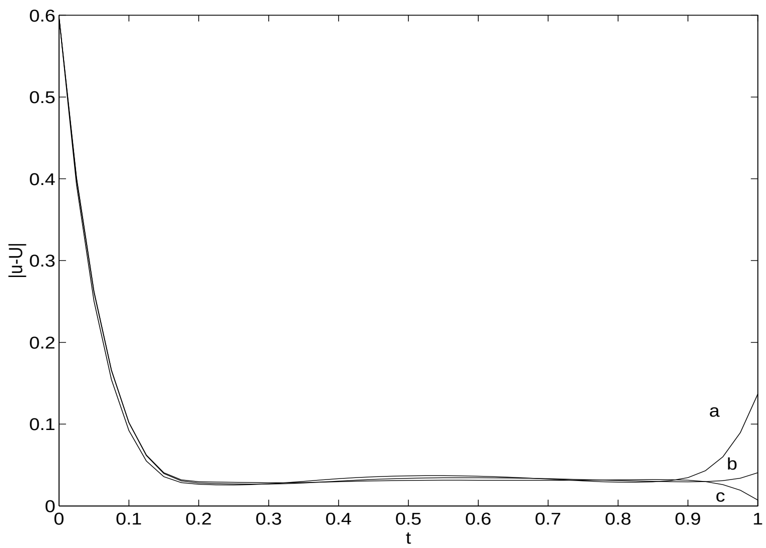

Figure 2.12: Test 1. Error $\|\vec{u} - \vec{U}\|$ for different γ

of γ the optimal control solution tends to go far from the desired flow at $t = T$. This is not acceptable and not desirable. This effect is more important if β is small. Generally a good control involves small values of β that are completely disregarded with low γ as the final error can be greater than the initial one and thus the tracking is not acceptable. In Fig.2.11 we have the norm of the control $\vec{g}$ for the corresponding values of γ (.01 (a), .1 (b) and .5 (c)). As expected the control $\vec{g}$ approaches zero at $t = T$ when γ goes to zero. The error is not controllable in this situation. The final zero condition gives freedom to the system around T. The conclusion of these remarks is that the final term in the functional is necessary. A good value of γ is about .5 for almost all β.

Convergence for different values of h and T.

In Fig.2.13 we can see the error $\|\vec{u} - \vec{U}\|$ between the controlled flow $\vec{u}$ and the target flow $\vec{U}$ for h equal to 1/8, 1/12 and 1/16.

In Fig.2.14 we can see the corresponding norm of the control $\vec{g}$. The initial velocity has been chosen to be zero and β and γ equal to 1/3000 and .5 respectively. Again $\Delta t = 0.0125$. Because of the simple form of the target velocity spatial convergence is achieved in all these cases. Some differences are visible around $t = T$ since this point is very sensitive to the discretization. For $\beta = .0001$ and $\gamma = .1$ Figure 2.15 shows the error for different values of

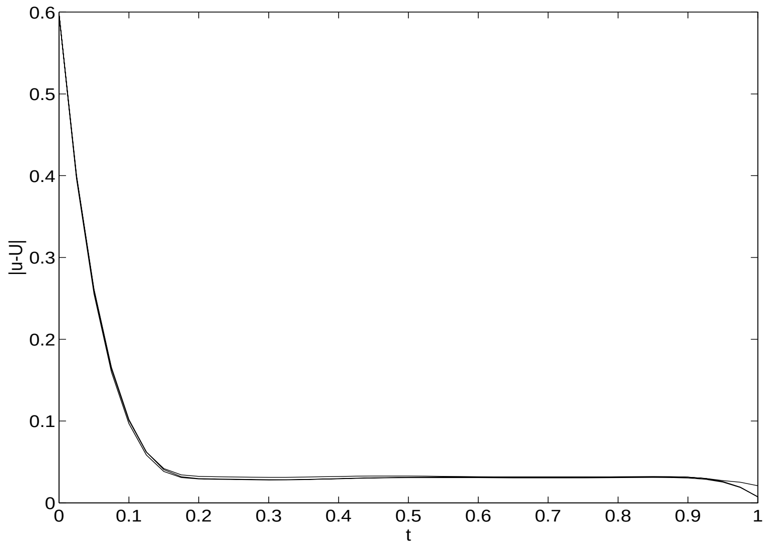

Figure 2.13: Test 1. Error $\|\vec{u} - \vec{U}\|$ for different h

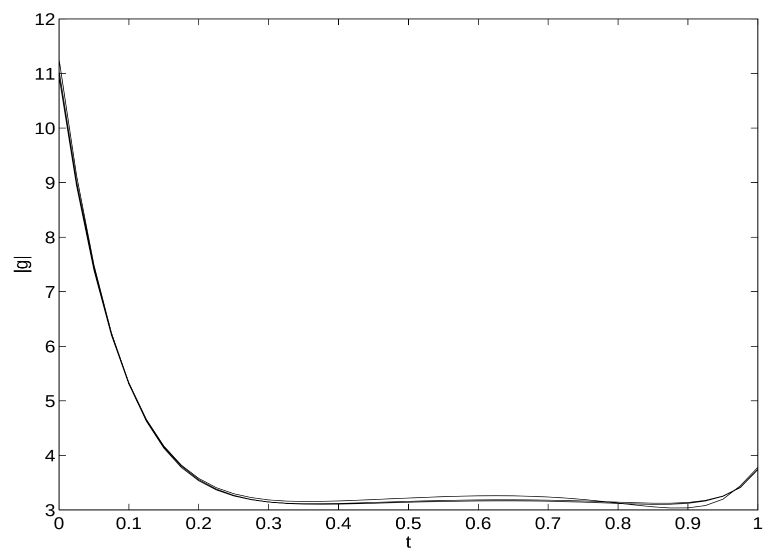

Figure 2.14: Test 1. Control norm $\|g\|$ for different h

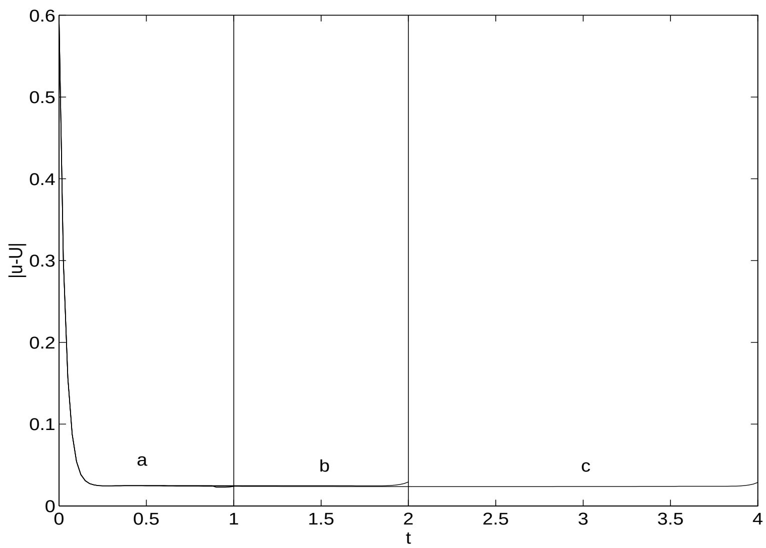

Figure 2.15: Test 1. Error $\|\vec{u} - \vec{U}\|$ for different T

T ($T = 1$ (a), $T = 2$ (b) and $T = 4$ (c)).

2.5.4 Test 2

We consider a unit square domain $(0,1) \times (0,1) \subset \mathbb{R}^2$. We assume that the time interval [0,1] is divided into equal intervals of time $\Delta t = 1/N$. The Taylor-Hood finite elements are used in this calculation on a rectangular mesh. We report only the final result but calculations with varying mesh sizes have been performed.

The target velocity $\vec{U}$ for this test is a solution of the Stokes system of equations with body force $\vec{f} = (f_1, f_2)$ equal to

$$\phi(t,x,y) = (1 - cos(4\pi tx)) \times (1-x)^2 \, (1 - cos(4\pi ty)) \times (1-y)^2$$
$$a(t,x,y) = 10\frac{d\phi(t,x,y)}{dy} \qquad b(t,x,y) = -10\frac{d\phi(t,x,y)}{dx}$$
$$f_1 = a(.4,x,y) - e^{-2*t}a(.6,x,y)$$
$$f_2 = b(.4,x,y) - e^{-2*t}b(.6,x,y).$$

and zero initial velocity.

With this body force we have the superposition of two flows. One vortex at the center of the domain with large radius and another vortex with small radius centered in the lower

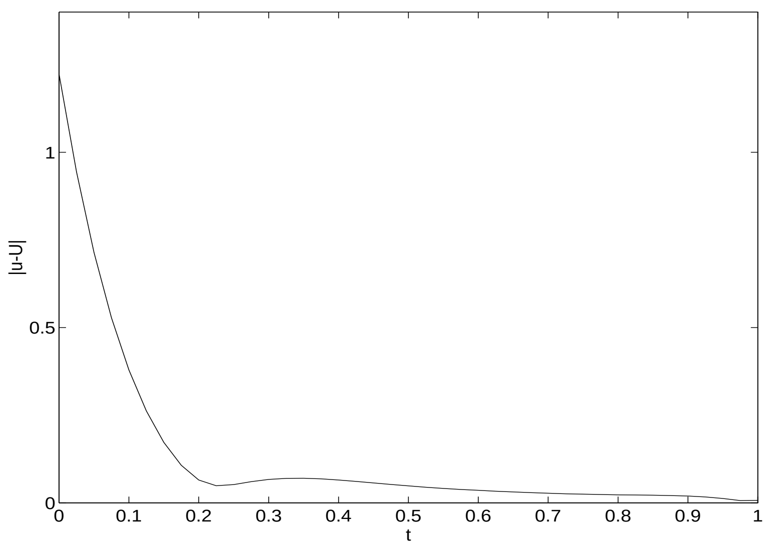

Figure 2.16: Test 2. Error $\|\vec{u} - \vec{U}\|$

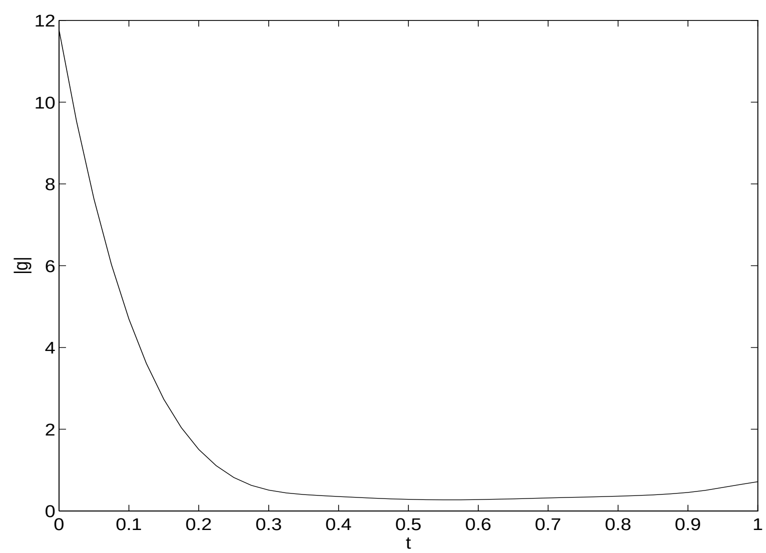

Figure 2.17: Test 2. Control norm $\|g\|$

left corner. Each of these flows prevails at different times of the evolution. The initial velocity for the controlled flow is

$$u_0(x, y) = (cos(2\pi * x) - 1)\, sin(2\pi * y)$$
$$v_0(x, y) = -\,(cos(2\pi * y) - 1)\, sin(2\pi * x).$$

Fig.2.16 shows the error $\|\vec{u} - \vec{U}\|$ between the controlled flow $\vec{u}$ and the target flow $\vec{U}$. As we can see the error rapidly goes to zero. For the same flow, Fig.2.17 shows the values of the norm of the control $\vec{g}$ as a function of time. The evolution is given in Fig.2.18 - Fig.2.21. The controlled fluid is on the left and the desired flow is on the right. As we can see at $t = 0.5$ we reach perfect match. Again the control works hard at the beginning in order to steer the controlled flow to the desired one and then remains flat. Near $t = T$ the control strength increases and minimizes the error. In all these calculations α has been set to 1, β to 1/5000 and γ to .5. Δt is 0.025 and $h = 1/16$.

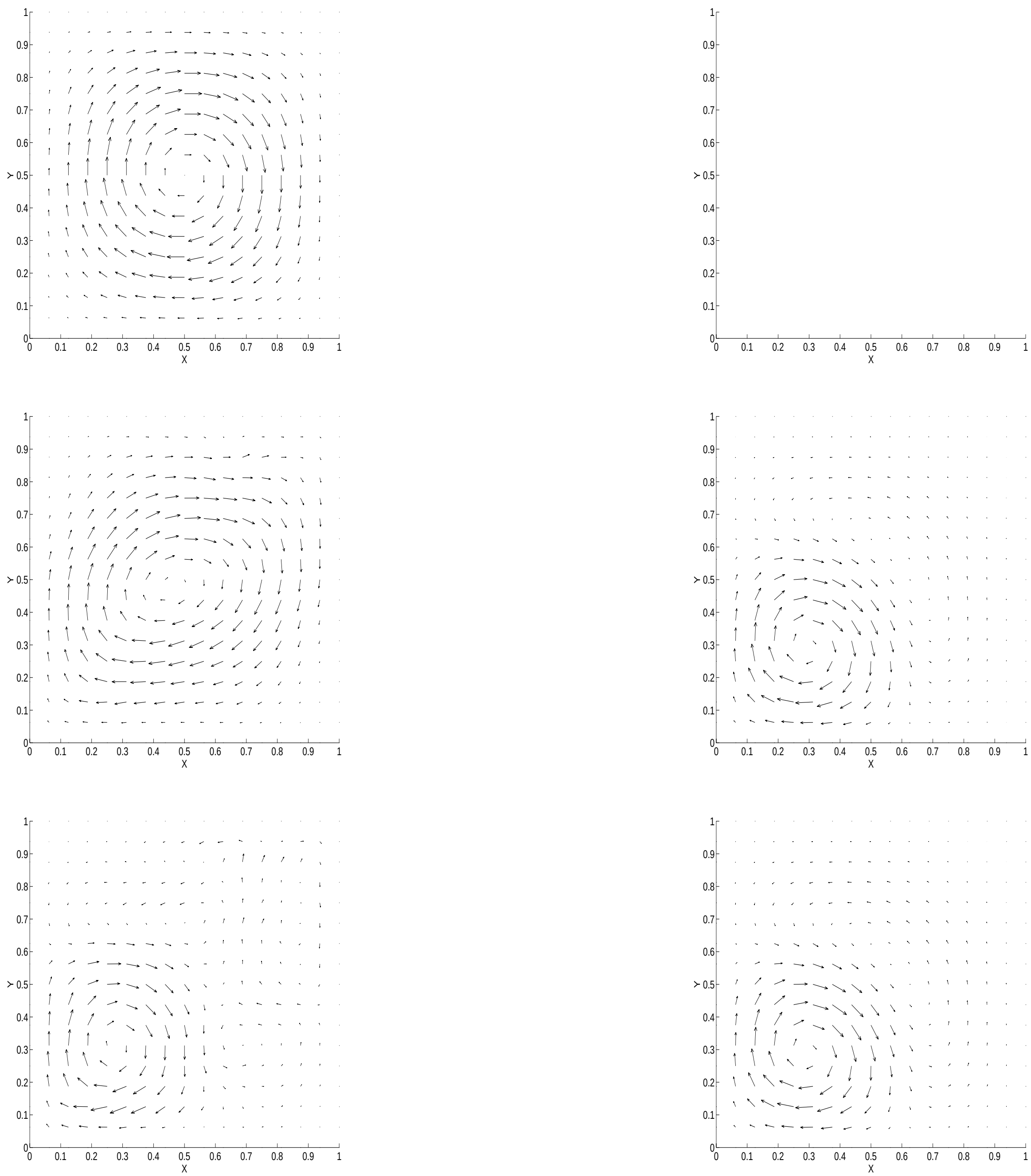

Figure 2.18: Test 2.Controlled(left) and desired(right) flow at $t = 0$ (top), $t = .1$ (middle) and $t = .2$ (bottom)

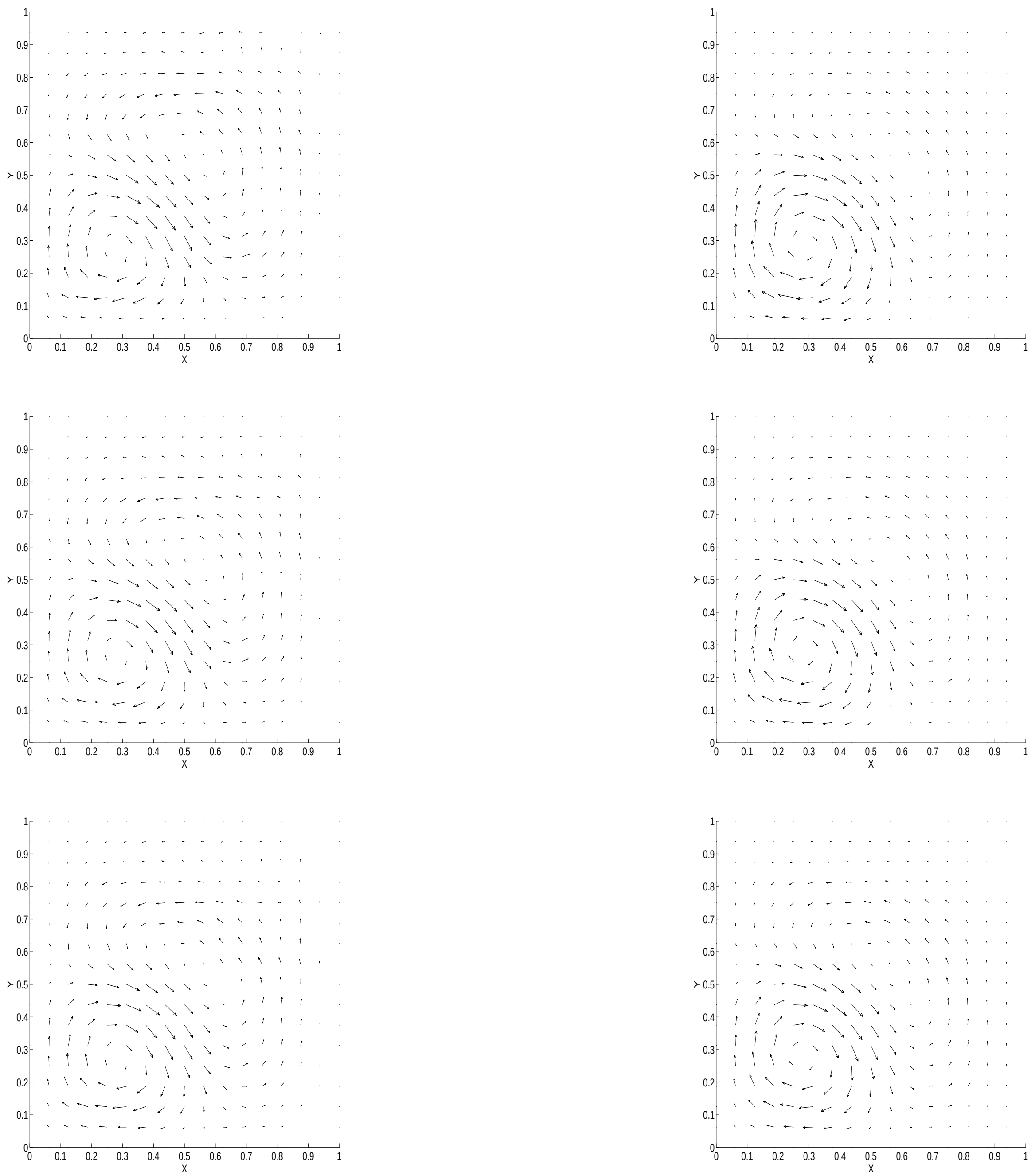

Figure 2.19: Test 2. Controlled(left) and desired(right) flow at $t = .3$ (top), $t = .4$ (middle) and $t = .5$ (bottom)

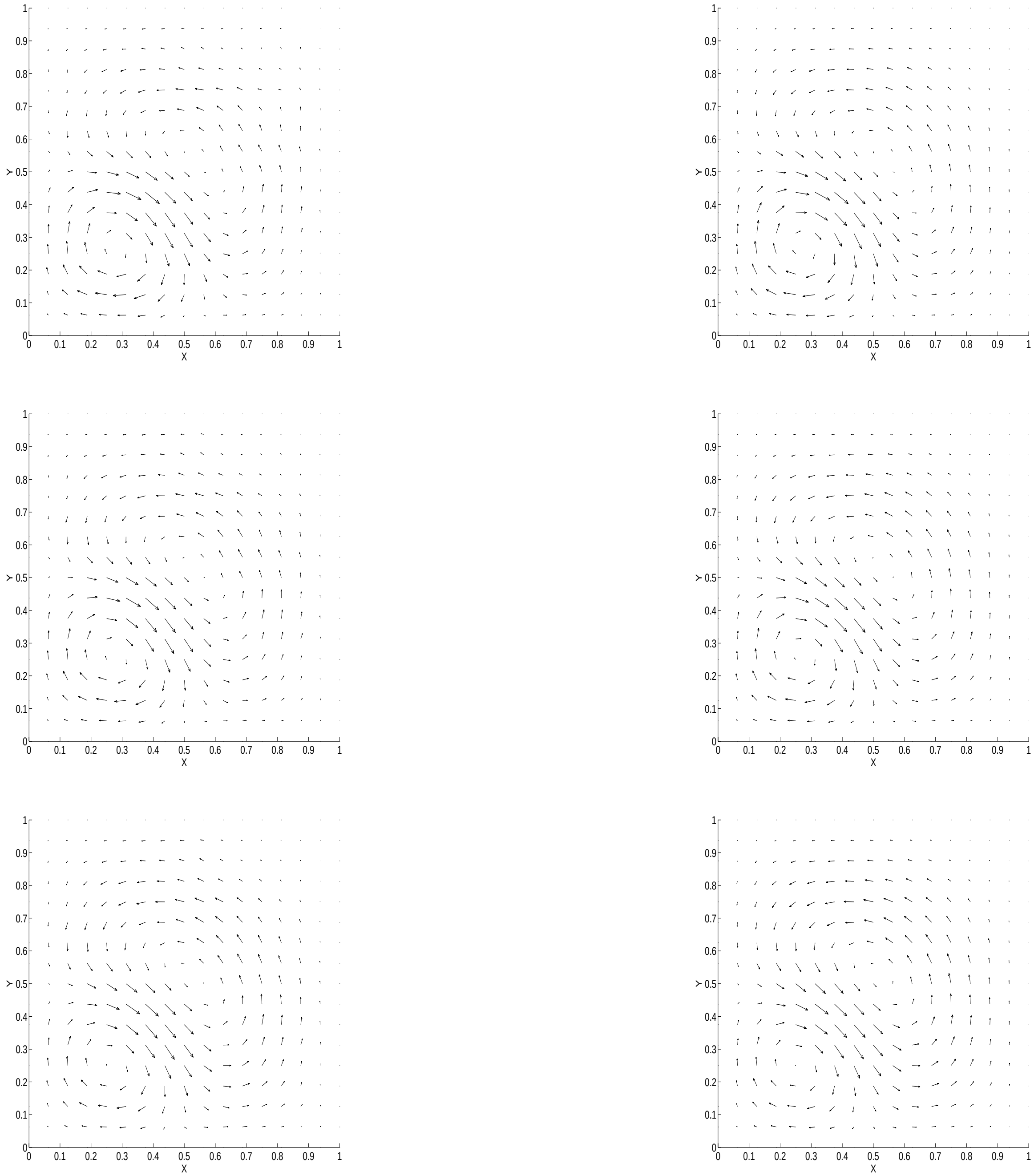

Figure 2.20: Test 2. Controlled(left) and desired(right) flow at $t = .6$ (top), $t = .7$ (middle) and $t = .8$ (bottom)

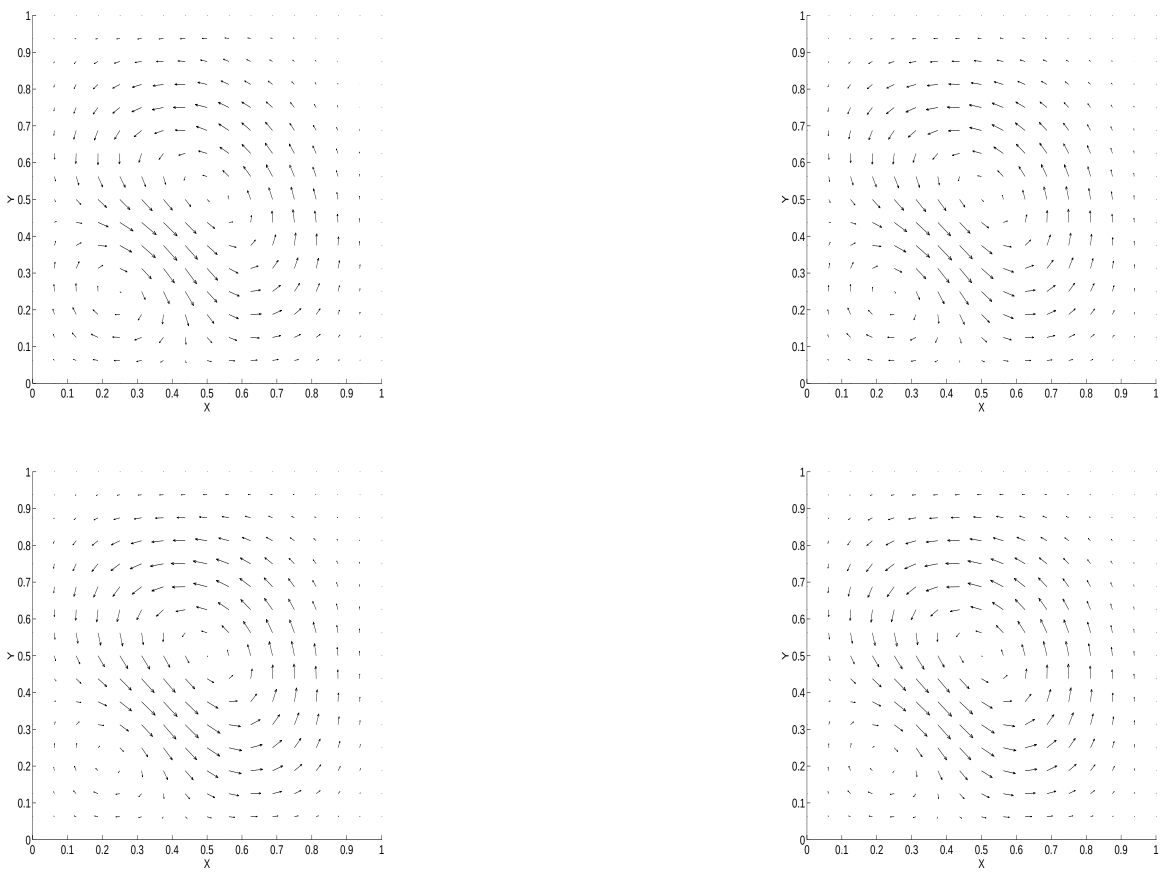

Figure 2.21: Test 2. Controlled(left) and desired(right) flow at $t = .9$ (top) and $t = 1$ (middle)

Chapter 3

Velocity tracking problem with bounded control

3.1 Introduction

Distributed controls are technological difficult and sometimes impossible to realize in practice. However if they are available, the size of the controls would be limited by technological constraints. For example, one way to effect such control is through a magnetic field acting on an ionized fluid or on a liquid metal. If one wishes to control such a system the amount of energy we can use is limited. The size is limited by technological constraints. The simplest method to describe this limit is to impose an a priori bound. In this chapter we present an approach to the mathematical formulation and numerical resolution of the time dependent problem of tracking velocity for Navier- Stokes flow in a bounded two-dimensional domain with bounded control.

In this chapter we are led to consider this problem in the framework of the optimal control theory of distributed systems. The control is the bounded distributed force, the state of the system $(\vec{u}, p)$ is the solution of the Navier-Stokes system of partial differential equations modeling the flow evolution and the cost function is a quadratic functional involving the state variables. The minimum of this functional corresponds to the minimum possible distance in L^2 norm between the target velocity $\vec{U}$ and the state velocity $\vec{u}$.

In this chapter we will formulate the problem in a convenient mathematical way, then we will prove the existence of an optimal control, and characterize such an optimal control by deriving the first-order necessary optimal conditions associated with the problem. Finally, once the optimality conditions are available, we will write down a gradient algorithm and find numerically solutions through suitable approximations. The minimization of this quadratic functional in state variables can drive the state velocity $\vec{u}$ to the target velocity $\vec{U}$ over a period of time with bounded control size.

In section 2.2 we will treat the continuous optimal control problem. In section 2.3 and

2.4 we will analyze the semidiscrete approximation and the fully discrete space-time approximation respectively. Finally, in section 2.5, some numerical experiments are performed and compared.

3.2 Distributed control problem

3.2.1 Formulation of the optimal control problem

In this section we describe the problem of the time distributed control for the Navier-Stokes equations that models the velocity tracking problem with a bounded control. This problem reflects the desire to steer over the time a candidate velocity field $\vec{u}$ to a target velocity field $\vec{U}$ by appropriately shaping the body force with bounded control.

Classical formulation

We consider a two-dimensional flow over the physical domain Ω with boundary Γ. The equations considered here are the the nondimensional incompressible Navier-Stokes equations on the interval of time $[0, T]$

$$\begin{cases} \vec{u}_t(t,\vec{x}) + (\vec{u}.\vec{\nabla})\vec{u}(t,\vec{x}) - \nu\nabla^2\vec{u}(t,\vec{x}) + \vec{\nabla}p(t,\vec{x}) = \vec{g}(t,\vec{x}) \\ \vec{\nabla}\cdot\vec{u}(t,\vec{x}) = 0 \end{cases} \tag{3.1}$$

with initial velocity $\vec{u}(0,\vec{x}) = \vec{u}_0(\vec{x})$ and boundary condition $\vec{u}(t,\vec{x}) = 0$ for $\vec{x} \in \Gamma$ and $t \in [0,T]$. The vector $\vec{u} = (u_1, u_2)$ is the velocity vector, p is the pressure and ν is the kinematics viscosity. We note that $Re = 1/\nu$. The function $\vec{u}_0$ must be divergence free and must satisfy the boundary conditions. The control is a bounded function in energy. This means that when we act upon the system, the body force can not exceed a fixed value in strength. The optimal control problem is formulated as follows:

Given $M > 0$ find $\vec{u}$ minimizing the cost function

$$L(\vec{f}) = \frac{\alpha}{2}\int_0^T\int_\Omega(\vec{u}-\vec{U})^2 d\vec{x}dt + \frac{\gamma}{2}\int_\Omega(\vec{u}(T,\vec{x}) - \vec{U}(T,\vec{x}))^2 d\vec{x} \tag{3.2}$$

and solution of eq(3.1) with $\|\vec{f}\| \le M$ for all t in [0,T].

M is a positive real number and represents the maximum power that can be used to control the system. The minimization of the $\int_0^T\int_\Omega(\vec{u}-\vec{U})^2 d\vec{x}dt$ term is the goal of the velocity tracking problem. The $\int_\Omega(\vec{u}(T,\vec{x}) - \vec{U}(T,\vec{x}))^2 d\vec{x}$ term is necessary in order to keep the solution $\vec{u}$ close to $\vec{U}$ at the time T.

Weak formulation of the bounded control problem

We consider an open bounded set $\Omega \subset \mathbb{R}^2$ with a Lipschitz-continuous boundary Γ. We begin by defining the particular target field $\vec{U}$. For the target field we make some assumptions.

$\vec{U}$ is said to be in the set of admissible target velocities U_{ad} if

$$\begin{cases} \vec{U} = \vec{U}(t,\vec{x}) \in C([0,T]; H^2(\Omega)) \\ \vec{\nabla}_{\vec{x}}.\vec{U}(t,\vec{x}) = 0 \quad \forall \vec{x} \in \Omega \\ \vec{U}(t,\vec{x}) = 0 \quad \forall \vec{x} \in \Gamma \\ \vec{F}_{\vec{U}}(t,\vec{x}) = \vec{U}_t(t,\vec{x}) - \nu \nabla^2 \vec{U}(t,\vec{x}) + (\vec{U}(t,\vec{x}).\vec{\nabla})\vec{U}(t,\vec{x}) \end{cases} \tag{3.3}$$

where $\vec{F}_{\vec{U}} \in L^\infty((0,T); L^2(\Omega))$.

Let $\vec{u} \in L^2((0,T); H^1_0(\Omega))$ and $p \in L^2((0,T); L^2_0(\Omega))$ denote the state variables and $\vec{f} \in L^2((0,T); K_M(\Omega))$ denote the distributed control where $K_M(\Omega) = \{\vec{f} \in L^2(\Omega) : \|\vec{f}\| \leq M\}$. The state variables are constrained to satisfy the weak form of the Navier-Stokes equations for a.e. t in $[0,T]$, i.e

$$\begin{cases} (\vec{u}_t, \vec{v}) + \nu a(\vec{u}, \vec{v}) + c(\vec{u}; \vec{u}, \vec{v}) + b(\vec{v}, p) = (\vec{f}, \vec{v}) \qquad \forall v \in H^1_0(\Omega) \\ b(\vec{u}, q) = 0 \quad \forall q \in L^2_0(\Omega) \\ \vec{u} = 0 \qquad \vec{x} \in \Gamma \\ \vec{u}(0,\vec{x}) = \vec{u}_0(\vec{x}) \qquad \vec{\nabla} \cdot \vec{u}_0 = 0 \qquad \vec{u}_0 \in L^2(\Omega) \end{cases} \tag{3.4}$$

An admissible solution for our optimal control problem can be defined as follows.

Given T , $\vec{f} \in L^2((0,T); K_M(\Omega))$, $\vec{u}_0$ a free divergence vector in $H^1_0(\Omega)$ and $\vec{U} \in U_{ad}$, then a weak solution of the eq(3.4) $(\vec{u}, p, \vec{g})$ is called an admissible solution for our optimal control problem if $u \in \mathcal{H}((0,T) \times \Omega)$, $p \in L^2((0,T); L^2_0(\Omega))$, and the functional in eq(3.5) is bounded.

The set of all admissible solution is denoted by A_{ad}.

P_B form of weak formulation

Given a positive number M the optimal control problem, in the P_B form, can be formulated as follows:

Let $\vec{f}$ belong to $L^2((0,T); K_M(\Omega))$ and $\vec{u}_0$ belong to $V(\Omega)$. Given $\vec{U} \in U_{ad}$ find $(\vec{u}, p, \vec{g}) \in A_{ad}$ such that the cost function

$$L(\vec{g}) = \int_0^T \int_\Omega [\frac{\alpha}{2}(\vec{u} - \vec{U})^2 + \frac{\gamma}{2} \int_\Omega (\vec{u}(T) - \vec{U}(T))^2 d\vec{x} \tag{3.5}$$

reaches its minimum value.

Again let us define the operators A, B and C in the usual way (see [36]):

$$(A\vec{u}, \vec{v}) = a(\vec{u}, \vec{v}) \qquad \forall \vec{u}, \vec{v} \in H^1_0(\Omega) \tag{3.6}$$

$$(C(\vec{w}) \cdot \vec{u}, \vec{v}) = c(\vec{w}, \vec{u}; \vec{v}) \qquad \forall \vec{u}, \vec{v} \in H_0^1(\Omega) \tag{3.7}$$
$$(B\vec{v}, q) = b(\vec{v}, q) \qquad \forall \vec{v} \in H_0^1(\Omega) \qquad \forall q \in L_0^2(\Omega) \tag{3.8}$$

and denote the linear functional $(\vec{g}, \vec{v})$ again with $\vec{g}$. Let $B(\Omega)$ be the unit ball in $L^2(\Omega)$.

With this notation the formulation of the problem P_B becomes:
given M a positive real number, $\vec{u}_0$ in $V(\Omega)$ and $\vec{U} \in U_{ad}$, find $(\vec{u}, p, \vec{g})$ in $L^2((0,T); H_0^1(\Omega)) \times L^2((0,T); L_0^2(\Omega)) \times L^2((0,T); B(\Omega))$ such that $\vec{g}$ minimizes the cost function in eq(3.5) and $(\vec{u}, p)$ is the solution of

$$P_B \left\{ \begin{array}{l} \vec{u}_t + \nu A\vec{u} + C(\vec{u})\vec{u} + B^* p = M\vec{g} \\ B\vec{u} = 0 \end{array} \right. \tag{3.9}$$

with initial velocity $\vec{u}(0, \vec{x}) = \vec{u}_0(\vec{x})$ and homogeneous boundary condition.
In this formulation we set $\vec{f} = M\vec{g}$. Working in the unit ball improves the geometric comprehension of the problem.

<u>Q_B form of the weak formulation</u>

Finally it is very useful to associate with the optimal control problem P_B an optimal control problem Q_B in $V(\Omega)$. To this purpose, we set $V(\Omega) = Ker(B)$ and denote the usual projections of the operators A and C on the space $V(\Omega)$ by πA and πC respectively. The auxiliary optimal control problem Q_B can be formulated as follows:
given $\vec{u}_0 \in V(\Omega)$ and $\vec{U} \in U_{ad}$, find $(\vec{u}, \vec{g})$ in $L^2((0,T); V(\Omega)) \times L^2((0,T); W(\Omega) \cap B(\Omega))$, such that $\vec{u}$ is the solution of

$$Q_B \left\{ \begin{array}{l} \vec{u}_t + \nu(\pi A)\vec{u} + (\pi C)(\vec{u})\vec{u} = M\pi\vec{g} \\ \vec{u}(0, \vec{x}) = \vec{u}_0(\vec{x}) \qquad \vec{u}_0 \in V(\Omega) \end{array} \right. \tag{3.10}$$

and minimizes the cost function in eq(3.5).
We can recall that if $\Gamma \in C^2$ and also $\vec{u}_0 \in V(\Omega)$, then the solution of the eq(3.10) belongs to $C([0,T]; V(\Omega)) \cap L^\infty((0,T); H^2(\Omega))$ and $\vec{u}_t \in L^2((0,T); W(\Omega))$ [35]
The real difficulty here lies in solving the problem Q_B. Of course, a solution of the optimal control problem Q_B is also a solution for the homogeneous optimal control problem P_B. The converse can be also established provided the inf-sup condition holds (see [36]).

3.2.2 Existence of the optimal control solution

We prove the existence for the problem Q_B with a Lipschitz-continuous boundary Γ. If Γ is in C^2 we can prove that $\vec{u}$ is in $L^\infty((0,T); H^2)$.

Theorem 3.1 *Given $\vec{u}_0 \in W(\Omega)$ then there exists at least one solution $\vec{g} \in L^2((0,T); B(\Omega))$ and $\vec{u} \in L^\infty((0,T); W(\Omega)) \cap L^2((0,T); V(\Omega))$ of the problem Q_B defined in (3.10).*

Proof: Let $\vec{g}_k$ be a minimizing sequence for the problem (3.10) in $L^\infty((0,T);B(\Omega))$. We set $\vec{u}_k = \vec{u}(\vec{g}_k)$. Of course the sequence $\{\vec{g}_k\}$ is bounded in $L^2((0,T);L^2(\Omega))$ and the corresponding solution $\vec{u}_n$ is bounded in $L^\infty((0,T);V(\Omega))$. Hence there is a subsequence of $(\vec{u}_n, \vec{g}_n)$ such that

$$\begin{aligned}
&\lim_{n\to\infty} \vec{u}_n = \vec{u} \quad \text{weakly in } L^2((0,T);V(\Omega)) \\
&\lim_{n\to\infty} \vec{u}_n = \vec{u} \quad \text{*-weakly in } L^\infty((0,T);V(\Omega)) \\
&\lim_{n\to\infty} \vec{g}_n = \vec{g} \quad \text{weakly in } L^2((0,T);L^2(\Omega)).
\end{aligned}$$

We note that $\vec{g}$ is in the weak closure of of the closed convex set $L^2((0,T);B(\Omega))$. The proof is complete proceeding in the same way as for the quadratic functional case showing that $(\vec{u}, \vec{g})$ satisfies the optimal control problem . $\sqcup$.

3.2.3 First-order necessary condition

In this section we proceed to derive the first-order necessary condition associated with the problem (3.9). We will show that the optimal solution must satisfy the first-order necessary condition.

Theorem 3.2 *Let $\vec{u}_0$ be in $V(\Omega)$. If $(\vec{u}, \vec{g})$ is an optimal pair then the necessary condition for $\vec{g}$ to minimize $L(\vec{g})$ over $L^2((0,T);B(\Omega))$ is*

$$\left(\frac{D\vec{u}}{D\vec{g}} \cdot (\vec{v} - \vec{g})\right) \geq 0 \quad \forall \vec{v} \in L^2((0,T);B(\Omega)). \tag{3.11}$$

Proof: Let $(\vec{u}, \vec{g})$ be an optimal pair. For every $0 \leq \lambda \leq 1$ and $\vec{v} \in L^2((0,T);B(\Omega))$ the function $\vec{h} = (1-\lambda)\vec{g} + \lambda\vec{v}$ is in the convex set $L^2((0,T);B(\Omega))$. Hence

$$L(\vec{h}) = L(\vec{g} + \lambda(\vec{v} - \vec{g})) \geq L(\vec{g}) \tag{3.12}$$

and in the limit of $\lambda \to 0$

$$\lim_{\lambda\to 0} \frac{L(\vec{g} + \lambda(\vec{v} - \vec{g})) - L(\vec{g})}{\lambda} \geq 0. \tag{3.13}$$

If the Gateaux derivative exists then

$$\left(\frac{DL(\vec{g})}{D\vec{g}} \cdot (\vec{v} - \vec{g})\right) \geq 0 \tag{3.14}$$

for all $\vec{v}$ in $L^2((0,T);B(\Omega))$. $\sqcup$

From the theorem 2.2 we know that the mapping $\vec{u} = \vec{u}(\vec{g})$ from $L^2((0,T);L^2(\Omega))$ to $L^2((0,T):W(\Omega))$ has a Gateaux derivatives $\frac{D\vec{u}}{D\vec{g}} \cdot \vec{h}$ in every direction $\vec{h}$ in $L^2((0,T);L^2(\Omega))$ and that $\tilde{w}(h) = \frac{D\vec{u}}{D\vec{g}} \cdot \vec{h}$ is the solution of

$$\begin{cases} \tilde{w}_t + \nu\nabla^2\tilde{w} + (\tilde{w}\cdot\nabla)\vec{u} + (\vec{u}\cdot\nabla)\tilde{w} + \nabla\tilde{q} = \vec{h} \\ \tilde{w} \in V(\Omega) \end{cases} \tag{3.15}$$

with zero initial value and homogeneous boundary condition. Of course $\vec{g}+L^2((0,T);B(\Omega)) \subset L^2((0,T);L^2(\Omega))$ and so the Frechet map is differentiable for all $\vec{v}-\vec{g}$ with $\vec{v}$ in the admissible set $L^2((0,T);B(\Omega))$.

Theorem 3.3 *Let $\Omega \in C^2$, $\vec{u}_0 \in V(\Omega)$ and $\vec{w} \in V(\Omega)$ be the solution of this adjoint problem*

$$-\vec{w}_t + \nu\nabla^2\vec{w} + (\nabla\vec{u})^T\vec{w} - (\vec{u}\cdot\nabla)\vec{w} + \nabla q = \alpha(\vec{u} - \vec{U}) \tag{3.16}$$

with final condition $\vec{w}(T,\vec{x}) = \gamma(\vec{u}(T) - \vec{U}(T))$ and homogeneous boundary condition. If $(\vec{u},\vec{g})$ is an optimal pair for the problem in eq(3.9) then we have for t a.e. in (0,T)
i) $\|\vec{g}\| = 1$;
ii) $\|\vec{w}\|\vec{g} = -\vec{w}$.

Proof: Let $(\vec{u},\vec{g})$ be an optimal pair solution of the problem defined in eq (3.9). We can compute the Gateaux derivative of the functional $L(\vec{g})$ in the direction of $\vec{h}$. We have

$$\frac{DL(\vec{g})}{D\vec{g}} \cdot \vec{h} = \alpha \int_0^T \int_\Omega (\vec{u} - \vec{U}) \left(\frac{D\vec{u}}{D\vec{g}} \cdot \vec{h}\right) d\vec{x}dt +$$
$$\int_\Omega \gamma(\vec{u}(T,\vec{x}) - \vec{U}(T,\vec{x})) \left(\frac{D\vec{u}}{D\vec{g}} \cdot \vec{h}\right)(T,\vec{x})d\vec{x} =$$
$$\alpha \int_0^T \int_\Omega (\vec{u} - \vec{U})\tilde{w}d\vec{x}dt + \gamma \int_\Omega (\vec{u}(T) - \vec{U}(T))\tilde{w}(T)d\vec{x}$$

Now using the function $\vec{w}$ in eq(3.16) we can integrate by parts. We have

$$\alpha \int_0^T \int_\Omega (\vec{u} - \vec{U})\tilde{w}d\vec{x}dt + \gamma \int_\Omega (\vec{u}(T) - \vec{U}(T))\tilde{w}(T)d\vec{x}$$
$$\int_0^T \int_\Omega \tilde{w}\left(-\vec{w}_t + \nu(\pi A)\vec{w} + (\pi C)'^*(\vec{u}(\vec{g}))\vec{w}\right)\vec{x}dt + \int_\Omega [\vec{w}\tilde{w}]_0^T d\vec{x} =$$
$$\int_0^T \int_\Omega \vec{w}\left(\tilde{w}_t + \nu(\pi A)\tilde{w} + (\pi C)'(\vec{u}(\vec{g}))\tilde{w}\right)\vec{x}dt.$$

Hence we get

$$\frac{DL(\vec{g})}{D\vec{g}} \cdot \vec{h} = \int_0^T \int_\Omega \vec{w}\vec{h}d\vec{x}dt = < \vec{w},\vec{h} >_{L^2((0,T);L^2)} \tag{3.17}$$

where $\vec{w}$ is the solution of the system in eq(3.16). In order to show i) and ii) we use the first order necessary condition.
i) Let $\vec{w}$ be different from zero and suppose $(\vec{u},\vec{g})$ to be a solution of the optimal problem with $\|g\| < 1$ for $t \in J \subset (0,T)$. We set ϵ equal to

$$\epsilon = \frac{1}{2}\left(\int_\Omega \vec{g}\frac{\vec{w}}{\|\vec{w}\|}d\vec{x} + \sqrt{\left(\int_\Omega \vec{g}\frac{\vec{w}}{\|\vec{w}\|}d\vec{x}\right)^2 + 1 - \|\vec{g}\|^2}\right) > 0,$$

$\vec{v} = -\epsilon\frac{\vec{w}}{\|\vec{w}\|} + \vec{g}$ on J and $\vec{v} = \vec{g}$ elsewhere In this case we have $\|\vec{v}\| \le 1$ and $< \vec{w}, \vec{v} - \vec{g} >_{L^2((0,T);L^2)} = -\epsilon\|\vec{w}\|_{L^2((0,T);L^2)} < 0$, which contradicts the first order necessary condition. The set J must have zero measure and the norm of $\vec{g}$ can not be less than 1 a.e on $(0,T)$.
ii) Again let $\vec{w}$ be different from zero and $\vec{g}$, according to part i), be an optimal unit vector for t a.e. in $(0,T)$. Let $I \subset [0,T]$ be a set on which the vector $\vec{g}$ is different from $-\vec{w}/\|\vec{w}\|$. Let $\vec{v} = -\vec{w}/\|\vec{w}\|$. The first order necessary condition in eq(3.11) implies

$$-\int_I (\vec{w},\vec{g})dt \ge \int_I \|\vec{w}\|dt. \tag{3.18}$$

We can note that from the positively of $< \vec{w}, -\vec{g} >_{L^2((0,T);L^2)}$ the control $\vec{g}$ can not be equal to $+\frac{\vec{w}}{\|\vec{w}\|}$ and so the set I is the set on which $\vec{g}$ is not proportional to $\vec{w}$. From the Schwartz inequality we have $|(\vec{w},\vec{g})| < \|\vec{w}\|$ for all t in I. Now

$$-\int_I\int_\Omega \vec{w}\vec{g}d\vec{x}dt \le \int_I |(\vec{w},\vec{g})|dt < \int_I \|\vec{w}\|dt. \tag{3.19}$$

But this contradicts eq(3.18). The set I must have zero measure. Hence if $\vec{w}$ is not equal to zero we have

$$\|\vec{w}\|\vec{g} = -\vec{w}.$$

for t a.e in $(0,T)$. $\sqcup$

Now in order to get the solution of our optimal control problem we have to solve the Navier-Stokes system and the adjoint system

$$\begin{cases} (\vec{u}_t,\vec{v}) + \nu a(\vec{u},\vec{v}) + c(\vec{u};\vec{u},\vec{v}) + b(\vec{v},p) = M(\vec{g},\vec{v}) \qquad \forall v \in H_0^1(\Omega) \\ -(\vec{w}_t,\vec{v}) + \nu a(\vec{w},\vec{v}) + c(\vec{v};\vec{w},\vec{v}) + c(\vec{u};\vec{v},\vec{w}) = -\alpha(\vec{u} - \vec{U}) \\ b(\vec{u},q) = 0 \quad \forall q \in L_0^2(\Omega) \\ b(\vec{w},q) = 0 \\ \|\vec{w}\|\vec{g} = \vec{w} \end{cases} \tag{3.20}$$

with initial velocity $\vec{u}(0,\vec{x}) = \vec{u}_0(\vec{x})$, final condition $\vec{w}(T,\vec{x}) = -\gamma(\vec{u}(T) - \vec{U}(T))$ and homogenous boundary condition on Γ. The above system of equations is a weak formulation of the following system

$$\begin{cases} \vec{u}_t(t,\vec{x}) + (\vec{u}.\vec{\nabla})\vec{u} - \nu\nabla^2\vec{u} + \vec{\nabla}p(t,\vec{x}) = M\vec{g} \\ \vec{\nabla}\cdot\vec{u} = 0 \\ -\vec{w}_t(t,\vec{x}) + \nu\nabla^2\vec{w} + (\nabla\vec{u})^T\vec{w} - (\vec{u}\cdot\nabla)\vec{w} + \vec{\nabla}\sigma = -\alpha(\vec{u} - \vec{U}) \\ \vec{\nabla}\cdot\vec{w} = 0 \\ \|\vec{w}\|\vec{g} = \vec{w} \end{cases} \tag{3.21}$$

with the same initial, final and boundary condition. From a computational standpoint, this is a very difficult system to solve. Therefore, how one solves this system is a rather important question. In the next sections we will try to discretize in time and then in time-space.

3.3 Semidiscrete time approximation

3.3.1 Formulation of the semidiscrete time approximation optimal control

Let $\sigma_N = \{t_n\}_{n=0}^N$ be a partition of $[0,T]$ into equal intervals $\Delta t = T/N$ with $t_0 = 0$ and $t_N = T$. For each fixed Δt (or N) and for every involved quantity $q(t,\vec{x})$ we associate the corresponding set $\{q^{(n)}(\vec{x})\}_{n=0}^N$. We will denote with bold vector $\mathbf{q}$ the vector $(q^{(1)}, q^{(2)}, ..., q^{(N)})$ of the discrete time components in $\mathbf{X} = X^N$. The associated continuous linear function $q^N = q^N(t,\vec{x})$ is defined by $q^N(t_n,\vec{x}) = q^{(n)}(\vec{x})$ for all $n = 0,1,..N$. For example, $\vec{U}^N$ is a continuous linear function in the variable t defined by $\vec{U}^N(t_n,\vec{x})=$ $\vec{U}^{(n)}(\vec{x}) =\vec{U}(t_n,\vec{x})$ for all $n = 0,1,..N (\vec{U} \in U_{ad})$.

The state variables $\vec{u}^{(n)} \in H_0^1(\Omega)$ and $p^{(n)} \in L_0^2(\Omega)$ are constrained to satisfy the semidiscrete Navier-Stokes equations

$$\begin{cases} \frac{1}{\Delta t}(\vec{u}^{(n)} - \vec{u}^{(n-1)}) + \nu A\vec{u}^{(n)} + C(\vec{u}^{(n)})\vec{u}^{(n)} + B^*p^{(n)} = M\vec{g}^{(n)} \\ Bq = 0 \\ \vec{u}^{(n)}(\vec{x}) = 0 \qquad \vec{x} \in \Gamma \end{cases} \tag{3.22}$$

for $n = 1,2,..N$ with initial velocity $\vec{u}^{(0)} = \vec{u}_0(\vec{x}) \in V(\Omega)$. The control $\vec{f}$ is bounded in norm by a real positive number M and thus $\|\vec{g}^{(n)}\| \leq 1$ for each $n = 1,2,...N$ $(\vec{f} = M\vec{g})$.

The optimization is achieved by mean of the minimization of the functional

$$L^N(\vec{\mathbf{g}}) = \frac{\alpha\Delta t}{2}\sum_{n=1}^{N}\|\vec{u}^{(n)} - \vec{U}^{(n)}\|^2 + \frac{\gamma}{2}\|\vec{u}^{(N)} - \vec{U}^N\|^2. \tag{3.23}$$

The discretization of this functional corresponds to the right-point discretization rule for the integral in time, which tends to the corresponding continuous functional when Δt tends to zero.

The formulation of the problem P_B in the semidiscrete approximation becomes: *given* $\Delta t = T/N$, $\vec{u}_0 \in V(\Omega)$ *and* $\vec{U} \in U_{ad}$, *find a sequence* $(\vec{\mathbf{u}}, \mathbf{p}, \vec{\mathbf{g}})$ *in* $(\mathbf{H_0^1(\Omega)} \times \mathbf{L_0^2(\Omega)} \times \mathbf{B(\Omega)})$ *such that* $(\vec{\mathbf{u}}, \mathbf{p})$ *is the solution of eq(3.22) and minimizes the cost function in eq(3.23).* We note that in this formulation the value of $\vec{g}^{(0)}$ is not defined from the solution of the Navier-Stokes problem and can be arbitrarily chosen as an extension of the corresponding continuous linear function $\vec{g}^N(t, \vec{x})$ in $C((0,T); L^2(\Omega))$. More complicated discretizations of the functional can also involve the value $\vec{g}^{(0)}$.

The auxiliary optimal control problem Q_B can be formulated as follows: *given* $\Delta t = T/N$, $\vec{u}_0 \in V(\Omega)$ *and* $\vec{U} \in U_{ad}$, *find* $(\vec{\mathbf{u}}, \vec{\mathbf{g}})$ *in* $\Big(V(\Omega)^N \times \mathbf{B(\Omega)}\Big)$ *such that* $\vec{u}^{(n)}$ *is the solution of*

$$\frac{1}{\Delta t}(\vec{u}^{(n)} - \vec{u}^{(n-1)}) + \nu(\pi A)\vec{u}^{(n)} + (\pi C)(\vec{u}^{(n)})\vec{u}^{(n)} = M\pi\vec{g}^{(n)} \tag{3.24}$$

with initial value $\vec{u}^{(0)} = \vec{u}_0 \in V(\Omega)$ *and minimizes the cost function.*

3.3.2 Existence and consistency of the semidiscrete optimal control problem

Now we state and prove the existence for the semidiscrete optimal control problem in a open bounded domain Ω with Lipschitz-continuous boundary Γ.

Theorem 3.4 *Given* $\Delta t = T/N$, $\vec{u}_0 \in V(\Omega)$ *and* $\vec{U} \in U_{ad}$, *there exists a pair* $(\vec{\mathbf{u}}, \vec{\mathbf{g}})$ *in* $(\mathbf{V(\Omega)} \times \mathbf{L^2(\Omega)})$ *such that* $\vec{\mathbf{u}}$ *is the solution of eq(3.24) and minimizes the cost function.*

Proof: Given N let $\{\vec{\mathbf{g}}_k\}_{k=1}^{\infty}$ be a minimizing sequence. Each component of the minimizing sequence is uniformly bounded by 1 and the corresponding solution $\vec{\mathbf{u}}_k$ is uniformly bounded in $\mathbf{V(\Omega)}$. Now we can proceed with a weakly convergent subsequence for each component and show that these subsequences converge to the solution of the optimal control problem in the semidiscrete approximation. Using the fact that the injection of $V(\Omega)$ into $L^2(\Omega)$ is compact the subsequence $\vec{\mathbf{u}}_k$ converges strongly. This allows us to pass to the limit in the semidiscrete Navier-Stokes equation and the proof follows as in the continuous case. The components of the sequence $\vec{\mathbf{g}}$ are bounded in the unit ball and so the weak limit satisfies the bound imposed on the control. $\sqcup$

Now we state the consistency of our semidiscrete optimal control problem.

Theorem 3.5 *Given* $\Delta t = T/N$, $\vec{U} \in U_{ad}$ *and* $\vec{u}_0 \in V(\Omega)$. *For* $\Delta T \to 0$ $(N \to \infty)$ *the solution* $\{(\vec{u}^{(n)}, \vec{g}^{(n)})\}_{n=1}^{N}$ *of the problem described in eq(3.24) tends to the optimal control pair* $(\vec{u}, \vec{g})$ *solution of the corresponding continuous optimal control problem.*

Proof: Let $\Delta t = T/N$ and $\vec{u}'^N$ be the linear function associated to $\vec{\mathbf{u}}'$ whose components are defined by $\vec{u}'^{(n)} = (\vec{u}^{(n)} - \vec{u}^{(n-1)})/\Delta t$. From the well known theorems on the time semidiscrete Navier-Stokes eqautions the sequences $\{\vec{u}^N\}_{N=1}^{\infty}$, $\{\vec{g}^N\}_{N=1}^{\infty}$ and $\{\vec{u}'^N\}_{N=1}^{\infty}$ are uniformly bounded in $L^2((0,T);V(\Omega)) \cap L^\infty((0,T);W(\Omega))$, $L^2((0,T);B(\Omega))$ and $L^2((0,T);V^*(\Omega))$ respectively. Then we can extract some subsequences, which we denote $\vec{u}^K$ and $\vec{g}^K$ such as

$$\begin{cases} \vec{u}^K \to \vec{u} \quad L^2(0,T,V(\Omega)) \text{ weakly} \\ \vec{u}^K \to \vec{u} \quad L^\infty(0,T,H(\Omega)) \text{ *-weakly} \\ \vec{g}^K \to \vec{g} \quad L^2(0,T,L^2(\Omega)) \text{ weakly} \\ \vec{u}'^K \to \vec{u}' \quad L^2(0,T,V^*(\Omega)) \text{ weakly.} \end{cases} \tag{3.25}$$

The set $L^2((0,T);B(\Omega))$ is a convex closed set and thus weakly closed, that is the limit $\vec{g}$ satisfies the bound. As a consequence of the compactness theorem 2.7 the convergence of the sequence $\vec{u}^K$ is a strong convergence in $L^2((0,T);W(\Omega))$. Now we can pass to the limit in the system of equations and in the functional using the standard technique. The fact that the sequence converges weakly in $L^2((0,T);V(\Omega))$ and strongly in $L^2((0,T);W(\Omega))$ allows us to pass to the limit in the nonlinear term. Thus the semidiscrete optimal control problem for $N \to \infty$ is consistent with the continuous one. $\sqcup$

3.3.3 First-order necessary condition

In this section we proceed to derive the first-order necessary condition in term of Gateaux derivative for the semidiscrete approximation. We will show that the first-order necessary condition shapes the optimal control solution. We proceed similarly to the continuous case in order to restate and prove the same theorems for the semidiscrete approximation.

Theorem 3.6 *Let $\Delta t = T/N$ and $\vec{u}_0$ be in $V(\Omega)$. If $(\vec{\mathbf{u}}, \vec{\mathbf{g}})$ is an optimal control solution then the necessary condition for $\vec{\mathbf{g}}$ to minimize $L^N(\vec{\mathbf{g}})$ over $\mathbf{B}(\boldsymbol{\Omega})$ is*

$$\sum_{k=1}^{N} \left(\frac{DL^N}{D\vec{g}^{(k)}} \cdot (\vec{v}^{(k)} - \vec{g}^{(k)}) \right) = \left(\frac{DL^N}{D\vec{\mathbf{g}}} \cdot (\vec{\mathbf{v}} - \vec{\mathbf{g}}) \right) \geq 0 \quad \forall \vec{\mathbf{v}} \in \mathbf{B}(\boldsymbol{\Omega}). \tag{3.26}$$

Proof: Let $(\vec{\mathbf{u}}, \vec{\mathbf{g}})$ be an optimal pair. For every $0 \leq \lambda \leq 1$ and $\vec{\mathbf{v}} \in \mathbf{B}(\boldsymbol{\Omega})$ the function $\vec{\mathbf{h}} = (1-\lambda)\vec{\mathbf{g}} + \lambda \vec{\mathbf{v}}$ is in the convex set $\mathbf{B}(\boldsymbol{\Omega})$. Hence

$$\frac{L^N(\vec{\mathbf{h}}) - L^N(\vec{\mathbf{g}})}{\lambda} = L^N \frac{\vec{\mathbf{g}} + \lambda(\vec{\mathbf{v}} - \vec{\mathbf{g}})) - L^N(\vec{\mathbf{g}})}{\lambda} \geq 0 \tag{3.27}$$

and in the limit of $\lambda \to 0$

$$\left(\frac{DL^N(\vec{\mathbf{g}})}{D\vec{\mathbf{g}}} \cdot (\vec{\mathbf{v}} - \vec{\mathbf{g}}) \right) \geq 0 \tag{3.28}$$

for all $\vec{\mathbf{v}}$ in $\mathbf{B}(\boldsymbol{\Omega})$. $\sqcup$

From theorem 2.10, if $\delta\vec{g}^N$ belongs to in $L^\infty((0,T);L^2(\Omega))$ there exists a small $(\Delta t)^*$ such that the mapping $\vec{u}^{(n)}(\vec{\mathbf{g}})$ from $\mathbf{L^2}(\boldsymbol{\Omega})$ to $\mathbf{H_0^1}(\boldsymbol{\Omega})$ has Gateaux derivative $\frac{D\vec{\mathbf{u}}}{D\vec{\mathbf{g}}}\cdot\vec{\mathbf{h}}$ for all $\Delta t\leq(\Delta t)^*$. Furthermore $\tilde{w}^{(n)}=\frac{D\vec{u}^{(n)}}{D\vec{\mathbf{g}}}\cdot\delta\vec{\mathbf{g}}$ and is the solution of the system of equations

$$\begin{cases} \frac{1}{\Delta t}(\tilde{w}^{(n)}-\tilde{w}^{(n-1)})+\nu(\pi A)\tilde{w}^{(n)}+(\pi C)'(\vec{u}^{(n)}(\vec{g}))\cdot\tilde{w}^{(n)}=\delta\vec{g}^{(n)} \\ \tilde{w}^{(n)}\in V(\Omega) \end{cases} \tag{3.29}$$

for $n=1,2..N$ with initial value $\tilde{w}^{(0)}(\vec{x})=0$ and homogeneous boundary condition. In order to estimate the optimal control $\vec{\mathbf{g}}$ we need to evaluate the Gateaux derivative of the functional and finally show that the control must be proportional to the solution of the adjoint system of equations as in the continuous case.

Theorem 3.7 *Given $\Delta t=T/N$, $\Omega\in C^2$ and $\vec{u}_0\in V(\Omega)$. If $(\vec{\mathbf{u}},\vec{\mathbf{g}})$ is an optimal pair for the problem in eq(3.24) then there exists an N_1 such that for each $N\geq N_1$ we have for all n=1,2,...N*
i) $\|\vec{g}^{(n)}\|=1$;
ii) $\|\vec{w}^{(n)}\|\vec{g}^{(n)}=-\vec{w}^{(n)}$;
where $\vec{w}^{(n)}$ is solution of this adjoint problem

$$\begin{cases} -\frac{1}{\Delta t}(\vec{w}^{(n+1)}-\vec{w}^{(n)})+\nu\nabla^2\vec{w}^{(n)}+(\nabla\vec{u}^{(n)})^T\vec{w}^{(n)}-(\vec{u}^{(n)}\cdot\nabla)\vec{w}^{(n)}+ \\ \nabla q=\alpha(\vec{u}^{(n)}-\vec{U}^{(n)}) \end{cases} \tag{3.30}$$

with final condition $\vec{w}^{(N+1)}(\vec{x})=\gamma(\vec{u}^N-\vec{U}^N)$ and homogeneous boundary condition.

Proof: Let $(\vec{\mathbf{u}},\vec{\mathbf{g}})$ be an optimal pair solution of the problem defined in eq (3.24) and $\Delta t\leq\Delta t_1$. We can compute the Gateux derivative of the functional $L^N(\vec{\mathbf{g}})$ in the direction of $\vec{\mathbf{h}}\in\mathbf{L^2}(\boldsymbol{\Omega})$. We have

$$\frac{DL}{D\vec{\mathbf{g}}}\cdot\vec{\mathbf{h}}=\alpha\sum_{n=1}^{N}\int_\Omega(\vec{u}^{(n)}-\vec{U}^{(n)})\left(\frac{D\vec{u}^{(n)}}{D\vec{\mathbf{g}}}\cdot\vec{\mathbf{h}}\right)d\vec{x}+$$
$$\gamma\int_\Omega(\vec{u}^{(N)}-\vec{U}^{(N)})\left(\frac{D\vec{u}^{(N)}}{D\vec{\mathbf{g}}}\cdot\vec{\mathbf{h}}\right)d\vec{x}=$$
$$\alpha\sum_{n=1}^{N}\int_\Omega(\vec{u}^{(n)}-\vec{U}^{(n)})\tilde{w}^{(n)}d\vec{x}+\gamma\int_\Omega(\vec{u}^{(N)}-\vec{U}^{(N)})\tilde{w}^{(N)}d\vec{x}$$

We note that $\alpha(\vec{u}^N-\vec{U}^N)$ belongs to $L^\infty((0,T);L^2(\Omega))$. Now using eq(3.30) we have

$$\alpha\sum_{n=1}^{N}\Delta t\int_\Omega(\vec{u}^{(n)}-\vec{U}^{(n)})\tilde{w}^{(n)}d\vec{x}+\int_\Omega\gamma(\vec{u}^{(N)}-\vec{U}^{(N)})\tilde{w}^{(N)}d\vec{x}=$$

$$\sum_{n=1}^{N}\int_\Omega \tilde{w}^{(n)}\Delta t\left(-\frac{1}{\Delta t}(\vec{w}^{(n+1)}-\vec{w}^{(n)})+\nu(\pi A)\vec{w}+(\pi C)'^{*}\vec{w}^{(n)}\right)d\vec{x}+$$
$$\int_\Omega \vec{w}^{N+1}\tilde{w}^{N}d\vec{x}=-\sum_{n=2}^{N+1}\int_\Omega \tilde{w}^{(n-1)}\vec{w}^{(n)}d\vec{x}+\int_\Omega \vec{w}^{N+1}\tilde{w}^{N}d\vec{x}+$$
$$\Delta t\sum_{n=1}^{N}\int_\Omega \tilde{w}^{(n)}\left(\frac{1}{\Delta t}\vec{w}^{(n)}+\nu(\pi A)\vec{w}^{(n)}+(\pi C)'^{*}\vec{w}^{(n)}\right)d\vec{x}=$$
$$\sum_{n=1}^{N}\Delta t\int_\Omega \vec{w}^{(n)}\left(\frac{1}{\Delta t}(\tilde{w}^{(n)}-\tilde{w}^{(n+1)})+\nu(\pi A)\tilde{w}^{(n)}+(\pi C)'\tilde{w}^{(n)}\right)d\vec{x}=$$
$$\sum_{n=1}^{N}\Delta t\int_\Omega \vec{w}^{(n)}\vec{h}^{(n)}d\vec{x}$$

where $\vec{w}^{(n)}$ is the solution of the system in eq(3.30).
i) the proof is the same as in theorem 3.3.
ii) Let $\vec{\mathbf{w}}$ be different from zero and $\vec{g}^{(n)}$ be an optimal unit vector for all $n=1,2,...N$. The first order necessary condition in eq(3.26) ($\vec{v}^{(n)}=-\vec{w}^{(n)}/\|\vec{w}^{(n)}\|$ for all $k=1,2...N$) implies that

$$-\sum_{k=1}^{N}\int_\Omega \vec{w}^{(k)}\vec{g}^{(k)}\Delta t\geq\sum_{k=1}^{N}\int_\Omega \|\vec{w}^{(k)}\|\Delta t. \tag{3.31}$$

We suppose that there exists an $m\leq N$ such that $\vec{g}^{(m)}$ is not equal to $-\vec{w}^{(m)}/\|\vec{w}^{(m)}\|$. From the Schwartz inequality we have $|(\vec{w}^{(m)},\vec{g}^{(m)})|<\|\vec{w}^{(m)}\|$ and $|(\vec{w}^{(n)},\vec{g}^{(n)})|\leq\|\vec{w}^{(n)}\|$ for n different from m. Now we have

$$-\sum_{n=1}^{N}\int_\Omega \vec{w}^{(n)}\vec{g}^{(n)}\Delta t d\vec{x}\leq\sum_{n=1}^{N}|(\vec{w}^{(n)},\vec{g}^{(n)})|\Delta t<\sum_{n=1}^{N}\|\vec{w}^{(n)}\|\Delta t. \tag{3.32}$$

But this contradicts eq(3.31). Hence if $\vec{w}^{(n)}$ can not be equal to zero we have

$$\vec{g}^{(n)}=-\frac{\vec{w}^{(n)}}{\|\vec{w}^{(n)}\|}.\quad\sqcup$$

Now in order to get the solution of our optimal control problem we have to solve the Navier-Stokes system and the adjoint system

$$\begin{cases}\frac{1}{\Delta t}(\vec{u}^{(n)}-\vec{u}^{(n-1)},\vec{v})+\nu a(\vec{u}^{(n)},\vec{v})+c(\vec{u}^{(n)};\vec{u}^{(n)},\vec{v})+\\ b(\vec{v},p^{(n)})=M(\vec{g}^{(n)},\vec{v})\\ -\frac{1}{\Delta t}(\vec{w}^{(n+1)}-\vec{w}^{(n)},\vec{v})+\nu a(\vec{w}^{(n)},\vec{v})+c(\vec{u}^{(n)};\vec{v},\vec{w}^{(n)})+c(\vec{v};\vec{u}^{(n)},\vec{w}^{(n)})+\\ b(\vec{v},\sigma^{(n)})=-\alpha(\vec{u}^{(n)}-\vec{U}^{(n)},\vec{v})\quad \vec{v}\in H_0^1(\Omega)\\ b(\vec{u}^{(n)},q)=0\\ b(\vec{w}^{(n)},q)=0\quad\forall q\in L_0^2(\Omega)\\ \|\vec{w}\|\vec{g}^{(n)}=\vec{w}\end{cases}$$

for $n = 1, 2...N$ with final condition $\vec{g}^{(N+1)}(\vec{x}) = -\gamma(\vec{u}^{(N)} - \vec{U}^{(N)})$ and initial velocity $\vec{u}^{(0)}(\vec{x}) = \vec{u}_0(\vec{x})$. The above system of equations is the weak formulation of the following system

$$\begin{cases} \frac{1}{\Delta t}(\vec{u}^{(n)} - \vec{u}^{(n-1)} + (\vec{u}^{(n)}.\vec{\nabla})\vec{u}^{(n)} - \nu\nabla^2\vec{u}^{(n)} + \vec{\nabla}p^{(n)} = \vec{g}^{(n)} \\ \vec{\nabla}\cdot\vec{u} = 0 \\ -\frac{1}{\Delta t}(\vec{w}^{(n+1)} - \vec{w}^{(n)} + \nu\nabla^2\vec{g}^{(n)} + (\nabla\vec{u})^T\vec{w} - (\vec{u}^{(n)}\cdot\nabla)\vec{w}^{(n)} + \\ \vec{\nabla}\sigma^{(n)} = -\alpha(\vec{u}^{(n)} - \vec{U}^{(n)}) \\ \vec{\nabla}\cdot\vec{w}^{(n)} = 0 \\ \|\vec{w}\|\vec{g}^{(n)} = \vec{w} \end{cases} \tag{3.33}$$

$forn = 1, 2...N$ with the same initial, final and homogeneous boundary condition.

3.4 Fully discrete time-space approximation

3.4.1 Formulation of the fully discrete optimal control aproximation

Let $\sigma_N = \{t_n\}_{n=0}^N$ be a partition of $[0, T]$ into equal intervals $\Delta t = T/N$ with $t_0 = 0$ and $t_N = T$. On the finite element spaces $X^h \subset H_0^1(\Omega)$ and $S^h \subset L^2(\Omega)$ we assume all the hypotheses made in subsection 2.4.1. Again for each fixed Δt (or N) and for every involved quantity $q(t, \vec{x})$ we associate the corresponding set $\{q_h^{(n)}\}_{n=1}^N$ and a continuous linear function $\vec{q}_h^N(t, \vec{x})$ such as $\vec{q}_h^N(t_n, \vec{x}) = q_h(t_n, \vec{x})$ for all n=0,1,2...N. We denote with bold vector $\mathbf{q}$ the vector $(q^{(1)}, q^{(2)}, ..., q^{(N)})$ of the discrete time components.

Given $\Delta t = T/N$, $\{\vec{g}^{(n)}\}_{n=1}^N \in \mathbf{B}(\boldsymbol{\Omega})$ and $\vec{u}_0 \in V(\Omega)$ then $(\vec{\mathbf{u}}_h, \mathbf{p}_h) \in (\mathbf{X^h}\times\mathbf{S_0^h})$ is said to be generalized solution for the Navier-Stokes fully discrete time-space approximation if $(u_h^{(n)}, p_h^{(n)})$ satisfies the following system of equations

$$\begin{cases} \frac{1}{\Delta t}(\vec{u}_h^{(n)} - \vec{u}_h^{(n-1)}, \vec{v}_h) + \nu a(\vec{u}_h^{(n)}, \vec{v}_h) + \tilde{c}(\vec{u}_h^{(n)}; \vec{u}_h^{(n)}, \vec{v}_h) + \\ b(\vec{v}_h, p^{(n)}) = M(\vec{g}^{(n)}, \vec{v}_h) \quad \forall \vec{v}_h \in X^h(\Omega) \\ b(\vec{v}_h^{(n)}, q_h) = 0 \quad \forall q_h \in S_0^h(\Omega) \end{cases} \tag{3.34}$$

for n=1,2,...N with initial velocity $\vec{u}_h^{(0)} = \pi^h\vec{u}_0(\vec{x})$ and homogeneous boundary condition.
We assume that the control $\vec{f}^{(n)}$ is bounded by M so $\|\vec{g}^{(n)}\| \le 1$ for all $n = 1, 2, ...N(\vec{f}^{(n)} = M\vec{g}^{(n)})$.

The optimal control is achieved through the functional

$$L_h^N(\vec{\mathbf{g}}) = \frac{\alpha}{2}\sum_{n=1}^{N}\|\vec{u}_h^{(n)} - \vec{U}^{(n)}\|^2\Delta t + \frac{\gamma}{2}\|\vec{u}_h^{(N)} - \vec{U}^{(N)}\|^2. \tag{3.35}$$

The formulation of the problem P_B in the fully discrete approximation becomes: *given* $\Delta t = T/N$, $\vec{u}_0 \in V(\Omega)$ *and* $\vec{U} \in U_{ad}$*, find* $(\vec{\mathbf{u}}_h, \mathbf{p}_h, \vec{\mathbf{g}})$*, a sequence in* $(\mathbf{X^h(\Omega)}\times\mathbf{S_0^h(\Omega)}\times\mathbf{B(\Omega)})$*, such that* $(\vec{\mathbf{u}}_h, \mathbf{p}_h)$ *is the solution of eq(3.34) and minimizes the cost function in eq(3.35).*

3.4.2 Existence and consistency of the fully discrete optimal control solution

We can state the existence and the consistency of the optimal control problem in the fully discrete approximation. The proof is similar to the semidiscrete case.

Theorem 3.8 *Given* $\Delta t = T/N$, $\vec{u}_0 \in V(\Omega)$ *and* $\vec{U} \in U_{ad}$*, there exists a sequence* $(\vec{\mathbf{u}}_h, \vec{\mathbf{g}})$ *in* $\mathbf{X^h}\times\mathbf{L^2(\Omega)}$ *such that* $\vec{\mathbf{u}}_h$ *is the solution of eq(3.34) and minimizes the cost function in eq(3.35).*

Theorem 3.9 *Let* $\Delta t = T/N$ *and* $\vec{u}_0^{(n)}$ *be in* $V(\Omega)$*. The solution* $\{(\vec{u}_h^{(n)}, \{\vec{g}^{(n)})\}_{n=1}^N$ *for the fully discrete optimal control problem tends to the optimal solution* $(\vec{u}, \vec{g})$ *of the continuous problem for* $\Delta t \to 0$ *(*$N \to \infty$*) and* $h \to 0$*.*

3.4.3 First-order necessary condition

The first necessary condition stated in the semidiscrete approximation can be applied also in the fully discrete approximation. By using the theorem 2.13 i) we can compute the Gateux derivative of the functional $L_h^N(\vec{\mathbf{g}})$ in the direction $\delta\vec{\mathbf{g}}$. There exists a small $(\Delta t)^*$ such that the mapping $\vec{\mathbf{u}}_h = \vec{\mathbf{u}}_h(\vec{\mathbf{g}})$ from $\mathbf{B(\Omega)}$ to $\mathbf{X^h}$ has Gateaux derivative $\frac{D\vec{\mathbf{u}}_h}{D\vec{\mathbf{g}}}\cdot\delta\vec{\mathbf{g}}$ for all $\Delta t \le (\Delta t)^*$ and $\delta\vec{\mathbf{g}} \in \mathbf{L^2(\Omega)}$. The function $\tilde{w}_h^{(n)} = \frac{D\vec{u}_h^{(n)}}{D\vec{\mathbf{g}}}\cdot\delta\vec{\mathbf{g}}$ is solution of this system of equations

$$\begin{cases} \frac{1}{\Delta t}(\tilde{w}_h^{(n)} - \tilde{w}_h^{(n-1)}, \vec{v}_h) + \nu a(\tilde{w}_h^{(n)}, \vec{v}_h) + \tilde{c}'(\tilde{w}_h^{(n)}; \vec{u}_h^{(n)}, \vec{v}_h) + \\ b(\vec{v}_h, \sigma_h^{(n)}) = (\delta\vec{g}^{(n)}, \vec{v}_h) \qquad \forall\vec{v}_h \in X^h(\Omega) \\ b(\tilde{w}_h^{(n)}, q_h) = 0 \qquad \forall q_h \in S_0^h(\Omega) \end{cases} \tag{3.36}$$

for $n = 1, 2, ...N$ with initial value $\tilde{w}_h^{(0)}(\vec{x}) = 0$ and homogeneous boundary condition. We have for every $\delta\vec{\mathbf{g}} \in \mathbf{L^2(\Omega)}$

$$\frac{DL_h^N}{D\vec{\mathbf{g}}}\cdot\delta\vec{\mathbf{g}} = \alpha\sum_{n=1}^{N}\int_\Omega(\vec{u}_h^{(n)} - \vec{U}^{(n)})\left(\frac{D\vec{u}_h^{(n)}}{D\vec{\mathbf{g}}}\cdot\delta\vec{\mathbf{g}}\right)d\vec{x} +$$

$$\gamma \int_\Omega (\vec{u}_h^{(N)} - \vec{U}^{(N)}) \left(\frac{D\vec{u}_h^{(n)}}{D\vec{\mathbf{g}}} \cdot \delta\vec{g}^{(N)} d\vec{x} \right) =$$
$$\alpha \sum_{n=1}^{N} \int_\Omega (\vec{u}_h^{(n)} - \vec{U}^{(n)}) \tilde{w}_h^{(n)} d\vec{x} + \gamma \int_\Omega (\vec{u}_h^{(N)} - \vec{U}^{(N)}) \tilde{w}_h^{(n)} d\vec{x}.$$

After using the Gateaux derivative given by eq(3.36) we get

$$\frac{DL_h^N}{D\vec{\mathbf{g}}} \cdot \delta\vec{\mathbf{g}} = \alpha \sum_{n=1}^{N} \int_\Omega (\vec{u}_h^{(n)} - \vec{U}_h^{(n)}) \tilde{w}_h^{(n)} d\vec{x} +$$
$$\gamma \int_\Omega (\vec{u}_h^{(N)} - \vec{U}^{(N)}) \tilde{w}_h^{(N)} d\vec{x} = \sum_{n=1}^{N} \int_\Omega \vec{w}_h^{(n)} \delta\vec{g}^{(n)} d\vec{x}.$$

where $\vec{w}_h^{(n)}$ is the solution of

$$\begin{cases} \frac{1}{\Delta t}(\vec{w}_h^{(n)} - \vec{w}_h^{(n+1)}, \vec{v}_h) + \nu a(\vec{w}_h^{(n)}, \vec{v}_h) + \tilde{c}(\vec{u}_h^{(n)}; \vec{v}_h, \vec{w}_h^{(n)}) + \\ \tilde{c}(\vec{v}_h; \vec{u}_h^{(n)}, \vec{w}_h^{(n)}) + b(\vec{v}_h, \sigma_h^{(n)}) = \alpha(\vec{u}_h^{(n)} - \vec{U}^{(n)}), \vec{v}_h) \quad \vec{v}_h \in X^h(\Omega) \\ b(\vec{w}_h^{(n)}, q_h) = 0 \quad q_h \in S_0^h(\Omega) \end{cases} \tag{3.37}$$

for $n = 1, 2, ...N$ with final value $\vec{w}^{(N+1)}(\vec{x}) = \gamma(\vec{u}^N - \vec{U}^N)$ and homogeneous boundary condition. Now we can state the following theorem in which the statements in i) and ii) are already contained in the theorem 3.7 as $X^h \subset L^2(\Omega)$.

Theorem 3.10 *Let $\Delta t = T/N$ and $\vec{u}_0 \in V(\Omega)$. If $(\vec{\mathbf{u}}_h, \vec{\mathbf{g}})$ is a solution of the fully discrete optimal control problem, then there exists a $(\Delta t)^*$ such that for each $\Delta t \leq (\Delta t)^*$ and for all n=1,2,...N we have*
i) $\|\vec{g}^{(n)}\| = 1$
ii) $\|\vec{w}_h^{(n)}\| \vec{g}^{(n)} = -\vec{w}_h^{(n)}$ if $\vec{w}_h^{(n)}$ is different from zero.
The function $\vec{w}_h^{(n)}$ is the solution of the system in eq(3.37).

3.5 Numerical results

3.5.1 Introduction

In order to get the optimal control solution we have to solve the following system of equations in the state and control variables $(\vec{u}, p, \vec{f}, \sigma)$

$$\begin{cases} \vec{u}_t + (\vec{u}.\vec{\nabla})\vec{u} - \nu\nabla^2\vec{u} + \vec{\nabla}p = \vec{f} \\ \vec{\nabla}\cdot\vec{u} = 0 \\ -\vec{w}_t + \nu\nabla^2\vec{w} + (\nabla\vec{u})^T\vec{w} - (\vec{u}\cdot\nabla)\vec{w} + \vec{\nabla}\sigma = -\frac{\alpha}{\beta}(\vec{u} - \vec{U}) \\ \vec{\nabla}\cdot\vec{w} = 0 \\ \|\vec{w}\|\vec{f} = M\vec{w} \end{cases} \tag{3.38}$$

on Ω with final condition $\vec{w}(T,\vec{x}) = -\gamma(\vec{u}(T,\vec{x}) - \vec{U}(T,\vec{x}))$, initial velocity $\vec{u}^{(0)}(\vec{x}) = \vec{u}_0(\vec{x})$ and homogeneous boundary condition.

Let $\sigma_N = \{t_n\}_{n=0}^N$ be a partition of $[0,T]$ into equal intervals $\Delta t = T/N$ with $t_0 = 0$ and $t_N = T$. For a fixed Δt (or N) let $X^h \subset H_0^1(\Omega)$ and $S_0^h \subset L^2(\Omega)$ be two families of finite dimensional subspaces parameterized by h that tends to zero. The eq(3.38) becomes

$$\begin{cases} \frac{1}{\Delta t}(\vec{u}_h^{(n)} - \vec{u}_h^{(n-1)}, \vec{v}_h) + \nu a(\vec{u}_h^{(n)}, \vec{v}_h) + \tilde{c}(\vec{u}_h^{(n)}; \vec{u}_h^{(n)}, \vec{v}_h) + \\ b(\vec{v}_h, p_h^{(n)}) = (\vec{f}_h^{(n)}, \vec{v}_h) \\ -\frac{1}{\Delta t}(\vec{w}_h^{(n+1)} - \vec{w}_h^{(n)}, \vec{v}_h) + \nu a(\vec{w}_h^{(n)}, \vec{v}_h) + \tilde{c}(\vec{u}_h^{(n)}; \vec{v}_h, \vec{w}_h^{(n)}) + \\ \tilde{c}(\vec{v}_h; \vec{u}_h^{(n)}, \vec{w}_h^{(n)}) + b(\vec{v}_h, \sigma_h^{(n)}) = -\alpha(\vec{u}_h^{(n)} - \vec{U}^{(n)}, \vec{v}_h) \\ b(\vec{u}_h^{(n)}, q_h) = 0 \\ b(\vec{w}_h^{(n)}, q_h) = 0 \\ \|\vec{w}_h^{(n)}\|(\vec{f}_h^{(n)}, \vec{v}_h) = M(\vec{w}_h^{(n)}, \vec{v}_h) \qquad \forall q_h \in S_0^h(\Omega) \quad \vec{v}_h \in X^h(\Omega) \end{cases} \tag{3.39}$$

for $n = 1, 2, ...N$ with final condition $\vec{w}_h^{(N+1)}(\vec{x}) = -\gamma(\vec{u}_h^{(N)} - \vec{U}^{(N)})$, initial velocity $\vec{u}_h^{(0)}(\vec{x}) = \pi^h\vec{u}_0(\vec{x})$ and homogeneous boundary conditions.

3.5.2 Numerical algorithm

Let us consider a gradient method for the optimal control problem. We have to split the system in three parts in order to apply the algorithm. Now the fully discrete system consists of:

a) Navier-Stokes equation

$$\begin{cases} \frac{1}{\Delta t}(\vec{u}_h^{(n)} - \vec{u}_h^{(n-1)}, \vec{v}_h) + \nu a(\vec{u}_h^{(n)}, \vec{v}_h) + c(\vec{u}_h^{(n)}; \vec{u}_h^{(n)}, \vec{v}_h) + \\ b(\vec{v}_h, p_h^{(n)}) = M(\vec{g}_h^{(n)}, \vec{v}_h) \\ b(\vec{u}_h^{(n)}, q_h) = 0 \qquad \forall q_h \in S_0^h(\Omega) \quad \vec{v}_h \in X^h(\Omega) \end{cases} \tag{3.40}$$

for $n = 1, 2, ...N$ with initial velocity $\vec{u}^{(0)}(\vec{x})_h = \pi^h \vec{u}_0(\vec{x})$ and homogeneous boundary condition;
b) adjoint equation :

$$\begin{cases} -\frac{1}{\Delta t}(\vec{w}_h^{(n+1)} - \vec{w}_h^{(n)}, \vec{v}_h) + \nu a(\vec{w}_h^{(n)}, \vec{v}_h) + c(\vec{u}_h^{(n)}; \vec{v}_h, \vec{w}_h^{(n)}) + \\ c(\vec{v}_h; \vec{u}_h^{(n)}; \vec{w}_h^{(n)}) + b(\vec{v}_h, \sigma_h^{(n)}) = -\alpha(\vec{u}_h^{(n)} - \vec{U}^{(n)}, \vec{v}_h) \\ b(\vec{w}_h^{(n)}, q_h) = 0 \qquad \forall q_h \in S_0^h(\Omega) \quad \vec{v}_h \in X^h(\Omega) \end{cases} \tag{3.41}$$

for $n = 1, 2...N$ with final condition $\vec{w}_h^{(N+1)}(\vec{x}) = -\gamma(\vec{u}_h^{(N)} - \vec{U}^{(N)})$ and homogeneous boundary condition.
c) control equation

$$\|\vec{w}_h^{(n)}\|(\vec{g}_h^{(n)}, \vec{v}_h) = (\vec{w}_h^{(n)}, \vec{v}_h) \qquad \forall \vec{v}_h \in X^h(\Omega) \tag{3.42}$$

for n=1,2,...N.

In the gradient algorithm we satisfy this relation only when convergence is reached. Let $L^{(k)} = L(\vec{\mathbf{g}}_h(k))$ and τ be the tolerance required for the functional convergence. We set

$$N_h^{(n)}(k) = 1/\sqrt{\beta^2\|\vec{g}_h^{(n)}(k)\|^2 - 2\beta(\vec{g}_h^{(n)}(k), \vec{w}_h^{(n)}(k+1)) + \|\vec{w}_h^{(n)}(k+1)\|^2}$$

and

$$\beta^{(n)}(k) = (\vec{g}_h^{(n)}(k), \vec{w}_h^{(n)}(k+1)) + \epsilon\sqrt{\frac{\|\vec{w}_h^{(n)}(k+1)\|^2 - (\vec{g}_h^{(n)}(k), \vec{w}_h^{(n)}(k+1))^2}{1 + 4\epsilon^2}}.$$

The gradient algorithm in this case proceeds as follows:

a) initial configuration:
i) given $\vec{\mathbf{g}}_h(0), \tau$ and $\epsilon = 1$; $(\|\vec{g}_h^{(n)}(0)\| = 1;\ n = 1, 2, ..N)$
ii) solve for $\vec{\mathbf{u}}_h(0)$ in eq(3.40) with $\vec{\mathbf{g}}_h(0)$;
iii) evaluation of $L^{(0)}$;

b) main loop :
iv) solve for $\vec{\mathbf{w}}_h(k)$ in eq(3.41) with $\vec{\mathbf{u}}_h(k-1)$;

c) optimization loop:
v) with $\vec{g}_h^{(n)}(k) = \vec{g}_h^{(n)}(k-1) - \epsilon N_h^{(n)}(k-1)\left(\beta^{(n)}(k-1)\vec{g}_h^{(n)}(k-1) - \vec{w}_h^{(n)}(k)\right)$ solve for $\vec{\mathbf{u}}_h(k)$ in eq(3.40);

vi) check if $L^{(k)}$ is less than $L^{(k-1)}$: if $L^{(k)} \leq L^{(k-1)}$ then $\epsilon = 1.5\epsilon$ and go to b); if $L^{(k)} > L^{(k-1)}$ then $\epsilon = .5\epsilon$ and go to c).

The algorithm stops when $|L^{(k)} - L^{(k-1)}|/L^{(k)} \leq \tau$. Of course with this choice of $\beta^{(n)}$ we have $\|\vec{g}_h^{(n)}(k)\| = 1$ for all k. The idea stems from the expansion in Taylor's series of the functional. The convergence of the algorithm is a direct consequence of the lemma 2.9. In fact, using this lemma, we can show that the gradient algorithm converges to the solution.

Theorem 3.11 *Let $(\vec{\mathbf{u}}_h(k), \vec{\mathbf{w}}_h(k), \mathbf{p}_h(k), \sigma_h(k), \vec{\mathbf{g}}_h(k))$ be the k-th step solution of the gradient algorithm and $(\vec{\mathbf{u}}_h, \vec{\mathbf{w}}_h, \mathbf{p}_h, \sigma_h, \vec{\mathbf{g}}_h)$ be the solution of the eq(3.39) for $\Delta t = T/N$. Then there exists a Δt_1 such that if $\Delta t \leq \Delta t_1$ the solution of the gradient algorithm converges to $(\vec{\mathbf{u}}_h, \vec{\mathbf{w}}_h, \mathbf{p}_h, \sigma_h, \vec{\mathbf{g}}_h)$ for any initial guess $\vec{\mathbf{g}}_h(0)$ when $k \to \infty$.*

Proof: In order to prove this theorem we have to satisfy the hypotheses of the lemma 2.9. Let Δt be equal to T/N. For each $\vec{\mathbf{g}}_h$ in $\mathbf{X^h}$ and such that $\|\vec{g}_h^{(n)}\|^2 \leq 1$ for $n = 1, 2, ...N$ the second Frechet derivative $\frac{D^2 J}{D\vec{\mathbf{g}}_{1h} D\vec{\mathbf{g}}_{2h}} \cdot \delta\vec{\mathbf{g}}_{1h}\delta\vec{\mathbf{g}}_{2h}$ can be computed as

$$\frac{D^2 J}{D^2\vec{\mathbf{g}}_h} \cdot \delta\vec{\mathbf{g}}_{1h} \cdot \delta\vec{\mathbf{g}}_{1h} = \alpha \sum_{n=1}^{N} \int_\Omega \left(\frac{D\vec{u}_h^{(n)}}{D\vec{\mathbf{g}}_h} \cdot \delta\vec{\mathbf{g}}_{1h} \right) \left(\frac{D\vec{u}_h^{(n)}}{D\vec{\mathbf{g}}_h} \cdot \delta\vec{\mathbf{g}}_{2h} \right) d\vec{x}\Delta t +$$
$$\gamma \int_\Omega \left(\frac{D\vec{u}_h^{(N)}}{D\vec{\mathbf{g}}_h} \cdot \delta\vec{\mathbf{g}}_{1h} \right) \left(\frac{D\vec{u}_h^{(N)}}{D\vec{\mathbf{g}}_h} \cdot \delta\vec{\mathbf{g}}_{2h} \right) d\vec{x}.$$

where $\tilde{w}_{h1}^{(n)} = \frac{D\vec{u}_h^{(n)}}{D\vec{\mathbf{g}}_h} \cdot \delta\vec{\mathbf{g}}_{1h}$ and $\tilde{w}_{h2}^{(n)} = \frac{D\vec{u}_h^{(n)}}{D\vec{\mathbf{g}}_h} \cdot \delta\vec{\mathbf{g}}_{2h}$ are the Gateaux derivatives. Besides, from lemma 2.8 we have an estimate for the Gateaux derivatives. We know that there exists a Δt_1 such that if $\Delta t \leq \Delta t_1$ we have

$$\sum_{n=1}^{N} \|\tilde{w}_{h1}^{(n)}\|^2 \Delta t \leq C_1 \sum_{n=1}^{N} \|\delta\vec{g}_{h1}^{(n)}\|_1^2 \Delta t$$
$$\sum_{n=1}^{N} \|\tilde{w}_{h2}^{(n)}\|^2 \Delta t \leq C_1 \sum_{n=1}^{N} \|\delta\vec{g}_{h2}^{(n)}\|_1^2 \Delta t \tag{3.43}$$

where C_1 is constant.

Thus it follows that there exists a constant r such that

$$\frac{D^2\vec{u}_h^{(n)}}{D\vec{\mathbf{g}}_{1h} D\vec{\mathbf{g}}_{2h}} \cdot \delta\vec{\mathbf{g}}_1 \cdot \delta\vec{\mathbf{g}}_1 \leq r \|\delta\vec{g}_{h1}^{(n)}\|_{L^2((0,T);X^h)} \|\delta\vec{g}_{h2}^{(n)}\|_{L^2((0,T);X^h)}$$

if $\Delta t \leq \Delta t_1$.

There exists also a constant s such that

$$\frac{D^2 J}{D\vec{\mathbf{g}}_{1h} D\vec{\mathbf{g}}_{1h}} \cdot \delta\vec{\mathbf{g}}_{1h} \cdot \delta\vec{\mathbf{g}}_{1h} =$$
$$\alpha \sum_{n=1}^{N} \|\tilde{w}_{h1}^{(n)}\|^2 \Delta t + \sum_{n=1}^{N} \beta^{(n)} \|\delta\vec{g}_{1h}^{(n)}\|^2 \Delta t + \gamma \|\tilde{w}_{h1}^{(N)}\|^2 \geq s \|\delta\vec{g}_{1h}^{(n)}\|^2_{L^2((0,T);X^h)}.$$

The fact that no limitations are imposed assures that for every initial guess the gradient algorithm converges for small time steps. $\sqcup$
We remark that $\int_\Omega \vec{g}_h^{(n)} \delta\vec{g}_h^{(n)} d\vec{x} = 0$, which is a direct consequence of the constraint $\int_\Omega |\vec{g}_h^{(n)}|^2 d\vec{x} = 1$ for all $n = 1, 2, ...N$. The gradient of the functional can be written

$$\frac{\partial L_h^N(\vec{\mathbf{g}})}{\partial \vec{\mathbf{g}}} \cdot \delta\vec{\mathbf{g}} = \sum_{n=1}^{N} \int_\Omega \vec{w}^{(n)} \cdot \delta\vec{\mathbf{g}} d\vec{x} = \sum_{n=1}^{N} \int_\Omega (\vec{w}^{(n)} - \beta^{(n)} \vec{g}^{(n)}) \cdot \delta\vec{\mathbf{g}} d\vec{x}$$

where $\beta^{(n)}$ can be arbitrarily choosen in order to satisfy the constraint. We also note that the time step choice in this algorithm can be critical for high values of M.

3.5.3 Test 1

We consider a unit square domain $(0,1) \times (0,1) \subset \mathcal{R}^2$. We assume that the time interval [0,1] is divided into equal intervals of time $\Delta t = 1/N$. The finite element spaces are chosen to be piecewise quadratic for the on velocity and linear for the pressure. The Taylor-Hood finite elements are used in this calculation on a rectangular mesh. The mesh size is h and calculations with varying mesh sizes have been performed. In this first test we are interested in the convergence history for all the parameters involved and so a simple stationary target velocity $\vec{U} = (U, V)$ is chosen. The target velocity for this test is defined by

$$\phi(t,z) = (1 - cos(2\pi tz)) \times (1-z)^2$$
$$U(x,y) = 10\frac{d}{dy}\left(\phi(0.4,x)\phi(0.4,y)\right) \qquad V(x,y) = -10\frac{d}{dx}\left(\phi(0.4,x)\phi(0.4,y)\right).$$

<u>Velocity tracking evolution</u>

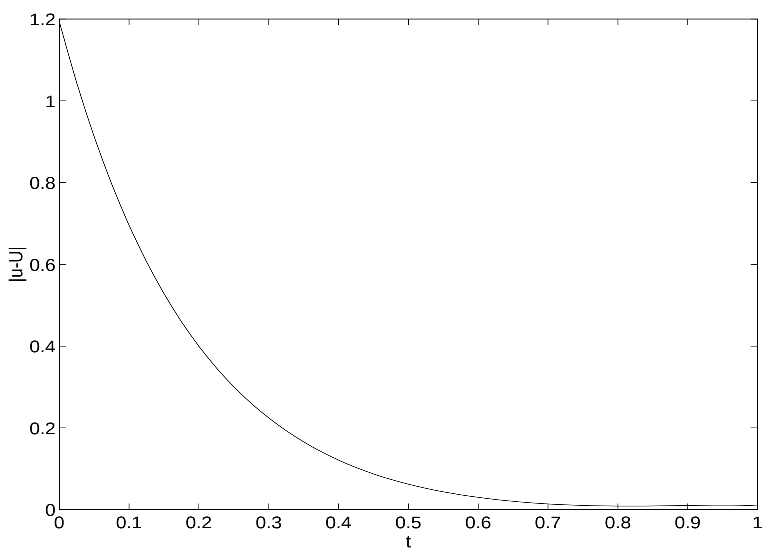

Figure 3.1: Test 1. Error $\|\vec{u} - \vec{U}\|$

We can see a first example of control where the initial velocity is

$$u_0(x,y) = -U(x,y) \qquad v_0(x,y) = -V(x,y).$$

All the pictures are normalized by the maximum value. Fig.3.1 shows the error $\|\vec{u} - \vec{U}\|$ between the controlled flow $\vec{u}$ and the target flow $\vec{U}$. As we can see the error does not go

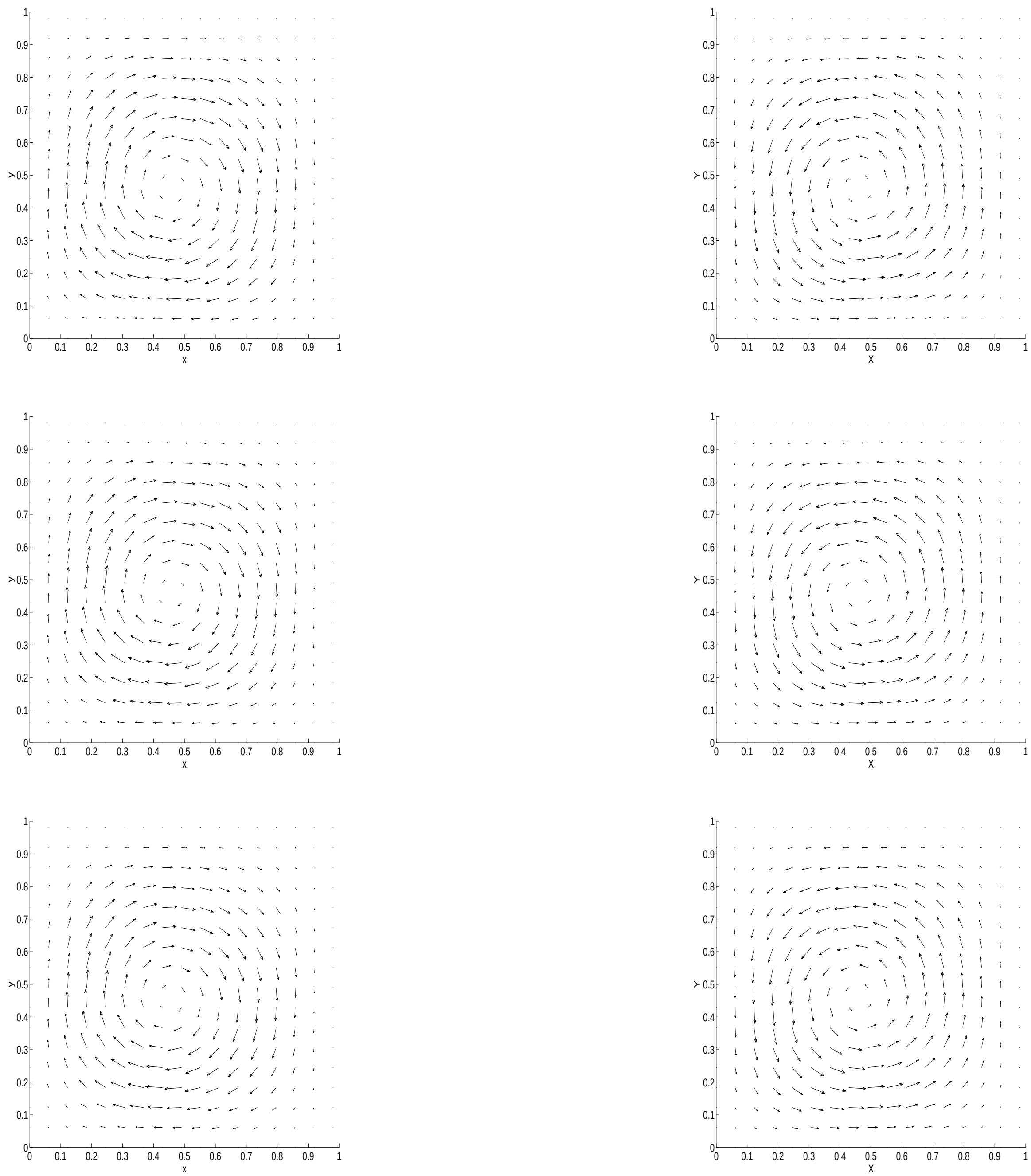

Figure 3.2: Test 1. Controlled(right) and desired(left) flow at $t = 0$ (top), $t = .05$ (middle) and $t = .1$ (bottom)

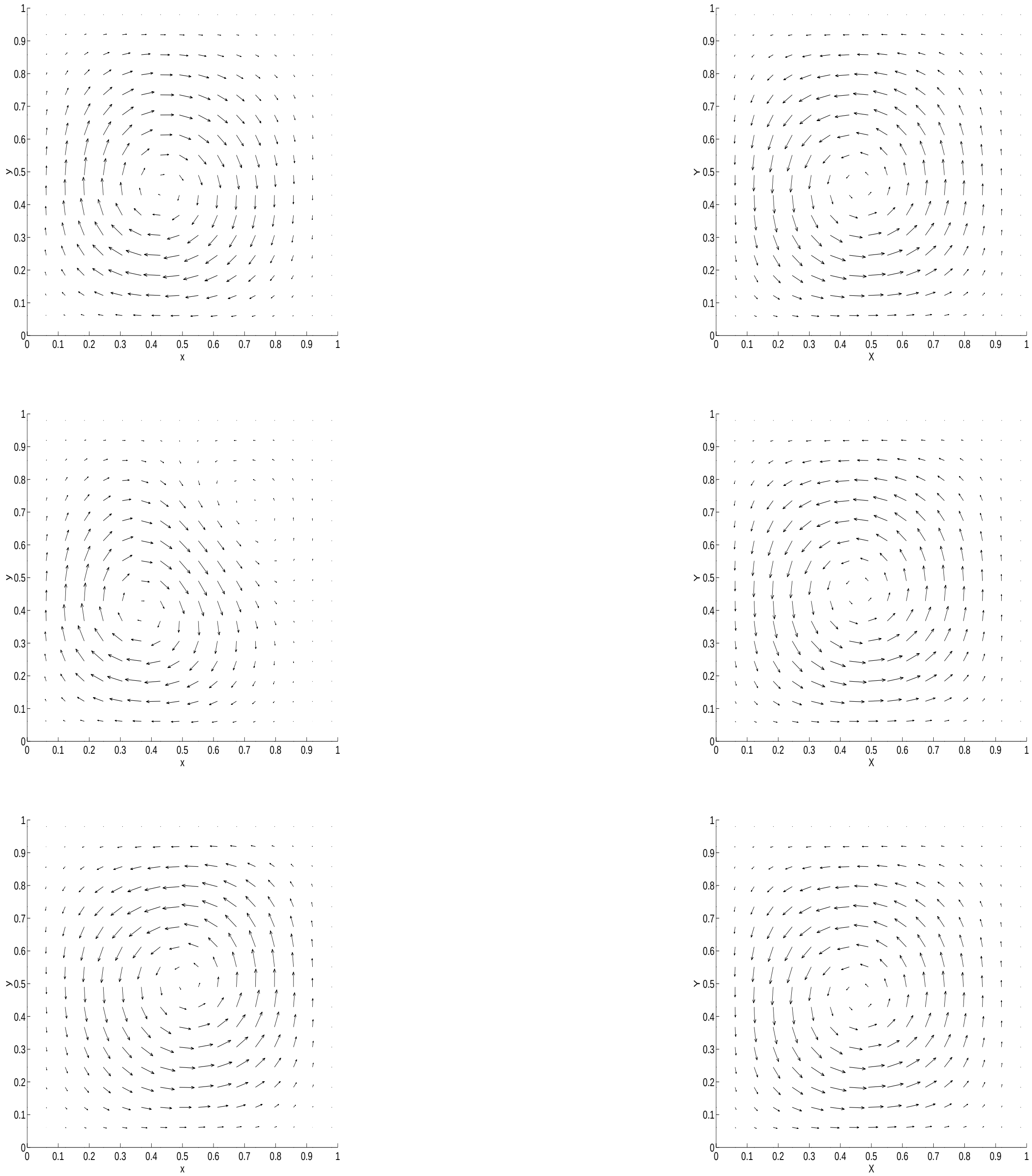

Figure 3.3: Test 1. Controlled(left) and desired(right) flow at $t = .112$ (top), $t = .125$ (middle) and $t = .135$ (bottom)

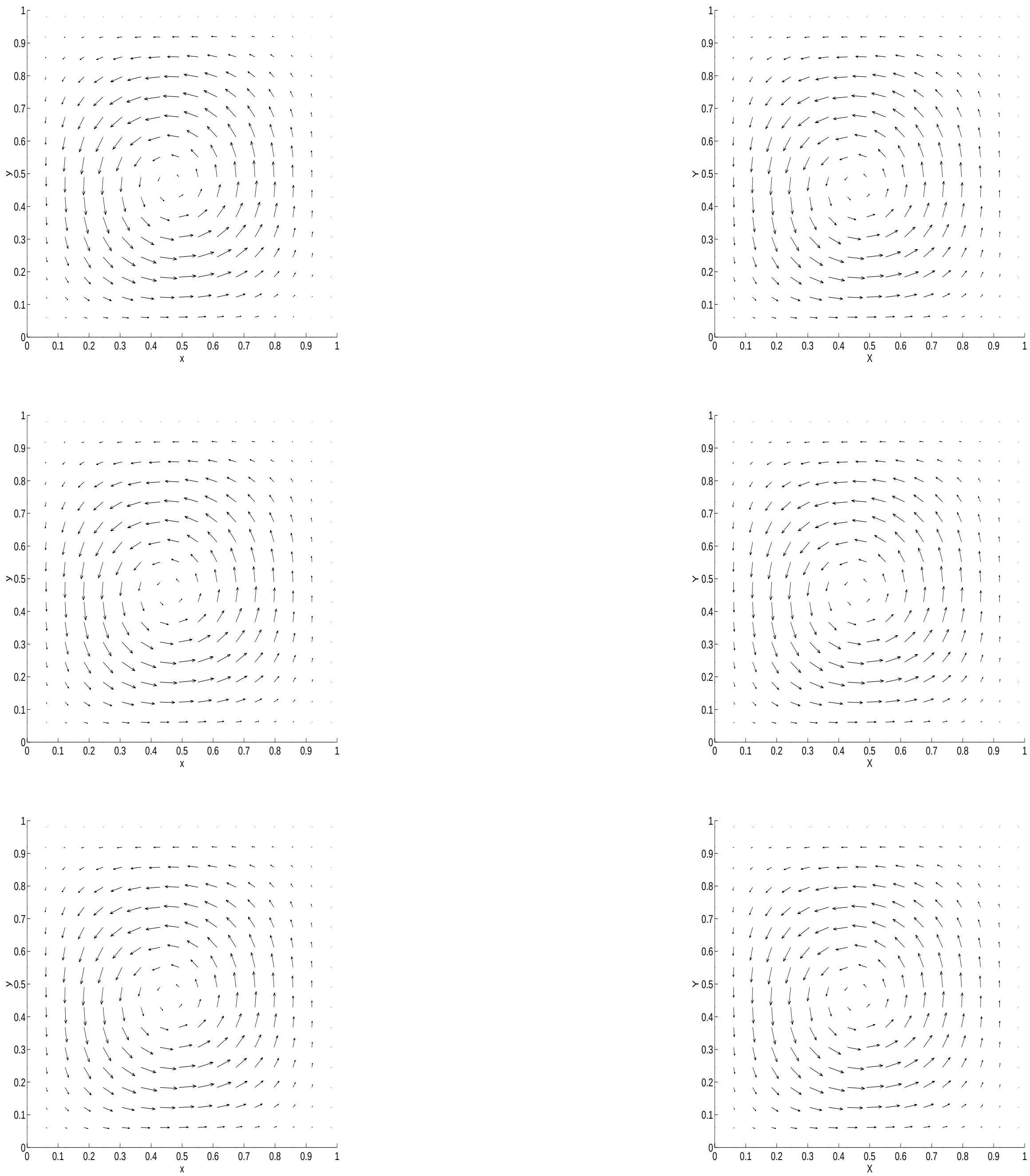

Figure 3.4: Test 1. Controlled(left) and desired(right) flow at $t = .2$ (top), $t = .5$ (middle) and $t = 1$ (bottom)

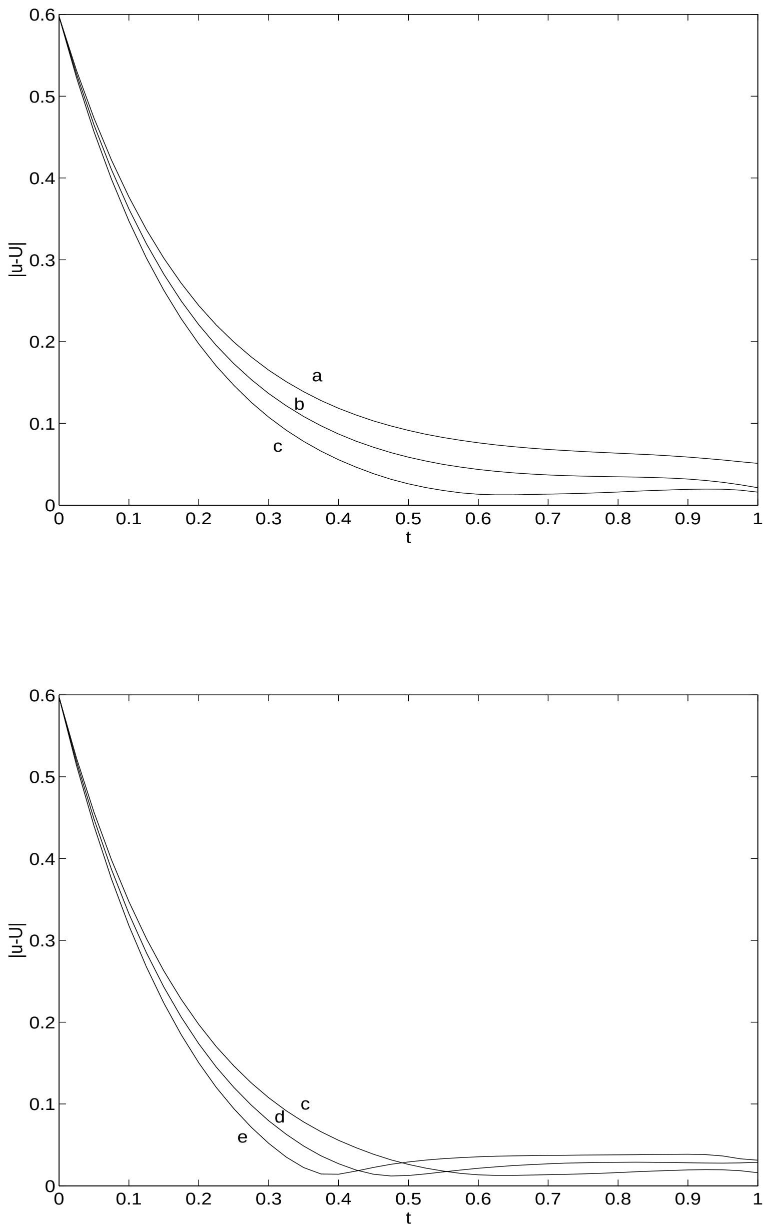

Figure 3.5: Test 1. Error $\|\vec{u} - \vec{U}\|$ for different values of M

to zero very rapidly due to the bounded control. The evolution is in Fig.3.2 - Fig.3.4. The controlled fluid is on the left and the desired flow is on the right. As we can see at t=0.15 we reach a match in shape and at t=0.6 we reach match in magnitude. For this calculation $\Delta t = 0.025$ and $h = 1/16$. Also α has been set to 1, $\|\vec{f}\|$ to 3.2 and γ to .5.

Velocity tracking with different control norm.

We want to analyse what happens if we change the norm of the control $\|\vec{f}\| = M$. The initial velocity is set to zero. In Fig.3.5 we have the error $\|\vec{u} - \vec{U}\|$ between the controlled

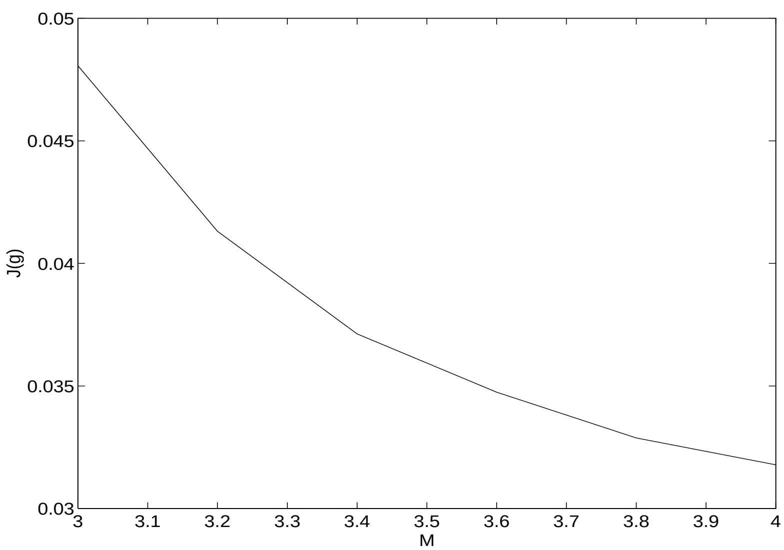

Figure 3.6: Test 1. Value of the functional for different values of M ($\Delta t = .025$)

flow $\vec{u}$ and the target flow $\vec{U}$ for different value of M. On the top and bottom figure we have M equals 3.0 (a), 3.2 (b), 3.4 (c) and 3.4 (c), 3.6 (d), 3.8 (e) respectively. The value of γ in this calculation is constant and equal to .5. The time step Δt is again 0.025 and $h = 1/16$. The initial velocity is set to zero. We can note that the control flow matches better over all the time interval for increasing value of M less than 3.4. For $M > 3.4$ there is a gain for low t but the controlled flow goes far from the desired flow on the remaining part of the time interval with respect to lower M. As we can see in Fig.3.6 the net gain in the functional $J = L(\vec{g})$ is very little for these values of M and Δt fixed. For high values of M the time step should be reduced.

The $\beta(t)$ function agrees with the intuitive behavior of the norm error. This function is

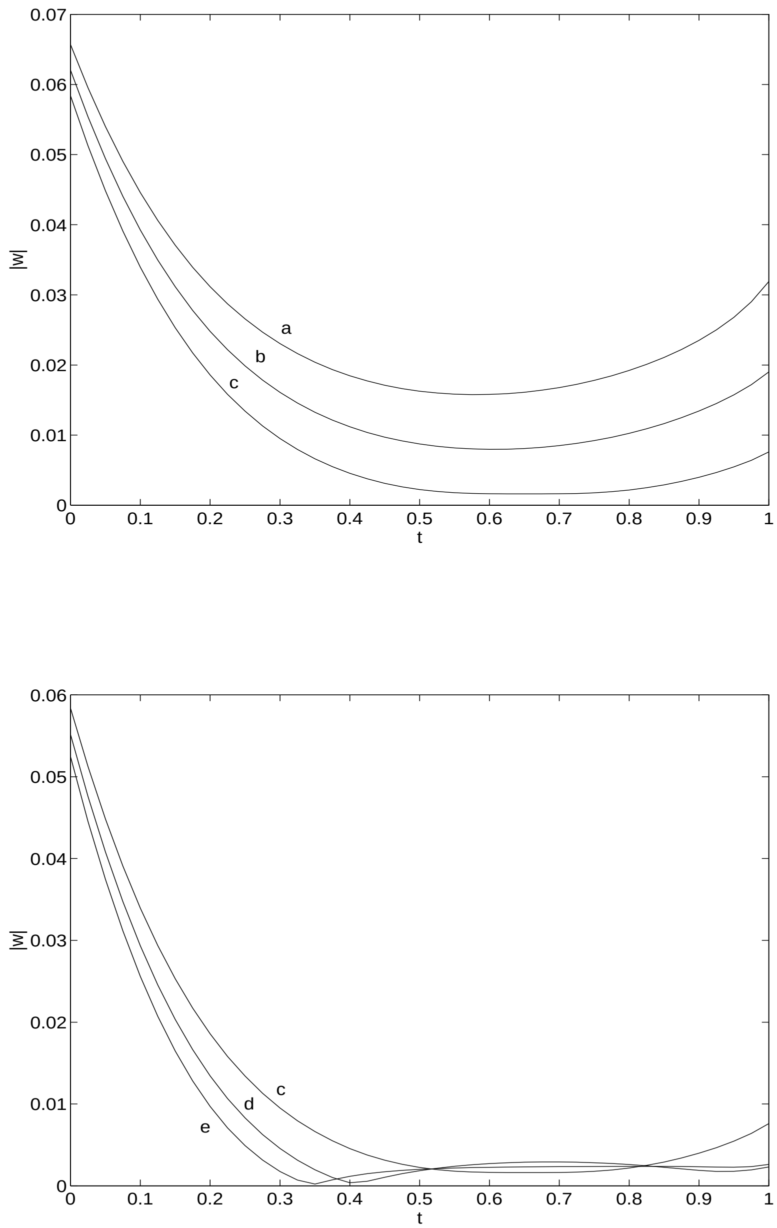

Figure 3.7: Test 1. Adjoint function norm $\|\vec{w}\|$

shown in Fig.3.7. Starting on the top the function $\beta(t)$ is shown for M equals 3.0, 3.2, 3.4 and 3.4, 3.6, 3.6. We can recall that the function $\beta(t)$ takes the values of the norm of the adjoint function $\vec{w}$ or also $(-\vec{g}, \vec{w})$.

3.5.4 Test 2

We consider a unit square domain $(0,1)\times(0,1) \subset \mathcal{R}^2$. We assume that the time interval [0,1] is divided in equal intervals of time $\Delta t = 1/N$. The Taylor-Hood finite elements are used in this calculation on a rectangular mesh. We report only the final result with $h = 1/16$ but calculations with varying mesh sizes has been performed. The target velocity $\vec{U}$ for this test is equal to

$$\phi(k,t,z) = (1 - cos(2k\pi tz)) \times (1-z)^2$$
$$a(k,t,x,y) = \frac{d}{dy}\left(\phi(k,t,x)\phi(k,t,y)\right) \qquad b(k,t,x,y) = -\frac{d}{dx}\left(\phi(k,t,x)\phi(k,t,y)\right)$$
$$U = a(1,.4,x,y) + a(2,t,x,y)/(4\pi t + 1) \qquad V = b(1,.4,x,y) + b(2,t,x,y)/(4\pi t + 1).$$

With this velocity field we have the superposition of two flows. One flow with a vortex at the center of the domain and another flow with four vortices. Each of these flows prevails at different times of the evolution. The initial velocity for the controlled flow is

$$u_0(x,y) = -8U(1/4,x,y) \qquad v_0(x,y) = -8V(1/4,x,y)$$

The evolution is in Fig.3.8 - Fig.3.11. In this computation α has been set to 1 and γ to .5. The control $\vec{g}$ in norm must be less than 1.6 ($M = 1.6$). The controlled fluid is on the right, the desired flow is on the left and all the pictures are normalized. As we can see at $t = 0.3$ the controlled flow looks like the desired flow. Fig.3.12 shows the error $\|\vec{u} - \vec{U}\|$ between the controlled flow $\vec{u}$ and the target flow $\vec{U}$. At the beginning the error rapidly decreases but after this initial interval of time this error increases due to changes in the desired flow. The magnitude of the bounded control is less then 1.6. This power is not enough to match perfectly the time evolution of the desired flow. For the same flow Fig.3.13 shows the values of the norm of the adjoint variable $\vec{w}$ in function of time ($\beta(t)$).

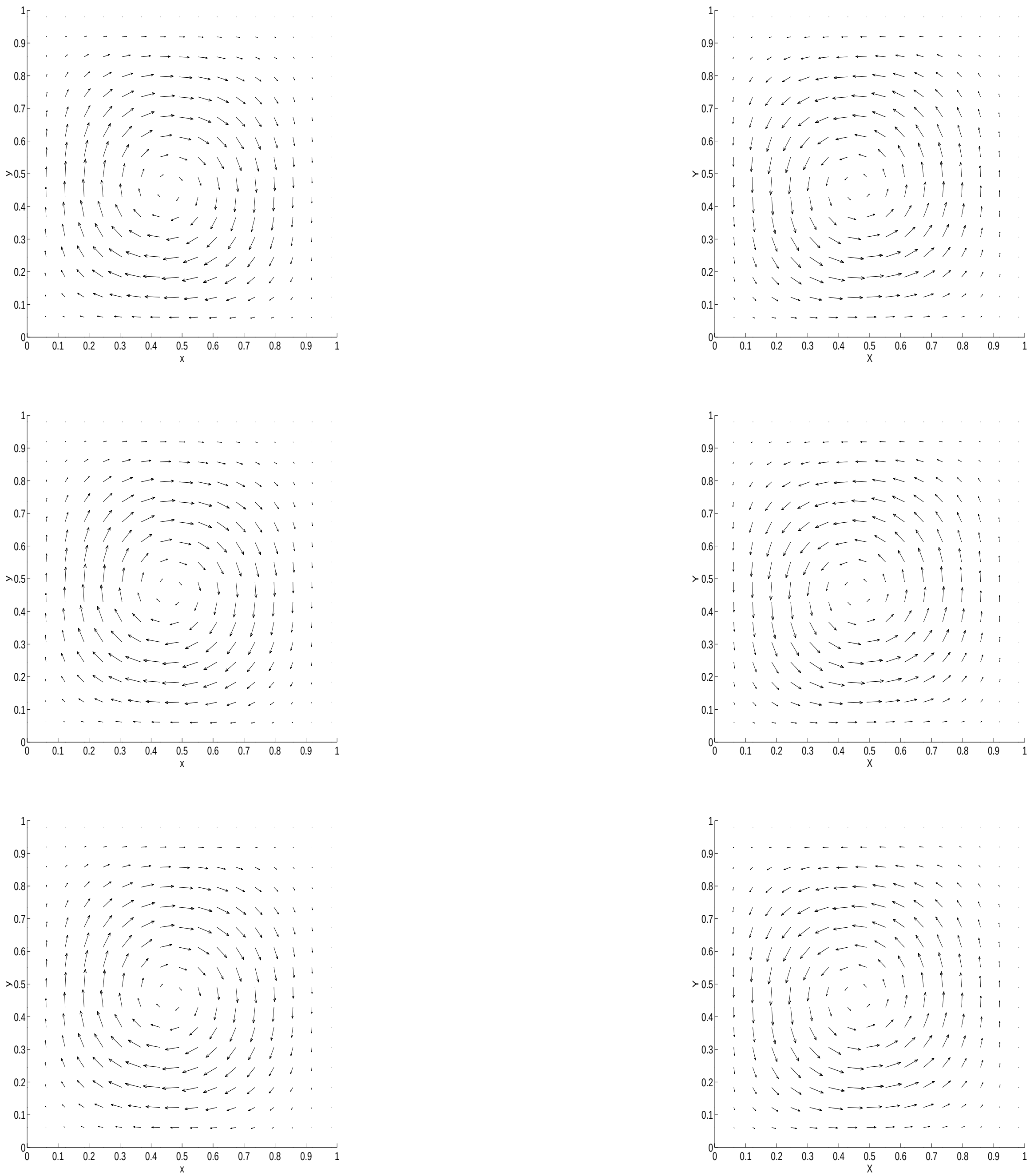

Figure 3.8: Test 2. Controlled(left) and desired(right) flow at $t = 0$ (top), $t = .1$ (middle) and $t = .2$ (bottom)

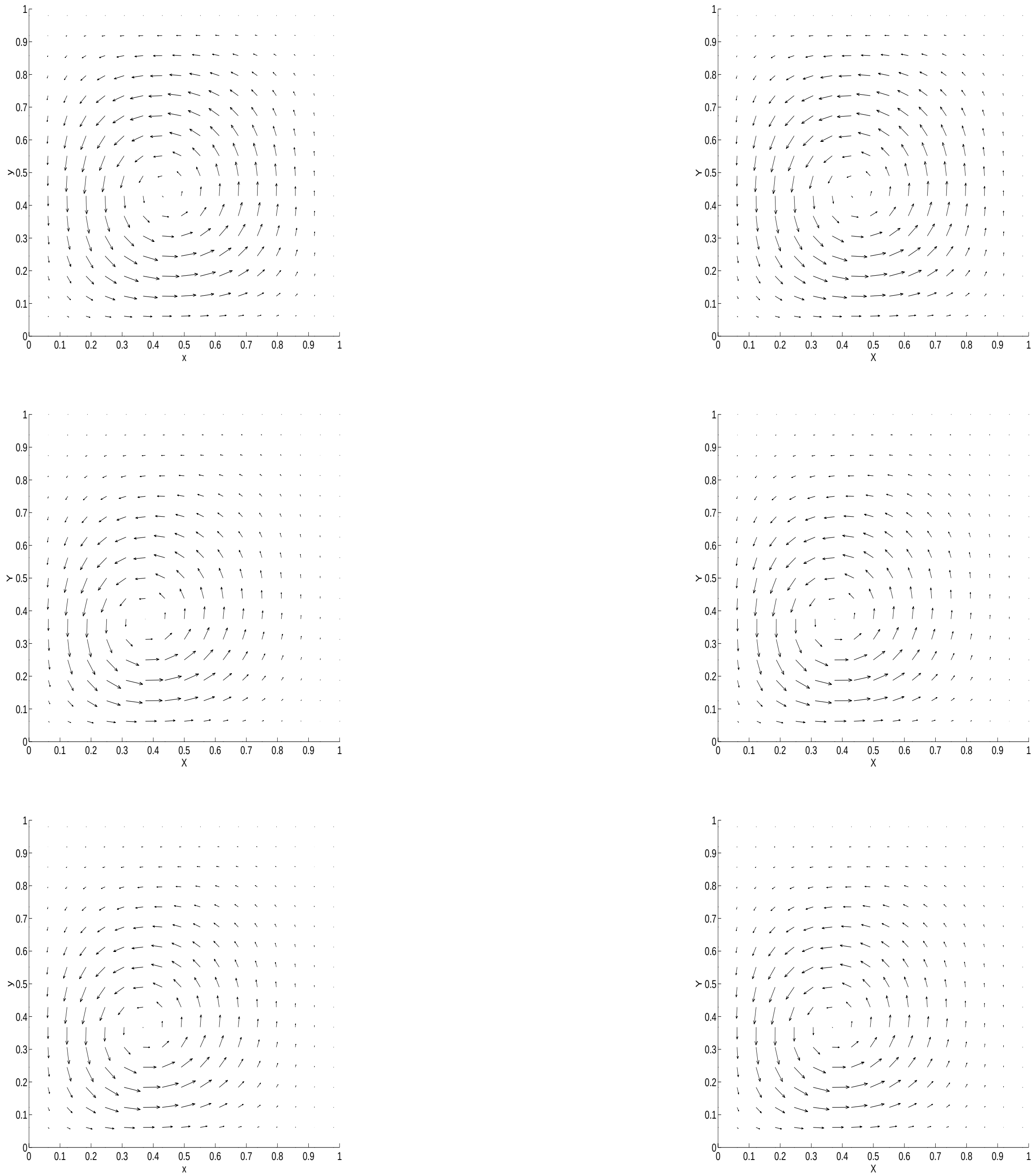

Figure 3.9: Test 2. Controlled(left) and desired(right) flow at $t = .3$ (top), $t = .4$ (middle) and $t = .5$ (top)

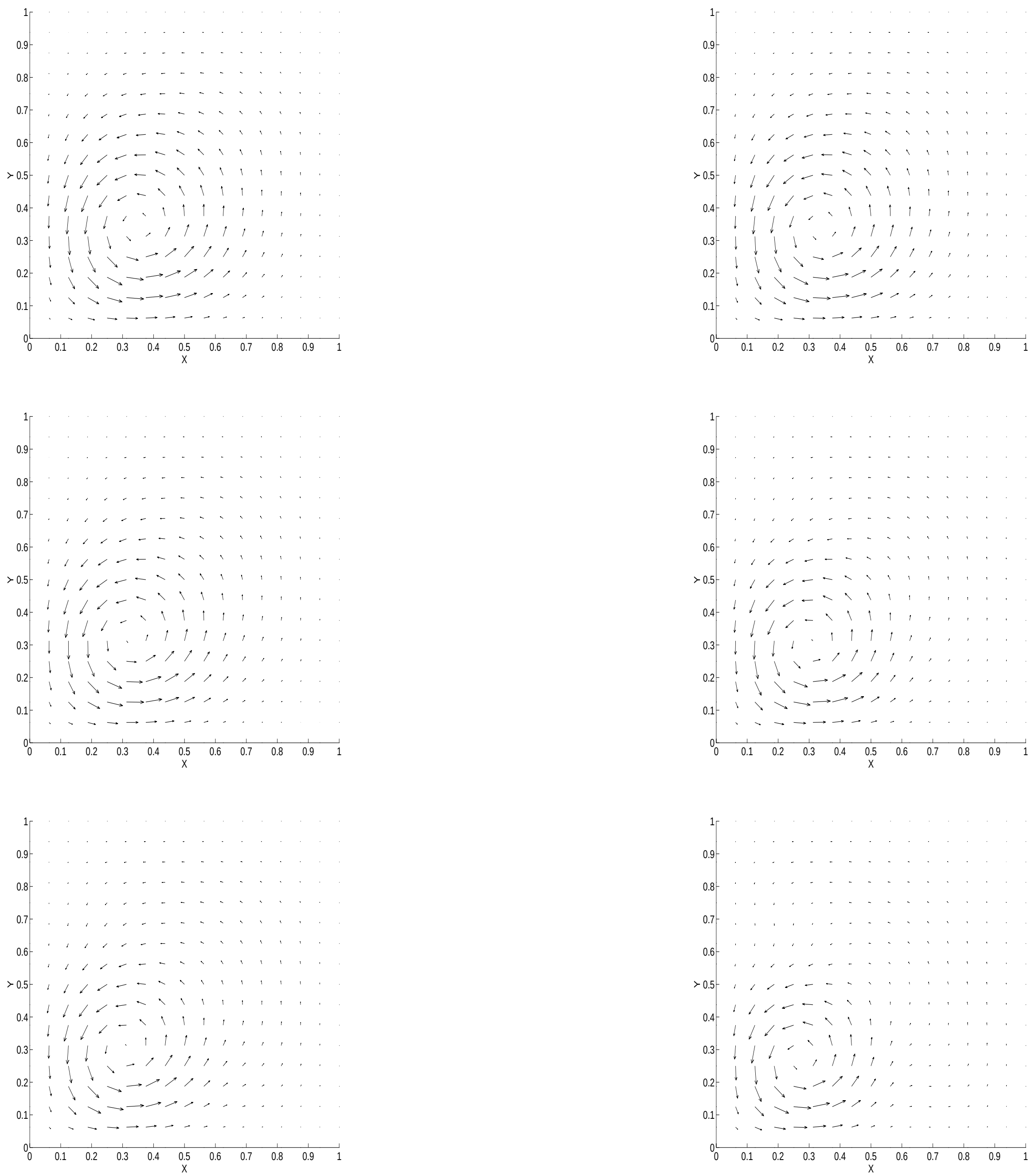

Figure 3.10: Test 2. Controlled(left) and desired(right) flow at $t = .6$ (top), $t = .7$ (middle) and $t = .8$ (bottom)

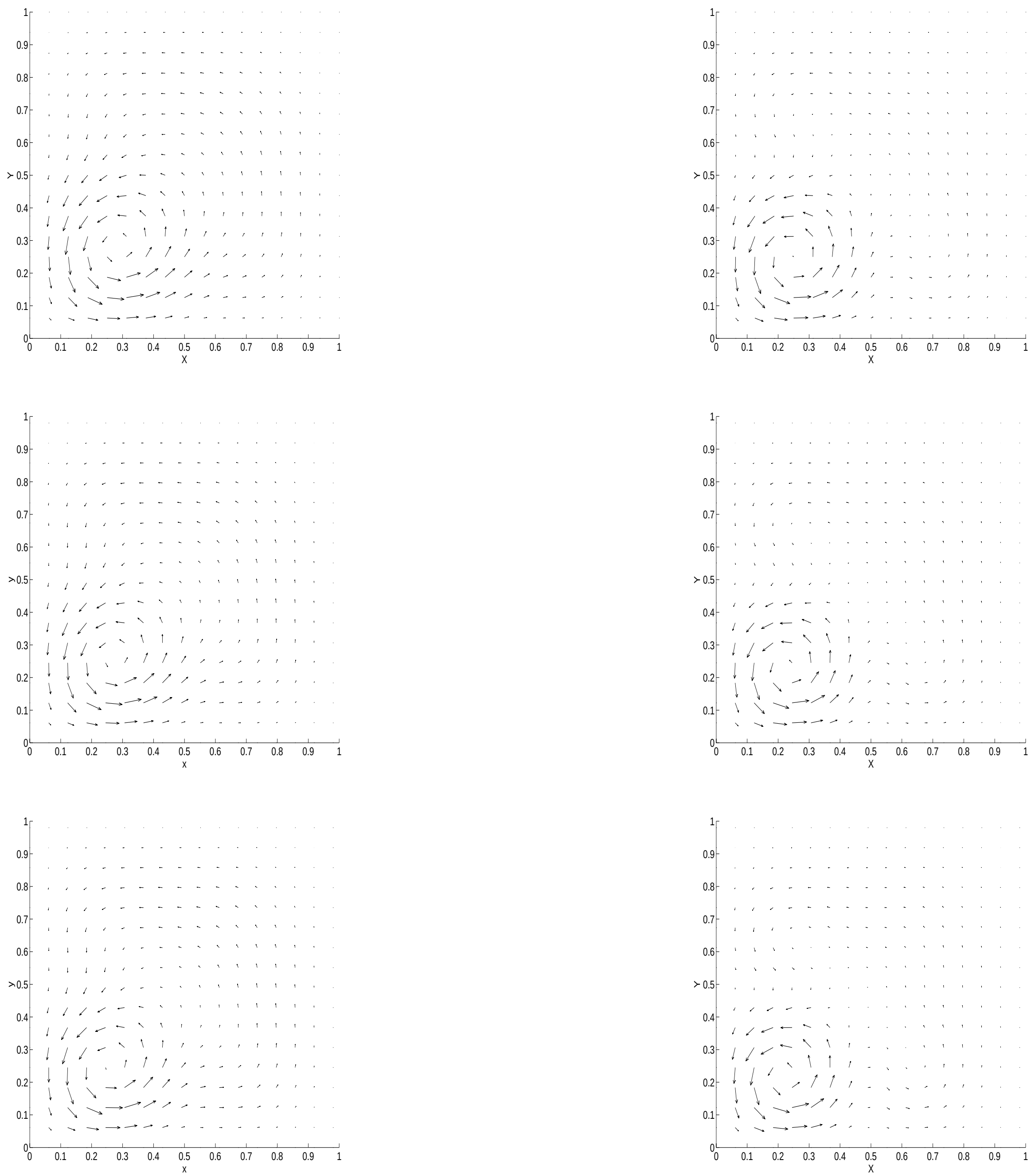

Figure 3.11: Test 2.Controlled(left) and desired(right) at Flow at $t = .9$ (top), $t = .95$ (middle) and $t = 1$ (bottom)

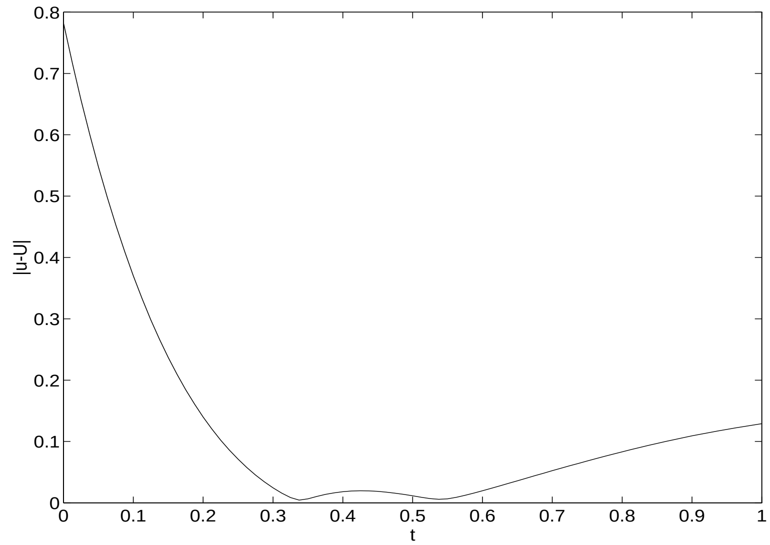

Figure 3.12: Test 2. Error $\|\vec{u} - \vec{U}\|$

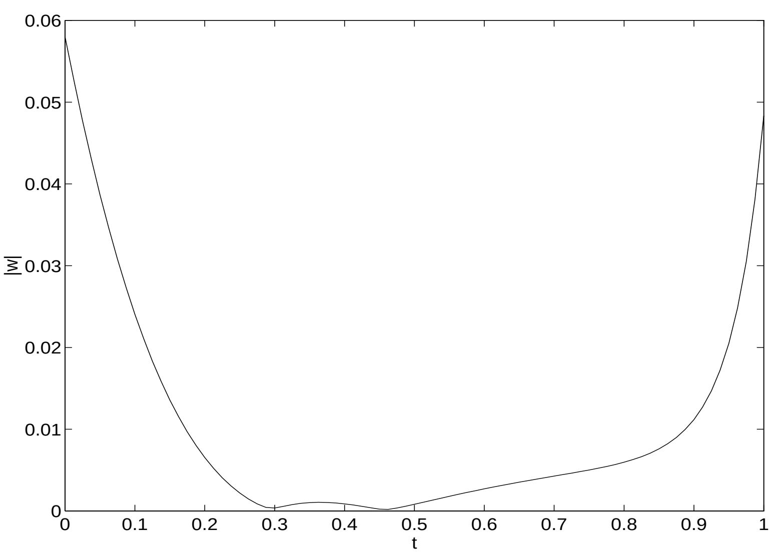

Figure 3.13: Test 2. Adjoint function norm $\|\vec{w}\|$

Chapter 4

Velocity tracking problem for Navier-Stokes flow with linear feedback control

4.1 Introduction

The results obtained in the previous chapters have convinced us that the computation of the controlled flows based on solving the derived optimality system is expensive in terms of CPU time and memory space. This is due to the fact that it involves a couple system of the state and the adjoint variables with initial and final conditions. The system has to be solved on the entire space-time domain and can not be solved by marching in time. A gradient algorithm is needed and so the final algorithm is complex and the convergence is slow.
It is natural to seek for a less expensive solution that gives similar results. The feedback control has proven to be a good control on the tracking velocity and at the same time the solution can be obtained step by step in time. Of course we do not have the optimal control on the system but the control is stronger. The purpose of the linear feedback control is to design an appropriate control mechanism with the physical objective of tracking the velocity field over time and to design a control that can be solved numerically by a marching in time algorithm. Besides, we required that the error in L^2 norm between the controlled velocity and the target velocity decays exponentially to zero as time increases. The linear feedback control could also be used as initial guess for the optimal control gradient algorithm. In fact all the computations in the previous chapters have a linear feedback control solution as initial guess. This improves enormously the performance of the gradient algorithm.

In section 4.2 we define the linear feedback control and prove the exponential decaying property. In section 4.3 and 4.4, we will analyze the semidiscrete approximation and the finite element approximation, respectively. In section 4.5, some numerical tests are performed. Finally, in section 4.6 the results found in the previous chapters are discussed and compared with the linear feedback control solutions.

4.2 Distributed linear feedback control problem

4.2.1 Formulation of the control problem

We start with a precise definition of admissible target velocity. $\vec{U}$ is said to be in the set of admissible target velocity U_{ad} if $\vec{U}$ is a divergence free vector in the set $\{\vec{v} : \vec{v} \in C((0,T); H^2(\Omega) \cap H_0^1(\Omega)) : \partial_t \vec{v} \in C((0,T); H^1(\Omega))\}$. The corresponding body force generated by $\vec{U}$ is defined as

$$\vec{F}(t,\vec{x}) = \vec{U}_t(t,\vec{x}) - \nu \nabla^2 \vec{U}(t,\vec{x}) + (\vec{U}(t,\vec{x}).\vec{\nabla})\vec{U}(t,\vec{x}) \tag{4.1}$$

Let $\vec{u} \in L^2((0,T); H_0^1(\Omega))$ and $p \in L^2((0,T); L_0^2(\Omega))$ denote the state variables, i.e. the velocity and pressure field respectively. Let $\vec{f}(t,\vec{x}) \in L^2((0,T); L^2(\Omega))$ denote the distributed control. The state variables are constrained to satisfy the weak form of the Navier-Stokes equations a.e. for t in $(0,T)$, i.e $\forall v \in H_0^1(\Omega)$ and $\forall q \in L_0^2(\Omega)$ we have

$$\begin{cases} (\vec{u}_t, \vec{v}) + \nu a(\vec{u}, \vec{v}) + c(\vec{u}; \vec{u}, \vec{v}) + b(\vec{v}, p) = (\vec{f}, \vec{v}) \\ b(\vec{u}, q) = 0 \end{cases} \tag{4.2}$$

with initial velocity $\vec{u}_0(\vec{x}) \in V(\Omega)$ and homogeneous boundary condition.
Given T , $\vec{f} \in L^2((0,T); H^{-1}(\Omega))$ and $\vec{u}_0 \in V_0(\Omega)$ then $(\vec{u}, p)$ is called a generalized solution for the Navier-Stokes equations if $u \in \mathcal{H}^1((0,T) \times \Omega)$, $p \in L^2((0,T); L_0^2(\Omega))$ and (u,p) satisfies the equations (4.2) with initial velocity $\vec{u}_0$.

An admissible solution for our control problem can be defined as follows.
Given T , $\vec{f} \in L^2((0,T); L^2(\Omega))$, $\vec{u}_0 \in V_0(\Omega)$ and $\vec{U} \in U_{ad}$, the solution $(\vec{u}, p, \vec{f})$ of eq(4.2) is called an admissible solution for the control problem if $u \in \mathcal{H}^1((0,T) \times \Omega)$, $p \in L^2((0,T); L_0^2(\Omega))$ and

$$\frac{d}{dt}\|\vec{u}\|^2 \leq 0 \qquad a.e. \quad t \in (0,T). \tag{4.3}$$

The set of all admissible solutions is defined as A_{ad}. The constraint in eq (4.3) arises from the desire to have the control of the system for a.e. t in $(0,T)$. If this constraint is violated we have that the derivative in time of the square of the norm of $\vec{w}$ is positive and the velocity $\vec{u}$ is driven far from the target velocity $\vec{U}$. This is not admissible especially if $\vec{u}$ has already reached the desired velocity $\vec{U}$.

We now turn to the definition of our control problem. The control is achieved by mean of a linear feedback body force, i.e.

$$\vec{f} = \vec{F} - \gamma(\vec{u} - \vec{U}) \tag{4.4}$$

with $\gamma > M$, $M = max\{0, -C_0[\nu - K_0\|U\|_{L^\infty[(0,T);L^4(\Omega)]}]\}$ and $\vec{F}$ defined in eq(4.1), given.

4.2.2 Dynamics of the control problem

We wish the controlled solution $\vec{u}$ to match $\vec{U}$ over time, i.e. we wish our solution to be is in the set of admissible solutions and $\|\vec{u} - \vec{U}\|$ tends to zero as t increases. We will prove that this decay property is true and furthermore that this decay is exponential.

Theorem 4.1 *If $(\vec{u}, p, \vec{f})$ is a solution of the control problem in (4.2) and (4.4) then $(\vec{u}, p, \vec{f})$ is in the set of admissible solution, i.e. $\frac{d}{dt}\|\vec{w}\|^2 \leq 0 \quad a.e \quad t \in (0, T)$.*

Proof: We set $\vec{g} = \vec{F} - \vec{f}$. Using the definition of $\vec{F}$ and the fact that, $\forall \vec{v} \in H_0^1(\Omega)$, we have

$$c(\vec{u}; \vec{u}, \vec{v}) - c(\vec{U}; \vec{U}, \vec{v}) = c(\vec{w}; \vec{w}, \vec{v}) + c(\vec{w}; \vec{U}, \vec{v}) + c(\vec{U}; \vec{w}, \vec{v}). \tag{4.5}$$

Now eq(4.2) can be written in the following convenient way

$$(\vec{w}_t, \vec{v}) + \nu a(\vec{w}, \vec{v}) + c(\vec{w}; \vec{w}, \vec{v}) + c(\vec{w}; \vec{U}, \vec{v}) + c(\vec{U}; \vec{w}, \vec{v}) + b(\vec{w}, q) = (\vec{g}, \vec{w}). \tag{4.6}$$

By using the fact that

$$c(\vec{v}; \vec{w}, \vec{w}) = 0 \qquad \forall \vec{w}, \vec{v} \in H_0^1(\Omega) \tag{4.7}$$

we see that if $\vec{v} = \vec{w}$ and $r = q$ we can write eq(4.6) as

$$\frac{1}{2}\frac{d}{dt}\|\vec{w}(t)\|^2 + \nu\|\nabla\vec{w}(t)\|^2 + c(\vec{w}; \vec{U}, \vec{w}) = (\vec{g}, \vec{w}). \tag{4.8}$$

Now we use the continuity of the trilinear form and the definition of the linear feedback control $\vec{g}$ in order to obtain

$$\frac{1}{2}\frac{d}{dt}\|w\vec{(t)}\|^2 + \gamma\|\vec{w}\|^2 + [\nu - K_0\|\vec{U}\|_{L^4(\Omega)}]\|\nabla\vec{w}(t)\|^2 \leq 0.$$

The theorem follows from the Poincarè inequality since then

$$\frac{1}{2}\frac{d}{dt}\|\vec{w}(t)\|^2 \leq -[\gamma + C_0(\nu - K_0\|\vec{U}\|_{L^4(\Omega)})]\|\vec{w}(t)\|^2 \leq 0 \qquad \sqcup \, .$$

The solution of our control system is in the admissible set and $\|\vec{w}\|^2$ is driven to zero as time increases. Furthermore the decay of this solution is exponential as it is proved in the following theorem.

Theorem 4.2 *If $(\vec{u}, p, \vec{f})$ is a solution of the control problem in (4.2) and (4.4) then*

$$\|\vec{u}(t) - \vec{U}(t)\|^2 \leq \|\vec{u}_0 - \vec{U}_0\|^2 e^{-2rt} \qquad a.e. \quad t \in (0, T) \tag{4.9}$$

with $r = \gamma - M > 0$, where $M = max\{0, -C_0(\nu - K_0\|\vec{U}\|_{L^\infty[(0,T);L^4(\Omega)]})\}$.

Proof: By applying the same technique as in the theorem 4.1 we arrive at eq(4.8) or

$$\frac{1}{2}\frac{d}{dt}\|\vec{w}(t)\|^2 \leq (\vec{g}, \vec{w}) - \|\nabla\vec{w}(t)\|^2[\nu - K_0\|\vec{U}\|_{L^4(\Omega)}].$$

From the control force in (4.4) we have

$$(\vec{g}, \vec{w}) - [\nu - K_0\|\vec{U}\|_{L^4(\Omega)}]\|\vec{w}(t)\|^2 \leq -\gamma\|\vec{w}(t)\|^2 -$$

$$C_0[\nu - K_0\|\vec{U}\|_{L^\infty[(0,T);L^4(\Omega)]}]\|\vec{w}(t)\|^2 \leq -r\|\vec{w}(t)\|^2$$

and thus

$$\frac{1}{2}\frac{d}{dt}\|\vec{w}(t)\|^2 + r\|\vec{w}\|^2 \leq 0$$

Now Gronwall's inequality gives

$$\|\vec{u}(t) - \vec{U}(t)\|^2 \leq \|\vec{u}_0 - \vec{U}_0\|^2 e^{-2rt} \quad \sqcup \, .$$

4.2.3 The control system

In order to solve our problem we have to solve the following set of equations

$$\begin{cases} (\vec{u}_t, \vec{v}) + \nu a(\vec{u}, \vec{v}) + c(\vec{u}; \vec{u}, \vec{v}) + b(\vec{v}, p) = (\vec{F} - \gamma\vec{w}, \vec{v}) \\ b(\vec{u}, q) = 0 \end{cases} \tag{4.10}$$

$\forall v \in H_0^1(\Omega)$, $\forall q \in L_0^2(\Omega)$ with initial velocity $\vec{u}_0(\vec{x}) \in V(\Omega)$ and homogeneous boundary condition. The existence of the solution can be proved by standard techniques (see [35]).

By integrating by part it is possible to show that this system is the weak formulation of the following system of equations

$$\begin{cases} \vec{u}_t + (\vec{u}\cdot\nabla)\vec{u} - \nu\nabla^2\vec{u} + \vec{\nabla}p = F - \gamma(\vec{u}-\vec{U}) \\ \nabla.\vec{u} = 0 \quad on \quad \Omega \\ \vec{u}(t,\vec{x}) = 0 \qquad \forall \vec{x}\in\partial\Omega \\ \vec{u}(0,\vec{x}) = \vec{u}_0. \end{cases} \tag{4.11}$$

It is very convenient to write this system in terms of the variables $\vec{w} = \vec{u} - \vec{U}$ and $\vec{g} = \vec{f} - \vec{F}$. The Navier-Stokes system of equations can be written as

$$\begin{cases} (\vec{w}_t,\vec{v}) + \nu a(\vec{w},\vec{v}) + c(\vec{w};\vec{w},\vec{v}) + c(\vec{w};\vec{U},\vec{v}) + \\ c(\vec{U};\vec{w},\vec{v}) + b(\vec{w},q) = -\gamma(\vec{w},\vec{v}) \qquad \forall v\in H_0^1(\Omega) \\ b(\vec{w},r) = 0 \quad \forall r\in L_0^2(\Omega) \end{cases} \tag{4.12}$$

with initial velocity $\vec{w}(0,\vec{x}) = (\vec{u}_0 - \vec{U}_0)$ and homogeneous boundary condition.

4.3 The control problem in the semidiscrete time approximation

4.3.1 Definition of the semidiscrete time approximation problem

In order to compute the solution discussed in the previous section we need to discretize this problem in time and in space. Here we give some considerations on the semidiscrete time approximation. After discretization in time this solution can be determined by marching in time with a sequence of stationary problems.

Let $\sigma_N = \{t_n\}_{n=0}^N$ be a partition of $(0,T)$ into equal intervals with $t_0 = 0$ and $t_N = T$. We will denote the vector $(\vec{q}^{(1)}, \vec{q}^{(2)}, ..., \vec{q}^{(N)})$ and the relative space X^N with $\vec{\mathbf{q}}$ and $\mathbf{X}$ respectively. The discrete target velocity $\vec{U}$ is defined by $\vec{U}^{(n)}(\vec{x}) = \vec{U}^{(n)}(t_n,\vec{x})$ for all $n = 1,2,...,N$. Of course $\vec{U}\in U_{ad}$ and its fixed body force is defined as

$$\vec{F}^{(n)}(\vec{x}) = \frac{\vec{U}^{(n)} - \vec{U}^{(n-1)}}{\Delta t} - \nu\nabla^2\vec{U}^{(n)} + (\vec{U}^{(n)}.\nabla)\vec{U}^{(n)} \tag{4.13}$$

The state variables $\vec{\mathbf{u}}\in\mathbf{H_0^1(\Omega)}$ and $\mathbf{p}\in\mathbf{L_0^2(\Omega)}$ are constrained to satisfy the weak form of the Navier-Stokes equations for $n = 1,2,..N$ i.e

$$\begin{cases} \frac{1}{\Delta t}(\vec{u}^{(n)} - \vec{u}^{(n-1)},\vec{v}) + \nu a(\vec{u}^{(n)},\vec{v}) + c(\vec{u}^{(n)};\vec{u}^{(n)},\vec{v}) + b(\vec{v},p^{(n)}) = \\ (\vec{f}^{(n)},\vec{v}) \quad \forall v\in H_0^1(\Omega) \\ b(\vec{u}^{(n)},q) = 0 \quad \forall q\in L_0^2(\Omega) \end{cases} \tag{4.14}$$

with initial velocity $\vec{u}^{(0)}(\vec{x}) = \vec{u}_0 \in V(\Omega)$ and homogeneous boundary condition.

Given T , $\vec{\mathbf{f}} \in \mathbf{H^{-1}(\Omega)}$ *and* $\vec{u}_0 \in V_0(\Omega)$ *then* $(\vec{\mathbf{u}}, \mathbf{p})$ *is called a generalized solution for Navier-Stokes semidiscrete time approximation (4.14) if* $\vec{\mathbf{u}} \in \mathbf{H^1_0(\Omega)}$, $\mathbf{p} \in \mathbf{L^2_0(\Omega)}$ *and* $(\mathbf{u}, \mathbf{p})$ *satisfies the semidiscrete Navier-Stokes eq(4.14) with initial velocity* $\vec{u}^{(0)} = \vec{u}_0$.

Given T and $\vec{u}_0 \in V_0(\Omega)$ then $(\vec{\mathbf{u}}, \mathbf{p}, \vec{\mathbf{f}})$ is called an admissible solution for the control problem in eq(4.14) if $(\vec{\mathbf{u}}, \mathbf{p}, \vec{\mathbf{f}}) \in (\mathbf{H^1_0(\Omega)} \times \mathbf{L^2_0(\Omega)} \times \mathbf{L^2(\Omega)})$ is a solution of eq(4.14) and $\|\vec{w}_h^{(n)}\|^2 \leq \|\vec{w}_h^{(n-1)}\|^2$ for $n = 1, 2...N$.

In order to track the velocity target $\vec{U}^{(n)} \in U_{ad}^N$ we assume discrete linear feedback control for the bodyforce, i.e.

$$\vec{f}^{(n)} = \vec{F}^{(n)} - \gamma(\vec{u}^{(n)} - \vec{U}^{(n)}) \tag{4.15}$$

with $\gamma \geq M$.

4.3.2 Dynamics of the semidiscrete approximation

It can be shown, using techniques similar to those employed for the continuous problem, that $\|\vec{u}^{(n)} - \vec{U}^{(n)}\|$ decays exponentially to zero for sufficient small time step Δt.

Theorem 4.3 *If* $(\vec{\mathbf{u}}, \mathbf{p}, \vec{\mathbf{f}})$ *is a solution of eq(4.14) then*
a) $(\vec{\mathbf{u}}, \mathbf{p}, \vec{\mathbf{f}})$ *is in* A_{ad}, *i.e.* $\|\vec{w}^{(n)}\|^2 - \|\vec{w}^{(n-1)}\|^2 \leq 0 \quad n = 1, ..N$
b) $\|\vec{u}^{(n)}(t) - \vec{U}^{(n)}(t)\|^2 \leq (\frac{1}{1+2r\Delta t})^n \|\vec{u}_0 - \vec{U}_0\| \qquad n = 1, ...N$
with $r = \gamma - M > 0$ *and* $M = max\{0, -C_0(\nu - K_0\|\vec{U}\|_{L^\infty[(0,T);L^4(\Omega)]})\}$.

Proof:
If we use the definition of $\vec{f}^{(n)}$, eq(4.15) and eq(4.7), then eq(4.14) for $\vec{v} = \vec{w}^{(n)}$ and $r = q^{(n)}$ can be written as

$$\begin{aligned} &\frac{1}{2\Delta t}\|\vec{w}^{(n)}\|^2 + \frac{1}{2\Delta t}\|\vec{w}^{(n)} - \vec{w}^{(n-1)}\|^2 + \nu\|\vec{\nabla}\vec{w}^{(n)}\|^2 + \\ &c(\vec{w}^{(n)}; \vec{U}(t_n), \vec{w}^{(n)}) = \frac{1}{2\Delta t}\|\vec{w}^{(n-1)}\|^2 - \gamma\|\vec{w}^{(n)}\|^2 \end{aligned} \tag{4.16}$$

and thus

$$\frac{1}{2\Delta t}\|\vec{w}^{(n)}\|^2 + \nu\|\vec{\nabla}\vec{w}^{(n)}\|^2 + c(\vec{w}^{(n)}; \vec{U}(t_n), \vec{w}^{(n)}) \leq \frac{1}{2\Delta t}\|\vec{w}^{(n-1)}\|^2 - \gamma\|\vec{w}^{(n)}\|^2. \tag{4.17}$$

From the continuity of the trilinear form we get

$$\frac{1}{2\Delta t}\|\vec{w}^{(n)}\|^2 + \gamma\|\vec{w}^{(n)}\|^2 + (\nu - K_0\|\vec{U}\|_{L^4(\Omega)})\|\nabla\vec{w}^{(n)}\|^2 \le \frac{1}{2\Delta t}\|\vec{w}^{(n-1)}\|^2$$

By applying Poincarè's inequality we have

$$\gamma\|\vec{w}^{(n)}\|^2 - [\nu - K_0\|\vec{U}\|_{L^4(\Omega)}]\|\nabla\vec{w}^{(n)}\|^2 \le$$
$$\gamma\|\vec{w}^{(n)}\|^2 - C_0[\nu - K_0\|\vec{U}\|_{L^\infty[(0,T);L^4(\Omega)]}]\|\vec{w}^{(n)}\|^2 \le -r\|\vec{w}^{(n)}\|^2$$

or

$$\frac{1}{2\Delta t}\|\vec{w}^{(n)}\|^2 + r\|\vec{w}^{(n)}\|^2 \le \frac{1}{2\Delta t}\|\vec{w}^{(n-1)}\|^2$$

and thus the result in a) is obtained. Now in order to prove b) we can use the recursive inequality to get

$$\|\vec{w}^{(n)}\|^2 \le (\frac{1}{(1+2r\Delta t)})^n\|\vec{w}^{(0)}\|^2 \quad \sqcup\,.$$

We can note that using $\vec{F}^{(n)}$, defined in eq(4.15), instead of $\vec{F}(t_n,\vec{x})$ is crucial in the estimate. The use of $\vec{F}(t_n,\vec{x})$ does not seem appropriate. In fact, if the derivative is approximated in this way

$$U_t(\vec{x},t_n) = \frac{U(\vec{x},t_n) - U_t(\vec{x},t_{(n-1)})}{\Delta t} + \frac{\partial^2 U(\vec{x},\gamma)}{\partial t^2}\Delta t \tag{4.18}$$

with $\gamma \in (t_n, t_{n-1})$, the eq(4.16) takes the form

$$\frac{1}{2\Delta t}\|\vec{w}^{(n)}\|^2 + \frac{1}{2\Delta t}\|\vec{w}^{(n)} - \vec{w}^{(n-1)}\|^2 + \nu\|\vec{\nabla}\vec{w}^{(n)}\|^2 + c(\vec{w}^{(n)};\vec{U}(t_n),\vec{w}^{(n)}) =$$
$$\frac{1}{2\Delta t}\|\vec{w}^{(n-1)}\|^2 + (\frac{\partial^2 U}{\partial t^2}\Delta t, \vec{w}^{(n)}) - \gamma\|\vec{w}^{(n)}\|^2.$$

By using Schwartz' and Young's inequality we have

$$|(\frac{\partial^2 U}{\partial t^2}\Delta t, \vec{w}^{(n)})| \le \|\frac{\partial^2 U}{\partial t^2}\Delta t\|_{L^\infty((0,T)\times\Omega)}\|\vec{w}^{(n)}\| \le$$
$$\frac{2\Delta t^2}{\nu C_0}\|\frac{\partial^2 U}{\partial t^2}\|^2_{L^\infty((0,T)\times\Omega)} + \frac{\nu C_0}{4}\|\vec{w}^{(n)}\|^2$$

and thus

$$\|\vec{w}^{(n)}\|^2 \leq (1+2r\Delta t)^{-1}\|\vec{w}^{(n-1)}\|^2 + \frac{2(\Delta t)^3}{\nu C_0}\|\frac{\partial^2 U}{\partial t^2}\|^2_{L^\infty((0,T)\times\Omega)}.$$

By substituting the recursive inequality in itself we find

$$\|\vec{w}^{(n)}\|^2 \leq (1+2r\Delta t)^{-n}\|\vec{w}^{(0)}\|^2 + 2\frac{(\Delta t)^3}{\nu C_0}\|\frac{\partial^2 U}{\partial t^2}\|^2_{L^\infty((0,T)\times\Omega)}\sum_{j=0}^{n-1}(1+2r\Delta t)^{-j} \leq$$
$$(1+2r\Delta t)^{-n}\|\vec{w}^{(0)}\|^2 + \frac{2(\Delta t)^3}{\nu C_0}\frac{[1-(1+2r\Delta t)^{-n}]}{2r\Delta t}\|\frac{\partial^2 U}{\partial t^2}\|^2_{L^\infty((0,T)\times\Omega)} \leq$$
$$(1+2r\Delta t)^{-n}\|\vec{w}^{(0)}\|^2 + \frac{\Delta t^2}{r\nu C_0}\|\frac{\partial^2 U}{\partial t^2}\|^2_{L^\infty((0,T)\times\Omega)} \tag{4.19}$$

Hence finally we have

$$\|\vec{w}^{(n)}\|^2 \leq (1+2r\Delta t)^{-n}\|\vec{u}_0 - \vec{U}_0\|^2 + \frac{\Delta t^2}{r\nu C_0}\|\frac{\partial^2 U}{\partial t^2}\|^2_{L^\infty((0,T)\times\Omega)}$$

While the estimate in the theorem 4.3 was independent from the magnitude of Δt, now for large Δt the error is significant.

4.3.3 Solution of the control problem

In order to solve the semidiscrete control problem we have to solve the following system

$$\begin{cases} \frac{1}{\Delta t}(\vec{u}^{(n)} - \vec{u}^{(n-1)}, \vec{v}) + \nu a(\vec{u}^{(n)}, \vec{v}) + c(\vec{u}^{(n)}; \vec{u}^{(n)}, \vec{v}) + b(\vec{v}, p^{(n)}) = \\ (\vec{F}^{(n)} - \gamma \vec{w}^{(n)}, \vec{v}) \quad \forall v \in H_0^1(\Omega) \\ b(\vec{u}^{(n)}, q) = 0 \quad \forall q \in L_0^2(\Omega) \end{cases} \tag{4.20}$$

$forn = 1, 2, ...N$ with initial velocity $\vec{u}_0(\vec{x}) \in V(\Omega)$ and homogeneous boundary condition. The existence of the solution of this problem can be proved with the standard techniques used for the stationary Navier-Stokes equations. We can remark that the problem in eq(4.20) is a finite collection of stationary problems that can be solved step by step until $\|\vec{u}^{(n)} - \vec{U}^{(n)}\|$ is sufficient small. Since the solution is in the admissible set after this time the controlled flow $\vec{u}$ matches the desired flow $\vec{U}$ and never departs from it.

4.4 The control problem in the fully discrete time space approximation

4.4.1 The control problem in the fully discrete time-space approximation

In this subsection we consider the finite element space $X^h \subset H_0^1(\Omega)$ and $S_0^h \subset L_0^2(\Omega)$ and the hypotheses in subsection 2.4.1 are made. Since we know the evolution of the function $\vec{U}$ it is more convenient tracking the approximation in X^h than trying to follow the continuous function. If we chose the optimal approximation in X^h as the interpolant for $\vec{U}$, the discrete solution of the Navier-Stokes system can not be better than this optimal approximation. In this case the optimal approximation can be matched exactly. For interpolation in particular points different from the optimal approximation we get a very accurate result in these points but not elsewhere. In general we need to track the velocity in a particular region or points and thus we can focus the tracking on that area with a suitable interpolation.

We approximate $\vec{U} \in U_{ad}$ with its projection $\vec{U}_h^{(n)} = \pi^h \vec{U}(t_n, \vec{x}) \in X^h$. As $\vec{U}$ is in $H^2 \cap H_0^1$, from the theory of approximation (see [34]), there exists a constant C_1 independent from $\vec{U}$ and h such that

$$\begin{cases} \|\vec{U}^{(n)} - \vec{U}_h^{(n)}\| \le C_1 h^2 \|\vec{U}^{(n)}\|_2 \\ \|\vec{U}^{(n)} - \vec{U}_h^{(n)}\|_1 \le C_1 h \|\vec{U}^{(n)}\|_2 . \end{cases} \tag{4.21}$$

For the fixed body force generated by $\vec{U}$ we define

$$(\vec{F}_h^{(n)}, \vec{v}_h) = \frac{1}{\Delta t}(\vec{U}_h^{(n)} - \vec{U}_h^{(n-1)}, \vec{v}_h) + \nu a(\vec{U}_h^{(n)}, \vec{v}_h) + \tilde{c}(\vec{U}_h^{(n)}; \vec{U}_h^{(n)}, \vec{v}_h). \tag{4.22}$$

Given T , $\vec{\mathbf{f}} \in \mathbf{X^h}$ and $\vec{u}_0 \in V_0(\Omega)$ then $(\vec{\mathbf{u}}_h, \mathbf{p}_h)$ is said a generalized solution for Navier-Stokes fully discrete time space approximation if $(\mathbf{u}_h, \mathbf{p}_h) \in (\mathbf{X^h(\Omega)} \times \mathbf{S_h(\Omega)})$ and $(u_h^{(n)}, p_h^{(n)})$ satisfies the following system of equations

$$\begin{cases} \frac{1}{\Delta t}(\vec{u}_h^{(n)} - \vec{u}_h^{(n-1)}, \vec{v}_h) + \nu a(\vec{u}_h^{(n)}, \vec{v}_h) + \tilde{c}(\vec{u}_h^{(n)}; \vec{u}_h^{(n)}, \vec{v}_h) + \\ b(\vec{v}_h, p^{(n)}) = (\vec{f}_h^{(n)}, \vec{v}_h) \\ b(\vec{u}_h, q_h) = 0 \quad \forall q_h \in S_h(\Omega) \quad \forall \vec{v}_h \in X^h(\Omega) \end{cases} \tag{4.23}$$

with initial velocity $\vec{u}_h^{(0)} = I^h \vec{u}_0$ and homogeneous boundary condition.

Given T,$\vec{U} \in U_{ad}$ and $\vec{u}_0 \in V_0(\Omega)$, then $(\vec{\mathbf{u}}_h, \mathbf{p_h}, \vec{\mathbf{g}}_h)$ in $(\mathbf{X^h} \times \mathbf{S_h} \times \mathbf{X^h}$ is called an admissible solution for the fully discrete control problem if $\|\vec{w}_h^{(n)}\|^2 \le \|\vec{w}_h^{(n-1)}\|^2$ for $n = 1, 2...N$. We can remark that if h is not a function of Δt it is impossible to have $\|\vec{u}_h^{(n)} - \vec{U}^{(n)}\|^2 \le$

$\|\vec{u}_h^{(n-1)} - \vec{U}^{(n-1)}\|^2$ for all n due to the space approximation in X^h. In order to track the velocity field $\vec{U}$ we use a linear feedback control and define

$$\vec{f}_h^{(n)} = \vec{F}_h^{(n)} - \gamma \vec{w}_h^{(n)} \qquad \gamma > M \tag{4.24}$$

with $M = max\{0, -C_0(\nu - K_0\|\vec{U}\|_{L^\infty[(0,T);L^4(\Omega)]})\}$. The existence of the solution of the eq(4.23) with $\vec{f}_h^{(n)}$ defined by eq(4.24) can be proved using the same techniques used for stationary Navier-Stokes equations.

4.4.2 Dynamics for the fully discrete control problem

Now we study briefly the behaviour of the fully discrete solution of the problem in eq(4.23). The purpose of this control is to take appropriately action in order to track the velocity field over time. With this control the goal is achieved and $\|\vec{u}_h^{(n)} - \vec{U}_h^{(n)}\|$ decays to zero exponentially as the time increases.

Theorem 4.4 *The solution of the problem defined in eq(4.23) is in the admissible set. We have also that*

$$\|\vec{u}_h^{(n)} - \vec{U}_h^{(n)}\|^2 \leq (1 + 2r(\Delta t))^{-n} \|\pi^h \vec{u}_0 - \pi^h \vec{U}_0)\|^2 \tag{4.25}$$

where $r = \gamma + C_p(\nu - K_0\|\vec{U}\|_{L^\infty[(0,T);L^4]})$ *(C_p is the Poincarè constant).*

Proof: Let $(\vec{\mathbf{u}}_h, \mathbf{p}_h)$ be a generalized solution of the fully discrete Navier-Stokes approximation for $\vec{\mathbf{f}}_h = \vec{\mathbf{F}}_h - \gamma \vec{\mathbf{w}}_h$. For this particular $(\vec{\mathbf{u}}_h, \mathbf{q}_h, \vec{\mathbf{f}}_h)$ we have

$$\frac{1}{\Delta t}(\vec{u}_h^{(n)} - \vec{u}_h^{(n-1)}, \vec{v}_h) + \nu a(\vec{u}_h^{(n)}, \vec{v}_h) + \tilde{c}(\vec{u}_h^{(n)}; \vec{u}_h^{(n)}, \vec{v}_h) + b(\vec{v}_h, p^{(n)}) = (\vec{f}_h^{(n)}, \vec{v}_h)$$

We set $\vec{w}_h^{(n)} = \vec{u}_h^{(n)} - \vec{U}_h^{(n)}$ and rewrite the above equations in the following way

$$\begin{aligned}&\frac{1}{\Delta t}(\vec{w}_h^{(n)} - \vec{w}_h^{(n-1)}, \vec{v}_h) + \nu a(\vec{w}_h^{(n)}, \vec{v}_h) + \tilde{c}(\vec{w}_h^{(n)}; \vec{w}_h^{(n)}, \vec{v}_h) + \\ &\tilde{c}(\vec{w}_h^{(n)}; \vec{U}_h^{(n)}, \vec{v}_h) + \tilde{c}(\vec{U}_h^{(n)}; \vec{w}_h^{(n)}, \vec{v}_h) + b(\vec{w}_h, q_h^{(n)}) = -\gamma(\vec{w}_h^{(n)}, \vec{v}_h)\end{aligned}$$

Now if $\vec{v}_h = \vec{w}_h^{(n)}$ and $r_h = q_h^{(n)}$ the above equation gives

$$\begin{aligned}&\frac{1}{2\Delta t}\|\vec{w}_h^{(n)}\|^2 + \frac{1}{2\Delta t}\|\vec{w}_h^{(n)} - \vec{w}_h^{(n-1)}\|^2 + \nu\|\vec{\nabla}\vec{w}_h^{(n)}\|^2 + \\ &\tilde{c}(\vec{w}_h^{(n)}; \vec{U}_h^{(n)}, \vec{w}_h^{(n)}) = \frac{1}{2\Delta t}\|\vec{w}^{(n-1)}\|^2 - \gamma\|\vec{w}_h^{(n)}\|^2\end{aligned}$$

and thus from the continuity of the trilinear form we have

$$\frac{1}{2\Delta t}\|\vec{w}_h^{(n)}\|^2 + \|\vec{\nabla}\vec{w}_h^{(n)}\|^2(\nu - K_0\|\vec{U}\|_{L^\infty((0,T);L^4)}) \leq$$
$$\frac{1}{2\Delta t}\|\vec{w}_h^{(n-1)}\|^2 - \gamma\|\vec{w}_h^{(n)}\|^2.$$

By applying Poincarè's inequality we have

$$[\gamma + C_0(\nu - K_0\|\vec{U}\|_{L^\infty((0,T);L^4)}) + \frac{1}{2\Delta t}]\|\vec{w}_h^{(n)}\|^2 \leq \frac{1}{2\Delta t}\|\vec{w}_h^{(n-1)}\|^2$$

or

$$\|\vec{w}_h^{(n)}\|^2 \leq (1 + 2r\Delta t)^{-1}\|\vec{w}_h^{(n-1)}\|^2$$

and by substituting the recursive inequality in itself we find

$$\|\vec{w}_h^{(n)}\|^2 \leq (1 + 2r\Delta t)^{-n}\|\vec{w}_h^{(0)}\|^2 \quad \sqcup .$$

If $\Delta t << 1$ we have that

$$\|\vec{u}_h^{(n)} - \vec{U}_h^{(n)}\| \leq exp[-rC_0 t]\|\pi^h\vec{u}_0 - \pi^h\vec{U}_0)\|$$

and we have an exponential decay in time. We can note that a large Δt does not affect the error.

Theorem 4.5 *For the solution of the problem defined in eq(4.23) we have that*

$$\|\vec{u}_h^{(n)} - \vec{U}^{(n)}\|^2 \leq 2(1 + r\Delta t)^{-n}\|\pi^h\vec{u}_0 - \pi^h\vec{U}_0\|^2 + 2h^4C_1^2\|\vec{U}\|_2^2 \tag{4.26}$$

for all $n = 1, 2, ...N$ *where* $r = \gamma - C_0(\nu - K_0\|\vec{U}\|_{L^\infty[(0,T);L^4]} > 0$.

Proof: This is a direct consequence of the theorem 4.4 and the inequality

$$\|\vec{u}_h^{(n)} - \vec{U}^{(n)}\|^2 \leq 2\|\vec{u}_h^{(n)} - \vec{U}_h^{(n)}\|^2 + 2\|\vec{U}_h^{(n)} - \vec{U}^{(n)}\|^2 \quad \sqcup .$$

4.5 Numerical algorithm and computation

4.5.1 Introduction

In order to get the optimal control solution we have to solve the following system of equations

$$\begin{cases} \vec{u}_t + (\vec{u}.\vec{\nabla})\vec{u} - \nu\nabla^2\vec{u} + \vec{\nabla}p = \vec{f} \\ \vec{\nabla}\cdot\vec{u} = 0 \\ \vec{f} = \vec{F} - \gamma(\vec{u} - \vec{U}) \end{cases} \tag{4.27}$$

with initial velocity $\vec{u}^{(0)}(\vec{x}) = \vec{u}_0(\vec{x})$ and homogeneous boundary condition.

Let $\sigma_N = \{t_n\}_{n=0}^{N}$ be a partition of $[0,T]$ into equal intervals $\Delta t = T/N$ with $t_0 = 0$ and $t_N = T$. For a fixed Δt (or N) let $X^h \subset H_0^1(\Omega)$ and $S_0^h \subset L^2(\Omega)$ be some families of finite dimensional subspaces parameterized by h that tends to zero. Then eq(4.27) becomes

$$\begin{cases} \frac{1}{\Delta t}(\vec{u}_h^{(n)} - \vec{u}_h^{(n-1)}, \vec{v}_h) + \nu a(\vec{u}_h^{(n)}, \vec{v}_h) + \tilde{c}(\vec{u}_h^{(n)}; \vec{u}_h^{(n)}, \vec{v}_h) + \\ b(\vec{v}_h, p_h^{(n)}) = (\vec{F}^{(n)} - \gamma(\vec{u}_h^{(n)} - \vec{U}^{(n)}), \vec{v}_h) \\ b(\vec{u}_h^{(n)}, q_h) = 0 \qquad \forall q_h \in S_0^h(\Omega) \quad \vec{v}_h \in X^h(\Omega) \end{cases} \tag{4.28}$$

for $n = 1, 2, ...N$ initial velocity $\vec{u}_h^{(0)}(\vec{x}) = \pi^h \vec{u}_0(\vec{x})$ and homogeneous boundary condition. If $\vec{U}^{(n)} = \vec{U}_h^{(n)} = \pi^h \vec{U}^{(n)}$ and

$$(\vec{F}^{(n)}, \vec{v}_h) = \frac{1}{\Delta t}(\vec{U}_h^{(n)} - \vec{U}_h^{(n-1)}, \vec{v}_h) + \nu a(\vec{U}_h^{(n)}, \vec{v}_h) + \tilde{c}(\vec{U}_h^{(n)}; \vec{U}_h^{(n)}, \vec{v}_h) \tag{4.29}$$

we can write

$$\begin{cases} \frac{1}{\Delta t}(\vec{w}_h^{(n)} - \vec{w}_h^{(n-1)}, \vec{v}_h) + \nu a(\vec{w}_h^{(n)}, \vec{v}_h) + \tilde{c}(\vec{w}_h^{(n)}; \vec{w}_h^{(n)}, \vec{v}_h) + \\ \tilde{c}(\vec{w}_h^{(n)}; \vec{U}_h^{(n)}, \vec{v}_h) + \tilde{c}(\vec{U}_h^{(n)}; \vec{w}_h^{(n)}, \vec{v}_h) + b(\vec{v}_h, p_h^{(n)}) + \gamma(\vec{w}_h^{(n)}, \vec{v}_h) = 0 \\ b(\vec{u}_h^{(n)}, q_h) = 0 \qquad \forall q_h \in S_0^h(\Omega) \quad \vec{v}_h \in X^h(\Omega) \end{cases} \tag{4.30}$$

for $n = 1, 2, ...N$ initial velocity $\vec{w}_h^{(0)}(\vec{x}) = \pi^h(\vec{u}_0 - \vec{U}_0)$ and homogeneous boundary condition.

4.5.2 Numerical algorithm

The algorithm is based on the following equations

$$\begin{cases} \frac{1}{\Delta t}(\vec{u}_h^{(n)}(k) + \nu a(\vec{u}_h^{(n)}(k), \vec{v}_h) + \sigma \tilde{c}(\vec{u}_h^{(n)}(k); \vec{u}_h^{(n)}(k-1), \vec{v}_h) + \\ \tilde{c}(\vec{u}_h^{(n)}(k-1); \vec{u}_h^{(n)}(k), \vec{v}_h) + b(\vec{v}_h, p^{(n)}(k)) + \gamma(\vec{u}_h^{(n)}(k), \vec{v}_h) = \\ \frac{1}{\Delta t}(\vec{u}_h^{(n-1)}(k), \vec{v}_h) + \sigma \tilde{c}(\vec{u}_h^{(n)}(k-1); \vec{u}_h^{(n)}(k-1), \vec{v}_h) + \\ \gamma(\vec{U}^{(n)}, \vec{v}_h) + (\vec{F}^{(n)}, \vec{v}_h) \\ b(\vec{u}_h^{(n)}(k), q_h) = 0 \quad \forall q_h \in S_h(\Omega) \quad \forall \vec{v}_h \in X^h(\Omega) \end{cases} \tag{4.31}$$

with initial velocity $\vec{u}_h^{(0)} = \pi^h \vec{u}_0(\vec{x})$ and homogeneous boundary condition. If $\vec{U}^{(n)} = \pi^h \vec{U}^{(n)}$ and $\vec{F}^{(n)}$ is given by eq(4.29) the algorithm can be based on the following set of equations

$$\begin{cases} \frac{1}{\Delta t}(\vec{w}_h^{(n)}(k) + \nu a(\vec{w}_h^{(n)}(k), \vec{v}_h) + \sigma \tilde{c}(\vec{w}_h^{(n)}(k); \vec{w}_h^{(n)}(k-1), \vec{v}_h) + \\ \tilde{c}(\vec{w}_h^{(n)}(k-1); \vec{w}_h^{(n)}(k), \vec{v}_h) + \tilde{c}(\vec{U}_h^{(n)}; \vec{w}_h^{(n)}(k), \vec{v}_h) + \\ \tilde{c}(\vec{w}_h^{(n)}(k); \vec{U}_h^{(n)}, \vec{v}_h) + b(\vec{v}_h, p^{(n)}(k)) + \gamma(\vec{w}_h^{(n)}(k), \vec{v}_h) = \\ \frac{1}{\Delta t}(\vec{w}_h^{(n-1)}(k), \vec{v}_h) + \sigma \tilde{c}(\vec{w}_h^{(n)}(k-1); \vec{w}_h^{(n)}(k-1), \vec{v}_h) \\ b(\vec{u}_h^{(n)}(k), q_h) = 0 \quad \forall q_h \in S_h(\Omega) \quad \forall \vec{v}_h \in X^h(\Omega) \end{cases} \tag{4.32}$$

with initial velocity $\vec{w}_h^{(0)} = \pi^h(\vec{u}_0 - \vec{U}_0)$ and homogeneous boundary condition.

Given the velocity $\vec{u}^{(n-1)}$ and an integer m one can generate the sequences $\{\vec{u}^{(n)}(k), \vec{u}^{(n)}(k-1)\}$ for $k = 1, 2..$ by solving the linear problem with $\sigma = 0$ for $k \leq m$ and $\sigma = 1$ for $k > m$. The algorithm stops when the maximum value of $|u_h^{(n)}(k) - u_h^{(n)}(k)|/|u_h^{(n)}(k)|\}$ is less or equal to τ on Ω. When convergence is reached this procedure is repeated for the successive n. The global convergence of the simple iteration method and the rapid convergence of Newton's method are combined in this numerical algorithm. The value of m is set to be two or tree in relation to the time step Δt. Some convergence theorems can be found in [35].

4.5.3 Test 1

We consider a unit square domain $(0,1) \times (0,1) \subset \mathcal{R}^2$ We assume that the time interval [0,1] is divided into equal intervals of time $\Delta t = 1/N$. The finite element spaces are chosen to be piecewise quadratic for velocity and linear for the pressure. The Taylor-Hood finite elements are used in this calculation on a rectangular mesh. The mesh size is h and calculations with varying mesh sizes have been performed. In this first test we are interest in the convergence history for all the parameters involved and so a simple stationary target velocity $\vec{U} = (U, V)$ is chosen. The target velocity for this test is defined by

$$\phi(t,x,y) = (1 - cos(2\pi tx))(1-x)^2 \times (1 - cos(2\pi ty))(1-y)^2$$
$$U(x,y) = 10\frac{d\phi(0.4,x,y)}{dy} \qquad V(x,y) = -10\frac{d\phi(0.4,x,y)}{dx}.$$

Velocity tracking evolution

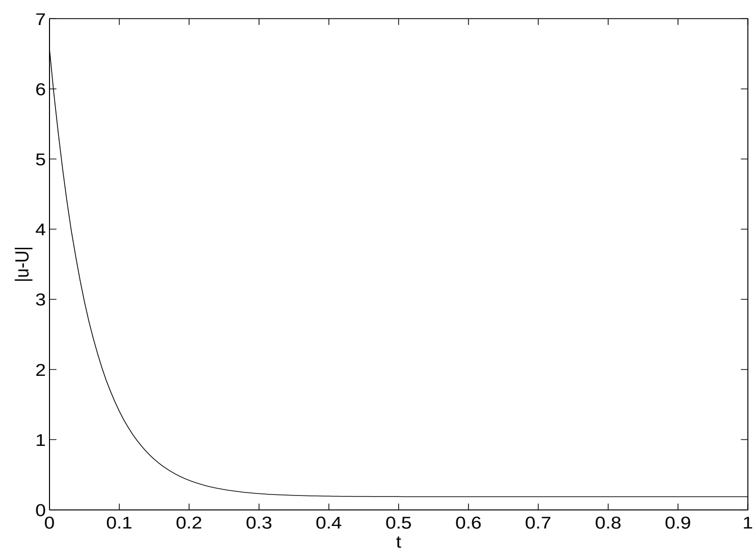

Figure 4.1: Test 1. Error $\|\vec{u} - \vec{U}\|$

We can see a first example of control where the initial velocity is

$$u_0(x,y) = -10U(x,y) \qquad v_0(x,y) = -10V(x,y). \tag{4.33}$$

This initial velocity rotates in opposite direction to the initial target velocity $\vec{U}_0$. Fig.4.1 shows the error $\|\vec{u} - \vec{U}\|$ between the controlled flow $\vec{u}$ and the target flow $\vec{U}$. For the same

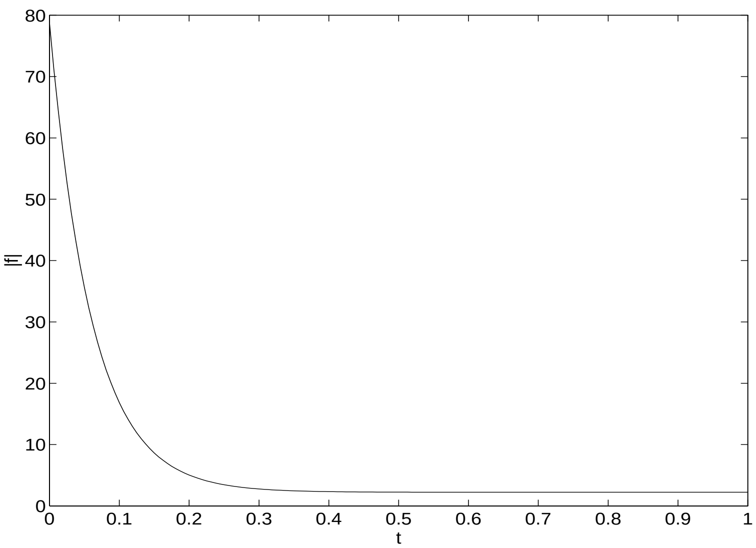

Figure 4.2: Test 1. Control norm $\|\vec{f}\|$

flow Fig.4.2 shows the corresponding value of the norm of the control $\vec{f} = \vec{f} - \gamma(\vec{u} - \vec{U})$ as a function of time. The flow evolution is in Fig.4.3 - Fig.4.6. The controlled fluid is on the left and the desired flow is on the right. All the pictures are normalized. At the beginning we have a reduction in magnitude, then we have a change in shape and finally a change in magnitude again. This evolution is typical of linear feedback control. The change in shape is over a short time interval when the velocity field is approximately zero. As shown in Fig.4.4, the change in shape is so quick that it is difficult to see the full evolution with small time steps. The error $\|\vec{u} - \vec{U}\|$ rapidly goes to zero and a perfect match is reached at $t = 0.15$. For this calculation $\Delta t = 0.00625$, $h = 1/16$, $\alpha = 1$ and γ has been set to 30.
The control $\vec{f}$ norm goes rapidly to a constant, which is $\|\vec{F}\|$. The control $\vec{f}$ works hard at the beginning in order to steer the controlled flow to the desired one and then remains flat.

Velocity tracking with different values of γ

We want to analyse what happens if we change the parameter γ. The initial velocity is set to zero.

In Fig.4.7 we have the error $\|\vec{u} - \vec{U}_h\|$ and in Fig.4.8 the error $\|\vec{u} - \vec{U}\|$ between the controlled flow $\vec{u}$ and the target flow $\vec{U}$ for different value of γ. Starting from the bottom we have γ equals 30 (A), 20 (B), 10 (C) and 1 (D). The error $\|\vec{u} - \vec{U}_h\|$ goes to zero while $\|\vec{u} - \vec{U}\|$ goes to a constant value depending on the space-time discretization. In this case the space-time discretization error is very small and almost negligible. In fact, Fig.4.7

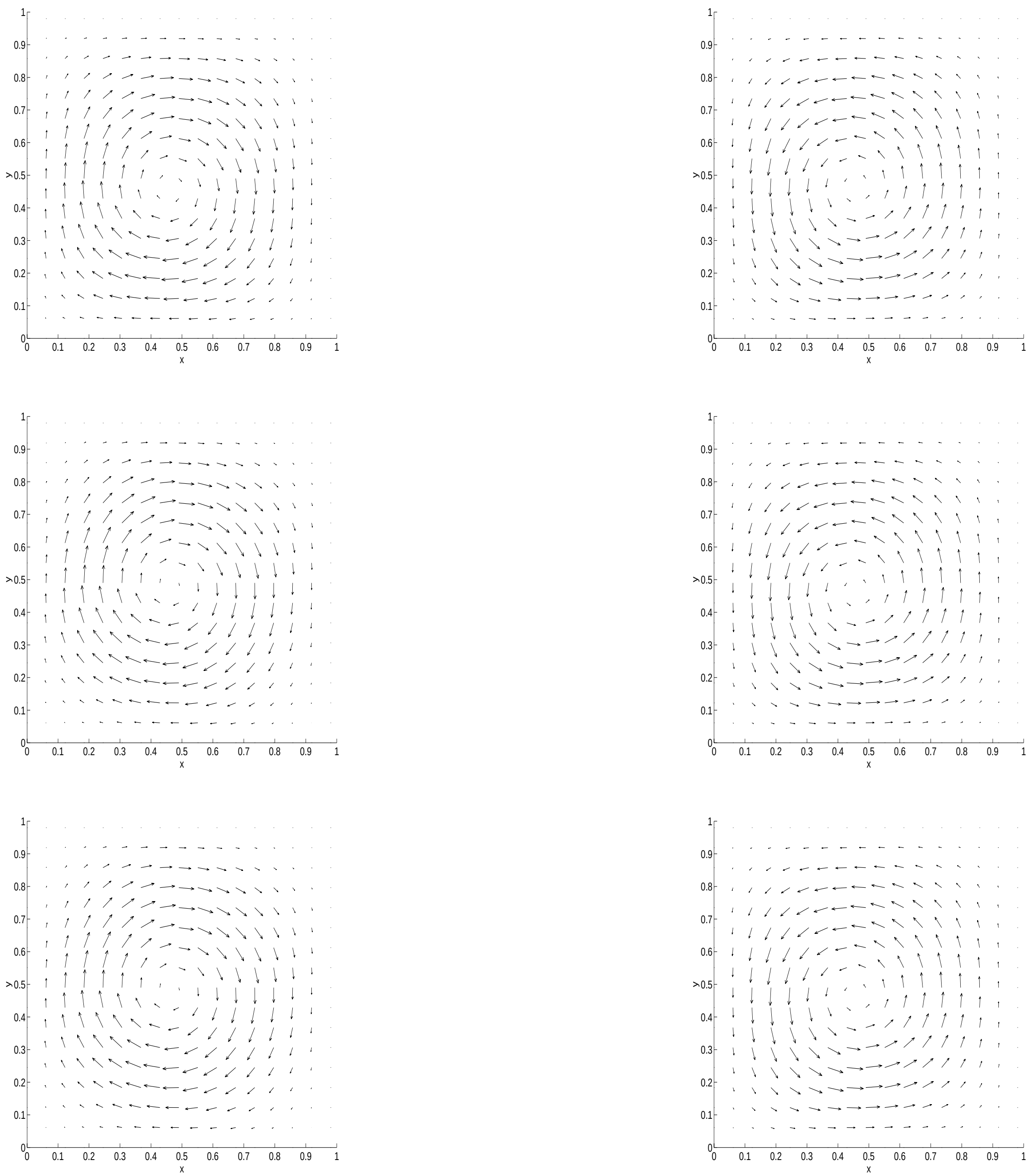

Figure 4.3: Test 1. Controlled(right) and desired(left) flow at $t = 0$ (top), $t = .1$ (middle) and $t = .125$ (bottom)

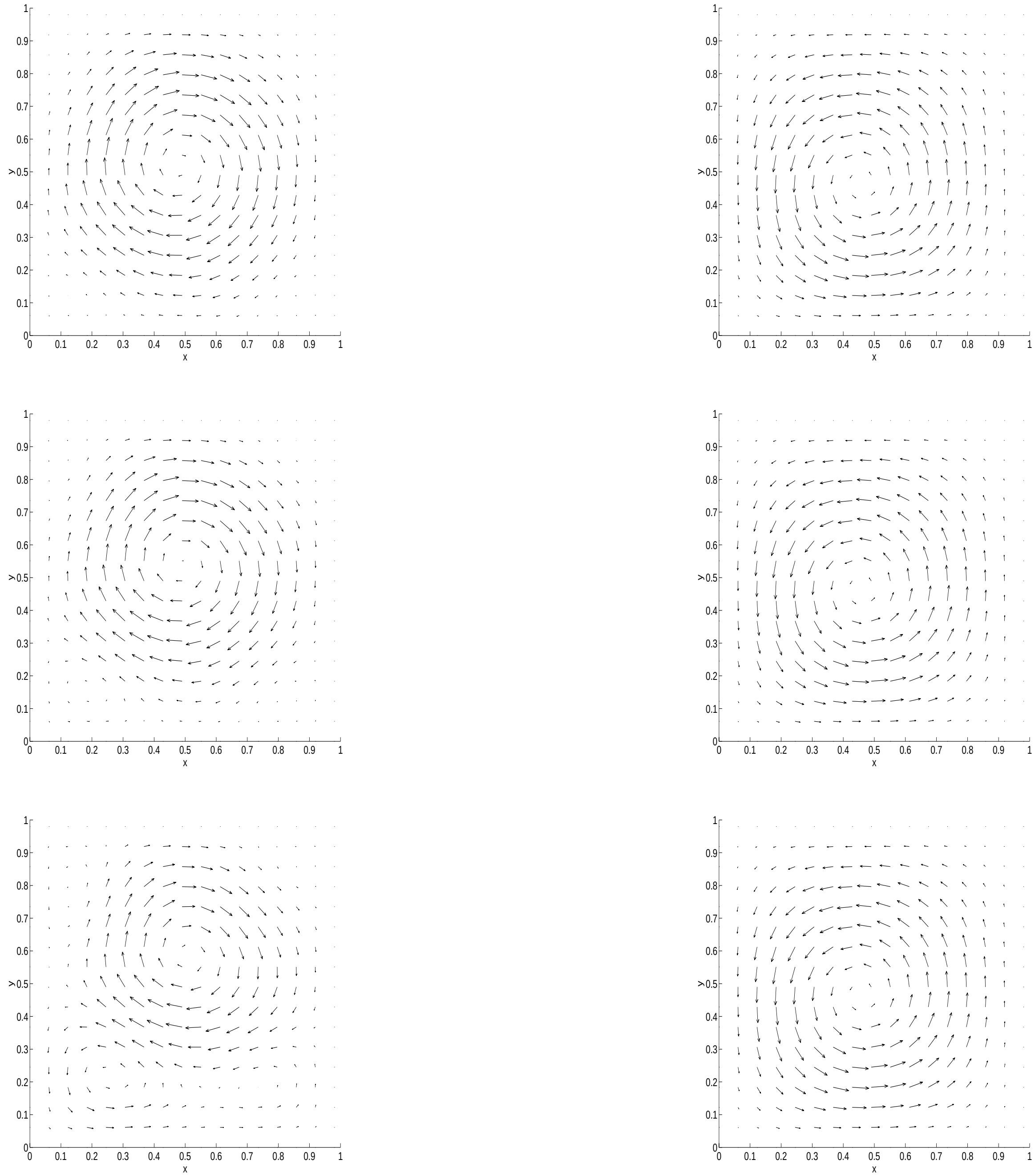

Figure 4.4: Test 1. Controlled(right) and desired(left) flow at $t = .15$ (top), $t = .156$ (middle) and $t = .162$ (bottom)

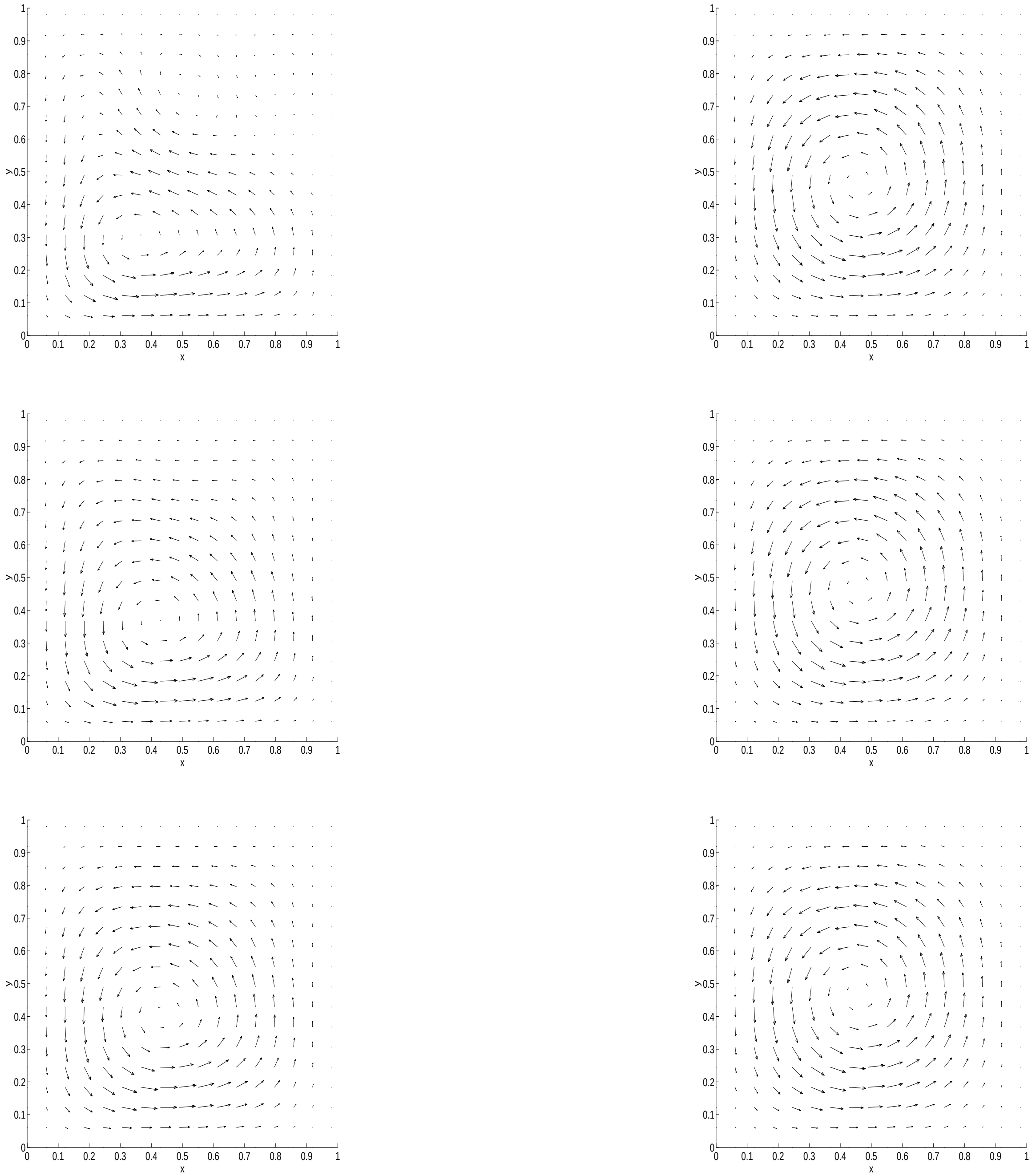

Figure 4.5: Test 1. Controlled(right) and desired(left) flow at $t = .169$ (top), $t = .175$ (middle) and $t = .181$ (bottom)

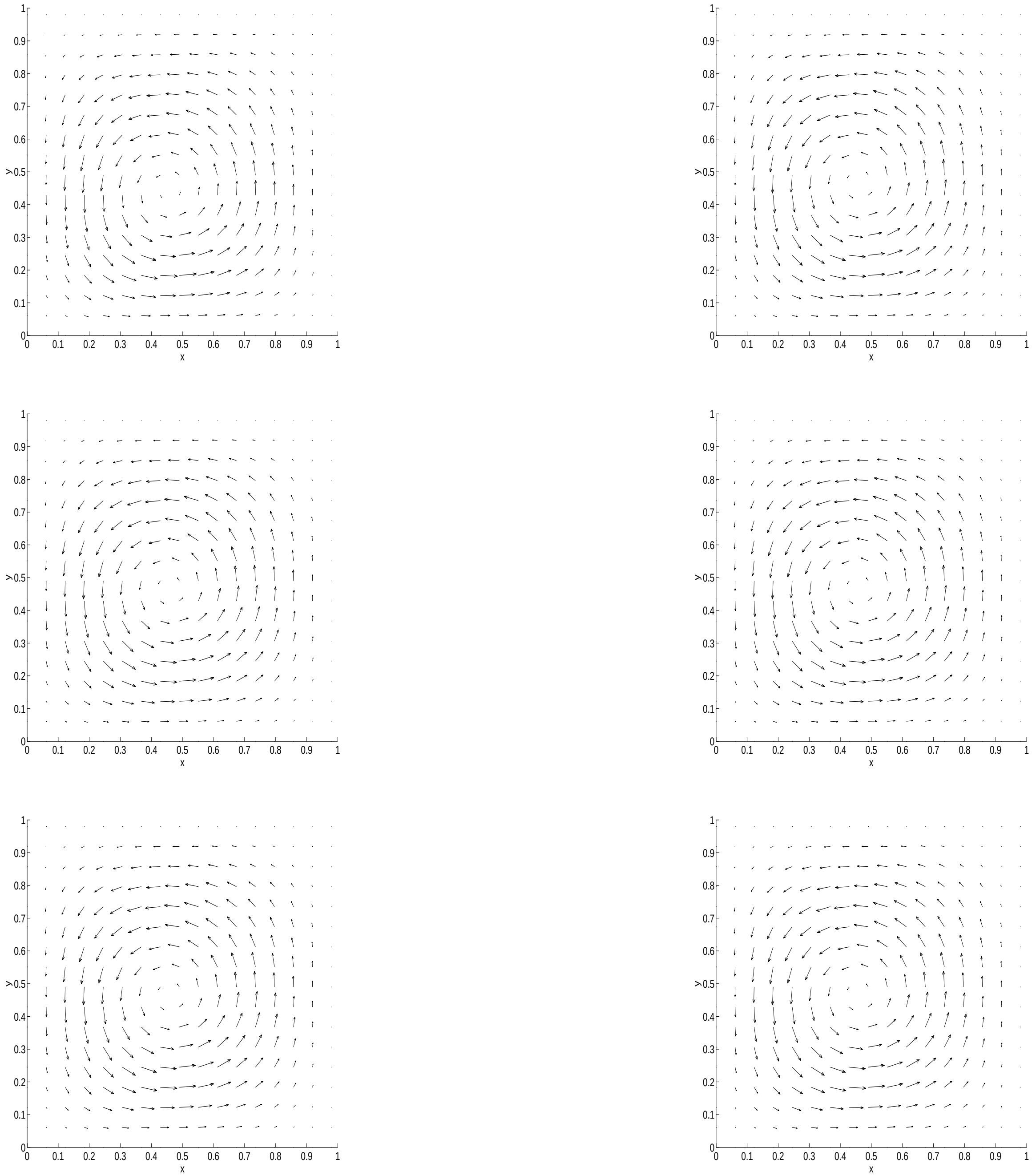

Figure 4.6: Test 1. Controlled(right) and desired(left) flow at $t = .2$ (top), $t = .5$ (middle) and $t = 1$ (bottom)

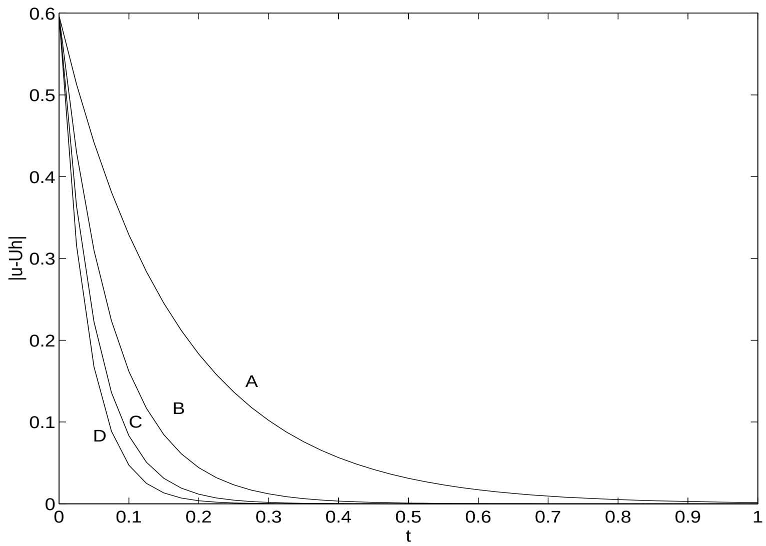

Figure 4.7: Test 1. Error $\|\vec{u}_h - \vec{U}_h\|$ for different γ

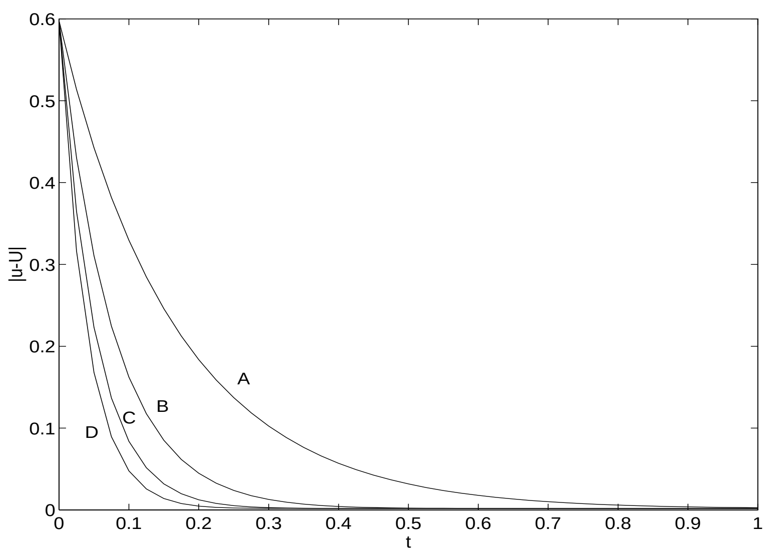

Figure 4.8: Test 1. Error $\|\vec{u}_h - \vec{U}\|$ for different γ

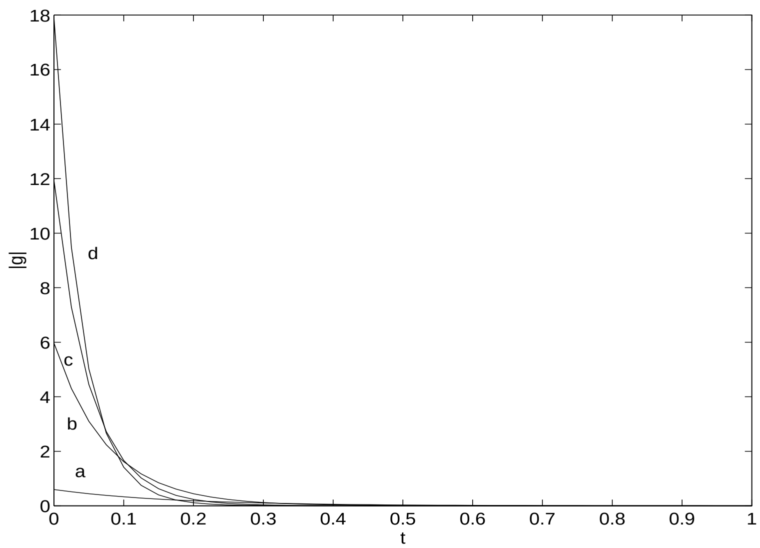

Figure 4.9: Test 1. Control norm $\|\vec{g}\|$ for different γ

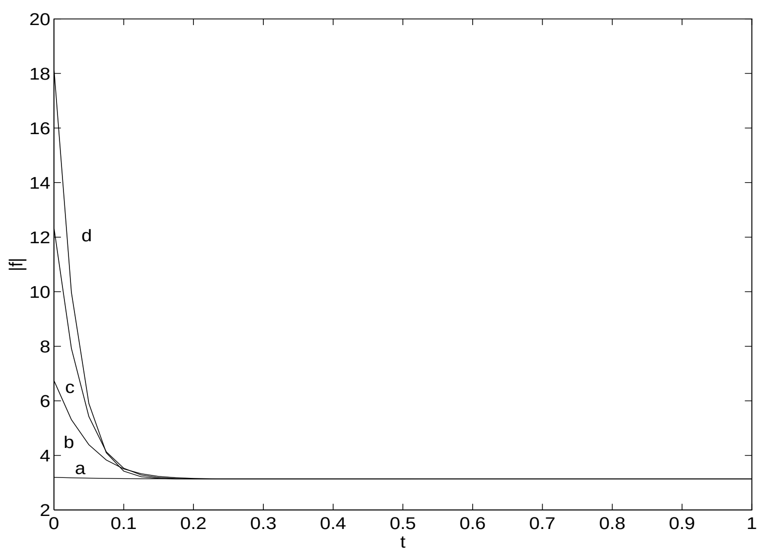

Figure 4.10: Test 1. Control norm $\|\vec{f}\|$ for different γ

appears to be the same figure as Fig. 4.8. We can note that the control flow matches very well for all values. The reduction of the error in the stationary part and around $t = T$ is accompanied by a reduction of the error near $t = 0$ when γ increases. It is not surprising that the controlled flow matches the target flow for $\gamma = 1$ since $\|\vec{U}\|_{L^\infty(0,T;L^4)}$ is small. For particular target flow with high norm values it is possible to see that there is a limit value for γ. Under this limit the controlled function can drive away from the target velocity around $t = T$. As shown the previous sections, this is a typical behavior of the nonlinear term. For linear equation γ need only to be positive. In all this computation the time step Δt is 0.025 and $h = 1/16$. The norm of $\vec{g}$ goes monotonically to zero as expected while the $\vec{f}$ norm goes to $\|\vec{F}\|$. The norm of the control agrees with the intuitive behaviour of the error. For high values of γ the control resembles a delta function plus the body force generated by the target velocity $\vec{U}$. The norm of $\vec{g}$ is shown in Fig.4.9 and the norm of $\vec{f}$ in Fig.4.10.

4.5.4 Test 2

We consider a unit square domain $(0,1) \times (0,1) \subset \mathcal{R}^2$. We assume that the time interval [0,1] is divided into equal intervals of time $\Delta t = 1/N$. The Taylor-Hood finite elements are used in this calculation on a rectangular mesh. We report only the final result with $h = 1/16$ but calculations with varying mesh sizes has been performed. The target velocity $\vec{U}$ for this test is equal to

$$\phi(k,t,z) = (1 - cos(2k\pi tz)) \times (1-z)^2$$
$$a(k,t,x,y) = \frac{d}{dy}\left(\phi(k,t,x)\phi(k,t,y)\right) \qquad b(k,t,x,y) = -\frac{d}{dx}\left(\phi(k,t,x)\phi(k,t,y)\right)$$
$$U = a(1,.4,x,y) + a(2,t,x,y)/(1+4\pi t) \qquad V = b(1,.4,x,y) + b(2,t,x,y)/(1+4\pi t).$$

With these velocity field we have the superposition of two flows. One flow with a vortex at the center of the domain and another flow with four vortices. Only one, in the lower left corner, is visible due to the different magnitude of the flow in different regions of the domain. Each of these flows prevails at different time of the evolution. The initial velocity for the controlled flow is

$$u_0(x,y) = -8U(1/4,x,y) \qquad v_0(x,y) = -8V(1/4,x,y)$$

The evolution is given in Fig.4.11 - Fig.4.16. The controlled fluid is on the right and the desired flow is on the left. In these calculations α has been set to 1 and γ to 10. The mesh length h is equal to 1/16. Again after a reduction in magnitude we have a quick change in shape and a perfect match is reached at $t = 0.3$. The Fig.4.17 and Fig.4.18 show the error $\|\vec{u} - \vec{U}_h\|$ and $\|\vec{u} - \vec{U}\|$ respectively. The error $\|\vec{u} - \vec{U}_h\|$ goes rapidly and monotonically to zero as expected while the error $\|\vec{u} - \vec{U}\|$ tends to a time dependent constant. For the same flow Fig.4.19 shows the values of the norm of the control $\vec{g}$ in function of time. At the beginning the control works hard in order to steer the controlled flow to the desired one and then it follows the error to zero. Fig.4.20 shows the norm of the control function $\vec{f}$. The norm of $\vec{f}$ looks like $\|\vec{g}\|$ for small t and then tends to $\|\vec{F}\|$.

4.6 Optimal controls and linear feedback control

4.6.1 Optimal quadratic control and linear feedback control

The quadratic control described in chapter 2 minimizes this quadratic functional

$$L(\vec{g}) = \frac{\alpha}{2}\int_0^T \int_\Omega (\vec{u} - \vec{U})^2 d\vec{x}dt +$$

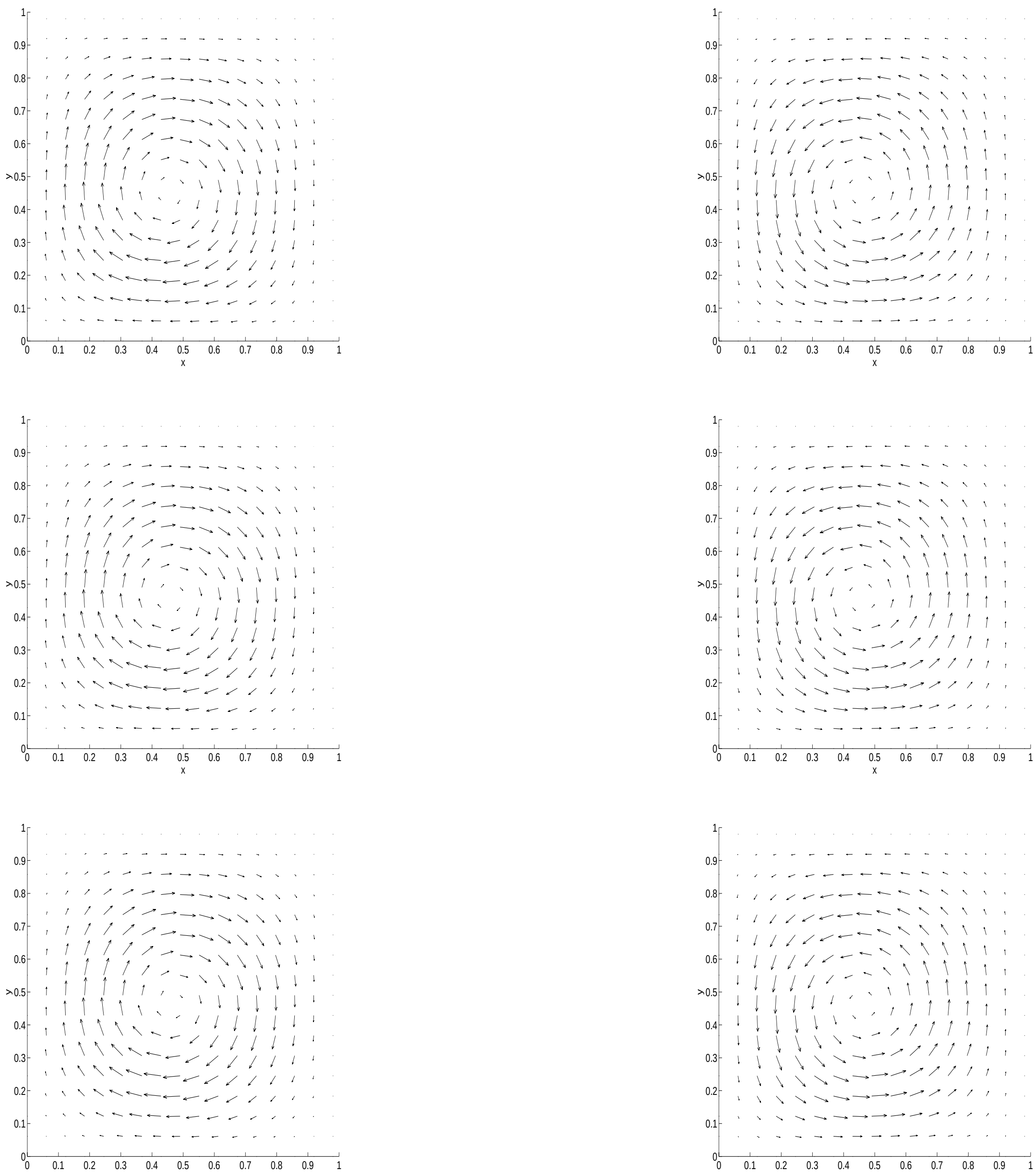

Figure 4.11: Test 2. Controlled(left) and desired(right) flow at $t = 0$ (top), $t = .05$ (middle) and $t = .1$ (bottom)

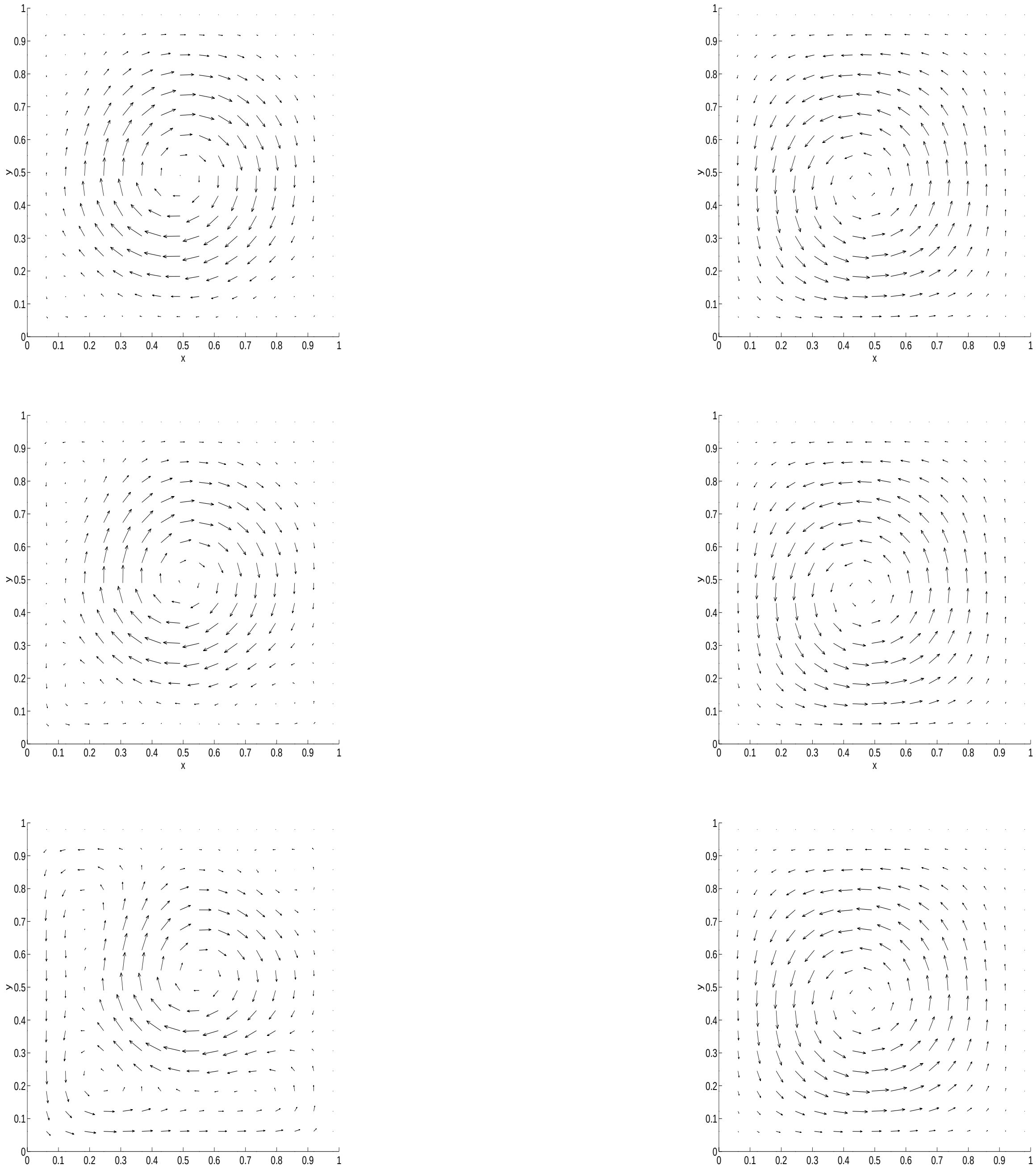

Figure 4.12: Test 2. Controlled(left) and desired(right) flow at $t = .15$ (top) $t = .156$ (middle) and $t = .162$ (bottom)

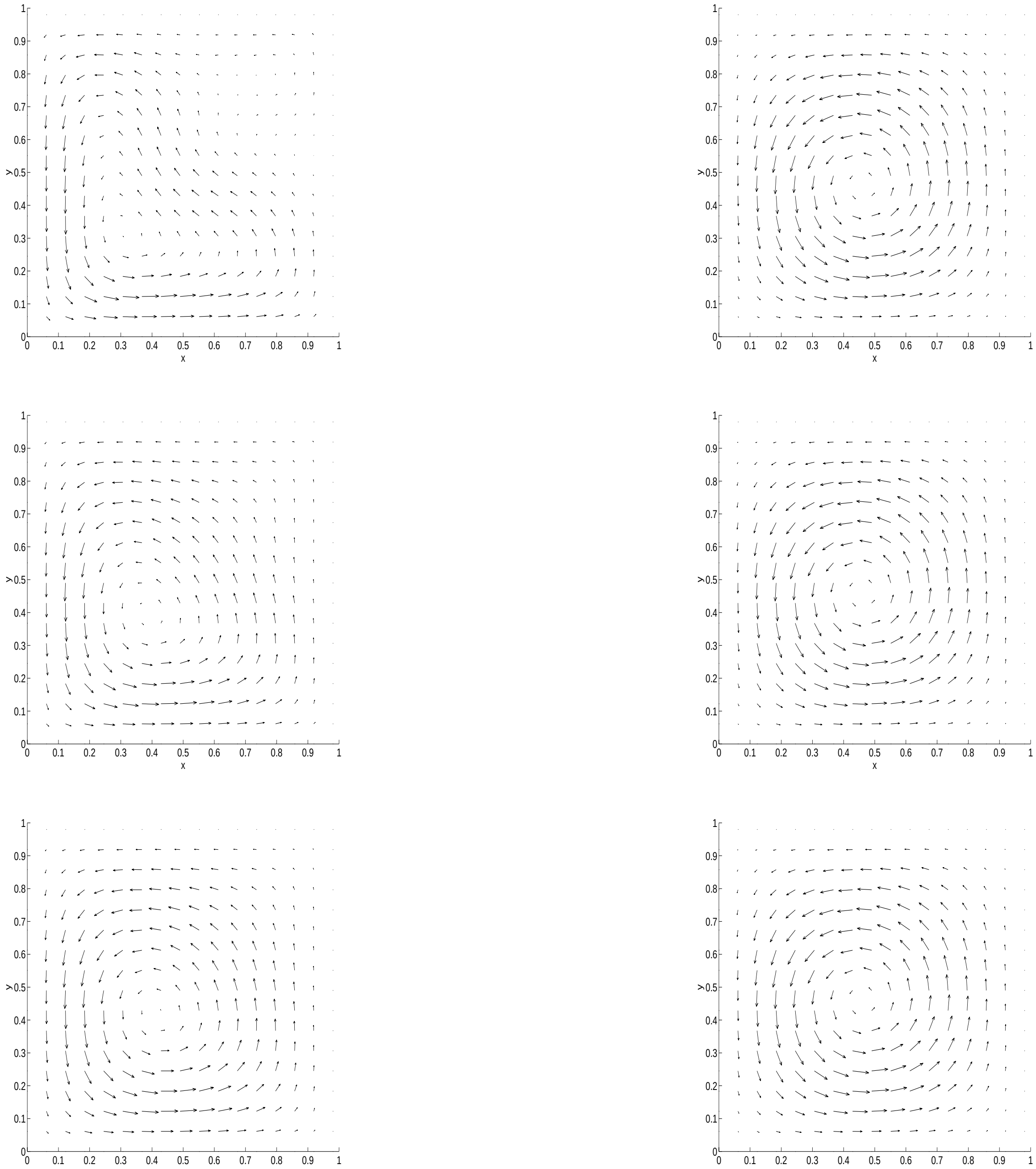

Figure 4.13: Test 2. Controlled(left) and desired(right) flow at $t = .169$ (top) $t = .175$ (middle) and $t = .181$ (bottom)

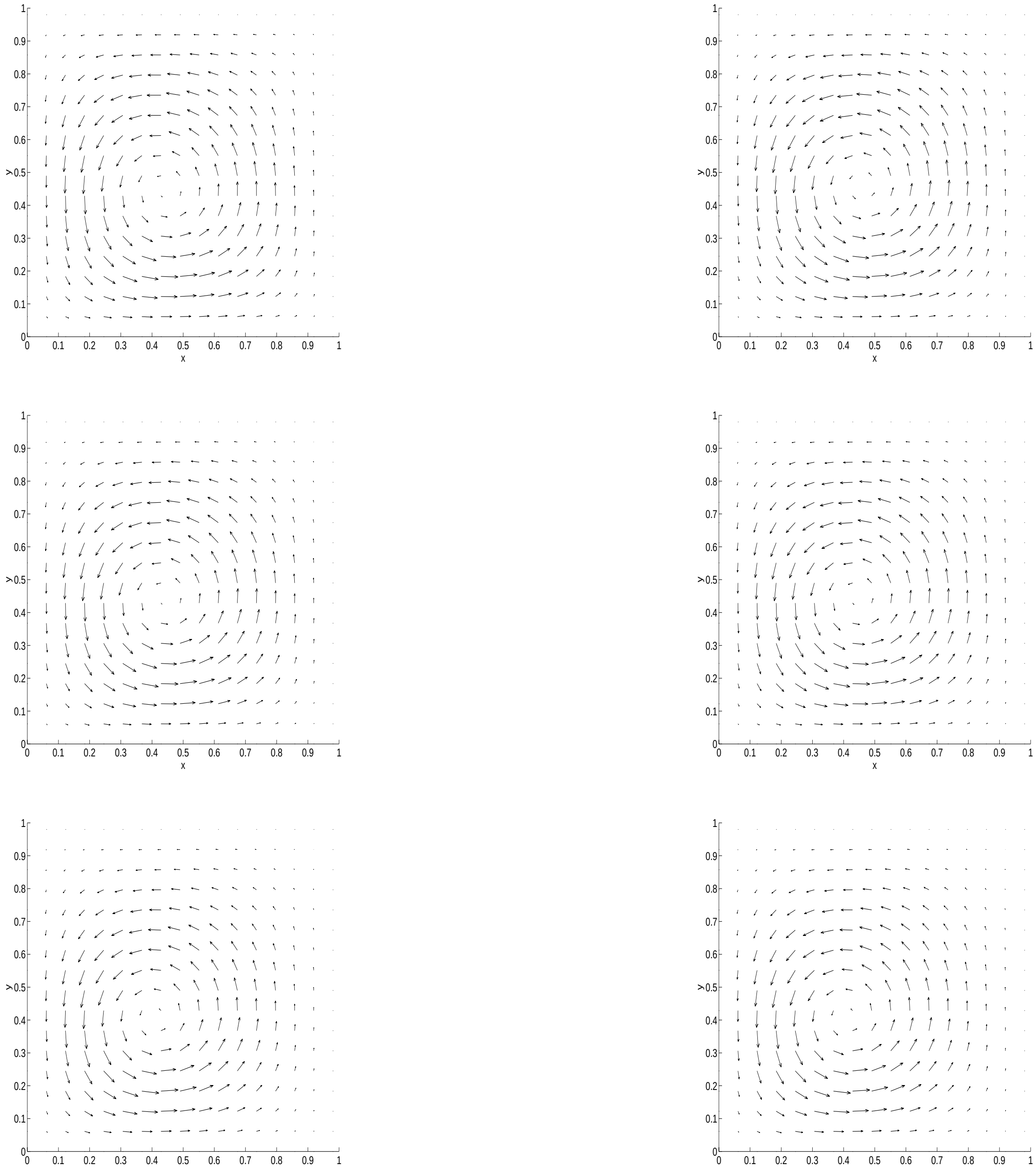

Figure 4.14: Test 2. Controlled(left) and desired(right) flow at $t = .2$ (top), $t = .3$ (middle) and $t = .4$ (bottom)

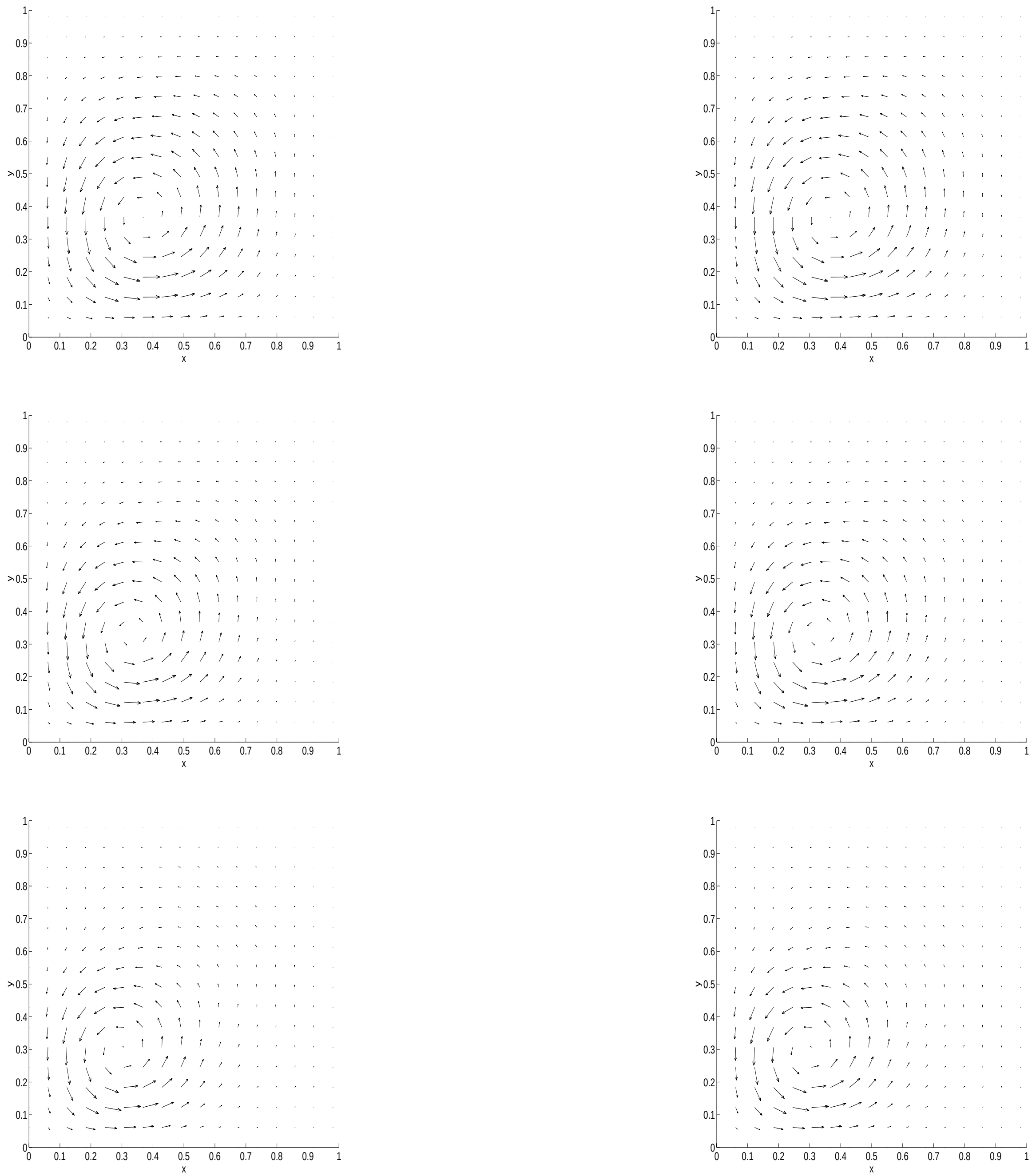

Figure 4.15: Test 2. Controlled(left) and desired(right) flow at $t = .5$ (top), $t = .6$ (middle) and $t = .7$ (bottom)

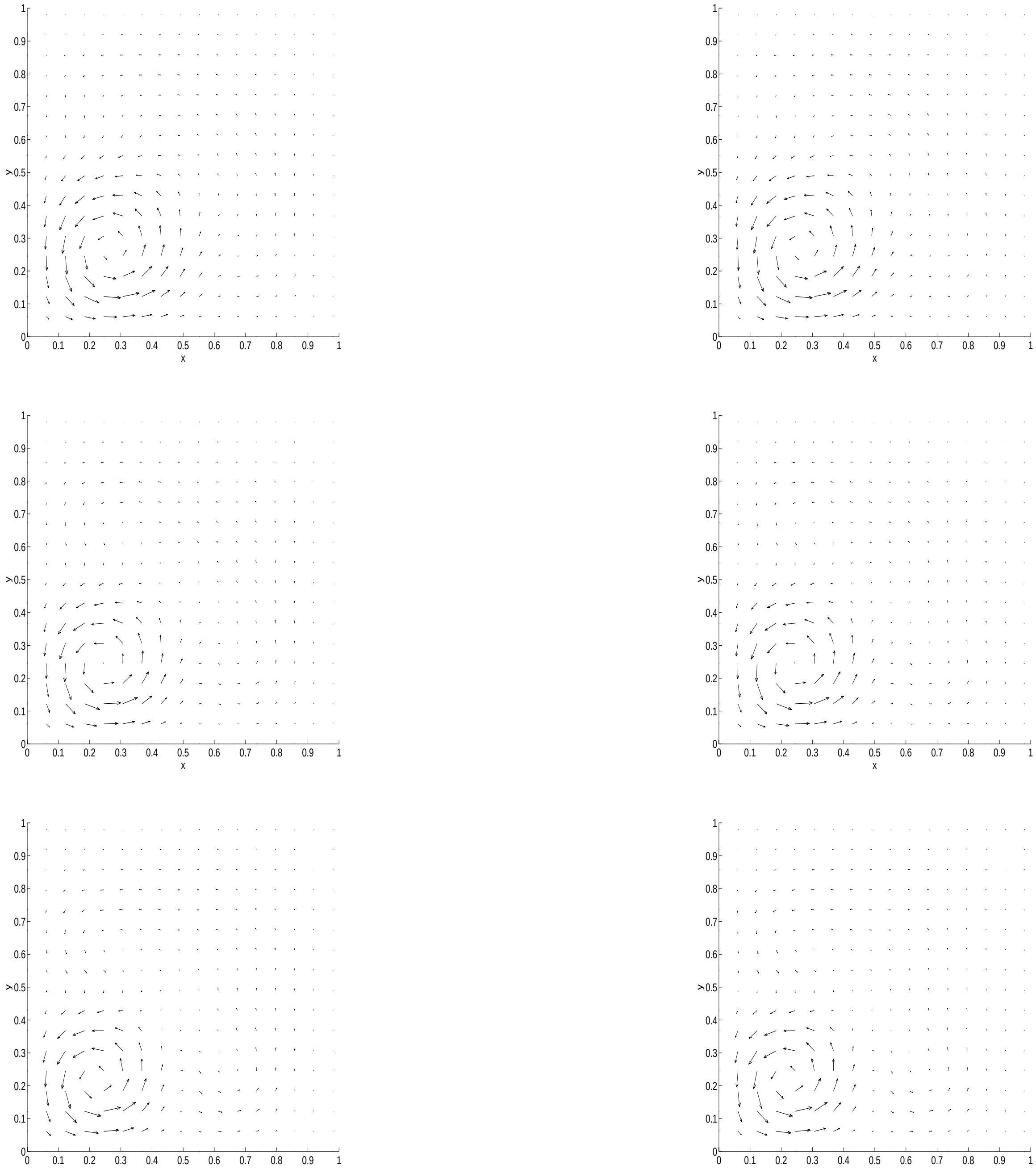

Figure 4.16: Test 2. Controlled(left) and desired(right) flow at $t = .8$ (top), $t = .9$ (middle) and $t = 1$ (botttom)

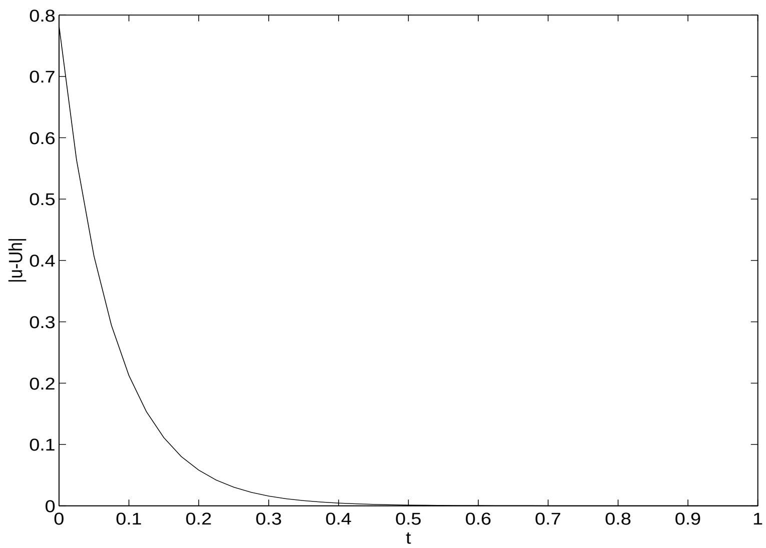

Figure 4.17: Test 2. Error $\|\vec{u} - \vec{U}_h\|$

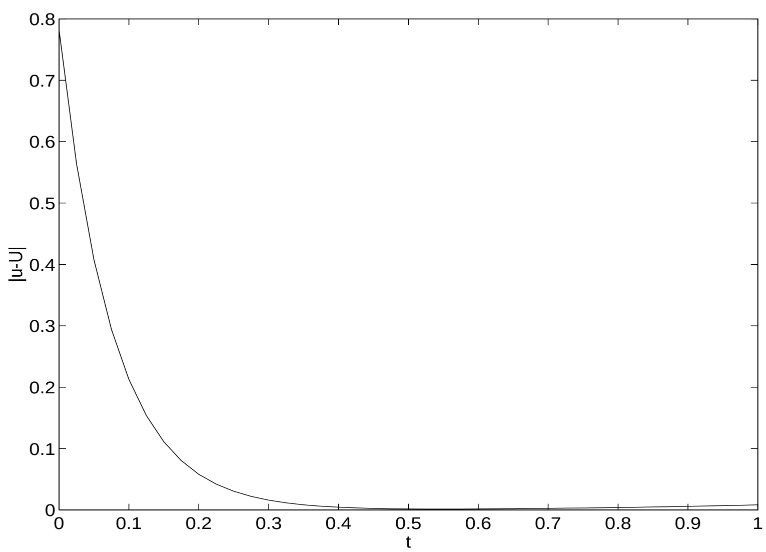

Figure 4.18: Test 2. Error $\|\vec{u} - \vec{U}\|$

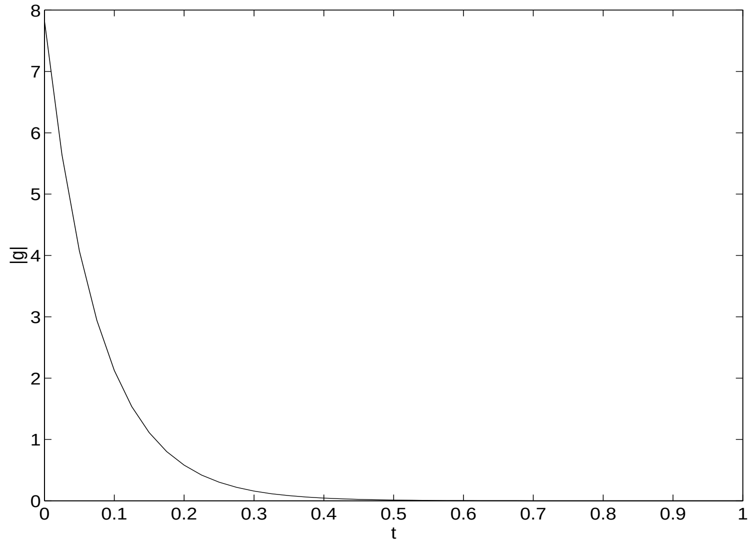

Figure 4.19: Test 2.Control norm $\|g\|$

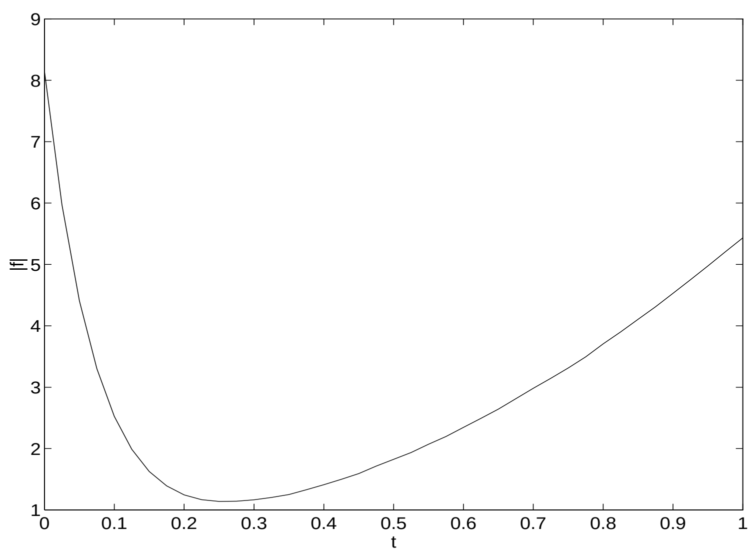

Figure 4.20: Test 2.Control norm $\|f\|$

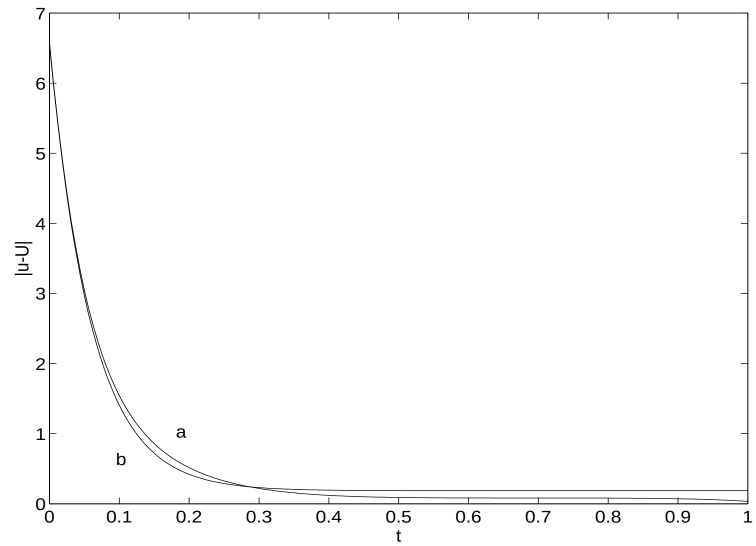

Figure 4.21: Test 1. Error norm $\|u - U\|$ for LQF and LF control

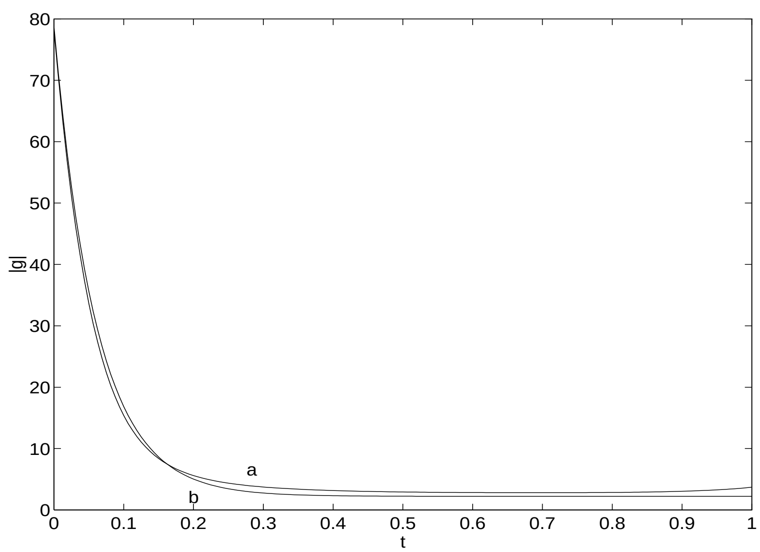

Figure 4.22: Test 1. Control norm $\|g\|$ for LQF and LF control

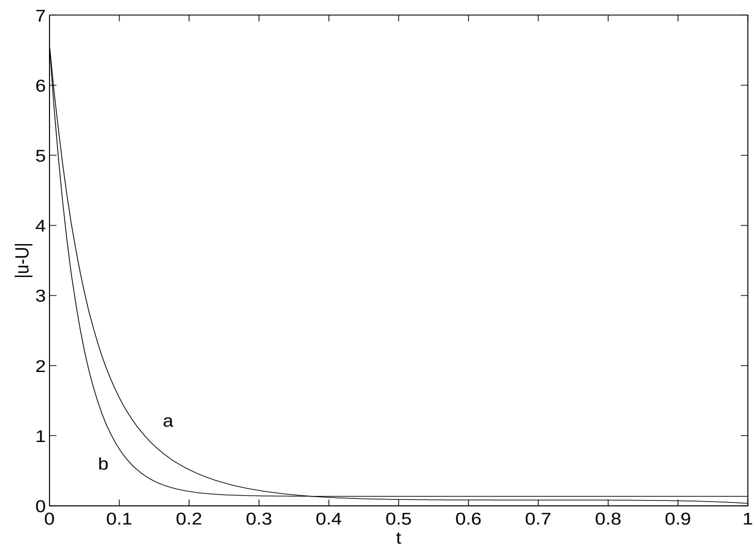

Figure 4.23: Error norm $\|u - U\|$ for LQF and LF control

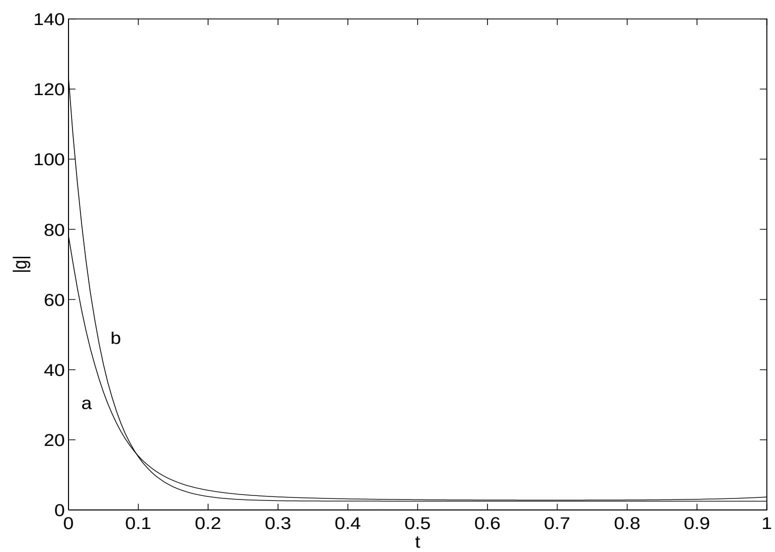

Figure 4.24: Control norm $\|g\|$ for LQF and LF control

$$\frac{\beta}{2}\int_0^T\int_\Omega \vec{g}^2 d\vec{x}dt + \frac{\gamma}{2}\int_\Omega (\vec{u}(T)-\vec{U}(T))^2 d\vec{x}.$$

The term in $\vec{g}$ limits the control performance. For value of $\beta > .05$ there is not control over the flow because the term in $\vec{g}$ may be predominant. In some cases, for particular target $\vec{U}$ and initial velocity $\vec{u}_0$ (for example when $\|\vec{u}_0\| >> \|\vec{U}\|$), the error $\|\vec{u}-\vec{U}\|$ decreases monotonically and $\vec{g}$ is almost zero but in general this is not true and a large value of β is needed. For high values of β (around .0001) the control is effective and we can drive easily the flow to the target flow. In same particular cases we can approximate very well some optimal singular controls. With high values of β the control $\vec{g}$ is finite but the bound for the control is not predictable. The bound depend of the initial velocity $\vec{u}_0$ and the target velocity $\vec{U}$. The computation of the controlled flow is hard and the convergence of the gradient algorithm slow. The optimal control system needs to be solved on the entire time space cylinder domain and it can not be solved marching in time. Besides the solution of the system involves a coupled system of the state and adjoint variables with initial and final conditions. The computation of the linear feedback control algorithm is simple and it can be solved as a standard Navier-Stokes system marching in time. In Fig.4.21 we can see the error $\|\vec{u}-\vec{U}\|$ between the controlled velocity $\vec{u}$ and the target velocity $\vec{U}$ for the optimal solution (a) and the linear feedback solution (b). This picture refers to Test 1 ($\beta = .0001$) in chapter 2. The optimal solution is optimal but in this case the difference is almost negligible. Fig.4.22 shows the norm of the control for the same flow. The choice of γ is the choice such that they have the same initial control norm ($\gamma = \|\vec{g}_0\|/\|\vec{u}_0-\vec{U}\|$). The computation for the linear feedback control has been performed by using the system in eq(6.37) and the function $\vec{F}$ has been set to zero so that the linear feedback control is defined by $\vec{f}=\vec{g}=-\gamma(\vec{u}-\vec{U})$. As shown in Fig.4.23, it is easy to find a value of γ for the linear feedback control and to obtain better tracking. In this case $\gamma = 20$. But a larger γ gives a larger initial control norm $\|\vec{g}_0\|$. The control norm is shown in Fig.4.24. The curve (a) is for the optimal quadratic control and (b) is for the linear feedback control. As in the case of the optimal quadratic control, the solution tends to be unbounded when γ tends to infinity. But in this case it easy to predict the bound and clearly we have $\|\vec{g}_0\| \leq \gamma\|\vec{u}_0-\vec{U}_0\|$ ($\|\vec{g}_0\| \leq 20\times\ 6 = 120$).

4.6.2 Optimal bounded control and linear feedback control

The bounded optimal control described in chapter 3 minimizes the quadratic functional

$$L = \frac{\alpha}{2}\int_0^T\int_\Omega (\vec{u}-\vec{U})^2 d\vec{x}dt + \frac{\gamma}{2}\int_\Omega (\vec{u}(T)-\vec{U}(T))^2 d\vec{x}$$

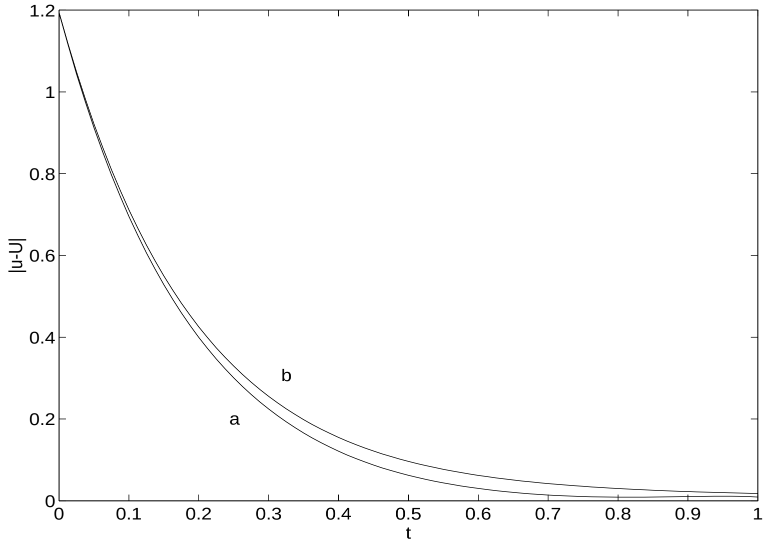

Figure 4.25: Test 1. Error norm $\|\vec{u} - \vec{U}\|$ for bounded and LF control

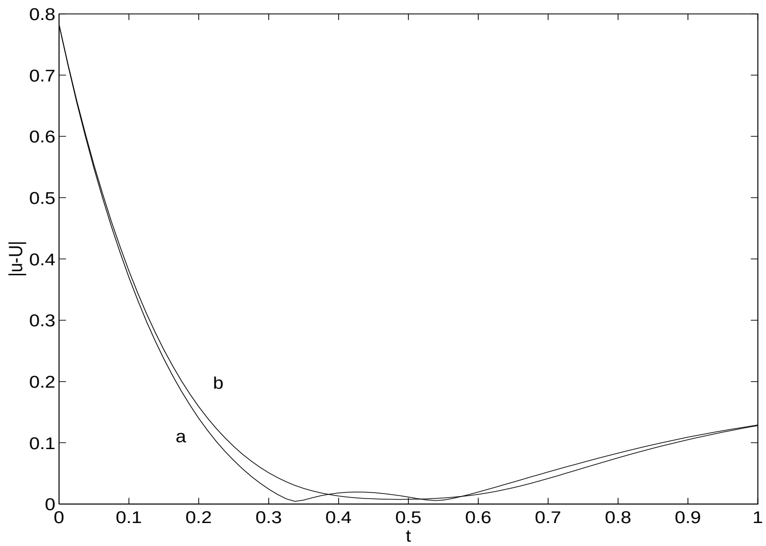

Figure 4.26: Test 2. Error norm $\|\vec{u} - \vec{U}\|$ for bounded and LF control

with bounded control. In this case the limit on the control is explicit. The solution is always optimal and, inside this bound, the solution is better then the optimal quadratic control solution and the linear feedback control solution. In some particular cases the term in γ can be predominant and the linear feedback control can perform better then the bounded optimal control solution. The computation of the optimal controlled flow is hard and the convergence of the gradient algorithm very slow. The control has fixed norm but the optimal control system, which involves a coupled system of the state and adjoint variables with initial and final conditions, needs again to be solved on the entire time space cylinder domain. As in the quadratic optimal control case, the gradient feedback algorithm can be adapted to obtain a solution with bounded control. The feedback control can be chosen to be in this form

$$\vec{f} = -\gamma(\vec{u} - \vec{U})/(\|\vec{u} - \vec{U}\|)$$

where γ is the bound for the control. Now we have a nonlinear system that can be solved as a standard Navier-Stokes system with a few iterations. Another choice can be

$$\vec{f}^{(n)} = -\gamma(\vec{u}^{(n)} - \vec{U}(n))/(\|\vec{u}^{(n)} - \vec{U}^{(n)}\|).$$

In this case the norm $\|\vec{g}\|$ is not strictly constant but for small Δt it gives almost the same results.

We can see in Fig.4.25 the error $\|\vec{u} - \vec{U}\|$ between the controlled velocity $\vec{u}$ and the target velocity $\vec{U}$ for the optimal solution (a) and the linear feedback solution (b). This figure refers to the test 1 ($\|\vec{g}\| \leq 3.2$) and Fig.4.26 to test 2 ($\|\vec{g}\| \leq 1.6$) described in chapter 3. The optimal solution is optimal but again the difference is wry small.

Chapter 5

Velocity tracking problem with boundary control

5.1 Introduction

In this chapter, we study a class of flow control problems where the fluid is controlled by a distributed forcing at a portion of the boundary and the cost functional is quadratic with respect to the control variable. Optimal control theory of distributed systems, which has been analyzed in the previous chapters, can have several applications. However, in this analysis the control term appears as a distributed force in the momentum equation and hence is not realizable in many practical situations. In this chapter, we try to formulate an optimal control problem covering a broad class of practical boundary control problems in bounded domains. This chapter is organized as follows. Section 5.2 introduces the flow model, objective functions and the control problem in their continuous form. First-order necessary conditions are derived and an optimal solution is characterized by a partial differential equation. Semidiscretization in time is discussed in Section 5.3 and a full discretization is treated in Section 5.4. The numerical implementation is discussed and tested in Section 5.5.

5.2 Distributed control problem

5.2.1 Formulation of the optimal control problem

Notations

In order to approach the problem with boundary control in $H^1(\Gamma)$ we have to extend some proprieties of the continuous trilinear form

$$c(\vec{u};\vec{v},\vec{w})=\sum_{i=1}^{2}\sum_{i=1}^{2}\int_{\Omega}u_i\frac{\partial v_j}{x_i}w_j d\vec{x}\quad \forall \vec{u},\vec{v},\vec{w}\in H^1(\Omega). \tag{5.1}$$

We limit our domain Ω to be an open bounded set of class C^2 in $\mathbb{R}^2$. We define

$$\begin{aligned}
&curl(H^2)(\Omega)=\{\vec{v}\in H^1(\Omega):\quad \nabla\cdot\vec{v}=0,\quad \int_{\Gamma}\vec{v}\cdot\vec{n}d\vec{x}=0\}\\
&H^1_n(\Gamma)=\{\vec{g}\in H^1(\Gamma):\int_{\Gamma}\vec{g}\cdot\vec{n}d\vec{x}=0\}\\
&H^1_{n0}(\Gamma)=H^1_0(\Gamma)\cap H^1_n(\Gamma).
\end{aligned}$$

$curl(H^2)(\Omega)$ and $H^1_n(\Gamma), H^1_{n0}(\Gamma)$ are a closed subspaces of $H^1(\Omega)$ and $H^1(\Gamma)$ respectively. For details concerning these subspaces see [38]. We remark that the space $H^1(\Gamma)$ can be decomposed in $H^1_n(\Gamma)\oplus(H^1_n)^{\perp}(\Gamma)$ where $(H^1_n)^{\perp}(\Gamma)$ is the space of constant vectors normal to the surface. If $\vec{g}\in H^1(\Gamma)$ one can write $\vec{g}=\vec{g}_1+\vec{g}_2$ where

$$\begin{aligned}
&\vec{g}_2=a\vec{n}\qquad a=\frac{\int_{\Gamma}\vec{g}\cdot\vec{n}d\vec{x}}{\mu(\Gamma)}\\
&\vec{g}_1=\vec{t}(\vec{g}\cdot\vec{t})+\vec{n}(\vec{g}\cdot\vec{n}-a).
\end{aligned}$$

We have

$$\int_{\Gamma}\vec{g}_1\cdot\vec{n}d\vec{x}=\int_{\Gamma}(\vec{g}\cdot\vec{n}-a)d\vec{x}=\int_{\Gamma}\vec{g}\cdot\vec{n}d\vec{x}-a\mu(\Gamma)=0.$$

Hence $\vec{g}_1\in H^1_n(\Gamma)$ and $\vec{g}_2\in(H^1_n)^{\perp}(\Gamma)$ can be regarded as the projection of $\vec{g}$ into the space $H^1_n(\Gamma)$ and $(H^1_n)^{\perp}(\Gamma)$ respectively.

Lemma 5.1 *We have the following useful properties for the trilinear form:*
i)

$$\begin{cases} c(\vec{u};\vec{v},\vec{w})=-c(\vec{u};\vec{w},\vec{v}) & \forall\vec{u}\in V(\Omega),\quad \forall\vec{v},\vec{w}\in H^1(\Omega)\\ c(\vec{u};\vec{v},\vec{w})=-c(\vec{u};\vec{w},\vec{v}) & \forall\vec{u}\in curl(H^2)(\Omega),\quad \forall\vec{v}\in H^1(\Omega),\quad \forall\vec{w}\in H^1_0(\Omega)\end{cases} \tag{5.2}$$

ii)

$$\begin{cases} c(\vec{u};\vec{v},\vec{v})=0 & \forall\vec{u}\in V(\Omega),\quad \forall\vec{v}\in H^1(\Omega)\\ c(\vec{u};\vec{v},\vec{v})=0 & \forall\vec{u}\in curl(H^2)(\Omega),\quad \forall\vec{v}\in H^1_0(\Omega).\end{cases} \tag{5.3}$$

iii)

$$\begin{cases} |c(\vec{u};\vec{v},\vec{w})|\le\sqrt{2}\|\vec{u}\|^{1/2}\|\nabla\vec{u}\|^{1/2}\|\vec{w}\|^{1/2}\|\nabla\vec{w}\|^{1/2}\|\nabla\vec{v}\| & \forall\vec{u},\vec{v},\vec{w}\in H^1(\Omega)\\ |c(\vec{u};\vec{v},\vec{w})|\le C\|\vec{u}\|_1\cdot\|\vec{v}\|_1\cdot\|\vec{w}\|_1 & \forall\vec{u},\vec{v},\vec{w}\in H^1(\Omega).\end{cases} \tag{5.4}$$

where K and C are independent of the functions $\vec{u},\vec{w}$ and $\vec{v}$.

Proof: i) Let $\vec{u} \in \mathcal{V}(\Omega)$

$$c(\vec{u};\vec{v},\vec{w}) = \sum_{i=1}^{2}\sum_{i=1}^{2}\int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j d\vec{x} = -\sum_{i=1}^{2}\sum_{i=1}^{2}\int_\Omega u_i v_j \frac{\partial w_j}{\partial x_i} d\vec{x} = -c(\vec{u};\vec{w},\vec{v}) \tag{5.5}$$

for all $\vec{u} \in \mathcal{V}(\Omega)$ and $\vec{v},\vec{w} \in H^1(\Omega)$. We note that the integration by parts is well defined by the regularity of $\vec{u}$. The density of $\mathcal{V}(\Omega)$ in $V(\Omega)$ implies that the equation is true for all $\vec{u} \in V(\Omega)$. Using a density argument for $\vec{w} \in \mathcal{D}(\Omega)$ ($\mathcal{D}(\Omega)$ is dense in $H^1_0(\Omega)$) the second equality can be proved with the same techniques.
ii) It is a direct consequence of i).
iii) By repeated application of the Schwartz and Holder inequalities we find:

$$|c(\vec{u};\vec{w},\vec{v})| = \sum_{i=1}^{2}\sum_{j=1}^{2}\int_\Omega u_i \frac{\partial v_j}{x_i} w_j d\vec{x} \le \sum_{i=1}^{2}\sum_{j=1}^{2} \|u_i\|_{L^4}\cdot\|\partial_{x_i} v_j\|_{L^2}\cdot\|w_j\|_{L^4} \le$$
$$\left(\sum_{i=1}^{2}\|u_i\|^2_{L^4}\right)^{1/2}\left(\sum_{j=1}^{2}\sum_{i=1}^{2}\|\partial_{x_i} v_j\|^2_{L^2}\right)^{1/2}\left(\sum_{j=1}^{2}\|w_j\|^2_{L^4}\right)^{1/2}. \tag{5.6}$$

Now we will show that

$$\sum_{i=1}^{2}\|z_i\|^2_{L^4} \le \sqrt{2}\sum_{i=1}^{2}\|z_i\|\cdot\|\nabla z_i\|$$

for all $z_i \in H^1(\Omega)$. As $\mathcal{D}(\bar{\Omega})$ is a dense set in $H^1(\Omega)$ we write for all $z_i \in \mathcal{D}(\bar{\Omega})$

$$|z_i(\vec{x})|^2 = 2\int_{-\infty}^{x_1} u_i(x_1', x_2)\partial_{x_1'} z_i(x_1', x_2)dx_1' \le 2\|z_i\|_{L^2(\mathbb{R}^2)}\|\partial_{x_1'} z_i(x_1', x_2)\|_{L^2(\mathbb{R}^2)}$$

and

$$|z_i(\vec{x})|^2 = 2\int_{-\infty}^{x_2} u_i(x_1, x_2')\partial_{x_2'} z_i(x_1, x_2')dx_2' \le 2\|z_i\|_{L^2(\mathbb{R}^2)}\|\partial_{x_2'} z_i(x_1, x_2')\|_{L^2(\mathbb{R}^2)}$$

so that

$$\int_{\mathbb{R}^2}|z_i(\vec{x})|^4 d\vec{x} \le 4\|z_i\|^2_{L^2(\mathbb{R}^2)}\|\partial_{x_1'} z_i\|_{L^2(\mathbb{R}^2)}\|\partial_{x_2'} z_i\|_{L^2(\mathbb{R}^2)} \le 2\|z_i\|^2_{L^2(\mathbb{R}^2)}\|\nabla z_i\|^2_{L^2(\mathbb{R}^2)}.$$

Hence

$$|c(\vec{u};\vec{w},\vec{v})| \le \sqrt{2}\|\nabla\vec{v}\|\left(\sum_{i=1}^{2}\|u_i\|\cdot\|\nabla u_i\|\right)^{1/2}\left(\sum_{j=1}^{2}\|w_j\|\cdot\|\nabla w_j\|\right)^{1/2} \le$$
$$\sqrt{2}\|\vec{u}\|^{1/2}\|\nabla\vec{u}\|^{1/2}\|\vec{w}\|^{1/2}\|\nabla\vec{w}\|^{1/2}\|\nabla\vec{v}\| \quad \forall \vec{u},\vec{v},\vec{w} \in H^1(\Omega)$$

as

$$\sum_{i=1}^{2}\|z_i\|\cdot\|\nabla z_i\| \le \|\vec{z}\|\cdot\|\nabla\vec{z}\|. \quad \sqcup$$

Part iii) a) of the Lemma 5.1 cannot be extended to a three-dimensional domain.

We will use the following standard notations. Let Ω be a bounded connected open set in $\mathbb{R}^2$ with $\Gamma \in C^2$ and denote by $\vec{n} = (n_1, n_2)$ the unit normal vector defined on the boundary. The normal and the tangent to the boundary Γ are denoted by $\vec{n}$ and $\vec{\tau}$. Γ_c denotes the controlled part of the boundary. Let $I =]0, T[$ and $Q = I \times \Omega$, and let $S = I \times \Gamma$ ($S_c = I \times \Gamma_c$). Denote $\gamma_0 \vec{f} = \vec{f}_\Gamma$ and $\gamma_0 \partial_n^k \vec{f} = \gamma_k \vec{f}$, where $\partial_n \vec{f} = n_1 \partial_1 \vec{f} + n_2 \partial_2 \vec{f}$, with $\partial_j \vec{f} = \partial \vec{f}/\partial x_j$. We use the notation r_0 for the restriction operator of a function $\vec{u}(\vec{x}, t)$ to $t = 0$,(i.e., $r_0 \vec{u} = \vec{u}_0$) and denote $\Omega \times \{0\} = \Omega_0$ and $\Gamma \times \{0\} = \Gamma_0$. The following statements are well known (see [47], theorem 4.2.1 and proposition 4.2.3):

Lemma 5.2 *Let $r > 0$ and $s > 0$, and let $a(\vec{x}, t) \in C^\infty(\bar{Q})$. The mappings $a(\vec{x}, t)$,D_x^α, D_t^j, γ_0 and r_0 going from $C^\infty(\bar{Q})$ to $C^\infty(\bar{Q})$,$C^\infty(\bar{Q})$, $C^\infty(\bar{Q})$, $C^\infty(\bar{S})$, $C^\infty(\Omega_0)$, respectively, extend to continuous mappings:*

$$
\begin{array}{lll}
a(\vec{x}, t) : H^{r,s}(Q) \to H^{r,s}(Q); & & \\
D_x^\alpha : H^{r,s}(Q) \to H^{r-|\alpha|,(r-|\alpha|)s/r}(Q) & for & r > 0, |\alpha| \le r; \\
D_t^j : H^{r,s}(Q) \to H^{(s-j)r/s, s-j}(Q) & for & s > 0, j \le s; \\
\gamma_0 : H^{r,s}(Q) \to H^{r-1/2,(r-1/2)s/r}(S) & for & r > 1/2; \\
r_0 : H^{r,s}(Q) \to H^{(s-1/2)r/s}(\Omega_0) & for & s > 1/2;
\end{array}
\tag{5.7}
$$

where γ_0 and r_0 described here are surjective.

It is interesting to note that one has the continuity and surjectiveness of the system of trace operators (see [47] theorem 4.2.3 and its proof):

$$\{\gamma_0, ..., \gamma_m\} : H^{r,r/q}(Q) \to \Pi_{j=0}^m H^{r-j-1/2,(r-j-1/2)/q}(S)$$

for $r > m + 1/2$,

$$\{r_0, ..., r_0 \partial_t^m\} : H^{r,r/q}(Q) \to \Pi_{j=0}^m H^{r-jq-q/2}(\Omega_0)$$

for $r/q > m + 1/2$. These operators also have continuous right inverses.

Lemma 5.3 *Let $r > 0$, $q > 0$. The mapping*

$$u \to \mathcal{B}u = \{\{\gamma_j u\}_{0 \le j < r-1/2}, \{r_0 \partial_t^l u\}_{0 \le l < r/q-1/2}\} \tag{5.8}$$

is continuous and surjective from $H^{r,r/q}(Q)$ to the space of vectors

$$\Phi = \{\{\phi_j\}_{0 \le j < r-1/2}, \{v_l\}_{0 \le l < r/q-1/2}\}$$

satisfying

$$
\begin{array}{lll}
\phi_j \in H^{r-j-1/2,(r-j-1/2)/q}(S) & for & 0 \le j < r - 1/2, \\
v_l \in H^{r-lq-q/2}(\Omega_0) & for & 0 \le l < r/q - 1/2, \\
r_0 \partial_t^l \phi_j = \gamma_j v_l & for & j + lq < r - (q+1)/2, \\
I[\partial_t^l \phi_j, D_\nu^j \chi v_l] < \infty & for & j + lq < r - (q+1)/2,
\end{array}
\tag{5.9}
$$

provided with the norm $\|\Phi\|^2$ *defined by*

$$\|\Phi\|^2 = \sum_{0\le j<r-1/2} \|\phi_j\|^2_{H^{r-j-1/2,(r-j-1/2)/q}(S)} + \sum_{0\le l<r/q-1/2} \|v_l\|^2_{H^{r-lq-q/2}(\Omega_0)} + \tag{5.10}$$
$$\sum_{j+lq=r-(q+1)/2} I[\partial_t^l \phi_j, D_\nu^j \chi v_l]; \quad I[\psi, v] = \int_{t\in I}\int_{x'\in\Gamma}\int_{y\in\Omega_0} \frac{|\psi(x',t)-v(y)|^2}{(|x'-y|^q+t)^{1+n/q}} dy d\sigma_{x'} dt$$

and $\mathcal{B}$ *has a continuous right inverse in these spaces.*

For the proof, see [51] and [47].

Theorem 5.1 *Let* V *be an bounded open set with boundary in* C^2. *Let* $\vec{v} \in L^2((0,T); H^1(V))$ *(or* $\vec{v} \in L^2((0,T); H^1_0(V))$*) and* $\vec{v}' \in L^2((0,T); (H^1(V))^*)$ *(or* $\vec{v}' \in L^2((0,T); H^{-1}(V))$*) then, the function* $\vec{v}$ *is almost everywhere equal to a continuous function in* $L^2((0,T); L^2(V))$. *Further this set is continuously imbedded in* $C^0([0,T]; L^2(V))$, *where the space* $C^0([0,T]; L^2(V))$ *is equipped with the norm of uniform convergence.*

This theorem can be found in XVIII.2 Vol.5 [38]. This allows us to state that every function in $H^{1,1}(Q)$ is almost everywhere equal to a continuous function.

In the next section we will use the following operators:

$$A : H^1(\Omega) \to H^{-1}(\Omega) \tag{5.11}$$
$$< A\vec{u}, \vec{v} >= a(\vec{u}, \vec{v}) \quad \forall \vec{u} \in H^1(\Omega) \quad \forall \vec{v} \in H^1_0(\Omega)$$
$$C : H^1(\Omega) \times H^1(\Omega) \to H^{-1}(\Omega) \tag{5.12}$$
$$< C(\vec{w})\vec{u}, \vec{v} >= c(\vec{w}; \vec{u}, \vec{v}) \quad \forall \vec{w}, \vec{u} \in H^1(\Omega) \quad \forall \vec{v} \in H^1_0(\Omega)$$
$$B : H^1(\Omega) \to L^2_0(\Omega) \tag{5.13}$$
$$< B\vec{u}, p >= b(\vec{u}, p) \quad \forall p \in L^2_0(\Omega) \quad \vec{u} \in H^1(\Omega)$$
$$B^* : L^2_0(\Omega) \to H^{-1}(\Omega) \tag{5.14}$$
$$< \vec{u}, B^* p >= b(\vec{u}, p) \quad \forall p \in L^2_0(\Omega) \quad \vec{u} \in H^1_0(\Omega).$$

We will denote with πA and πC the projections of these operators on $V(\Omega)$.

Classical formulation

In this section, we describe the problem of time boundary control for the Navier-Stokes equations that models the velocity tracking problem through a quadratic functional. This problem reflects the desire to steer, over time, a candidate velocity field $\vec{u}$ to a target velocity field $\vec{U}$ by appropriately controlling the boundary. We consider a two-dimensional flow over the physical domain Ω with boundary Γ and control over Γ_c. The equations considered here are the nondimensional incompressible Navier-Stokes equations on the interval of time

$[0, T]$.

$$\begin{cases} \vec{u}_t(t,\vec{x}) + (\vec{u}\cdot\vec{\nabla})\vec{u}(t,\vec{x}) - \nu\nabla^2\vec{u}(t,\vec{x}) + \vec{\nabla}p(t,\vec{x}) = 0 \\ \vec{u} = \vec{g}(t,\vec{x}) \quad \vec{x}\in\Gamma_c \quad t\in(0,T) \\ \vec{u} = 0 \quad \vec{x}\in\Gamma\setminus\Gamma_c \quad t\in(0,T) \end{cases} \tag{5.15}$$

with initial velocity $\vec{u}(0,\vec{x}) = \vec{u}_0(\vec{x})$. The vector $\vec{u} = (u_1, u_2)$ is the velocity vector, p is the pressure and ν is the kinematics viscosity. We note that the Reynolds number is equal to $1/\nu$. The vector $\vec{u}_0$ must be a divergence free vector and must satisfy the compatibility conditions. The control is the boundary velocity which must satisfy the mass conservation law given by

$$\int_\Omega \nabla\cdot\vec{u}(t,\vec{x})d\vec{x} = \int_{\Gamma_c} \vec{g}(t,\vec{x})\cdot\vec{n}d\vec{x} = 0. \tag{5.16}$$

The optimal control problem is formulated as follows:
find a boundary control $\vec{g}$ minimizing the cost function

$$L(\vec{g}) = \frac{\alpha}{2}\int_0^T\int_\Omega(\vec{u}-\vec{U})^2 d\vec{x}dt + \frac{\beta}{2}\int_0^T\int_{\Gamma_c}(\vec{g}^2 + \beta_1\vec{g}_x^2 + \beta_2\vec{g}_t^2)d\vec{x}dt \tag{5.17}$$

where $\vec{u}$ is solution of eq(5.15).

The minimization of the $\int_0^T\int_\Omega(\vec{u}-\vec{U})^2 d\vec{x}dt$ term is the real goal of the velocity tracking problem; the $\beta\int_0^T\int_{\Gamma_c}(\vec{g}^2 + \beta_1\vec{g}_x^2 + \beta_2\vec{g}_t^2)d\vec{x}dt$ term has been introduced in order to bound the control function and to prove the existence of an optimal control. We can limit the size of the control with an appropriate choice of the coefficients β_j.

Weak formulation of the LQF control problem

We consider an open bounded set $\Omega\subset\mathbb{R}^2$ with a boundary $\Gamma\in C^2$. $\vec{U}$ is said to be in the set of admissible target velocities U_{ad} if

$$\begin{cases} \vec{U} = \vec{U}(t,\vec{x}) \in C([0,T]; H^2(\Omega)) \\ \vec{F}_{\vec{U}}(t,\vec{x}) \in L^\infty((0,T); L^2(\Omega)) \end{cases} \tag{5.18}$$

where $\vec{F}_{\vec{U}} = \vec{U}_t(t,\vec{x}) - \nu\nabla^2\vec{U}(t,\vec{x}) + (\vec{U}(t,\vec{x})\cdot\vec{\nabla})\vec{U}(t,\vec{x})$.

Let $\vec{u}\in L^2((0,T); H^1(\Omega))$ and $p\in L^2((0,T); L_0^2(\Omega))$ denote the state variables, i.e. the velocity and pressure field, respectively. Let the boundary control $\vec{g}$ be in $L^2((0,T); H_{n0}^1(\Gamma_c))$ and $\vec{g}_t\in L^2((0,T); L^2(\Gamma_c))$. The state variables are constrained to satisfy the weak form of the Navier-Stokes equations for almost all t in $(0,T)$, i.e.,

$$\begin{cases} <\vec{u}_t,\vec{v}> +\nu a(\vec{u},\vec{v}) + c(\vec{u};\vec{u},\vec{v}) + b(\vec{v},p) = 0 \qquad \forall\vec{v}\in H_0^1(\Omega) \\ b(\vec{u},q) = 0 \quad \forall q\in L_0^2(\Omega) \\ (\vec{u},\vec{s})_\Gamma = (\vec{g}(t,\vec{x}),\vec{s})_{\Gamma_c} \quad \forall\vec{s}\in H^{-1/2}(\Gamma) \\ \vec{u} = 0 \quad \vec{x}\in\Gamma\setminus\Gamma_c \\ \vec{u}(0,\vec{x}) = \vec{u}_0(\vec{x}) \in V(\Omega) \end{cases} . \tag{5.19}$$

More precisely, *Let* $\vec{g} \in H^{1,1}(S_c) \cap L^2((0,T); H^1_{n0}(\Gamma_c))$ *and* $\vec{u}_0, \in curl(H^2)(\Omega)$ *then,* $(\vec{u}, p) \in L^2((0,T); H^1(\Omega)) \times L^2((0,T); L^2_0(\Omega))$ *is called a weak solution for the Navier-Stokes equations if it satisfies the equations (5.19) with initial velocity* $\vec{u}_0$ *and boundary velocity* $\vec{g}$*.*

If $\vec{u}$ is a solution of the eq(5.15), then it is also solution of the weak formulation in eq(5.19). If $\vec{u}$ is solution of eq(5.19), then it satisfies eq(5.15) in the distribution sense on $(0,T)$. If $\vec{g}, \vec{u}_0$ are given as above, then we can show that there exists a unique admissible weak solution $(\vec{u}, p)$ of (5.19), such that $\vec{u} \in L^\infty((0,T); W(\Omega)) \cap L^2((0,T); H^1(\Omega))$ and $\vec{u}_t \in L^2((0,T); H^{-1}(\Omega))$, i.e, it is a.e. equal to a continuous function.

Theorem 5.2 *Let* Ω *be an open bounded of class* C^2 *in* $\mathbb{R}^2$*. Let* $\vec{g}(t, \vec{x})$ *be a function in the anisotropic Sobolev space* $H^{1/2,1}(S)$ *satisfying the compatibility condition:*

$$\int_\Gamma \vec{g} \cdot \vec{n} d\Gamma = 0. \tag{5.20}$$

Then there exists a unique $\vec{u} \in L^2((0,T); H^1(\Omega)) \cap L^\infty((0,T); L^2(\Omega))$ *and a* $p \in L^2((0,T); L^2_0(\Omega))$ *which is the solution of the nonhomogeneous Stokes problem:*

$$\begin{cases} \vec{u}_t - \nu \nabla^2 \vec{u} + \nabla p = 0 \quad on \quad \Omega \\ \nabla \cdot \vec{u} = 0 \quad on \quad \Omega \\ \vec{u} = g(t, \vec{x}) \quad in \quad \Gamma \\ \vec{u}(0, \vec{x}) = \vec{u}_0(\vec{x}) \in curl(H^2)(\Omega). \end{cases} \tag{5.21}$$

With this hypothesis we have

$$\|\vec{u}\|^2_{L^2((0,T);H^1)} + \|\vec{u}\|^2_{L^\infty((0,T);L^2)} \leq K \|\vec{g}\|^2_{H^{1/2,1}(S)} \tag{5.22}$$

where the constant K *is independent of* $\vec{g}$*.*

Proof: This proof follows [38] theorem XIX.1.11. We first show that there exists a $\tilde{u} \in L^2((0,T); H^1(\Omega))$ such that $\hat{f} = \tilde{u}_t - \nu \nabla^2 \tilde{u} \in L^2((0,T); H^{-1}(\Omega))$ and

$$\begin{cases} \nabla \cdot \tilde{u} = 0 \quad in \quad \Omega \\ \tilde{u} = g \quad on \quad \partial\Omega. \end{cases} \tag{5.23}$$

Such a function $\tilde{u}$ can be defined by:

$$\tilde{u} = \nabla w_1 + (D_2 w_2, -D_1 w_2). \tag{5.24}$$

The function w_1 is the solution of the Neumann problem:

$$\begin{cases} \nabla^2 w_1 = 0 \quad in \quad \Omega \\ \frac{\partial w_1}{\partial n} = \vec{g} \cdot \vec{n} \quad on \quad \Gamma \end{cases} \tag{5.25}$$

where $\vec{g}(\vec{x},t) \in H^{1/2,1}(S)$. The function w_2 is therefore chosen to satisfy

$$\begin{cases} \frac{\partial w_2}{\partial \tau} = 0 \\ \frac{\partial w_2}{\partial n} = g \cdot \tau - \frac{\partial w_1}{\partial \tau} \quad on \quad \Gamma. \end{cases} \tag{5.26}$$

The problem in (5.25) has a solution in $H^{2,1}(Q)$. In fact, from lemma 5.3, we note that the map

$$\mathcal{B} : H^{2,4}(Q) \to \{\gamma_0 \vec{u}, \gamma_1 \vec{u}, r_0 \vec{u}\} \subset H^{3/2,3}(S) \times H^{1/2,1}(S) \times H^{3/4}(\Omega)$$

is continuous and surjective and has a continuous right inverse. Since $\vec{g} \in H^{1/2,1}(S)$ there exists a function $w \in H^{2,4}(Q)$ such that $\gamma_1 w = \partial w / \partial n = \vec{g} \cdot \vec{n}$. Now from lemma 5.2 $dw/dt \in H^{3/2,3}(Q)$ (i.e ∇w_t and w_{tt} are in $L^2((0,T), L^2(\Omega)))$, $\nabla^2 w \in L^2((0,T); L^2(\Omega))$ and $\nabla^2 w_t \in L^2((0,T); H^{-1}(\Omega))$. Hence $f = \nabla^2 w$ is in $L^2((0,T), L^2(\Omega))$ and $f_t = \nabla^2 w_t$ is in $L^2((0,T), H^{-1}(\Omega))$. If we set $w_3 = w_1 - w$ the problem in (5.25) becomes a Laplacian problem with homogeneous Neumann boundary condition:

$$\begin{cases} \nu \nabla^2 w_3 = f \quad in \quad \Omega \\ \frac{\partial w_3}{\partial n} = 0 \quad on \quad \Gamma. \end{cases} \tag{5.27}$$

It is easy to check that the problem in eq(5.27) has a weak solution in $H^{2,1}(Q)$ and $w_{3t} \in L^2((0,T); H^1(\Omega))$. Hence the problem in eq(5.25) has a weak solution $w_1 = w + w_3 \in H^{2,1}(Q)$ with $w_{1t} \in L^2((0,T); H^1(\Omega))$. The last statement assures that $\partial w_1/\partial \tau$ is in $H^{1/2,1}(Q)$.

We can find a solution of the problem in eq(5.26) with $w_2 \in H^{2,2}(Q)$. In fact, as $\vec{g} \cdot \tau - \partial w_1/\partial \tau \in H^{1/2,1}(S)$, this problem is equivalent to find a function ∇w_2 when ∇w_2 is specified on the boundary in $H^{1/2,1}(S)$. From lemma 5.3 there exists such a function ∇w_2 in $H^{1,2}(Q)$ and so $w_2 \in H^{2,2}(Q)$.

We are reduced to the case of a problem with homogeneous boundary conditions. To this end, we set $\hat{u} = \vec{u} - \tilde{u}$, and the Stokes problem in eq(5.21) is equivalent to find a solution $(\hat{u}, p)$ of the problem:

$$\begin{cases} \hat{u}_t + \nu \nabla^2 \hat{u} + \nabla p = \hat{f} \\ \nabla \cdot \hat{u} = 0 \quad on \quad \Omega \\ \hat{u} = 0 \quad \vec{x} \in \Gamma \quad t \in (0,T) \\ \hat{u}(0, \vec{x}) = \hat{u}_0(\vec{x}) \in V(\Omega) \end{cases} \tag{5.28}$$

where

$$\hat{f} = \tilde{u}_t + \nu \nabla \tilde{u} = \nabla^2 w_{1t} + (D_1 w_{2t}, D_2 w_{2t}) + \nu \nabla^2 (D_1 w_2, D_2 w_2).$$

From the regularity of w_1 and w_2, applying again lemma 5.2, we have $\hat{f} \in L^2((0,T); H^{-1}(\Omega))$. In fact $\tilde{u} \in H^{1,1/2}(Q)$ thus $\nabla \tilde{u} \in L^2((0,T); L^2(\Omega))$ and consequently $\nabla^2 \tilde{u} \in L^2((0,T); H^{-1}(\Omega))$.

The function $\tilde{u}_t$ is in $L^2((0,T);H^{-1}(\Omega))$ as the time derivative of w_1 and w_2 are both in $L^2((0,T);L^2(\Omega))$. We know that for homogeneous boundary condition and $\hat{f} \in L^2((0,T);$ $H^{-1}(\Omega))$ there exists a unique weak solution $\hat{u} \in L^2((0,T);V(\Omega)) \cap L^\infty((0,T);W(\Omega))$, $\hat{u}_t \in L^2((0,T);\, H^{-1}(\Omega))$. The eq.(5.22) follows from the regularity of $\hat{f}$ and from the fact that the map $\mathcal{B}$ has a continuous right inverse. $\sqcup$

Theorem 5.3 *Let Ω be an open bounded of class C^2 in $\mathbb{R}^2$. Let $\vec{g}(t,\vec{x})$ be a function in $H^{1/2,1}(S)$ satisfying the compatibility condition:*

$$\int_\Gamma \vec{g}\cdot\vec{n}d\Gamma = 0. \tag{5.29}$$

Then there exists a unique $\vec{u} \in L^2((0,T);H^1(\Omega)) \cap L^\infty((0,T);L^2(\Omega))$ and a $p \in L^2((0,T);L^2_0(\Omega))$, that are the solution of the nonhomogeneous Navier-Stokes problem:

$$\begin{cases} <\vec{u}_t,\vec{v}> +\nu a(\vec{u},\vec{v}) + c(\vec{u};\vec{u},\vec{v}) + b(\vec{v},p) = 0 \qquad \forall\vec{v}\in H^1_0(\Omega) \\ b(\vec{u},q) = 0 \quad \forall q \in L^2_0(\Omega) \\ \vec{u} = \vec{g}(t,\vec{x}) \quad \forall\vec{x}\in\Gamma \\ \vec{u}(0,\vec{x}) = \vec{u}_0(\vec{x}) \in curl(H^2)(\Omega). \end{cases} \tag{5.30}$$

With this hypothesis we have

$$\|\vec{u}\|^2_{L^2((0,T);H^1)} + \|\vec{u}\|^2_{L^\infty((0,T);L^2)} \le K\|\vec{g}\|^2_{H^{1/2,1}(S)} \tag{5.31}$$

where the constant K is independent of $\vec{g}$.

Proof: We will reduce this problem to the case of a problem with homogeneous boundary conditions. To this end, we set $\hat{u} = \vec{u} - \tilde{u}$. The variable $\tilde{u}$ is the solution of the following corresponding linear Stokes problem

$$\begin{cases} <\tilde{u}_t,\vec{v}> +\nu a(\tilde{u},\vec{v}) + b(\vec{v},p) = 0 \qquad \forall\vec{v}\in H^1_0(\Omega) \\ b(\tilde{u},q) = 0 \quad \forall q\in L^2_0(\Omega) \\ \tilde{u} = \vec{g}(t,\vec{x}) \quad \forall\vec{x}\in\Gamma_c \quad t\in(0,T) \\ \tilde{u} = 0 \quad \forall\vec{x}\in\Gamma\setminus\Gamma_c \quad t\in(0,T) \\ \vec{u}(0,\vec{x}) = \vec{u}_0 \in curl(H^2)(\Omega) \quad \forall\vec{x}\in\Omega. \end{cases} \tag{5.32}$$

From theorem 5.2 $\tilde{u} \in L^2((0,T),H^1(\Omega)) \cap L^\infty((0,T);L^2(\Omega))$. The Navier-Stokes system for the variable $(\hat{u},\hat{p})$ is now homogeneous:

$$\begin{cases} <\hat{u}_t,\vec{v}> +\nu a(\hat{u},\vec{v}) + c(\hat{u};\hat{u},\vec{v}) + c(\tilde{u};\tilde{u},\vec{v}) + \\ c(\hat{u};\tilde{u},\vec{v}) + c(\tilde{u};\hat{u},\vec{v}) + b(\vec{v},p) = 0 \qquad \forall\vec{v}\in H^1_0(\Omega) \\ b(\hat{u},q) = 0 \quad \forall q\in L^2_0(\Omega) \\ \hat{u}(t,\vec{x}) = 0 \quad \vec{x}\in\Gamma \quad t\in(0,T) \\ \hat{u}(0,\vec{x}) = 0. \end{cases} \tag{5.33}$$

We note that if $\vec{v} = \hat{u}$, we have

$$\frac{d}{dt}\|\hat{u}\|^2 + \nu\|\nabla\hat{u}\|^2 + c(\tilde{u}, \tilde{u}, \hat{u}) + c(\hat{u}, \tilde{u}, \hat{u}) = 0 \tag{5.34}$$

or ($|c(\vec{u}, \vec{w}, \vec{u})| \le K\|\vec{u}\| \cdot \|\nabla\vec{u}\| \cdot \|\nabla\vec{w}\| \qquad \forall\vec{u}, \vec{v}, \vec{w} \in H^1(\Omega)$)

$$\frac{d}{dt}\|\hat{u}\|^2 + \nu\|\nabla\hat{u}\|^2 \le K_1(\|\tilde{u}\| \cdot \|\nabla\tilde{u}\| \cdot \|\nabla\hat{u}\| + \|\hat{u}\| \cdot \|\nabla\hat{u}\| \cdot \|\nabla\tilde{u}\|).$$

Applying Young inequality and using $\tilde{u} \in L^\infty((0,T); L^2(\Omega))$ we have

$$\frac{d}{dt}(\|\hat{u}\|^2 \exp - \int_0^t c\|\nabla\tilde{u}\|^2 d\tau) + \frac{\nu}{4}\|\nabla\hat{u}\|^2 \le c_1\|\nabla\tilde{u}\|^2$$

where c is a constant. With this observation the existence of the solution $\hat{u}$ can be proved by using standard techniques. The theorem follows from $\vec{u} = \hat{u} + \tilde{u}$. $\sqcup$

From the previous discussion we can define more precisely the set of admissible solutions. *Given T , $\vec{g} \in H^{1,1}(S_c) \cap H^1_{n0}(\Gamma_c)$, $\vec{u}_0 \in curl(H^2)(\Omega)$ and $\vec{U} \in U_{ad}$, then a weak solution of the eq(5.19) $(\vec{u}, p, \vec{g})$ is called an admissible solution for our optimal control problem if $u \in L^2((0,T); V(\Omega))$, $p \in L^2((0,T); L^2_0(\Omega))$, and the functional $L(\vec{g})$ is bounded.*
The set of all admissible solution is defined as A_d. The optimal control problem, in the "P_L form", can be formulated as follows:
Let $\vec{g} \in H^{1,1}(S_c) \cap H^1_{n0}(\Gamma_c)$ and $\vec{u}_0 \in curl(H^2)(\Omega)$. Given $\vec{U} \in U_{ad}$, find $(\vec{u}, p, \vec{g}) \in A_d$ such that the control $\vec{g}$ minimizes the cost function

$$L(\vec{g}) = \frac{\alpha}{2}\int_0^T \int_\Omega (\vec{u} - \vec{U})^2 d\vec{x}dt + \frac{\beta}{2}\int_0^T \int_{\Gamma_c} (\vec{g}^2 + \beta_1\vec{g}_x^2 + \beta_2\vec{g}_t^2) d\vec{x}dt \tag{5.35}$$

with β and $\beta_1, \beta_2 > 0$.

5.2.2 Existence of the optimal control solution

In this section we prove that the problem in P_L is well posed and a solution exists. Here Ω is an open bounded domain with boundary Γ in C^2.

Theorem 5.4 *Given $\vec{u}_0 \in curl(H^2)(\Omega)$, then there exists a function $\vec{g} \in H^{1,1}(S_c) \cap H^1_{n0}(\Gamma_c)$ and a solution $\vec{u} \in L^\infty((0,T); L^2(\Omega)) \cap L^2((0,T); L^2(\Omega))$ of the optimal control problem defined in eq(5.35).*

Proof: We can consider this equivalent problem. Let $\vec{g} \in H^{1,1}(S_c) \cap H^1_{n0}(\Gamma_c)$ and $\tilde{u}$ satisfy the linear Stokes equation

$$\begin{cases} < \tilde{u}_t, \vec{v} > + \nu a(\tilde{u}, \vec{v}) + b(\vec{v}, p) = 0 \qquad \forall\vec{v} \in H^1_0(\Omega) \\ b(\tilde{u}, q) = 0 \quad \forall q \in L^2_0(\Omega) \\ (\tilde{u}, \vec{s})_\Gamma = (\vec{g}(t, \vec{x}), \vec{s})_{\Gamma_c} \quad \forall\vec{s} \in H^{-1/2}(\Gamma) \\ \vec{u}(0, \vec{x}) = u_0 \in curl(H^2)(\Omega). \end{cases} \tag{5.36}$$

The problem is now to find $\tilde{u}$ and a solution $(\hat{u}, p)$ of the system

$$\begin{cases} <\hat{u}_t, \vec{v}> + \nu a(\hat{u}, \vec{v}) + c(\hat{u}; \hat{u}, \vec{v}) + c(\tilde{u}; \tilde{u}, \vec{v}) + \\ c(\hat{u}; \tilde{u}, \vec{v}) + c(\tilde{u}; \hat{u}, \vec{v}) + b(\vec{v}, p) = 0 \qquad \forall \vec{v} \in H_0^1(\Omega) \\ b(\hat{u}, q) = 0 \quad \forall q \in L_0^2(\Omega) \\ \hat{u}(t, \vec{x}) = 0 \quad \vec{x} \in \Gamma \quad t \in (0, T) \\ \hat{u}(0, \vec{x}) = 0 \end{cases} \tag{5.37}$$

such as the control $\vec{g}$ minimizes the cost function

$$L(\vec{g}) = \frac{\alpha}{2} \int_0^T \int_\Omega (\tilde{u} + \hat{u} - \vec{U})^2 d\vec{x}dt + \frac{\beta}{2} \int_0^T \int_{\Gamma_c} (\vec{g}^2 + \beta_1 \vec{g}_x^2 + \beta_2 \vec{g}_t^2) d\vec{x}dt. \tag{5.38}$$

As the admissible set is bounded and not empty ($(\hat{u} + \tilde{u}, p, 0) \in A_d$), let $\vec{g}_n$ be a minimizing sequence for the problem in eq(5.36-5.38) and set $\tilde{u}_n = \tilde{u}(\vec{g}_n)$, $\hat{u}_n = \hat{u}(\tilde{u}_n, \vec{g}_n)$.

The sequences $\{\vec{g}_n\}$ and $\{\vec{g}_n'\}$ are uniformly bounded in $L^2((0,T); H_{n0}^1(\Gamma_c))$ and $L^2((0,T); L^2(\Gamma_c))$ respectively (in fact we have $L(\vec{g}) \leq L(0)$). The boundary velocity $\tilde{u}$ is thus uniformly bounded in $L^2((0,T); H_0^{1/2}(\Gamma_c))$ and the corresponding solutions $\tilde{u}_n$ and $\hat{u}_n$ are uniformly bounded in the set $L^\infty((0,T); L^2(\Omega)) \cap L^2((0,T); H^1(\Omega))$. Hence there is a $(\tilde{u}, \hat{u}, \vec{g})$ and a subsequence of $(\tilde{u}_m, \hat{u}_m, \vec{g}_m)$ that converges weakly to $(\tilde{u}, \hat{u}, \vec{g})$. We write

$$\begin{array}{lll} \vec{g}_m \to \vec{g} & in \quad L^2((0,T); H_{n0}^1(\Gamma_c)) & weakly \\ \vec{g}_m' \to \vec{g}' & in \quad L^2((0,T); L^2(\Gamma_c)) & weakly \\ \tilde{u}_m \to \vec{u} & in \quad L^2((0,T); H^1(\Omega)) & weakly \\ \hat{u}_m \to \vec{u} & in \quad L^2((0,T); V(\Omega)) & weakly \\ \hat{u}_m \to \vec{u} & in \quad L^\infty((0,T); W(\Omega)) & * -weakly \end{array}$$

Now the pair $(\tilde{u}, \hat{u}, \vec{g})$ satisfies the system of eq (5.36-5.37) and minimizes the functional. In fact, by the lower semicontinuity of the functional in (5.38) we have

$$L(\tilde{u}, \hat{u}, \vec{g}) \leq \liminf_{m \to \infty} L(\tilde{u}_m, \hat{u}_m, \vec{g}_m).$$

Let $\vec{w}$ be in $\mathcal{V}(\Omega)$ and $\psi(t)$ be a continuously differentiable function on $[0,T]$ with $\psi(T) = 0$. We multiply eq(5.36) and eq(5.37) by $\psi(\tau)\vec{w}$ and then integrate by parts in τ

$$- \int_0^T (\hat{u}_m, \psi'(\tau)\vec{w}) d\tau + \nu \int_0^T a(\hat{u}_m, \psi(\tau)\vec{w}) d\tau + \int_0^T c(\hat{u}_m; \hat{u}_m, \psi(\tau)\vec{w}) d\tau =$$
$$\int_0^T (\hat{f}_m, \psi(\tau)\vec{w}) d\tau;$$
$$- \int_0^T (\tilde{u}_m, \psi'(\tau)\vec{w}) d\tau + \nu \int_0^T a(\tilde{u}_m, \psi(\tau)\vec{w}) d\tau = (\vec{u}_0, \psi(0)\vec{w}).$$

We can pass to the limit inside the linear and the nonlinear terms. In fact, the 'a priori estimate' (see [35] or [44]) for $\hat{u}$ in a fractional time order Sobolev space yields and $\hat{u}_m$

converges strongly to $\hat{u} \in L^2((0,T);V(\Omega))$. If $\psi \in \mathcal{D}((0,T))$ the limit $(\hat{u}, \tilde{u}, \vec{g})$ satisfies the Navier-Stokes equation (5.36) in the distribution sense. Since $\mathcal{V}(\Omega)$ is dense in $V(\Omega)$, then this is still true for any $\vec{w}$ in $V(\Omega)$ by a continuity argument. $\sqcup$

5.2.3 First-order necessary condition

In this section we proceed to derive the first-order necessary condition. First we show that that the optimal solution must satisfy the first-order necessary condition if there exists.

Theorem 5.5 *If $(\vec{u}, \vec{g})$ is an optimal pair for the problem in eq(5.21), then the Gateaux derivative of $L(\vec{g})$ vanishes at $(\vec{u}, \vec{g})$.*

Proof: If $(\vec{u}, \vec{g})$ is an optimal pair, then for every $\tilde{h} \in H^{1,1}(S_c) \cap H^1_{n0}(\Gamma_c)$ and for every $\lambda \in \mathbb{R}$ we have from the definition of an optimal solution

$$L(\vec{g} + \lambda\tilde{h}) \geq L(\vec{g}).$$

The above inequality implies

$$\frac{L(\vec{g} + \lambda\tilde{h}) - L(\vec{g})}{\lambda} \geq 0 \quad if \quad \lambda \geq 0$$

and

$$\frac{L(\vec{g} + \lambda\tilde{h}) - L(\vec{g})}{\lambda} \leq 0 \quad if \quad \lambda \leq 0,$$

that is

$$\frac{L(\vec{g} + \lambda\tilde{h}) - L(\vec{g})}{\lambda} = 0 \quad if \quad \lambda = 0,$$

and thus the Gateaux derivative must vanish. $\sqcup$

For all $\vec{g} \in H^{1,1}(S) \cap H^1_{n0}(\Gamma)$ the first order necessary condition is available if the map

$$\vec{u}(\vec{g}) : H^{1,1}(S) \cap H^1_{n0}(\Gamma) \to L^2((0,T);H^1(\Omega))$$

is differentiable. According to lemma 2.3, if $\vec{w}$ is the solution of this system of equations

$$\begin{cases} \vec{w}_t + \nu(\pi A)\vec{w} + \delta[(\pi C)(\vec{w})\vec{u} + (\pi C)(\vec{u})\vec{w}] + \sigma(\pi C)(\vec{w})\vec{w} = \vec{f} \\ \vec{w} \in V(\Omega) \end{cases} \tag{5.39}$$

with initial value $\vec{w}(0, \vec{x}) = 0$ and homogeneous boundary condition, then the solution $\vec{w}$ for all nonegative real values of δ and σ has the following property: if Ω is an open bounded set with Lipschitz-continuous boundary Γ, $\vec{f} \in L^2((0,T);V'(\Omega))$ and $\vec{u} \in L^\infty((0,T);W(\Omega)) \cap L^2((0,T);V(\Omega))$, then the solution $\vec{w}$ belongs to $L^\infty((0,T);W(\Omega)) \cap L^2((0,T);V(\Omega))$ and

$$\int_0^T \|\vec{w}\|_1^2 dt \leq C_1 \int_0^T \|\vec{f}\|_{V'}^2 dt \tag{5.40}$$

where C_1 is a constant depending on Ω and ν. By studying the case in which the Gateaux derivative vanishes we can get a possible candidate for the optimal control solution. The differentiability of the Frechet map $\vec{u}(\vec{g})$ is a focal point for the calculation of the optimal solution. Using the previous theorems we are now ready to state and prove the existence of the Gateaux derivative for this class of functions. It is useful to remark that the Gateaux derivatives makes sense whenever one is able to prove the uniqueness of the solution of the Navier-Stokes system.

Theorem 5.6 *Given $\Omega \in C^2$, $\vec{u}_0 \in curl(H^2)$ and $\vec{g} \in H^{1,1}(S) \cap H^1_{n0}(\Gamma)$. The mapping*

$$\vec{u}(\vec{g}) : H^{1,1}(S) \cap H^1_{n0}(\Gamma) \to L^2((0,T);H^1(\Omega))$$

has a Gateaux derivatives $\frac{D\vec{u}}{D\vec{g}} \cdot \tilde{h}$ in every direction $\tilde{h}$ in $H^{1,1}(S) \cap H^1_{n0}(\Gamma)$. Furthermore, $\tilde{w}(h) = \frac{D\vec{u}}{D^*} \cdot \tilde{h}$ is the solution of the problem

$$\begin{cases} < \tilde{w}_t, \vec{v} > + \nu a(\tilde{w}, \vec{v}) + c(\vec{u}; \tilde{w}, \vec{v}) + c(\vec{w}; \tilde{u}, \vec{v}) + b(\vec{v}, p) = 0 \quad \forall \vec{v} \in H^1_0(\Omega) \\ b(\tilde{w}, q) = 0 \quad \forall q \in L^2_0(\Omega) \\ (\tilde{w}(t,\vec{x}), \vec{s}) = (\tilde{h}(t,\vec{x}), \vec{s}) \quad \forall \vec{s} \in H^{-1/2}(\Gamma) \\ \tilde{w}(0,\vec{x}) = 0 \quad \vec{x} \in \Omega \end{cases} \tag{5.41}$$

where $\tilde{w} \in L^\infty((0,T);L^2(\Omega)) \cap L^2((0,T);H^1(\Omega))$.

Proof: Let $\vec{g}$ and $\tilde{h}$ be given in $H^{1,1}(S) \cap H^1_{n0}(\Gamma)$. We need to prove the following result:

$$\lim_{s \to 0} \left(\frac{\|(\vec{u}_{\vec{g}+s\tilde{h}} - \vec{u}_{\vec{g}}) - s\tilde{w}(\tilde{h})\|_{L^2((0,T);H^1)}}{|s|} \right) = 0.$$

We set $\tilde{u} = (\vec{u}_{\vec{g}+s\tilde{h}} - \vec{u}_{\vec{g}}) - s\tilde{w}(\tilde{h})$ so that $\tilde{u}$ is the solution of the evolution equation

$$\begin{cases} \frac{d\tilde{u}}{dt} + \nu(\pi A)\tilde{u} + (\pi C)(\vec{u}_{\vec{g}+s\tilde{h}})\vec{u}_{\vec{g}+s\tilde{h}} - (\pi C)(\vec{u}_{\vec{g}})\vec{u}_{\vec{g}} - (\pi C)'(\vec{u}_{\vec{g}})s\tilde{w} = 0 \\ \tilde{u} \in V(\Omega) \\ \tilde{u}(t,\vec{x}) = 0 \quad \vec{x} \in \Gamma \quad t \in [0,T] \\ \tilde{u}(0,\vec{x}) = 0 \quad \vec{x} \in \Omega. \end{cases} \tag{5.42}$$

If we define the function $\vec{k} \in L^2((0,T);H^{-1}(\Omega))$ as follows

$$\vec{k} = (\pi C)(\vec{u}_{\vec{g}+s\tilde{h}})\vec{u}_{\vec{g}+s\tilde{h}} - (\pi C)(\vec{u}_{\vec{g}})\vec{u}_{\vec{g}} - (\pi C)'(\vec{u}_{\vec{g}})(\vec{u}_{\vec{g}+s\tilde{h}} - \vec{u}_{\vec{g}})$$

then, the problem in eq(5.42) becomes

$$\begin{cases} \frac{d\tilde{u}}{dt} + \nu(\pi A)\tilde{u} + (\pi C)'(\vec{u}_{\vec{g}})\tilde{u} = \vec{k} \\ \tilde{u} \in V(\Omega) \\ \tilde{u}(t, \vec{x}) = 0 \quad \vec{x} \in \Gamma \quad t \in (0, T) \\ \tilde{u}(0, \vec{x}) = 0 \quad \vec{x} \in \Omega. \end{cases} \tag{5.43}$$

Using $(\pi C)'(\vec{u}) \cdot \tilde{u} = (\pi C)(\vec{u})\tilde{u} + (\pi C)(\tilde{u})\vec{u}$ and eq(5.40) with $\sigma = 0, \delta = 1, \vec{f} = \vec{k} \in L^2((0,T); H^{-1}(\Omega))$ we obtain

$$\int_0^T \|\tilde{u}\|_1^2 d\tau \le C_1 \int_0^T \|\vec{k}\|_{H^{-1}}^2 d\tau.$$

Now we need to evaluate the left-hand side term above. From the definition of norm in $H^{-1}(\Omega)$ we have

$$\|\vec{k}\|_{H^{-1}} = \sup_{\|\vec{v}\|_{H_0^1(\Omega)} \le 1} \frac{| < \vec{k}, \vec{v} > |}{\|\vec{v}\|_1}.$$

The evaluation of the duality pairing on $H^{-1} \times H_0^1$ gives

$$\begin{aligned} &| < \vec{k}, \vec{v} > | = |c(\vec{u}_{\vec{g}+s\tilde{h}}; \vec{u}_{\vec{g}+s\tilde{h}}, \vec{v}) - c(\vec{u}_{\vec{g}}; \vec{u}_{\vec{g}}, \vec{v}) - c'(\vec{u}_{\vec{g}}; \hat{u}, \vec{v})| = \\ &|c(\vec{u}_{\vec{g}+s\tilde{h}}, \vec{u}_{\vec{g}+s\tilde{h}}, \vec{v}) - c(\vec{u}_{\vec{g}}, \vec{u}_{\vec{g}}, \vec{v}) - c(\vec{u}_{\vec{g}}, \hat{u}, \vec{v}) - c(\hat{u}, \vec{u}_{\vec{g}}, \vec{v})| = \\ &|c(\hat{u}, \vec{u}_{\vec{g}+s\tilde{h}}, \vec{v}) - c(\hat{u}, \vec{u}_{\vec{g}}, \vec{v})| = |c(\hat{u}, \hat{u}, \vec{v})| \le K\|\nabla\hat{u}\| \cdot \|\hat{u}\| \cdot \|\nabla\vec{v}\| \end{aligned} \tag{5.44}$$

where $\hat{u} = \vec{u}_{\vec{g}+s\tilde{h}} - \vec{u}_{\vec{g}}$. Hence the estimate for $\tilde{u}$ gives

$$\int_0^T \|\tilde{u}(t)\|_1^2 dt \le C \int_0^T \|\vec{u}_{\vec{g}+s\tilde{h}} - \vec{u}_{\vec{g}}\|^2 \cdot \|\nabla\vec{u}_{\vec{g}+s\tilde{h}} - \nabla\vec{u}_{\vec{g}}\|^2 dt.$$

Now we need to estimate the norm of $\hat{u}$ in $L^2((0,T); L^2(\Omega))$ and in $L^2((0,T); H^1(\Omega))$. We set $\hat{u} = \vec{u}_{\vec{g}+s\tilde{h}} - \vec{u}_{\vec{g}}$ and hence we have

$$\begin{cases} < \hat{u}_t, \vec{v} > + \nu a(\hat{u}, \vec{v}) + c(\vec{u}_{\vec{g}}; \hat{u}, \vec{v}) + c(\hat{u}; \vec{u}_{\vec{g}}, \vec{v}) + c(\hat{u}; \hat{u}, \vec{v}) + b(\vec{v}, p) = 0 \quad \forall \vec{v} \in H^1(\Omega) \\ b(\hat{u}, q) = 0 \quad \forall q \in L_0^2(\Omega) \\ \hat{u}(t, \vec{x}) = s\tilde{h} \quad \forall \vec{x} \in \Gamma \quad t \in (0, T) \\ \hat{u}(0, \vec{x}) = 0 \quad \vec{x} \in \Omega. \end{cases} \tag{5.45}$$

Again in order to solve this system we set $\hat{u}_2 = \hat{u} - \hat{u}_1$ and decompose the nonhomogeneous Navier-Stokes equation into two systems: a linear system defined by a Stokes problem

$$\begin{cases} < \hat{u}_{1t}, \vec{v} > + \nu a(\hat{u}_1, \vec{v}) + b(\vec{v}, p) = 0 \qquad \forall \vec{v} \in H_0^1(\Omega) \\ b(\hat{u}_1, q) = 0 \quad \forall q \in L_0^2(\Omega) \\ \hat{u}_1 = s\tilde{h} \quad \forall \vec{x} \in \Gamma \quad t \in (0, T) \\ \hat{u}_1(0, \vec{x}) = u_0 \in H^1(\Omega). \end{cases} \tag{5.46}$$

and a Navier-Stokes system

$$\left\{\begin{array}{l} < \hat{u}_{2t}, \vec{v} > +\nu a(\hat{u}_2, \vec{v}) + c(\vec{u}_{\vec{g}} + \hat{u}_1; \hat{u}_2, \vec{v}) + c(\hat{u}_2; \vec{u}_{\vec{g}} + \hat{u}_1, \vec{v}) + \\ c(\hat{u}_2; \hat{u}_2, \vec{v}) = (\hat{f}, \vec{v}) \qquad \forall \vec{v} \in V(\Omega) \\ \hat{u}_2 \in V(\Omega) \\ \hat{u}_2 = 0 \quad on \quad \Gamma_c \end{array}\right. \tag{5.47}$$

where $\hat{f} = -c(\hat{u}_1; \hat{u}_1, \vec{v})$. We have $\hat{u}_1 \in L^\infty((0,T); L^2(\Omega))$ and

$$\int_0^T \|\hat{u}_1\|_1^2 dt \leq C_1 |s|^2 \|\tilde{h}\|^2_{H^{1,1}(S)} \quad \forall t \in (0,T)$$

and thus $\|\hat{f}\|_{L^2((0,T);H^{-1})} \leq C_2 |s|^2 \|\tilde{h}\|^2_{H^{1,1}(S)}$. By using eq(5.40), we have

$$\int_0^T \|\hat{u}_2\|_1^2 dt \leq C_3 \int_0^T \|\nabla \hat{u}_1\|^2 dt.$$

As both $\hat{u}_1$ and $\hat{u}_2$ are in $L^\infty((0,T); L^2(\Omega))$, our claim will follow from the estimate

$$\int_0^T \|\tilde{u}(t)\|_1^2 dt \leq C_4 \int_0^T \|\vec{u}_{\vec{g}+s\tilde{h}} - \vec{u}_{\vec{g}}\|_1^2 dt \leq C_5 |s|^2 \|\tilde{h}\|^2_{H^{1,1}(S)}. \tag{5.48}$$

From the regularity of $\tilde{h}$ it follows that $\tilde{w} \in L^\infty((0,T); L^2(\Omega)) \cap L^2((0,T); H^1(\Omega))$. $\sqcup$

The canonical extension $\tilde{h}_c \to \tilde{h}$ from $H^1_{n0}(\Gamma_c)$ to $H^1(\Gamma)$ where

$$\tilde{h} = \left\{\begin{array}{ll} \tilde{h}_c & \vec{x} \in \Gamma_c \\ 0 & \vec{x} \in \Gamma \setminus \Gamma_c \end{array}\right.$$

is a continuous mapping (see [38]). This allows us to take variations in subdomains of the boundary (i.e., $H^1(\Gamma_c)$) and to claim the existence of the Gateaux derivative for such configurations. For a variation $\tilde{h}_c \in H^{1,1}(S_c) \cap H^1_{n0}(\Gamma_c)$ (i.e., $\tilde{h}_c \in L^2((0,T); H^1_{n0}(\Gamma_c))$ and $\tilde{h}_{ct} \in L^2((0,T); L^2(\Gamma_c))$) of the control $\vec{g}$, the Gateaux derivative of the Navier-Stokes system can be written in this following form

$$\left\{\begin{array}{l} < \tilde{w}_t, \vec{v} > +\nu a(\tilde{w}, \vec{v}) + c(\tilde{w}; \vec{u}, \vec{v}) + c(\vec{u}; \tilde{w}, \vec{v}) + b(\vec{v}, p_1) = 0 \quad \forall \vec{v} \in H^1_0(\Omega) \\ b(\tilde{w}, q) = 0 \quad \forall q \in L^2_0(\Omega) \\ (\tilde{w}, \vec{s})_\Gamma = (\tilde{h}_c(t, \vec{x}), \vec{s})_{\Gamma_c} \quad \forall \vec{s} \in H^{-1/2}(\Gamma) \\ \tilde{w}(0, \vec{x}) = 0 \quad \vec{x} \in \Omega. \end{array}\right. \tag{5.49}$$

We have this interesting preliminary result.

Lemma 5.4 *Given $\Omega \in C^2$ and $\vec{u}_0 \in curl(H^2)(\Omega)$. Let $\tilde{h}_c$ be given in $H^{1,1}(S_c) \cap H^1_{n0}(\Gamma_c)$ and let $\tilde{w}(\tilde{h}_c)$ be defined in eq(5.49). For every $\tilde{h}_2$ in $L^2((0,T);H^1(\Omega))$, we have*

$$\int_0^T \int_\Omega \tilde{h}_2 \tilde{w}(\tilde{h}_c) d\vec{x}dt = -\int_0^T \int_{\Gamma_c} (\nu \tilde{h}_c \cdot \gamma_1 \vec{w}(\tilde{h}_2) + \sigma \tilde{h}_c \cdot \vec{n}) d\vec{x}dt$$

where $\vec{w}$ is the solution of the adjoint linearized problem

$$\begin{cases} -(\vec{w}_t, \vec{v}) + \nu a(\vec{w}, \vec{v}) + c(\vec{v}; \vec{u}, \vec{w}) + c(\vec{u}; \vec{v}, \vec{w}) + b(\vec{v}, \sigma) = (\tilde{h}_2, \vec{v}) \quad \forall \vec{v} \in H^1_0(\Omega) \\ b(\vec{w}, q) = 0 \quad \forall q \in L^2_0(\Omega) \\ \vec{w} = 0 \quad \forall \vec{w} \in \Gamma \\ \vec{w}(T, \vec{x}) = 0. \end{cases} \tag{5.50}$$

The function $\vec{\tau} = -(\gamma_1 \vec{w} + \sigma \vec{n}) \in L^2((0,T); H^{-1/2}(\Gamma_c))$ is defined by

$$\int_{\Gamma_c} \vec{\tau} \cdot \vec{v} d\vec{x} = -(\vec{w}_t, \vec{v}) - \nu a(\vec{w}, \vec{v}) + c'^*(\vec{w}; \vec{u}, \vec{v}) + b(\vec{v}, \sigma) - (\tilde{h}_2, \vec{v}) \quad \forall \vec{v} \in H^1(\Omega). \tag{5.51}$$

Proof: We remark that $\tilde{h}_2$ is in $L^2((0,T);H^1(\Omega))$ then, the eq(5.50) has a solution $\vec{w}$ in $L^\infty((0,T);W(\Omega)) \cap L^2((0,T);V(\Omega))$ and $\sigma \in L^2((0,T);L^2_0(\Omega))$. The definition in eq(5.51) makes sense as the weak formulation of the adjoint equation tested against $H^1(\Omega)$ has solution for all $\tilde{h}_2 \in (H^1)^*(\Omega)$ (one can use the same techiniques in [31]). Hence there exists a $\vec{\tau} \in H^{-1/2}(\Gamma_c)$ defined by eq(5.51). We need to evaluate the integral over time of $(\tilde{w}, \tilde{h}_2)$. The integral contains $\tilde{h}_2$ for which we can use eq(5.51). We set $\vec{v} = \tilde{w}$ in that equation and then, we can proceed by integration by parts with respect to the time variable. Therefore we have

$$\begin{aligned} &\int_0^T (\tilde{w}, \tilde{h}_2) dt = \\ &\int_0^T [-(\tilde{w}, \vec{w}_t) + \nu a(\tilde{w}, \vec{w}) + c'^*(\vec{w}; \vec{u}; \tilde{w}) + b(\tilde{w}, \sigma)] dt - \int_0^T \int_{\Gamma_c} \vec{\tau} \cdot \tilde{h}_c d\vec{x}dt = \\ &\int_0^T [(\tilde{w}_t, \vec{w}) + \nu a(\tilde{w}, \vec{w}) + c'(\tilde{w}, \vec{u}, \vec{w})] dt - \int_0^T \int_{\Gamma_c} \vec{\tau} \cdot \tilde{h}_c d\vec{x}dt \end{aligned}$$

as we have $c'(\tilde{w}, \vec{u}, \vec{w}) = c'^*(\vec{w}, \vec{u}, \tilde{w})$ and $b(\tilde{w}, \sigma) = 0$ from eq(5.49). The proof follows from the fact that the first term vanishes satisfying the weak equation for the Gateaux derivative with $\vec{v} = \vec{w}$ ($b(\vec{w}, p_1) = 0$ from eq.(5.50)). $\sqcup$

In the next theorem we will show that if the Gateaux derivative vanishes, then $\vec{g}$ must be solution of a differential equation.

Theorem 5.7 *If $(\vec{u}, \vec{g})$ is an optimal pair for the problem in eq(5.21), then $\vec{g} \in H^{1,1}(S_c) \cap H^1_{n0}(\Gamma_c)$ is solution of this following equation*

$$\int_0^T \int_{\Gamma_c} [\vec{g} \cdot \tilde{h} + \beta_1 \vec{g}_t \cdot \tilde{h}_t + \beta_2 \nabla_s \vec{g} \cdot \nabla_s \tilde{h} - \frac{1}{\beta} (\vec{\tau} \cdot \tilde{h})] d\vec{x}dt = 0 \tag{5.52}$$

for all $\tilde{h} \in H^{1,1}(S_c) \cap H^1_{n0}(\Gamma_c)$. *The function* $\vec{w}$ *is in* $L^\infty((0,T); L^2(\Omega)) \cap L^2((0,T); H^1(\Omega))$ *and solution of this adjoint linearized problem*

$$\begin{cases} -(\vec{w}_t, \vec{v}) + \nu a(\vec{w}, \vec{v}) + c(\vec{u}; \vec{v}, \vec{w}) + c(\vec{v}; \vec{u}, \vec{w}) + b(\vec{v}, \sigma) = \alpha(\vec{u} - \vec{U}, \vec{v}) \quad \forall \vec{w} \in H^1_0(\Omega) \\ b(\vec{w}, q) = 0 \quad \forall q \in L^2_0(\Omega) \\ \vec{w} = 0 \quad \forall \vec{x} \in \Gamma \\ \vec{w}(T, \vec{x}) = 0 \end{cases} \tag{5.53}$$

and the quantity $\vec{\tau} = -(\gamma_1 \vec{w} + \sigma \vec{n})$ *defined by*

$$\begin{aligned} &\int_{\Gamma_c} \vec{\tau} \cdot \vec{v} d\vec{x} = -\int_{\Gamma_c} (\nu \gamma_1 \vec{w} \cdot \gamma_0 \vec{v} + \sigma \gamma_0 \vec{v} \cdot \vec{n}) d\vec{x} = \\ &-(\vec{w}_t, \vec{v}) + \nu a(\vec{w}, \vec{v}) + c(\vec{u}; \vec{v}, \vec{w}) + c(\vec{v}; \vec{u}, \vec{w}) + b(\vec{v}, \sigma) - (\tilde{h}_2, \vec{v}) \quad \forall \vec{v} \in H^1(\Omega). \end{aligned} \tag{5.54}$$

Proof: Let $(\vec{u}, \vec{g})$ be an optimal pair solution of the problem defined in eq (5.21). We compute the Gateaux derivative of the functional $L(\vec{g})$ in the direction of $\tilde{h}$, and then lemma 5.4 completes the proof. We have

$$\begin{aligned} &\frac{DL(\vec{g})}{D\vec{g}} \cdot \tilde{h} = \\ &\alpha \int_0^T \int_\Omega (\vec{u} - \vec{U}) \left(\frac{D\vec{u}}{D\vec{g}} \cdot \tilde{h} \right) d\vec{x} dt + \beta \int_0^T \int_{\Gamma_c} [\vec{g} \cdot \tilde{h} + \beta_1 \vec{g}_t \cdot \tilde{h}_t + \beta_2 \nabla_s \vec{g} \cdot \nabla_s \tilde{h}] d\vec{x} dt \end{aligned}$$

Now using lemma 5.4, we can integrate by parts to obtain

$$\begin{aligned} &\frac{DL(\vec{g})}{D\vec{g}} \cdot \tilde{h} = \alpha \int_0^T \int_\Omega (\vec{u} - \vec{U}) \tilde{w} d\vec{x} dt + \beta \int_0^T \int_{\Gamma_c} [\vec{g} \cdot \tilde{h} + \beta_1 \vec{g}_t \cdot \tilde{h}_t + \beta_2 \nabla_s \vec{g} \cdot \nabla_s \tilde{h}] d\vec{x} dt = \\ &\int_0^T \int_{\Gamma_c} [\beta(\vec{g} \cdot \tilde{h} + \beta_1 \vec{g}_t \cdot \tilde{h}_t + \beta_2 \nabla_s \vec{g} \cdot \nabla_s \tilde{h}) - (\vec{\tau} \cdot \tilde{h})] d\vec{x} dt \end{aligned}$$

where $\vec{w}$ is the solution of the system in eq(5.53). Now from Theorem 5.5, if $(\vec{u}, \vec{g})$ is a solution of the optimal problem the Gateaux derivative must be zero. The regularity of $\vec{g}$ follows from the regularity proprieties shown by $\gamma_1 \vec{w}$ $\sqcup$.

The eq(5.52) gives the solution for the boundary control. Since $\tilde{h} \in H^{1,1}(S_c) \cap H^1_{n0}(\Gamma_c)$ we can take $\tilde{h} = \psi(t) \vec{r}(\vec{x})$ where $\psi \in \mathcal{D}((0,T))$ with $\psi(0) = 0$ and $\vec{r}(\vec{x})$ in $H^1_{n0}(\Gamma_c)$. After integration by parts we have

$$\int_0^T \psi(t) [(\vec{g}, \vec{r}) - \beta_1 (\vec{g}_{tt}, \vec{r}) + \beta_2 (\nabla_s \vec{g}, \nabla_s \vec{r}) - \frac{1}{\beta} (\vec{\tau}, \vec{r})] dt = 0 \tag{5.55}$$

for all $\psi \in \mathcal{D}((0,T))$ and for all $\vec{r} \in H^1_{n0}(\Gamma_c)$. In the integration by parts we assume that $\vec{g}(0,\vec{x}) = \gamma_0 \vec{u}_0$ ($\tilde{h}(0,\vec{x}) = 0$) and $\vec{g}_t(T,\vec{x}) = 0$. In this way the eq(5.52) is equivalent to

$$(\vec{g},\vec{r}) - \beta_1(\vec{g}_{tt},\vec{r}) + \beta_2(\nabla_s\vec{g},\nabla_s\vec{r}) - \frac{1}{\beta}(\vec{\tau},\vec{r}) = 0 \quad \forall \vec{r} \in H^1_{n0}(\Gamma_c) \tag{5.56}$$

in a distribution sense over $(0,T)$ with $\vec{g}(0,\vec{x}) = \gamma_0\vec{u}_0(\vec{x})$ and $\vec{g}_t(T,\vec{x}) = 0$. Now we can use the space $H^1_n(\Gamma_c)$ to test the equation. Because of the orthogonality between $H^1_n(\Gamma_c)$ and $(H^1_n)^\perp(\Gamma_c)$ we can write a weak formuation of the eq(5.56) with test functions in $H^1(\Gamma_c)$ by adding an arbitrary costant vector in the normal direction. We recall that $H^1(\Gamma_c) = H^1_n(\Gamma_c) \oplus (H^1_n)^\perp(\Gamma_c)$ and thus

$$\vec{r} = \vec{r}_1 - \vec{n}\frac{\int_{\Gamma_c}\vec{r}_1\cdot\vec{n}d\vec{x}}{\mu(\Gamma_c)}$$

where $\vec{r}_1 \in H^1(\Gamma_c)$. Now the equation can be tested against $\vec{r}_1 \in H^1_0(\Gamma_c)$ in the following weak form

$$(\vec{g},\vec{r}_1) - \beta_1(\vec{g}_{tt},\vec{r}_1) + \beta_2(\nabla_s\vec{g},\nabla_s\vec{r}_1) + k(t)(\vec{n},\vec{r}_1) = \frac{1}{\beta}(\vec{\tau},\vec{r}_1) \quad \forall \vec{r}_1 \in H^1_0(\Gamma_c) \tag{5.57}$$

where $k(t)$ is specified by the costraint

$$\int_\Gamma \vec{g}\cdot\vec{n}d\vec{x} = 0.$$

Finally in order to obtain the solution of our optimal control problem we have to solve the Navier-Stokes system

$$\begin{cases} <\vec{u}_t,\vec{v}> + \nu a(\vec{u},\vec{v}) + c(\vec{u};\vec{u},\vec{v}) + b(\vec{v},p) = 0 \quad \forall \vec{v} \in H^1_0(\Omega) \\ b(\vec{u},q) = 0 \quad \forall q \in L^2_0(\Omega) \\ (\vec{u},\vec{s})_\Gamma = (g(t,\vec{x}),\vec{s})_{\Gamma_c} \quad \forall \vec{s} \in H^{-1/2}(\Gamma) \\ \vec{u}(0,\vec{x}) = \vec{u}_0(\vec{x}) \quad \vec{x} \in \Omega, \end{cases} \tag{5.58}$$

the adjoint system

$$\begin{cases} - <\vec{w}_t,\vec{v}> + \nu a(\vec{w},\vec{v}) + c(\vec{w};\vec{u},\vec{v}) + c(\vec{u};\vec{w},\vec{v}) = -\alpha(\vec{u}-\vec{U},\vec{v}) \quad \forall \vec{v} \in H^1_0(\Omega) \\ b(\vec{w},q) = 0 \quad \forall q \in L^2_0(\Omega) \\ \vec{w} = 0 \quad \forall \vec{x} \in \Gamma \quad t \in (0,T) \\ \vec{w}(T,\vec{x}) = 0 \quad \vec{x} \in \Omega, \end{cases} \tag{5.59}$$

the boundary control equation

$$\begin{cases} (\vec{g},\vec{r}) - \beta_1(\vec{g}_{tt},\vec{r}) + \beta_2(\nabla_s\vec{g},\nabla_s\vec{r}) + k(t)(\vec{n},\vec{r}) = -\frac{1}{\beta}[(\gamma_1\vec{w},\vec{r}) + (\vec{n}\sigma,\vec{r})] \quad \forall \vec{r} \in H^1_0(\Gamma_c) \\ \beta_1\vec{g}(0,\vec{x}) = \gamma_0\vec{u}_0 \quad \vec{x} \in \Gamma_c \\ \beta_1\vec{g}_t(T,\vec{x}) = 0 \quad \vec{x} \in \Gamma_c \\ \beta_2\vec{g} = 0 \quad \forall \vec{x} \in \partial\Gamma_c \end{cases} \tag{5.60}$$

and the compatibility equation

$$(\vec{g}, \vec{n})_\Gamma = 0. \tag{5.61}$$

The eq(5.61) is needed in order to calculate the variable $k(t)$. The above system of equations is a weak formulation of the following system

$$\begin{cases} \vec{u}_t(t,\vec{x}) + (\vec{u}.\vec{\nabla})\vec{u} - \nu\nabla^2\vec{u} + \vec{\nabla}p(t,\vec{x}) = 0 \quad in \quad \Omega \\ \nabla \cdot \vec{u} = 0 \quad in \quad \Omega \\ \vec{u} = g(t,\vec{x}) \quad on \quad \Gamma_c \\ \vec{u} = 0 \quad on \quad \Gamma \setminus \Gamma_c \\ \vec{u}(0,\vec{x}) = \vec{u}_0(\vec{x}) \quad on \quad \Omega \end{cases} \tag{5.62}$$

$$\begin{cases} -\vec{w}_t(t,\vec{x}) + \nu\nabla^2\vec{w} + (\nabla\vec{u})^T\vec{w} - (\vec{u}\cdot\nabla)\vec{w} + \vec{\nabla}\sigma = -\alpha(\vec{u} - \vec{U}) \quad in \quad \Omega \\ \nabla \cdot \vec{w} = 0 \quad in \quad \Omega \\ \vec{w} = 0 \quad on \quad \Gamma \\ \vec{w}(T,\vec{x}) = 0 \quad in \quad \Omega \end{cases} \tag{5.63}$$

$$\begin{cases} -\beta_1\vec{g}_{tt} - \beta_2\nabla_s^2\vec{g} + \vec{g} + k(t)\vec{n} = -\frac{1}{\beta}(\frac{\partial\vec{w}}{\partial n} + \sigma\vec{n}) \quad on \quad \Gamma_c \\ \int_\Gamma \vec{g}\cdot\vec{n}d\vec{x} = 0 \\ \beta_1\vec{g}(0,\vec{x}) = \vec{u}_0 \quad on \quad \Gamma_c \\ \beta_1\vec{g}_t(T,\vec{x}) = 0 \quad on \quad \Gamma_c \\ \beta_2\vec{g} = 0 \quad \forall\vec{x} \in \partial\Gamma_c. \end{cases} \tag{5.64}$$

If the control is a tangential control then, the adjoint pressure and the term with $k(t)$ can be neglected.

5.3 Semidiscrete time approximation

5.3.1 Formulation of the semidiscrete time approximation optimal control

Let $\sigma_N = \{t_n\}_{n=0}^N$ be a partition of $[0,T]$ into equal intervals $\Delta t = T/N$ with $t_0 = 0$ and $t_N = T$. For each fixed Δt (or N) and for every quantity $q(t,\vec{x})$ we associate the corresponding set $\{q^{(n)}(\vec{x})\}_{n=0}^N$ and a continuous piecewise linear function $q^N = q^N(t,\vec{x})$ such as $q^N(t_n,\vec{x}) = q^{(n)}(\vec{x})$ for all $n = 0,1,..N$. We will denote with bold letters $\mathbf{q}$ the vector $(q^{(1)}, q^{(2)}, ...q^{(N)})$ of the discrete time components. Also the space X^N will be denoted as $\mathbf{X}$. On this partition we define the discrete target velocity as $\vec{U}^{(n)}(\vec{x}) = \vec{U}(t_n,\vec{x})$ for $n = 0,1...N$ when $\vec{U} \in U_{ad}$. Let Γ_c be part of the boundary on which we apply the boundary control $\vec{g}$ and

$$\mathbf{H^1_n}(\Gamma) = \{\vec{\mathbf{g}} \in \mathbf{H^1}(\Gamma) : \int_\Gamma \vec{g}^{(n)} \cdot \vec{n}d\vec{x} \quad n = 1,2,...N\},$$

$\mathbf{H^1_{n0}}(\Gamma) = \mathbf{H^1_0}(\Gamma_c) \cap \mathbf{H^1_n}(\Gamma_c)$ denote the spaces of all the functions that are compatible with the divergence free motion of the fluid. We remark that the subspaces $H^1_{n0}(\Gamma)$ and $H^1_n(\Gamma)$ are closed subspaces of $H^1(\Gamma)$. The space $H^1(\Gamma)$ can be decomposed in $H^1_n(\Gamma) \oplus (H^1_n)^{\perp}(\Gamma)$ where $(H^1_n)^{\perp}(\Gamma)$ is the space of constant vectors normal to the surface. We assume no slip boundary conditions on the rest of the boundary $\Gamma \setminus \Gamma_c$. Hence the component of the velocity $\vec{u}^{(n)}$ on the boundary is the canonical extension of $\vec{g}^{(n)}$ from $H^1_0(\Gamma_c)$ to $H^1(\Gamma)$. We recall that this extension is a continuous map. The state variables $\vec{u}^{(n)} \in H^1_0(\Omega)$ and $p^{(n)} \in L^2_0(\Omega)$ are constrained to satisfy the semidiscrete Navier-Stokes equations

$$\begin{cases} \frac{1}{\Delta t}(\vec{u}^{(n)} - \vec{u}^{(n-1)}, \vec{v}) + \nu a(\vec{u}^{(n)}, \vec{v}) + c(\vec{u}^{(n)}, \vec{u}^{(n)}, \vec{v}) + b(\vec{v}, p^{(n)}) = 0 \quad \forall \vec{v} \in H^1_0(\Omega) \\ b(\vec{u}^{(n)}, q) = 0 \quad \forall q \in L^2_0(\Omega) \\ (\vec{u}^{(n)}(\vec{x}), \vec{s})_\Gamma = (\vec{g}^{(n)}(\vec{x}), \vec{s})_{\Gamma_c} \quad \forall \vec{s} \in H^{-1/2}(\Gamma) \\ \text{for } n = 1, 2, ..N \\ \vec{u}^{(0)} = \vec{u}_0(\vec{x}) \in curl(H^2)(\Omega). \end{cases} \tag{5.65}$$

The admissibility set A_{ad} is defined by

$$A_{ad} = \{(\vec{\mathbf{u}}, \mathbf{p}, \vec{\mathbf{g}}) \in \mathbf{H^1}(\Omega) \times \mathbf{L^2_0}(\Omega) \times \mathbf{H^1_{n0}}(\Gamma_c) \quad \textit{such that eq(5.65) is satisfied}\}.$$

The optimization is achieved by mean of the minimization of the discretized functional

$$\begin{aligned} L^N = \quad & \frac{\alpha}{2} \sum_{n=1}^{N} \|\vec{u}^{(n)} - \vec{U}^{(n)}\|^2_\Omega \Delta t + \\ & \frac{\beta}{2} \sum_{n=1}^{N} [\|\vec{g}^{(n)}\|^2_{\Gamma_c} \Delta t + \beta_1 \|\nabla_s \vec{g}^{(n)}\|^2_{\Gamma_c} \Delta t + \beta_2 \|(\vec{g}^{(n)} - \vec{g}^{(n-1)})\|^2_{\Gamma_c}]. \end{aligned} \tag{5.66}$$

Of course, if Δt tends to zero, this functional tends to the corresponding continuous functional. The formulation of the problem P_L in the semidiscrete approximation becomes: *given $\Delta t = T/N$, $\vec{u}_0 \in curl(H^2)(\Omega)$ and $\vec{U} \in U_{ad}$, then $(\vec{\mathbf{u}}, \mathbf{p}, \vec{\mathbf{g}}) \in A_{ad}$ is called an optimal solution if there exists $\epsilon > 0$ such that*

$$L^N(\vec{\mathbf{g}}) \le L^N(\tilde{\mathbf{h}}) \quad \forall \tilde{h} \in \mathbf{H^1_{n0}} \quad \|\vec{g}^{(n)} - \tilde{h}^{(n)}\|_{\Gamma_c} \le \epsilon \quad n = 1, 2, ...N. \tag{5.67}$$

For the semidiscrete Navier-Stokes nonhomogeneous boundary problem one can prove the following theorem

Theorem 5.8 *Let $\Delta t = T/N$ and $\vec{u}_0 \in curl(H^2)(\Omega)$. Let $\epsilon > 0$ $\vec{\mathbf{g}} \in \mathbf{H^1_{n0}}(\Omega)$ such that $\sum_{i=1}^{N}(\|\vec{g}^{(n)}\|^2_1 \Delta t + \|\vec{g}^{(n)} - \vec{g}^{(n-1)}\|^2) \le \epsilon$ (i.e. $\vec{g}^N$ and $\vec{g}'^N$ are uniformly bounded by ϵ in $L^2((0,T); H^1(\Gamma_c)$ and in $L^2((0,T); L^2(\Omega))$ respectively). 7 Then, there exists a function*

$\vec{\mathbf{u}} \in \mathbf{H^1(\Omega)}$ *that is a solution of the system*

$$\begin{cases} \frac{1}{\Delta t}(\vec{u}^{(n)} - \vec{u}^{(n-1)}, \vec{v}) + \nu a(\vec{u}^{(n)}, \vec{v}) + c(\vec{u}^{(n)}, \vec{u}^{(n)}, \vec{v}) + b(\vec{v}, p^{(n)}) = 0 \quad \forall \vec{v} \in H_0^1(\Omega) \\ b(\vec{u}^{(n)}, q) = 0 \quad \forall q \in L_0^2(\Omega) \\ \vec{u}^{(n)}(\vec{x}) = \vec{g}^{(n)} \quad \vec{x} \in \Gamma_c \\ \vec{u}^{(n)}(\vec{x}) = 0 \quad \vec{x} \in \Gamma \setminus \Gamma_c \\ \ for\ n = 1, 2, ..N \\ \vec{u}^{(0)} = \vec{u}_0(\vec{x}) \in curl(H^2)(\Omega) \end{cases} \tag{5.68}$$

with the following estimates

$$\|\vec{u}^{(n)}\|_1^2 \leq K \quad n = 1, 2.., N \tag{5.69}$$

$$\sum_{n=1}^{N} \|\nabla \vec{u}^{(n)}\|^2 \Delta t \leq K \tag{5.70}$$

$$\sum_{n=1}^{N} \|\vec{u}^{(n)} - \vec{u}^{(n-1)}\|_{H^{-1}}^2 \leq K \tag{5.71}$$

where the constant K is independent of Δt.

Proof: We set $\hat{u} = \vec{u} - \tilde{u}$. The variable $\tilde{u}$ satisfies a linear system

$$\begin{cases} \frac{1}{\Delta t}(\tilde{u}^{(n)} - \tilde{u}^{(n-1)}, \vec{v}) + \nu a(\tilde{u}^{(n)}, \vec{v}) + b(\vec{v}, p^{(n)}) = 0 \quad \forall \vec{v} \in H_0^1(\Omega) \\ b(\tilde{u}^{(n)}, q) = 0 \quad \forall q \in L_0^2(\Omega) \\ \tilde{u}^{(n)}(\vec{x}) = \vec{g}^{(n)} \quad \vec{x} \in \Gamma_c \\ \tilde{u}^{(n)}(\vec{x}) = 0 \quad \vec{x} \in \Gamma \setminus \Gamma_c \\ \ \text{for } n = 1, 2, ..N \\ \tilde{u}^{(0)} = \vec{u}_0(\vec{x}) \in curl(H^2)(\Omega) \end{cases} \tag{5.72}$$

and the function $\hat{u}$ is the solution of

$$\begin{cases} \frac{1}{\Delta t}(\hat{u}^{(n)} - \hat{u}^{(n-1)}, \vec{v}) + \nu a(\hat{u}^{(n)}, \vec{v}) + c(\hat{u}^{(n)}, \hat{u}^{(n)}, \vec{v}) + \quad \forall \vec{v} \in H_0^1(\Omega) \\ c(\vec{u}^{(n)}, \vec{u}^{(n)}, \vec{v}) + c(\vec{u}^{(n)}, \vec{u}^{(n)}, \vec{v}) + b(\vec{v}, p^{(n)}) = (\hat{f}, \vec{v}) \quad \forall \vec{v} \in H_0^1(\Omega) \\ b(\hat{u}^{(n)}, q) = 0 \quad \forall q \in L_0^2(\Omega) \\ \hat{u}^{(n)}(\vec{x}) = 0 \quad \vec{x} \in \Gamma \\ \ \text{for } n = 1, 2, ..N \\ \tilde{u}^{(0)} = 0 \end{cases} \tag{5.73}$$

when $\hat{f} = -c(\tilde{u}, \tilde{u}, \vec{v})$. We want to solve the linear system and for this purpose we lift the boundary condition. As $\vec{g}^N \in L^2((0,T); H^{1/2}(\Omega))$ and $\vec{g}'^N \in L^2((0,T); L^2(\Omega))$ there exists a function $\tilde{w}^N \in H^{1,1}(Q)$ such that $\gamma_0 \tilde{w}^N = \vec{g}^N$ and $r_0 \tilde{w}^N = \vec{u}_0^N$. We set $\vec{w} = \tilde{u} - \tilde{w}$ and write

$$\begin{cases} \frac{1}{\Delta t}(\vec{w}^{(n)} - \vec{w}^{(n-1)}, \vec{v}) + \nu a(\vec{w}^{(n)}, \vec{v}) + b(\vec{v}, p^{(n)}) = (\tilde{f}, \vec{v}) \quad \forall \vec{v} \in H_0^1(\Omega) \\ b(\vec{w}^{(n)}, q) = 0 \quad \forall q \in L_0^2(\Omega) \\ \vec{w}^{(n)}(\vec{x}) = 0 \quad \vec{x} \in \Gamma \\ \text{ for } n = 1, 2, ..N \\ \vec{w}^{(0)} = 0 \end{cases}$$

where $\tilde{f}^{(n)} \in H^{-1}(\Omega)$ for all $n = 1, 2, ...N$. As $\vec{w}^N$ is in $H^{1,1}(Q)$ is also in $L^\infty((0,T); L^2(\Omega))$. By using the continuity of the lift we find that $\|\tilde{f}^N\|_{L^2((0,T);H^{-1})} \leq C\|\vec{g}^N\|_{H^{1/2,1}}$ and $\|\vec{w}^{(n)}\|$, $\sum_{n=1}^N \|\nabla\vec{u}^{(n)}\|\Delta t$ are uniformly bounded. The lift satisfies these estimates and thus $\tilde{u} = \tilde{w} + \vec{w}$. Now by applying the standard techniques to the non linear system we can prove eq(5.69 - 5.71) as $\sum_{n=1}^N \|\hat{f}^{(n)}\|^2_{H^{-1}(\Omega)}\Delta t \leq K \sum_{n=1}^N \|\hat{g}^{(n)}\|^2_{H^{1/2}(\Gamma)}\Delta t$. $\sqcup$.

Hence $\vec{u}^N \in L^\infty((0,T); H^1(\Omega))$ and $\vec{u}'^N \in L^2((0,T); H^{-1}(\Omega))$ for all N. Also if $\vec{g}^N \to \vec{g} \in L^2((0,T); H^1(\Omega)) \cap L^\infty((0,T); L^2(\Omega))$ then $\vec{u}^N \to \vec{u}$ where $\vec{u}$ is the solution of the continuous Navier-Stokes system of equations.

5.3.2 Existence and consistency for the semidiscrete optimal control problem

If $\vec{\mathbf{g}}$ and its time derivative are uniformly bounded then, the existence of solutions of the optimal control problem can be proved. This fact is an easy consequence of the definition of the optimal control problem and the boundness of the functional.

Lemma 5.5 *Let $\Delta t = T/N$, $\vec{u}_0 \in curl(H^2)(\Omega)$ and $\vec{U} \in U_{ad}$. If $(\vec{\mathbf{u}}, \vec{\mathbf{g}})$ is the solution of the semidiscrete optimal control problem then, for all β_1 and $\beta_2 > 0$ there exists a constant C independent of Δt such that*

$$\sum_{n=1}^N \|\vec{g}^{(n)}\|^2_{1,\Gamma}\Delta t \leq C \tag{5.74}$$

$$\sum_{n=1}^N \|\vec{g}^{(n)} - \vec{g}^{(n-1)}\|^2_\Gamma \leq C \tag{5.75}$$

$$\sum_{n=1}^N \|\vec{u}^{(n)}\|^2_\Omega\Delta t \leq C. \tag{5.76}$$

Hence we have $\vec{g}^N \in L^2((0,T); H^1(\Gamma))$, $\vec{g}'^N \in L^2((0,T); L^2(\Gamma))$ and $\vec{u}^N \in L^2((0,T); W(\Omega))$ for all N.

Proof: Let $\vec{\mathbf{g}}$ be zero and $\vec{\mathbf{u}}$ be the solution of eq(5.65). Using the result in eq(5.70) the functional yields

$$\begin{aligned} L^N(0) &= \frac{\alpha\Delta t}{2}\sum_{n=1}^{N}\|\vec{u}^{(n)}-\vec{U}^{(n)}\|^2 \leq \\ &\frac{\alpha T}{2}\|\vec{U}\|^2_{L^\infty((0,T);V)}+\frac{\alpha\Delta t}{2}\sum_{n=1}^{N}\|\vec{u}^{(n)}\|^2 \leq \frac{T\alpha}{2}\left(\|\vec{U}\|^2_{L^\infty((0,T);V)}+\|\vec{u}_0\|^2\right)=C_1. \end{aligned}$$

where C_1 is independent of Δt. Now if $(\vec{\mathbf{u}},\vec{\mathbf{g}})$ is a solution of our optimal control problem then $L(\vec{\mathbf{g}}) \leq L(0)$. From this inequality we have

$$\frac{\alpha\Delta t}{2}\sum_{n=1}^{N}\|\vec{u}^{(n)}-\vec{U}^{(n)}\|^2+\frac{\beta\Delta t}{2}\sum_{n=1}^{N}(\|\vec{g}^{(n)}\|^2_\Gamma+\beta_1\|\nabla_s\vec{g}^{(n)}\|^2_\Gamma+\beta_2\|\vec{g}^{(n)}-\vec{g}^{(n-1)}\|^2_\Gamma)\leq L(0)\leq C_1.$$

From the above inequality follows eq(5.74 - 5.76). $\sqcup$

We can recall that if the norm of $\vec{g}^N \in L^2((0,T);H^1(\Gamma))$ and the norm of $\vec{g}'^N$ in $L^2((0,T);L^2(\Gamma))$ are uniformly bounded for all N then, $\vec{g}^N$ is uniformly bounded in $L^2((0,T);L^2(\Gamma))$ for all N. Now we can state and prove the existence for the optimal control problem in an open bounded domain Ω with boundary Γ in C^2.

Theorem 5.9 *Given $\Delta t = T/N$, $\vec{u}_0 \in curl(H^2)(\Omega)$ and $\vec{U} \in U_{ad}$, there exists a pair $(\vec{\mathbf{u}},\vec{\mathbf{g}})$ in $(\mathbf{H^1}(\Omega)\times\mathbf{H^1_{n0}}(\Gamma))$ such that $\vec{\mathbf{u}}$ is the solution of eq(5.65) and $\vec{\mathbf{g}}$ minimizes the cost functional.*

Proof: Let $\Delta t = T/N$ and $\{\vec{\mathbf{g}}_k\}_{k=1}^\infty$ be a minimizing sequence in $\mathbf{H^1_{n0}}(\Gamma_c)$. Using Theorem 5.8 and the result in eq(5.74 - 5.75), we find that the corresponding sequence $\vec{\mathbf{u}}_k$ is uniformly bounded in $\mathbf{H^1}(\Omega)$. Now we can proceed with a weakly convergent subsequence and show that this subsequence converges to the solution of the optimal control problem in the semidiscrete approximation. We can write

$$\begin{array}{llll} \vec{g}_k^{(n)} \rightarrow \vec{g}^{(n)} & in & H_0^1(\Gamma_c) & weakly \\ \vec{u}_k^{(n)} \rightarrow \hat{u}^{(n)} & in & H^1(\Omega) & weakly \end{array}$$

for $n = 1,2,...N$. By using the fact that the injection of $H^1(\Omega)$ into $L^2(\Omega)$ is compact, the subsequence converges strongly. The lower semicontinuity of the functional in eq(5.66) allows the pair $(\vec{\mathbf{u}},\vec{\mathbf{g}})$ to minimize the functional. Since we can pass to the limit in the linear and the nonlinear term, the pair also satisfies the Navier-Stokes eq(5.65). In fact, since $\vec{\mathbf{u}}_k$ converges to $\vec{\mathbf{u}}$ strongly in $\mathbf{L^2}(\Omega)$, then for any $\vec{\mathbf{z}} \in \mathcal{V}(\mathbf{\Omega})$ we have

$$\lim_{k\rightarrow\infty} c(\vec{\mathbf{u}}_k;\vec{\mathbf{u}}_k,\vec{\mathbf{z}}) = c(\vec{\mathbf{u}};\vec{\mathbf{u}},\vec{\mathbf{z}}).$$

Since $\mathcal{V}(\mathbf{\Omega})$ is dense in $\mathbf{V}(\Omega)$, this is still true for any $\vec{w}$ in $\mathbf{V}(\Omega)$ by a continuity argument. This allows us to pass to the limit in the semidiscrete equation and complete the proof. $\sqcup$

We recall that, since $H^1(\Omega) \subset L^2(\Omega) \subset H^{1*}(\Omega)$, where the injections are continuous and $H^1(\Omega) \to L^2(\Omega)$ is compact from the Sobolev imbedding theorem, then the injection from $\mathcal{Y}((0,T); 2, 2, H^1, H^{1*})$ into $L^2((0,T); L^2(\Omega))$ is compact. Hence if a sequence $\vec{v}_k$ converges weakly in $L^2((0,T); H^1(\Omega))$ and $\vec{v}_k'$ in $L^2((0,T); H^{1*}(\Omega))$ then $\vec{v}_k$ converges strongly in $L^2((0,T); L^2(\Omega))$. Now we can prove the consistency of our semidiscrete optimal control problem.

Theorem 5.10 *Given $\Delta t = T/N$, $\vec{U} \in U_{ad}$ and $\vec{u}_0 \in V(\Omega)$. For $\Delta T \to 0$ ($N \to \infty$) the solution $\{(\vec{u}^{(n)}, \vec{g}^{(n)})\}_{n=1}^N$ tends to the optimal control pair $(\vec{u}, \vec{g})$ solution of the corresponding continuous optimal control problem.*

Proof: Let $\Delta t = T/N$, $\vec{u}'^{(n)} = (\vec{u}^{(n)} - \vec{u}^{(n-1)})/\Delta t$, and $\vec{u}'^N$ the corresponding linear function. From Lemma 5.5 the sequences $\{\vec{g}^N\}_{N=1}^\infty \in L^2((0,T); H^1(\Gamma_c))$ and $\{\vec{g}'^N\}_{N=1}^\infty \in L^2((0,T); L^2(\Gamma_c))$ are are uniformly bounded. Therefore the functions $\{\vec{g}^N\}_{N=1}^\infty$, $\{\vec{u}^N\}_{N=1}^\infty$, and $\{\vec{u}'^N\}_{N=1}^\infty$ are uniformly bounded in $L^\infty((0,T); H^1_{n0}(\Gamma_c))$, $L^2((0,T); H^1(\Omega))$ and $L^2((0,T); H^{-1}(\Omega))$ respectively. Hence we can extract from these sequences some subsequences such as

$$\begin{cases} \vec{u}^K \to \vec{u} & L^2(0,T,H^1(\Omega)) \text{ weakly} \\ \vec{u}^K \to \vec{u} & L^\infty(0,T,L^2(\Omega)) \text{ *-weakly} \\ \vec{g}^K \to \vec{g} & L^2(0,T,H^1_{n0}(\Gamma_c)) \text{ weakly} \\ \vec{g}'^K \to \vec{g}' & L^2(0,T,L^2(\Gamma_c)) \text{ weakly} \\ \vec{u}'^K \to \vec{u}' & L^2((0,T);H^{-1}(\Omega)) \text{ weakly} \end{cases} \tag{5.77}$$

As a consequence of the compactness theorem the convergence of the sequence $\{\vec{u}^N\}_{N=1}^\infty$ is strong in $L^2((0,T); L^2(\Omega))$. Now we can pass to the limit in the system of equations and in the functional. The linear terms do not give problems. Using the fact that the sequence converges weakly in $L^2((0,T); H^1(\Omega))$ and strongly in $L^2((0,T); L^2(\Omega))$ we can pass to the limit in the nonlinear term. Thus the semidiscrete optimal control problem for $N \to \infty$ is consistent with the continuous one. $\sqcup$

5.3.3 First-order necessary condition

In this section we proceed to derive the first-order necessary condition in a different way. Let denote with $\mathbf{B_1}, \mathbf{B_2}$ the following sets

$$\begin{cases} \mathbf{B_1} = \mathbf{H^1}(\Omega) \times \mathbf{L^2_0}(\Omega) \times \mathbf{H^1_{n0}}(\Gamma_c) \\ \mathbf{B_2} = \mathbf{H^{-1}}(\Omega) \times \mathbf{L^2_0}(\Omega) \times \mathbf{H^{1/2}}(\Gamma_c) \end{cases} \tag{5.78}$$

We define the non linear map

$$M(\vec{\mathbf{u}}, \mathbf{p}, \vec{\mathbf{g}}) : \mathbf{B_1} \to \mathbf{B_2}$$

as $M(\vec{\mathbf{u}}, \mathbf{p}, \vec{\mathbf{g}}) = (\vec{\mathbf{f}}, \mathbf{z}, \vec{\mathbf{b}})$ if and only if

$$\begin{cases} \frac{1}{\Delta t}(\vec{u}^{(n)} - \vec{u}^{(n-1)}, \vec{v}) + \nu a(\vec{u}^{(n)}, \vec{v}) + c(\vec{u}^{(n)}, \vec{u}^{(n)}, \vec{v}) + b(\vec{v}, p^{(n)}) = (\vec{f}^{(n)}, \vec{v}) \quad \forall \vec{v} \in H_0^1(\Omega) \\ b(\vec{u}^{(n)}, q) = (z^{(n)}, q) \quad \forall q \in L_0^2(\Omega) \\ (\vec{u}^{(n)}, \vec{s})_\Gamma - (\vec{g}^{(n)}, \vec{s})_{\Gamma_c} = (\vec{b}^{(n)}, \vec{s})_\Gamma \quad \vec{s} \in H^{-1/2}(\Gamma) \\ \text{for } n = 1, 2, ..N \\ \vec{u}^{(0)} = \vec{u}_0(\vec{x}) \in curl(H^2)(\Omega). \end{cases} \tag{5.79}$$

In the same manner let $\hat{\mathbf{g}}$ be an optimal solution and define

$$N(\vec{\mathbf{u}}, \mathbf{p}, \vec{\mathbf{g}}) : \mathbf{B_1} \to \mathbb{R} \times \mathbf{B_2}$$

as $N(\vec{\mathbf{u}}, \mathbf{p}, \vec{\mathbf{g}}) = (a, \vec{\mathbf{f}}, \mathbf{z}, \vec{\mathbf{b}})$ if and only if

$$\begin{pmatrix} L(\vec{\mathbf{g}}) - L(\hat{\mathbf{g}}) \\ M(\vec{\mathbf{u}}, \mathbf{p}, \vec{\mathbf{g}}) \end{pmatrix} = \begin{pmatrix} a \\ (\vec{\mathbf{f}}, \mathbf{z}, \vec{\mathbf{b}}). \end{pmatrix} \tag{5.80}$$

Thus, the constraints can be expressed as $M(\vec{\mathbf{u}}, \mathbf{p}, \vec{\mathbf{g}}) = (0, 0, 0)$ and the optimal problem can be reformulated as:
find $(\hat{\mathbf{u}}, \hat{\mathbf{p}}, \hat{\mathbf{g}})$, $\epsilon > 0$ *and* $a \leq 0$ *such that the equation* $N(\vec{\mathbf{u}}, \mathbf{p}, \hat{\mathbf{g}}) = (a, 0, 0, 0)$ *is satisfied* $\forall \vec{\mathbf{g}}$ *such that* $\|\vec{g}^{(n)} - \hat{g}^{(n)}\| \leq \epsilon$ *for* $n = 1, 2, ...N$. Since we are looking for local minimum points it is natural to define for small perturbations the operator $M'(\vec{\mathbf{u}}, \mathbf{p}, \vec{\mathbf{g}})$ and $N'(\vec{\mathbf{u}}, \mathbf{p}, \hat{\mathbf{g}})$. Given a $(\vec{\mathbf{u}}, \mathbf{p}, \vec{\mathbf{g}})$ we define the linear operator

$$M'(\vec{\mathbf{u}}, \mathbf{p}, \vec{\mathbf{g}}) : \mathbf{B_1} \to \mathbf{B_2}$$

as $M'(\vec{\mathbf{u}}, \mathbf{p}, \vec{\mathbf{g}}) \cdot (\tilde{\mathbf{w}}, \mathbf{r}, \tilde{\mathbf{h}}) = (\bar{\mathbf{f}}, \bar{\mathbf{z}}, \bar{\mathbf{b}})$ if and only if

$$\begin{cases} \frac{1}{\Delta t}(\tilde{w}^{(n)} - \tilde{w}^{(n-1)}, \vec{v}) + \nu a(\tilde{w}^{(n)}, \vec{v}) + c(\tilde{w}^{(n)}, \vec{u}^{(n)}, \vec{v}) + \\ c(\vec{u}^{(n)}, \tilde{w}^{(n)}, \vec{v}) + b(\vec{v}, r^{(n)}) = (\bar{f}^{(n)}, \vec{v}) \quad \forall \vec{v} \in H_0^1(\Omega) \\ b(\tilde{w}^{(n)}, q) = (\bar{z}^{(n)}, q) \quad \forall q \in L_0^2(\Omega) \\ (\tilde{w}^{(n)}, \vec{s})_\Gamma - (\tilde{h}^{(n)}, \vec{s})_{\Gamma_c} = (\bar{b}^{(n)}, \vec{s})_\Gamma \quad \vec{s} \in H^{-1/2}(\Gamma) \\ \text{for } n = 1, 2, ..N \\ \vec{u}^{(0)} = \vec{u}_0(\vec{x}) \in curl(H^2)(\Omega). \end{cases} \tag{5.81}$$

Let

$$N'(\vec{\mathbf{u}}, \mathbf{p}, \vec{\mathbf{g}}) : \mathbf{B_1} \to \mathbb{R} \times \mathbf{B_2}$$

be defined as $N'(\vec{\mathbf{u}}, \mathbf{p}, \vec{\mathbf{g}}) \cdot (\tilde{a}, \tilde{\mathbf{w}}, \mathbf{r}, \tilde{\mathbf{h}}) = (\bar{a}, \bar{\mathbf{f}}, \bar{\mathbf{z}}, \bar{\mathbf{b}})$ if and only if

$$\left(\begin{array}{l} L'(\vec{\mathbf{u}}, \mathbf{p}, \vec{\mathbf{g}})) \cdot (\tilde{a}, \tilde{\mathbf{w}}, \mathbf{r}, \tilde{\mathbf{h}})) \\ M(\vec{\mathbf{u}}, \mathbf{p}, \vec{\mathbf{g}}) \cdot (\tilde{a}, \tilde{\mathbf{w}}, \mathbf{r}, \tilde{\mathbf{h}})) \end{array} \right) = \left(\begin{array}{l} \bar{a} \\ (\bar{\mathbf{f}}, \bar{\mathbf{z}}, \bar{\mathbf{b}}). \end{array} \right) \tag{5.82}$$

Now we have to prove that these operators are well defined, i.e., the equations for the Gateaux derivatives are well posed and have solutions.

Lemma 5.6 *Given $\Delta t = T/N$, $\vec{u}_0 \in curl(H^2)(\Omega)$ and $\vec{\mathbf{u}} \in \mathbf{H^1}(\Omega)$. Then, we have:*
i) The operator $M'(\vec{\mathbf{u}}, \mathbf{p}, \vec{\mathbf{g}})$ has closed range and is onto in $\mathbf{B_2}$.
ii) The operator $N'(\vec{\mathbf{u}}, \mathbf{p}, \vec{\mathbf{g}})$ has closed range in $\mathrm{I\!R} \times \mathbf{B_2}$.

Proof:
i) We set

$$\begin{cases} \nu\tilde{a}(\tilde{w}^{(n)}, \vec{v}) = \nu a(\tilde{w}^{(n)}, \vec{v}) + \frac{1}{\Delta t}(\tilde{w}^{(n)}, \vec{v}) \quad \forall \vec{v} \in H_0^1(\Omega) \quad n = 1, 2, ...N \\ (\tilde{f}^{(n)}, \vec{v}) = (\vec{f}^{(n)}, \vec{v}) + \frac{1}{\Delta t}(\tilde{w}^{(n-1)}, \vec{v}) \quad \forall \vec{v} \in H_0^1(\Omega) \quad n = 1, 2, ...N. \end{cases}$$

With this notation the operator $M'(\vec{\mathbf{u}}, \mathbf{p}, \vec{\mathbf{g}})$ can be written as

$$\begin{cases} \nu\tilde{a}(\tilde{w}^{(n)}, \vec{v}) + c(\tilde{w}^{(n)}, \vec{u}^{(n)}, \vec{v}) + \\ c(\vec{u}^{(n)}, \tilde{w}^{(n)}, \vec{v}) + b(\vec{v}, r^{(n)}) = (\tilde{f}^{(n)}, \vec{v}) \quad \forall \vec{v} \in H_0^1(\Omega) \\ b(\tilde{w}^{(n)}, q) = (\bar{z}^{(n)}, q) \quad \forall q \in L_0^2(\Omega) \\ (\tilde{w}^{(n)}, \vec{s})_\Gamma - (\tilde{h}^{(n)}, \vec{s})_{\Gamma_c} = (\bar{b}^{(n)}, \vec{s})_\Gamma \quad \vec{s} \in H^{-1/2}(\Gamma) \\ \text{ for } n = 1, 2, ..N \\ \vec{u}^{(0)} = \vec{u}_0(\vec{x}) \in curl(H^2)(\Omega). \end{cases} \tag{5.83}$$

The function $\tilde{f}$ is still in $H^{-1}(\Omega)$ and the range of M' is still the same if $Ran(M'(\vec{\mathbf{u}}, \mathbf{p}, \vec{\mathbf{g}})) = \mathbf{B_2}$. This is a steady system and one can apply the standard techniques for the stationary case (see [31]). Let S be the stokes operator

$$S = \begin{pmatrix} A & B^* \\ B & 0 \\ \gamma_0 & 0 \end{pmatrix}.$$

By the trace theorem, using the ellipticity of A and the inf sup property one can see ([49] , [36], [37]) that the Stokes operator is an isomorphism from $H^1(\Omega) \times L_0^2(\Omega) \to H^{-1}(\Omega) \times L_0^2(\Omega) \times H^{-1/2}(\Gamma)$. The operators $C(\vec{w}^{(n)})\vec{u}^{(n)}$ is continuous in $\vec{w}^{(n)}$ from $H^{1/2}(\Gamma)$ into $H^{-1}(\Omega)$ for all $\vec{u}^{(n)} \in H^1(\Omega)$ and $n = 1, 2, ...N$ and thus compact from $H^1(\Omega)$ into $H^{-1}(\Omega)$. The operator $C(\vec{u}^{(n)})\vec{w}^{(n)}$ is continuous for all $\vec{u} \in H^1(\Omega)$ and all $n = 1, 2, ...N$

from $H^1(\Omega)$ into $H^{-1/2}(\Gamma)$ and thus compact from $H^1(\Omega)$ into $H^{-1}(\Omega)$. The perturbation operator

$$M'^{(n)}(\vec{\mathbf{u}},\mathbf{p},\vec{\mathbf{g}}) = S^{(n)} + \begin{pmatrix} C(\vec{u}^{(n)})\vec{w}^{(n)} + C(\vec{w}^{(n)})\vec{u}^{(n)} \\ 0 \\ 0 \end{pmatrix}$$

is a Fredholm operator for all $n = 1,2,...N$ i.e., has a closed range and a finite-dimensional kernel.
The operator $M'(\vec{\mathbf{u}},\mathbf{p},\vec{\mathbf{g}})$ is into. In fact if we assume that it is not into then, there exists a non zero element $(\vec{\mu},\phi,\vec{\tau}) \in \mathbf{B_2}^*$ such that

$$\Big\langle(\bar{\mathbf{f}},\bar{\mathbf{z}},\bar{\mathbf{b}}),(\vec{\mu},\phi,\vec{\tau})\Big\rangle = 0 \quad \forall(\bar{\mathbf{f}},\bar{\mathbf{z}},\bar{\mathbf{b}}) \in Ran(M'(\vec{\mathbf{u}},\mathbf{p},\vec{\mathbf{g}})$$

or

$$\begin{cases} \nu\tilde{a}(\tilde{w}^{(n)},\vec{\mu}) + c(\tilde{w}^{(n)},\vec{u}^{(n)},\vec{\mu}) + \\ c(\vec{u}^{(n)},\tilde{w}^{(n)},\vec{\mu}) + b(\tilde{w},\phi^{(n)}) = 0 \quad \forall\tilde{w} \in H_0^1(\Omega) \\ b(\vec{\mu}^{(n)},q) = 0 \quad \forall q \in L_0^2(\Omega) \\ (\vec{\mu}^{(n)},\vec{y})_\Gamma = 0 \quad \vec{y} \in H_\Gamma^{-1/2} \\ (\tilde{h}^{(n)},\vec{\tau})_{\Gamma_c} = 0 \quad \tilde{h} \in H_{n0}^1(\Gamma_c) \\ \text{ for } n = 1,2,..N \\ \vec{u}^{(0)} = \vec{u}_0(\vec{x}) \in curl(H^2)(\Omega) \end{cases}$$

which implies $(\vec{\mu},\phi,\vec{\tau}) = (0,0,0)$. For details see [31].
ii) The operator $M'(\vec{\mathbf{u}},\mathbf{p},\vec{\mathbf{g}})$ belongs to $\mathcal{L}(\mathbf{B_1},\mathbf{B_2})$ and therefore the kernel is a closed subspace. We recall that a linear functional $\vec{f}$ on a Banach space can have either $Ran(\vec{f}) = \{0\}$ or $Ran(\vec{f}) = \{\mathbb{R}\}$. Now $N'(\vec{\mathbf{u}},\mathbf{p},\vec{\mathbf{g}})$ acting on the kernel is either identically zero or into $\mathbb{R}$. Let X, Y, Z be Banach spaces and $A : X \to Y$ and $B : X \to Z$ be linear continuous operators. If the range of B is closed in Z and the subspace $A \cdot ker((B)$ is closed in Y, then, if we define $C : X \to Y \times Z$ by $Cx = (Ax, Bx)$, the range of C is closed in $Y \times Z$. Applying this result we prove that $Ran(N'(\vec{\mathbf{u}},\mathbf{p},\vec{\mathbf{g}}))$ is a closed set. $\sqcup$

The optimality implies that the operator $N'(\vec{\mathbf{u}},\mathbf{p},\vec{\mathbf{g}})$ can not be onto which implies a first order condition.

Theorem 5.11 *Given* $\Delta t = T/N$ *and* $\vec{u}_0 \in curl(H^2)(\Omega)$. *If* $(\hat{\mathbf{u}},\hat{\mathbf{p}},\hat{\mathbf{g}}) \in (\mathbf{H^1}(\Omega) \times \mathbf{L_0^2}(\Omega) \times \mathbf{H_{n0}^1}(\Gamma_c))$ *is a solution of the semidiscrete optimal control problem then, the operator* $N'(\vec{\mathbf{u}},\mathbf{p},\vec{\mathbf{g}})$ *is not onto in* $\mathbb{R} \times \mathbf{B_2}$.

Proof: Now the operator $N'(\vec{\mathbf{u}},\mathbf{p},\vec{\mathbf{g}})$ can not be onto. If it were, by the implicit function Theorem, we would have that there exists a solution, which is different from the optimal solution, that minimizes the functional for every small neighborhood of $(\hat{\mathbf{u}}, \hat{\mathbf{p}}, \hat{\mathbf{g}})$. This contradicts the hypothesis that $(\hat{\mathbf{u}}, \hat{\mathbf{p}}, \hat{\mathbf{g}})$ is an optimal solution. $\sqcup$

The optimality implies a first order necessary condition over the operator M' and N'. In the next theorem we will write the first order necessary condition in a differential form and the optimal control solution as the solution of the corresponding Euler system of equations.

Theorem 5.12 *Given $\Delta t = T/N$ and $\vec{u}_0 \in curl(H^2)(\Omega)$. If $(\hat{\mathbf{u}}, \hat{\mathbf{g}}) \in (\mathbf{H^1}(\Omega), \mathbf{H^1_{n0}}(\Gamma_c))$ is an optimal control solution (i.e., the operator $N'(\vec{\mathbf{u}}, \mathbf{p}, \vec{\mathbf{g}})$ is not onto) then, there exists a non zero Lagrangian multiplier $(\vec{\mathbf{w}}, \sigma, \vec{\tau}) \in (\mathbf{H^1_0}(\Omega) \times \mathbf{L^1_0}(\Omega) \times \mathbf{H^{-1/2}}(\Gamma))$ satisfying the Euler equations*

$$L'(\hat{\mathbf{u}}, \hat{\mathbf{g}}) \cdot (\tilde{\mathbf{w}}, \mathbf{r}, \tilde{\mathbf{h}}) + \Big\langle (\vec{\mathbf{w}}, \sigma, \vec{\tau}), M'(\hat{\mathbf{u}}, \hat{\mathbf{p}}, \hat{\mathbf{g}}) \cdot (\tilde{\mathbf{w}}, \mathbf{r}, \tilde{\mathbf{h}}) \Big\rangle = 0$$
$$\forall (\tilde{\mathbf{w}}, \mathbf{r}, \tilde{\mathbf{h}}) \in \mathbf{H^1}(\Omega) \times \mathbf{L^2_0}(\Omega) \times \mathbf{H^1_{n0}}(\Gamma_c)) \tag{5.84}$$

where $< .,. >$ denotes the duality pairing between $\mathbb{R} \times \mathbf{B_2}$ and $\mathbb{R} \times \mathbf{B_2}^*$.

Proof: From Lemma 5.6, the range of $N'(\vec{\mathbf{u}}, \mathbf{p}, \vec{\mathbf{g}})$ is a closed set and from Theorem 5.11 this range is a closed proper subspace of $\mathbb{R} \times \mathbf{B_2}$. The Hahn-Banach theorem implies that there exists a nonzero element of $\mathbb{R} \times \mathbf{B_2}^* = \mathbb{R} \times \mathbf{H^1_0}(\Omega) \times \mathbf{L^2_0}(\Omega) \times \mathbf{H^{1/2}}(\Gamma)$ that annihilates the range of $N'(\hat{\mathbf{u}}, \hat{\mathbf{p}}, \hat{\mathbf{g}})$. One can find a $(\hat{a}, \vec{\mathbf{w}}, \sigma, \vec{\mathbf{h}}) \in \mathbb{R} \times \mathbf{B_2}^*$ such that

$$\Big\langle (\tilde{a}, \tilde{\mathbf{f}}, \tilde{\mathbf{z}}, \tilde{\mathbf{b}}), (\hat{a}, \vec{\mathbf{w}}, \sigma, \vec{\mathbf{h}}) \Big\rangle = 0 \quad \forall (\tilde{a}, \tilde{\mathbf{f}}, \tilde{\mathbf{z}}, \tilde{\mathbf{b}}) \in Ran(N'(\hat{\mathbf{u}}, \hat{\mathbf{p}}, \hat{\mathbf{g}})) \tag{5.85}$$

where $\hat{a}$ is different from zero as this solution is non trivial ($M'(\hat{\mathbf{u}}, \hat{\mathbf{p}}, \hat{\mathbf{g}})$ is onto). If we set $\hat{a} = 1$ we have eq.(5.84).

The optimality system

From the first order necessary condition we can write the optimal control as a solution of an integral-differential equation.

Theorem 5.13 *Given $\Delta t = T/N$ and $\vec{u}_0 \in curl(H^2)(\Omega)$. Let $(\hat{\mathbf{u}}, \hat{\mathbf{p}}, \hat{\mathbf{g}}) \in (\mathbf{H^1}(\Omega) \times \mathbf{L^2_0}(\Omega) \times \mathbf{H^1_{n0}}(\Gamma_c))$ denote an optimal control solution. Then, the control $\vec{g}^{(n)}$, satisfying the costraint*

$$\int_\Gamma \vec{g}^{(n)} \cdot \vec{n} d\vec{x} = 0$$

is solution of the following system

$$\int_{\Gamma_c} [\vec{g}^{(n)} \cdot \tilde{h} - \frac{\beta_1}{\Delta t^2} (\vec{g}^{(n+1)} - 2\vec{g}^{(n)} + \vec{g}^{(n-1)}) \cdot \tilde{h} + \beta_2 \nabla_s \vec{g}^{(n)} \cdot \nabla_s \tilde{h} + k^{(n)} \vec{n} \cdot \tilde{h}] d\vec{x} =$$
$$\frac{1}{\beta} \int_{\Gamma_c} (\vec{\tau}^{(n)} \cdot \tilde{h}) d\vec{x} \quad \forall \tilde{h} \in H^1_0(\Gamma_c)$$

for all $n = 1, 2, ...N$, *(* $\vec{\lambda}^0 = \gamma_0 \vec{u}_0$ *and* $\vec{\lambda}^N = \vec{\lambda}^{N-1}$ *). The function* $\vec{\tau} \in \mathbf{H^{-1/2}}(\Gamma_c)$ *is defined by*

$$(\vec{\tau}^{(n)}, \gamma_0 \tilde{v})_{\Gamma_c} = -\alpha(\vec{u}^{(n)} - \vec{U}^{(n)}, \vec{v})_\Omega - \frac{1}{\Delta t}(\vec{w}^{(n+1)} - \vec{w}^{(n)}, \tilde{v})_\Omega + \nu a(\vec{w}^{(n)}, \tilde{v}) + c(\tilde{v}; \vec{u}^{(n)}, \vec{w}^{(n)}) + c(\vec{u}^{(n)}; \tilde{v}, \vec{w}^{(n)}) + b(\tilde{v}, r^{(n)}) \quad \forall \tilde{v} \in H^1(\Omega) \tag{5.86}$$

and $\vec{\mathbf{w}}$ *satisfies*

$$\begin{cases} -\frac{1}{\Delta t}(\vec{w}^{(n+1)} - \vec{w}^{(n)}, \vec{v}) + \nu a(\vec{w}^{(n)}, \vec{v}) + c(\vec{w}^{(n)}, \vec{u}^{(n)}, \vec{v}) + \\ c(\vec{u}^{(n)}, \vec{w}^{(n)}, \vec{v}) + b(\vec{v}, r^{(n)}) = -\alpha(\vec{u}^{(n)} - \vec{U}^{(n)}, \vec{v}) \quad \forall \vec{v} \in H_0^1(\Omega) \\ b(\vec{w}^{(n)}, q) = 0 \quad \forall q \in L_0^2(\Omega) \\ (\vec{u}^{(n)}, \vec{s}) = 0 \quad \forall \vec{s} \in H^{-1/2}(\Gamma) \\ \textit{ for } n = 1, 2, ..N \\ \vec{w}^{(N+1)} = 0 \end{cases} \tag{5.87}$$

Proof: Given $\tilde{\mathbf{h}} \in \mathbf{H^1_{n0}}(\Gamma_c)$ then, the first necessary condition

$$L'(\hat{\mathbf{u}}, \hat{\mathbf{g}}) \cdot (\tilde{\mathbf{w}}, \mathbf{r}, \tilde{\mathbf{h}}) + \left\langle (\vec{\mathbf{w}}, \sigma, \vec{\tau}), M'(\hat{\mathbf{u}}, \hat{\mathbf{p}}, \hat{\mathbf{g}}) \cdot (\tilde{\mathbf{w}}, \vec{\mathbf{r}}, \tilde{\mathbf{h}}) \right\rangle = 0$$
$$\forall (\tilde{\mathbf{w}}, \vec{\mathbf{r}}, \tilde{\mathbf{h}}) \in \mathbf{H^1}(\Omega) \times \mathbf{L^2_0}(\Omega)) \times \mathbf{H^1_{n0}}(\Gamma_c)$$

can be written as

$$\alpha \sum_{n=1}^{N}((\vec{u}^{(n)} - \vec{U}^{(n)})\tilde{w}^{(n)}, 1)_\Omega \Delta t +$$
$$\beta \sum_{n=1}^{N} \int_{\Gamma_c} [\vec{g}^{(n)} \cdot \tilde{h}^{(n)} - \frac{\beta_1}{\Delta t^2}(\vec{g}^{(n)} - 2\vec{g}^{(n)} + \vec{g}^{(n)})\tilde{h}^{(n)} + \beta_2 \nabla_s \vec{g}^{(n)} \cdot \nabla_s \tilde{h}^{(n)}] d\vec{x} \Delta t +$$
$$\sum_{n=1}^{N}(\bar{f}^{(n)}, \vec{w}^{(n)})_\Omega \Delta t + \sum_{n=1}^{N}(\bar{z}^{(n)}, \sigma^{(n)})_\Omega \Delta t + \sum_{n=1}^{N}(\bar{b}^{(n)}, \vec{\tau}^{(n)})_\Gamma \Delta t = 0$$

for all $(\bar{\mathbf{f}}, \bar{\mathbf{z}}, \bar{\mathbf{b}}) \in \mathbf{H^{-1}}(\Omega) \times \mathbf{L^2_0}(\Omega)) \times \mathbf{H^{-1/2}}(\Gamma)$. We have

$$\beta \sum_{n=1}^{N} \int_{\Gamma_c} [\vec{g}^{(n)} \cdot \tilde{h}^{(n)} - \frac{\beta_1}{\Delta t^2}(\vec{g}^{(n)} - 2\vec{g}^{(n)} + \vec{g}^{(n)})\tilde{h}^{(n)} + \beta_2 \nabla_s \vec{g}^{(n)} \cdot \nabla_s \tilde{h}^{(n)}] d\vec{x} \Delta t -$$
$$\sum_{n=1}^{N}[-\frac{1}{\Delta t}(\tilde{w}^{(n)} - \tilde{w}^{(n-1)}, \vec{w}^{(n)}) + \nu a(\tilde{w}^{(n)}, \vec{w}^{(n)}) + c(\tilde{w}^{(n)}; \vec{u}^{(n)}, \vec{w}^{(n)}) +$$
$$c(\vec{u}^{(n)}; \tilde{w}^{(n)}, \vec{w}^{(n)}) + b(\vec{w}^{(n)}, r^{(n)})]\Delta t + \sum_{n=1}^{N} b(\tilde{w}^{(n)}, \sigma^{(n)})\Delta t +$$
$$\sum_{n=1}^{N}[(\vec{w}^{(n)}, \vec{\tau}^{(n)})_\Gamma - (\tilde{h}^{(n)}, \vec{\tau}^{(n)})_{\Gamma_c}]\Delta t = 0$$

for all $(\tilde{\mathbf{w}}, \vec{\mathbf{r}}, \tilde{\mathbf{h}}) \in \mathbf{H^1}(\Omega) \times \mathbf{L_0^2}(\Omega)) \times \mathbf{H^1_{n0}}(\Gamma)$. We assume $\vec{g}^{(N)} = \vec{g}^{(N-1)}$, $\vec{g}^{(0)} = \gamma_0 \vec{u}_0$. After integration by parts we can define the quantity $\vec{\tau}$ by

$$\begin{aligned}
&\int_{\Gamma_c} \vec{\tau}^{(n)} \cdot \tilde{v} d\vec{x} = -\int_{\Gamma_c} (\gamma_1 \vec{w}^{(n)} + \vec{n}\sigma^{(n)}) \cdot \tilde{v} d\vec{x} = \\
&-\alpha(\vec{u}^{(n)} - \vec{U}^{(n)}, \tilde{v})_\Omega - \frac{1}{\Delta t}(\vec{w}^{(n+1)} - \vec{w}^{(n)}, \tilde{v})_\Omega + \nu a(\vec{w}^{(n)}, \tilde{v}) + \\
&c(\tilde{v}, \vec{u}^{(n)}, \vec{w}^{(n)}) + c(\vec{u}^{(n)}, \tilde{v}, \vec{w}^{(n)}) + b(\tilde{v}, \sigma^{(n)}) \quad \forall \tilde{v} \in H^1(\Omega)
\end{aligned} \tag{5.88}$$

where $\vec{\mathbf{w}}$ satisfies the adjoint equations

$$\begin{cases}
-\frac{1}{\Delta t}(\vec{w}^{(n+1)} - \vec{w}^{(n)}, \vec{v}) + \nu a(\vec{w}^{(n)}, \vec{v}) + c(\vec{w}^{(n)}, \vec{u}^{(n)}, \vec{v}) + \\
c(\vec{u}^{(n)}, \vec{w}^{(n)}, \vec{v}) + b(\vec{v}, r^{(n)}) = (\vec{f}^{(n)}, \vec{v}) \quad \forall \vec{v} \in H_0^1(\Omega) \\
b(\vec{w}^{(n)}, q) = 0 \quad \forall q \in L_0^2(\Omega) \\
(\vec{w}^{(n)}, \vec{s})_\Gamma = 0 \quad \vec{s} \in H^{-1/2}(\Gamma) \\
\text{ for } n = 1, 2, ..N \\
\vec{w}^{(N+1)} = 0.
\end{cases} \tag{5.89}$$

Thus we write

$$\sum_{n=1}^{N} \int_{\Gamma_c} [\vec{g}^{(n)} \cdot \tilde{h}^{(n)} - \frac{\beta_1}{\Delta t^2}(\vec{g}^{(n)} - 2\vec{g}^{(n)} + \vec{g}^{(n)})\tilde{h}^{(n)} + \beta_2 \nabla_s \vec{g}^{(n)} \cdot \nabla_s \tilde{h}^{(n)} - \frac{1}{\beta}\vec{\tau}^{(n)} \cdot \tilde{h}^{(n)}] d\vec{x} \Delta t = 0.$$

As the variation $\tilde{h}$ is independent in the space $\mathbf{H^1_{n0}}(\Gamma_c)$ we have

$$\int_{\Gamma_c} [\vec{g}^{(n)} \cdot \tilde{h} - \frac{\beta_1}{\Delta t^2}(\vec{g}^{(n)} - 2\vec{g}^{(n)} + \vec{g}^{(n)}) \cdot \tilde{h} + \beta_2 \nabla_s \vec{g}^{(n)} \cdot \nabla_s \tilde{h} - \frac{1}{\beta}\vec{\tau}^{(n)} \cdot \tilde{h}] d\vec{x} = 0 \tag{5.90}$$

for $n = 1, 2, .., N$ and for all $\tilde{h} \in H^1_{n0}(\Gamma_c)$. We recall that $H^1(\Gamma_c) = H^1_n(\Gamma_c) \oplus (H^1_n)^\perp(\Gamma_c)$ and thus

$$\tilde{h} = \tilde{h}_1 - \vec{n}\frac{\int_{\Gamma_c} \tilde{h}_1 \cdot \vec{n} d\vec{x}}{\mu(\Gamma_c)}$$

where $\tilde{h}_1 \in H^1(\Gamma_c)$. We can write a weak formulation of the eq(5.90) in the space $H^1_n(\Gamma_c)$ then, by using the above decomposition, write a new formulation in $H^1(\Gamma_c)$. In this new formulation a costant vector in the normal direction appears. The weak formulation in the $H^1_{n0}(\Gamma_c)$ space reads

$$\begin{aligned}
&\int_{\Gamma_c} [\vec{g}_1^{(n)} \cdot \tilde{h}_1 - \frac{\beta_1}{\Delta t^2}(\vec{g}_1^{(n)} - 2\vec{g}_1^{(n)} + \vec{g}_1^{(n)}) \cdot \tilde{h}_1 + \beta_2 \nabla_s \vec{g}_1^{(n)} \cdot \nabla_s \tilde{h}_1 + k^{(n)}\vec{n} \cdot \tilde{h}_1 d\vec{x} = \\
&\frac{1}{\beta}\int_{\Gamma_c} (\vec{\tau}^{(n)} \cdot \tilde{h}_1) d\vec{x}
\end{aligned}$$

for all $\vec{h}_1 \in H_0^1(\Gamma_c)$ and $n = 1, 2, ..N$. The costant $k^{(n)}$ can be calculated by using the costraint

$$\int_\Gamma \vec{g}^{(n)} \cdot \vec{n} d\vec{x} = 0.$$

$\sqcup$

Now in order to get the solution of our optimal control problem we have to solve the Navier-Stokes system

$$\begin{aligned}
&\frac{1}{\Delta t}(\vec{u}^{(n)} - \vec{u}^{(n-1)}, \vec{v}) + \nu a(\vec{u}^{(n)}, \vec{v}) + c(\vec{u}^{(n)}; \vec{u}^{(n)}, \vec{v}) + b(\vec{v}, p^{(n)}) = 0 \quad \vec{v} \in H_0^1(\Omega)\\
&b(\vec{u}^{(n)}, q) = 0 \quad \forall q \in L_0^2(\Omega)\\
&(\vec{u}^{(n)}, \vec{s})_\Gamma = (\vec{g}^{(n)}, \vec{s})_{\Gamma_c} \quad \forall \vec{s} \in H^{-1/2}(\Gamma)\\
&\vec{w}^{(0)} = \vec{u}_0 \quad in \quad \Omega,
\end{aligned}$$

the adjoint system

$$\begin{aligned}
&-\frac{1}{\Delta t}(\vec{w}^{(n+1)} - \vec{w}^{(n)}, \vec{v}) + \nu a(\vec{w}^{(n)}, \vec{v}) + c(\vec{u}^{(n)}; \vec{v}, \vec{w}^{(n)}) + c(\vec{v}; \vec{u}^{(n)}, \vec{w}^{(n)}) +\\
&b(\vec{v}, \sigma^{(n)}) = -\alpha(\vec{u}^{(n)} - \vec{U}^{(n)}, \vec{v}) \quad \forall \vec{v} \in H_0^1(\Omega)\\
&b(\vec{w}^{(n)}, q) = 0 \quad \forall q \in L_0^2(\Omega)\\
&\vec{w}^{(n)} = 0 \quad on \quad \Gamma\\
&\vec{w}^{(N+1)} = 0 \quad in \quad \Omega
\end{aligned}$$

and the control equations

$$\begin{aligned}
&-\frac{\beta_1}{\Delta t^2}(\vec{g}^{(n+1)} - 2\vec{g}^{(n)} + \vec{g}^{(n-1)}, \tilde{h}) + \beta_2(\nabla_s \vec{g}^{(n)}, \nabla_s \tilde{h}) + (\vec{g}^{(n)}, \tilde{h}) + k^{(n)}(\vec{n}, \tilde{h}) =\\
&-\frac{1}{\beta}(\gamma_1 \vec{w}^{(n)} + \vec{n}\sigma^{(n)}, \tilde{h}) \quad \forall \tilde{h} \in H_0^1(\Gamma_c)\\
&(\vec{g}^{(n)}, \vec{n}) = 0\\
&\vec{g}^{(n)} = 0 \quad on \quad \Gamma \setminus \Gamma_c\\
&\vec{g}^{(0)} = \gamma_0 \vec{u}_0 \quad on \quad \Gamma_c\\
&\vec{g}^{(N)} = \vec{g}^{(N-1)} \quad on \quad \Gamma_c
\end{aligned}$$

for $n = 1, 2, ...N$. The above system of equations is the weak formulation of the following system

$$\begin{aligned}
&\frac{1}{\Delta t}(\vec{u}^{(n)} - \vec{u}^{(n-1)}) + \nu \nabla^2 \vec{u}^{(n)} + (\vec{u}^{(n)} \cdot \nabla)\vec{u}^{(n)} + \nabla p^{(n)} = 0 \quad in \quad \Omega\\
&\nabla \cdot \vec{u}^{(n)} = 0 \quad in \quad \Omega\\
&\vec{u}^{(n)} = \vec{g}^{(n)} \quad on \quad \Gamma_c\\
&\vec{u}^{(n)} = 0 \quad on \quad \Gamma \setminus \Gamma_c\\
&\vec{w}^{(0)} = \vec{u}_0 \quad in \quad \Omega
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{\Delta t}(\vec{w}^{(n+1)}-\vec{w}^{(n)})+\nu\nabla^2\vec{w}^{(n)}+(\nabla\vec{u})^T\vec{w}-(\vec{u}^{(n)}\cdot\nabla)\vec{w}^{(n)}+\vec{\nabla}\sigma^{(n)}=-\alpha(\vec{u}^{(n)}-\vec{U}^{(n)})\\
&\nabla\cdot\vec{g}^{(n)}=0 \quad in \quad \Omega\\
&\vec{w}^{(n)}=0 \quad on \quad \Gamma\\
&\vec{w}^{(N)}=0 \quad in \quad \Omega
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{\beta_1}{\Delta t^2}(\vec{g}^{(n+1)}-2\vec{g}^{(n)}+\vec{g}^{(n-1)})-\beta_2\nabla_s^2\vec{g}^{(n)}+\vec{g}^{(n)}+k^{(n)}\vec{n}=-\frac{1}{\beta}(\vec{n}\sigma^{(n)}+\frac{\partial\vec{w}^{(n)}}{\partial n})\\
&\int_\Gamma \vec{g}^{(n)}\cdot\vec{n}d\vec{x}=0\\
&\vec{g}^{(n)}=0 \quad on \quad \Gamma\setminus\Gamma_c\\
&\vec{g}^{(0)}=\gamma_0\vec{u}_0 \quad on \quad \Gamma_c\\
&\vec{g}^{(N)}=\vec{g}^{(N-1)} \quad on \quad \Gamma_c
\end{aligned}
$$

for $n=1,2,...N$.

5.4 Fully discrete time-space approximation

5.4.1 Assumptions on the finite element spaces

We consider only conforming finite element approximations. Let $X^h\subset H_0^1(\Omega)$ and $S^h\subset L^2(\Omega)$ be two families of finite dimensional subspaces parameterized by h that tends to zero. We also denote $S_0^h=S^h\cap L_0^2(\Omega)$ We make the following assumptions on X^h and S^h :
a) the approximation hypotheses:
there exists an integer l and a constant C, independent of $h,\vec{u}$ and p, such that for $1\le k\le l$ we have

$$\inf_{\vec{u}_h\in X^h}\|\vec{u}_h-\vec{u}\|_1\le Ch^k\|\vec{u}\|_{k+1} \quad \forall\vec{u}\in H^{k+1}(\Omega)\cap H_0^1(\Omega) \tag{5.91}$$

$$\inf_{p_h\in S^h}\|p-p_h\|\le Ch^k\|p\|_k \qquad \forall p\in H^k(\Omega)\cap L_0^2(\Omega). \tag{5.92}$$

b) the inf-sup condition or L-B-B condition:
there exists a constant C' , independent of h such that

$$\inf_{0\neq q_h\in S^h}\sup_{0\neq\vec{u}_h\in X^h}\frac{\int_\Omega q_h div\vec{u}_h}{\|\vec{u}_h\|_1\|q_h\|}\ge C'>0. \tag{5.93}$$

This condition assures the stability of the Navier-Stokes discrete solutions.

To preserve the antisymmetry of the trilinear form $c(\vec{u};\vec{v},\vec{w})$ on the finite element spaces we introduce the modified trilinear form (see [35])

$$\tilde{c}(\vec{u};\vec{v},\vec{w}) = \frac{1}{2}\{c(\vec{u};\vec{v},\vec{w}) - c(\vec{u};\vec{w},\vec{v})\} \qquad \forall \vec{u},\vec{v},\vec{w} \in H_0^1(\Omega)$$

We can recall some useful formulas and inequalities in a two-dimensional domain Ω :

$$c(\vec{u};\vec{v},\vec{w}) = \tilde{c}(\vec{u};\vec{v},\vec{w}) \qquad \forall \vec{u} \in H_0^1(\Omega) \cap W(\Omega) \quad \forall \vec{v},\vec{w} \in H_0^1(\Omega)$$

$$\begin{cases} \tilde{c}(\vec{u};\vec{v},\vec{w}) = -\tilde{c}(\vec{u};\vec{w},\vec{v}) \\ \tilde{c}(\vec{u};\vec{v},\vec{v}) = 0 \end{cases} \tag{5.94}$$

and [35]

$$\begin{cases} |\tilde{c}(\vec{u};\vec{v},\vec{w})| \leq K_1\|\vec{u}\|_1 \cdot \|\vec{v}\|_1 \cdot \|\vec{w}^{(n)}\|_1 \\ |\tilde{c}(\vec{u};\vec{v},\vec{w})| \leq K_2\|\vec{u}\|^{\frac{1}{2}}\|\nabla\vec{u}\|^{\frac{1}{2}}\|\nabla\vec{v}\| \cdot \|\vec{w}\|^{\frac{1}{2}}\|\nabla\vec{w}\|^{\frac{1}{2}} \end{cases} \tag{5.95}$$

for all $\vec{u},\vec{v},\vec{w} \in H^1(\Omega)$. We remark that the inequality in eq(5.95) is true in the framework of the conforming finite element approximation and only in the two-dimensional case (see [35]). Next, let $P^h = X^h|_\Gamma$, i.e. P_h consists of the restriction, to the boundary Γ, belonging to P^h. For all choises of conforming finite element space X^h we then have that $P^h \subset H^{-1/2}(\Gamma)$. For the subspaces $P^h = X^h|_\Gamma$, we assume the approximation property:
there exists an integer l and a constant C, independent of $h, \vec{s}$ such that for $1 \leq k \leq l$ we have

$$\inf_{\vec{s}_h \in p_h} \|\vec{s}_h - \vec{s}\|_{-1/2,\Gamma} \leq Ch^k\|\vec{u}\|_{k-1/2} \quad \forall \vec{s} \in H^{k-1/2}(\Gamma). \tag{5.96}$$

Now, let $Q^h = X^h|_\Gamma$, i.e., Q^h consists of the restriction, to the boundary segment Γ_c, of the functions belonging to X^h. For all choises of conforming finite element spaces X^h we have that $Q^h \subset H^1(\Gamma_c)$. We define $Q_0^h = Q^h \cap H_{n0}^1(\Gamma_c)$. If the same type of polynomials are used in $Q_0^h = Q^h \cap (\Gamma_c)$ we have the approximation propriety:
there exists an integer k and a constant C, independent of $h, \vec{s}$ such that for $1 \leq m \leq k$ we have

$$\inf_{\vec{s}_h \in Q_0^h} \|\vec{s}_h - \vec{s}\|_{s,\Gamma_c} \leq Ch^{m-s+1/2}\|\vec{s}\|_{m+1/2} \quad \forall \vec{s} \in H_{n0}^1(\Gamma_c), \quad 0 \leq s \leq 1. \tag{5.97}$$

See [53] and [54] for details concerning the boundary approximations.

5.4.2 Formulation of the fully discrete optimal control approximation

Let $\sigma_N = \{t_n\}_{n=0}^N$ be a partition of $[0,T]$ in equal intervals $\Delta t = T/N$ with $t_0 = 0$ and $t_N = T$. For each fixed Δt (or N) and for every quantity $q(t,\vec{x})$, we associate the corresponding set $\{q_h^{(n)}\}_{n=1}^N$. We will denote the vector $(q_h^{(1)}, q_h^{(2)}, ...q_h^{(N)})$ with bold letter $\mathbf{q}_h$ and the space Y^N as $\mathbf{Y}$. The continuous linear function $\vec{q}_h^N(t,\vec{x})$ is defined by $\vec{q}_h^N(t_n,\vec{x}) = q_h(t_n,\vec{x})$ for all $n = 0,1,2...N$.

Given $\Delta t = T/N$, $\vec{\mathbf{g}} \in \mathbf{H^{1/2}}(\Gamma)$ and $\vec{u}_0 \in curl(H^2)(\Omega)$, then $(\vec{\mathbf{u}}_h, \mathbf{p}_h)$ is called a generalized solution for the Navier-Stokes fully discrete time-space approximation if $u_h^{(n)} \in X^h$, $p_h^{(n)} \in S_0^h$ and $(u_h^{(n)}, p_h^{(n)})$ satisfies the following system of equations

$$\begin{cases} \frac{1}{\Delta t}(\vec{u}_h^{(n)} - \vec{u}_h^{(n-1)}, \vec{v}_h) + \nu a(\vec{u}_h^{(n)}, \vec{v}_h) + \tilde{c}(\vec{u}_h^{(n)}; \vec{u}_h^{(n)}, \vec{v}_h) + b(\vec{v}_h, p^{(n)}) = 0 \quad \forall \vec{v}_h \in X^h(\Omega) \\ b(\vec{v}_h^{(n)}, q_h) = 0 \quad \forall q_h \in S_0^h(\Omega) \\ (\vec{u}_h^{(n)}, \vec{s})_\Gamma = (\vec{g}^{(n)}, \vec{s})_{\Gamma_c} \quad \forall \vec{s} \in P^h(\Gamma) \end{cases} \tag{5.98}$$

for n=1,2,...N with initial velocity $\vec{u}_h^{(0)} = \pi^h \vec{u}_0(\vec{x})$.
The optimal control is achieved with the functional

$$\begin{aligned} L_h^N \;=\; & \frac{\alpha}{2}\sum_{n=1}^N \|\vec{u}_h^{(n)} - \vec{U}^{(n)}\|^2 \Delta t + \\ & \frac{\beta}{2}\sum_{n=1}^N [\|\vec{g}^{(n)}\|^2 \Delta t + \beta_1 \|\nabla_s \vec{g}^{(n)}\|_{\Gamma_c}^2 \Delta t + \beta_2 \|(\vec{g}^{(n)} - \vec{g}^{(n-1)})\|_{\Gamma_c}^2]. \end{aligned} \tag{5.99}$$

The formulation of the problem P_L in the fully discrete approximation becomes:
given $\Delta t = T/N$, $\vec{u}_0 \in curl(H^2)(\Omega)$ and $\vec{U} \in U_{ad}$, find $(\vec{\mathbf{u}}_h, \mathbf{p}_h, \vec{\mathbf{g}})$ in $\mathbf{X^h(\Omega)} \times \mathbf{S_0^h(\Omega)} \times \mathbf{H^1_{n0}}(\Gamma_c)$ such that $(\vec{u}_h^{(n)}, p_h^{(n)})$ is the solution of eq(5.98) and $\vec{\mathbf{g}}$ minimizes the cost function in eq(5.99).

In analogy with the semidiscrete case for the problem defined in eq(5.98) we have these useful theorems [35].

Theorem 5.14 *Given $\Delta t = T/N$ if there exists $\epsilon > 0$ such that $\sum_{i=1}^N (\|\vec{g}^{(n)}\|_1^2 \Delta t + \|\vec{g}^{(n)} - \vec{g}^{(n-1)}\|^2) \leq \epsilon$ then, there exists at least one $\vec{\mathbf{u}}_h$ satisfying the eq(5.98) with the following estimates*

$$\|\vec{u}_h^{(n)}\|^2 \leq K \quad n = 1,2,...N \tag{5.100}$$

$$\Delta t \sum_{n=1}^N \|\nabla \vec{u}_h^{(n)}\|^2 \leq K \tag{5.101}$$

$$\sum_{n=1}^N \|\vec{u}_h^{(n)} - \vec{u}_h^{(n-1)}\|^2 \leq K \tag{5.102}$$

where K is a constant independent of Δt and h.

Hence $\vec{u}_h^N \in L^\infty((0,T);X^h)$ and $\vec{u}_h'^N \in L^2((0,T);L^2(\Omega))$. Also if $\vec{g}^N \to \vec{g} \in L^2((0,T);L^2(\Omega))$ then $\vec{u}_h^N \to \vec{u}$ where $\vec{u}$ is the solution of the continuous Navier-Stokes system of equations.

5.4.3 Existence and consistency of the fully discrete optimal control solution

In order to prove the existence and consistency of the fully discrete optimal control problem we need to show that $\vec{g}^N$ is uniformly bounded in $L^2((0,T);H^1_{n0}(\Gamma_c))$.

Lemma 5.7 *Let $\vec{u}_0 \in curl(H^2)(\Omega)$ and $\vec{U} \in U_{ad}$. If $(\vec{\mathbf{u}}_h, \vec{\mathbf{g}})$ is the solution of the fully discrete optimal control problem then there exists a constant C independent of Δt and h such that*

$$\sum_{n=1}^{N} \|\vec{g}^{(n)}\|^2_{1,\Gamma_c}\Delta t \leq C \tag{5.103}$$

$$\sum_{n=1}^{N} \|\vec{g}^{(n)} - \vec{g}^{(n)}\|^2_{\Gamma_c} \leq C \tag{5.104}$$

$$\sum_{n=1}^{N} \|\vec{u}_h^{(n)}\|^2_{\Omega}\Delta t \leq C. \tag{5.105}$$

Hence we have $\vec{g}^N \in L^2((0,T);L^2(\Omega))$ and $\vec{u}_h^N \in L^2((0,T);L^2(\Omega))$ for all N and h.

Proof: The proof is substantially the same as in the semidiscrete case. If $\vec{\mathbf{g}} = 0$, then the functional can be bounded as

$$\begin{aligned} &L(\vec{\mathbf{g}}) \leq L(\mathbf{0}) = \frac{T\alpha+\gamma}{2}\left(\|\vec{U}\|^2_{L^\infty((0,T);W)} + \|\pi^h\vec{u}_0\|^2\right) \leq \\ &\frac{T\alpha+\gamma}{2}\left(\|\vec{U}\|^2_{L^\infty((0,T);V)} + \|\vec{u}_0\|^2\right) = C \end{aligned} \tag{5.106}$$

which implies eq(5.103 - 5.105). ⊔

In the framework of conforming finite elements, the existence and the consistency theorems can be proved by using the same standard techniques. We state both theorems for completeness.

Theorem 5.15 *Given $\Delta t = T/N$, $\vec{u}_0 \in curl(H^2)(\Omega)$ and $\vec{U} \in U_{ad}$, there exists a sequence $(\vec{\mathbf{u}}_h, \vec{\mathbf{g}})$ in $\mathbf{X^h} \times \mathbf{H^1_{n0}}(\Gamma_c)$ such that $\vec{\mathbf{u}}_h$ is the solution of eq(5.98) and $\vec{\mathbf{g}}$ minimizes the cost function in eq(5.99).*

Theorem 5.16 *Let $\Delta t = T/N$ and $\vec{u}_0$ belong to $curl(H^2)(\Omega)$. The solution $(\vec{\mathbf{u}}_{\mathbf{h}}, \vec{\mathbf{g}})$ for the fully discrete optimal control problem tends to the optimal control solution $(\vec{u}, \vec{g})$ of the continuous problem as $\Delta t \to 0$ ($N \to \infty$) and $h \to 0$.*

5.4.4 First-order necessary condition and the optimality system

We can derive the first-order necessary condition, the Euler equation and the final equation for the optimal control. All these results can be obtained proceeding in a similar to the semi-discrete case. For conforming finite elements we can state the following theorem

Theorem 5.17 *Given $\Delta t = T/N$ and $\vec{u}_0 \in curl(H^2)(\Omega)$. Let $(\hat{\mathbf{u}}_h, \hat{\sigma}_h, \hat{\mathbf{g}}_h) \in \mathbf{X}^h(\Omega)\times$ $\mathbf{S_0^h}(\Omega)\times$ $\mathbf{Q_0^h}(\Gamma_c)$ denote an optimal control solution. Then, the control $\vec{g}_h^{(n)}$ satisfies the following system*

$$
\begin{aligned}
&\int_{\Gamma_c}[\vec{g}_h^{(n)}\cdot\tilde{r}_h - \frac{\beta_1}{\Delta^2}(\vec{g}_h^{(n+1)} - 2\vec{g}_h^{(n)} + \vec{g}_h^{(n-1)})\cdot\tilde{r}_h + \beta_2\nabla_s\vec{g}_h^{(n)}\cdot\nabla_s\tilde{r}_h + \\
&k^{(n)}\vec{n}\cdot\tilde{r}_h - \frac{1}{\beta}(\vec{\tau}_h^{(n)}\cdot\tilde{r}_h)]d\vec{x}dt = 0 \quad \forall\tilde{k}_h \in Q_0^h(\Gamma_c)\\
&\int_{\Gamma}\vec{g}_h^{(n)}\cdot\vec{n}d\vec{x} = 0\\
&\vec{g}^{(0)} = \gamma_0\vec{u}_0 \quad onquad\Gamma_c\\
&\vec{g}^{(N)} = \vec{g}^{(N-1)} \quad on \quad \Gamma_c
\end{aligned}
\tag{5.107}
$$

for all $n = 1, 2, ...N$, where $\vec{\tau}_h \in \mathbf{P}^h(\Gamma)$ is defined by

$$
\begin{aligned}
&(\vec{\tau}_h^{(n)}, \tilde{v}_h)_{\Gamma_c} = -(\frac{\partial\vec{w}^{(n)}}{\partial n} + \sigma^{(n)}\vec{n}, \tilde{v}_h)_{\Gamma_c} = \\
&-\alpha(\vec{u}_h^{(n)} - \vec{U}^{(n)}, \vec{v}_h)_\Omega - \frac{1}{\Delta t}(\vec{w}_h^{(n)} - \vec{w}_h^{(n-1)}, \tilde{v}_h)_\Omega + \nu a(\vec{w}_h^{(n)}, \tilde{v}_h) + \\
&\tilde{c}(\tilde{v}_h, \vec{u}_h^{(n)}, \vec{w}_h^{(n)}) + \tilde{c}(\vec{u}_h^{(n)}, \tilde{v}_h, \vec{w}_h^{(n)}) + b(\tilde{v}_h, r_h^{(n)}) \quad \forall\tilde{v}_h \in X^h(\Omega)
\end{aligned}
$$

and $\vec{\mathbf{w}}$ satisfies

$$
\begin{cases}
\frac{1}{\Delta t}(\vec{w}_h^{(n)} - \vec{w}_h^{(n-1)}, \vec{v}_h) + \nu a(\vec{w}_h^{(n)}, \vec{v}_h) + \tilde{c}(\vec{w}_h^{(n)}, \vec{u}_h^{(n)}, \vec{v}_h) + \\
\tilde{c}(\vec{u}_h^{(n)}, \vec{w}_h^{(n)}, \vec{v}_h) + b(\vec{v}_h, r_h^{(n)}) = -\alpha(\vec{u}_h^{(n)} - \vec{U}^{(n)}, \vec{v}_h) \quad \forall\vec{v}_h \in X^h(\Omega)\\
b(\vec{w}_h^{(n)}, q_h) = 0 \quad \forall q \in S_0^h(\Omega)\\
\vec{u}_h^{(n)} = 0 \quad on \quad \Gamma\\
\text{ for } n = 1, 2, ..N\\
\vec{w}_h^{(N+1)} = 0
\end{cases}
$$

5.5 Numerical results

5.5.1 Introduction

In order to determine the optimal control solution we have to solve the following system of equations in the state and control variables $(\vec{u}, p, \vec{w}, \sigma, \vec{g}, k)$. We have the non linear Navier-Stokes equations

$$\begin{aligned}
&< \vec{u}_t, \vec{v} > +\nu a(\vec{u}, \vec{v}) + c(\vec{u}; \vec{u}, \vec{v}) + b(\vec{v}, p) = 0 \quad \forall \vec{v} \in H_0^1(\Omega) \\
&b(\vec{u}, q) = 0 \quad \forall q \in L_0^2(\Omega) \\
&(\vec{u}, \vec{s})_\Gamma = (g(t, \vec{x}), \vec{s})_{\Gamma_c} \quad \forall \vec{s} \in H^{-1/2}(\Gamma) \\
&\vec{u} = 0 \quad \forall \vec{x} \in \Gamma \setminus \Gamma_c \\
&\vec{u}(0, \vec{x}) = \vec{u}_0(\vec{x}) \quad \vec{x} \in \Omega,
\end{aligned} \tag{5.108}$$

coupled with the adjoint system

$$\begin{aligned}
&- < \vec{w}_t, \vec{v} > +\nu a(\vec{w}, \vec{v}) + c(\vec{w}; \vec{u}, \vec{v}) + c(\vec{u}; \vec{w}, \vec{v}) = -\alpha(\vec{u} - \vec{U}, \vec{v}) \quad \forall \vec{v} \in H_0^1(\Omega) \\
&b(\vec{w}, q) = 0 \quad \forall q \in L_0^2(\Omega) \\
&\vec{w} = 0 \quad \forall \vec{x} \in \Gamma \quad t \in (0, T) \\
&\vec{w}(T, \vec{x}) = 0 \quad \vec{x} \in \Omega,
\end{aligned} \tag{5.109}$$

and the boundary control equation

$$\begin{aligned}
&(\vec{g}, \vec{r}) - \beta_1(\vec{g}_{tt}, \vec{r}) + \beta_2(\nabla_s \vec{g}, \nabla_s \vec{r}) + k(t)(\vec{n}, \vec{r}) = -\frac{1}{\beta}(\frac{\partial \vec{w}}{\partial n} + \vec{n}\sigma, \vec{r}) \quad \forall \vec{r} \in H_0^1(\Gamma_c) \\
&\int_{\Gamma_c} \vec{g} \cdot \vec{n} d\vec{x} = 0 \\
&\beta_1 \vec{g}(0, \vec{x}) = \gamma_0 \vec{u}_0 \quad \vec{x} \in \Gamma_c \\
&\beta_1 \vec{g}_t(T, \vec{x}) = 0 \quad \vec{x} \in \Gamma_c \\
&\beta_2 \vec{g} = 0 \quad \forall \vec{x} \in \partial\Gamma_c.
\end{aligned} \tag{5.110}$$

Let $\sigma_N = \{t_n\}_{n=0}^N$ be a partition of $[0, T]$ into equal intervals $\Delta t = T/N$ with $t_0 = 0$ and $t_N = T$. For a fixed Δt (or N) let $X^h \subset H_0^1(\Omega)$ and $S_0^h \subset L^2(\Omega)$ be two families of finite dimensional subspaces parameterized by h that tends to zero. We consider only conforming finite elements. Next, let P^h be the restriction to the boundary Γ of X^h. For all choices of conforming finite element space X^h we then have that $P^h \subset H^{-1/2}(\Gamma)$. Now, let Q^h consist of restrictions, to the boundary segment Γ_c, of the functions belonging to X^h ($Q^h \subset H^1(\Gamma_c)$). We define $Q_0^h = Q^h \cap H_{n0}^1(\Gamma_c)$ and the same type of polynomials will be used.

The eq(5.108 - 5.110) for tangential control consist of:
a) the Navier-Stokes system

$$\begin{aligned}
&\frac{1}{\Delta t}(\vec{u}_h^{(n)} - \vec{u}_h^{(n-1)}, \vec{v}_h) + \nu a(\vec{u}_h^{(n)}, \vec{v}_h) + c(\vec{u}_h^{(n)}; \vec{u}_h^{(n)}, \vec{v}_h) + b(\vec{v}_h, p_h^{(n)}) = 0 \quad \forall \vec{v}_h \in X^h(\Omega)\\
&b(\vec{u}_h^{(n)}, q_h) = 0 \quad \forall q_h \in S_0^h(\Omega)\\
&(\vec{u}_h^{(n)}, \vec{s}_h)_\Gamma = (\vec{g}_h^{(n)}, \vec{s}_h)_{\Gamma_c} \quad \forall \vec{s}_h \in P^h(\Gamma)\\
&\vec{w}_h^{(0)} = \pi^h \vec{u}_0 \quad in \quad \Omega\\
&\text{for all } n = 1, 2, ...N
\end{aligned} \tag{5.111}$$

b) the adjoint system

$$\begin{aligned}
&-\frac{1}{\Delta t}(\vec{w}_h^{(n+1)} - \vec{w}_h^{(n)}, \vec{v}_h) + \nu a(\vec{w}_h^{(n)}, \vec{v}_h) + c(\vec{u}_h^{(n)}; \vec{v}_h, \vec{w}_h^{(n)}) + c(\vec{v}_h; \vec{u}_h^{(n)}, \vec{w}_h^{(n)}) +\\
&b(\vec{v}_h, \sigma_h^{(n)}) = -\alpha(\vec{u}_h^{(n)} - \vec{U}^{(n)}, \vec{v}_h) \quad \forall \vec{v}_h \in X^h(\Omega)\\
&b(\vec{w}_h^{(n)}, q_h) = 0 \quad \forall q_h \in S_0^h(\Omega)\\
&\vec{w}_h^{(n)} = 0 \quad on \quad \Gamma\\
&\vec{w}_h^{(N+1)} = 0 \quad in \quad \Omega\\
&\text{for all } n = 1, 2, ...N
\end{aligned} \tag{5.112}$$

c) the control equation

$$\begin{aligned}
&-\frac{\beta_1}{\Delta t^2}(\vec{\lambda}_h^{(n+1)} - 2\vec{\lambda}_h^{(n)} + \vec{\lambda}_h^{(n-1)}, \vec{r}_h) + \beta_2(\nabla_s \vec{\lambda}_h^{(n)}, \nabla_s \vec{r}_h) + (\vec{\lambda}_h^{(n)}, \vec{r}_h) =\\
&-\frac{1}{\beta}(\frac{\partial \vec{w}^{(n)}}{\partial n}, \vec{r}_h) = 0 \quad \forall \vec{r}_h \in Q_0^h(\Gamma_c)\\
&\vec{\lambda}_h^{(n)} = 0 \quad on \quad \Gamma \setminus \Gamma_c\\
&\vec{\lambda}_h^{(0)} = \gamma_0 \vec{u}_0 \quad on \quad \Gamma_c\\
&\vec{\lambda}_h^{(N)} = \vec{\lambda}_h^{(N-1)} \quad on \quad \Gamma_c\\
&\text{for all } n = 1, 2, ...N-1
\end{aligned} \tag{5.113}$$

d) the compatibility boundary equation

$$\begin{aligned}
&\vec{\lambda}^{(n)} = \vec{g}^{(n)}\\
&\text{for all } n = 1, 2, ...N \; .
\end{aligned} \tag{5.114}$$

5.5.2 Numerical algorithm

Let us consider the gradient method for the optimal control problem. We have to split the system in three parts in order to apply the algorithm: the Navier-Stokes system (eq.(5.111)) and the adjoint system (eq.(5.112)) and the control equation (eq.(5.113)) . In the gradient algorithm we satisfy the relation in eq (5.114) only when convergence is reached. Let $L^{(k)} = L(\vec{\lambda}_h(k))$ and τ be the tolerance required for the convergence of the functional.

The gradient algorithm proceeds as follows:

a) initial configuration:
i) given $\vec{\mathbf{g}}_h(0), \tau$ and $\epsilon = 1$;
ii) solve for $\vec{\mathbf{u}}_h(0)$ in eq(5.111) with $\vec{\mathbf{g}}_h(0)$;
iii) evaluate $L^{(0)}$;

b) main loop :
iv) solve for $\vec{\mathbf{w}}_h(k)$ in eq(5.112) with $\vec{\mathbf{u}}_h(k-1)$;
v) solve for $\vec{\lambda}_h(k)$ in eq(5.113) with $\vec{\mathbf{w}}_h(k)$;

c) optimization loop:
vi) with $\vec{g}_h^{(n)}(k) = \vec{g}_h^{(n)}(k-1) - \epsilon\left(\vec{g}_h^{(n)}(k-1) - \vec{\lambda}_h^{(n)}(k)\right)$ solve for $\vec{\mathbf{u}}_h(k)$ in eq(5.111);
vii) check if $L^{(k)}$ is less than $L^{(k-1)}$; if $L^{(k)} \leq L^{(k-1)}$ then $\epsilon = 1.5\epsilon$ and go to b); if $L^{(k)} > L^{(k-1)}$ then $\epsilon = .5\epsilon$ and go to c).

The algorithm stops when $|L^{(k)} - L^{(k-1)}|/L^{(k)} \leq \tau$. Using this Lemma 2.9 we can show that the gradient algorithm converges to the solution.

Theorem 5.18 *Let $(\vec{\mathbf{u}}_h(k), \vec{\mathbf{w}}_h(k), \mathbf{p}_h(k), \sigma_h(k), \vec{\mathbf{g}}_h(k))$ be the k-th step solution of the gradient algorithm and $(\vec{\mathbf{u}}_h, \vec{\mathbf{w}}_h, \mathbf{p}_h, \sigma_h, \vec{\mathbf{g}}_h)$ be the solution of the eq(5.111 - 5.113). Then, the solution of the gradient algorithm converges to $(\vec{\mathbf{u}}_h, \vec{\mathbf{w}}_h, \mathbf{p}_h, \sigma_h, \vec{\mathbf{g}}_h)$ for any initial guess $\vec{\mathbf{g}}_h(0)$ when $k \to \infty$.*

Proof: In order to prove this theorem we have to satisfy the hypotheses of the Lemma 2.9. Let Δt be equal to T/N then for each $\vec{\mathbf{g}}_h$ in $L^2((0,T); X^h)$ the second Frechet derivative $\frac{D^2 L_h^N(\vec{\mathbf{g}}_h)}{D\vec{\mathbf{g}}_{1h} D\vec{\mathbf{g}}_{2h}} \cdot \delta\vec{\mathbf{g}}_{1h}\delta\vec{\mathbf{g}}_{2h}$ can be computed by

$$\frac{D^2 L_h^N(\vec{\mathbf{g}}_h)}{D\vec{\mathbf{g}}_{1h} D\vec{\mathbf{g}}_{2h}} \cdot \delta\vec{\mathbf{g}}_{1h} \cdot \delta\vec{\mathbf{g}}_{2h} = \alpha \sum_{n=1}^{N} \int_\Omega \left(\frac{D\vec{u}_h^{(n)}}{D\vec{\mathbf{g}}_h} \cdot \delta\vec{\mathbf{g}}_{1h}\right)\left(\frac{D\vec{u}_h^{(n)}}{D\vec{\mathbf{g}}_h} \cdot \delta\vec{\mathbf{g}}_{2h}\right) d\vec{x}\Delta t +$$
$$\beta \sum_{n=1}^{N} \int_{\Gamma_c} [\vec{g}_{1h}^{(n)} \delta\vec{g}_{2h}^{(n)} + \beta_2 \nabla_s \delta\vec{g}_{1h}^{(n)} \nabla_s \delta\vec{g}_{2h}^{(n)} + \frac{\beta_1}{\Delta t^2}(\delta g_{1h}^{(n)} - \delta g_{1h}^{(n-1)})(\delta g_{2h}^{(n)} - \delta g_{2h}^{(n-1)})] d\vec{x}\Delta t$$

where $\vec{w}_{h1}^{(n)} = \frac{D\vec{u}_h^{(n)}}{D\vec{\mathbf{g}}_h} \cdot \delta\vec{\mathbf{g}}_{1h}$ and $\vec{w}_{h2}^{(n)} = \frac{D\vec{u}_h^{(n)}}{D\vec{\mathbf{g}}_h} \cdot \delta\vec{\mathbf{g}}_{2h}$ are the Gateaux derivatives. It is easy to show that

$$\sum_{n=1}^{N} \|\vec{w}_{1h}^{(n)}\|^2 \Delta t \leq C_1 \sum_{n=1}^{N} \|\delta \vec{g}_{1h}^{(n)}\|_1^2 \Delta t$$
$$\sum_{n=1}^{N} \|\vec{w}_{2h}^{(n)}\|^2 \Delta t \leq C_1 \sum_{n=1}^{N} \|\delta \vec{g}_{2h}^{(n)}\|_1^2 \Delta t \tag{5.115}$$

where C_1 is a constant. It follows that there exists a constant r such that

$$\frac{D^2 L_h^N(\vec{\mathbf{g}}_h)}{D\vec{\mathbf{g}}_{1h} D\vec{\mathbf{g}}_{2h}} \cdot \delta\vec{\mathbf{g}}_{1h} \cdot \delta\vec{\mathbf{g}}_{2h} \leq r \sum_{n=1}^{N} \|\delta \vec{g}_{1h}^{(n)}\|_1^2 \Delta t \sum_{n=1}^{N} \|\delta \vec{g}_{2h}^{(n)}\|_1^2 \Delta t.$$

Also there exists a constant s such that

$$\frac{D^2 L_h^N(\vec{\mathbf{g}}_h)}{D\vec{\mathbf{g}}_{1h} D\vec{\mathbf{g}}_{2h}} \delta\vec{\mathbf{g}}_{1h} \delta\vec{\mathbf{g}}_{1h} = \alpha \sum_{n=1}^{N} \|\vec{w}_{h1}^{(n)}\|^2 \Delta t +$$
$$\beta \sum_{n=1}^{N} [\|\delta \vec{g}_{1h}^{(n)}\|^2 \Delta t + \beta_2 \|\delta \nabla_s \vec{g}_{1h}^{(n)}\|^2 \Delta t + \beta_1 \|\vec{g}_{1h}^{(n)} - \vec{g}_{1h}^{(n-1)}\|^2] \geq s \sum_{n=1}^{N} \|\delta \vec{\mathbf{g}}_{h2}\|_1^2 \Delta t.$$

The fact that no limitations are imposed assures that for every initial guess the gradient algorithm converges for small Δt. $\sqcup$

5.5.3 Test 1

We consider a unit square domain $(0,1) \times (0,1) \subset \mathbb{R}^2$. We assume that the time interval $[0,T]$ is divided into N equal intervals ($\Delta t = T/N$). The finite element spaces are chosen to be piecewise quadratic for the velocity and linear on the pressure. The Taylor-Hood finite elements are used in this calculation on a rectangular mesh. The same polynomials are used for the restriction on the boundary. The mesh size is h and calculations with varying mesh sizes have been performed. In this first test the control is the tangential velocity on the boundary and the target velocity is defined by

$$\phi(t,z) = (1 - cos(2\pi t z)) \times (1-z)^2$$
$$U(x,y) = 10\frac{d}{dy}\left(\phi(.4,x)\phi(.4,y)\right) \qquad V(x,y) = -10\frac{d}{dx}\left(\phi(.4,x)\phi(.4,y)\right).$$

<u>Velocity tracking evolution</u>

In this velocity tracking evolution the initial velocity has been chosen to be

$$u_0(x,y) = -5U(x,y) \qquad v_0(x,y) = -5V(x,y),$$

that is a high energy flow rotating in opposite direction with respect to the target flow. The control is the right side tangential boundary velocity. This evolution is described in Fig.5.1 - Fig.5.7. with the controlled fluid on the left and the desired flow on the right. All the pictures are normalized by the maximum values. At the beginning (Fig.5.1), the control is small and acts in a very limited area because the high energy initial flow prevails in the central part of the domain. Then, the control gains force and progressively reaches the whole domain (Fig.5.2 - Fig.5.4). In Fig.5.5-Fig.5.7 the controlled flow reaches the optimal approximation and keeps this stationary configuration. Fig.5.8 shows the error $\|\vec{u} - \vec{U}\|$ between the controlled flow $\vec{u}$ and the target flow $\vec{U}$. As we can see the error rapidly goes to a constant value which represents the optimal steady approximation. In this test $\Delta t = 0.05$, α has been set to 1, β to .0001 and $\beta_1 = \beta_2$ to .1. For the same flow Fig.5.9, Fig. 5.10 and Fig. 5.11 show the corresponding values of the norm of the control $\vec{g}$ and its time and space derivatives as a function of time. The control works hard at the beginning in order to steer the controlled flow to the desired one then, decreases and remains flat. Near $t = T$ it is not necessary to drive the flow and a decreasing in boundary velocity minimizes the functional. This small change does not affect the norm error in Fig. 5.8 due to the scale of the graph but we can see a small improvement in the match in Fig. 5.7 at $t = 4$. We remark that the optimal stationary controlled flow is very different from the target flow. One side control is not enough and the result can be considered poor. Smaller values of β can not help and for a better control we need to extend the controlled boundary.

<u>Velocity tracking with different values of β β_1 and β_2</u>

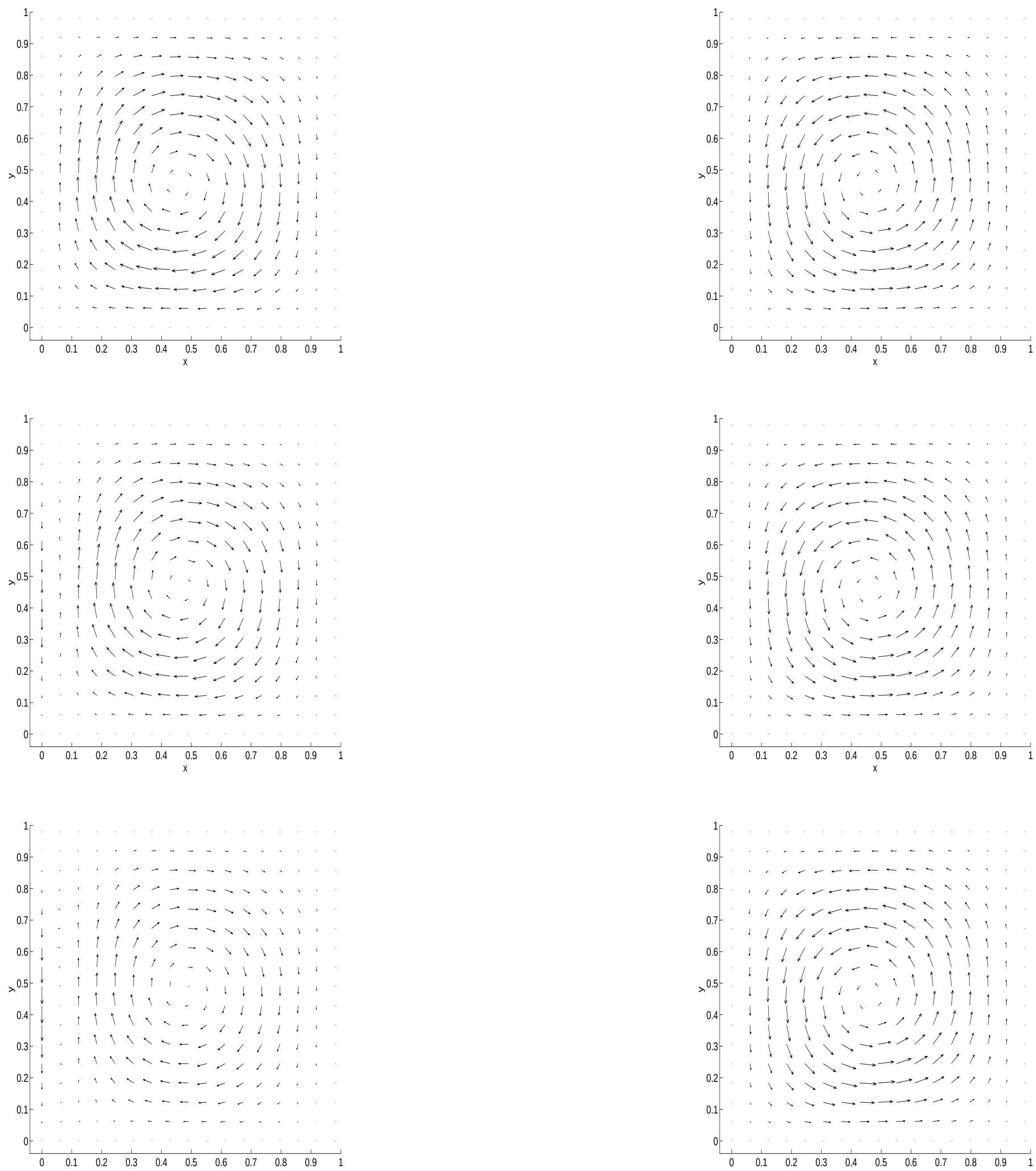

Figure 5.1: Test 1.1s Controlled(right) and desired(left) flow at $t = 0$ (top), $t = .05$ (middle) and $t = .1$ (bottom)

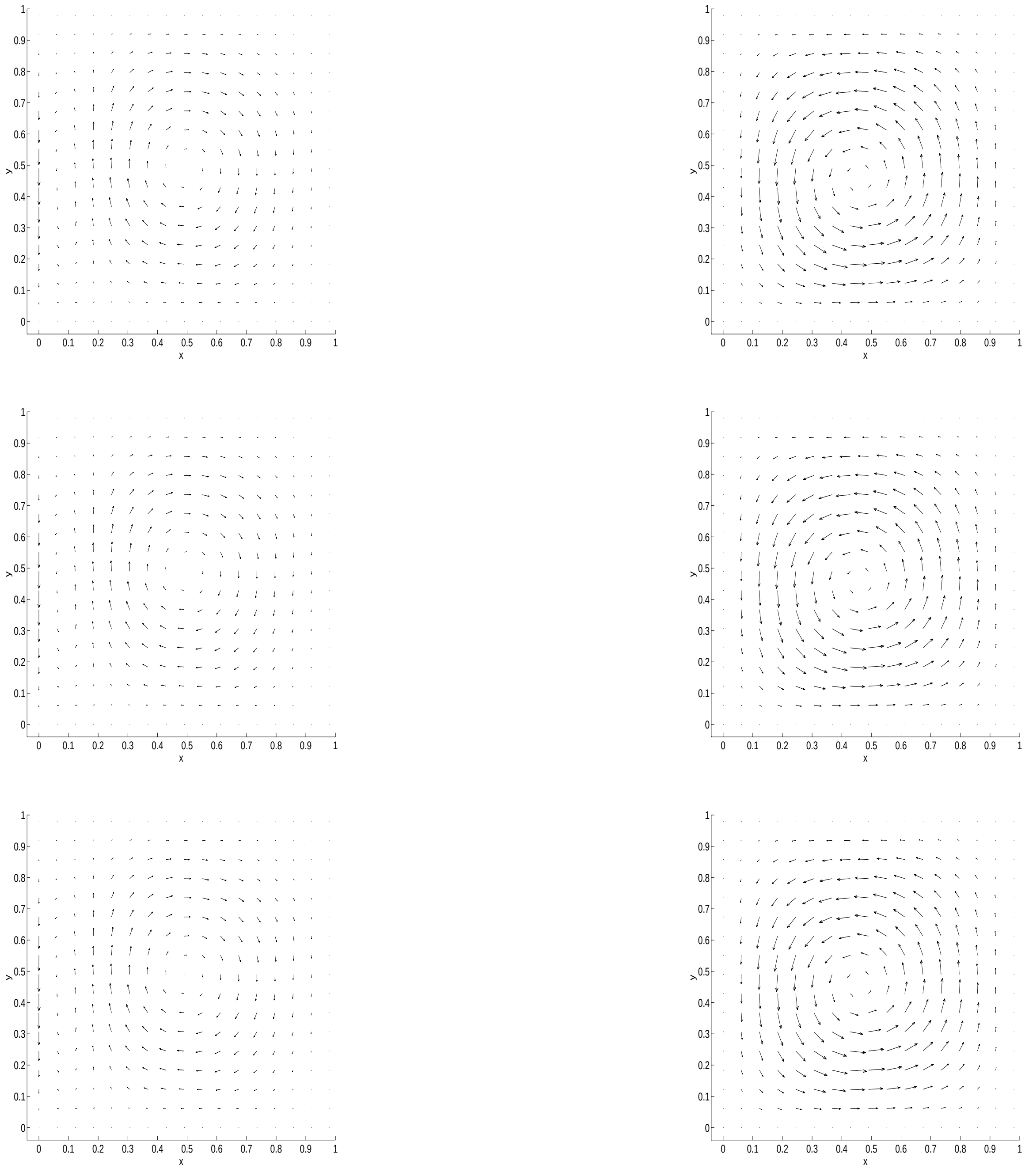

Figure 5.2: Test 1.1s Controlled(left) and desired(right) flow at $t = .15$ (top), $t = .20$ (middle) and $t = .25$ (bottom)

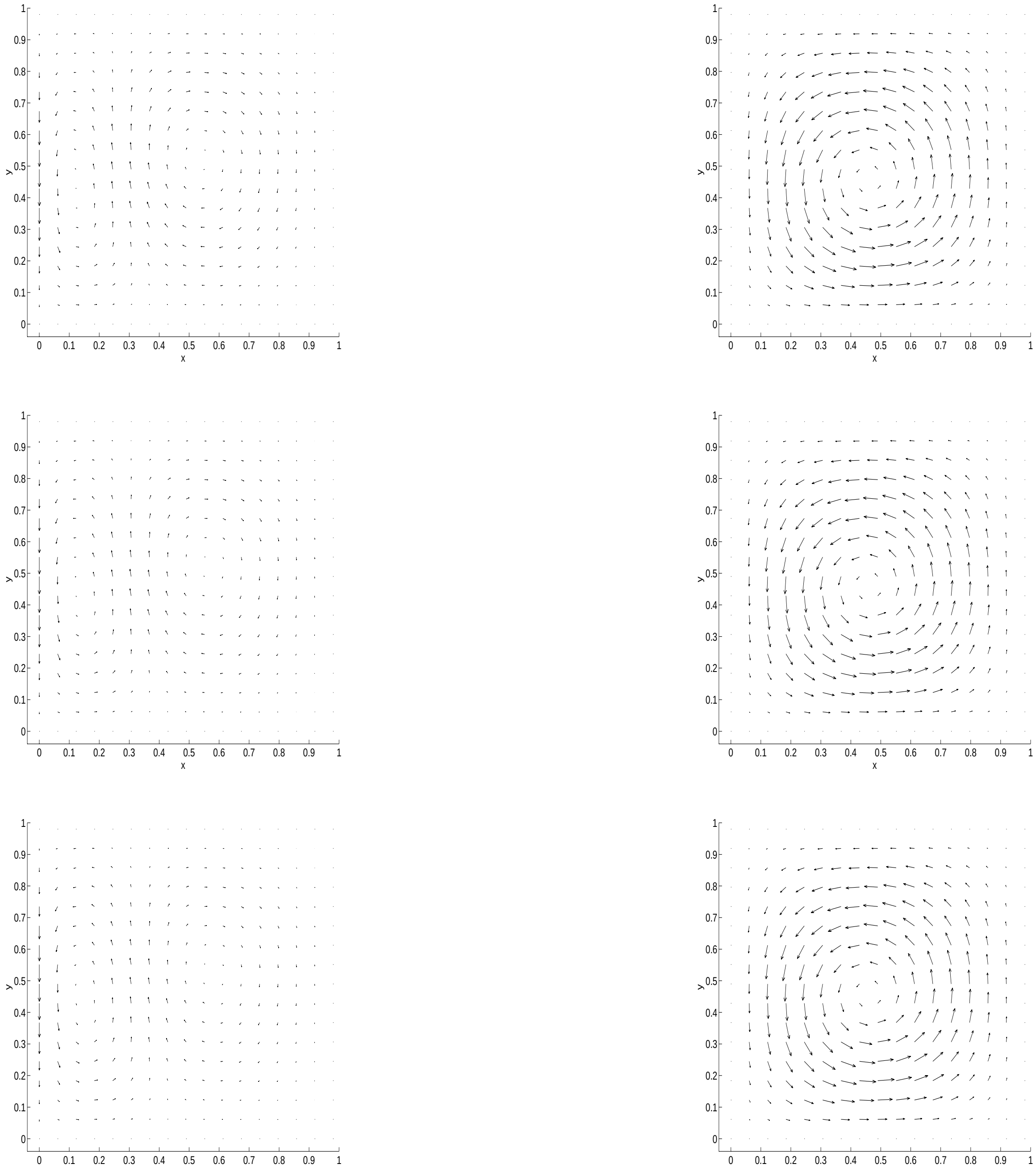

Figure 5.3: Test 1.1s Controlled(left) and desired(right) flow at $t = .3$ (top), $t = .35$ (middle) and $t = .4$ (bottom)

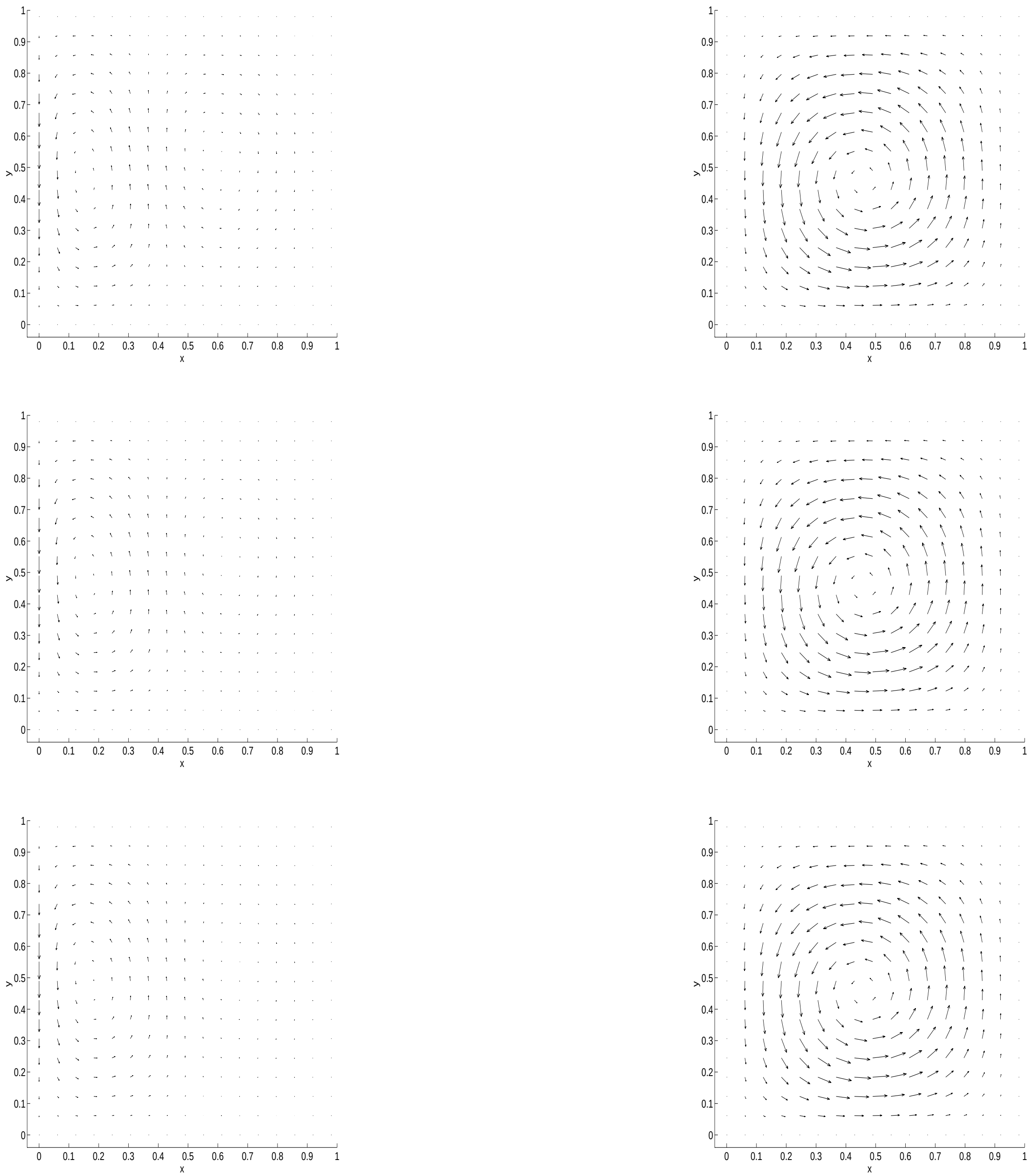

Figure 5.4: Test 1.1s Controlled(left) and desired(right) flow at $t = .45$ (top), $t = .5$ (middle) and $t = .55$ (bottom)

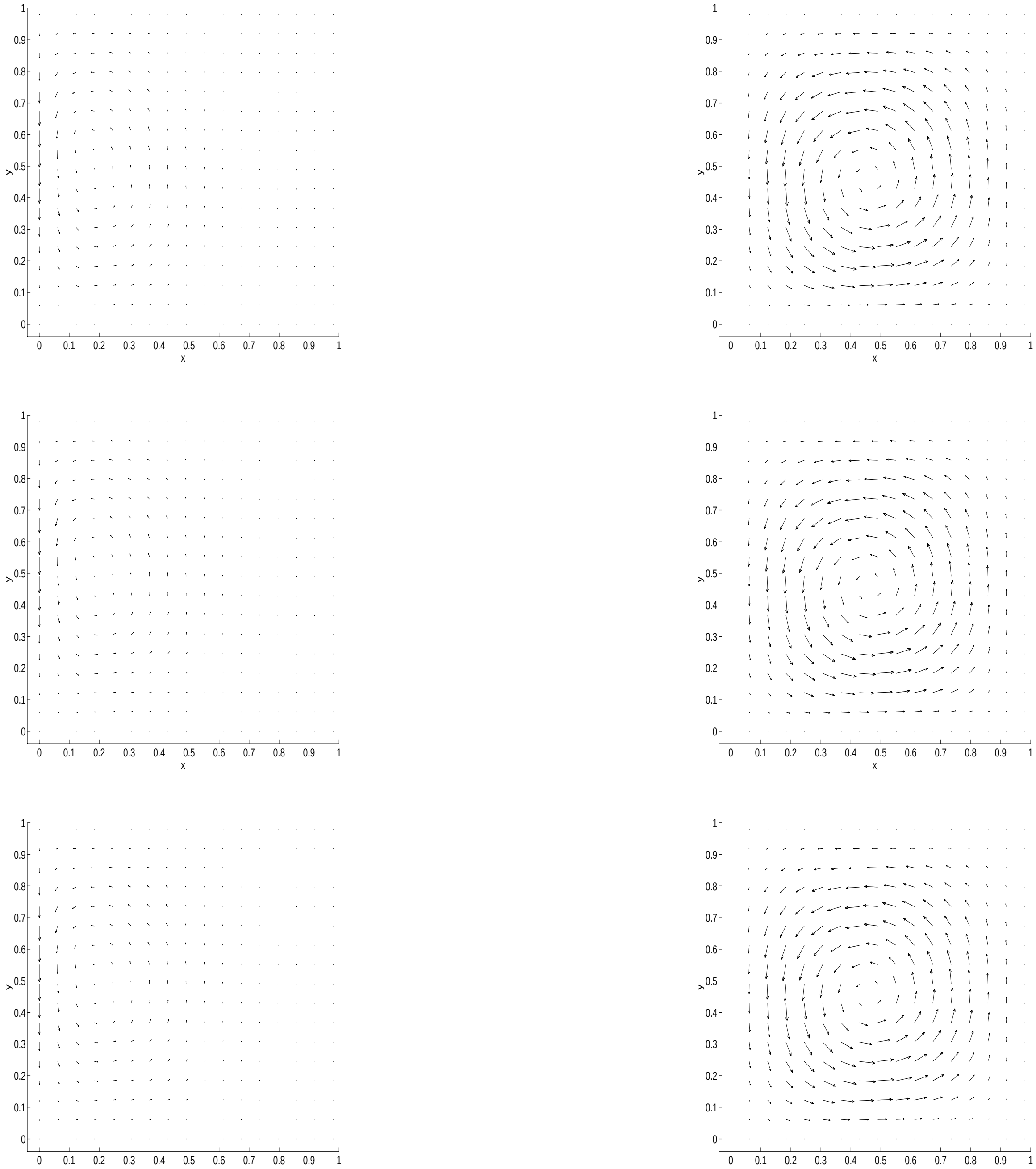

Figure 5.5: Test 1.1s Controlled(left) and desired(right) flow at $t = .6$ (top), $t = .65$ (middle) and $t = .7$ (bottom)

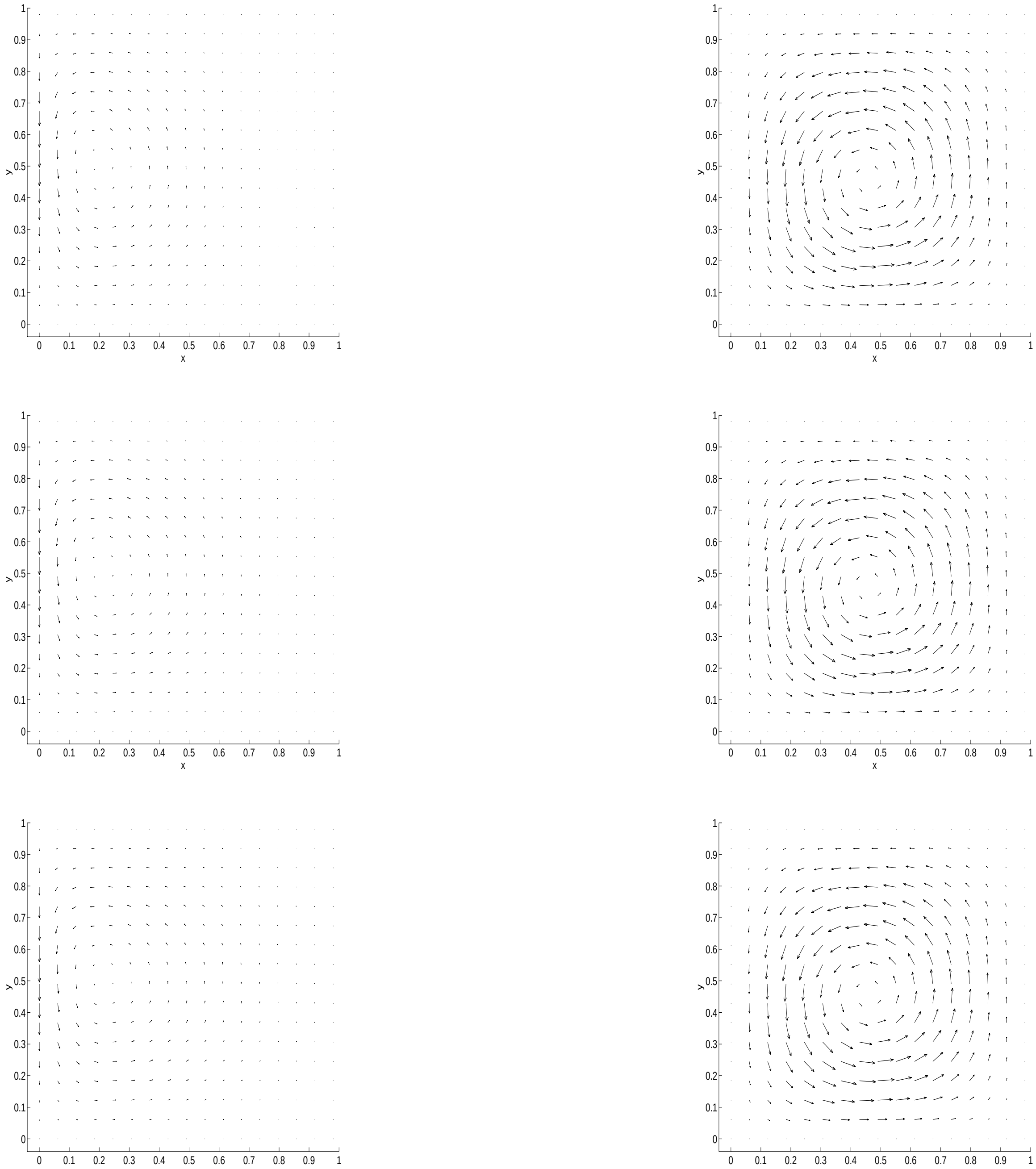

Figure 5.6: Test 1.1s Controlled(left) and desired(right) flow at $t = .75$ (top), $t = 1.$ (middle) and $t = 2.$ (bottom)

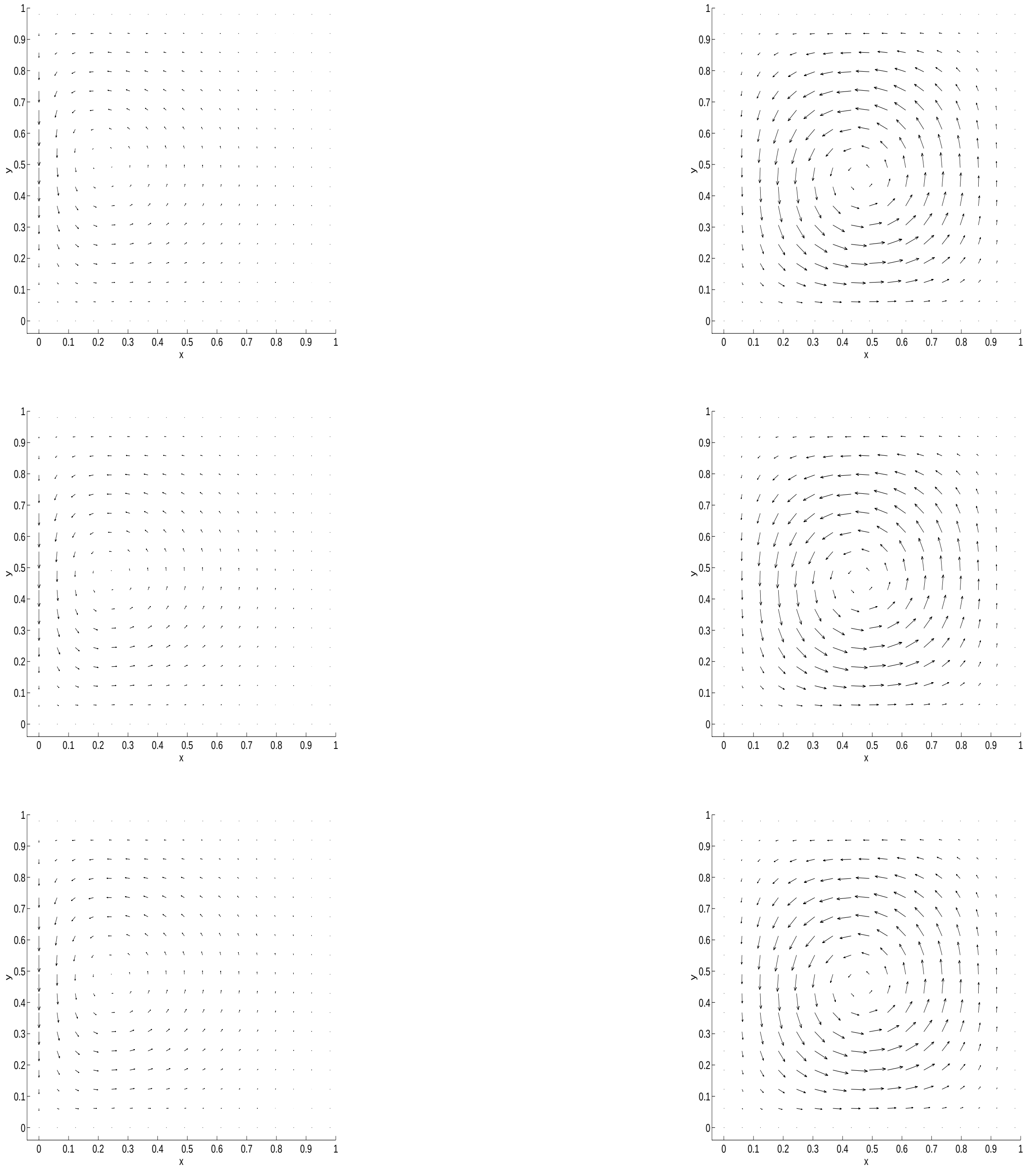

Figure 5.7: Test 1.1s Controlled(left) and desired(right) flow at $t = 3.$ (top), $t = 3.9$ (middle) and $t = 4.$ (bottom)

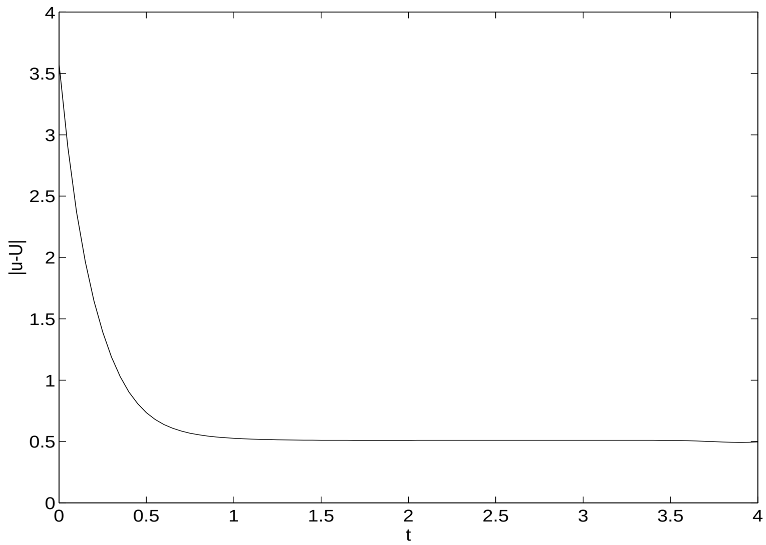

Figure 5.8: Test 1.1s Error $\|\vec{u} - \vec{U}\|$

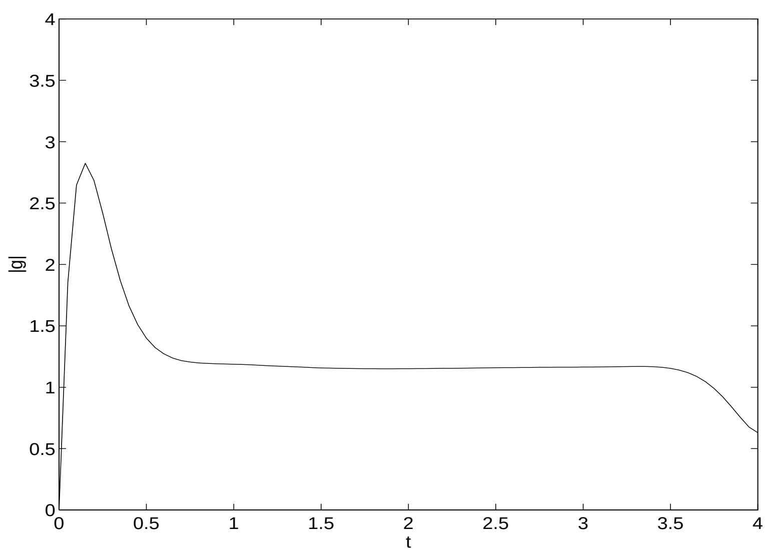

Figure 5.9: Test 1.1s Control norm $\|g\|$

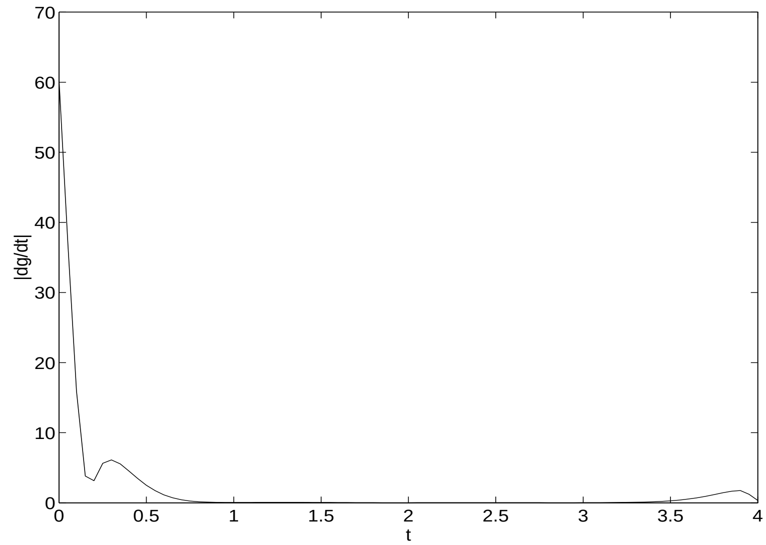

Figure 5.10: Test 1.1s Control norm $\|\vec{g}_t\|$

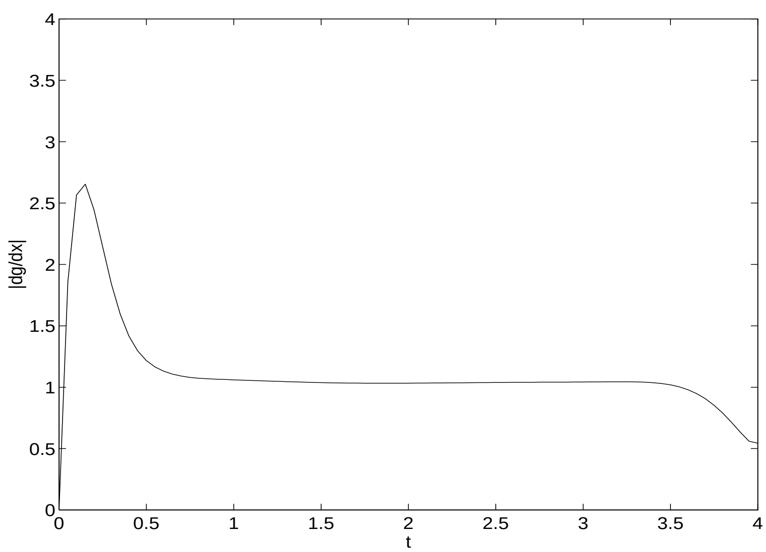

Figure 5.11: Test 1.1s Control norm $\|g_x\|$

We want to analyse what happens if we change the form of the functional by changing the parameters β and β_1 ($\beta_2 = \beta_1$). The initial velocity field is set to be zero.

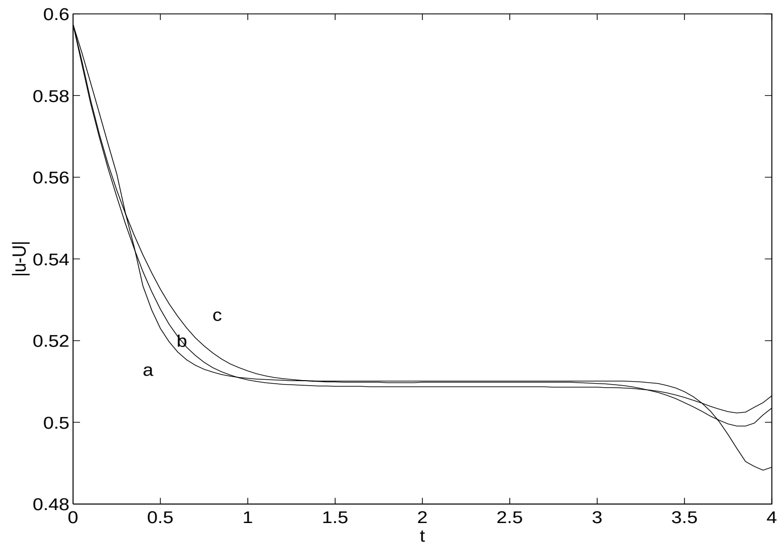

Figure 5.12: Test 1. Error for different β_1

In Fig.5.12 we have the error $\|\vec{u} - \vec{U}\|$ between the controlled flow $\vec{u}$ and the target flow $\vec{U}$ for different values of $\beta_1 = \beta_2$. We have β_1 ($\beta_2 = \beta_1$) equals .01 (a), .1 (b) and 1. (c). The value of β in this computation is held constant at .0001. The time step Δt is again 0.05 and $h = 1/16$. We can note that the control flow matches very well the optimal stationary flow for all the values of β_1. Of course the reduction is quicker when β_1 is lower. The norm of $\vec{g}$ and its derivatives are shown in Fig. 5.13 - 5.15. For low values of β_1 the control is allowed to move quicker and the sensibility of the system is greater. We note that there is a small change in control magnitude but the optimal stationary flow is still approximately the same. Differences can be noted in the non steady part of the evolution flow.

For different values of β we have different controls. In Fig.5.16 we see the error $\|\vec{u} - \vec{U}\|$ between the controlled flow $\vec{u}$ and the target flow $\vec{U}$ for β equal to .0001 (a), .001 (b) and .01 (c). For small values of β the optimal control solution is limited by the magnitude of the control and its derivatives. For values of $\beta > .005$ the control is poor. Generally a good control involves small values of β around .001. In Fig.5.17 - Fig. 5.19 we have the norm of the control $\vec{g}$ and its derivatives for the corresponding values of β. As expected the control $\vec{g}$ approaches zero for low values of β.

<u>Different number of controlled sides</u>

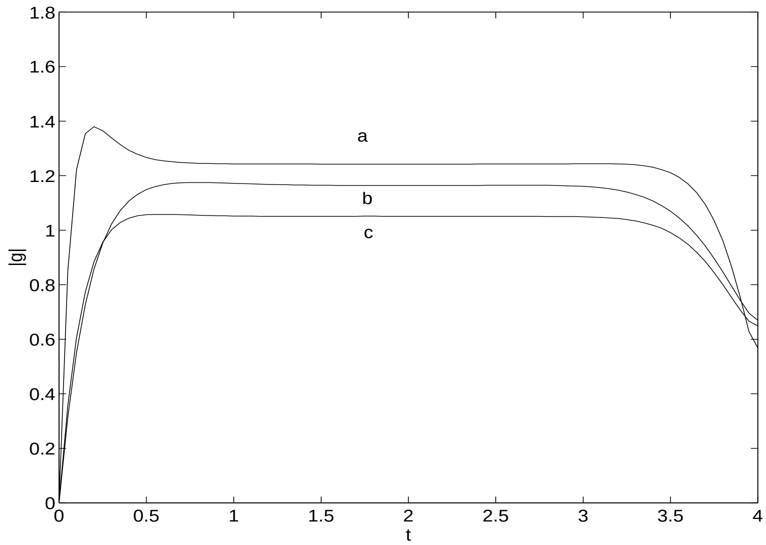

Figure 5.13: Test 1. Control norm $\|g\|$ for different β_1

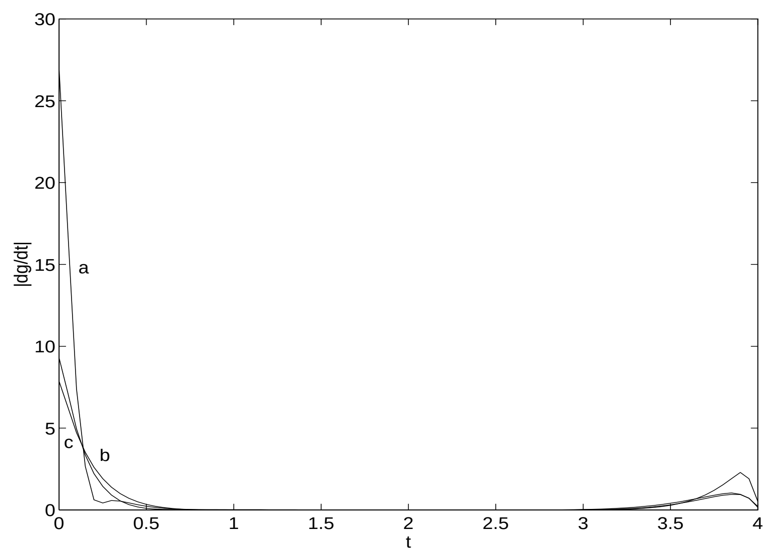

Figure 5.14: Test 1. Control norm $\|g_t\|$ for different β_1

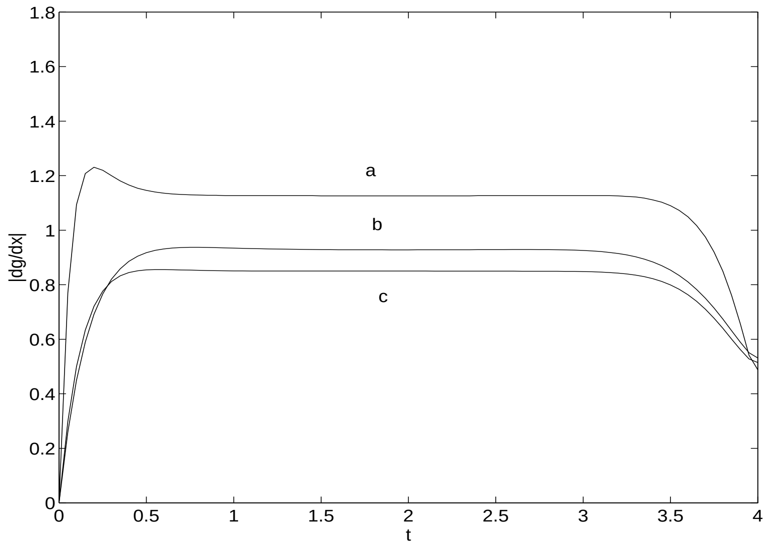

Figure 5.15: Test 1. control norm $\|g_x\|$ for different β_1

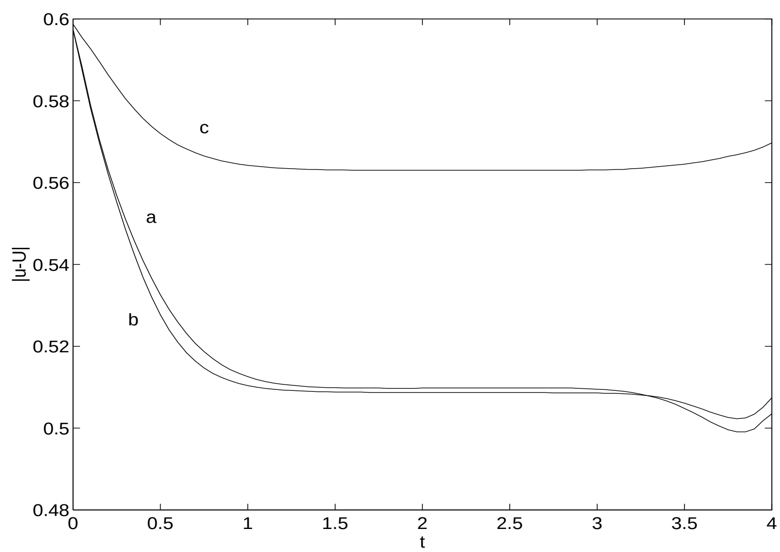

Figure 5.16: Test 1. Error $\|\vec{u} - \vec{U}\|$ for different β

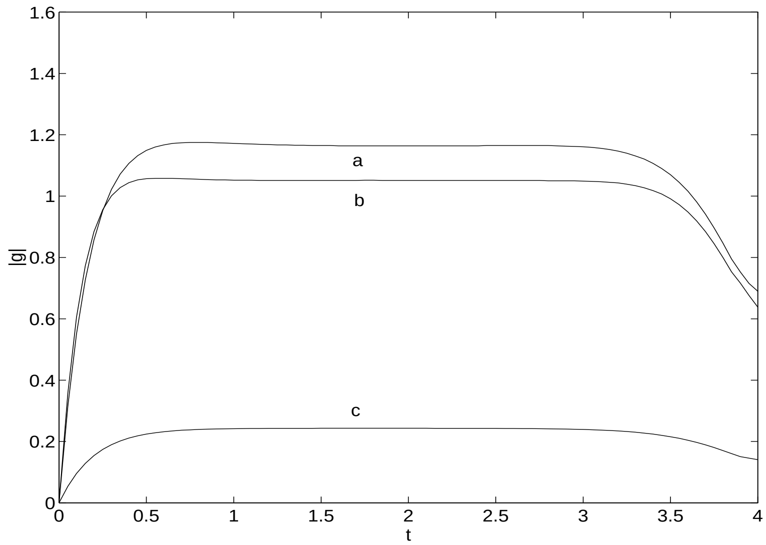

Figure 5.17: Test 1. Control norm $\|g\|$ for different β

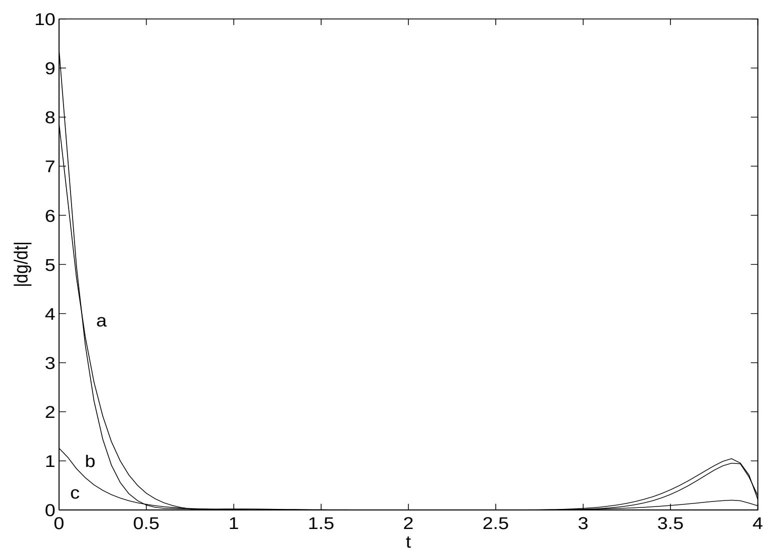

Figure 5.18: Test 1. Control norm $\|g_t\|$ for different β

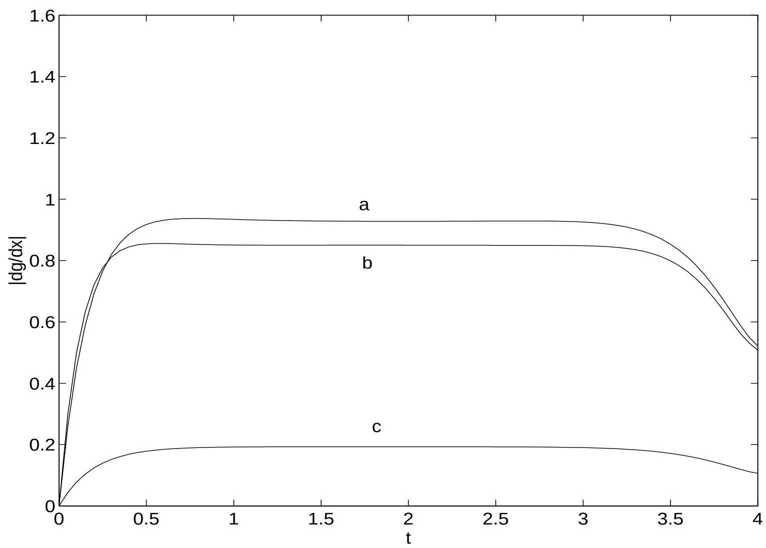

Figure 5.19: Test 1. Control norm $\|g_x\|$ for different β

Now the control is extended to all the four sides of the domain. We try to repeat the evolution problem changing the number of sides. In Fig.5.20, Fig. 5.21,Fig.5.22 and Fig. 5.23 we have the norm error, the control norm, the derivative norm in time and in space respectively with different number of sides. One-side control is in (a), two-side control in (b), three-side control in (c) and finally the control on the whole boundary in (d). Of course the match is improving with increasing number of controlled sides. As we can see in Fig. 5.21 the control behaves better and the maximum strength required decreases when the number of controlled sides increases. A picture of the stationary match can be found in Fig. 5.24. The improvement is evident. The tracking time evolution for the four-side case is reported in Fig.5.25, Fig. 5.31

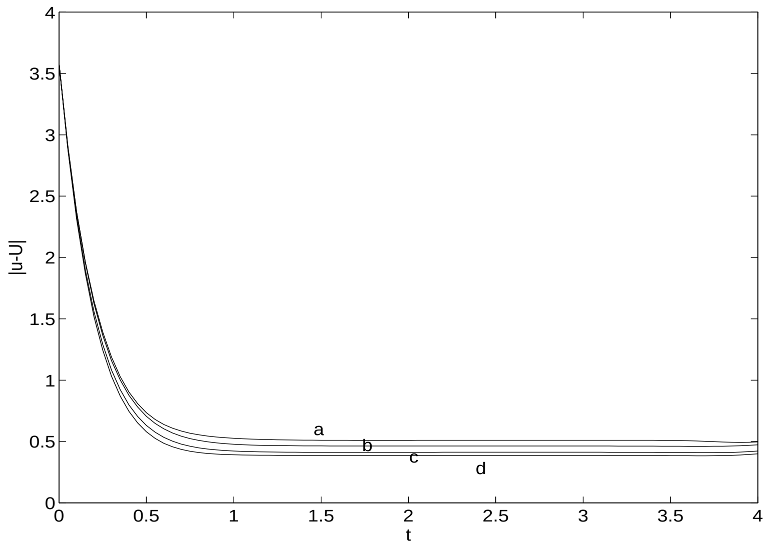

Figure 5.20: Test 1. Error $\|\vec{u} - \vec{U}\|$ for different controlled side number

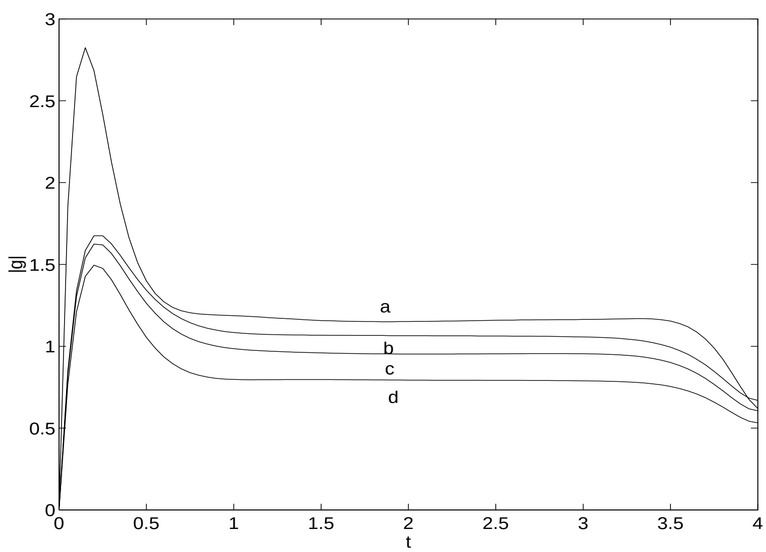

Figure 5.21: Test 1. Control norm $\|g\|$ for different controlled side number

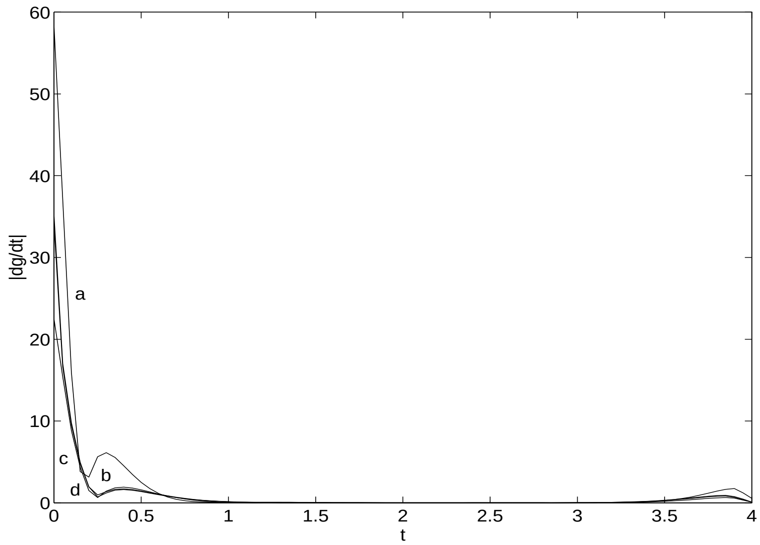

Figure 5.22: Test 1. Control norm $\|\vec{g}_t\|$ for different controlled side number

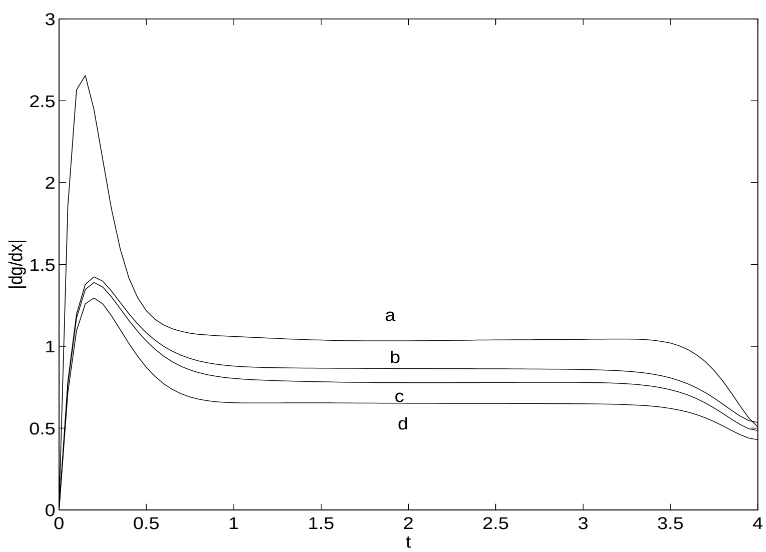

Figure 5.23: Test 1. Control norm $\|g_x\|$ for different controlled side number

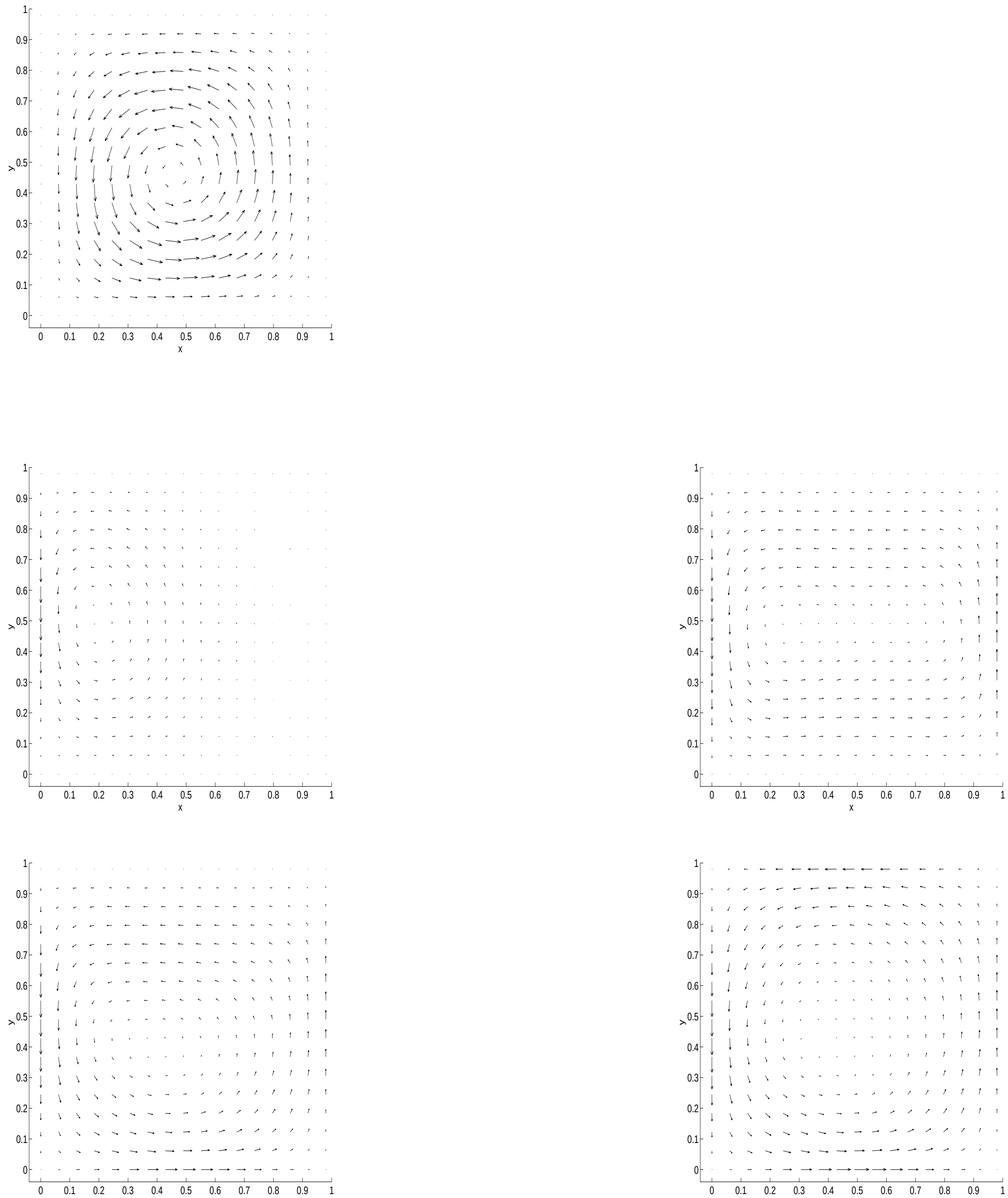

Figure 5.24: Test 1. Desired flow (top), stationary one-side (middle left), two-side (middle right), three-side (bottom left) and four side controlled flow (bottom right)

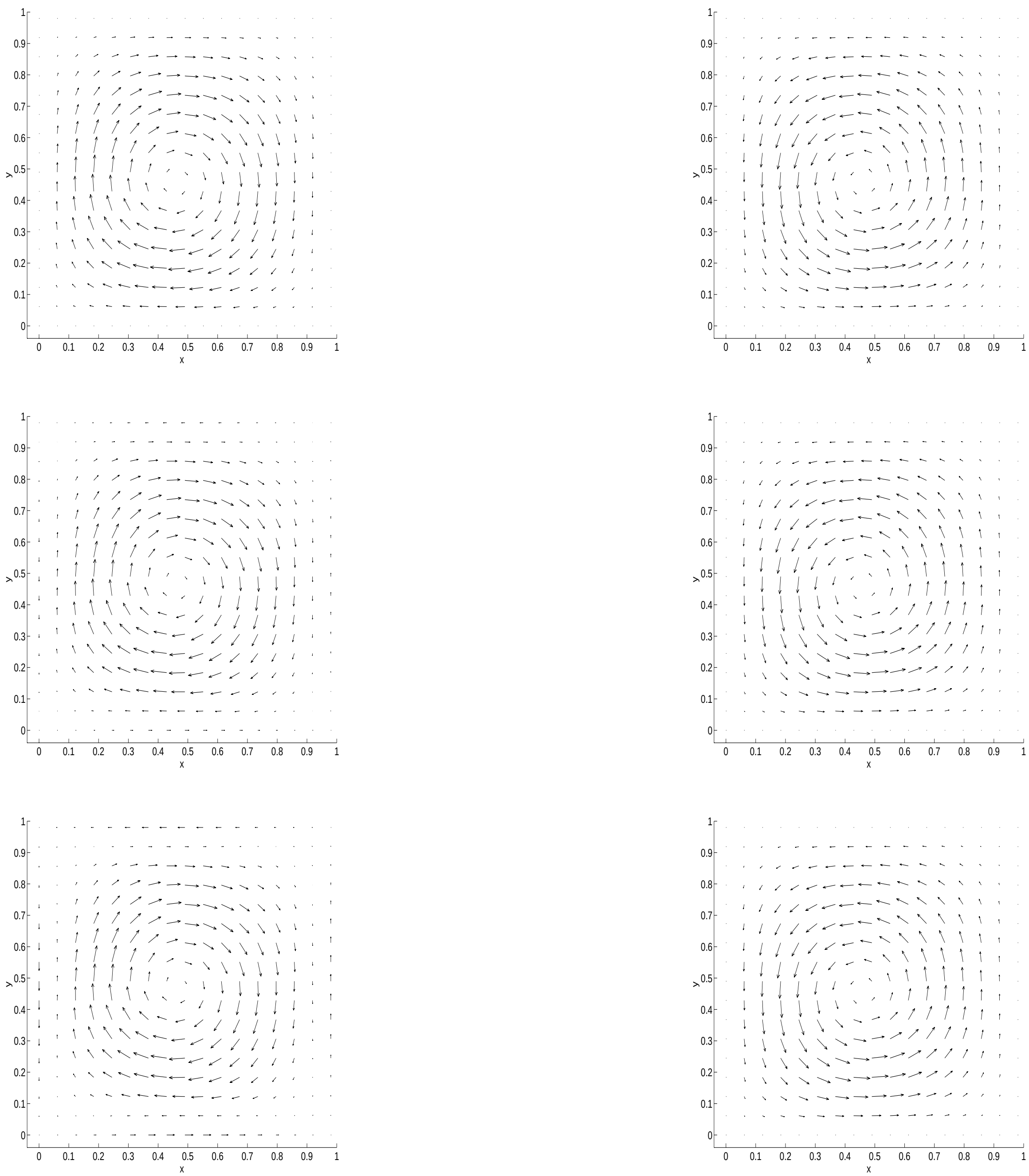

Figure 5.25: Test 1.4s Controlled(right) and desired(left) flow at $t = 0$ (top), $t = .05$ (middle) and $t = .1$ (bottom)

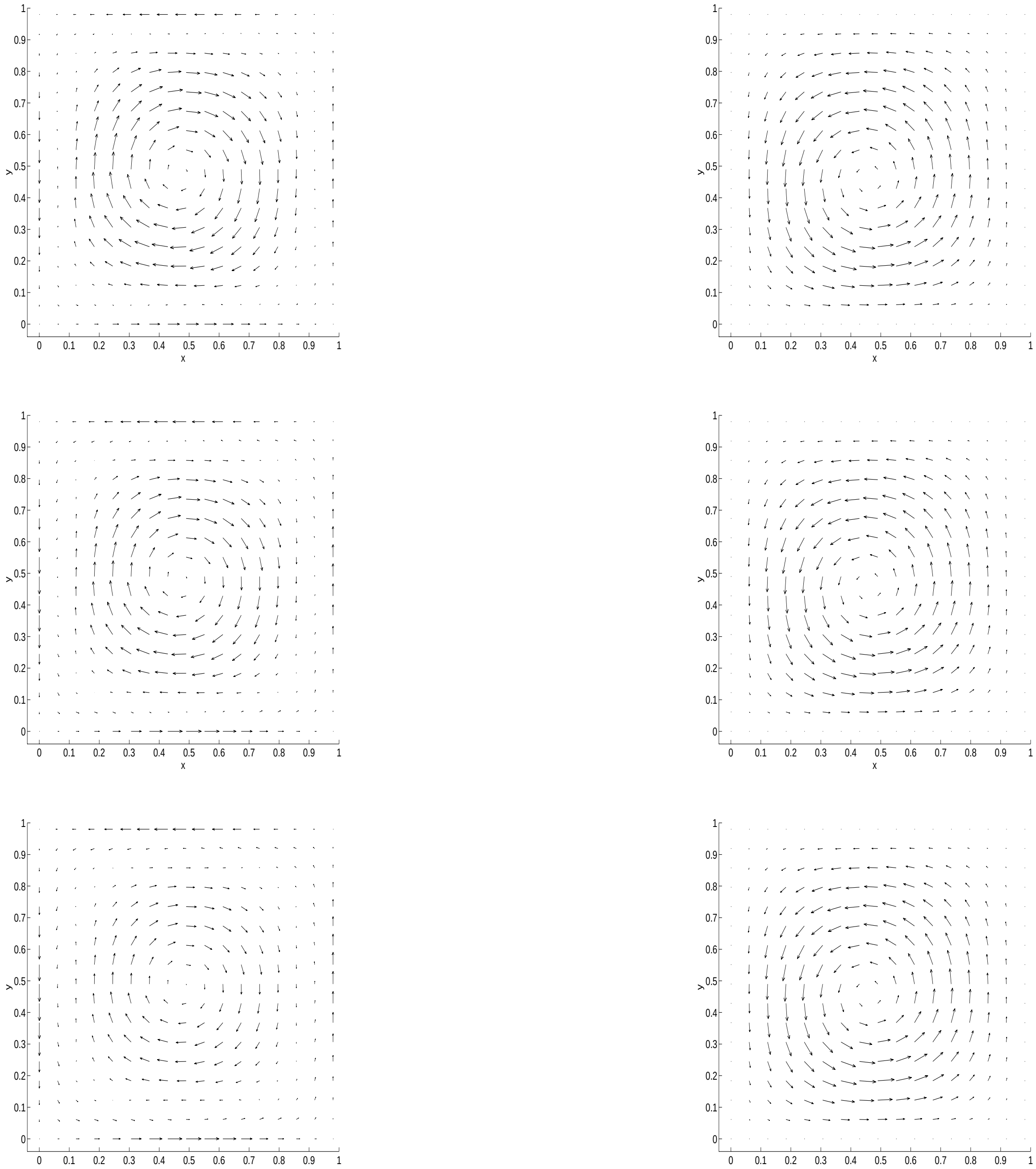

Figure 5.26: Test 1.4s Controlled(left) and desired(right) flow at $t = .15$ (top), $t = .20$ (middle) and $t = .25$ (bottom)

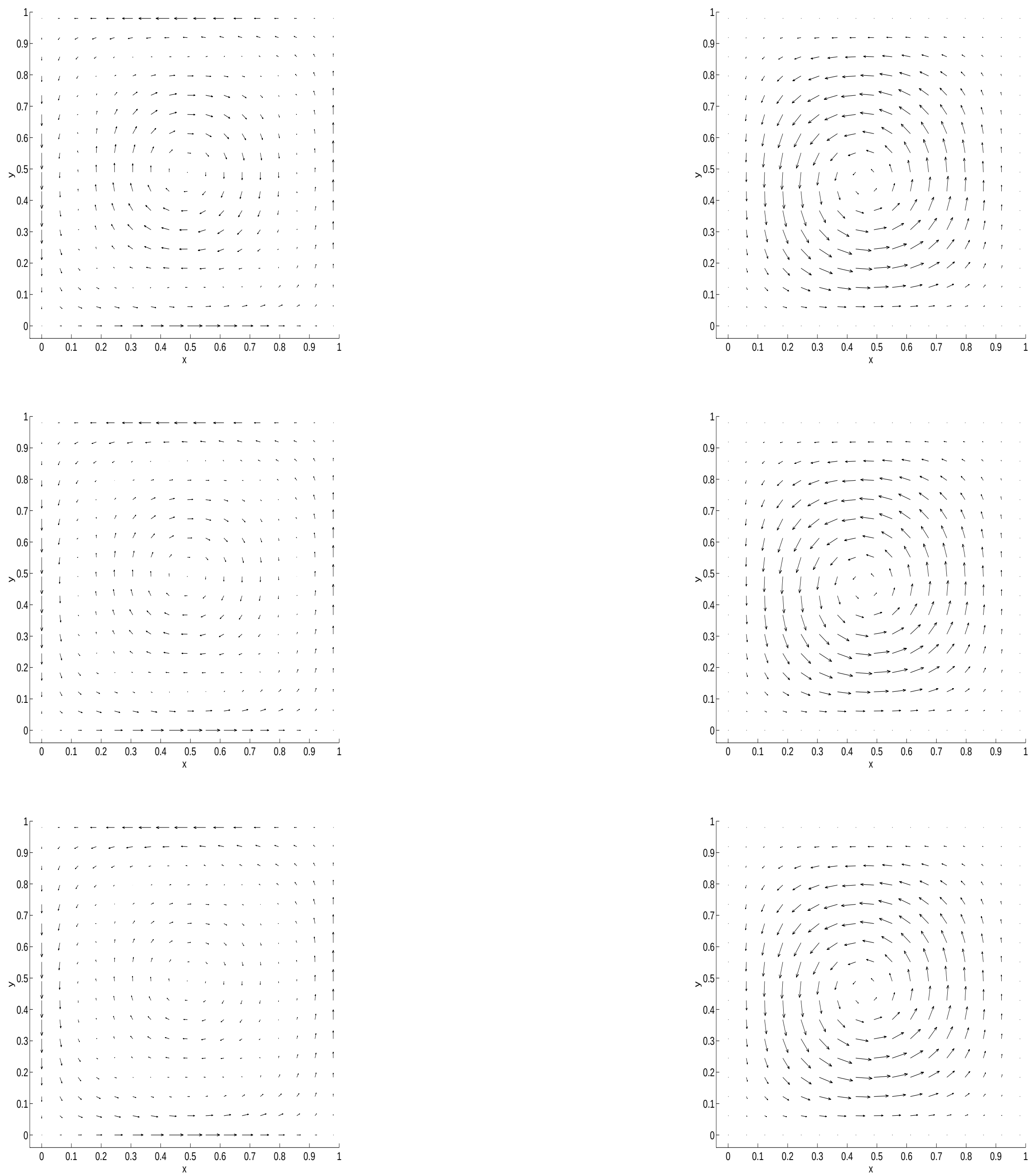

Figure 5.27: Test 1.4s Controlled(left) and desired(right) flow at $t = .3$ (top), $t = .35$ (middle) and $t = .4$ (bottom)

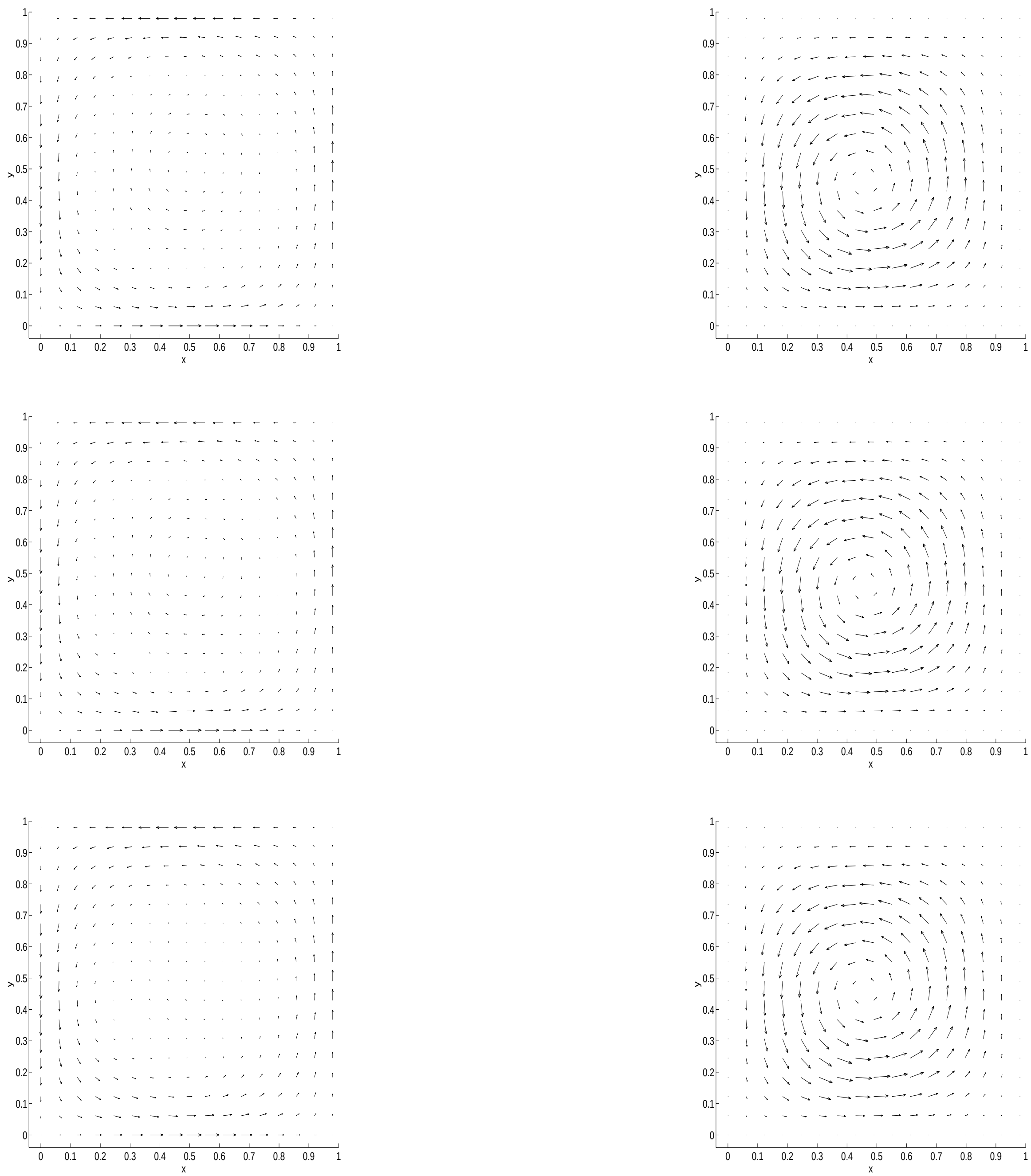

Figure 5.28: Test 1.4s Controlled(left) and desired(right) flow at $t = .45$ (top), $t = .5$ (middle) and $t = .55$ (bottom)

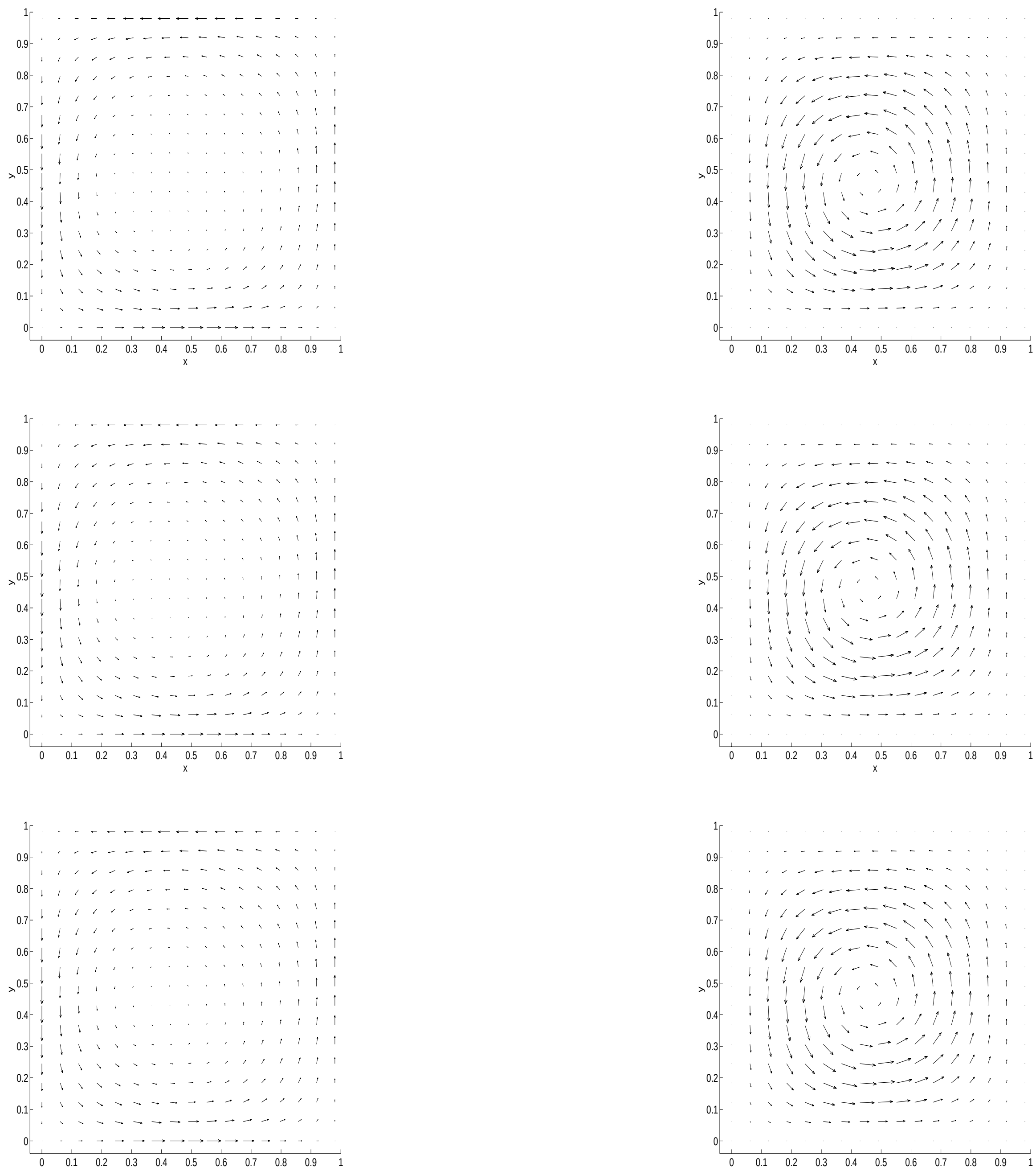

Figure 5.29: Test 1.4s Controlled(left) and desired(right) flow at $t = .6$ (top), $t = .65$ (middle) and $t = .7$ (bottom)

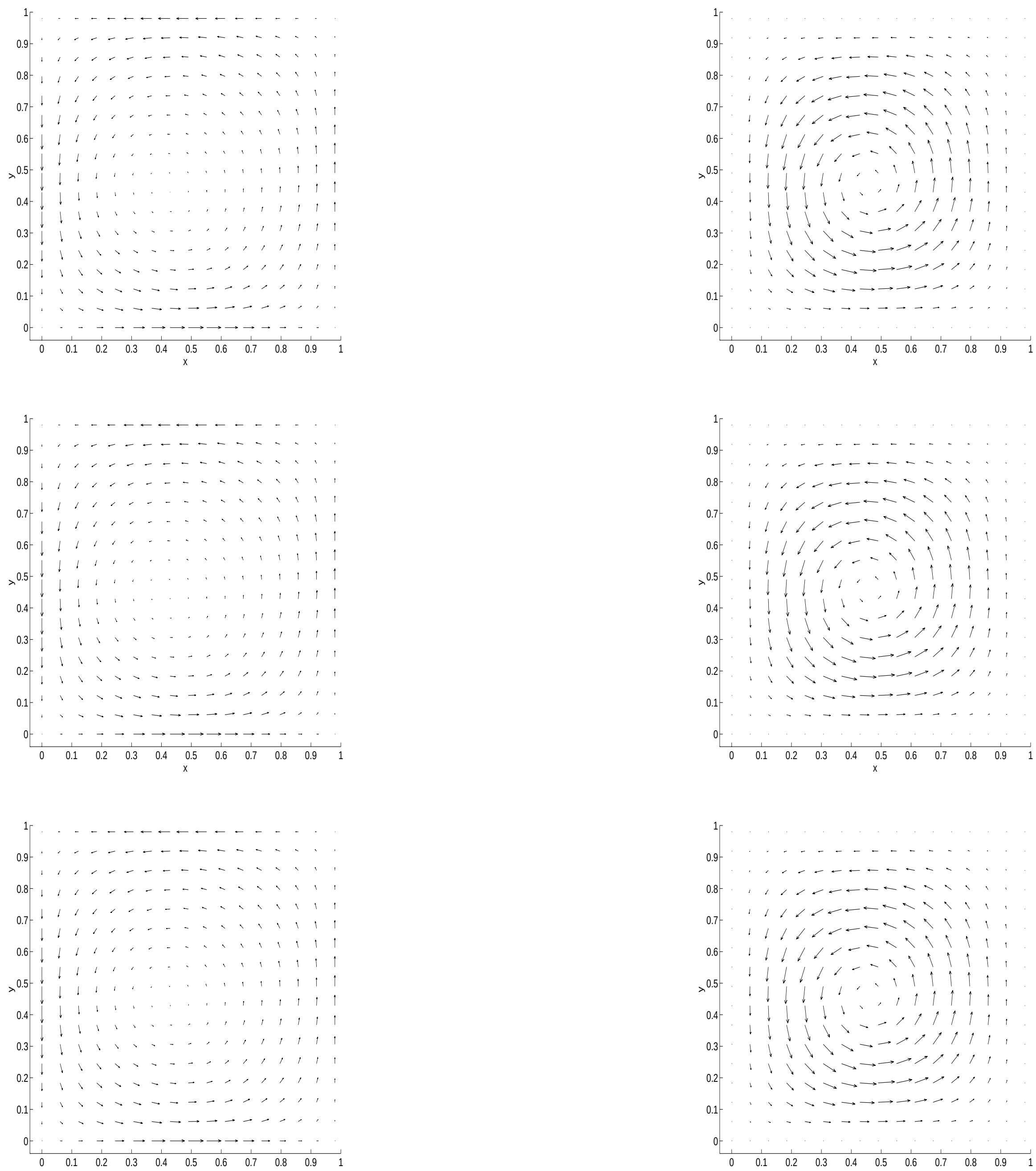

Figure 5.30: Test 1.4s Controlled(left) and desired(right) flow at $t = .8$ (top), $t = .9$ (middle) and $t = 1.$ (bottom)

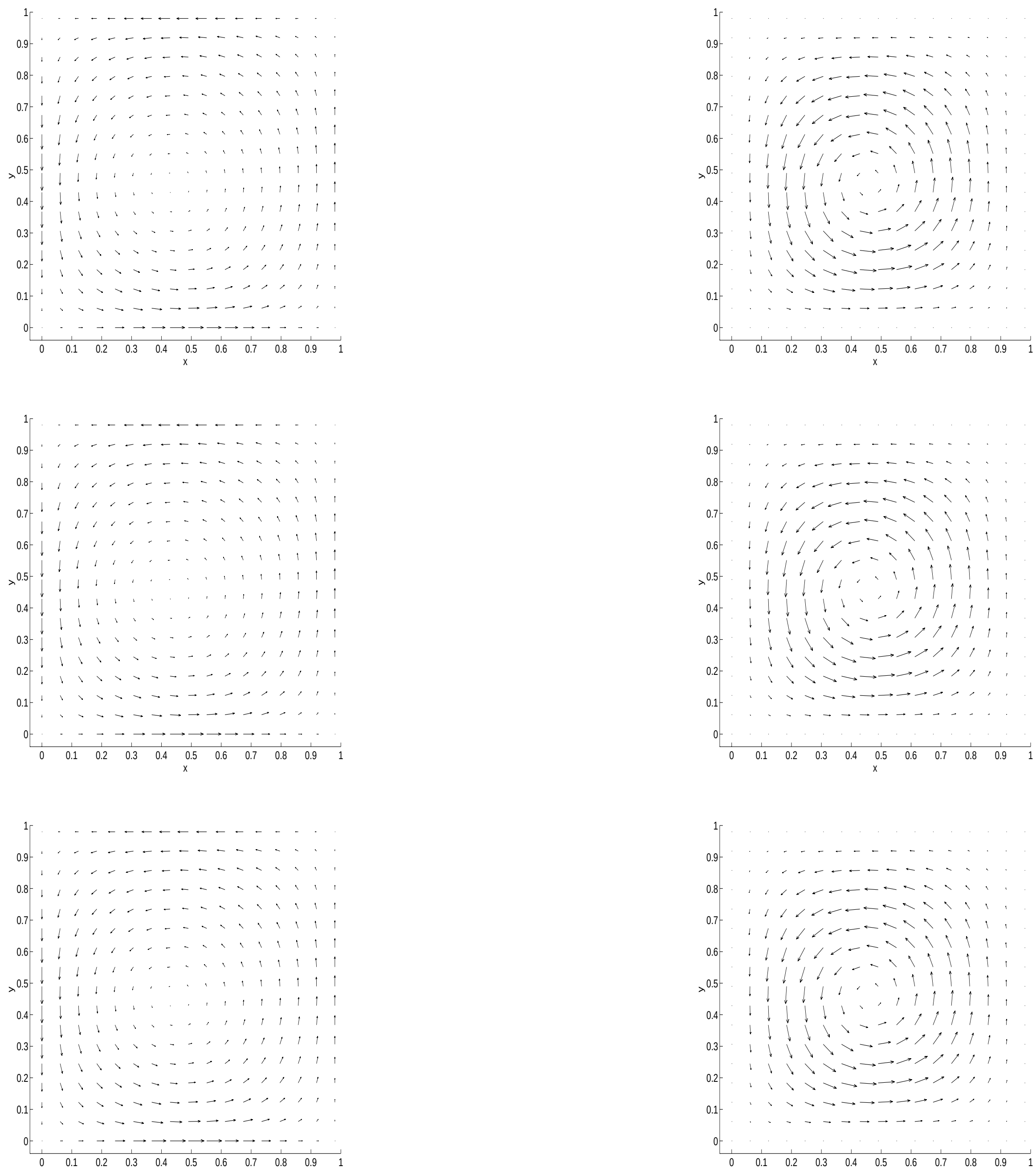

Figure 5.31: Test 1.4s Controlled(left) and desired(right) flow at $t = 2.$ (top), $t = 3.$ (middle) and $t = 4.$ (bottom)

5.5.4 Test 2

We consider a unit square domain $(0,1) \times (0,1) \subset \mathbb{R}^2$. We assume that the time interval [0,T] is divided in equal intervals of time $\Delta t = T/N$. The Taylor-Hood finite elements are used in this calculation on a rectangular mesh. We report only the final result with $h = 1/16$ but calculations with varying mesh sizes has been performed. The target velocity $\vec{U}$ for this test is equal to

$$\phi(k,t,z) = (1 - cos(2k\pi tz)) \times (1-z)^2$$
$$a(k,t,x,y) = \frac{d}{dy}\left(\phi(k,t,x)\phi(k,t,y)\right) \qquad b(k,t,x,y) = -\frac{d}{dx}\left(\phi(k,t,x)\phi(k,t,y)\right)$$
$$U = a(.25,.4,x,y) + a(.5,t,x,y)/(4\pi t + 1)$$
$$V = b(.25,.4,x,y) + b(.5,t,x,y)/(4\pi t + 1).$$

With this velocity field we have the superposition of two flows. One flow with a vortex at the center of the domain and another flow with four vortices. Each of these flows prevails at different times of the evolution. The initial velocity for the controlled flow is

$$u_0(x,y) = -5U(1,x,y) \qquad v_0(x,y) = -5V(1,x,y) \tag{5.116}$$

The evolution is in Fig.5.32 - Fig.5.37. In this computation α has been set to 1, β to .001 and $\beta_1 = \beta_2 = 0.1$. The control $\vec{g}$ is a four side control and covers the whole boundary Γ. The controlled fluid is on the left, the desired flow is on the right and all the pictures are normalized. As we can see at $t = .5$ the controlled flow reaches the optimal approximation and follows the motion of the target fluid. Fig.5.38 shows the error $\|\vec{u} - \vec{U}\|$ between the controlled flow $\vec{u}$ and the target flow $\vec{U}$. At the beginnig the error rapidly decreases but after this initial interval of time this error increases due to changes in the desired flow. The boundary velocity can not control the interior of the domain if the desired flow moves rapidly. However this represents the optimium that can be done with this energy available. For the same flow, Fig.5.39 - Fig.5.41 show the values of the norm of the control variable $\vec{g}$ and its derivatives.

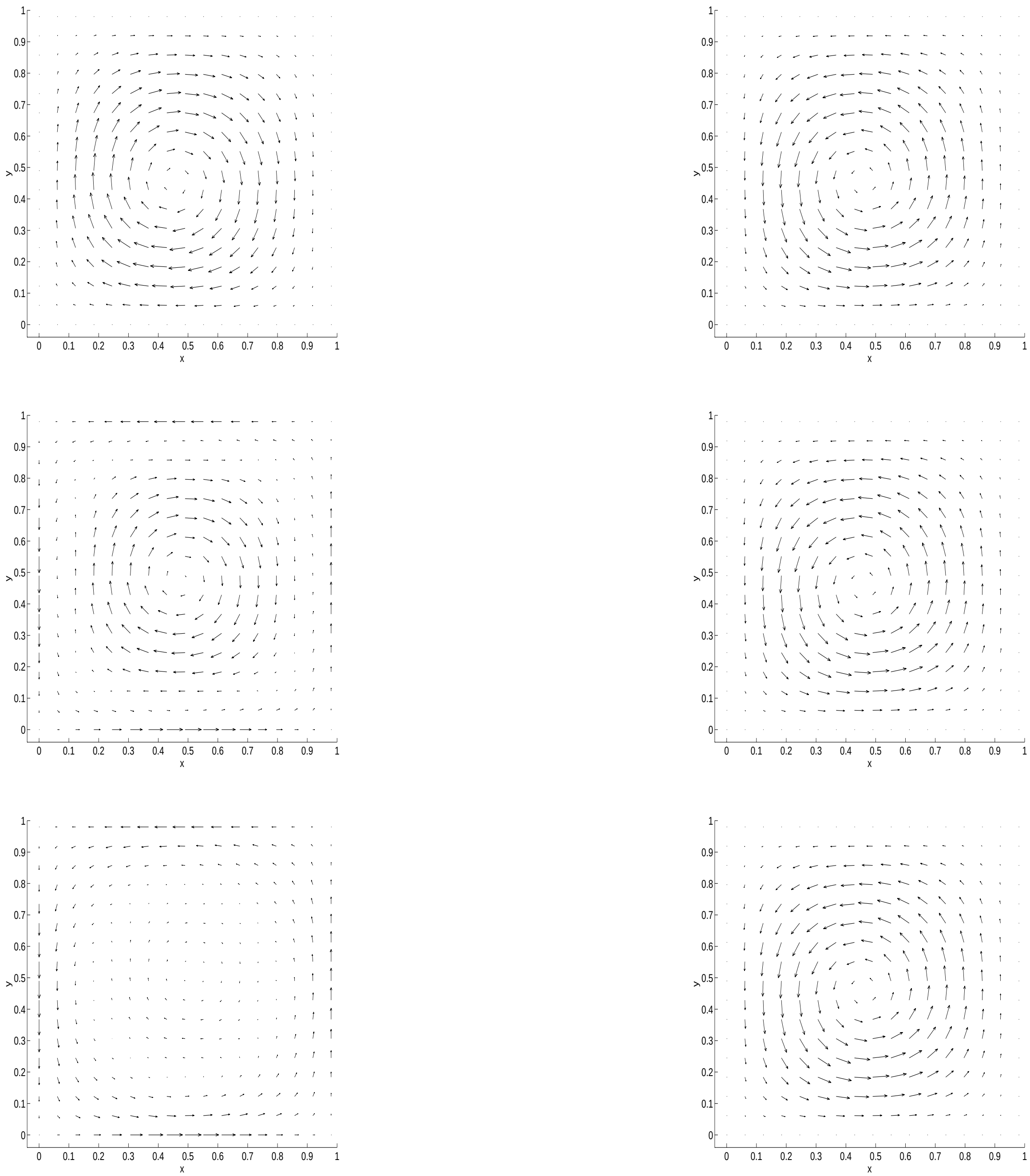

Figure 5.32: Test 2. Controlled(right) and desired(left) flow at $t = 0$ (top), $t = .25$ (middle) and $t = .5$ (bottom)

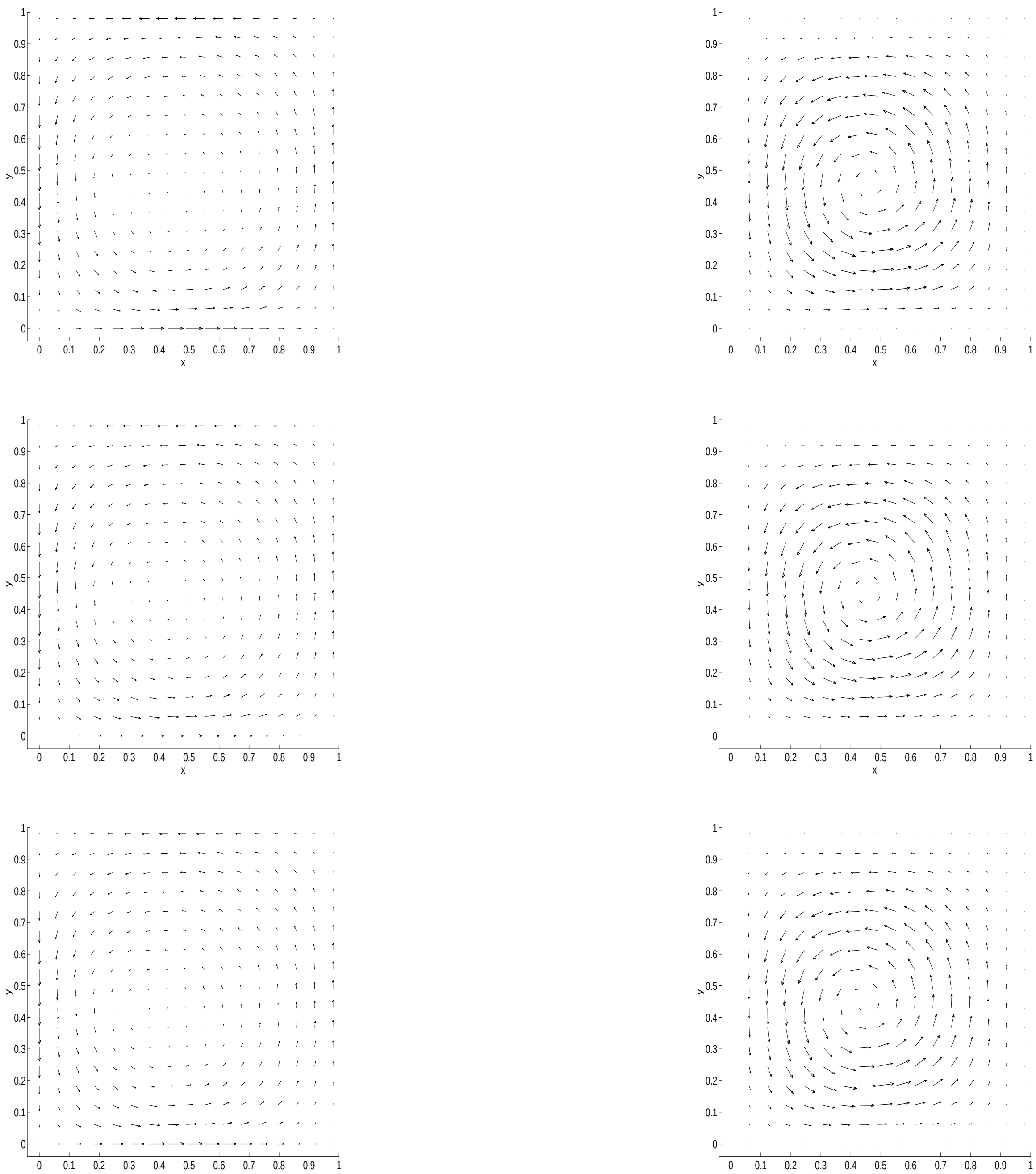

Figure 5.33: Test 2. Controlled(left) and desired(right) flow at $t = .75$ (top), $t = 1.$ (middle) and $t = 1.25$ (bottom)

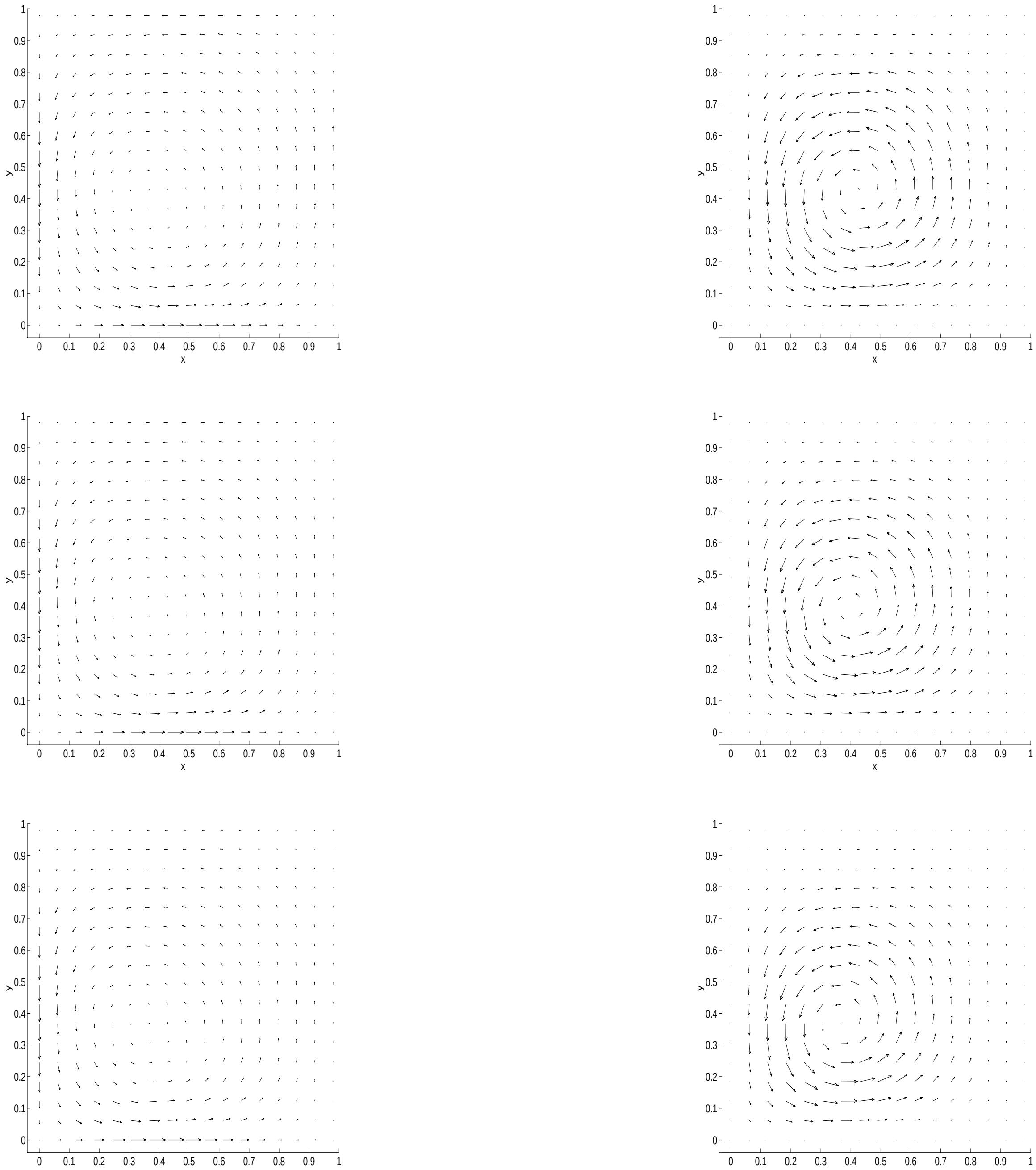

Figure 5.34: Test 2. Controlled(left) and desired(right) flow at $t = 1.5$ (top), $t = 1.75$ (middle) and $t = 2.$ (bottom)

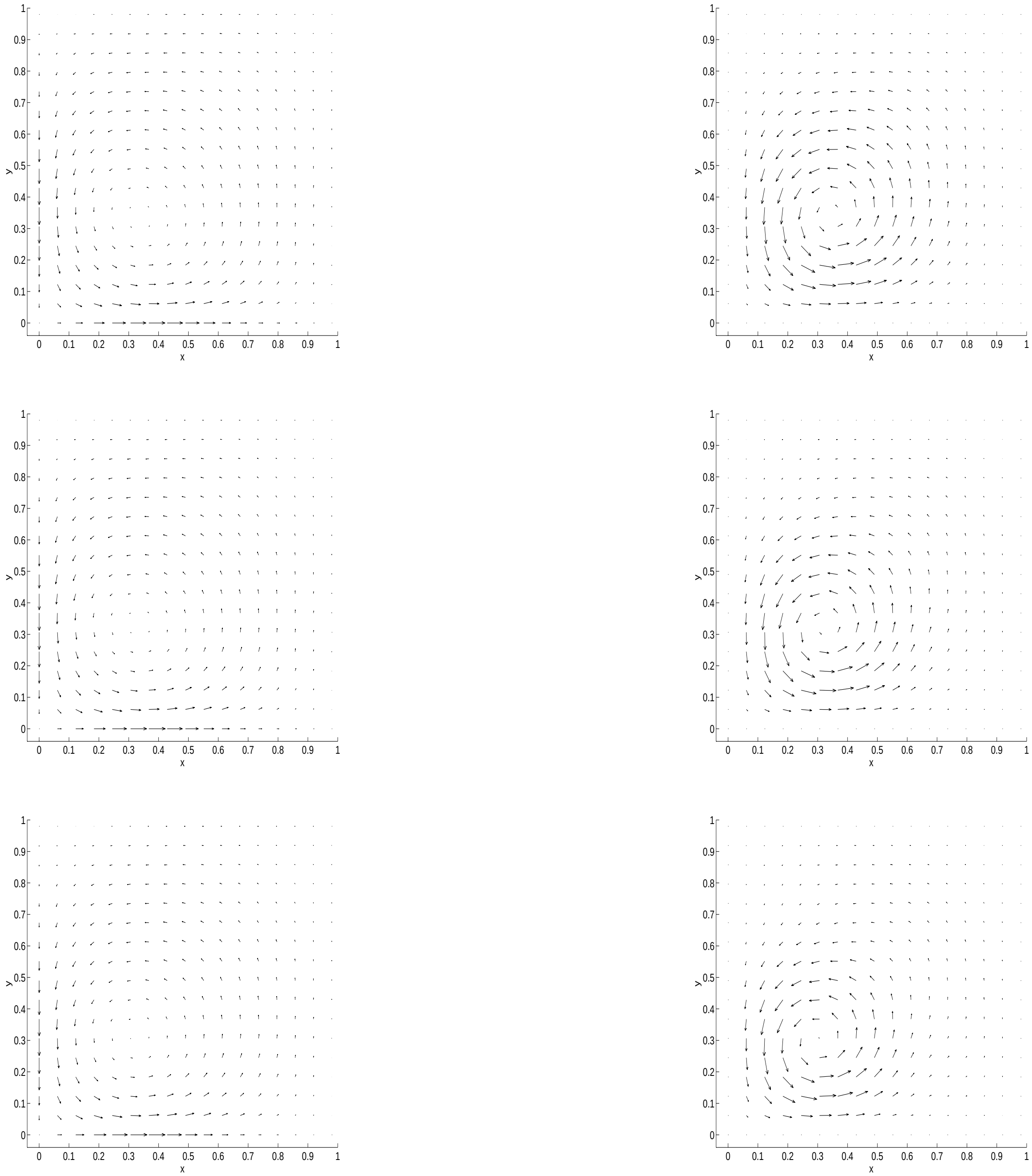

Figure 5.35: Test 2. Controlled(left) and desired(right) flow at $t = 2.25$ (top), $t = 2.5$ (middle) and $t = 2.75$ (bottom)

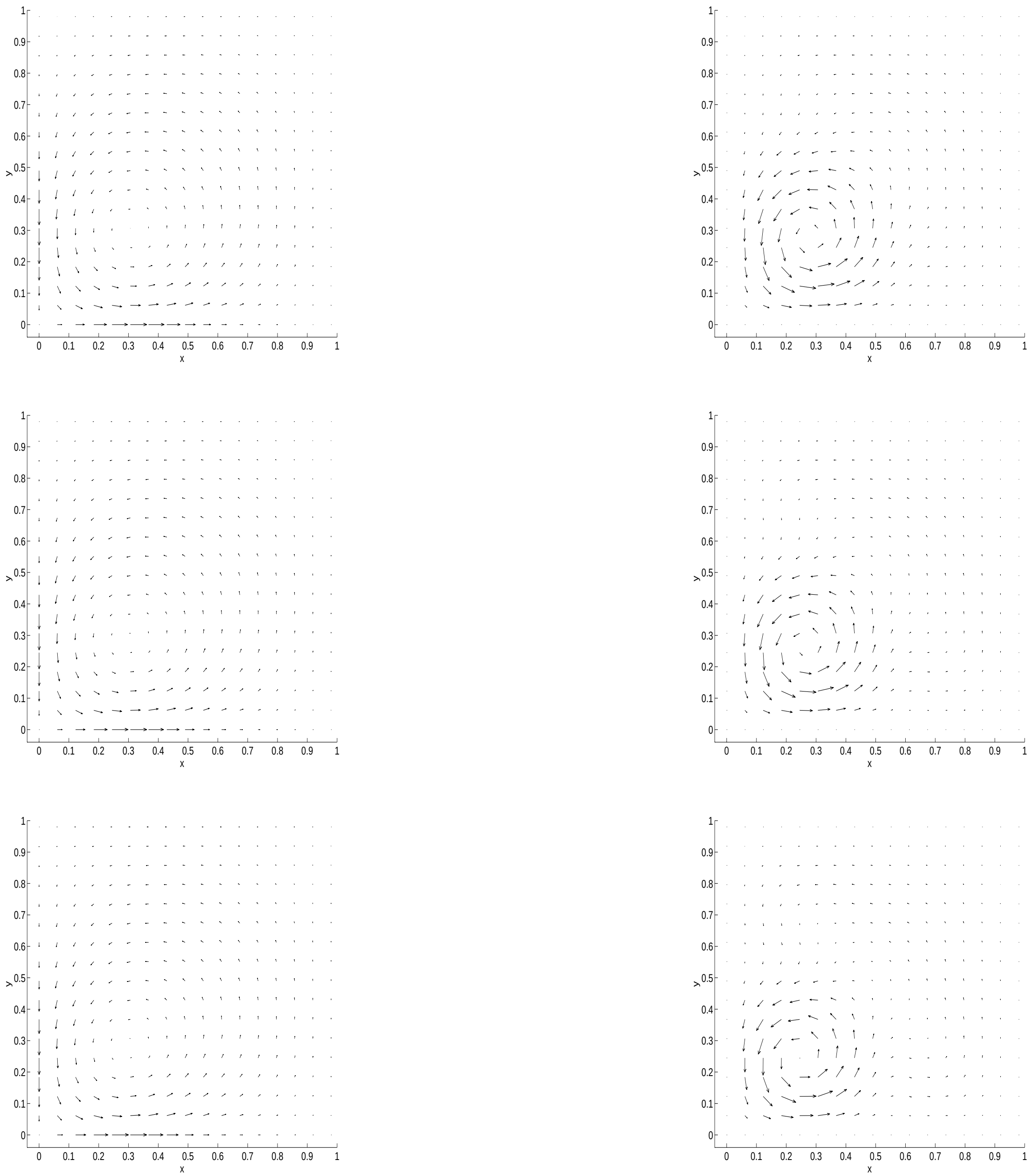

Figure 5.36: Test 2. Controlled(left) and desired(right) flow at $t = 3.$ (top), $t = 3.25$ (middle) and $t = 3.5$ (bottom)

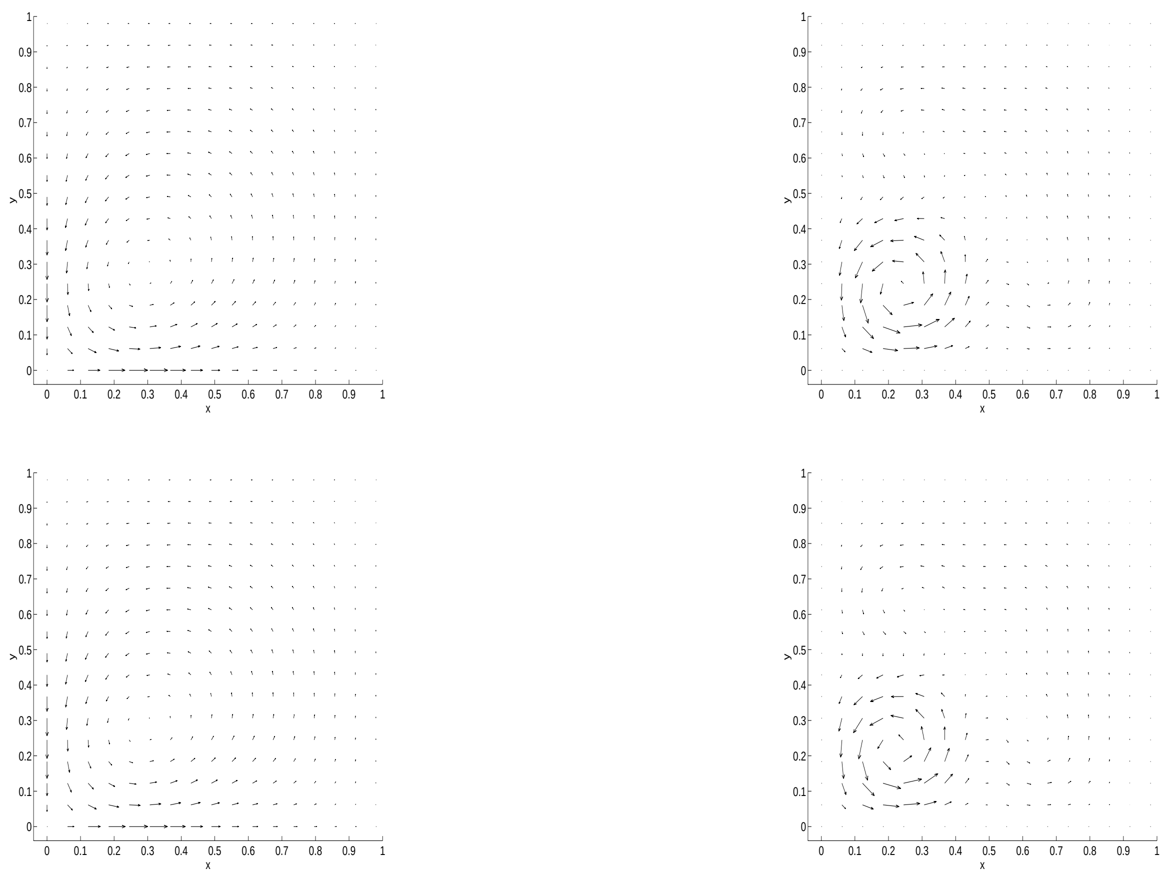

Figure 5.37: Test 2. Controlled(left) and desired(right) flow at $t = 3.5$ (top), $t = 3.75$ (middle) and $t = 4.$ (bottom)

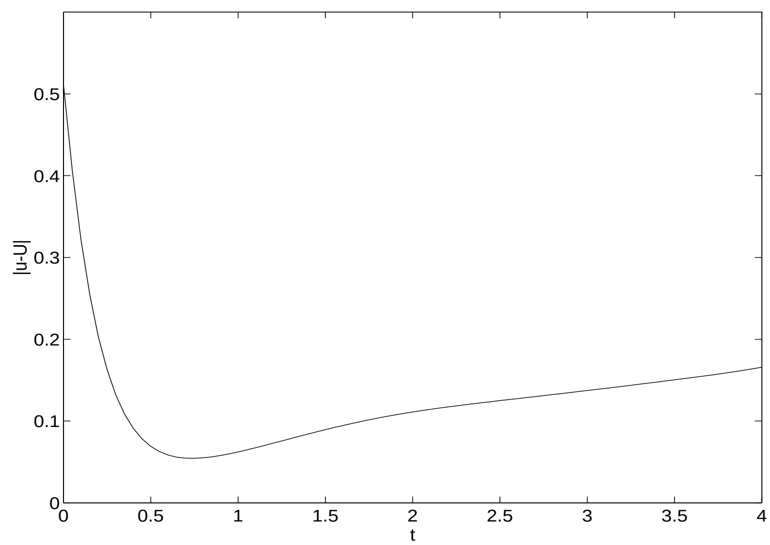

Figure 5.38: Test 2. Error $\|\vec{u} - \vec{U}\|$

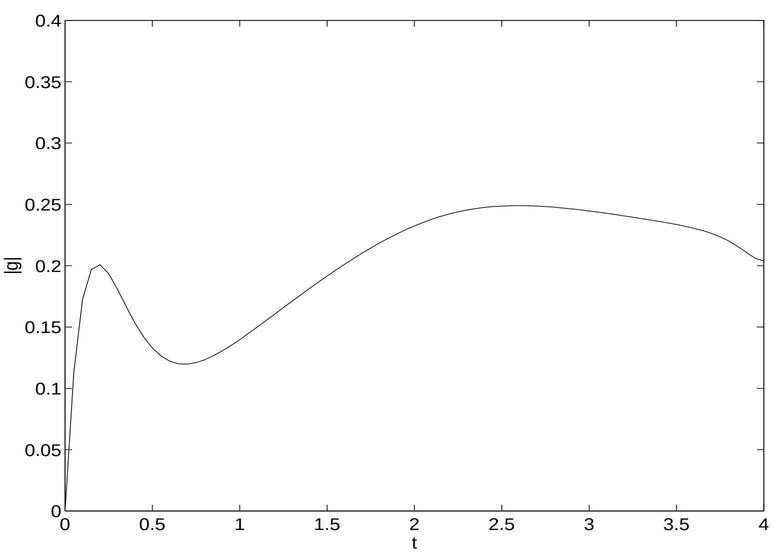

Figure 5.39: Test 2. Control norm $\|g\|$

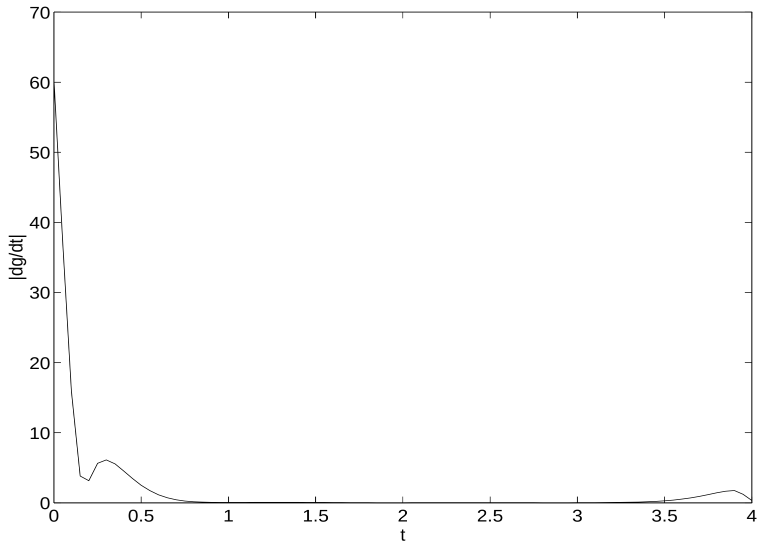

Figure 5.40: Test 2. Control norm $\|\vec{g}_t\|$

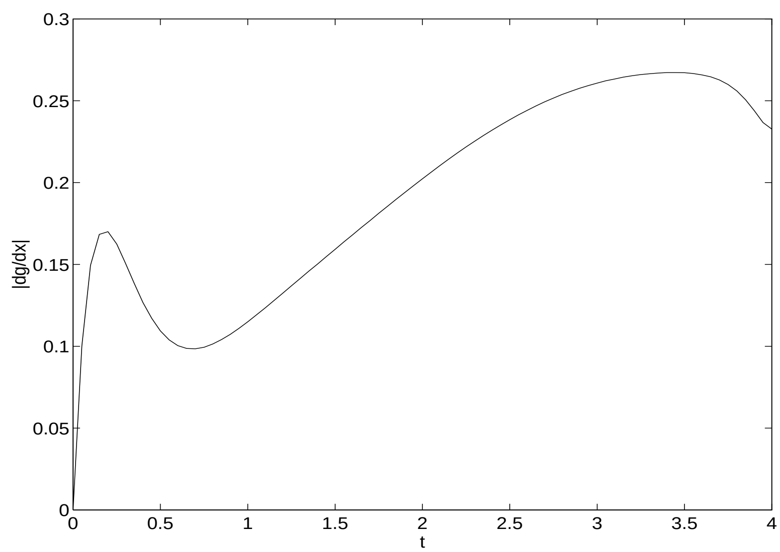

Figure 5.41: Test 2. Control norm $\|g_x\|$

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