CHAPTER 6
A MODE SHAPE SENSING TECHNIQUE

In this chapter we discuss the development of a method to extract the mode shapes of structures using the adaptive modal filter.

6.1 Sensor Gain Matrices and Mode Shapes

The sensor gain matrix in Fig. 1.2 in the Introduction give an idea that the sensor gain matrix of a spatial modal filter is closely related to the mode shapes of the structure to which the sensors are attached. Figures 3.1 and 3.13 Indicates that the rows of the sensor gain matrix are similar to the mode shapes of the structure. These examples suggest that structural mode shapes can be recovered from sensor gain matrices.

In this chapter we assume that the sensor gain matrix of the adaptive modal filter has converged to the ideal gain matrix. Only then can we compute the mode shape function using the procedure developed below. Notice that the m^{th} row of the sensor gain matrix **W** contains the information on the m^{th} mode shape function, f_m. This information is in discrete form, i.e., if we use a sensor with N segments, each row of the sensor gain matrix contains only N numbers whose relationship to the mode shape function is not always obvious. In most cases where we need mode shape functions, what we need is the values of the mode shape functions at given positions on the structure.

The problem now is how to process the sensor gain matrix (for example, the one in Fig. 6.1) to extract mode shapes as continuous functions of positions on the structure (for example, functions shown in Fig. 6.2).

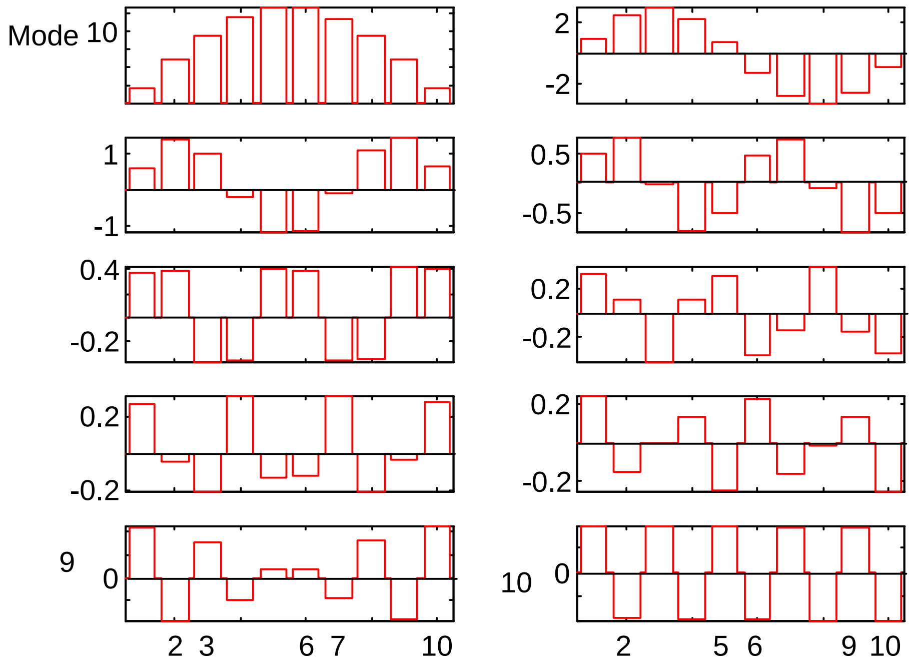

Figure 6.1 Sensor gain matrix .

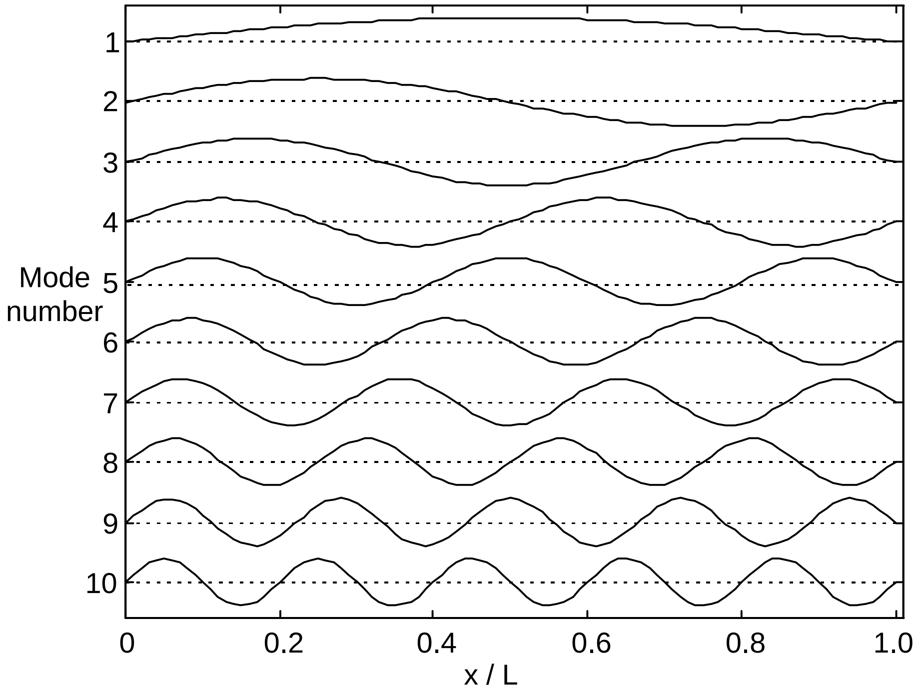

Figure 6.2 Mode shapes of beam, f *x*).

6.2 Lagrange Interpolation

Based on the observation of the above sensor gain matrix and the mode shape functions, we can make an approximating assumption that the mode shape functions are related to the rows of the sensor gain matrix. Under this approximation, we assume that the m^{th} row of the sensor gain matrix are the values of the m^{th} mode shape at points c_n in the middle of the segments for n = 1, … N. These positions are shown in Fig. 6.3. Additionally, we know from the boundary conditions in this example that the values of the mode shapes at the ends of the beam are equal to zero.

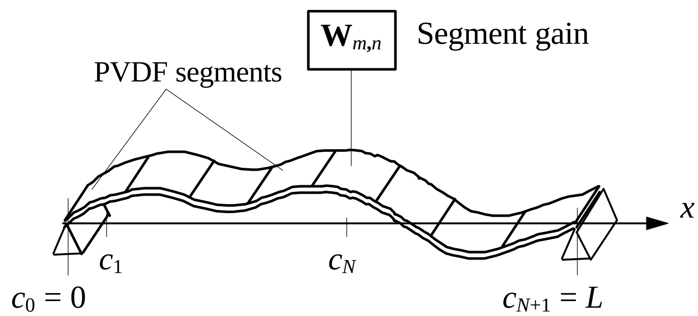

Figure 6.3 Beam with strain sensor segments.

We only have a limited amount of (discrete) data, i.e., the third row of the sensor gain matrix, while the domain of the mode shape function $\phi(x)$ is continuous, i.e., it has an infinite number of possible values. For structures in general $\phi(x)$ is not known to take the form of any specific function. Therefore, we can only *estimate* the deflection. The quality of our estimation depends on the function we assume for the deflection. In static cases, the deflection is often a polynomial function of x. In many cases, polynomials provide close approximation to other functions provided that the polynomials are of sufficiently high order. Therefore, we choose polynomials as approximating functions. The estimated mode shapes will be inherently biased. However, if the estimate is good, the bias can be made negligibly small for many purposes.

For simplicity and easier understanding, we limit our discussion to the extraction of the third mode shape function (m = 3) of the beam. We will use the third row of the sensor gain matrix accordingly. Of course, the method we will develop can be applied to other mode shapes as well. For m = 3, we assume that the value of the mode shape function in the middle of the n^{th} segment is equal to the n^{th} element of the third row of the sensor gain matrix, that is,

$$\phi_3(c_n) = \mathbf{W}_{3,n} \tag{6.1}$$

We can estimate the mode shape function between the above points by interpolating the known
N+1) order polynomial from N+
interpolation formula. (For explanation of this formula, see Burden and Faires, 1985).

$$f_{3}(x)=\left[\begin{array}{ccccc} \dfrac{\prod\limits_{\substack{i=0\\ i\neq 0}}^{N+1}(x-c_i)}{\prod\limits_{\substack{i=0\\ i\neq 0}}^{N+1}(c_0-c_i)} & \dfrac{\prod\limits_{\substack{i=0\\ i\neq 1}}^{N+1}(x-c_i)}{\prod\limits_{\substack{i=0\\ i\neq 1}}^{N+1}(c_1-c_i)} & \cdots & \dfrac{\prod\limits_{\substack{i=0\\ i\neq N}}^{N+1}(x-c_i)}{\prod\limits_{\substack{i=0\\ i\neq N}}^{N+1}(c_N-c_i)} & \dfrac{\prod\limits_{\substack{i=0\\ i\neq N+1}}^{N+1}(x-c_i)}{\prod\limits_{\substack{i=0\\ i\neq N+1}}^{N+1}(c_{N+1}-c_i)} \end{array}\right]\begin{Bmatrix} f_{3}(0)\\ \mathbf{W}_{3,1}\\ \vdots\\ \mathbf{W}_{3,N}\\ f_{3}(L) \end{Bmatrix} \tag{6.2}$$

In this example the mode shape functions at the ends of the beam are zero, so

$$f_{3}(0)=f_{3}(L)=0\,, \tag{6.3}$$

In the implementation of the segmented sensors to simulate a mode shape sensor at any given x, we shall pre-compute the ratios in Eqs. (6.2) to form the appropriate shape sensor weight coefficients. Therefore, as soon as the sensor gain matrices are available from the adaptive modal filters, the only real-time computation required to obtain the slope and the shape is multiplication of the sensor gain matrices by the pre-computed weight coefficients. The above formula results in a single (scalar) number for each position x. To obtain the mode shape function at an array of positions, we can enter a vector of values of desired positions.

6.3 Numerical Example

As an example, we will estimate the third mode shape of the beam, $f_3(x)$, using the third row of the sensor gain matrix, $\mathbf{W}_{3,n}$, where n = 1, …, 10 denotes the segment number. This row is shown in Fig. 6.4.

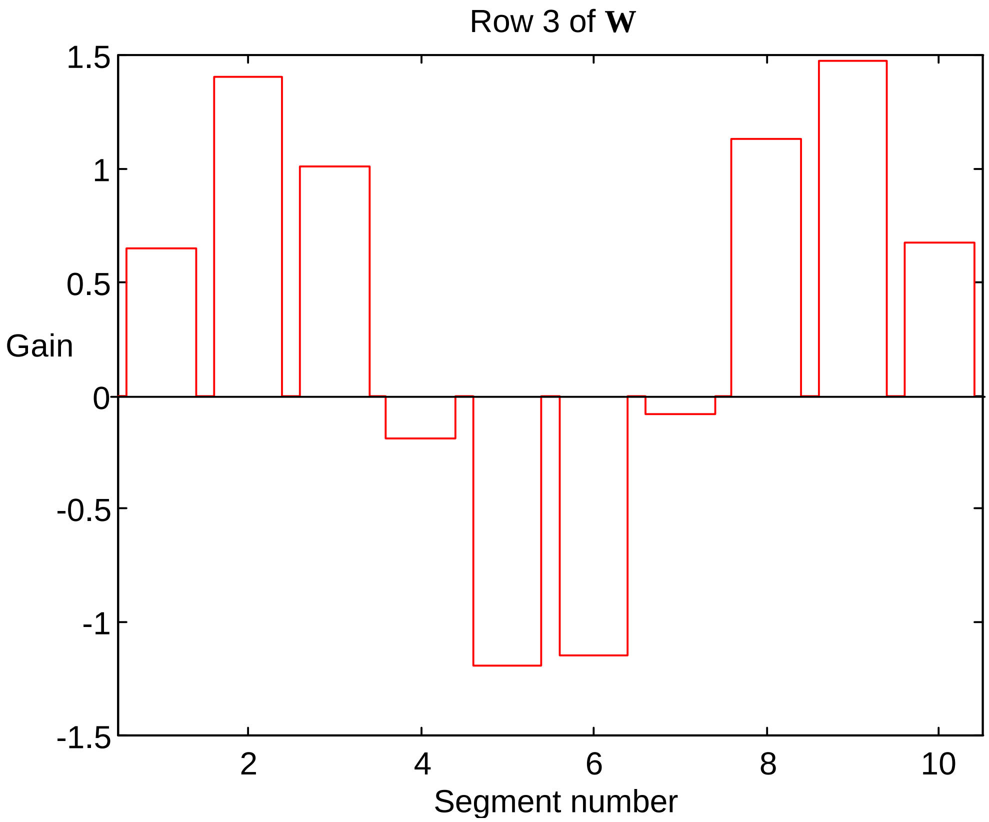

Figure 6.4 Third row of sensor gain matrix.

We apply Eq. (6.3) to the above $\mathbf{W}_{3,n}$ with an array of 65 points equally spaced between $x = 0$ and $x = L$ as an example. (The equal spacing is not necessary, since the mode shape value at any point can be evaluated.) The result of the reconstructed mode shape agrees well with the analytical mode shape calculated using Eq. (2.4). This agreement is shown in Fig. 6.5. The analytical mode shape has been normalized to a maximum value of 1 using Eq. (2.6). To make comparison easier, the recovered mode shape is also normalized so that its maximum absolute value = 1.

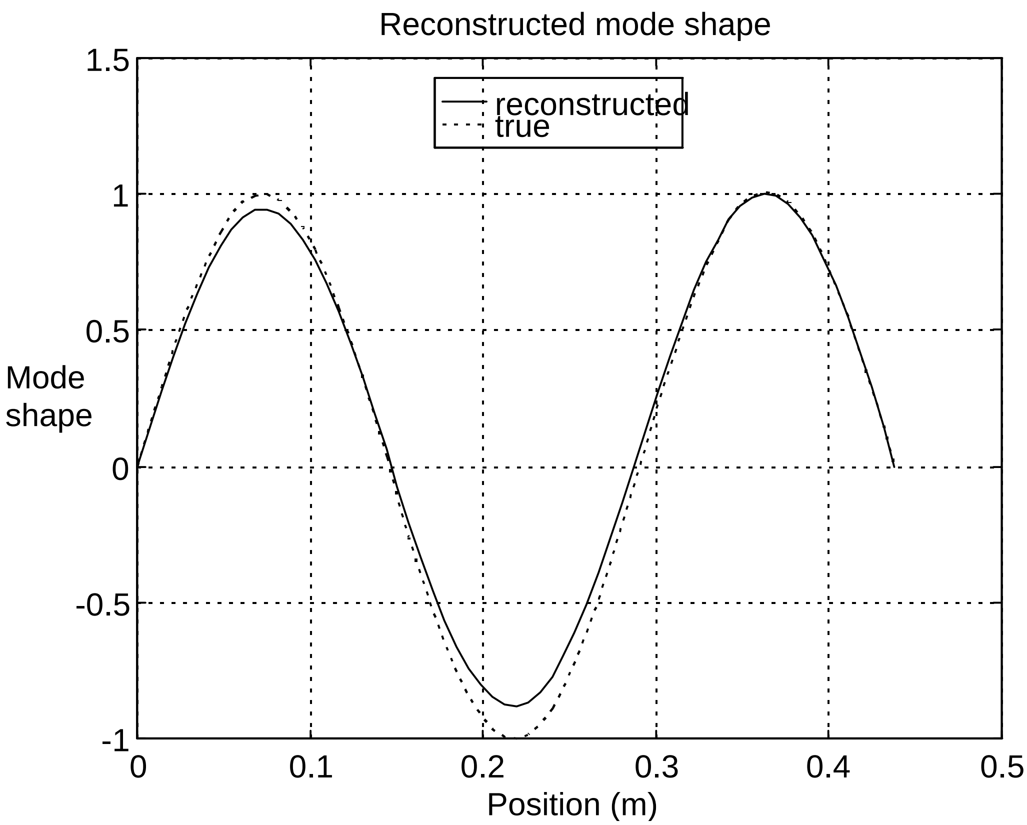

Figure 6.5 Reconstructed mode shape.

6.4 Chapter Summary

The m^{th} row of the sensor gain matrix **W** provides discrete spatial information that can be interpolated to recover the m^{th} mode shape of the structure. We used Lagrange interpolation formula to create a matrix that premultiplies the sensor gain matrix **W** to recover a mode shape of the structure. The recovered mode shape agrees well with the analytical mode shape.