

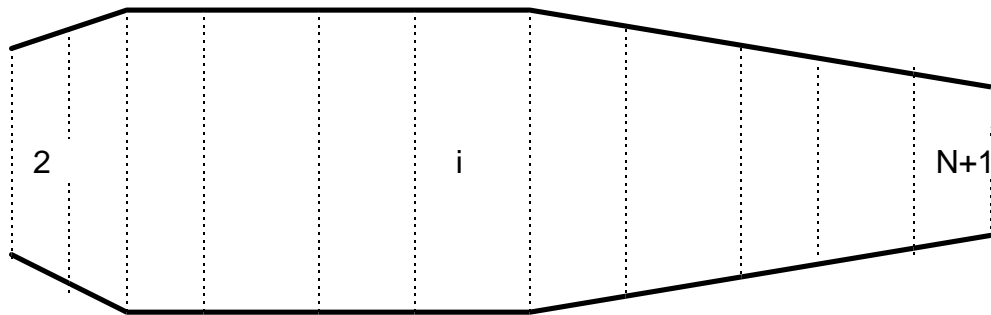
## **Chapter 2: Theoretical Foundations of the Model**

This chapter will establish the governing equations that define the present model and describe the resulting solution procedure. After introducing these equations the evaluation of its terms, particularly the fluxes, is emphasized. Next the solution procedure is explained, culminating in the time-integration of the equations. An analysis of the chemistry model follows, and a description of some of the source terms ends this chapter. The entire procedure is applied to a simply-connected domain. Extension to multiply connected domains will be explained in the following chapter.

## 2.1. Discretization of the Domain

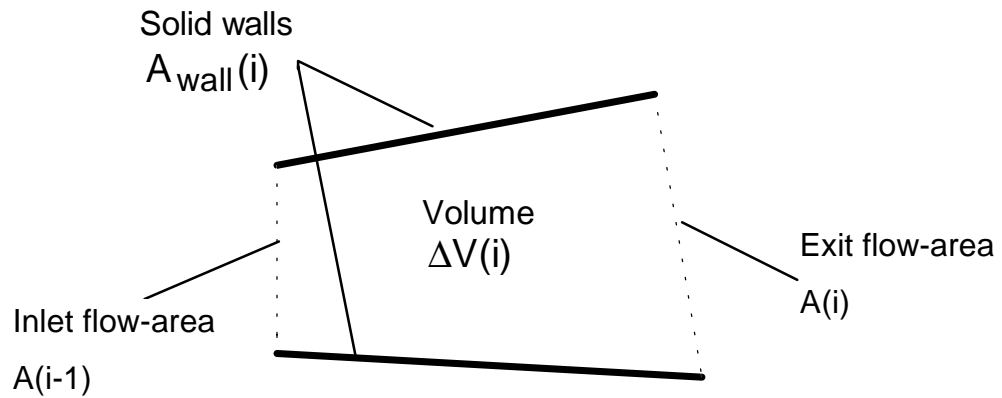
The basic idea of the present method (as with most CFD methods these days) is to divide the given domain into a discrete number of finite- or control-volumes (CVs). The integral form of the governing equations, to be introduced in the next section, is applied to each of these CVs.

Figure 2.1 shows a simply-connected domain discretized by one-dimensional CVs.  $N$  is the total number of CVs. For the purposes of presentation the CVs are numbered as shown in the figure, that is, the same index as computational nodes.



**Figure 2.1: Discretization of a simply-connected domain**

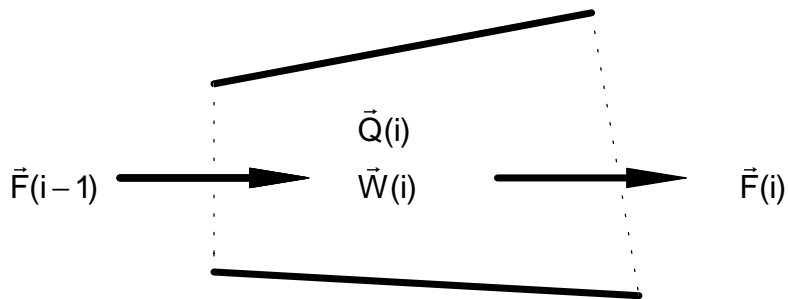
The details of a generic CV  $i$  are presented in Fig. 2.2. The CV is bound at top and bottom by solid **walls**, and on each side by inlet and exit **flow areas**. Geometrically the CV is defined by its volume  $\Delta V(i)$  and by the areas  $A(i)$  and  $A_{\text{wall}}(i)$  of the flow-areas and walls respectively.



**Figure 2. 2: Geometric elements of a generic CV.**

## ***2.2. Governing Equations at the CV Level***

The previous CV  $i$  is shown again in Fig 2.3, this time with the terms that will appear in the governing equations.



**Figure 2. 3: Terms in the governing equations.**

In the notation usually employed in CFD (Hirsch, 1990) the complete system of one-dimensional equations being solved can be represented by the column-vector equation

$$\Delta V(i) \frac{\partial}{\partial t} \vec{Q}(i) + \vec{F}(i) A(i) - \vec{F}(i-1) A(i-1) = \vec{W}(i). \quad (2.1)$$

$\vec{Q}(i)$  is the vector of **conservative variables** defined in the interior of the CV.  $\vec{F}(i)$  represents the **fluxes** through the flow-area  $A(i)$  of the conserved properties associated with  $\vec{Q}(i)$ .  $\vec{W}(i)$  is the vector of **source-terms**, i.e., additional fluxes of properties coming through the walls or created within the CV itself; this vector is also evaluated inside the CV.

Equation 2.1 represents any system of one-dimensional governing equations for fluid flow. It will now be particularized to continuum systems of multi-component, reacting gas mixtures (Williams, 1988). For the present application the following assumptions will be added to the general equations:

- zero diffusion velocities (velocity of each species equal to the mean or bulk velocity)
- zero diffusion gradients in the direction of the flow.

Therefore the equations to be solved are

- global mass
- axial momentum
- energy
- conservation of species  $j$  ( $j= 1, \dots, N_s -1$ )

(where  $N_s$  is the total number of species present in the system) and the resulting vectors  $\vec{Q}(i)$  and  $\vec{F}(i)$  are

$$\bar{Q} = \begin{bmatrix} \rho \\ \rho u \\ \rho e_t \\ \rho y_j \end{bmatrix} \quad \bar{F} = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho u h_t \\ \rho u y_j \end{bmatrix} \quad (j = 1, \dots, N_s - 1) \quad (2.2.1)$$

where  $\rho$  is the density,  $u$  the axial or mean-flow velocity,  $e_t$  the **absolute** (formation plus sensible) total internal energy,  $y_j$  the mass-fraction of species  $j$ ,  $p$  the static pressure and  $h_t$  the absolute total enthalpy.

The above is a system of  $N_s + 2$  equations in the  $N_s + 2$  unknowns

$$\bar{q} = \begin{bmatrix} \rho \\ u \\ p \\ \rho y_j \end{bmatrix} \quad (j = 1, \dots, N_s - 1) \quad (2.2.2)$$

known as **primitive** variables ( $\bar{q}$  being the primitive-variables vector). To close the system it is necessary to relate  $e_t$  and  $h_t$  to the primitive variables. We start with the definitions of absolute total enthalpy and energy:

$$h_t \equiv e_t + \frac{p}{\rho} \quad (2.3.1)$$

$$e_t \equiv \sum_{k=1}^{N_s} y_k e_k(T) + \frac{1}{2} u^2 \quad (2.3.2)$$

where  $e_k$  is the static energy of species  $k$ . This energy in turn is given by

$$e_k(T) = (h_f^0)_k + \int_0^T C_{v_k}(\tau) d\tau \quad (2.3.3)$$

with  $(h_f^0)_k$  being the energy of formation of species  $k$  at  $0^\circ\text{K}$ . The constant-volume specific heat of species  $k$ ,  $C_{v_k}$  is usually given in terms of empirical polynomial correlations (Kee et al., 1994; Turns, 1996):

$$C_{v_k}(T) = \sum_n A_{n_k} T^{n-1}. \quad (2.3.4)$$

Implicit in the above is the assumption of each species being a thermally-perfect gas (specific heats function of temperature only).

The previous expressions introduced a new unknown, the temperature  $T$ . The system is closed by relating  $T$  to the main variables via the equation of state for a mixture of gases:

$$p = \rho \left[ \sum_{k=1}^{N_s} y_k R_k \right] T \quad (2.3.5)$$

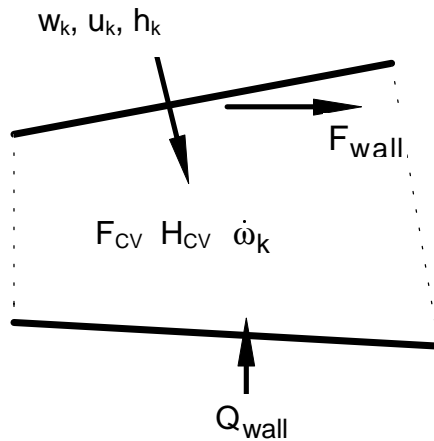
where  $R_k$  is the specific gas-constant of species  $k$ .

Finally the remaining mass-fraction is obtained from

$$y_{N_s} = 1 - \sum_{k=1}^{N_s-1} y_k \quad (2.3.6)$$

### **2.3. Source Terms**

The governing equations so far presented were valid for any application involving multi-species flows, reacting or not. It is up to the source terms to specify the equations to a particular case. As already mentioned these sources represent additional mass, momentum, energy and species **into** the CV, either as fluxes through the solid boundaries or created within the CV itself.



**Figure 2. 4: Sources for CV i**

Figure 2.4 shows the source terms that may be present in a typical CV as could be found in combustor applications. Through the solid walls there may be mass flow-rates  $w_k$  of species  $k$  ( $k= 1, \dots, N_s$ ) with associated axial velocity  $u_k$  and absolute enthalpy  $h_k$ . The solid walls themselves will in general exert an axial-component force  $F_{wall}$  and may have a heat-transfer  $\dot{Q}_{wall}$  into the CV. Within the CV itself there could be a force  $F_{CV}$  due to obstacles that may be present inside the CV. Energy  $\dot{H}_{CV}$  may also be produced (**not** including energy released during chemical reactions). Finally there may be production of species  $k$  due to chemical reactions, represented by the mass production-rate per unit volume  $\dot{\omega}_k$  within the CV volume  $\Delta V$ .

Therefore the source vector  $\vec{W}$  in equation 2.1 will look like

$$\vec{W} = \begin{bmatrix} \sum_{k=1}^{N_s} w_k \\ \sum_{k=1}^{N_s} w_k u_k + F_{\text{wall}} + F_{\text{CV}} \\ \sum_{k=1}^{N_s} w_k h_k + \dot{Q}_{\text{wall}} + \dot{H}_{\text{CV}} \\ w_k + \dot{\omega}_k \Delta V \end{bmatrix} \quad (2.4)$$

In the context of simply-connected domains, the flow-rates  $w_k$  and associated properties may be assumed as given. The remaining terms will in general be functions of the main variables  $\vec{q}$ . However, before dealing with their evaluation in detail, attention will be turned next to the evaluation of the fluxes and the integration of the governing equations.

## 2.4. Evaluation of Fluxes

From equation 2.2 it would appear that the calculation of the fluxes would be a simple matter of evaluating the vector  $\vec{F}$  as a function of either  $\vec{Q}$  or  $\vec{q}$ . The problem is that these last two vectors are evaluated **inside** the CV while the fluxes have to be evaluated at the flow-areas. This is equivalent to the problem in finite-differences of discretizing the spatial derivatives in the differential form of the governing equations.

Therefore the evaluation of the fluxes will involve two successive stages (Hirsch, 1990):

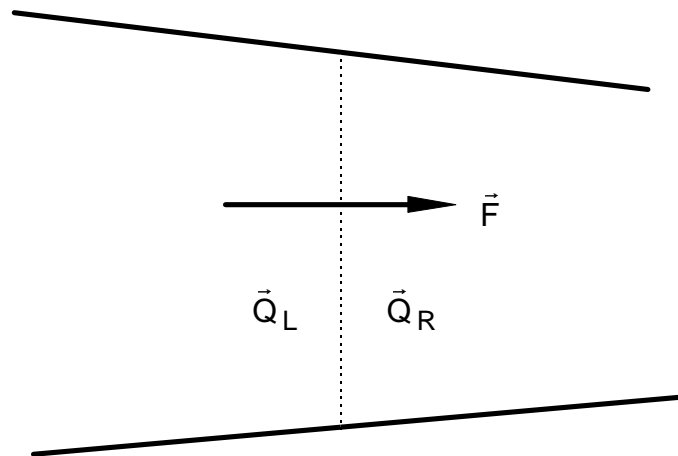
- extrapolation of the main variables from the interior of the CV to the faces or flow-areas (**projection** stage),
- evaluation of the fluxes at the faces (**physical** stage).



For presentation purposes the physical stage will be described first, followed by the projection stage

## Physical Stage

The algorithm used in the present method for calculating the fluxes is Roe's flux-difference split scheme (Hirsch, 1990). Originally developed for perfect gases and later extended to equilibrium flows, it was finally applied to flows with finite-rate chemistry. Here the approach of Grossman and Cinnella (1990) is followed. Only the outline of the scheme will be presented. The details can be found in the excellent reference just mentioned. Hirsch (1990) should be consulted for the generalities of the method. For completeness the principal equations for the method are given without proof in Appendix B.



**Figure 2. 5: Flux through an interface.**

Assume it is desired to calculate the flux  $\vec{F}$  through a cross-section or interface separating two different flow states as represented by the main-variables vectors

$\vec{Q}_L$  (left) and  $\vec{Q}_R$  (right). As an example it could be the boundary between two CVs at different states. Then the flux will be approximated with the following expression:

$$\vec{F} = \frac{1}{2}(F_L + F_R) + \Delta\vec{F} \quad (2.5)$$

where

$$F_{L,R} = \vec{F}(\vec{Q}_{L,R}). \quad (2.6)$$

The above can be recognized as a central-difference approximation with “artificial viscosity” added for numerical stability. This viscosity or correction can be written as

$$\Delta\vec{F} = -\frac{1}{2} \sum_{i=1}^{N_s+2} \alpha_i \lambda_i \vec{E}_i \quad (2.8)$$

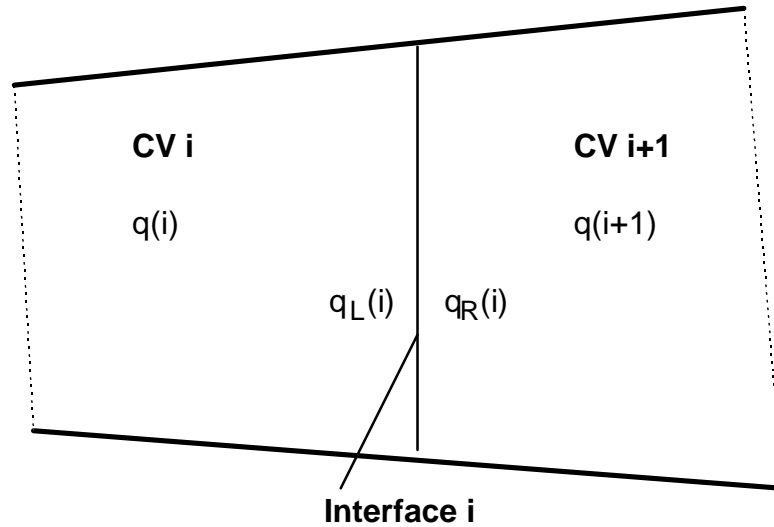
where  $\lambda_i$  and  $\vec{E}_i$  are the eigenvalues and eigenvectors of the jacobian of the fluxes, and  $\alpha_i$  the wave-strengths of the associated characteristics.

## **Projection Stage**

To apply Roe’s method for the numerical fluxes at a given flow-area a discontinuity in the state of the flow on each side is assumed. As mentioned before it could simply be the states of the adjacent CVs. It can be shown that the resulting procedure is first-order accurate in space (Hirsch, 1990).

To improve the spatial accuracy use will be made of interpolating schemes known as MUSCL (Monotone Upwind-centered Schemes for Conservation Laws) (Hirsch, 1990). The main idea is to obtain the left (or right) states needed for the flux-split method by mainly upstream (or downstream) interpolation of the main

variables which, as mentioned several times, are evaluated in the interior of the CVs.



**Figure 2. 6: MUSCL interpolation**

Consider interface  $i$  which separates states  $i$  and  $i+1$  (Fig. 2.6). Then the left and right states are given by (Hirsch, 1990) (the vector notation being dropped for convenience)

$$q_L = q(i) + 0.25 \left[ (1 - \kappa) \Delta^- + (1 + \kappa) \Delta^+ \right] q(i) \quad (2.8.1)$$

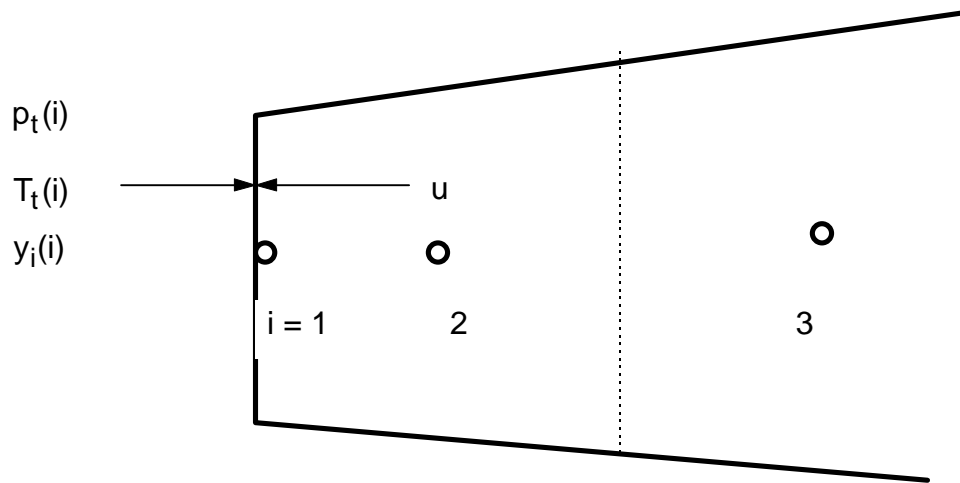
$$q_R = q(i+1) - 0.25 \left[ (1 + \kappa) \Delta^- + (1 - \kappa) \Delta^+ \right] q(i+1) \quad (2.8.2)$$

where  $\Delta^{-/+}$  are the backward/forward difference operator and  $\kappa$  that determines the order of the interpolation (and the resulting spatial differentiation). In particular  $\kappa = 1/3$  gives third-order accuracy.

It is standard practice to apply the above interpolation scheme to the primitive variables  $q$ .

## 2.5. Boundary Conditions

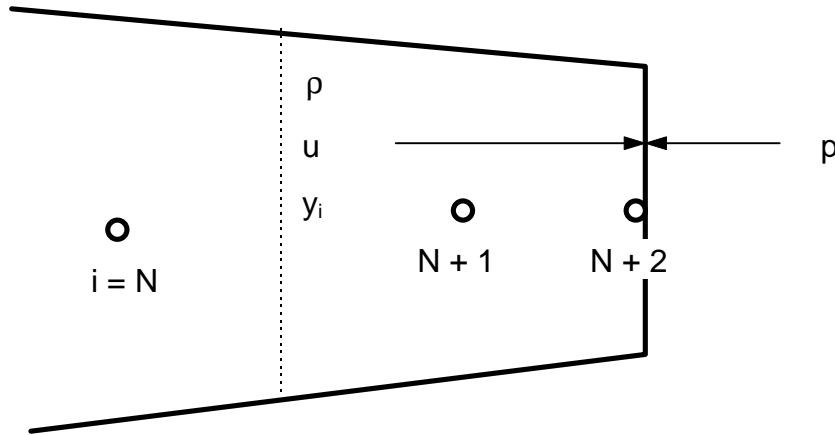
To calculate the fluxes on the faces at the inlet and exit of the domain boundary conditions have to be applied. This will give directly the value of the  $N_s + 2$  main variables at those faces. The following analysis will be restricted to subsonic flows.



**Figure 2. 7: Inlet boundary conditions**

At the inlet face a node (or zero-volume CV) with index  $i=1$  is assumed to be located (Fig 2.7). From theory of characteristics it can be shown (Hirsch, 1990) that  $N_s + 1$  variables have to be specified from outside the domain. These variables are called **physical boundary conditions** and usually represent user-given input (in general these physical conditions may be functions of time). The remaining variable has to come from the interior and is called a **numerical boundary condition**. The standard practice is to specify as physical boundary conditions the total pressure and temperature  $p_t$  and  $T_t$  and the  $N_s - 1$  mass-

fractions  $y_i$ . For the remaining numerical condition the velocity  $u$  is interpolated from the interior.



**Figure 2. 8: Exit boundary conditions**

At the exit face ( $i=N+2$ ) (see Fig. 2.7) one physical boundary condition has to be specified from outside the domain; the remaining  $N_s + 1$  numerical conditions are therefore imposed from inside. Usually the static pressure  $p$  is user-given while  $\rho, u, y_i$  ( $i= 1, \dots, N_s - 1$ ) are interpolated from the interior.

## ***2.6. Overall Procedure***

This section will put together in a single procedure all the elements introduced in previous sections. It begins by noting that the equation to solve, equation 2.1, is just an ordinary differential equation (ODE) in time. Furthermore it is parabolic in nature, i.e., the solution can be marched in time depending only on the solution at previous times. Therefore the time-variable is discretized as

$$t^n = n \Delta t \quad (n= 0, 1, 2, 3, \dots) \quad (2.9)$$

where  $\Delta t$  is the time step and integration is assumed to start at  $t = 0 = t^0$ .

In order to advance the solution from a time-level  $n$  to level  $n+1$  the following procedure is implemented:

1. Start with a known solution at time  $t = t^n$ .

To advance the solution to time-level  $n+1$  the solution has to be known at level  $n$  (in some methods the solution at  $n-1$  may also be needed - see next section). At time  $t = 0$  the solution will have to be provided by other means. In the present approach a steady-state solver (similar to the one described by Rodriguez and O'Brien [1997]) is used.

2. Apply boundary conditions at time  $t = t^{n+1}$ .

The physical boundary-conditions are evaluated from the user-provided functions. The numerical conditions are extrapolated from the solution at time  $n$ .

3. Project the solution to the interior interfaces.

The solution at time  $t^n$  will be known at the interior of the CVs. Using the MUSCL scheme this solution is projected to the interior interfaces.

4. Calculate fluxes.

With the values of the main variables known or projected at all faces the fluxes can be calculated directly or with Roe's method.

5. Calculate sources.

With the current solution known at the interior or at the faces of the CVs the source terms can be evaluated, as described in following sections.

6. Integration of the ODE in time.

Once the fluxes and sources are known the resulting equation is an ODE in time as explained before, and can be integrated with any of the methods to be described in the next section.

## **2.8. Time-Integration Procedures**

There are three basic procedures used in CFD to integrate the governing equations in time: explicit, implicit and semi-implicit. They differ in the time-level at which the **residual** (or steady-state part of the equation)

$$R(Q)(i) = \frac{1}{\Delta V(i)} [F(i) A(i) - F(i-1) A(i-1) - W(i)] \quad (2.10)$$

for CV  $i$  is evaluated (for the remainder of the section the vector notation will be dropped for convenience).

### **2.8.1. Explicit Methods**

In these methods the residual is evaluated at time  $t = t^n$ . Therefore in the usual CFD notation

$$\frac{\partial Q^n}{\partial t} = -R^n = -R(Q^n) \quad (2.11)$$

One of the most popular explicit schemes in CFD is Jameson's M-stage Runge-Kutta (Hirsch, 1990). It can be summarized as follows:

$$Q^0(i) = Q^n(i)$$

$$Q^m(i) = Q^n(i) - \frac{\Delta t}{M-m+1} R(Q^{m-1}) \quad (m= 1, \dots, M) \quad (2.12)$$

$$Q^{n+1}(i) = Q^M(i)$$

In most applications  $M = 4$ , which gives fourth-order accuracy in time. Note that the accuracy in space depends solely on the residual  $R$ .

The greatest advantage of the method is its simplicity: no matrix inversions are involved. This in turn implies that at any stage of the methods the CVs can be solved independently from each other. The main drawbacks are the restrictions on its time-step due to numerical-stability considerations. For linear systems without chemical reactions ( $\dot{\omega}_k = 0$  in equation 2.4) the limitation is given by  $CFL = 2\sqrt{2}$  where

$$CFL = \frac{\Delta t / \Delta x}{u + |a|} \quad (2.13)$$

is the Courant-Friedrichs-Lewy (CFL) number.

For reacting systems, however, it can be shown (Bussing and Murman, 1985) that the characteristic time for chemical processes is the limiting factor. This time-scale can be several orders of magnitude smaller than the fluid-dynamic time-step from the CFL condition. Therefore a purely-explicit scheme is not a practical proposition for chemically-reacting flows.

### 2.8.2. Implicit Methods

For these methods the residual is evaluated at the time-level where the solution is desired:

$$\frac{\partial Q^n}{\partial t} = -R^{n+1} = -R(Q^{n+1}) \quad (2.14)$$



A general implicit approximation is given by (Pulliam, 1993):

$$\left[ I + \frac{\nu \Delta t}{1 + \phi} \left( \frac{\partial R}{\partial Q} \right)^n \right] \Delta Q^n = - \frac{\Delta t}{1 + \phi} R(Q^n) + \frac{\phi}{1 + \phi} \Delta Q^{n-1} \quad (2.15)$$

where  $\Delta Q^n = Q^{n+1} - Q^n$ ,  $I$  is the identity matrix, and where the parameters  $\phi$  and  $\nu$  determine the accuracy of the scheme in time (the spatial accuracy is still given by  $R$ ). The left-hand side term in brackets represents a block  $N \times N$  tridiagonal matrix, each block being a  $(N_s + 2) \times (N_s + 2)$  sub-matrix. It can be shown that only the first-order accurate scheme is unconditionally stable for any CFL. For higher accuracy a sub-iteration scheme has to be used (Pulliam, 1993). More details can be found in Appendix A.

The inversion of the block tridiagonal matrix and the sub-iteration scheme require a considerable amount of computer time. This is the main drawback of implicit methods. On the other hand, in theory they are unconditionally stable, allowing the use of big time-steps. This is convenient when a steady-state solution is the main objective.

### 2.8.3. Semi-Implicit Methods

As explained before, the stiffness associated with the chemical-production source terms is the principal obstacle in using explicit methods for reacting flows. One alternative is to evaluate the residual at time  $n$  except for the chemical sources which are evaluated at time  $n+1$ :

$$\frac{\partial Q^n}{\partial t} = - \left[ (R')^n - W_c^{n+1} \right]$$

where  $R'$  is the residual minus the chemical sources  $W_c$ . Linearizing,

$$\left[ I - \Delta t \left( \frac{\partial W_c}{\partial Q} \right)^n \right] \Delta Q^n = - \Delta t R^n. \quad (2.16)$$

The left-hand side term is a block diagonal matrix, relatively easy to invert (for simple reaction mechanisms it can even be inverted analytically). The resulting equation can then be integrated using an explicit solver like equation 2.12 with its usual CFL restriction.

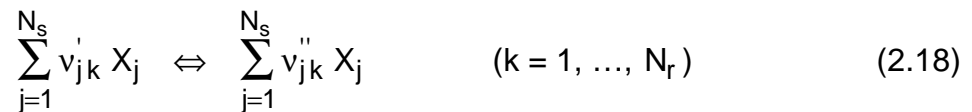
## **2.9. Chemical Source Terms - Chemistry Sub-Model**

The chemistry sub-model deals with the determination of the production rates sources in the species conservation equation:

$$\dot{\omega}_j = \frac{\partial \rho_j}{\partial t} \left[ \frac{\text{Kg}}{\text{m}^3 \text{s}} \right] \quad (2.17)$$

where  $\dot{\omega}_j$  is the rate of production of mass of species  $j$  per unit volume per unit time.

To define the sub-model it is necessary to specify the reaction mechanism or sequence of reactions assumed to be taking place. In general for a set of  $N_r$  reactions involving  $N_s$  species this mechanism can be represented by the following equation:



where  $v'_{jk}$ ,  $v''_{jk}$  are the stoichiometric coefficients of species  $j$  on the reactant and product side of the equation for reaction  $k$ , and  $X_j$  is the chemical symbol for species  $j$ .

The general form of the law of mass action (Williams, 1988; Turns, 1996) states the **molar** rate of production of species  $j$ ,

$$\hat{\omega}_j = \frac{\partial [X_j]}{\partial t} \left[ \frac{\text{Kmol}}{\text{m}^3 \text{ s}} \right] \quad (2.17)$$

(where  $[X_j]$  is the concentration or moles per unit volume of species  $j$ ) is given by

$$\hat{\omega}_j = \sum_{k=1}^{N_r} (v_{jk}'' - v_{jk}') q_k. \quad (2.20.1)$$

$q_k$  is known as the rate-of-progress variable or **reaction-rate** for reaction  $k$ , and can be obtained from

$$q_k = k_{f_k} \prod_{j=1}^{N_s} [X_j]^{a_{jk}} - k_{b_k} \prod_{j=1}^{N_s} [X_j]^{b_{jk}}. \quad (2.20.2)$$

$k_{f_k}$  and  $k_{b_k}$  are the forward and backward **reaction-rate constants**. Depending on the type of reaction the coefficients  $a_{jk}$ ,  $b_{jk}$  can be empirically-determined constants for global reactions, or stoichiometric coefficients for elementary reactions (Turns, 1996). The forward reaction rate is usually given by the empirical Arrhenius form

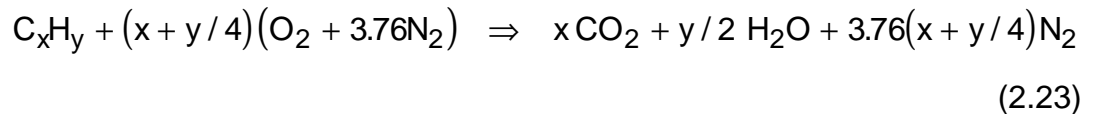
$$k_f = A T^b \exp\left(-\frac{E_a/R_u}{T}\right) \quad (2.21)$$

where  $A$  is the pre-exponential factor and  $E_a$  the activation energy; together with the temperature exponent  $b$  they are determined from experimental data.  $R_u$  is the universal gas-constant. The backward rate can be obtained by similar expressions.

Note that  $\dot{\omega}_j$  and  $\hat{\omega}_j$  are related by the molecular weight of species j:

$$\dot{\omega}_j = \hat{\omega}_j MW_j \quad (2.22)$$

The present dissertation is concerned with hydrocarbon (HC) fuels reacting with air to produce the heat-release in a combustion chamber. Since the main purpose of this model is to provide useful engineering results, the complex reaction mechanism involved will be approximated with a global, single-step irreversible reaction. For a generic HC  $C_xH_y$  the reaction equation on a molar basis will be (Turns, 1996)



Note that the above assumes **stoichiometric** reaction, i.e., no fuel or molecular oxygen among the products. For the reaction rate of equation 2.23 Westbrook and Dryer (1981) propose

$$q = A \exp\left(-\frac{E_a/R_u}{T}\right) [C_xH_y]^m [O_2]^n \quad (2.24)$$

where the authors tabulated values for A,  $E_a/R_u$ , m and n for several fuels in order to provide the best possible agreement with experimental and predicted flame-speeds and flammability limits.

It is convenient to express equation 2.23 in generic form and on a mass basis:



where

$$v = 4.76(x + y/4) \frac{MW_{air}}{MW_{fuel}} \quad (2.25)$$

is the **stoichiometric air/fuel mass ratio**.

Therefore the present model is assumed to consist of three species ( $N_s = 3$ ): fuel, air and products of combustion. Consequently two species equations are required, and those for fuel and air are chosen. From equation 2.19, and taking into account 2.22 and 2.25, it can be shown that the **mass** production rates of fuel and air are given by

$$\dot{\omega}_{\text{fuel}} = -\dot{\omega}; \quad \dot{\omega}_{\text{air}} = -\nu \dot{\omega} \quad (2.26a)$$

where

$$\dot{\omega} = MW_{\text{fuel}} q \quad (2.26b)$$

with  $q$  given by equation 2.24.

In order to calculate the thermodynamic properties of the products of combustion ( $\text{CO}_2$ ,  $\text{H}_2\text{O}$ ,  $\text{N}_2$ ) using equations 2.3.1-2.3.5 and since in the present model the individual constitutive species are not tracked separately, it is assumed that the constitution of the products is as given by equation 2.23. This assumption affects only the calculation of the mixture specific heats. Since the products are for the most part composed of  $\text{N}_2$  anyway, this does not appear to be a very restrictive assumption (some methods assume constant and equal specific heats for all species).

Before concluding this section it is convenient to introduce a useful parameter, the **equivalence ratio**:

$$\phi = \frac{f/a}{1/\nu} \quad (2.27)$$

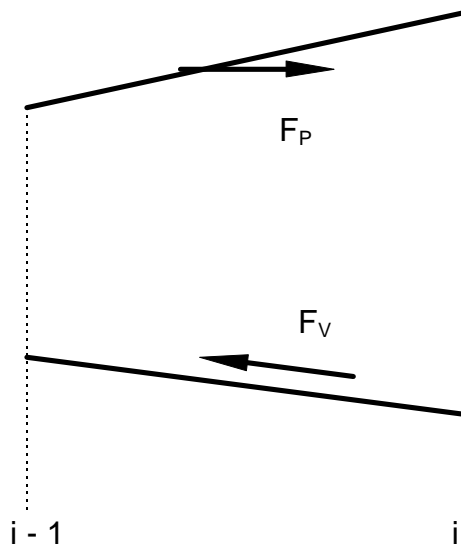
where  $f/a$  is the actual or local fuel/air mass ratio and  $\nu$  the stoichiometric ratio defined in equation 2.25. The equivalence ratio determines whether a mixture is lean ( $\phi < 1$ ), stoichiometric ( $\phi = 1$ ), or rich ( $\phi > 1$ ).

## 2.10. Remaining Source Terms

In this section the calculation of some of the non-chemically related source terms that may appear in equation 2.4 will be presented.

### 2.10.1. Wall Forces

The solid walls in Fig. 2.2 exerts normal or pressure and tangential or viscous forces on the fluid within the CV (see Fig. 2.9)



**Figure 2. 9: Wall forces acting within a CV**

It can be shown that the axial component of the pressure forces is

$$F_p = \bar{p} [A(i) - A(i-1)] \quad (2.28)$$

where  $\bar{p}$  is the pressure evaluated at some intermediate point between interfaces  $i$  and  $i+1$ . It should be noted that the analysis leading to equation 2.28 is exact. The approximation will involve the evaluation of  $\bar{p}$ . One such approximation is

$$\bar{p} = \frac{1}{2} [p(i-1) + p(i)] \quad (2.29)$$

with  $p(i)$  being the pressure at the free-flow area  $A(i)$  and obtained from the knowledge of the flux  $F(i)$  there.

The viscous forces are assumed parallel and acting counter to the direction of the flow and given by

$$F_v = -C_f \frac{1}{2} \rho(i) u(i) |u(i)| A_{\text{wall}}(i) \quad (2.30)$$

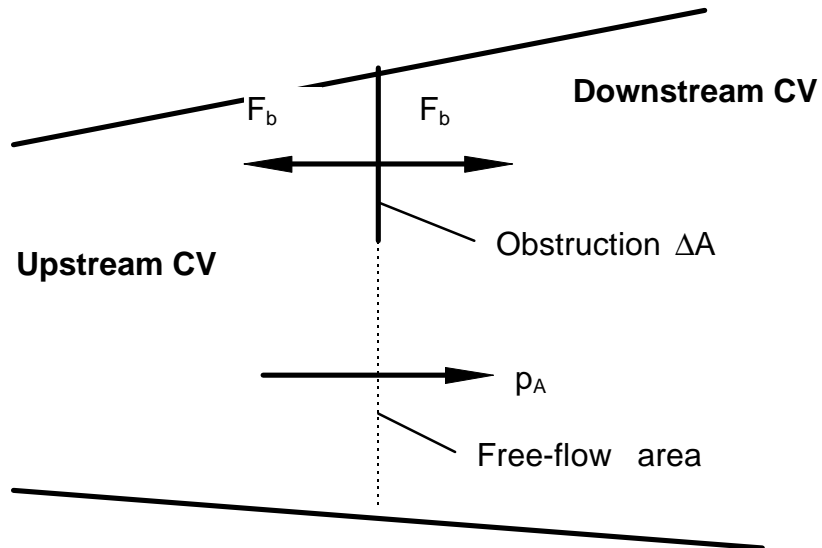
where  $A_{\text{wall}}$  is the wall area in Fig. 2.2,  $C_f$  a friction coefficient, and  $\rho(i)$  and  $u(i)$  the average density velocity in CV  $i$  obtained from vector  $\vec{q}(i)$ .

$C_f$  can be either estimated from correlations or from experimental data. In the latter case it may be convenient to express it in terms of pressure losses. From one-dimensional gas-dynamic relations it can be shown (Shapiro, 1952)

$$\frac{dp_t}{p_t} = -\gamma \frac{M^2}{2} C_f \frac{dx}{D_h/4} \quad (2.31)$$

where  $D_h$  is the hydraulic diameter.

### 2.10.2. Obstructions - Sudden Expansions and Contractions



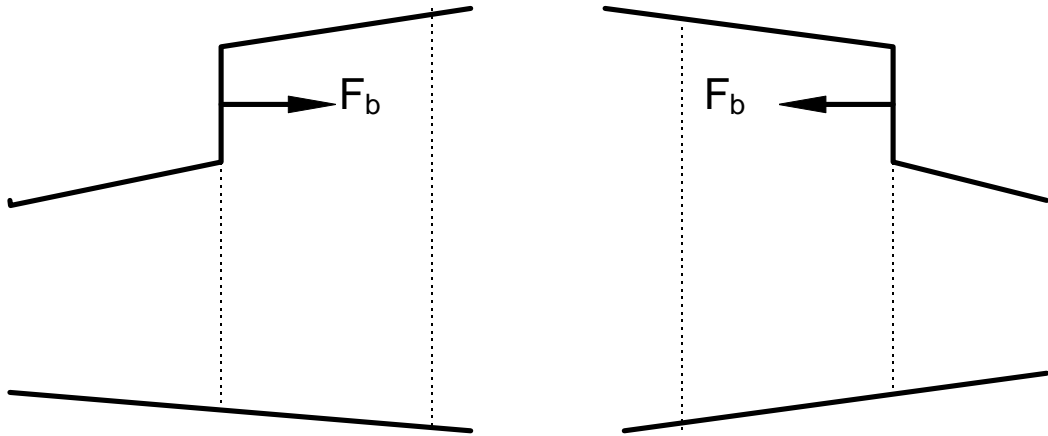
**Figure 2. 10: Forces due to a flow obstruction.**

Figure 2.10 shows an obstruction to the flow. Assume that the pressure is uniform on both sides of the obstruction and equal to the pressure in the free-flow area. Then the pressure acting on the blockage area  $\Delta A$  produces momentum source terms of the form

$$F_b = \mp p_A \Delta A$$

(-: upstream CV; +: downstream CV) which are added to the  $F_{\text{wall}}$  terms on the CVs on each side of the obstruction.





**Figure 2. 11: Sudden expansions and contractions**

The same reasoning can be used to calculate the source terms associated with sudden expansions and contractions (Fig. 2.11). In these cases the sources are added to the CVs on the side of the obstruction.