Chapter III

Equivalent Primal and Dual Differentiable Reformulations of the Euclidean Multifacility Location Problem

As introduced earlier, the Euclidean Multifacility Location Problem (**EMFLP**) seeks to locate n new facilities at some points (x_i, y_i), $i = 1, .., n$ in R^2, in the presence of some m existing facilities located at specified points (a_j, b_j), $j = 1, .., m$, given certain interaction weights $w_{ij} > 0$ between designated pairs (i, j) of new and existing facilities in some index-pair set A_{NE}, as well as certain interaction weights $v_{kl} > 0$ between designated pairs (k, l), $k < l$, of new facilities themselves in some index-pair set A_{NN}. The cost for each pair of interacting facilities is assumed to be directly proportional to the interaction weight and the Euclidean distance that separates these facilities. This problem may be mathematically stated as follows:

$$\textbf{EMFLP}: \text{Minimize } f(x, y) \equiv \sum\sum_{(i,j)\in A_{NE}} w_{ij}\{(x_i - a_j)^2 + (y_i - b_j)^2\}^{1/2}$$

$$+ \sum\sum_{(k,l)\in A_{NN}} \nu_{k\ell}\{(x_k - x_\ell)^2 + (y_k - y_\ell)^2\}^{1/2}. \tag{3.1}$$

According to our discussion on EMFLP in Chapter 1, the objective function (3.1) is convex and nondifferentiable. The points of nondifferentiability of the objective function occur when new and existing facility locations coincide, as well as on linear subspaces where the new facility locations themselves coincide. One popular global strategy to overcome the difficulties posed by this feature of the problem is to employ the hyperboloid approximation procedure (HAP) due to Eyster *et al.* (1973) in which a differentiable hyperboloid ε-approximation of the objective function is employed. However, this approach suffers from ill-conditioning effects if the point of convergence is nondifferentiable (Charalambous, (1985).

In this chapter, we present two equivalent, differentiable, convex reformulations for EMFLP to which standard nonlinear programming algorithms that are designed for smooth problems can be applied The first of these is constructed directly in the primal variable space. Here, although certain individual constraints are nonconvex, we show that the overall feasible region is in fact a convex set. Furthermore, in order for the Karush-Kuhn-Tucker (KKT) conditions for this problem to be able to closely (but not exactly) conform with the general necessary and sufficient optimality conditions for EMFLP, we show that our reformulation needs to incorporate certain implied linear inequalities. This is conformance important for enabling standard nonlinear programming algorithms that are guaranteed to converge to KKT solutions to more readily recover an optimum to EMFLP.

The second differentiable formulation derived in this chapter is based on a standard Lagrangian dual approach (see Bazaraa *et al.*, 1993, for example), through the use of a transformation that is related to the optimization of a linear function over a unit ball (circle). The resulting formulation turns out to be precisely the dual to EMFLP that is considered by Francis and Cabot (1972) and Xue *et al*. (1996). Hence, not only does this analysis recover all the theoretical relationships between EMFL and its dual via the standard, rich Lagrangian duality theory, but it also reveals possible algorithmic approaches for recovering the primal locational decisions via this dual problem. In particular, we show that any nonlinear programming algorithm for smooth problems that also produces optimal Lagrange multiplier values, can be used to directly yield an optimal solution for EMFLP as part of this set of multipliers.

The remainder of this chapter is organized as follows. As a preliminary to our analysis, we first present in Section 3.1 a characterization of subgradients of the objective function f given by (3.1), and state the associated necessary and sufficient optimality conditions for EMFLP. Based on this, we present in Sections 3.2 and 3.3 the aforementioned differentiable reformulations of EMFLP in the primal and dual spaces, respectively. Finally, Section 3.4 presents some preliminary computational results along with suggestions for further research on this topic.

3.1 Characterization of Subgradients and Optimality Conditions

Since EMFLP is an unconstrained, convex, nondifferentiable optimization problem, we know that $(\bar{x}, \bar{y})$ solves EMFLP if and only if there exists a zero of the subgradient of the objective function f at $(\bar{x}, \bar{y})$ (see Bazaraa *et al.*, 1993, or Rockafellar, 1970, for example). Lemma 1 below provides a characterization of the subgradients of the components of f, and as a direct consequence, Theorem 1 then states the readily obtained necessary and sufficient optimality conditions for EMFLP. Plastria (1992) has used a related approach to derive optimality conditions for a general distance norm location problem, and the stated specialized conditions have also been derived by Charalambous (1985) by using an ε-limiting form argument based on the hyperboloid approximation method. Our analysis is simpler and more direct, and sets the stage for the discussion to follow.

Lemma 1 *Let* $g_{ij}(x_i, y_i) \equiv \{(x_i - a_j)^2 + (y_i - b_j)^2\}^{1/2} : R^2 \rightarrow R$. *Then the subdifferential of* g_i *at any* $(\bar{x}_i, \bar{y}_i)$ *is given as follows, where* $\bar{\alpha}_{ij} = g_{ij}(\bar{x}_i, \bar{y}_i)$,

$$(\zeta_1, \zeta_2) = \left[\frac{(\bar{x}_i - a_j)}{\bar{\alpha}_{ij}}, \frac{(\bar{y}_i - b_j)}{\bar{\alpha}_{ij}}\right] \text{ is the unique subgradient if } \bar{\alpha}_{ij} \neq 0, \tag{3.2}$$

and otherwise any

(ζ_1, ζ_2) *such that* $\zeta_1^2 + \zeta_2^2 \leq 1$ *is a valid subgradient.*

Similarly, let $h_{kl}(x_k, y_k, x_l, y_l) \equiv \{(x_k - x_l)^2 + (y_k - y_l)^2\}^{1/2} : R^4 \rightarrow R$, *where* $k < \ell$. *Then the subdifferential of* h_{kl} *at* $(\bar{x}_k, \bar{y}_k, \bar{x}_\ell, \bar{y}_\ell)$ *is given as follows,*

where $\bar{\beta}_{k\ell} = h_{k\ell}(\bar{x}_k, \bar{y}_k, \bar{x}_\ell, \bar{y}_\ell)$.

$$(\zeta_1, \zeta_2, \zeta_3, \zeta_4) = \left[\frac{(\bar{x}_k - \bar{x}_\ell)}{\bar{\beta}_{k\ell}}, \frac{(\bar{y}_k - \bar{y}_\ell)}{\bar{\beta}_{k\ell}}, \frac{(\bar{x}_\ell - \bar{x}_k)}{\bar{\beta}_{k\ell}}, \frac{(\bar{y}_\ell - \bar{y}_k)}{\bar{\beta}_{k\ell}}\right] \text{ is the unique subgradient if} \tag{3.3a}$$

$\bar{\beta}_{k\ell} \neq 0$, *and otherwise any*

$$(\zeta_1, \zeta_2, \zeta_3, \zeta_4) \equiv (\zeta_1, \zeta_2, -\zeta_1, -\zeta_2) \text{ such that } \zeta_1^2 + \zeta_2^2 \leq 1 \text{ is a valid subgradient.} \tag{3.3b}$$

Proof. When $\bar{a}_{ij} \neq 0$, g_{ij} is differentiable at $(\bar{x}_i, \bar{y}_i)$ and its gradient (z_1, z_2) defined by (3.2) is the unique subgradient of g_{ij} at $(\bar{x}_i, \bar{y}_i)$. Otherwise, if $\bar{a}_{ij} = 0$, then (z_1, z_2) is a subgradient of g_{ij} at $(\bar{x}_i, \bar{y}_i)$ if and only if

$g_{ij}(x_i, y_i) \geq g_{ij}(\bar{x}_i, \bar{y}_i) + (x_i - \bar{x}_i, y_i - \bar{y}_i) \cdot (z_1, z_2) \quad \forall\ (x_i, y_i)$

i.e. $g_{ij}(x_i, y_i) \equiv \{(x_i - a_j)^2 + (y_i - b_j)^2\}^{1/2} \geq z_1(x_i - a_j) + z_2(y_i - b_j) \quad \forall\ (x_i, y_i).$ (3.4)

Denoting $X_i \equiv (x_i - a_j)$ and $Y_i \equiv (y_i - b_j)$, this is true if and only if

$(1 - z_1^2 - z_2^2)(X_i^2 + Y_i^2) + (X_i z_2 - Y_i z_1)^2 \geq 0 \qquad \forall\ (X_i, Y_i).$

which holds true if and only if $z_1^2 + z_2^2 \leq 1$.

Similarly, when $\bar{b}_{k\ell} \neq 0$, $h_{k\ell}$ is differentiable at $(\bar{x}_k, \bar{y}_k, \bar{x}_\ell, \bar{y}_\ell)$ and (3.3a) yields the unique corresponding subgradient. Otherwise, if $\bar{b}_{k\ell} = 0$, so that $\bar{x}_k = \bar{x}_\ell = s$, say, and $\bar{y}_k = \bar{y}_\ell = m$, say, and then, (z_1, z_2, z_3, z_4) is a subgradient of h if and only if

$$\{(x_k - x_l)^2 + (y_k - y_l)^2\}^{1/2} \geq z_1(x_k - s) + z_2(y_k - m) + z_3(x_l - s) + z_4(y_l - m)$$

$$\forall\ (x_k, y_k, x_l, y_l) \tag{3.5}$$

First, note that we must have $z_1 + z_3 = 0$ or else, if $z_1 + z_3 \neq 0$, then by selecting $y_k = y_l = \mu$, and $x_k = x_l = z_1 + z_3 + s$, (3.5) would yield $(z_1 + z_3)^2 \leq 0$, a contradiction. Similarly, we must have $z_2 + z_4 = 0$. Consequently, (3.5) reduces to

$$\{(x_k - x_l)^2 + (y_k - y_l)^2\}^{1/2} \geq z_1(x_k - x_l) + z_2(y_k - y_l) \quad \forall\ (x_k, y_k, x_l, y_l). \tag{3.6}$$

Noting the analogy with (3.4), we can similarly show that (3.6) is true if and only if $z_1^2 + z_2^2 \leq 1$, which establishes (3.3b). This completes the proof.

Theorem 1. *Given $(\bar{x}, \bar{y})$, let $\bar{a}_{ij}$ $\forall\ (i, j) \in A_{NE}$ and $\bar{b}_{k\ell}$ $\forall\ (k, \ell) \in A_{NN}$ be as defined in Lemma 1, and define the vectors*

$$\bar{\zeta}_x = \left[\sum_{\substack{\ell: a_j \neq \bar{x}_i \\ (i,j)\in A_{NE}}} \frac{(\bar{x}_i - a_j) w_{ij}}{\bar{\alpha}_{ij}} + \sum_{\substack{\ell: \bar{x}_\ell \neq \bar{x}_i \\ (i,l)\,\mathrm{or}\,(l,i)\in A_{NN}}} \frac{(\bar{x}_i - \bar{x}_\ell) v_{(il)}}{\bar{\beta}_{(il)}} \text{ for components } i = 1,\dots,n \right] \tag{3.7a}$$

$$\bar{\zeta}_y = \left[\sum_{\substack{\ell: b_j \neq \bar{y}_i \\ (i,j)\in A_{NE}}} \frac{(\bar{y}_i - b_j) w_{ij}}{\bar{\alpha}_{ij}} + \sum_{\substack{\ell: \bar{y}_l \neq \bar{y}_i \\ (i,l)\,\mathrm{or}\,(l,i)\in A_{NN}}} \frac{(\bar{y}_i - \bar{y}_\ell) v_{(il)}}{\bar{\beta}_{(il)}} \text{ for components } i = 1,\dots,n \right], \tag{3.7b}$$

where $\nu_{(i\ell)} \equiv \nu_{i\ell}$ *if* $i < \ell$ *and* $\nu_{(i\ell)} \equiv \nu_{\ell i}$*, otherwise,* $\forall\ i \neq \ell$*, and where* $\bar{\beta}_{(i\ell)}$ *is similarly defined. Then* $(\bar{x}, \bar{y})$ *solves EMFLP if and only if there exist* $(\zeta_{1ij}, \zeta_{2ij})$ *where* $\zeta_{1ij}^2, \zeta_{2ij}^2 \leq 1\ \forall (i,j) \in A_{NE}$ *such that* $\bar{\alpha}_{ij} = 0,$ *and there exist* $(\zeta_{3kl}, \zeta_{4kl})$*, where* $\zeta_{3kl}^2 + \zeta_{4kl}^2 \leq 1\ \ \forall\ (k,\ell) \in A_{NN}$ *such that* $\bar{\beta}_{k\ell} = 0$*, for which*

$$\bar{\zeta}_x + \sum_{j:\bar{\alpha}_{ij}=0} \bar{\zeta}_{1ij} w_{ij} + \sum_{\substack{\ell > i \\ \bar{\beta}_{i\ell}=0}} \bar{\zeta}_{3i\ell} \nu_{i\ell} - \sum_{\substack{\ell < i \\ \bar{\beta}_{\ell i}=0}} \bar{\zeta}_{3\ell i} \nu_{\ell i} = 0\ \ \forall\ i = 1,\dots,n, \tag{3.8a}$$

$$\bar{\zeta}_y + \sum_{j:\bar{\alpha}_{ij}=0} \bar{\zeta}_{2ij} w_{ij} + \sum_{\substack{\ell > i \\ \bar{\beta}_{i\ell}=0}} \bar{\zeta}_{4i\ell} \nu_{i\ell} - \sum_{\substack{\ell < i \\ \bar{\beta}_{\ell i}=0}} \bar{\zeta}_{4\ell i} \nu_{\ell i} = 0\ \ \forall\ i = 1,\dots,n, \tag{3.8b}$$

where $\bar{\zeta}_x$ *and* $\bar{\zeta}_y$ *are given by* (3.7a) *and* (3.7b), *respectively.*

Proof. Evident from Lamma 1 noting that $(\bar{x}, \bar{y})$ solves the convex program EMFLP if and only if there exist a zero subgradient of the objective function at the solution $(\bar{x}, \bar{y})$.