If we replace the redundant constraints (10d) and (10e) by new trigonometric constraints, we obtain the following differentiable reformulation

**REMFLP’**: Minimize \( \sum \sum w_{ij} \alpha_{ij} + \sum \sum v_{il} \beta_{il} \) \hfill (3.14a)

subject to \( \alpha_{ij}^2 \geq (x_i - a_j)^2 + (y_i - b_j)^2 \) \hfill (3.14b)

\( \beta_{il}^2 \geq (x_i - x_l)^2 + (y_i - y_l)^2 \) \hfill (3.14c)

\((\cos \delta_{ij}) \alpha_{ij} \geq | x_i - a_j |, (\sin \delta_{ij}) \alpha_{ij} \geq | y_i - b_j | \) \forall (i, j) \in A_{NE} \hfill (3.14d)

\((\cos \Delta_{il}) \beta_{il} \geq | x_i - x_l |, (\sin \Delta_{il}) \beta_{il} \geq | y_i - y_l | \) \forall (i, l) \in A_{NN} \hfill (3.14e)

\( \alpha_j \geq 0, \beta_{il} \geq 0 \) \hfill (3.14f)

To explore its utility for solving Problem EMFLP, we consider the following theorem.

**Theorem 3** If \((\bar{x}, \bar{y})\) solves Problem EMFLP, then there exists a KKT point \((\bar{x}, \bar{y}, \bar{\alpha}, \bar{\beta}, \bar{\delta}, \bar{\Delta})\) for REMFLP’, where \(\bar{\alpha}_{ij}\) and \(\bar{\beta}_{il}\) is given by (3.9).

**Proof.** Let \((\bar{x}, \bar{y})\) solve EMFLP. Consider the KKT conditions for REMFLP. Accordingly, associate Lagrange multipliers \(\theta_{ij}, \phi_{il}, \{\zeta_{ij}^+, \zeta_{ij}^-\}, \{\zeta_{3il}^+, \zeta_{3il}^-\}, \lambda_{ij}, \) and \(\gamma_{il}\) with each of \((3.14b), (3.14c), (3.14d), (3.14e),\) and \(3.14f\) respectively, where \(\zeta_{1ij}^+\) is associated with \((\cos \delta_{ij}) \alpha_{ij} \geq (x_i - a_j)\) and \(\zeta_{1ij}^-\) is associated with \((\cos \delta_{ij}) \alpha_{ij} \geq (a_j - x_i)\) and similarly for \(\zeta_{2ij}^+, \zeta_{3il}^+\), and \(\zeta_{4il}^+\).

This yields the following condition, in addition to primal feasibility.

\[
\sum_{j} 2(x_i - a_j) \theta_{ij} + \sum_{l \neq i} 2(x_i - x_l) \phi_{il} + \sum_{j} (\zeta_{1ij}^+ - \zeta_{1ij}^-) + \sum_{l \neq i} (\zeta_{3il}^+ - \zeta_{3il}^-) - \sum_{l \neq i} (\zeta_{3il}^+ - \zeta_{3il}^-) = 0 \quad \forall i = 1,\ldots, n \hfill (3.15a)
\]

\[
\sum_{j} 2(y_i - b_j) \theta_{ij} + \sum_{l \neq i} 2(y_i - y_l) \phi_{il} + \sum_{j} (\zeta_{2ij}^+ - \zeta_{2ij}^-) + \sum_{l \neq i} (\zeta_{4il}^+ - \zeta_{4il}^-) - \sum_{l \neq i} (\zeta_{4il}^+ - \zeta_{4il}^-) = 0 \quad \forall i = 1,\ldots, n \hfill (3.15b)
\]
2α_{ij} \theta_{ij} + \cos \delta_j (\zeta_{1ij}^+ + \zeta_{1ij}^-) + \sin \delta_j (\zeta_{2ij}^+ + \zeta_{2ij}^-) + \lambda_j = w_{ij} \quad \forall (i, j) \in A_{NE} \ (3.15c)

2\beta_{il} \phi_{(il)} + \cos \Delta_d (\zeta_{3il}^+ + \zeta_{3il}^-) + \sin \Delta_d (\zeta_{4il}^+ + \zeta_{4il}^-) + \nu_{il} = \nu_{il} \quad \forall (i, l) \in A_{NN} \ (3.15d)

\sin \delta_j \alpha_j (\zeta_{1ij}^+ + \zeta_{1ij}^-) - (\cos \delta_j) \alpha_j (\zeta_{2ij}^+ + \zeta_{2ij}^-) = 0 \quad \forall (i, j) \in A_{NE} \ (3.15e)

\sin \Delta_d \beta_{il} (\zeta_{3il}^+ + \zeta_{3il}^-) - (\cos \Delta_d) \beta_{il} (\zeta_{4il}^+ + \zeta_{4il}^-) = 0 \quad \forall (i, l) \in A_{NN} \ (3.15f)

Complementary Slackness conditions

\begin{align*}
(\ (x_i - a_j)^2 + (y_i - b_j)^2 - \alpha_{ij}^2 \ ) \theta_{ij} &= 0 \quad \forall (i, j) \in A_{NE} \ (3.15g) \\
(\ (x_i - x_l)^2 + (y_i - y_l)^2 - \beta_{il}^2 \ ) \phi_{(il)} &= 0 \quad \forall (i, l) \in A_{NN} \ (3.15h)
\end{align*}

\begin{align*}
[x_i - a_j - (\cos \delta_j) \alpha_j] \zeta_{1ij}^+ &= [a_j - x_i - (\cos \delta_j) \alpha_j] \zeta_{1ij}^- = 0 \ (3.15i) \\
[y_i - b_j - (\sin \delta_j) \alpha_j] \zeta_{2ij}^+ &= [b_j - y_i - (\sin \delta_j) \alpha_j] \zeta_{2ij}^- = 0 \ (3.15j)
\end{align*}

\begin{align*}
[x_i - x_l - (\cos \Delta_d) \beta_{il}] \zeta_{3il}^+ &= [x_i - x_l - (\cos \Delta_d) \beta_{il}] \zeta_{3il}^- = 0 \ (3.15k) \\
[y_i - y_l - (\sin \Delta_d) \beta_{il}] \zeta_{4il}^+ &= [y_i - y_l - (\sin \Delta_d) \beta_{il}] \zeta_{4il}^- = 0 \ (3.15l)
\end{align*}

\begin{align*}
\lambda_j \alpha_j &= 0 \ (3.15m) \\
\gamma_{il} \beta_{il} &= 0 \ (3.15n)
\end{align*}

where,

\begin{align*}
(\theta, \phi, \zeta_{1i}^\pm, \zeta_{2i}^\pm, \zeta_{3i}^\pm, \zeta_{4i}^\pm, \lambda, \gamma) &\geq 0 \ (3.15o)
\end{align*}

Now, select values for the Lagrange multipliers as follows in order to satisfy (3.15a), ..., (3.15n).
(1) For each \((i, j) \in A_{NE}\) if \(\alpha_{ij} \geq 0\), then from (3.15m), let \(\lambda_{ij} = 0\).

Select \(\zeta_{ij}^+ = \zeta_{ij}^- = 0\). Hence, from (3.15c), we have \(\theta_y = \frac{w_{ij}}{2\alpha_y}\).

(2) For each \((i, l) \in A_{NE}\) if \(\beta_{il} > 0\), then from (3.15n), let \(\gamma_{il} = 0\).

Select \(\zeta_{il}^+ = \zeta_{il}^- = 0\). Then from (3.15d), we have \(\phi_{(il)} = \frac{v_{il}}{2\beta_{il}}\).

(3) For each \((i, j) \in A_{NE}\) such that \(\alpha_{ij} = 0\), take \(q_{ij} = 0\).

From (3.15c), we get \(l_{ij} = w_{ij} - \cos \delta_{ij} (\zeta_{ij}^+ + \zeta_{ij}^-) - \sin \delta_{ij} (\zeta_{ij}^+ - \zeta_{ij}^-)\).

Since \((x, y)\) solves EMFLP, then we can select \(\zeta_{il}^+\) and \(\zeta_{il}^-\) such that

\[
\begin{align*}
\zeta_{ij}^+ - \zeta_{ij}^- & = w_{ij} (\zeta_{ij}^+ + \zeta_{ij}^-), \\
\zeta_{ij}^+ - \zeta_{ij}^- & = w_{ij} (\zeta_{ij}^+ - \zeta_{ij}^-).
\end{align*}
\]

where, \(\zeta_{ij}\) and \(\zeta_{2ij}\) are as given in Theorem (1). Now, for

\(\lambda_{ij}\) to be \(\geq 0\), we must have \(\cos \delta_{ij} w_{ij} |\zeta_{ij}| + \sin \delta_{ij} w_{ij} |\zeta_{2ij}| \leq w_{ij}\) \,(3.16)

Note that Theorem (1) asserts that \(\zeta_{1ij}^2 + \zeta_{2ij}^2 \leq 1\). Consider the following auxiliary problem:

\[
Max \ [(\cos \delta) p + (\sin \delta) q] \quad \text{s.t.} \quad p^2 + q^2 \leq w^2.\]

Since the optimal objective function value of this problem is \(w\), in deduce that \(\lambda_{ij}\) is nonnegative.

(4) For each \((i, l) \in A_{NN}\) such that \(\beta_{il} = 0\), take \(\phi_{(il)} = 0\). From (3.15d), we get

\(\gamma_{il} = v_{il} - (\cos \Delta_{il}) (\zeta_{3il}^+ + \zeta_{5il}^-) - (\sin \Delta_{il}) (\zeta_{4il}^+ + \zeta_{4il}^-)\).
Similar to the above argument, we can select nonnegative $\zeta_{3il}^\pm$ and $\zeta_{4il}^\pm \forall (i, l) \in A_{NN}$ such that

$$(\zeta_{3il}^+ + \zeta_{3il}^-) = v_{il} |\zeta_{3il}|, (\zeta_{4il}^+ + \zeta_{4il}^-) = v_{il} |\zeta_{4il}|, (\zeta_{3il}^+ - \zeta_{3il}^-) = v_{il} \zeta_{3il},$$

$$(\zeta_{4il}^+ - \zeta_{4il}^-) = v_{il} \zeta_{4il}, \zeta_{3ij}^+ \equiv \zeta_{3ij}^- = 0, \zeta_{4ij}^+ \equiv \zeta_{4ij}^- = 0,$$ where $\zeta_{3il}$ and $\zeta_{4il}$ are as given in Theorem 1. Now for $\gamma_{il} \geq 0$ to hold true, we must have $\cos D_{il} v_{il} |\zeta_{3il}| + \sin D_{il} |\zeta_{4il}|$ must be $0 \leq v_{il}$.

According to the approach of part [3.16], we similarly conclude that $\gamma_{il} \geq 0$.

Then examining the necessary optimality condition for EMFLP in Theorem (1), it is readily verified that the prescription given by [1] - [4] above satisfies (3.15a, ..., 3.15n), and so $(x, y, \alpha, \beta, \delta, \Delta)$ is a KKT solution for REMFLP'.

Note that the converse of this theorem is not necessarily true. By the KKT conditions and examining (3.16), we have $\cos \delta_{ij} p + \sin \delta_{ij} q = w_{ij}$, when $p = w_{ij} | \zeta_{ij} |$, $q = w_{ij} | \zeta_{2ij} |$. From Theorem 1, for $(\bar{x}, \bar{y})$ to be an optimal for EMFLP, it is necessary to have $\frac{p^2 + q^2}{w^2} \leq 1$. But since it is possible that $\max \left\{ \frac{p^2 + q^2}{w^2} : p \cos \delta + q \sin \delta \leq w \right\} > 1$, the Euclidean norm condition in Theorem 1 could be violated by a KKT point of this formulation. Hence, while REMFLP' captures any optimum to EMFLP among its KKT points, its nonconvexity introduces an additional burden of searching among its KKT solutions for a global optimum. Hence, we do not recommend its use in contrast with the other convex, differential reformulations of EMFLP developed in this chapter.