CONVERGENCE AND BOUNDEDNESS OF PROBABILITY-ONE HOMOTOPIES FOR MODEL ORDER REDUCTION

by

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(ABSTRACT)

The optimal model reduction problem, whether formulated in the H^2 or H^∞ norm frameworks, is an inherently nonconvex problem and thus provides a nontrivial computational challenge. This study systematically examines the requirements of probability-one homotopy methods to guarantee global convergence. Homotopy algorithms for nonlinear systems of equations construct a continuous family of systems, and solve the given system by tracking the continuous curve of solutions to the family. The main emphasis is on guaranteeing transversality for several homotopy maps based upon the pseudogramian formulation of the optimal projection equations and variations based upon canonical forms. These results are essential to the probability-one homotopy approach by guaranteeing good numerical properties in the computational implementation of the homotopy algorithms.

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TABLE OF CONTENTS

1. INTRODUCTION.

The optimal model reduction problem, whether formulated in the H^2 or H^∞ norm frameworks, is an inherently nonconvex problem and thus provides a nontrivial computational challenge ((Ge *et al.*, 1994a), (Haddad and Bernstein, 1989), (Hyland and Bernstein, 1985), (Žigić *et al.*, 1993)). The essential difficulties of the model reduction problem are of significant interest since techniques developed for model reduction find immediate application to the closely related problem of reduced-order controller synthesis ((Hyland and Bernstein, 1984), (Haddad and Bernstein, 1990)).

A variety of methods have been developed to address the difficulties of model order reduction. Balanced truncation and associated Hankel norm reduction theory are widely used in practice to provide H^∞-suboptimal solutions ((Moore, 1981), (Glover, 1984), (Zhou, 1995), (Kabamba, 1985b)). More recently, convex optimization methods have been employed iteratively to approximate solutions to the nonconvex model reduction problem (Grigoriadis, 1995). These techniques are inherently attractive since they rely only upon convexity-based procedures.

This study reconsiders the application of homotopy methods to optimal H^2 and H^2/H^∞ model reductions. In computational practice, homotopy methods are widely used for nonconvex optimization ((Watson, 1990), (Watson and Haftka, 1989)). Homotopy methods, in particular, probability-one homotopy methods, have global convergence properties that are often advantageous in comparison to locally convergent methods such as quasi-Newton methods ((Chow *et al.*, 1978), (Watson, 1989), (Watson, 1986)). Under suitable hypotheses, probability-one homotopy methods are guaranteed to converge globally (from an arbitrary starting point) to a solution of a nonlinear system of equations.

The goal of this study is to systematically examine the requirements of probability-one homotopy methods to guarantee global convergence. The crucial requirements are 1) transversality and 2) boundedness. As discussed in Section 3, transversality implies the existence of and the ability to track a zero curve of the homotopy map, while boundedness is equivalent to the existence of solutions to the model reduction problem. The existence of optimal reduced-order H^2 models follows from the results in (Spanos *et al.*, 1990). The main emphasis in the present study is on guaranteeing transversality for several homotopy maps based upon the pseudogramian formulation of the optimal projection equations and variations based upon canonical forms. These results are essential to the probability-one homotopy approach by guaranteeing good numerical properties in the computational implementation of the homotopy algorithms.

The contents of this dissertation are as follows. After stating the notations in Section 2, a brief review of probability-one homotopy theory is then provided in Section 3. The H^2 model reduction problem is described in Section 4. The transversality assumption of probability-one homotopy theory is then proven in Section 5 for several canonical forms. Next, it is shown by example in Section 6 that the boundedness assumption required by probability-one homotopy theory is not always satisfied by the pseudogramian formulation of the optimal projection equations and by some formulations based upon canonical forms. Then it is shown that for a reformulation of the pseudogramian optimal projection equations in a complex projective space using homogeneous transformations, the boundedness assumption holds and thus convergence of the homotopy algorithm to a solution (in complex projective space) is guaranteed. In practice, it is not necessary to track the homotopy zero curves in complex projective space. The combined H^2/H^∞ model order reduction problem is described in Section 7. The transversality assumption of probability-one homotopy theory for H^2/H^∞ model order reduction problem is then proven in Section 8 for two canonical forms. Section 9 states a H^∞ constrained LQG control problem. A brief discussion of the boundedness assumption for full order LQG synthesis is discussed in Section 10. Section 11 offers some conclusions and Section 12 suggests future work.

2. NOTATION

2.1 Description of the system.

A linear differential finite dimensional system which is continuous in time and time invariant can be represented by the equations

$$\dot{x}(t) = A\,x(t) + B\,u(t), \tag{2.1}$$

$$y(t) = C\,x(t), \tag{2.2}$$

where t is time, $x(t) \in \mathbf{R}^n$, $u(t) \in \mathbf{R}^m$, $y(t) \in \mathbf{R}^l$, $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$, and $C \in \mathbf{R}^{l\times n}$. $x(t)$ is the state of the system, $u(t)$ is the input and $y(t)$ is the output of the system. The presentation here follows that in (Wonham, 1974) and (Žigić *et al.*, 1993).

2.2 Controllability, observability and stability.

Definition 2.1.1. The pair (A, B) in (2.1) is a *controllable pair* if and only if for every $x_0 \in \mathbf{R}^{n\times 1}$ there exist $T > 0$ and a piecewise continuous input u on $[0, T]$ such that

$$0 = e^{A\,T}\,x_0 + \int_0^T e^{A(T-t)} B u(t)\, dt.$$

Definition 2.1.2. The *controllability matrix* of the matrix pair (A, B) is

$$M_c \equiv \begin{pmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{pmatrix}. \tag{2.3}$$

Definition 2.1.3. For given $\tau > 0$ the *controllability Gramian* of the matrix pair (A, B) is

$$W_c(\tau) \equiv \int_0^\tau e^{A(\tau-t)} BB^T e^{A^T(\tau-t)}\, dt. \tag{2.4}$$

W_c is obviously a symmetric matrix. Moreover, it is positive semidefinite since

$$x^T\,W_c(\tau)x = \int_0^\tau \|B^T e^{A^T(\tau-t)} x\|^2\, dt \geq 0.$$

The following theorem gives a characterization of controllability.

THEOREM 2.1 (Kwakernaak and Sivan, 1972). *(A, B) is a controllable pair if and only if* rank $(M_c) = n$, *if and only if* $W_c(\tau)$ *is positive definite for every* $\tau > 0$.

DEFINITION 2.2.1. A vector $x_0 \in \mathbf{R}^l$ is *unobservable* with respect to the matrix pair (A, C) in (2.1) and (2.2) if $Ce^{At}x_0 = 0$ for $t > 0$. The matrix pair (A, C) is *observable* if the only $x \in \mathbf{R}^l$ which is unobservable with respect to (A, C) is $x = 0$.

DEFINITION 2.2.2. The *observability matrix* of the matrix pair (A, C) is

$$M_0 \equiv \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}. \tag{2.5}$$

DEFINITION 2.2.3. For given $\tau > 0$ the *observability Gramian* of the matrix pair (A, C) is

$$W_0(\tau) \equiv \int_0^\tau e^{A^T t} C^T C e^{At} \, dt. \tag{2.6}$$

$W_0(\tau)$ is obviously a symmetric matrix. Moreover, it is positive semidefinite since

$$x^T W_0(\tau) x = \int_0^\tau x^T e^{A^T t} C^T C e^{At} x \, dt = \int_0^\tau \|Ce^{At}x\|^2 \, dt \geq 0.$$

The following theorem gives a test for observability.

THEOREM 2.2 (Kwakernaak and Sivan, 1972). *The matrix pair (A, C) is observable if and only if* rank$(M_0) = n$, *if and only if* $W_0(\tau)$ *is positive definite for every* $\tau > 0$.

The stability of the system (2.1) and (2.2) is determined by the matrix A, so it is sufficient to consider only the equation

$$\dot{x}(t) = Ax(t), \tag{2.7}$$

whose general solution is given by

$$x(t) = e^{A(t-t_0)} x_0. \tag{2.8}$$

DEFINITION 2.3.1. A solution $x_0(t)$ of the system (2.7), which is given by (2.8), is *stable* if for any t_0 and any $\epsilon > 0$ there exists a $\delta(\epsilon, t_0)$ such that for any other solution $x(t)$,

$$\|x(t_0) - x_0(t_0)\| \leq \delta \Longrightarrow \|x(t) - x_0(t)\| \leq \epsilon \qquad \text{for all} \quad t \geq t_0.$$

Furthermore, the solution $x_0(t)$ is *asymptotically stable* if it is stable and for any t_0 there exists a $\rho > 0$ such that for any other solution $x(t)$

$$\|x(t_0) - x_0(t_0)\| \leq \rho \Longrightarrow \|x(t) - x_0(t)\| \to 0 \qquad \text{as } t \to \infty.$$

The system (2.7) is asymptotically stable if every solution $x_0(t)$ is asymptotically stable, which occurs if any solution is asymptotically stable. In this case the matrix A is referred to as asymptotically stable.

THEOREM 2.3 (Kwakernaak and Sivan, 1972). *The system (2.7) is asymptotically stable if and only if all the eigenvalues of A have negative real parts.*

2.3. Systems driven by white noise.

DEFINITION 2.4.1. A *vector stochastic process* $v(t)$ is

$$v(t) = (v_1(t) \quad v_2(t) \quad \dots \quad v_n(t))^T, \tag{2.9}$$

where $v_2(t), v_2(t), ..., v_n(t)$ are scalar stochastic processes for $t \geq t_0$ for some t_0. A *stochastic process* is a collection of random variables $\{x(t) \mid t \in I\}$ indexed by a time parameter t varying in an interval I.

DEFINITION 2.4.2. The *mean* $m(t)$ of a vector stochastic process $v(t)$ is $m(t) \equiv E[v(t)]$. The *covariance matrix* $R(t_1, t_2)$ of $v(t)$ is $R(t_1, t_2) \equiv E\{[v(t_1) - m(t_1)][v(t_2) - m(t_2)]^T\}$. The *variance matrix* $Q(t)$ of $v(t)$ is $Q(t) \equiv R(t, t)$.

DEFINITION 2.4.3. Let $v(t)$ be a zero-mean stochastic process with the covariance matrix $R(t_1, t_2) = V(t_1)\delta(t_2 - t_1)$, where $V(t)$ is a positive semidefinite matrix and $\delta(t)$ is the delta function. The process $v(t)$ is a *white noise* process with intensity $V(t)$.

LEMMA 2.4. *Let $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{m\times m}$ and $C \in \mathbf{R}^{n\times m}$. If $\lambda_i, i = 1, ..., n$, are the eigenvalues of A, and $\mu_j, j = 1, ..., m$, are the eigenvalues of B, then the equation $AX + XB = C$ has a unique solution X if and only if $\lambda_i + \mu_j \neq 0$ for all i and j.*

A proof is given in (Gantmacher, 1960); this lemma will be used later.

3. PROBABILITY-ONE GLOBALLY CONVERGENT HOMOTOPIES

A *homotopy* is a continuous map from the interval [0,1] into a function space, where the continuity is with respect to the topology of the function space. Intuitively, a homotopy $\rho(\lambda)$ continuously deforms the function $\rho(0) = g$ into the function $\rho(1) = f$ as λ goes from 0 to 1. In this case, f and g are said to be *homotopic*. Homotopy maps are fundamental tools in topology, and provide a powerful mechanism for defining equivalence classes of functions.

Homotopies provide a mathematical formalism for describing an old procedure in numerical analysis, variously known as continuation, incremental loading, and embedding. The continuation procedure for solving a nonlinear system of equations $f(x) = 0$ starts with a (generally simpler) problem $g(x) = 0$ whose solution x_0 is known. The continuation procedure is to track the set of zeros of

$$\rho(\lambda, x) = \lambda f(x) + (1 - \lambda)g(x) \tag{3.1}$$

as λ is increased monotonically from 0 to 1, starting at the known initial point $(0, x_0)$ satisfying $\rho(0, x_0) = 0$. Each step of this tracking process is done by starting at a point $(\tilde{\lambda}, \tilde{x})$ on the zero set of ρ, fixing some $\Delta\lambda > 0$, and then solving $\rho(\tilde{\lambda} + \Delta\lambda, x) = 0$ for x using a locally convergent iterative procedure, which requires an invertible Jacobian matrix $D_x\rho(\tilde{\lambda} + \Delta\lambda, x)$. The process stops at $\lambda = 1$, since $f(\bar{x}) = \rho(1, \bar{x}) = 0$ gives a zero $\bar{x}$ of $f(x)$. Note that continuation assumes that the zeros of ρ connect the zero x_0 of g to a zero $\bar{x}$ of f, and that the Jacobian matrix $D_x\rho(\lambda, x)$ is invertible along the zero set of ρ; these are strong assumptions, which are frequently not satisfied in practice.

Continuation can fail because the curve γ of zeros of $\rho(\lambda, x)$ emanating from $(0, x_0)$ may (1) have turning points, (2) bifurcate, (3) fail to exist at some λ values, or (4) wander off to infinity without reaching $\lambda = 1$. Turning points and bifurcation correspond to singular $D_x\rho(\lambda, x)$. Generalizations of continuation known as *homotopy methods* attempt to deal with cases (1) and (2), and allow tracking of γ to continue through singularities. In particular, continuation monotonically increases λ, whereas homotopy methods permit λ to both increase and decrease along γ. Homotopy methods can also fail via cases (3) or (4).

The map $\rho(\lambda, x)$ connects the functions $g(x)$ and $f(x)$, hence the use of the word "homotopy." In general the homotopy map $\rho(\lambda, x)$ need not be a simple convex combination of g and f as in (3.1), and can involve λ nonlinearly. Sometimes λ is a physical parameter in the original problem $f(x; \lambda) = 0$, where $\lambda = 1$ is the (nondimensionalized) value of interest, although "artificial parameter" homotopies are generally more computationally efficient than "natural parameter" homotopies $\rho(\lambda, x) = f(x; \lambda)$. An example of an artificial parameter homotopy map is

$$\rho(\lambda, x) = \lambda f(x; \lambda) + (1 - \lambda)(x - a), \tag{3.2}$$

which satisfies $\rho(0, a) = 0$. The name "artificial" reflects the fact that solutions to $\rho(\lambda, x) = 0$ have no physical interpretation for $\lambda < 1$. Note that $\rho(\lambda, x)$ in (3.2) has a unique zero $x = a$ at $\lambda = 0$, regardless of the structure of $f(x; \lambda)$.

All four shortcomings of continuation and homotopy methods have been overcome by probability-one homotopies, proposed in 1976 by Chow, Mallet-Paret, and Yorke (Chow *et al.*, 1978). The supporting theory, based on differential geometry, will be reformulated in less technical jargon here.

DEFINITION 3.1.1. Let $U \subset \mathbf{R}^m$ and $V \subset \mathbf{R}^p$ be open sets, and let $\rho : U \times [0, 1) \times V \to \mathbf{R}^p$ be a C^2 map. ρ is said to be *transversal to zero* if the Jacobian matrix $D\rho$ has full rank on $\rho^{-1}(0)$.

The C^2 requirement is technical, and part of the definition of transversality. The basis for the probability-one homotopy theory is:

THEOREM 3.1 (Parametrized Sard's Theorem) (Chow *et al.*, 1978). *Let $\rho : U \times [0, 1) \times V \to \mathbf{R}^p$ be a C^2 map. If ρ is transversal to zero, then for almost all $a \in U$ the map*

$$\rho_a(\lambda, x) = \rho(a, \lambda, x)$$

is also transversal to zero.

To discuss the import of this theorem, take $U = \mathbf{R}^m, V = \mathbf{R}^p$, and suppose that the C^2 map $\rho : \mathbf{R}^m \times [0, 1) \times \mathbf{R}^p \to \mathbf{R}^p$ is transversal to zero. A straightforward application of the implicit function theorem yields that for almost all $a \in \mathbf{R}^m$, the zero set of ρ_a consists of smooth, nonintersecting curves which either (1) are closed loops lying entirely in $(0,1)\times\mathbf{R}^p$, (2) have both endpoints in $\{0\}\times\mathbf{R}^p$, (3) have both endpoints in $\{1\}\times\mathbf{R}^p$, (4) are unbounded with one endpoint in either $\{0\}\times\mathbf{R}^p$ or in $\{1\}\times\mathbf{R}^p$, or (5) have one endpoint in $\{0\}\times\mathbf{R}^p$ and the other in $\{1\}\times\mathbf{R}^p$. Furthermore, for almost all $a\in\mathbf{R}^m$, the Jacobian matrix $D\rho_a$ has full rank at *every* point in $\rho_a^{-1}(0)$. The goal is to construct a map ρ_a whose zero set has an endpoint in $\{0\}\times\mathbf{R}^p$, and which rules out (2) and (4). Then (5) obtains, and a zero curve starting at $(0,x_0)$ is *guaranteed* to reach a point $(1,\bar{x})$. All of this holds for almost all $a\in\mathbf{R}^m$, and hence with probability one (Chow *et al.*, 1978). Furthermore, since $a\in\mathbf{R}^m$ can be almost any point (and, indirectly, so can the starting point x_0), an algorithm based on tracking the zero curve in (5) is legitimately called *globally convergent.* This discussion is summarized in the following theorem.

THEOREM 3.2. *Let $f:\mathbf{R}^p\to\mathbf{R}^p$ be a C^2 map, $\rho:\mathbf{R}^m\times[0,1)\times\mathbf{R}^p\to\mathbf{R}^p$ a C^2 map, and $\rho_a(\lambda,x)=\rho(a,\lambda,x)$. Suppose that*

(1) ρ is transversal to zero, and, for each fixed $a\in\mathbf{R}^m$,

(2) $\rho_a(0,x)=0$ has a unique solution x_0,

(3) $\rho_a(1,x)=f(x)\qquad(x\in\mathbf{R}^p)$. Then, for almost all $a\in\mathbf{R}^m$, there exists a zero curve γ of ρ_a emanating from $(0,x_0)$, along which the Jacobian matrix $D\rho_a$ has full rank. If, in addition,

(4) $\rho_a^{-1}(0)$ is bounded, then γ reaches a point $(1,\bar{x})$, where $f(\bar{x})=0$. Furthermore, if $Df(\bar{x})$ is invertible, then γ has finite arc length.

Any algorithm for tracking γ from $(0,x_0)$ to $(1,\bar{x})$, based on a homotopy map satisfying the hypotheses of Theorem 3.2, is called a *globally convergent probability-one homotopy algorithm.* Of course the practical numerical details of tracking γ are nontrivial, and have been the subject of twenty years of research in numerical analysis. Production quality software called HOMPACK (Watson *et al.*, 1987) exists for tracking γ. The distinctions between continuation, homotopy methods, and probability-one homotopy methods are subtle but worth noting. Only the latter are provably globally convergent and (by construction) expressly avoid dealing with singularities numerically, unlike continuation and homotopy methods which must explicitly handle singularities numerically.

The purpose of this study is to prove or disprove properties (1)–(4) of Theorem 3.2 for some homotopy maps that have been proposed for the H^2 and H^2/H^∞ optimal model order reduction problems, and which have been successful in practice. Assumptions (2) and (3) in Theorem 3.2 are usually achieved by the construction of ρ (such as (3.2)), and are straightforward to verify. Although assumption (1) is trivial to verify for some maps, for the H^2 and H^2/H^∞ model order reduction homotopies the verification is nontrivial. Assumption (4) is typically very hard to verify, and often is a deep result, since (1)–(4) holding implies the *existence* of a solution to $f(x) = 0$.

Note that (1)–(4) are sufficient, but not necessary, for the existence of a solution to $f(x) = 0$, which is why homotopy maps not satisfying the hypotheses of Theorem 3.2 can still be very successful on practical problems. If (1)–(3) hold and a solution does *not* exist, then (4) must fail, and nonexistence manifests itself by γ going off to infinity. Properties (1)–(3) are important because they guarantee good numerical properties along the zero curve γ, which, if bounded, results in a *globally convergent* algorithm. If γ is unbounded, then either the homotopy approach (with this particular ρ) has failed or $f(x) = 0$ has no solution.

4. H^2 OPTIMAL MODEL ORDER REDUCTION PROBLEM.

4.1. Statement of the problem.

The optimal model order reduction problem can be formulated as follows: given the nth-order asymptotically stable, controllable and observable linear time-invariant continuous-time system

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{4.1}$$

$$y(t) = Cx(t), \tag{4.2}$$

where $A \in \mathbf{R}^{n \times n}$, $B \in \mathbf{R}^{n \times m}$, and $C \in \mathbf{R}^{l \times n}$, and given $n_m < n$, find an n_mth reduced-order model

$$\dot{x}_m(t) = A_m x_m(t) + B_m u(t), \tag{4.3}$$

$$y_m(t) = C_m x_m(t), \tag{4.4}$$

where $A_m \in \mathbf{R}^{n_m \times n_m}$ is asymptotically stable, $B_m \in \mathbf{R}^{n_m \times m}$, $C_m \in \mathbf{R}^{l \times n_m}$, which minimizes the quadratic model-reduction criterion

$$J(A_m, B_m, C_m) \equiv$$

$$\lim_{t \to \infty} E[(y(t) - y_m(t))^T R(y(t) - y_m(t))], \tag{4.5}$$

where the input $u(t)$ is white noise with positive definite intensity V, and R is a positive definite weighting matrix. Throughout, all positive semidefinite and positive definite matrices are assumed to be symmetric.

To guarantee that J is finite, a solution (A_m, B_m, C_m) is sought in the set $\mathcal{S} = \{(A_m, B_m, C_m) : A_m$ is asymptotically stable, (A_m, B_m) is controllable and (A_m, C_m) is observable$\}$.

4.2. (G, M, Γ)**–factorization.**

The results in this section are classical linear algebra results.

DEFINITION 4.1.1. A matrix $Q \in \mathbf{R}^{n\times n}$ is *positive definite (positive semidefinite)* if $x^TQx > 0$ ($x^TQx \geq 0$) for all $x \neq 0$. A matrix P is *positive semisimple* if P is similar to a positive definite matrix.

LEMMA 4.1. *Let symmetric positive semidefinite* $\hat{Q}, \hat{P} \in \mathbf{R}^{n\times n}$ *satisfy*

$$\mathrm{rank}(\hat{Q}) = \mathrm{rank}(\hat{P}) = \mathrm{rank}(\hat{Q}\hat{P}) = n_m, \tag{4.6}$$

where $n_m \leq n$*. Then, there exists a nonsingular* $W \in \mathbf{R}^{n\times n}$ *and positive definite diagonal* $\Sigma, \Omega \in \mathbf{R}^{n_m \times n_m}$ *such that*

$$\hat{Q} = W \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} W^T, \qquad \hat{P} = W^{-T} \begin{pmatrix} \Omega & 0 \\ 0 & 0 \end{pmatrix} W^{-1}. \tag{4.7}$$

REMARK 4.1.2. *Let*

$$U \equiv W \begin{pmatrix} D_2^{1/4} & 0 \\ 0 & I \end{pmatrix},$$

we have

$$\hat{Q} = U \begin{pmatrix} \Lambda & 0 \\ 0 & 0 \end{pmatrix} U^T, \qquad \hat{P} = U^{-T} \begin{pmatrix} \Lambda & 0 \\ 0 & 0 \end{pmatrix} U^{-1}. \tag{4.8}$$

LEMMA 4.2. *Let symmetric positive semidefinite* $\hat{Q}, \hat{P} \in \mathbf{R}^{n\times n}$ *satisfy the rank conditions* (4.6)*. Then, there exist* $G, \Gamma \in \mathbf{R}^{n_m \times n}$ *and positive semisimple* $M \in \mathbf{R}^{n_m \times n_m}$ *such that*

$$\hat{Q}\hat{P} = G^T M \Gamma, \tag{4.9}$$

$$\Gamma G^T = I_{n_m}. \tag{4.10}$$

DEFINITION 4.2.1. Matrices $G, M,$ and Γ from Lemma 4.2 are called a (G, M, Γ)-*factorization* of $(\hat{Q}, \hat{P})$.

4.3. The optimal projection theorem.

REMARK 4.3.1.

$$W_c \equiv \int_0^\infty e^{At} BVB^T e^{A^T t}\, dt$$

and

$$W_0 \equiv \int_0^\infty e^{A^T t} C^T RC e^{At}\, dt$$

satisfy the dual Lyapunov equations

$$0 = AW_c + W_c A^T + BVB^T, \tag{4.11}$$

$$0 = A^T W_0 + W_0 A + C^T RC. \tag{4.12}$$

The following theorem and its proof will be needed in Section 5.

THEOREM 4.3 (Haddad and Bernstein, 1989). *Suppose A_m is stable, (A_m, B_m) is controllable, (A_m, C_m) is observable, and (A_m, B_m, C_m) solves the optimal model-reduction problem. Then there exist symmetric positive semidefinite matrices $\hat{Q}$, $\hat{P} \in \mathbf{R}^{n\times n}$ such that for some (G, M, Γ)-factorization of $\hat{Q}\hat{P}$, A_m, B_m and C_m are given by*

$$A_m = \Gamma\, A\, G^T, \tag{4.13}$$

$$B_m = \Gamma\, B, \tag{4.14}$$

$$C_m = C\, G^T, \tag{4.15}$$

and such that, with $\tau \equiv G^T\,\Gamma$ the following conditions are satisfied:

$$\tau[A\,\hat{Q} + \hat{Q}\,A^T + B\,V\,B^T] = 0, \tag{4.16}$$

$$[A^T\,\hat{P} + \hat{P}\,A + C^T\,R\,C]\,\tau = 0, \tag{4.17}$$

$$\text{rank}\ (\hat{Q}) = \text{rank}\ (\hat{P}) = \text{rank}\ (\hat{Q}\,\hat{P}) = n_m. \tag{4.6}$$

Proof. Introduce the augmented system

$$\dot{\widetilde{x}} = \widetilde{A}\widetilde{x} + \widetilde{B}u,$$

$$\widetilde{y} = \widetilde{C}\widetilde{x},$$

where

$$\widetilde{x} \equiv \begin{pmatrix} x \\ x_m \end{pmatrix}, \qquad \widetilde{y} \equiv y - y_m,$$

$$\widetilde{A} \equiv \begin{pmatrix} A & 0 \\ 0 & A_m \end{pmatrix}, \qquad \widetilde{B} \equiv \begin{pmatrix} B \\ B_m \end{pmatrix}, \qquad \widetilde{C} \equiv (C \quad -C_m).$$

The cost function can be written as

$$\begin{aligned} J(A_m, B_m, C_m) &= \lim_{t\to\infty} E[(y - y_m)^T R(y - y_m)] \\ &= \lim_{t\to\infty} E[\widetilde{y}^T R\widetilde{y}] \\ &= \lim_{t\to\infty} E[\widetilde{x}^T \widetilde{C}^T R\widetilde{C}\widetilde{x}] \\ &= \lim_{t\to\infty} E[\widetilde{x}^T \widetilde{R}\widetilde{x}], \end{aligned}$$

where $\widetilde{R} \equiv \widetilde{C}^T R\widetilde{C}$.

Since the cost is a scalar, we have

$$\begin{aligned} J(A_m, B_m, C_m) &= \mathrm{tr}[\lim_{t\to\infty} E(\widetilde{x}^T \widetilde{R}\widetilde{x})] \\ &= \lim_{t\to\infty} E[\mathrm{tr}(\widetilde{x}^T \widetilde{R}\widetilde{x})] \\ &= \lim_{t\to\infty} E[\mathrm{tr}(\widetilde{x}\widetilde{x}^T \widetilde{R})] \\ &= \mathrm{tr}[\lim_{t\to\infty} E(\widetilde{x}\widetilde{x}^T)\widetilde{R}] \\ &= \mathrm{tr}(\widetilde{Q}\widetilde{R}), \end{aligned} \tag{4.18}$$

where $\widetilde{Q} = \lim_{t\to\infty} E(\widetilde{x}\widetilde{x}^T)$.

NOTE: $\widetilde{Q}$ is the variance matrix which tends to the constant symmetric positive semidefinite matrix

$$\widetilde{Q} = \int_0^\infty e^{\widetilde{A}t}\widetilde{B}V\widetilde{B}^T e^{\widetilde{A}^T t}\,dt,$$

which is the unique solution of the Lyapunov matrix equation (cf. Remark 4.3.1)

$$\widetilde{A}\widetilde{Q} + \widetilde{Q}\widetilde{A}^T + \widetilde{B}V\widetilde{B}^T = 0. \tag{4.19}$$

To minimize (4.18), i.e.,

$$J(A_m, B_m, C_m) = \mathrm{tr}(\widetilde{Q}\widetilde{R}),$$

subject to the constraint (4.19), form the Lagrangian

$$L(A_m, B_m, C_m, \widetilde{Q}, \widetilde{P}) \equiv \mathrm{tr}[\lambda \widetilde{Q}\widetilde{R} + (\widetilde{A}\widetilde{Q} + \widetilde{Q}\widetilde{A}^T + \widetilde{V})\widetilde{P}], \tag{4.20}$$

where $\widetilde{V} \equiv \widetilde{B}V\widetilde{B}^T, \lambda \geq 0$, and $\widetilde{P} \in \mathbf{R}^{(n+n_m)\times(n+n_m)}$.

Using formulas for computing partial derivatives of matrices to compute the partial derivatives of (4.20) with respect to $\widetilde{Q}$ it follows that

$$0 = \widetilde{A}^T\widetilde{P} + \widetilde{P}\widetilde{A} + \lambda\widetilde{R}. \tag{4.21}$$

Since $\lambda = 0$ implies $\widetilde{P} = 0$ (since $\widetilde{A}$ is stable), we can take $\lambda = 1$ without loss of generality. Hence $\widetilde{P}$ is the solution of

$$0 = \widetilde{A}^T\widetilde{P} + \widetilde{P}\widetilde{A} + \widetilde{R}. \tag{4.22}$$

Since $\widetilde{A}$ is asymptotically stable, it follows that

$$\widetilde{P} = \int_0^\infty e^{\widetilde{A}^T t}\widetilde{R}e^{\widetilde{A}t}dt, \tag{4.23}$$

so $\widetilde{P}$ is symmetric.

Let $\widetilde{Q}$ and $\widetilde{P}$ be partitioned as

$$\widetilde{Q} = \begin{pmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{pmatrix}, \qquad \widetilde{P} = \begin{pmatrix} P_1 & P_{12} \\ P_{12}^T & P_2 \end{pmatrix}, \tag{4.24}$$

where $Q_1, P_1 \in \mathbf{R}^{n\times n},\ Q_2, P_2 \in \mathbf{R}^{n_m\times n_m}$. The Lagrangian L is then given by

$$\begin{aligned} L = \mathrm{tr}\Bigg[& \begin{pmatrix} A & 0 \\ 0 & A_m \end{pmatrix}\begin{pmatrix} Q_1 & Q_{12} \\ Q_{12}^t & Q_2 \end{pmatrix}\begin{pmatrix} P_1 & P_{12} \\ P_{12}^t & P_2 \end{pmatrix} \\ & + \begin{pmatrix} Q_1 & Q_{12} \\ Q_{12}^t & Q_2 \end{pmatrix}\begin{pmatrix} A^t & 0 \\ 0 & A_m^t \end{pmatrix}\begin{pmatrix} P_1 & P_{12} \\ P_{12}^t & P_2 \end{pmatrix} + \begin{pmatrix} B\,V\,B^t & B\,V\,B_m^t \\ B_m\,V\,B^t & B_m\,V\,B_m^t \end{pmatrix}\begin{pmatrix} P_1 & P_{12} \\ P_{12}^t & P_2 \end{pmatrix} \\ & + \lambda\begin{pmatrix} Q_1 & Q_{12} \\ Q_{12}^t & Q_2 \end{pmatrix}\begin{pmatrix} C^t\,R\,C & -C^t\,R\,C_m \\ -C_m^t\,R\,C & C_m^t\,R\,C_m \end{pmatrix}\Bigg]. \end{aligned} \tag{4.25}$$

Again, the partial derivatives $\dfrac{\partial L}{\partial A_m}, \dfrac{\partial L}{\partial B_m}$, and $\dfrac{\partial L}{\partial C_m}$ can be computed. Setting these derivatives to zero gives

$$\frac{\partial L}{\partial A_m} = Q_{12}^T P_{12} + Q_2 P_2 = 0, \tag{4.26}$$
$$\frac{\partial L}{\partial B_m} = 2(P_{12}^T B + P_2 B_m)V = 0, \tag{4.27}$$
$$\frac{\partial L}{\partial C_m} = 2R(C_m Q_2 - C Q_{12}) = 0. \tag{4.28}$$

On the other hand, equations (4.19) and (4.22) can be written in the form

$$0 = \begin{pmatrix} A & 0 \\ 0 & A_m \end{pmatrix} \begin{pmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{pmatrix} + \begin{pmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{pmatrix} \begin{pmatrix} A^T & 0 \\ 0 & A_m^T \end{pmatrix} + \begin{pmatrix} BVB^T & BVB_m^T \\ B_m V B^T & B_m V B_m^T \end{pmatrix},$$

$$0 = \begin{pmatrix} A^T & 0 \\ 0 & A_m^T \end{pmatrix} \begin{pmatrix} P_1 & P_{12} \\ P_{12}^T & P_2 \end{pmatrix} + \begin{pmatrix} P_1 & P_{12} \\ P_{12}^T & P_1 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & A_m \end{pmatrix} + \begin{pmatrix} C^T R C & -C^T R C_m \\ -C_m^T R C & C_m^T R C_m \end{pmatrix}.$$

If they are expanded, the following equations are obtained:

$$AQ_1 + Q_1 A^T + BVB^T = 0, \tag{4.29}$$

$$AQ_{12} + Q_{12} A_m^T + BVB_m^T = 0, \tag{4.30}$$

$$A_m Q_2 + Q_2 A_m^T + B_m V B_m^T = 0, \tag{4.31}$$

$$A^T P_1 + P_1 A + C^T R C = 0, \tag{4.32}$$

$$A^T P_{12} + P_{12} A_m - C^T R C_m = 0, \tag{4.33}$$

$$A_m^T P_2 + P_2 A_m + C_m^T R C_m = 0. \tag{4.34}$$

At this point, A_m, B_m, and C_m are independent of Q_1 and P_1 and thus (4.29) and (4.32) can be ignored. Since A_m is stable and (A_m, B_m) is controllable, equation (4.31) implies that Q_2 is positive definite. Similarly, P_2 is positive definite (from (4.34)). We can define

$$G \equiv Q_2^{-1} Q_{12}^T, \qquad \Gamma \equiv -P_2^{-1} P_{12}^T, \tag{4.35}$$

so that (4.27) becomes

$$P_{12}^T B = -P_2 B_m,$$

$$B_m = -P_2^{-1} P_{12}^T B = \Gamma B,$$

which is (4.14). (4.28) becomes (since Q_2 is symmetric)

$$C_m = CQ_{12}Q_2^{-1} = CG^T,$$

which is (4.15). Next, since Q_2 and P_2 are nonsingular, the following can be defined:

$$\hat{Q} \equiv Q_{12}Q_2^{-1}Q_{12}^T, \tag{4.36}$$

$$\hat{P} \equiv P_{12}P_2^{-1}P_{12}^T, \tag{4.37}$$

$$M \equiv Q_2 P_2. \tag{4.38}$$

Since Q_2 and P_2 are nonsingular, from (4.26) it follows that Q_{12} and P_{12} have full column rank. So using (4.35)-(4.38) and (4.26), we have

$$\hat{Q}\hat{P} = Q_{12}Q_2^{-1}Q_{12}^T P_{12}P_2^{-1}P_{12}^T = G^T M\Gamma = -Q_{12}P_{12}^T, \tag{4.39}$$

satisfying (4.9) and

$$\mathrm{rank}(\hat{Q}\hat{P}) = \mathrm{rank}(Q_2) = \mathrm{rank}(P_2) = \mathrm{rank}(\hat{Q}) = \mathrm{rank}(\hat{P}) = n_m,$$

satisfying (4.6).

From (4.26) and (4.35) it follows that

$$\Gamma G^T = -P_2^{-1}P_{12}^T Q_{12}Q_2^{-1} = I_{n_m},$$

which is (4.10).

From the definition of $\hat{Q}, \hat{P}$, (4.36) and (4.37), we have

$$Q_{12} = \hat{Q}\Gamma^T, \qquad P_{12} = -\hat{P}G^T,$$

$$Q_2 = \Gamma\hat{Q}\Gamma^T, \qquad P_2 = G\hat{P}G^T,$$

Now substituting these identities, (4.14) and (4.15) into (4.30), (4.31), (4.33) and (4.34) yields

$$0 = A\hat{Q}\Gamma^T + \hat{Q}\Gamma^T A_m^T + BVB^T\Gamma^T, \tag{4.40}$$

$$0 = A_m\Gamma\hat{Q}\Gamma^T + \Gamma\hat{Q}\Gamma^T A_m^T + \Gamma BVB^T\Gamma^T, \tag{4.41}$$

$$0 = A^T\hat{P}G^T + \hat{P}G^T A_m + C^T RCG^T, \tag{4.42}$$

$$0 = A_m^T G\hat{P}G^T + G\hat{P}G^T A_m + GC^T RCG^T. \tag{4.43}$$

(4.40) and (4.41) imply

$$\begin{aligned} A_m &= \Gamma A\hat{Q}\Gamma^T(\Gamma\hat{Q}\Gamma^T)^{-1} \\ &= \Gamma A\hat{Q}\Gamma^T Q_2^{-1} \\ &= \Gamma A Q_{12} Q_2^{-1} \\ &= \Gamma A G^T. \end{aligned}$$

Here we have used $Q_2 = Q_2^T$.

NOTE: According to the definition in Theorem 4.3 for τ, we have

$$\begin{aligned} \tau\hat{Q} &= G^T\Gamma\hat{Q} \\ &= Q_{12}Q_2^{-1}(-P_2^{-1}P_{12}^T)Q_{12}Q_2^{-1}Q_{12}^T \\ &= Q_{12}Q_2^{-1}(-P_2^{-1})(-P_2Q_2)Q_2^{-1}Q_{12}^T \\ &= Q_{12}Q_2^{-1}Q_{12}^T \\ &= \hat{Q}, \end{aligned} \tag{4.44}$$

and similarly

$$\hat{P} = \hat{P}\tau. \tag{4.45}$$

We have

$$(4.41) = \Gamma\ (4.40), \qquad (4.43) = G\ (4.42),$$

thus, only (4.40) and (4.42) are independent.

Finally, from (4.40) we have

$$\begin{aligned} 0 &= G^T(\Gamma\hat{Q}^T A^T + A_m\Gamma\hat{Q}^T + \Gamma BVB^T) \\ &= G^T(\Gamma\hat{Q}^T A^T + \Gamma A G^T\Gamma\hat{Q}^T + \Gamma BVB^T) \\ &= \tau(\hat{Q}A^T + A\hat{Q} + BVB^T), \end{aligned}$$

which is (4.16).

From (4.42) we have

$$\begin{aligned}0 &= (A^T \hat{P} G^T + \hat{P} G^T A_m + C^T R C G^T)\Gamma \\ &= (A^T \hat{P} G^T + \hat{P} G^T \Gamma A G^T + C^T R C G^T)\Gamma \\ &= (A^T \hat{P} + \hat{P} A + C^T R C)\tau,\end{aligned}$$

which is (4.17). Q. E. D.

REMARK 4.3.3. *The quadratic model reduction criterion can be written in the form*

$$\begin{aligned}J(A_m, B_m, C_m) &= \lim E[(y - y_m)^T R (y - y_m)] \\ &= 2\,\mathrm{tr}[(\hat{Q}\hat{P} - W_c W_0)A] \\ &= \mathrm{tr}[\widetilde{Q}\widetilde{R}]. \end{aligned} \tag{4.46}$$

Proof. We already proved

$$J(A_m, B_m, C_m) = \mathrm{tr}(\widetilde{Q}\widetilde{R})$$

as (4.18), so we need only to prove the last step of (4.46).

Now, using the definition of $\widetilde{Q}$ and $\widetilde{R}$, we have

$$\begin{aligned}\mathrm{tr}(\widetilde{Q}\widetilde{R}) &= \mathrm{tr}\left[\begin{pmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{pmatrix}\begin{pmatrix} C^T R C & -C^T R C_m \\ -C_m^T R C & C_m^T R C_m \end{pmatrix}\right] \\ &= \mathrm{tr}(Q_1 C^T R C - 2 Q_{12} C_m^T R C + Q_2 C_m^T R C_m). \end{aligned} \tag{4.47}$$

Combining (4.15) and (4.47) then

$$\begin{aligned}\mathrm{tr}(2Q_{12}C_m^T R C - Q_2 C_m^T R C_m) &= \mathrm{tr}(2Q_{12}C_m^T R C - C_m^T R C_m Q_2) \\ &= \mathrm{tr}(Q_{12} C_m^T R C) \\ &= \mathrm{tr}(Q_{12}(Q_2^{-1})^T Q_{12}^T C^T R C) \\ &= \mathrm{tr}(\hat{Q}^T C^T R C) \\ &= \mathrm{tr}(C^T R C \hat{Q}), \end{aligned} \tag{4.48}$$

then using (4.48), (4.47) becomes

$$\begin{aligned}\operatorname{tr}(\widetilde{Q}\widetilde{R}) &= \operatorname{tr}(Q_1 C^T RC - C^T RC\hat{Q}) \\ &= \operatorname{tr}[C^T RC(Q_1 - \hat{Q})].\end{aligned} \tag{4.49}$$

On the other hand, using (4.17) and $\tau\hat{Q} = \hat{Q}$, we have

$$2\operatorname{tr}(\hat{Q}\hat{P}A) = -\operatorname{tr}(C^T RC\hat{Q}), \tag{4.50}$$

while using (4.12) and $W_c = Q_1$, we have

$$-2\operatorname{tr}(W_c W_0 A) = \operatorname{tr}(C^T RCQ_1). \tag{4.51}$$

Finally, combining (4.49), (4.50) and (4.51) yields

$$\begin{aligned}2\operatorname{tr}[(\hat{Q}\hat{P} - W_c W_0)A] &= \operatorname{tr}[C^T RC(Q_1 - \hat{Q})\} \\ &= \operatorname{tr}(\widetilde{Q}\widetilde{R}).\end{aligned}$$

Q. E. D.

5. TRANSVERSALITY OF HOMOTOPIES FOR H^2 OPTIMAL MODEL ORDER REDUCTION PROBLEM

This section proves that three homotopies $\rho(\lambda, x, a)$ which have been used in (Žigić *et al.*, 1993) and (Ge *et al.*, 1994a) for the H^2 optimal model order reduction problem are transversal to zero, the first requirement of Theorem 3.2.

5.1. Transversality of homotopies based on decompositions of pseudogramians.

Since $\hat{Q}$ and $\hat{P}$ satisfy (4.8), there exists invertible $W \in \mathbf{R}^{n\times n}$ and positive definite diagonal $\Sigma \in \mathbf{R}^{n_m \times n_m}$ such that (Hyland and Bernstein, 1985)

$$\hat{Q} = W \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} W^T = W_1 \, \Sigma \, W_1^T,$$

$$\hat{P} = W^{-T} \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} W^{-1} = U_1^T \, \Sigma \, U_1$$

where

$$W = \begin{pmatrix} \overbrace{W_1}^{n_m} & W_2 \end{pmatrix}, \quad W^{-1} = U = \begin{array}{c} n_m\{ \\ \end{array} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}.$$

Premultipling (4.16) by U_1, and (4.17) by W_1, yields

$$U_1 \, A \, W_1 \Sigma \, W_1^T + \Sigma \, W_1^T \, A^T + U_1 \, B \, V \, B^T = 0, \tag{5.1}$$

$$A^T \, U_1^T \, \Sigma + U_1^T \, \Sigma \, U_1 \, A \, W_1 + C^T \, R \, C \, W_1 = 0. \tag{5.2}$$

The third equation

$$U_1 \, W_1 - I = 0 \tag{5.3}$$

determines the relationship between W_1 and U_1.

The matrix equations (5.1)–(5.3) contain $2\,n\,n_m + n_m^2$ scalar equations. But, the only natural unknowns in (5.1)–(5.3), W_1, U_1, and diagonal Σ, contain $2\,n\,n_m + n_m$ variables. Hence, some other formulation is necessary in order to make an exact match between the number of equations and the number of unknowns. Following (Žigić *et al.*, 1993), all n_m^2 elements of Σ are considered as unknowns, giving the same number of equations as unknowns.

The structure of the problem is such that Σ will turn out to be symmetric, so it can be diagonalized to produce the decomposition of $\hat{Q}$ and $\hat{P}$ described above.

The approach in (Žigić *et al.*, 1993) uses the homotopy map

$$\rho_a \equiv \lambda f(x) + (1-\lambda) g(x;a), \tag{5.4}$$

where the initial problem

$$\rho_a(0,x) = g(x;a) = 0$$

has an easily obtained unique solution and the final problem (5.1)–(5.3) is

$$\rho_a(1,x) = f(x) = 0.$$

f and g are displayed in (5.4) simply to point out that the map $\rho_a(\lambda,x)$ can be viewed as a simple convex combination of two other maps. For notational convenience later when displaying Jacobian matrices the order of the variables is henceforth taken as λ, x, a. Let

$$\begin{aligned} A(\lambda) &= A, \\ B(\lambda) &= \lambda B V B^T + (1-\lambda) B_i, \\ C(\lambda) &= \lambda C^T R C + (1-\lambda) C_i, \end{aligned}$$

where B_i and C_i are matrices defining the initial problem at $\lambda = 0$, and correspond to the parameter vector a in Theorem 3.2. Define

$$\rho_a(\lambda,x) \equiv \rho_a(\lambda,x,a) \equiv \begin{pmatrix} F_1(\lambda,x,a) \\ F_2(\lambda,x,a) \\ F_3(\lambda,x,a) \end{pmatrix}$$

in (5.4) by

$$F_1(\lambda,x,a) \equiv U_1\, A(\lambda)\, W_1\, \Sigma\, W_1^T + \Sigma\, W_1^T\, A^T(\lambda) + U_1\, B(\lambda), \tag{5.5}$$

$$F_2(\lambda,x,a) \equiv A^T(\lambda)\, U_1^T\, \Sigma + U_1^T\, \Sigma\, U_1\, A(\lambda)\, W_1 + C(\lambda)\, W_1, \tag{5.6}$$

$$F_3(\lambda,x,a) \equiv U_1\, W_1 - I, \tag{5.7}$$

where

$$a \equiv \begin{pmatrix} \text{Vec }(B_i) \\ \text{Vec }(C_i) \end{pmatrix}$$

is the generic parameter vector in Theorem 3.2,

$$x \equiv \begin{pmatrix} \text{Vec }(W_1) \\ \text{Vec }(U_1) \\ \text{Vec }(\Sigma) \end{pmatrix}$$

denotes the independent variables $W_1 \in \mathbf{R}^{n \times n_m}$, $U_1 \in \mathbf{R}^{n_m \times n}$, $\Sigma \in \mathbf{R}^{n_m \times n_m}$ corresponding to x in Theorem 3.2, and A, B, C, V, R are constants as in Section 4.

The Jacobian matrix of $\rho(\lambda, x, a)$ has $2\,n\,n_m + n_m^2$ rows and $2\,n^2 + 2\,n\,n_m + n_m^2 + 1$ columns. Rows 1 through $n\,n_m$ correspond to equation (5.5), rows $n\,n_m + 1$ through $2\,n\,n_m$ correspond to equation (5.6), and rows $2\,n\,n_m + 1$ through $2\,n\,n_m + n_m^2$ correspond to equation (5.7). The first column corresponds to the derivatives with respect to λ, columns 2 through $n\,n_m + 1$ correspond to the derivatives with respect to W_1, columns $n\,n_m + 2$ through $2\,n\,n_m + 1$ correspond to the derivatives with respect to U_1, columns $2\,n\,n_m + 2$ through $2\,n\,n_m + n_m^2 + 1$ correspond to the derivatives with respect to Σ, columns $2\,n\,n_m + n_m^2 + 2$ through $2\,n\,n_m + n_m^2 + n^2 + 1$ correspond to the derivatives with respect to B_i, and columns $2\,n\,n_m + n_m^2 + n^2 + 2$ through $2\,n\,n_m + n_m^2 + 2\,n^2 + 1$ correspond to the derivatives with respect to C_i:

$$D\rho(\lambda, x, a) = \begin{pmatrix} D_\lambda \rho & D_{W_1}\rho & D_{U_1}\rho & D_\Sigma \rho & D_{B_i}\rho & D_{C_i}\rho \end{pmatrix}. \tag{5.8}$$

Since $F_3(\lambda, x, a)$ does not depend upon λ, B_i, and C_i,

$$D_\lambda F_3(\lambda, x, a) = 0,$$

$$D_{B_i} F_3(\lambda, x, a) = 0,$$

$$D_{C_i} F_3(\lambda, x, a) = 0,$$

and similarly

$$D_{C_i} F_1(\lambda, x, a) = D_{B_i} F_2(\lambda, x, a) = 0.$$

Thus

$$
\begin{aligned}
D\rho(\lambda, x, a) &= D\rho(\lambda, W_1, U_1, \Sigma, B_i, C_i) \\
&= \begin{pmatrix} D_\lambda F_1 & D_x F_1 & D_a F_1 \\ D_\lambda F_2 & D_x F_2 & D_a F_2 \\ 0 & D_x F_3 & 0 \end{pmatrix} \\
&= \begin{pmatrix} D_\lambda F_1 & D_{W_1} F_1 & D_{U_1} F_1 & D_\Sigma F_1 & D_{B_i} F_1 & 0 \\ D_\lambda F_2 & D_{W_1} F_2 & D_{U_1} F_2 & D_\Sigma F_2 & 0 & D_{C_i} F_2 \\ 0 & D_{W_1} F_3 & D_{U_1} F_3 & D_\Sigma F_3 & 0 & 0 \end{pmatrix}.
\end{aligned}
\tag{5.9}
$$

The following lemma will be used in the proof of Theorem 5.2.

LEMMA 5.1. *Let $X \in \mathbf{R}^{p\times q}$ and $A \in \mathbf{R}^{n\times m}$, $B \in \mathbf{R}^{m\times l}$ be differentiable with respect to x_{ij} for $1 \le i \le p$, $1 \le j \le q$. Then*

$$\frac{\partial}{\partial x_{ij}}(AB) = (\frac{\partial}{\partial x_{ij}}A)B + A(\frac{\partial}{\partial x_{ij}}B),$$

and for constant M, interpreting the derivative D_X as $D_{\mathrm{Vec}(X)}$,

$$D_X(MX) = I \otimes M, \qquad D_X(XM) = M^T \otimes I.$$

The proof of Lemma 5.1 is straightforward calculus.

THEOREM 5.2. *The homotopy map given by (5.5)–(5.7) is transversal to zero (for $0 \le \lambda < 1$).*

Proof. To prove that $D\rho(\lambda, x, a)$ given in (5.9) has full column rank, i.e.,

$$\text{rank } (D\rho(\lambda, x, a)) = 2nn_m + n_m^2,$$

it suffices to prove that

$$\text{rank}(D_x F_3) = \text{rank}(D_{W_1} F_3 \quad D_{U_1} F_3 \quad D_\Sigma F_3) = n_m^2, \tag{5.10}$$

$$\text{rank}(D_a F_1) = \text{rank}(D_{B_i} F_1 \quad 0) = nn_m, \tag{5.11}$$

$$\text{rank}(D_a F_2) = \text{rank}(0 \quad D_{C_i} F_2) = nn_m. \tag{5.12}$$

The meaning of expressions like $D_\Sigma F_3$ is ambiguous until some ordering is specified for the components of the matrices Σ and F_3. Hereafter, except when a consistent ordering is required for compatibility, whichever ordering is convenient is used. If unspecified, the standard ordering by columns (Vec) is assumed.

Using Lemma 5.1, ordering U_1 and F_3 by rows,

$$D_{U_1}F_3(\lambda, x, a) = D_{U_1}(U_1W_1) = \overbrace{\begin{pmatrix} W_1^T & 0 & \dots & 0 \\ 0 & W_1^T & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & W_1^T \end{pmatrix}}^{n_m \text{ times}}, \tag{5.13}$$

and ordering W_1 and F_3 by columns,

$$D_{W_1}F_3(\lambda, x, a) = D_{W_1}(U_1W_1) = \overbrace{\begin{pmatrix} U_1 & 0 & \dots & 0 \\ 0 & U_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & U_1 \end{pmatrix}}^{n_m \text{ times}}. \tag{5.14}$$

Since $U_1W_1 = I$, by Sylvester's inequality,

$$\text{rank }(U_1) = \text{rank }(W_1) = n_m,$$

and therefore

$$\text{rank}(D_xF_3) = \text{rank}(D_{U_1}F_3) = \text{rank}(D_{W_1}F_3) = n_m^2,$$

which is (5.10).

Using Lemma 5.1, ordering B_i and F_1 by columns,

$$\begin{aligned} D_{B_i}F_1(\lambda, x, a) &= D_{B_i}(U_1B(\lambda)) \\ &= (1-\lambda)D_{B_i}(U_1B_i) \\ &= (1-\lambda)\overbrace{\begin{pmatrix} U_1 & 0 & \dots & 0 \\ 0 & U_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & U_1 \end{pmatrix}}^{n \text{ times}}, \end{aligned} \tag{5.15}$$

and using (5.15) for $\lambda \neq 1$ yields

$$\text{rank}(D_{B_i}F_1) = nn_m.$$

Similarly, ordering C_i and F_2 by rows,

$$\begin{aligned}
D_{C_i}F_2(\lambda, x, a) &= D_{C_i}(C(\lambda)W_1) \\
&= (1-\lambda)D_{C_i}(C_iW_1) \\
&= (1-\lambda)\overbrace{\begin{pmatrix} W_1^T & 0 & \dots & 0 \\ 0 & W_1^T & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & W_1^T \end{pmatrix}}^{n \text{ times}},
\end{aligned} \tag{5.16}$$

so for $\lambda \neq 1$

$$\operatorname{rank}(D_{C_i}F_2) = nn_m.$$

This completes the proof of (5.10)–(5.12), and the proof that the homotopy map (5.5)–(5.7) is transversal to zero for all $0 \le \lambda < 1$. Q. E. D.

REMARK 5.2.1. *One can use more variables in the parameter vector a, e.g., $A(\lambda) = \lambda A + (1-\lambda)A_i$, without affecting the full rank properties.*

5.2. Transversality of homotopies based on input normal form.

The following theorem is needed to present the homotopy method for the input normal form.

THEOREM 5.3 (Kabamba, 1985a). *Suppose $(\bar{A}_m, \bar{B}_m, \bar{C}_m)$ is asymptotically stable and minimal. Then there exist a similarity transformation U and a positive definite matrix $\Omega = \operatorname{diag}(\omega_1, \cdots, \omega_{n_m})$ such that $A_m = U^{-1}\bar{A}_mU$, $B_m = U^{-1}\bar{B}_m$, and $C_m = \bar{C}_mU$ satisfy*

$$\begin{aligned}
A_m + A_m^T + B_mVB_m^T &= 0, \\
A_m^T\Omega + \Omega A_m + C_m^TRC_m &= 0.
\end{aligned} \tag{5.17}$$

In addition,

$$\begin{aligned}
(A_m)_{ii} &= -\frac{1}{2}(B_mVB_m^T)_{ii}, \\
\omega_i &= \frac{(C_m^TRC_m)_{ii}}{(B_mVB_m^T)_{ii}}, \\
(A_m)_{ij} &= \frac{(C_m^TRC_m)_{ij} - \omega_j(B_mVB_m^T)_{ij}}{\omega_j - \omega_i}, \qquad \text{if } \omega_i \neq \omega_j.
\end{aligned} \tag{5.18}$$

DEFINITION 5.3.1. The triple $\big(A_m, B_m, C_m\big)$ satisfying (5.17) and (5.18) is said to be in *input normal form.*

Note that the utility of the input normal form (5.17)–(5.18) lies in using B_m and C_m as the independent variables, and then being able to recover A_m uniquely from B_m and C_m. The number of variables in B_m and C_m is $n_m(m+l)$, the minimum number of variables possible to describe a reduced order model, and thus the input normal form parametrization is referred to as a "minimal parametrization." If $\omega_i = \omega_j$ for some $i \neq j$, then regardless of (5.17) holding, (5.18) fails to permit the unique recovery of A_m.

Under the assumption that the solution (A_m, B_m, C_m) being sought exists in input normal form, the only independent variables are B_m and C_m, and in this case the domain is

$$\begin{aligned}\big\{(A_m, B_m, C_m):\ & A_m \text{ is asymptotically stable,}\\ &(A_m, B_m, C_m) \text{ is minimal and in input normal form}\big\}.\end{aligned}$$

Now for (A_m, B_m, C_m) in input normal form, the cost function can be written as

$$J(A_m, B_m, C_m) = \operatorname{tr}\,\big(\tilde{Q}_I \tilde{R}_I\big), \tag{5.19}$$

where $\tilde{Q}_I$ is a symmetric and positive definite matrix satisfying

$$\tilde{A}_I \tilde{Q}_I + \tilde{Q}_I \tilde{A}_I^T + \tilde{V}_I = 0, \tag{5.20}$$

and

$$\begin{aligned}\tilde{A}_I &= \begin{pmatrix} A & 0 \\ 0 & A_m \end{pmatrix}\\ \tilde{R}_I &= \begin{pmatrix} C^TRC & -C^TRC_m \\ -C_m^TRC & C_m^TRC_m \end{pmatrix}\\ \tilde{V}_I &= \begin{pmatrix} BVB^T & BVB_m^T \\ B_mVB^T & B_mVB_m^T \end{pmatrix}.\end{aligned} \tag{5.21}$$

$\tilde{Q}_I$ can be written as

$$\tilde{Q}_I = \begin{pmatrix} \bar{Q}_1 & \bar{Q}_{12} \\ \bar{Q}_{12}^T & \bar{Q}_2 \end{pmatrix}, \tag{5.22}$$

where $\bar{Q}_1 \in \mathbf{R}^{n\times n}$, $\bar{Q}_{12} \in \mathbf{R}^{n\times n_m}$, and $\bar{Q}_2 \in \mathbf{R}^{n_m\times n_m}$.

To minimize (5.19) under the constraints (5.17) and (5.20) leads to the Lagrangian

$$L(A_m, B_m, C_m, \Omega, \tilde{Q}_I, M_c, M_0, \tilde{P}_I) = \mathrm{tr}\Big[\tilde{Q}_I \tilde{R}_I + \big(A_m + A_m^T + B_m V B_m^T\big) M_c$$
$$+\big(A_m^T \Omega + \Omega A_m + C_m^T R C_m\big) M_o + \big(\tilde{A}_I \tilde{Q}_I + \tilde{Q}_I \tilde{A}_I^T + \tilde{V}_I\big) \tilde{P}_I\Big],$$

where the symmetric matrices M_o, M_c, and $\tilde{P}_I$ are Lagrange multipliers.

Setting $\partial L/\partial \tilde{Q}_I = 0$ gives an equation for $\tilde{P}_I$ similar to (5.20) for $\tilde{P}$,

$$\tilde{A}_I^T \tilde{P}_I + \tilde{P}_I \tilde{A}_I + \tilde{R}_I = 0, \tag{5.23}$$

where $\tilde{P}_I$ is symmetric positive definite and can be partitioned similarly to $\tilde{Q}_I$ as

$$\tilde{P}_I = \begin{pmatrix} \bar{P}_1 & \bar{P}_{12} \\ \bar{P}_{12}^T & \bar{P}_2 \end{pmatrix}. \tag{5.24}$$

The matrices M_c and M_o satisfy (Davis *et al.*, 1992)

$$M_c = -\big(\frac{1}{2} S + \Omega M_o\big), \tag{5.25}$$

$$\big(M_o\big)_{ii} = -\frac{1}{\big(A_m\big)_{ii}} \sum_{\substack{j=1 \\ j \neq i}}^{n_m} \big(A_m\big)_{ij} \big(M_o\big)_{ji}, \tag{5.26}$$

$$\big(M_o\big)_{ij} = \frac{(S)_{ij} - (S)_{ji}}{2(\omega_j - \omega_i)}, \quad \text{if } \omega_j \neq \omega_i, \tag{5.27}$$

where

$$S = 2\big(\bar{P}_{12}^T \bar{Q}_{12} + \bar{P}_2 \bar{Q}_2\big).$$

Setting $\partial L/\partial B_m = 0$ and $\partial L/\partial C_m = 0$ gives

$$2\big(\bar{P}_{12}^T B + \bar{P}_2 B_m\big) V + 2 M_c B_m V = 0, \tag{5.28}$$

$$2R\big(C_m \bar{Q}_2 - C \bar{Q}_{12}\big) + 2 R C_m M_o = 0. \tag{5.29}$$

Observe that $\tilde{P}_I$ through (5.23) and $\tilde{Q}_I$ through (5.20) depend on B_m and C_m as does A_m through (5.18). Similarly M_c through (5.25) and M_0 through (5.26)–(5.27) depend on B_m and C_m. Thus everything in (5.28)–(5.29) is a function of B_m and C_m. Use the homotopy map structure of (5.4) and let

$$B(\lambda) = \lambda B + (1 - \lambda) B_i,$$

$$C(\lambda) = \lambda C + (1 - \lambda) C_i,$$

where B_i and C_i are matrices defining the initial problem at $\lambda = 0$, and correspond to the parameter vector a in Theorem 3.2. The structure of the homotopy map $\rho(\lambda, x, a)$ for the input normal form is now

$$F_1(\lambda, x, a) = (\bar{P}_{12}^T B(\lambda) + \bar{P}_2 B_m)V + M_c B_m V, \tag{5.30}$$

$$F_2(\lambda, x, a) = R\big(C_m \bar{Q}_2 - C(\lambda)\bar{Q}_{12}\big) + RC_m M_o, \tag{5.31}$$

where

$$a \equiv \begin{pmatrix} \text{Vec } (B_i) \\ \text{Vec } (C_i) \end{pmatrix}$$

denotes the parameter variables $B_i \in \mathbf{R}^{n\times m}$, $C_i \in \mathbf{R}^{l\times n}$,

$$x \equiv \begin{pmatrix} \text{Vec } (B_m) \\ \text{Vec } (C_m) \end{pmatrix}$$

denotes the independent variables B_m and C_m corresponding to x in Theorem 3.2, and A, B, C, V, R are constants as in Chapter 4.

The Jacobian matrix of $\rho(\lambda, x, a)$ has $n_m m + n_m l$ rows and $(n_m + n)(m + l) + 1$ columns.

Since $F_1(\lambda, x, a)$ does not involve C_i and $F_2(\lambda, x, a)$ does not involve B_i

$$D_{C_i} F_1(\lambda, x, a) = 0, \quad D_{B_i} F_2(\lambda, x, a) = 0.$$

The Jacobian matrix of $\rho(\lambda, x, a)$ is

$$D\rho(\lambda, x, a) = \begin{pmatrix} D_\lambda F_1 & D_{B_m} F_1 & D_{C_m} F_1 & D_{B_i} F_1 & 0 \\ D_\lambda F_2 & D_{B_m} F_2 & D_{C_m} F_2 & 0 & D_{C_i} F_2 \end{pmatrix}. \tag{5.32}$$

The following lemma will be used in the proof of Theorem 5.5.

LEMMA 5.4. *Let $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{A}_I$, $\tilde{B}_I$, $\tilde{C}_I$, $\tilde{P}$, $\tilde{Q}$, $\tilde{R}$, $\tilde{P}_I$, $\tilde{Q}_I$, $\tilde{R}_I$, Ω and U be defined as above. Then*

$$\bar{Q}_1 = Q_1, \qquad \bar{P}_1 = P_1, \tag{5.33}$$

$$\bar{Q}_{12} = Q_{12}U^{-T}, \qquad \bar{P}_{12} = P_{12}U, \tag{5.34}$$

$$\bar{Q}_2 = I, \qquad \bar{P}_2 = \Omega, \tag{5.35}$$

$$Q_2 = UU^T, \qquad P_2 = U^{-T}\Omega U^{-1}. \tag{5.36}$$

In addition, P_{12}, Q_{12}, $\bar{P}_{12}$, and $\bar{Q}_{12}$ have full column rank.

Proof. Equations (5.20) and (5.23) can be written in the form

$$\begin{pmatrix} A & 0 \\ 0 & A_m \end{pmatrix}\begin{pmatrix} \bar{Q}_1 & \bar{Q}_{12} \\ \bar{Q}_{12}^T & \bar{Q}_2 \end{pmatrix} + \begin{pmatrix} \bar{Q}_1 & \bar{Q}_{12} \\ \bar{Q}_{12}^T & \bar{Q}_2 \end{pmatrix}\begin{pmatrix} A^T & 0 \\ 0 & A_m^T \end{pmatrix} + \begin{pmatrix} B\,V\,B^T & B\,V B_m^T \\ B_m\,V\,B^T & B_m\,V\,B_m^T \end{pmatrix} = 0,$$
$$\begin{pmatrix} A^T & 0 \\ 0 & A_m^T \end{pmatrix}\begin{pmatrix} \bar{P}_1 & \bar{P}_{12} \\ \bar{P}_{12}^T & \bar{P}_2 \end{pmatrix} + \begin{pmatrix} \bar{P}_1 & \bar{P}_{12} \\ \bar{P}_{12}^T & \bar{P}_2 \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & A_m \end{pmatrix} + \begin{pmatrix} C^TRC & -C^TRC_m \\ -C_m^TRC & C_m^TRC_m \end{pmatrix} = 0.$$

Expanding these equations yields

$$A\bar{Q}_1 + \bar{Q}_1A^T + BVB^T = 0, \tag{5.37}$$

$$A\bar{Q}_{12} + \bar{Q}_{12}A_m^T + BVB_m^T = 0, \tag{5.38}$$

$$A_m\bar{Q}_2 + \bar{Q}_2A_m^T + B_mVB_m^T = 0, \tag{5.39}$$

$$A^T\bar{P}_1 + \bar{P}_1A + C^TRC = 0, \tag{5.40}$$

$$A^T\bar{P}_{12} + \bar{P}_{12}A_m - C^TRC_m = 0, \tag{5.41}$$

$$A_m^TP_2 + \bar{P}_2A_m + C_m^TRC_m = 0. \tag{5.42}$$

Comparing (4.29) with (5.37), and (4.32) with (5.40) yields (5.33).

If the definitions $A_m = U^{-1}\bar{A}_mU$, $B_m = U^{-1}\bar{B}_m$, and $C_m = \bar{C}_mU$ in Theorem 5.3 are substituted into (5.17) then (5.17) becomes

$$\bar{A}_mUU^T + UU^T\bar{A}_m^T + \bar{B}_mV\bar{B}_m^T = 0, \tag{5.43}$$

$$\bar{A}_m^TU^{-T}\Omega U^{-1} + U^{-T}\Omega U^{-1}\bar{A}_m + \bar{C}_m^TR\bar{C}_m = 0. \tag{5.44}$$

Comparing (4.31) and (4.34) with (5.43) and (5.44) yields (5.36).

If $A_m = U^{-1}\bar{A}_mU$, $B_m = U^{-1}\bar{B}_m$, and $C_m = \bar{C}_mU$ are substituted into (5.38) and (5.41) and the resulting equations are compared with (4.30) and (4.33), then (5.34) follows. Comparing (5.17) and (5.18) with (5.39) and (5.42) yields (5.35).

Finally, since Q_2 and P_2 are nonsingular, from (4.26) it follows that Q_{12} and P_{12} have full column rank. Since U is nonsingular, from (5.34) it follows that $\bar{Q}_{12}$ and $\bar{P}_{12}$ also have full rank. Q. E. D.

THEOREM 5.5. *Let $\tilde{P}_I$ and $\tilde{Q}_I$ be defined as above. Then $D\rho(\lambda, x, a)$ given by (5.32) has full column rank for $0 \le \lambda < 1$, i.e., the homotopy map (5.30)–(5.31) is transversal to zero for $0 \le \lambda < 1$.*

Proof. To prove $D\rho(\lambda, x, a)$ given by (5.32) has full column rank, i.e.,

$$\text{rank } (D\rho(\lambda, x, a)) = n_m m + n_m l,$$

it suffices to prove that

$$\text{rank}(D_a F_1) = \text{rank}(D_{B_i} F_1) = n_m m, \tag{5.45}$$

$$\text{rank}(D_a F_2) = \text{rank}(D_{C_i} F_2) = n_m l. \tag{5.46}$$

Since V and R are constant symmetric positive definite matrices, without loss of generality set $V = I$ in (5.30) and $R = I$ in (5.31). Using Lemma 5.1 to compute $D_{B_i} F_1(\lambda, x, a)$, ordering B_i and F_1 by columns,

$$\begin{aligned} D_{B_i} F_1(\lambda, x, a) &= D_{B_i}(\bar{P}_{12}^T B(\lambda)) \\ &= (1-\lambda) D_{B_i}(\bar{P}_{12}^T B_i) \\ &= (1-\lambda) \overbrace{\begin{pmatrix} \bar{P}_{12}^T & 0 & \dots & 0 \\ 0 & \bar{P}_{12}^T & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \bar{P}_{12}^T \end{pmatrix}}^{m \text{ times}}. \end{aligned} \tag{5.47}$$

Ordering C_i and F_2 by rows gives

$$\begin{aligned} D_{C_i} F_2(\lambda, x, a) &= D_{C_i}(-C(\lambda) \bar{Q}_{12}) \\ &= (\lambda - 1) D_{C_i}(C_i \bar{Q}_{12}) \\ &= (\lambda - 1) \overbrace{\begin{pmatrix} \bar{Q}_{12}^T & 0 & \dots & 0 \\ 0 & \bar{Q}_{12}^T & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \bar{Q}_{12}^T \end{pmatrix}}^{l \text{ times}}. \end{aligned} \tag{5.48}$$

Now finally, using Lemma 5.4, (5.47), and (5.48), the rank statements of (5.45) and (5.46) follow.

So the homotopy map (5.30)–(5.31) for the input normal form parametrization of (A_m, B_m, C_m) for the H^2 model order reduction problem is transversal to zero. Q. E. D.

5.3. Transversality of homotopies based on Ly's formulation .

In Ly's formulation (Ly *et al.*, 1985), the reduced order model is represented with respect to a basis such that A_m is a 2×2 block-diagonal matrix (2×2 blocks with an additional 1×1 block if n_m is odd) with 2×2 blocks in the form

$$\begin{pmatrix} 0 & 1 \\ * & * \end{pmatrix},$$

B_m is a full matrix, and

$$C_m = \big(\, (C_m)_1 \quad (C_m)_2 \quad \cdots \quad (C_m)_i \quad \cdots \quad (C_m)_r \, \big),$$

where

$$(C_m)_i = \begin{pmatrix} 1 & * & \cdots & * \\ 0 & * & \cdots & * \end{pmatrix}^T,$$

$$(C_m)_r = \big(1 \quad * \quad \cdots \quad * \big)^T, \quad \text{if } n_m \text{ is odd.}$$

Let $\mathcal{S}$ be the set of indices of those elements of A_m which are independent variables , i.e.,

$$\mathcal{S} \equiv \{(2,1),(2,2),\ldots,(2i,2i-1),(2i,2i),\ldots,(n_m,n_m)\}.$$

To minimize the cost function $J(A_m, B_m, C_m)$, consider the Lagrangian

$$L(A_m, B_m, C_m, \tilde{Q}) = \operatorname{tr}\big[\tilde{Q}\tilde{R} + \big(\tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}^T + \tilde{V}\big)\tilde{P}\big], \tag{5.49}$$

where the symmetric matrix $\tilde{P}$ is a Lagrange multiplier, $\tilde{Q}$ satisfies (5.20), $\tilde{A}$, $\tilde{R}$, and $\tilde{V}$ are defined in Section 5.2. Setting $\partial L/\partial\tilde{Q} = 0$ gives (5.22); $\tilde{Q}$ and $\tilde{P}$ are symmetric positive definite and can be partitioned as in (4.24). A straightforward calculation shows

$$\begin{aligned}
\frac{\partial L}{\partial (A_m)_{ij}} &= 2\big(P_{12}^T Q_{12} + P_2 Q_2\big)_{ij}, \quad (i,j) \in \mathcal{S}, \\
\frac{\partial L}{\partial B_m} &= 2\big(P_{12}^T B + P_2 B_m\big)V, \\
\frac{\partial L}{\partial (C_m)_{ij}} &= 2\frac{\partial}{\partial (C_m)_{ij}}\big[\ \operatorname{tr}\ \big(-Q_{12}^T C^T R C_m\big) + \ \operatorname{tr}\ \big(Q_2 C_m^T R C_m\big)\big] \\
&= 2R\big(C_m Q_2 - C Q_{12}\big)_{ij}, \quad i > 1.
\end{aligned} \tag{5.50}$$

Let

$$A(\lambda) = A,$$
$$B(\lambda) = \lambda B + (1-\lambda)B_i,$$
$$C(\lambda) = \lambda C + (1-\lambda)C_i,$$

where B_i and C_i play the same role as in Section 5.1. Let

$$\begin{aligned} H_{A_m}(\lambda, x) &= \frac{1}{2}\frac{\partial L}{\partial A_m} = \big(P_{12}^T Q_{12} + P_2 Q_2\big), \\ H_{B_m}(\lambda, x, B_i) &= \frac{1}{2}\frac{\partial L}{\partial B_m} = \big(P_{12}^T B(\lambda) + P_2 B_m\big)V, \\ H_{C_m}(\lambda, x, C_i) &= \frac{1}{2}\frac{\partial L}{\partial C_m} = R\big(C_m Q_2 - C(\lambda)Q_{12}\big), \end{aligned} \tag{5.51}$$

where in H_{A_m} only those elements corresponding to the independent variables of A_m are nonzero and

$$x \equiv \begin{pmatrix} (A_m)_{\mathcal{S}} \\ \text{Vec }(B_m) \\ \text{Vec }(C_m)_{\mathcal{T}\cdot} \end{pmatrix} \tag{5.52}$$

denotes the independent variables, $(A_m)_{\mathcal{S}}$ is a vector consisting of those elements in A_m with indices in the set $\mathcal{S}$, i.e.,

$$(A_m)_{\mathcal{S}} = \big((A_m)_{21}, (A_m)_{22}, \cdots, (A_m)_{n_m n_m}\big)^T,$$

$(C_m)_{\mathcal{T}\cdot}$ is the matrix obtained from rows $\mathcal{T}\cdot = \{2, \dots, l\}$ of C_m.

The homotopy map $\rho(\lambda, x, a)$ for Ly's formulation is now defined as

$$F_1(\lambda, x, a) = \Big[H_{A_m}(\lambda, x)\Big]_{\mathcal{S}}, \tag{5.53}$$
$$F_2(\lambda, x, a) = \text{Vec }\Big[H_{B_m}(\lambda, x, B_i)\Big], \tag{5.54}$$
$$F_3(\lambda, x, a) = \text{Vec }\Big[H_{C_m}(\lambda, x, C_i)\Big]_{\mathcal{T}\cdot}, \tag{5.55}$$

where again the subscripts $\mathcal{S}$ and $\mathcal{T}\cdot$ pick out the appropriate matrix elements, and

$$a \equiv \begin{pmatrix} \text{Vec }(B_i) \\ \text{Vec }(C_i) \end{pmatrix} \tag{5.56}$$

denotes the parameter variables. As discussed in Section 5.2, without loss of generality set $V = I$ in (5.54) and $R = I$ in (5.55).

The Jacobian matrix of $\rho(\lambda, x, a)$ is

$$D\rho(\lambda, x, a) = \begin{pmatrix} D_\lambda F_1 & D_x F_1 & 0 & 0 \\ D_\lambda F_2 & D_x F_2 & D_{B_i} F_2 & 0 \\ D_\lambda F_3 & D_x F_3 & 0 & D_{C_i} F_3 \end{pmatrix}. \tag{5.57}$$

LEMMA 5.6. *Suppose* $\operatorname{rank}(D_x F_1) = n_m$. *Then the Jacobian matrix (5.57) has full column rank for all* $0 \le \lambda < 1$, *i.e., the homotopy map (5.53)–(5.55) is transversal to zero for all* $0 \le \lambda < 1$.

Proof. A similar proof to that in Section 5.2 yields

$$\operatorname{rank}(D_{B_i} F_2) = m n_m \quad \text{for} \quad \lambda \ne 1. \tag{5.58}$$

Ordering C_i and F_3 by rows gives

$$\begin{aligned} D_{C_i} F_3(\lambda, x, a) &= D_{C_i}(-C(\lambda) Q_{12})_{\mathcal{T}\cdot} \\ &= (\lambda - 1) D_{C_i}(C_i Q_{12})_{\mathcal{T}\cdot} \\ &= (\lambda - 1) D_{C_i}[(C_i)_{\mathcal{T}\cdot} Q_{12}] \\ &= (1 - \lambda) \overbrace{\begin{pmatrix} 0 & Q_{12}^T & 0 & \dots & 0 \\ 0 & 0 & Q_{12}^T & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & Q_{12}^T \end{pmatrix}}^{l \text{ times}}, \end{aligned} \tag{5.59}$$

and then as before

$$\operatorname{rank}(D_{C_i} F_3) = (l - 1) n_m \quad \text{for} \quad \lambda \ne 1. \tag{5.60}$$

Note that

$$\operatorname{rank}(D_x F_1) = n_m,$$

which completes the proof. Q. E. D.

Note that there are only n_m functions in F_1 but $(l + m) n_m + 1$ independent variables in x and λ. As $l + m \gg 1$ usually in real problems which have been considered previously, all Jacobian matrices of F_1 in those problems satisfied the full rank condition. Since each of Q_{12}, P_{12}, Q_2, and P_2 are implicit functions of x and $A(\lambda)$, and one can not give explicit expressions for $D_x F_1$ or $D_{A_i} F_1$ as (5.59) for $D_{C_i} F_3$, which show clearly the rank conditions, it was necessary to *assume* that $\text{rank}(D_x F_1) = n_m$ in Lemma 5.6. To guarantee the full rank of $D\rho$ without this assumption, instead of using (5.53), let $x = (\eta, \zeta)$, $\eta \in \mathbf{E}^{n_m}$,

$$F_1(\lambda, x, a) = \lambda \Big[H_{A_m}(\lambda, x) \Big]_{\mathcal{S}} + (1 - \lambda)(\eta - \eta_0), \tag{5.61}$$

with n_m independent parameter variables in η_0, which gives

$$D_{\eta_0} F_1 = (1 - \lambda) I_{n_m} \quad \text{for} \quad \lambda \neq 1. \tag{5.62}$$

Combining (5.58), (5.60), and (5.62) completes the proof that the map (5.61), (5.54), and (5.55) is transversal to zero. Note that the homotopy construction in (5.61) is a theoretical convenience, and in practice the choice (5.53) has been entirely satisfactory.

6. BOUNDEDNESS OF $\rho_a^{-1}(0)$ FOR H^2 OPTIMAL MODEL ORDER REDUCTION PROBLEM

6.1. Counterexample for optimal projection homotopies.

The zero set $\rho_a^{-1}(0)$ of a given homotopy map based on the optimal projection equations (5.1)–(5.3) is not always bounded, as shown by the following 2-dimensional example.

The system (Kabamba, 1985b) is given by

$$A = \begin{pmatrix} -0.25 & -0.4 \\ -0.4 & -0.72 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 1.2 \end{pmatrix}, \quad C = (1 \quad 1.2) . \tag{6.1}$$

For the system (4.1)–(4.4) defined by (6.1), the solution set of the optimal projection equations (5.1)–(5.3) contains an isolated solution and a one-dimensional manifold of solutions.

The isolated solution of this system is

$$A_m = (-0.838521), \quad B_m = (1.537575), \quad C_m = (1.537575),$$

which was obtained by both POLSYS from HOMPACK (Watson *et al.*, 1987) and by a homotopy approach (Žigić *et al.*, 1993). The one-dimensional manifold of solutions can be derived directly from equations (5.1)–(5.3) as follows.

Let $W_1 = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$, $U_1 = (u_1, u_2)$, $\Sigma = \sigma$, $V = I$, and $R = I$. The optimal projection equations (5.1)–(5.3) for this problem can be written as

$$\begin{aligned}
0 &= -0.25 w_1^2 u_1 \sigma - 0.4 w_1 w_2 u_1 \sigma - 0.4 w_1^2 u_2 \sigma \\
&\quad - 0.72 w_1 w_2 u_2 \sigma - 0.25 w_1 \sigma - 0.4 w_2 \sigma + u_1 + 1.2 u_2, \\
0 &= -0.25 w_1 w_2 u_1 \sigma - 0.4 w_2^2 u_1 \sigma - 0.4 w_1 w_2 u_2 \sigma \\
&\quad - 0.72 w_2^2 u_2 \sigma - 0.4 w_1 \sigma - 0.72 w_2 \sigma + 1.2 u_1 + 1.44 u_2, \\
0 &= -0.25 w_1 u_1^2 \sigma - 0.4 w_2 u_1^2 \sigma - 0.4 w_1 u_1 u_2 \sigma \\
&\quad - 0.72 w_2 u_1 u_2 \sigma - 0.25 u_1 \sigma - 0.4 u_2 \sigma + w_1 + 1.2 w_2, \\
0 &= -0.25 w_1 u_1 u_2 \sigma - 0.4 w_2 u_1 u_2 \sigma - 0.4 w_1 u_2^2 \sigma \\
&\quad - 0.72 w_2 u_2^2 \sigma - 0.4 u_1 \sigma - 0.72 u_2 \sigma + 1.2 w_1 + 1.44 w_2, \\
0 &= w_1 u_1 + w_2 u_2 - 1.
\end{aligned} \tag{6.2}$$

The triple (A_m, B_m, C_m) is given by

$$\begin{aligned} A_m &= \Gamma A G^T = (u_1 \; u_2) \begin{pmatrix} -0.25 & -0.4 \\ -0.4 & -0.72 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \\ &= w_1(-0.25u_1 - 0.4u_2) + w_2(-0.4u_1 - 0.72u_2), \\ B_m &= \Gamma B = (u_1 \; u_2) \begin{pmatrix} 1 \\ 1.2 \end{pmatrix} = u_1 + 1.2u_2, \\ C_m &= C G^T = (1 \quad 1.2) \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = w_1 + 1.2w_2, \end{aligned} \tag{6.3}$$

where $\Gamma = U_1$ and $G = W_1^T$.

The zero set of (6.2) contains

$$\{(W_1, U_1, \Sigma) : \; w_1 = -1.2w_2, \quad u_1 = -1.2u_2, \quad u_2 = \frac{1}{2.44w_2}, \quad \sigma = 0\}$$

which is unbounded. Every point in this set corresponds to the same triple (A_m, B_m, C_m):

$$A_m = -.0491803, \quad B_m = 0, \quad C_m = 0.$$

The homotopy map based on the optimal projection equations is

$$\begin{aligned} &U_1\, A(\lambda)\, W_1\, \Sigma\, W_1^T + \Sigma\, W_1^T\, A^T(\lambda) + U_1\, B V B^T = 0, \\ &A^T(\lambda)\, U_1^T\, \Sigma + U_1^T\, \Sigma\, U_1\, A(\lambda)\, W_1 + C^T R C\, W_1 = 0, \\ &U_1\, W_1 - I = 0, \end{aligned} \tag{6.4}$$

where $A(\lambda) = \lambda A + (1-\lambda)D$, and D is part of the parameter vector a in Theorem 3.2. The zero set $\rho_a^{-1}(0)$ of this homotopy map for the system (6.1) includes the subset

$$\{(\lambda, W_1, U_1, \Sigma) : \quad 0 \le \lambda < 1, \quad w_1 = -1.2w_2, \quad u_1 = -1.2u_2, \quad u_2 = \frac{1}{2.44w_2}, \quad \sigma = 0\}, \tag{6.5}$$

which is unbounded. This example shows that the zero set $\rho_a^{-1}(0)$ of a homotopy map can be unbounded and yet some zero curves may still converge to isolated solutions.

Note that, in practice, the algorithm in (Žigić *et al.*, 1993) always maintains $\mathrm{rank}(\Sigma) = n_m$. $n_m = 1$ in the above example. Solutions with $\Sigma = 0$ in the above example never come into play. Boundedness of $\rho_a^{-1}(0)$ for the optimal projection equations (5.1)–(5.3) can indeed be guaranteed with more sophisticated mathematics, a slightly different homotopy map from the one used in practice, and complex arithmetic for the curve tracking. This is pursued in Section 6.3.

6.2. Simplification and example for input normal form homotopy.

The following corollary is needed to simplify the homotopy map based on the input normal form formulation for the H^2 optimal model order reduction problem.

COROLLARY 6.1. *Let $\tilde{A}_I$, $\tilde{R}_I$, $\tilde{V}_I$ be defined as in Section 5.2, partitioned as in (5.21), let A_m be stable, and $\tilde{Q}_I$ satisfy (5.20). To minimize (5.19) under the constraints (5.17) and (5.20), the following two Lagrangians are equivalent:*

$$L_1(A_m, B_m, C_m, \Omega, \tilde{Q}_I, M_c, M_o, \tilde{P}_I) = \operatorname{tr}\Big[\tilde{Q}_I\tilde{R}_I + \big(A_m + A_m^T + B_m V B_m^T\big)M_c \\ + \big(A_m^T\Omega + \Omega A_m + C_m^T R C_m\big)M_o + \big(\tilde{A}_I\tilde{Q}_I + \tilde{Q}_I\tilde{A}_I^T + \tilde{V}_I\big)\tilde{P}_I\Big], \tag{6.6}$$

where the symmetric matrices M_o, M_c, and $\tilde{P}_I$ are Lagrange multipliers introduced in Section 5.2, and

$$L_2(A_m, B_m, C_m, \tilde{Q}_I, \tilde{P}_I) = \operatorname{tr}\Big[\tilde{Q}_I\tilde{R}_I + \big(\tilde{A}_I\tilde{Q}_I + \tilde{Q}_I\tilde{A}_I^T + \tilde{V}_I\big)\tilde{P}_I\Big], \tag{6.7}$$

where $\tilde{Q}_I$ is restricted to the form

$$\tilde{Q}_I = \begin{pmatrix} \bar{Q}_1 & \bar{Q}_{12} \\ \bar{Q}_{12}^T & I_{n_m} \end{pmatrix},$$

the Lagrange multiplier $\tilde{P}_I$ is restricted to the form

$$\tilde{P}_I = \begin{pmatrix} \bar{P}_1 & \bar{P}_{12} \\ \bar{P}_{12}^T & \Omega \end{pmatrix},$$

and $\Omega = diag(\omega_1, \cdots, \omega_{n_m})$ is a positive definite matrix.

Proof. The proof is straightforward. Setting $\partial L/\partial \tilde{Q}_I = 0$ gives the same equation

$$\tilde{A}_I^T\tilde{P}_I + \tilde{P}_I\tilde{A}_I + \tilde{R}_I = 0 \tag{6.8}$$

in both cases. Expanding (5.20) and (6.8) yields the equations for $\bar{Q}_2$ and $\bar{P}_2$. In the first case

$$A_m\bar{Q}_2 + \bar{Q}_2A_m^T + B_mVB_m^T = 0,$$

$$A_m^T\bar{P}_2 + \bar{P}_2A_m + C_m^TRC_m = 0.$$

Since the constraints (5.17) and (5.20) should be satisfied and A_m is stable, it follows that at a constrained minimum

$$\bar{Q}_2 = I_{n_m}, \qquad \bar{P}_2 = \Omega.$$

Q. E. D.

The partial derivatives $\dfrac{\partial L_2}{\partial B_m}$, and $\dfrac{\partial L_2}{\partial C_m}$ of L_2 can be computed as

$$\begin{aligned}\frac{\partial L_2}{\partial B_m} &= 2(\bar{P}_{12}^T B + \Omega B_m)V,\\ \frac{\partial L_2}{\partial C_m} &= 2R(C_m - C\bar{Q}_{12}).\end{aligned}$$

The corresponding homotopy map (5.30) and (5.31) is now simplified as

$$\rho(\lambda, x, a) = \begin{pmatrix} \text{Vec } (H_{B_m}(\lambda, x, a)) \\ \text{Vec } (H_{C_m}(\lambda, x, a)) \end{pmatrix},$$

where

$$\begin{aligned}H_{B_m}(\lambda, x, a) &= (\bar{P}_{12}^T B(\lambda) + \Omega B_m)V,\\ H_{C_m}(\lambda, x, a) &= R\big(C_m - C(\lambda)\bar{Q}_{12}\big).\end{aligned}$$

The zero set $\rho_a^{-1}(0)$ of a homotopy map based on the input normal form formulation given by (Ge *et al.*, 1994a) is not always bounded, as shown by the following 2-dimensional example.

The system is given by

$$A = \begin{pmatrix} -0.895116 & 0.612237 \\ 0.612237 & -0.447393 \end{pmatrix}, \quad B = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \quad C = (\,-2 \quad 1\,). \tag{6.9}$$

According to (Ge *et al.*, 1994a), the initial point and the triple $(A(\lambda), B(\lambda), C(\lambda))$ are chosen as follows:

(1) Transform the given triple (A, B, C) to balanced form (A_b, B_b, C_b), such that $A_b = T^{-1}AT$, $B_b = T^{-1}B$, and $C_b = CT$ satisfy

$$0 = A_b\Lambda + \Lambda A_b^T + B_b V B_b^T,$$

$$0 = A_b^T\Lambda + \Lambda A_b + C_b^T R C_b,$$

with a positive definite matrix $\Lambda = \text{diag}\,\big(d_1,\, d_2,\, \cdots,\, d_n\big),\; d_i \geq d_{i+1}$.

The balanced form of (6.9) is

$$A_b = \begin{pmatrix} -0.25297 & -0.5 \\ -0.5 & -1.0896 \end{pmatrix},\; B_b = \begin{pmatrix} -1.232 \\ -1.866 \end{pmatrix},\; C_b = \begin{pmatrix} -1.232 & -1.866 \end{pmatrix},$$

with

$$T = \begin{pmatrix} 0.866 & 0.5 \\ 0.5 & -0.866 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 3 & 0 \\ 0 & 1.5978 \end{pmatrix}.$$

(2) For $n_m = 1$, the parametrization $(A(\lambda), B(\lambda), C(\lambda))$ is chosen as

$$\begin{aligned}
A(\lambda) &= \lambda A + (1-\lambda)A_i \\
&= \begin{pmatrix} a_1(\lambda) & a_2(\lambda) \\ a_2(\lambda) & a_3(\lambda) \end{pmatrix} \\
&= \begin{pmatrix} -0.6422\lambda - 0.25297 & 0.612237\lambda \\ 0.612237\lambda & 0.6431\lambda - 1.0896 \end{pmatrix}, \\
B(\lambda) &= \lambda B + (1-\lambda)B_i \\
&= \begin{pmatrix} b_1(\lambda) \\ b_2(\lambda) \end{pmatrix} \\
&= \begin{pmatrix} -1.232 - 0.768\lambda \\ \lambda \end{pmatrix}, \\
C(\lambda) &= \lambda C + (1-\lambda)C_i \\
&= \begin{pmatrix} c_1(\lambda) & c_2(\lambda) \end{pmatrix} \\
&= \begin{pmatrix} -1.232 - 0.768\lambda & \lambda \end{pmatrix} \\
&= B^T(\lambda).
\end{aligned}$$

where

$$A_i = \begin{pmatrix} -0.25297 & 0 \\ 0 & -1.0896 \end{pmatrix}, \quad B_i = \begin{pmatrix} -1.232 \\ 0 \end{pmatrix}, \quad C_i = \begin{pmatrix} -1.232 & 0 \end{pmatrix}.$$

For brevity, $a_1(\lambda)$, $a_2(\lambda)$, $a_3(\lambda)$, $b_1(\lambda)$, $b_2(\lambda)$, $c_1(\lambda)$, and $c_2(\lambda)$ will be denoted by a_1, a_2, a_3, b_1, b_2, c_1, and c_2 respectively in the following. As discussed in Section 5.2, without loss of generality, set $V = I$ and $R = I$. For any $0 < \lambda < 1$, $B_m \in \mathbf{R}, B_m \neq 0$, let

$$
\begin{aligned}
A_m &= \frac{-B_m^2}{2}, \quad C_m = -\sqrt{\Omega} B_m, \\
\bar{P}_2 \equiv \Omega &= \left[\frac{M(b_1 - b_2)b_1}{a_1 + A_m - M a_2}\right]^2, \\
M &= \frac{a_2 b_1 - b_2(b_1 - A_m)}{b_1(a_3 + A_m) - b_2 a_2}, \\
(\bar{P}_{12})_{12} &= \frac{C_m(a_2 b_1 - a_1 b_2 - A_m b_2)}{a_2^2 - (a_1 + A_m)(a_3 + A_m)}, \\
(\bar{P}_{12})_{11} &= \frac{b_2 C_m - (\bar{P}_{12})_{12}(a_3 + A_m)}{a_2}, \\
(\bar{Q}_{12})_{11} &= \frac{(\bar{P}_{12})_{11}}{\sqrt{\Omega}}, \quad (\bar{Q}_{12})_{12} = \frac{(\bar{P}_{12})_{12}}{\sqrt{\Omega}}.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\rho(\lambda, x, a) = 0, \\
&\tilde{A}_I(\lambda)\tilde{Q}_I + \tilde{Q}_I \tilde{A}_I^T(\lambda) + \tilde{V}_I(\lambda) = 0, \\
&\tilde{A}_I^T(\lambda)\tilde{P}_I + \tilde{P}_I \tilde{A}_I(\lambda) + \tilde{R}_I(\lambda) = 0
\end{aligned}
$$

are satisfied. The zero set $\rho_a^{-1}(0)$ of this homotopy map includes

$$
\{(\lambda, B_m, C_m) : 0 < \lambda < 1, \, C_m = -\sqrt{\Omega} B_m\}. \tag{6.10}
$$

Clearly, (6.10) is unbounded. If $B_m \neq 0$, A_m is stable, (A_m, B_m) is controllable, and (A_m, C_m) is observable.

6.3. Homogeneous transformation to avoid solutions at infinity.

As shown by the examples in Sections 6.1 and 6.2, the polynomial systems (5.1)–(5.3) or (5.30)–(5.31) may have solutions at infinity, and $\rho_a^{-1}(0)$ contains paths that diverge to infinity as λ approaches 1. Solutions at infinity can be avoided via the following transformation (Morgan and Sommese, 1989), (Morgan and Sommese, 1987a), which will be used in Section 6.4.

Let $f(z)=0$ be a polynomial system of N equations in N unknowns, where $z \in \mathbf{C}^N$, and define $f'(z')$ as the homogenization of $f(z)$:

$$f'_j(z') = z_0^{d_j} f_j(z_1/z_0, \ldots, z_N/z_0), \quad j = 1, \ldots, N, \tag{6.11}$$

where $d_j = \deg(f_j)$. $f'(z') = 0$ is a system of N homogeneous equations in $N+1$ unknowns.

Note that, if $f'(z^0) = 0$, then $f'(cz^0) = 0$ for any complex scalar c. Therefore, we may take "solutions" of $f'(z') = 0$ to be (complex) lines through the origin in $\mathbf{C}^{N+1}$. The set of these lines is called complex projective space, denoted by $\mathbf{P}^N$, a smooth compact N-complex-dimensional manifold. It is natural to view $\mathbf{P}^N$ as a disjoint union of points $[(z_0, \ldots, z_N)]$ with $z_0 \neq 0$ and the "points at infinity", the points $[(z_0, \ldots, z_N)]$ with $z_0 = 0$. The solutions of $f'(z') = 0$ in $\mathbf{P}^N$ are identified with the solutions and solutions at infinity of $f(z) = 0$ as follows.

First, the solutions to $f(z) = 0$ can be identified with the solutions to $f'(z') = 0$ with $z_0 \neq 0$. Explicitly, if $L \in \mathbf{P}^N$ is a solution to $f'(z') = 0$, and $z' \in L$, with $z' = (z_0, \ldots, z_N)$ and $z_0 \neq 0$, then $z = (z_1/z_0, z_2/z_0, \ldots, z_N/z_0)$ is a solution to $f(z) = 0$. On the other hand, if $z \in \mathbf{C}^N$ is a solution to $f(z) = 0$, then the line through $z' = (1, z)$ is a solution to $f'(z') = 0$ with $z_0 = 1 \neq 0$. A "solution to $f(z) = 0$ at infinity" is simply a solution to $f'(z') = 0$ (in $\mathbf{P}^N$) generated by z' with $z_0 = 0$.

Define a homotopy map (in $\mathbf{P}^N$)

$$h(z', \lambda) = (1-\lambda)\gamma g'(z') + \lambda f'(z'), \tag{6.12}$$

where g' is a homogeneous system of N polynomials in $N+1$ variables, and γ is a randomly chosen complex number. Intuitively, let g' be chosen so that its homogeneous structure matches that of f'. Precisely, let $S \in \mathbf{P}^N$ be the set of common solutions of $f'(z') = 0$ and $g'(z') = 0$. Then for each $s \in S$ the following conditions must hold. For $s \in S$ let K denote the full connected component of solutions of $g'(z') = 0$ with $s \in K$.

If s is a geometrically isolated solution of $g'(z') = 0$, assume that: a) s is also a geometrically isolated solution of $f'(z') = 0$, and b) the multiplicity of s as a solution of $g'(z') = 0$ is less than or equal to the multiplicity of s as a solution of $f'(z') = 0$.

If s is not a geometrically isolated solution of $g'(z') = 0$, assume that: a) K is contained in S, b) K is the full solution component of $f'(z') = 0$ containing s, c) K is a smooth manifold, and d) at each point $z^0 \in K$ the rank of $\nabla g'(z^0)$ is the codimension of K.

Let S' denote the solution set of $g'(z') = 0$ in $\mathbf{P}^N - S$. Under these assumptions, the basic result is the following theorem.

THEOREM 6.2 (Morgan and Sommese, 1987b). *Assume the points in S' are nonsingular solutions of $g'(z') = 0$. For any positive r and for all but a finite number of angles θ, if $\gamma = re^{i\theta}$, then $h^{-1}(0) \cap ((\mathbf{P}^N - S) \times [0, 1))$ consists of smooth paths and every geometrically isolated solution of $f'(z') = 0$ not in S has a path in $(\mathbf{P}^N - S) \times [0, 1)$ converging to it.*

Let

$$L(z') = \sum_{i=0}^{N} b_i z_i,$$

where $b_i \neq 0$ for some i.

$$U_L = \{[z'] \in \mathbf{P}^N \mid L(z') \neq 0\}$$

is the Euclidean coordinate patch on $\mathbf{P}^N$ defined by L. Note that U_L, which is an open dense submanifold of $\mathbf{P}^N$, can be identified with $\mathbf{C}^N$ via

$$[(z_0, \ldots, z_N)] \to \frac{1}{L(z')}(z_0, \ldots, z_{i-1}, z_{i+1}, \ldots, z_N),$$

where $b_i \neq 0$.

The following theorem (Morgan and Sommese, 1987b) shows how to keep the homotopy process in complex Euclidean space, even though the basic theorem is formulated in $\mathbf{P}^N$.

THEOREM 6.3. *Assume the points in S' are nonsingular solutions of $g'(z') = 0$, then*

$$\overline{h^{-1}(0) \cap ((\mathbf{P}^N - S) \times [0, 1))} \subset U_L \times [0, 1],$$

for almost all U_L and all but a finite number of angles θ.

For computations, the coordinate patch U_L is realized via a projective transformation as follows. With homogeneous h in the variables z_i for $i = 0$ to N, let

$$z_0 = \sum_{i=1}^{N} \beta_i z_i + \beta_0, \tag{6.13}$$

where the β_i are constants and $\beta_i \neq 0$ for all i. The projective transformation of h is the system H of N polynomials in the N variables z_i for $i = 1$ to N where $H_j = h_j$, with (6.13) defining z_0 in terms of the other variables. By Theorem 6.3, the homotopy paths, including end points, are completely represented in $\mathbf{C}^N$ via H. The finite solutions of $f(x) = 0$ are recovered via $z_i \leftarrow z_i/z_0$ for $i = 1$ to N. If $z_0 = 0$, then the solution is at infinity. This concludes the background discussion of polynomial system theory.

6.4. Homogeneous transformation of optimal projection homotopies.

In this section the homogeneous transformation introduced in Section 6.3 is used to prevent unbounded zero sets for optimal projection homotopies. Consider the polynomial system given by (5.1)–(5.3) and the corresponding optimal projection homotopies defined in Section 5.1. The start system at $\lambda = 0$ is taken as

$$\begin{aligned}
&U_1 A(0) W_1 \Sigma W_1^T + \Sigma W_1^T A(0)^T + U_1 B(0) = 0,\\
&A(0)^T U_1^T \Sigma + U_1^T \Sigma U_1 A(0) W_1 + C(0) W_1 = 0,\\
&U_1 W_1 - I_{n_m} = 0,
\end{aligned} \tag{6.14}$$

where $A(0) = D = A - \epsilon I_n$, ϵ is a constant, $A(\lambda) = \lambda A + (1-\lambda) D$. The target system (at $\lambda = 1$) is (5.1)–(5.3).

According to Section 5.3, the homogenization of the target system (5.1)–(5.3) can be taken as

$$\begin{aligned}
&U_1' A W_1' \Sigma' {W_1'}^T + z_0^2 \Sigma' {W_1'}^T A^T + z_0^3 U_1' B V B^T = 0,\\
&z_0^2 A^T {U_1'}^T \Sigma' + {U_1'}^T \Sigma' U_1' A W_1' + z_0^3 C^T R C W_1' = 0,\\
&U_1' W_1' - z_0^2 I_{n_m} = 0,
\end{aligned} \tag{6.15}$$

where $z = (\mathrm{vec}(U_1), \mathrm{vec}(W_1), \mathrm{vec}(\Sigma))$,

$$\begin{aligned}
U_1'(z_0, \ldots, z_N) &= z_0 U_1(z_1/z_0, \ldots, z_N/z_0),\\
W_1'(z_0, \ldots, z_N) &= z_0 W_1(z_1/z_0, \ldots, z_N/z_0),\\
\Sigma'(z_0, \ldots, z_N) &= z_0 \Sigma(z_1/z_0, \ldots, z_N/z_0).
\end{aligned}$$

The corresponding homogenization of the start system is

$$
\begin{aligned}
&U_1' D W_1' \Sigma' {W_1'}^T + z_0^2 \Sigma' {W_1'}^T D^T + z_0^3 U_1' B_i = 0, \\
&z_0^2 D^T {U_1'}^T \Sigma' + {U_1'}^T \Sigma' U_1' D W_1' + z_0^3 C_i W_1' = 0, \\
&U_1' W_1' - z_0^2 I_{n_m} = 0, \qquad (6.16)
\end{aligned}
$$

where $B_i = B(0)$ and $C_i = C(0)$.

THEOREM 6.4. *If B_i, C_i, and ϵ can be chosen such that (6.16) and (6.15) have no common $z_0 \neq 0$, $\Sigma' \neq 0$ solutions, and all $z_0 \neq 0$, $\Sigma' \neq 0$ solutions of (6.16) are nonsingular, then every geometrically isolated solution of (6.15) has a path in $\mathbf{P}^N$ converging to it.*

Proof. If $\epsilon = 0$, (6.15) and (6.16) have the same $z_0 = 0$ solution set (corresponding to solutions of (5.1)–(5.3) at infinity). Since B_i and C_i can be chosen such that (6.15) and (6.16) have no common $z_0 \neq 0$, $\Sigma' \neq 0$ solutions and all $z_0 \neq 0$, $\Sigma' \neq 0$ solutions of (6.16) are nonsingular, then all the conditions of Theorems 6.1 and 6.2 hold. For each point in S', the associated path in $H^{-1}(0)$ can be tracked from $\lambda = 0$ to $\lambda = 1$. This will yield the full list of geometrically isolated solutions to $H(z,1) = 0$. No paths diverge to infinity.

If $\epsilon \neq 0$, $B(\lambda) = BVB^T$, and $C(\lambda) = C^T RC$ for $0 \leq \lambda \leq 1$ as in (Žigić *et al.*, 1993), using the fact $U_1' W_1' = 0$ (when $z_0 = 0$), it is clear that the $z_0 = 0$ solution set of (6.16) is the same as that of (6.15). Similarly, (6.15) and (6.16) have the same $z_0 \neq 0$ solutions when $\Sigma' = 0$. Note that this case corresponds to the counterexample of Section 6.1. Take S be all the $z_0 = 0$ solutions and any solutions corresponding to $z_0 \neq 0$ and $\Sigma' = 0$. Now ϵ can be chosen such that (6.15) and (6.16) have no other common solutions and all other $z_0 \neq 0$ solutions of (6.16) are nonsingular. Then the technical assumptions of Theorem 6.2 can clearly be met for the common solution set S. Thus Theorem 6.2 and Theorem 6.3 hold for the start system (6.16) in this case ($\epsilon \neq 0$) also. Q. E. D.

The import of all this is that the real solutions of (5.1)–(5.3), which satisfy the rank condition

$$\mathrm{rank}(W_1) = \mathrm{rank}(U_1) = \mathrm{rank}(\Sigma) = n_m,$$

if they exist, must be connected to the solutions of (6.16) in $\mathbf{P}^N - S$. Technically, this is guaranteed only with a complex multiplier γ in (6.16), and if complex arithmetic is used and the homotopy curve tracking is done in $\mathbf{P}^N$, but all this has never been necessary in practice. Furthermore, observe that the solution set (6.15) includes all solutions with rank $\Sigma' \leq n_m$, and thus one is guaranteed of finding a reduced order model of order *no greater than* n_m. Since (6.15) represents the optimal projection equations (5.1)–(5.3) for some stable $A(\lambda)$ for every λ, $0 \leq \lambda \leq 1$, it is clear why real arithmetic suffices generically. Generically, the real solutions are isolated, have constant rank, and vary smoothly with respect to λ.

Finally, for the target system (6.15), it is always possible to take the starting homogeneous system as

$$p_j z_j^4 - q_j z_0^4 = 0, \quad j = 1, \ldots, N, \tag{6.17}$$

where p_j and q_j are positive constants such that (6.17) has no common solution with (6.15). Since all solutions to (6.17) are nonsingular, all conditions of Theorem 6.2 and Theorem 6.3 are satisfied. The drawback is that the starting system (6.17) is totally unrelated to (6.15), requires complex arithmetic, and may take more steps to converge.

7. THE COMBINED H^2/H^∞ MODEL ORDER REDUCTION PROBLEM

This section introduces the model order reduction problem with a constrained H^∞ norm. Results of the deduction will be used in Section 8.

7.1. Statement of the problem.

The problem can be formulated as: given the controllable and observable, time invariant, continuous time system

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,Du(t), \\ y(t) &= C\,x(t), \end{aligned} \tag{7.1}$$

where $t \in [0, \infty)$, $A \in \mathbf{R}^{n\times n}$ is asymptotically stable, $B \in \mathbf{R}^{n\times m}$, $C \in \mathbf{R}^{l\times n}$, $D \in \mathbf{R}^{m\times p}$ $(m \leq p)$ and the input $Du(t)$ is white noise with symmetric and positive definite intensity $V \equiv DD^T$, determine a n_m-th order model

$$\begin{aligned} \dot{x}_m(t) &= A_m\,x_m(t) + B_m\,Du(t), \\ y_m(t) &= C_m\,x_m(t), \end{aligned} \tag{7.2}$$

where $A_m \in \mathbf{R}^{n_m\times n_m}$, $B_m \in \mathbf{R}^{n_m\times m}$, $C_m \in \mathbf{R}^{l\times n_m}$, and $n_m < n$, which satisfies the following criteria:

(1) A_m is asymptotically stable;

(2) the transfer function of the reduced order model lies within a radius-γ H^∞ neighborhood of the full order model, i.e.,

$$\|H(s) - H_m(s)\|_\infty \leq \gamma, \tag{7.3}$$

where

$$H(s) \equiv EC(sI_n - A)^{-1}BD, \qquad H_m(s) \equiv EC_m(sI_{n_m} - A_m)^{-1}B_mD,$$

$\gamma > 0$ is a given constant, $E \in \mathbf{R}^{q\times l}$ $(q \geq l)$ is a given constant matrix; and

(3) the H^2 model reduction criterion

$$J(A_m, B_m, C_m) \equiv \lim_{t\to\infty} \mathcal{E}\left[(y - y_m)^T R(y - y_m)\right] \tag{7.4}$$

is minimized, where $\mathcal{E}$ is the expected value and $R = E^T E$ is a symmetric and positive definite weighting matrix.

7.2. The auxiliary minimization problem .

Note that the full and reduced order systems (7.1)–(7.2) can be rewritten as a single augmented system (Haddad and Bernstein, 1989)

$$\begin{aligned} \dot{\tilde{x}}(t) &= \tilde{A}\,\tilde{x}(t) + \tilde{D}\,u(t), \\ \tilde{y}(t) &= \tilde{C}\,\tilde{x}(t), \end{aligned} \tag{7.5}$$

where $\tilde{D} \equiv \tilde{B}D$, and the $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, and $\tilde{n}$ have the same definitions as in Section 4.

The cost $J(A_m, B_m, C_m)$ satisfies

$$J(A_m, B_m, C_m) = \mathrm{tr}[\widetilde{Q}\widetilde{R}], \tag{7.6}$$

where the steady-state convariance $\tilde{Q}$ satisfies

$$\widetilde{A}\widetilde{Q} + \widetilde{Q}\widetilde{A}^T + \widetilde{B}V\widetilde{B}^T = 0. \tag{7.7}$$

The key step in enforcing (7.3) is to replace the algebraic Lyapunov equation (7.7) by an algebraic Riccati equation which is given by the following lemma.

LEMMA 7.1 (Haddad and Bernstein, 1989). *Let (A_m, B_m, C_m) be given and assume there exists $\mathcal{Q} \in \mathbf{R}^{\tilde{n}\times\tilde{n}}$ satisfying*

$$\mathcal{Q} \quad \text{is symmetric and nonnegative definite} \tag{7.8}$$

and

$$\tilde{A}\,\mathcal{Q} + \mathcal{Q}\tilde{A}^T + \gamma^{-2}\mathcal{Q}\tilde{R}\mathcal{Q} + \tilde{V} = 0. \tag{7.9}$$

Then

$$(\tilde{A}, \tilde{D}) \quad \text{is stabilizable} \tag{7.10}$$

if and only if

$$A_m \quad \text{is asymptotically stable.}$$

Furthermore, in this case

$$\|H(s) - H_m(s)\|_\infty \leq \gamma, \tag{7.11}$$

$$\tilde{Q} \leq \mathcal{Q},$$

and

$$\text{tr } \tilde{Q}\tilde{R} \equiv J(A_m, B_m, C_m) \leq \mathcal{J}(A_m, B_m, C_m) \equiv \text{ tr } \mathcal{Q}\tilde{R}.$$

Determining (A_m, B_m, C_m) that minimizes $\mathcal{J}(A_m, B_m, C_m)$ subject to (7.8)–(7.10) leads to (1) A_m stable; (2) a bound on the H_∞ distance between the full order and reduced order systems; and (3) an upper bound for the H^2 model-reduction criterion. Hence, a *necessary* condition for solving the auxiliary minimization problem is that (A_m, B_m, C_m) be restricted to the open set

$$\begin{aligned} \mathcal{S} \equiv \big\{(A_m, B_m, C_m, \mathcal{Q}) : \tilde{A} \text{ and } & \tilde{A} + \gamma^{-2}\mathcal{Q}\tilde{R} \text{ are asymptotically stable,} \\ & \mathcal{Q} \text{ are symmetric positive definite,} \\ & \text{and } (A_m, B_m, C_m) \text{ is controllable and observable } \big\}. \end{aligned}$$

7.3 Necessary and sufficient conditions for solving the problem .

THEOREM 7.2 (Haddad and Bernstein, 1989). *If* $(A_m, B_m, C_m) \in \mathcal{S}$ *solves the auxiliary minimization problem then there exist nonnegative definite* Q, P, $\hat{Q}$, $\hat{P} \in \mathbf{R}^{n\times n}$ *such that*

$$A_m = \Gamma(A - \gamma^{-4}Q\bar{\Sigma}QPS)G^T, \tag{7.12}$$

$$B_m = \Gamma B, \tag{7.13}$$

$$C_m = C(I_n + \gamma^{-2}QPS)G^T, \tag{7.14}$$

$$\mathcal{Q} = \begin{pmatrix} Q + \hat{Q} & \hat{Q}\Gamma^T \\ \Gamma\hat{Q} & \Gamma\hat{Q}\Gamma^T \end{pmatrix}, \tag{7.15}$$

and Q, P, $\hat{Q}$, $\hat{P}$ satisfy

$$
\begin{aligned}
0 &= AQ + QA^T + \gamma^{-2}Q\bar{\Sigma}Q + \tau_\perp \Sigma \tau_\perp^T, && (7.16)\\
0 &= A^TP + PA - \gamma^{-4}S^TPQ\bar{\Sigma}QPS \\
&\quad + \tau_\perp(I_n + \gamma^{-2}QPS)\bar{\Sigma}(I_n + \gamma^{-2}QPS)\tau_\perp, && (7.17)\\
0 &= (A - \gamma^{-4}Q\bar{\Sigma}QPS)\hat{Q} + \hat{Q}(A - \gamma^{-4}Q\bar{\Sigma}QPS)^T \\
&\quad + \gamma^{-6}\hat{Q}S^TPQ\bar{\Sigma}QPS\hat{Q} + \Sigma - \tau_\perp\Sigma\tau_\perp^T, && (7.18)\\
0 &= (A + \gamma^{-2}Q\bar{\Sigma})^T\hat{P} + \hat{P}(A + \gamma^{-2}Q\bar{\Sigma}) + (I_n + \gamma^{-2}QPS)^T\bar{\Sigma}(I_n + \gamma^{-2}QPS) \\
&\quad - \tau_\perp^T(I_n + \gamma^{-2}QPS)^T\bar{\Sigma}(I_n + \gamma^{-2}QPS)\tau_\perp, && (7.19)
\end{aligned}
$$

$$\operatorname{rank}(\hat{Q}) = \operatorname{rank}(\hat{P}) = \operatorname{rank}(\hat{Q}\hat{P}) = n_m, \quad (7.20)$$

where Σ, $\bar{\Sigma}$, S are defined as

$$\Sigma \equiv BVB^T, \quad (7.21)$$

$$\bar{\Sigma} \equiv C^TRC, \quad (7.22)$$

$$S \equiv (I_n + \gamma^{-2}\hat{Q}P)^{-1}, \quad (7.23)$$

and $\Gamma, G \in \mathbf{R}^{n_m \times n}$, $M \in \mathbf{R}^{n_m \times n_m}$ are the (G, M, Γ)-factorization matrices of $(\hat{Q}\hat{P})$:

$$\hat{Q}\hat{P} = G^TM\Gamma, \quad (7.24)$$

$$\Gamma G^T = I_{n_m}, \quad (7.25)$$

$$\tau \equiv G^T\Gamma, \quad (7.26)$$

$$\tau_\perp \equiv I_n - \tau. \quad (7.27)$$

Furthermore, the auxiliary cost is given by

$$\mathcal{J}(A_m, B_m, C_m) = \operatorname{tr}(\mathcal{Q}\tilde{R}) = \operatorname{tr}\bar{\Sigma}(Q + \gamma^{-4}QPS\hat{Q}S^TPQ). \quad (7.28)$$

Conversely, if there exist nonnegative definite Q, P, $\hat{Q}$, $\hat{P} \in R^{n \times n}$ satisfying (7.16)-(7.20), then $(A_m, B_m, C_m, \mathcal{Q})$ given by (7.12)-(7.15) satisfy (7.9) and (7.10) with the auxiliary cost given by (7.28).

8. TRANSVERSALITY OF HOMOTOPIES FOR COMBINED H^2/H^∞ OPTIMAL MODEL ORDER REDUCTION PROBLEM

This section proves that two homotopies $\rho(\lambda, x, a)$ which have been used in (Ge *et al.*, 1994b) for the H^2/H^∞ optimal model order reduction problem are transversal to zero, the first requirement of Theorem 3.2.

8.1. Transversality of H^2/H^∞ homotopies based on input normal form.

To optimize $\mathcal{J}(A_m, B_m, C_m)$ over the open set $\mathcal{S}$ defined in Section 7 under the constraint that symmetric positive definite $\mathcal{Q}$ satisfies (7.9), and (A_m, B_m, C_m) is in input normal form, the following Lagrangian is formed:

$$\mathcal{L}(A_m, B_m, C_m, \Omega, \mathcal{Q}, \mathcal{P}, M_c, M_o) \equiv \text{tr } \big[\mathcal{Q}\tilde{R} + (\tilde{A}\mathcal{Q} + \mathcal{Q}\tilde{A}^T + \gamma^{-2}\mathcal{Q}\tilde{R}\mathcal{Q} + \tilde{V})\mathcal{P} + \big(A_m + A_m^T + B_m V B_m^T\big)M_c + \big(A_m^T\Omega + \Omega A_m + C_m^T R C_m\big)M_o\big],$$

where the symmetric matrices M_c, M_o, and $\mathcal{P} \in \mathbf{R}^{\tilde{n}\times\tilde{n}}$ are Lagrange multipliers, $\Omega =$ diag $(\omega_1, \ldots, \omega_{n_m})$ is related to the input normal form constraint.

Setting $\partial\mathcal{L}/\partial\mathcal{Q} = 0$ yields

$$0 = \big(\tilde{A} + \gamma^{-2}\mathcal{Q}\tilde{R}\big)^T\mathcal{P} + \mathcal{P}\big(\tilde{A} + \gamma^{-2}\mathcal{Q}\tilde{R}\big) + \tilde{R}. \tag{8.1}$$

Partition $\mathcal{Q}$, $\mathcal{P} \in \mathbf{R}^{\tilde{n}\times\tilde{n}}$ into

$$\mathcal{Q} = \begin{pmatrix} \mathcal{Q}_1 & \mathcal{Q}_{12} \\ \mathcal{Q}_{12}^T & \mathcal{Q}_2 \end{pmatrix}, \qquad \mathcal{P} = \begin{pmatrix} \mathcal{P}_1 & \mathcal{P}_{12} \\ \mathcal{P}_{12}^T & \mathcal{P}_2 \end{pmatrix}, \tag{8.2}$$

where $\mathcal{Q}_1$, $\mathcal{P}_1 \in \mathbf{R}^{n\times n}$ and $\mathcal{Q}_2$, $\mathcal{P}_2 \in \mathbf{R}^{n_m\times n_m}$. Define

$$\mathcal{P}\mathcal{Q} \equiv Z = \begin{pmatrix} Z_1 & Z_{12} \\ Z_{21} & Z_2 \end{pmatrix}, \tag{8.3}$$

where

$$Z_1 \equiv \mathcal{P}_1\mathcal{Q}_1 + \mathcal{P}_{12}\mathcal{Q}_{12}^T, \qquad Z_{12} \equiv \mathcal{P}_1\mathcal{Q}_{12} + \mathcal{P}_{12}\mathcal{Q}_2,$$

$$Z_{21} \equiv \mathcal{P}_{12}^T\mathcal{Q}_1 + \mathcal{P}_2\mathcal{Q}_{12}^T, \qquad Z_2 \equiv \mathcal{P}_{12}^T\mathcal{Q}_{12} + \mathcal{P}_2\mathcal{Q}_2.$$

A straightforward calculation shows

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial B_m} &= 2(\mathcal{P}_{12}^T B + \mathcal{P}_2 B_m + M_c B_m)V, \\
\frac{\partial \mathcal{L}}{\partial C_m} &= 2R(C_m \mathcal{Q}_2 - C\mathcal{Q}_{12} + C_m M_o) \\
&\quad + \gamma^{-2} R\Big[-C(Z_1^T \mathcal{Q}_{12} + Z_{21}^T \mathcal{Q}_2 + \mathcal{Q}_1 Z_{12} + \mathcal{Q}_{12} Z_2) \\
&\qquad + C_m(\mathcal{Q}_{12}^T Z_{12} + Z_{12}^T \mathcal{Q}_{12} + \mathcal{Q}_2 Z_2 + Z_2^T \mathcal{Q}_2)\Big].
\end{aligned} \tag{8.4}$$

The matrices M_c and M_0 in (8.4) satisfy (5.25)–(5.27).

Let $B(\lambda)$ and $C(\lambda)$ be defined as

$$B(\lambda) = \lambda B + (1-\lambda)B_i,$$

$$C(\lambda) = \lambda C + (1-\lambda)C_i,$$

and let

$$H_{B_m}(\lambda, x, B_i) = \Big(\mathcal{P}_{12}^T B(\lambda) + \mathcal{P}_2 B_m + M_c B_m\Big)V, \tag{8.5}$$

$$\begin{aligned}
H_{C_m}(\lambda, x, C_i) &= R\Big(C_m \mathcal{Q}_2 - C(\lambda)\mathcal{Q}_{12} + C_m M_o\Big) \\
&\quad + \frac{1}{2}\gamma^{-2} R\Big[-C(Z_1^T \mathcal{Q}_{12} + Z_{21}^T \mathcal{Q}_2 + \mathcal{Q}_1 Z_{12} + \mathcal{Q}_{12} Z_2) \\
&\qquad + C_m(\mathcal{Q}_{12}^T Z_{12} + Z_{12}^T \mathcal{Q}_{12} + \mathcal{Q}_2 Z_2 + Z_2^T \mathcal{Q}_2)\Big].
\end{aligned} \tag{8.6}$$

Note the use of C instead of $C(\lambda)$ in the second term of H_{C_m}. The homotopy map $\rho(\lambda, x, a)$ is now defined by

$$F_1(\lambda, x, a) = \text{Vec}\ \Big[H_{B_m}(\lambda, x, B_i)\Big], \tag{8.7}$$

$$F_2(\lambda, x, a) = \text{Vec}\ \Big[H_{C_m}(\lambda, x, C_i)\Big], \tag{8.8}$$

where

$$x \equiv \begin{pmatrix} \text{Vec}\ (B_m) \\ \text{Vec}\ (C_m) \end{pmatrix}$$

denotes the independent variables B_m and C_m,

$$a \equiv \begin{pmatrix} \text{Vec}\ (B_i) \\ \text{Vec}\ (C_i) \end{pmatrix}$$

denotes the parameter variables B_i, C_i . As discussed in Section 5.2, without loss of generality set $V = I$ and $R = I$. The Jacobian matrix of $\rho(\lambda, x, a)$ is

$$D\rho(\lambda, x, a) = \begin{pmatrix} D_\lambda F_1 & D_x F_1 & D_{B_i} F_1 & 0 \\ D_\lambda F_2 & D_x F_2 & 0 & D_{C_i} F_2 \end{pmatrix}. \tag{8.9}$$

THEOREM 8.1. *The homotopy map given by (8.7)–(8.8) is transversal to zero (for $0 \le \lambda < 1$).*

Proof. A proof similar to that given in Section 5.2 yields

$$\operatorname{rank}(D_{B_i} F_1) = m n_m \quad \text{for} \quad \lambda \neq 1. \tag{8.10}$$

Ordering C_i and F_2 by rows gives

$$\begin{aligned} D_{C_i} F_2(\lambda, x, a) &= D_{C_i}(-C(\lambda)\mathcal{Q}_{12}) \\ &= (\lambda - 1) D_{C_i}(C_i \mathcal{Q}_{12}) \\ &= (\lambda - 1) \overbrace{\begin{pmatrix} \mathcal{Q}_{12}^T & 0 & \dots & 0 \\ 0 & \mathcal{Q}_{12}^T & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathcal{Q}_{12}^T \end{pmatrix}}^{l \text{ times}}. \end{aligned} \tag{8.11}$$

Again, a proof similar to that given in Section 5.2 yields

$$\operatorname{rank}(\mathcal{Q}_{12}) = n_m,$$

and

$$\operatorname{rank}(D_{C_i} F_2) = l n_m \quad \text{for} \quad \lambda \neq 1. \tag{8.12}$$

Finally, combining (8.10) with (8.12) yields the full rank property of $D\rho(\lambda, x, a)$.

Q. E. D.

8.2. Transversality of H^2/H^∞ homotopies based on Ly's formulation .

To optimize $\mathcal{J}(A_m, B_m, C_m)$ under the constraint that symmetric positive definite $\mathcal{Q}$ satisfies (7.9), and (A_m, B_m, C_m) is in Ly's form (refer to Section 5.3), the following Lagrangian is formed:

$$\mathcal{L}(A_m, B_m, C_m, \mathcal{P}, \mathcal{Q}) \equiv \text{tr } \left[\mathcal{Q}\tilde{R} + (\tilde{A}\mathcal{Q} + \mathcal{Q}\tilde{A}^T + \gamma^{-2}\mathcal{Q}\tilde{R}\mathcal{Q} + \tilde{V})\mathcal{P}\right],$$

where $\mathcal{P} \in \mathbf{R}^{\tilde{n}\times\tilde{n}}$ is a Lagrange multiplier. Setting $\partial\mathcal{L}/\partial\mathcal{Q} = 0$ yields (8.1). Partition $\mathcal{Q}$, $\mathcal{P} \in \mathbf{R}^{\tilde{n}\times\tilde{n}}$ as in (8.2) and define $\mathcal{P}\mathcal{Q} = Z$ as in (8.3). The partial derivatives of $\mathcal{L}$ can be computed as

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial(A_m)_{ij}} &= 2\big(\mathcal{P}_{12}^T\,\mathcal{Q}_{12} + \mathcal{P}_2\,\mathcal{Q}_2,\big)_{ij}, \qquad (i,j) \in \mathcal{S}\\
\frac{\partial\mathcal{L}}{\partial B_m} &= 2\big(\mathcal{P}_{12}^T B V + \mathcal{P}_2 B_m V\big),\\
\frac{\partial\mathcal{L}}{\partial(C_m)_{ij}} &= 2\big(RC_m\mathcal{Q}_2 - RC\mathcal{Q}_{12}\big)_{ij} +\\
&\quad + \gamma^{-2}\big[-RC(Z_1^T\mathcal{Q}_{12} + Z_{21}^T\mathcal{Q}_2 + \mathcal{Q}_1 Z_{12} + \mathcal{Q}_{12}Z_2)\\
&\qquad + RC_m(\mathcal{Q}_{12}^T Z_{12} + Z_{12}^T\mathcal{Q}_{12} + \mathcal{Q}_2 Z_2 + Z_2^T\mathcal{Q}_2)\big]_{ij}, \quad i > 1,
\end{aligned}$$

where $\mathcal{S}$ is the set of indices of the independent variables in A_m defined in Section 5.3. Define $B(\lambda)$ and $C(\lambda)$ as in Section 8.1 and let

$$\begin{aligned}
H_{A_m}(\lambda, x) &= \mathcal{P}_{12}^T\,\mathcal{Q}_{12} + \mathcal{P}_2\,\mathcal{Q}_2,\\
H_{B_m}(\lambda, x, B_i) &= \mathcal{P}_{12}^T B(\lambda)V + \mathcal{P}_2 B_m V,\\
H_{C_m}(\lambda, x, C_i) &= R\Big(C_m\mathcal{Q}_2 - C(\lambda)\mathcal{Q}_{12}\Big)\\
&\quad + \frac{1}{2}\gamma^{-2}R\Big[-C(Z_1^T\mathcal{Q}_{12} + Z_{21}^T\mathcal{Q}_2 + \mathcal{Q}_1 Z_{12} + \mathcal{Q}_{12}Z_2)\\
&\qquad + C_m(\mathcal{Q}_{12}^T Z_{12} + Z_{12}^T\mathcal{Q}_{12} + \mathcal{Q}_2 Z_2 + Z_2^T\mathcal{Q}_2)\Big],
\end{aligned}$$

where again C instead of $C(\lambda)$ is used in the second term of H_{c_m}. The homotopy map $\rho(\lambda, x, a)$ for Ly's formulation is now defined as

$$F_1(\lambda, x, \eta_0) = \lambda\Big[H_{A_m}(\lambda, x)\Big]_{\mathcal{S}} + (1-\lambda)(\eta - \eta_0), \tag{8.13}$$

where $x = (\eta, \zeta)$ denotes a partitioning of the independent variables and $\eta_0 \in \mathbf{R}^{n_m}$ denotes the independent parameter variables as in Section 5.3, and

$$F_2(\lambda, x, B_i) = \text{Vec}\left[H_{B_m}(\lambda, x, B_i)\right], \tag{8.14}$$

$$F_3(\lambda, x, C_i) = \text{Vec}\left[H_{C_m}(\lambda, x, C_i)\right]_{\mathcal{T}\cdot}, \tag{8.15}$$

where B_i, and C_i denote the parameter variables defined in Section 8.1, and $\mathcal{T}\cdot$ picks out rows $2, \cdots, l$ from the matrix H_{C_m}. The motivation for (8.13) is similar to that given in Section 5.3.

THEOREM 8.2. *The homotopy map $\rho(\lambda, x, a)$ given by (8.13)–(8.15) is transversal to zero (for $0 \le \lambda < 1$).*

Proof. The Jacobian matrix of $\rho(\lambda, x, a)$ is

$$D(\rho(\lambda, x, a)) = \begin{pmatrix} D_\lambda F_1 & D_x F_1 & D_{\eta_0 F_1} & 0 & 0 \\ D_\lambda F_2 & D_x F_2 & 0 & D_{B_i} F_2 & 0 \\ D_\lambda F_3 & D_x F_3 & 0 & 0 & D_{(C_i)_{\mathcal{T}\cdot}} F_3 \end{pmatrix}.$$

A similar proof to that in Sections 5.3 and 8.1 yields

$$\operatorname{rank}(D_{\eta_0} F_1) = n_m \quad \text{for} \quad \lambda \ne 1,$$

$$\operatorname{rank}(D_{B_i} F_2) = m n_m \quad \text{for} \quad \lambda \ne 1,$$

$$\operatorname{rank}(D_{(C_i)_{\mathcal{T}\cdot}} F_3) = (l-1) n_m \quad \text{for} \quad \lambda \ne 1,$$

which proves the full rank property of $D(\rho(\lambda, x, a))$ for all $0 \le \lambda < 1$. Q. E. D.

Note that the homotopy construction (8.13) is again a theoretical convenience, while in practice (Ge *et al.*, 1994b)

$$F_1(\lambda, x) = \left[H_{A_m}(\lambda, x)\right]_{\mathcal{S}}$$

has been used and $D_x F_1(\lambda, x)$ retains full rank (since $l + m \gg 1$ usually).

9. H^∞–CONSTRAINED LQG CONTROL PROBLEM .

This section introduces the LQG controller synthesis problem with an H^∞ performance bound. Propositions and results given in this section will be employed in Section 10.

9.1. Statement of the problem.

The problem can be stated as: given the n-th order stabilizable and detectable plant

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t) + D_1 w(t), \\ y(t) &= C\,x(t) + D_2 w(t), \end{aligned} \tag{9.1}$$

where $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$, $C \in \mathbf{R}^{l\times n}$, $D_1 \in \mathbf{R}^{n\times p}$, $D_2 \in \mathbf{R}^{l\times p}$, $D_1 D_2^T = 0$, and $w(t)$ is p-dimensional white noise, find a n_c-th order dynamic compensator

$$\begin{aligned} \dot{x}_c(t) &= A_c\,x_c(t) + B_c\,y(t), \\ u(t) &= C_c\,x_c(t), \end{aligned} \tag{9.2}$$

where $A_c \in \mathbf{R}^{n_c\times n_c}$, $B_c \in \mathbf{R}^{n_c\times l}$, $C_c \in \mathbf{R}^{m\times n_c}$, and $n_c \le n$, which satisfies the following design criteria:

(1) $\tilde{A} = \begin{pmatrix} A & B\,C_c \\ B_c C & A_c \end{pmatrix}$ is asymptotically stable;

(2) the $q_\infty \times p$ closed-loop transfer function $H(s) \equiv \tilde{E}_\infty (sI_{\tilde{n}} - \tilde{A})^{-1}\tilde{D}$ satisfies the constraint

$$\|H(s)\|_\infty \le \gamma, \tag{9.3}$$

where $\gamma > 0$ is a given constant, $\tilde{n} = n + n_c$, $\tilde{D} = \begin{pmatrix} D_1 \\ B_c D_2 \end{pmatrix}$, and $\tilde{E}_\infty = \begin{pmatrix} E_{1\infty} & E_{2\infty} C_c \end{pmatrix}$ with $E_{1\infty} \in \mathbf{R}^{q_\infty \times n}$, $E_{2\infty} \in \mathbf{R}^{q_\infty \times m}$, and $E_{1\infty}^T E_{2\infty} = 0$;

(3) the performance functional

$$J(A_c, B_c, C_c) \equiv \lim_{t\to\infty} \mathcal{E}\left[x^T(t) R_1 x(t) + u^T(t) R_2 u(t)\right]$$

is minimized, where $\mathcal{E}$ is the expected value, $R_1 = E_1^T E_1 \in \mathbf{R}^{n\times n}$ and $R_2 = E_2^T E_2 \in \mathbf{R}^{m\times m}$ ($E_1 \in \mathbf{R}^{q\times n}$, $E_2 \in \mathbf{R}^{q\times m}$, $E_1^T E_2 = 0$) are respectively symmetric nonnegative definite and symmetric positive definite weighting matrices.

Note that the problem statement involves both H^2 and H^∞ performance weights, the matrices $R_{1\infty} \equiv E_{1\infty}^T E_{1\infty}$ and $R_{2\infty} \equiv E_{2\infty}^T E_{2\infty}$ are the H^∞ counterparts of the H^2 weights

R_1 and R_2. We assume in the following sections that $R_{2\infty} = \beta R_2$, where the nonnegative scalar β is a design variable.

9.2. The auxiliary minimization problem .

Again, the closed-loop system (9.1)–(9.2) can be written as

$$\dot{\tilde{x}}(t) = \tilde{A}\,\tilde{x}(t) + \tilde{D}\,w(t).$$

Using the notation above and under the condition that $\tilde{A}$ is asymptotically stable, for a given compensator (A_c, B_c, C_c), the cost function satisfies

$$J(A_c, B_c, C_c) = \text{tr}\ (\tilde{Q}\,\tilde{R}),$$

where $\tilde{R} = \begin{pmatrix} R_1 & 0 \\ 0 & C_c^T R_2 C_c \end{pmatrix}$ and $\tilde{Q}$ satisfies the Lyapunov equation

$$\tilde{A}\,\tilde{Q} + \tilde{Q}\,\tilde{A}^T + \tilde{V} = 0, \tag{9.4}$$

with

$$\tilde{V} = \begin{pmatrix} V_1 & 0 \\ 0 & B_c V_2 B_c^T \end{pmatrix},$$

where $V_1 = D_1 D_1^T$ is symmetric nonnegative definite and $V_2 = D_2 D_2^T$ is symmetric positive definite.

The key step in enforcing (9.3) is to replace the algebraic Lyapunov equation (9.4) by an algebraic Riccati equation which is given by the following lemma.

LEMMA 9.1 (Bernstein and Haddad, 1989). *Let (A_c, B_c, C_c) be given and assume there exists $\mathcal{Q} \in \mathbf{R}^{\tilde{n}\times\tilde{n}}$ satisfying*

$$\mathcal{Q} \quad \text{is symmetric and positive definite} \tag{9.5}$$

and

$$\tilde{A}\mathcal{Q} + \mathcal{Q}\tilde{A}^T + \gamma^{-2}\mathcal{Q}\tilde{R}_\infty\mathcal{Q} + \tilde{V} = 0, \tag{9.6}$$

where $\tilde{R}_\infty = \begin{pmatrix} R_{1\infty} & 0 \\ 0 & C_c^T R_{2\infty} C_c \end{pmatrix}$, $R_{1\infty} = E_{1\infty}^T E_{1\infty}$, and $R_{2\infty} = E_{2\infty}^T E_{2\infty}$ are symmetric nonnegative definite matrices. Then

$$(\tilde{A}, \tilde{D}) \quad \text{is stabilizable}$$

if and only if

$$\tilde{A} \quad \text{is asymptotically stable.}$$

In this case

$$\|H(s)\|_\infty \le \gamma,$$

$$\tilde{Q} \le \mathcal{Q},$$

and

$$\text{tr } \tilde{Q}\tilde{R} \equiv J(A_c, B_c, C_c) \le \mathcal{J}(A_c, B_c, C_c) \equiv \text{ tr } \mathcal{Q}\tilde{R}.$$

9.3. Necessary conditions for solving the problem .

According to Lemma 9.1, we restrict $(A_c, B_c, C_c, \mathcal{Q})$ to the open set

$$\begin{aligned}\Upsilon \equiv \big\{(A_c, B_c, C_c, \mathcal{Q}) : \tilde{A} \text{ and } \tilde{A} + \gamma^{-2}\mathcal{Q}\tilde{R} \text{ are asymptotically stable,}\\ \mathcal{Q} \text{ is symmetric positive definite,}\\ \text{and } (A_c, B_c, C_c) \text{ is controllable and observable } \big\}.\end{aligned}$$

The following theorem presents the necessary conditions for optimality in the auxiliary minimization problem.

Theorem 9.2 (Bernstein and Haddad, 1989). *Suppose there exist nonnegative definite* Q, P, $\hat{Q}$, $\hat{P} \in \mathbf{R}^{n\times n}$ *satisfying*

$$\begin{aligned}
&0 = AQ + QA^T + V_1 + \gamma^{-2}QR_{1\infty}Q - Q\bar{\Sigma}Q + \tau_\perp Q\bar{\Sigma}Q\tau_\perp^T,\\
&0 = (A + \gamma^{-2}[Q+\hat{Q}]R_{1\infty})^T P + P(A + \gamma^{-2}[Q+\hat{Q}]R_{1\infty})\\
&\quad + R_1 - S^T P\Sigma PS + \tau_\perp S^T P\Sigma PS\tau_\perp,\\
&0 = (A - \Sigma PS + \gamma^{-2}QR_{1\infty})\hat{Q} + \hat{Q}(A - \Sigma PS + \gamma^{-2}QR_{1\infty})^T\\
&\quad + \gamma^{-2}\hat{Q}(R_{1\infty} + \beta^2 S^T P\Sigma PS)\hat{Q} + Q\bar{\Sigma}Q - \tau_\perp Q\bar{\Sigma}Q\tau_\perp^T, \qquad (9.6)\\
&0 = (A - Q\bar{\Sigma} + \gamma^{-2}QR_{1\infty})^T\hat{P} + \hat{P}(A - Q\bar{\Sigma} + \gamma^{-2}QR_{1\infty})\\
&\quad + S^T P\Sigma PS - \tau_\perp S^T P\Sigma PS\tau_\perp,\\
&\text{rank}(\hat{Q}) = \text{rank}(\hat{P}) = \text{rank}(\hat{Q}\hat{P}) = n_c,\\
&\tau = G^T\Gamma, \quad \tau_\perp = I - \tau,
\end{aligned}$$

and let A_c, B_c, C_c, and $\mathcal{Q}$ be given by

$$A_c = \Gamma(A - Q\bar{\Sigma} - \Sigma PS + \gamma^{-2}QR_{1\infty})G^T,$$

$$B_c = \Gamma QC^T V_2^{-1},$$

$$C_c = -R_2^{-1}B^T PSG^T,$$

$$\mathcal{Q} = \begin{pmatrix} Q + \hat{Q} & \hat{Q}\Gamma^T \\ \Gamma\hat{Q} & \Gamma\hat{Q}\Gamma^T \end{pmatrix},$$

where G and Γ are from the (G, M, Γ)–factorization of $\hat{Q}\hat{P}$, and Σ, $\bar{\Sigma}$, S are defined as

$$\Sigma \equiv BR_2^{-1}B^T,$$

$$\bar{\Sigma} \equiv C^T V_2^{-1}C,$$

$$S \equiv (I_n + \beta^2\gamma^{-2}\hat{Q}P)^{-1}.$$

Then, $(\tilde{A}, \tilde{D})$ is stabilizable if and only if $\tilde{A}$ is asymptotically stable. In this case the closed-loop transfer function $H(s)$ satisfies the H^∞ approximation constraint

$$\|H(s)\|_\infty \leq \gamma,$$

and the H^2 performance criterion satisfies the bound

$$J(A_c, B_c, C_c) \leq \text{ tr } [(Q + \hat{Q})R_1 + \hat{Q}S^T P\Sigma PS]. \tag{9.7}$$

10. THE PROPERTIES OF HOMOTOPIES FOR FULL ORDER LQG SYNTHESIS WITH AN H^∞ PERFORMANCE BOUND

Let R_1, R_2, $R_{1\infty}$, $R_{2\infty}$, Σ, $\bar{\Sigma}$, and S be defined the same as in Section 9 and $\rho(M)$ be the spectral radius of a matrix M.

When $n_c = n$, it is possible to achieve a simplification. Specifically, consider the case in which the H^2 and H^∞ weights are equalized, i.e.

$$R_{1\infty} = R_1, \quad \beta = 1. \tag{10.1}$$

The following lemma gives a simplified version of Theorem 9.2 for this case.

LEMMA 10.1 (Bernstein and Haddad, 1989). *Assume (10.1) is satisfied and suppose there exist nonnegative definite $Q \in \mathbf{R}^{n\times n}$ and positive definite $Y_\infty \in \mathbf{R}^{n\times n}$ satisfying*

$$0 = AQ + QA^T + Q(\gamma^{-2}R_{1\infty} - \bar{\Sigma})Q + V_1, \tag{10.2}$$

$$0 = A^TY_\infty + Y_\infty A + Y_\infty(\gamma^{-2}V_1 - \Sigma)Y_\infty + R_{1\infty}, \tag{10.3}$$

$$\rho(QY_\infty) < \gamma^2$$

and such that

$$A + (\gamma^{-2}V_1 - \Sigma)Y_\infty \quad \text{is asymptotically stable}$$

and

$$\left(A + Y_\infty^{-1}R_{1\infty}, \quad \gamma^{-1}[R_{1\infty} + (Y_\infty^{-1} - \gamma^{-2}Q)^{-1}\Sigma(Y_\infty^{-1} - \gamma^{-2}Q)]^{1/2}\right) \quad \text{is observable.}$$

Furthermore, let (A_c, B_c, C_c) be given by

$$A_c = A - Q\bar{\Sigma} - \Sigma(Y_\infty^{-1} - \gamma^{-2}Q)^{-1} + \gamma^{-2}QR_{1\infty},$$

$$B_c = QC^TV_2^{-1},$$

$$C_c = -R_{2\infty}^{-1}B^T(Y_\infty^{-1} - \gamma^{-2}Q)^{-1}.$$

Then $(\tilde{A}, \tilde{D})$ is stabilizable if and only if $\tilde{A}$ is asymptotically stable. In this case the closed-loop transfer function $H(s)$ satisfies the H^∞ disturbance attenuation constraint

$$\|H(s)\|_\infty \leq \gamma$$

and the H^2 performance criterion satisfies the bound

$$J(A_c, B_c, C_c) \leq \text{ tr } [QR_{1\infty} + Q\bar{\Sigma}Q(Y_\infty^{-1} - \gamma^{-2}Q)^{-1}].$$

Similarly as in Section 5.1, let

$$A(\lambda) = A,$$

$$\Sigma(\lambda) = B(\lambda)R_2^{-1}B^T(\lambda) = \lambda\Sigma + (1-\lambda)\Sigma_i,$$

$$\bar{\Sigma}(\lambda) = C(\lambda)^T V_2^{-1} C(\lambda) = \lambda\bar{\Sigma} + (1-\lambda)\bar{\Sigma}_i,$$

where Σ_i, $\bar{\Sigma}_i \in \mathbf{R}^{n\times n}$ correspond to the parameter a in Theorem 3.2. The homotopy map based on Riccati equations is now defined as

$$\rho(\lambda, x, a) = \begin{pmatrix} \text{Vec } (H_1(\lambda, x, a)) \\ \text{Vec } (H_2(\lambda, x, a)) \end{pmatrix},$$

where

$$H_1(\lambda, x, a) \equiv AQ + QA^T + Q(\gamma^{-2}R_{1\infty} - \bar{\Sigma}(\lambda))Q + V_1, \tag{10.4}$$

$$H_2(\lambda, x, a) \equiv A^T Y_\infty + Y_\infty A + Y_\infty(\gamma^{-2}V_1 - \Sigma(\lambda))Y_\infty + R_{1\infty} \tag{10.5}$$

and

$$x \equiv \begin{pmatrix} \text{Vec } (Q) \\ \text{Vec } (Y_\infty) \end{pmatrix}.$$

THEOREM 10.2. *If* $\left(A, (\gamma^{-2}V_1 - \Sigma(\lambda))^{1/2}\right)$ *and* $\left(A, (\gamma^{-2}R_{1\infty} - \bar{\Sigma}(\lambda))^{1/2}\right)$ *are controllable,* $(A, R_{1\infty}^{1/2})$ *and* $(A, V_1^{1/2})$ *are observable, then The homotopy map given by (10.4)–(10.11) is transversal to zero (for* $0 \leq \lambda < 1$*) and the zero set* $\rho_a^{-1}(0)$ *of (10.4) – (10.11) is bounded.*

Proof. Note that since the Q equation (10.4) is decoupled from the Y_∞ equation (10.5), they can be analyzed separately. Equations (10.4) $= 0$ and (10.5) $= 0$ are standard algebraic Riccati equations, where $R_{1\infty}$, V_1, $\bar{\Sigma}(\lambda)$, and $\Sigma(\lambda)$ are symmetric. Under the assumptions that $\left(A, (\gamma^{-2}V_1 - \Sigma(\lambda))^{1/2}\right)$ is controllable and $(A, R_{1\infty}^{1/2})$ is observable, it follows that, for given λ, a, and γ, (10.5) $= 0$ possesses exactly one positive definite solution Y_∞ such that $\left(A + (\gamma^{-2}V_1 - \Sigma(\lambda))Y_\infty\right)$ is stable (Kailath, 1980) (Bittanti *et al.*, 1991). Similarly, if $\left(A, (\gamma^{-2}R_{1\infty} - \bar{\Sigma}(\lambda))^{1/2}\right)$ is controllable and $(A, V_1^{1/2})$ is observable, (10.4) $= 0$ has exactly one positive definite solution Q such that $\left(A + (\gamma^{-2}R_{1\infty} - \bar{\Sigma}(\lambda))Q\right)$ is stable. Q. E. D.

So, under suitable conditions (described above), the zero set of ρ_a consists of exactly one smooth curve that does not intersect itself and has endpoints only at $\lambda = 0$ or $\lambda = 1$. The requirements of Theorem 3.2 are satisfied automatically. To find the solutions of (10.2)–(10.3), one needs only to follow the zero curve emanating from $\lambda = 0$ and check those conditions at each step until a zero Q, Y_∞ of (10.2)–(10.3) is reached (at $\lambda = 1$).

11. CONCLUSIONS

Probability-one homotopy methods were considered for the problem of H^2 model reduction, H^2/H^∞ model reduction, and full order LQG synthesis. The crucial requirement of transversality was verified for several homotopy maps for model reduction including the pseudogramian formulation of the optimal projection equations as well as variations based upon canonical forms. These results guarantee good numerical properties in the computational implementation of probability-one homotopy algorithms. Counterexamples to the boundedness requirement of probability-one homotopy theory were provided for the pseudogramian formulation of the optimal projection equations and for some formulations based upon canonical forms. Since a solution may not exist in any particular canonical form, these results are sharp for canonical forms, where unboundedness corresponds to nonexistence of solutions. However, for a reformulation of the pseudogramian optimal projection equations in complex projective space using homogeneous transformations, the boundedness assumption holds and thus global convergence of the homotopy algorithm to a solution (in complex projective space) is guaranteed. Considerable computational experience (Žigić *et al.*, 1993) indicates that real-valued homotopies are effective in practice and thus it is not necessary to track the homotopy zero curves in complex projective space. The boundedness assumption for full order LQG synthesis is verified without reformulation.

12. FUTURE WORK

Furure work includes: (1) boundedness of $\rho_a^{-1}(0)$ for H^2 input normal form homotopies, (2) boundedness of $\rho_a^{-1}(0)$ for H^2/H^∞ homotopies, (3) transversality and boundedness for reduced order LQG synthesis.

The following suggests an approach for proving boundedness of $\rho_a^{-1}(0)$ for the H^2 input normal form homotopy. As shown by the counterexample in Section 6.2, the zero set of a given homotopy map based on H^2 input normal form is not always bounded. $\rho_a^{-1}(0)$ may contain paths that diverge to infinity as λ approaches 1.

According to the discussion in Section 6.2, the homotopy map is simplified as

$$\begin{aligned}(\bar{P}_{12}^T B(\lambda) + \Omega B_m)V &= 0,\\ R(C_m - C(\lambda)\bar{Q}_{12}) &= 0,\end{aligned} \tag{12.1}$$

where $\bar{P}_{12}$ and $\bar{Q}_{12}$ satisfy

$$\begin{aligned}A(\lambda)\bar{Q}_{12} + \bar{Q}_{12}A_m^T + B(\lambda)VB_m^T &= 0,\\ A^T(\lambda)\bar{P}_{12} + \bar{P}_{12}A_m - C^T(\lambda)RC_m &= 0,\end{aligned} \tag{12.2}$$

$\Omega = \mathrm{diag}\big(\omega_1, \cdots, \omega_{n_m}\big)$ and A_m satisfies (4.18).The target system is defined by (12.1), (12.2), and (4.18) with $\lambda = 1$ and the corresponding start system is (12.1), (12.2) and (4.18) with $\lambda = 0$.

According to Lemma 2.4, if $\lambda_i, i = 1, ..., n_m$, are the eigenvalues of A_m and $\mu_j, j = 1, ..., n$, are the eigenvalues of $A(\lambda)$, then (12.2) has a unique solution $\bar{Q}_{12}$ and $\bar{P}_{12}$ if and only if $\lambda_i + \mu_j \neq 0$ for all i and j. Note that since A_m and $A(\lambda)$ are stable for both the start and target systems, it is clear that (12.2) has a unique solution for both cases. Furthermore, the equation (12.2) is equivalent to a system of $2nn_m$ scalar equations in the elements of $\bar{Q}_{12}$ and $\bar{P}_{12}$:

$$\sum_{j=1}^{n} A(\lambda)_{ij}(\bar{Q}_{12})_{jk} + \sum_{j'=1}^{n_m} (\bar{Q}_{12})_{ij'}(A_m^T)_{j'k} = -\Big(B(\lambda)VB_m^T\Big)_{ik},$$
$$\sum_{j=1}^{n} A^T(\lambda)_{ij}(\bar{P}_{12})_{jk} + \sum_{j'=1}^{n_m} (\bar{P}_{12})_{ij'}(A_m)_{j'k} = \Big(C^T(\lambda)RC_m\Big)_{ik},$$

for $1 \leq i \leq n$ and $1 \leq k \leq n_m$. It is clear that the solutions $\bar{Q}_{12}$ and $\bar{P}_{12}$ are rational functions of $(B_m)_{ij}$ and $(C_m)_{ki}$, for $1 \leq i \leq n_m$, $1 \leq j \leq m$, and $1 \leq k \leq l$.

Now, let $z = (z_1, \cdots, z_N) = \Big(\mathrm{Vec}(B_m), \mathrm{Vec}(C_m)\Big)$, $N = n_m(l+m)$, $z' = (z_0, z)$, and

$$
\begin{aligned}
B'_m(z_0, \ldots, z_N) &= z_0 B_m(z_1/z_0, \ldots, z_N/z_0),\\
C'_m(z_0, \ldots, z_N) &= z_0 C_m(z_1/z_0, \ldots, z_N/z_0).
\end{aligned}
$$

Let $M(\lambda, z_1, \ldots, z_N)$ be the lowest common denominator of $(\bar{Q}_{12})_{ij}$ and $(\bar{P}_{12})_{ij}$, for $1 \leq i \leq n$ and $1 \leq j \leq n_m$, $\deg(M) = d^M$, then define

$$
\begin{aligned}
(\bar{P}'_{12})_{ij} &= z_0^{d_1^j}\Big(M(\lambda, z_1/z_0, \ldots, z_N/z_0)\bar{P}_{12}(z_1/z_0, \ldots, z_N/z_0)\Big)_{ij},\\
(\bar{Q}'_{12})_{ij} &= z_0^{d_2^j}\Big(M(\lambda, z_1/z_0, \ldots, z_N/z_0)\bar{Q}_{12}(z_1/z_0, \ldots, z_N/z_0)\Big)_{ij},\\
M' &= z_0^{d^M} M(\lambda, z_1/z_0, \ldots, z_N/z_0),
\end{aligned}
$$

where

$$
\begin{aligned}
d_1^j &= \max\{d_1^{1j}, \cdots, d_1^{nj}\},\\
d_2^j &= \max\{d_2^{1j}, \cdots, d_2^{nj}\},\\
d_1^{ij} &= \deg\Big(M(\lambda, z)(\bar{P}_{12})_{ij}\Big),\\
d_2^{ij} &= \deg\Big(M(\lambda, z)(\bar{Q}_{12})_{ij}\Big).
\end{aligned}
$$

Now, consider the homogenization of (12.1) for both start and target systems. Let

$$
\begin{aligned}
\Omega_1 &= \mathrm{diag}\Big((B_m V B_m^T)_{11}, \cdots, (B_m V B_m^T)_{n_m n_m}\Big),\\
\Omega_2 &= \mathrm{diag}\Big((C_m^T R C_m)_{11}, \cdots, (C_m^T R C_m)_{n_m n_m}\Big),\\
\Omega'_1 &= \mathrm{diag}\Big((B'_m V {B'_m}^T)_{11}, \cdots, (B'_m V {B'_m}^T)_{n_m n_m}\Big),\\
\Omega'_2 &= \mathrm{diag}\Big(({C'_m}^T R C'_m)_{11}, \cdots, ({C'_m}^T R C'_m)_{n_m n_m}\Big).
\end{aligned}
$$

Let the first equation of (12.1) be multipled by $M(\lambda, z)\Omega_1$ and the second by $M(\lambda, z)$, then (12.1) becomes a system of polynomial equations

$$
\begin{aligned}
\Big(\Omega_1(M\bar{P}_{12}^T)B(\lambda) + M\Omega_2 B_m\Big)V &= 0,\\
R\Big(MC_m - C(\lambda)(M\bar{Q}_{12})\Big) &= 0,
\end{aligned}
\tag{12.3}
$$

The homogenization of (12.3) is

$$\begin{aligned} \Big(\Omega_1'(\bar{P}_{12}')^T B(\lambda) + Z_0(d_1^M)M'\Omega_2' B_m'\Big)V &= 0, \\ R\Big(M'C_m' Z_0(d_2^M) - C(\lambda)\bar{Q}_{12}'\Big) &= 0, \end{aligned} \tag{12.4}$$

where

$$\begin{aligned} Z_0(d_1^M) &= \mathrm{diag}(z_0^{d_1^1 - d^M - 1}, \cdots, z_0^{d_1^{nm} - d^M - 1}), \\ Z_0(d_2^M) &= \mathrm{diag}(z_0^{d_2^1 - d^M - 1}, \cdots, z_0^{d_2^{nm} - d^M - 1}). \end{aligned}$$

The homogenizations of the target and start systems are defined by (12.4) with $\lambda = 1$ and $\lambda = 0$ respectively.

(12.4) is a system of $n_m(l+m)$ homogeneous polynomial scalar equations. For $z_0 = 0$, in each scalar equation, only those terms which have the highest degree of $z = (z_1, \cdots, z_N)$ are left. So future studies include:

- Find a starting point $\Big(A(0), B(0), C(0)\Big)$ such that the start system has its homogeneous structure matching that of the target system and all its $z_0 = 0$ solutions are $z_0 = 0$ solutions to the target system (corresponding to solutions to (12.1) at infinity).

To guarantee that the curve tracking can recover all geometrically isolated solutions to the target system, except those ruled out above, the starting point $\Big(A(0), B(0), C(0)\Big)$ should satisfy

- all $z_0 \neq 0$ solutions to the start system are nonsingular, and
- if any start system solution is a solution to the target system too, it must be geometrically isolated solution to the target system.

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