Appendix B:

Ratio Constancy Proof for F-Optimal Designs under the Uniform Prior

Ratio Constancy of F-Optimal Designs under the Uniform Prior

The F-optimal ratio constancy proof is very much like the D-optimal case. When $\beta_1 \sim U(c,d)$, the Bayes risk is

$$\min_{\delta \in \mathcal{D}} \int R(\delta,\beta)\pi(\beta)d\beta = \min\left[\frac{1}{(d-c)(x_1-x_2)^2}\left(\frac{\left(e^{-cx_2}-e^{-dx_2}\right)}{n_2 x_2}+\frac{\left(e^{-cx_2}-e^{-dx_2}\right)}{n_1 x_1}\right)\right]. \tag{1}$$

When $\beta_1 \sim U(ac,ad)$, the Bayes risk is

$$\min_{\delta \in \mathcal{D}} \int R(\delta,\beta)\pi(\beta)d\beta = \min\left[\frac{1}{(ad-ac)(x_1^*-x_1^*)^2}\left(\frac{\left(e^{-cx_1^*}-e^{-dx_1^*}\right)}{n_2 x_1^*}+\frac{\left(e^{-cx_1^*}-e^{-dx_1^*}\right)}{n_1 x_1^*}\right)\right]. \tag{2}$$

Let x_1 and x_2 be the solution that minimizes expression (1) when $\beta_1 \sim U(c,d)$. Let x_1^* and x_2^* be the solution that minimizes expression (2) when $\beta_1 \sim U(ac,ad)$. Multiplying the right hand side of (2) by $\frac{a^2}{a^2}$ gives

$$\min_{\delta \in \mathcal{D}} \int R(\delta,\beta)\pi(\beta)d\beta = \min\left[\frac{1}{(d-c)(ax_1^*-ax_2^*)^2}\left(\frac{\left(e^{-cx_2^*}-e^{-dx_2^*}\right)}{n_2 ax_2^*}+\frac{\left(e^{-cx_1^*}-e^{-dx_1^*}\right)}{n_1 ax_1^*}\right)\right]. \tag{3}$$

By grouping $\left(ax_i^*\right)$, it is obvious that $x_i^* = \frac{x_i}{a}$ and these designs are ratio constant both in the selection of EC's. The following steps verify that equality of ECs holds among F-optimal Bayesian designs based on the uniform prior.

$$q_i = e^{\left(\frac{c+d}{2}\right)x_i} = e^{a\left(\frac{c+d}{2}\right)x_i^*} = e^{\left(\frac{c+d}{2}\right)a\left(\frac{x_i}{a}\right)} = e^{\left(\frac{c+d}{2}\right)x_i} = q_i^* \tag{4}$$

Note that while it is not shown here, it can be verified using calculus techniques that the allocation percentages remain the same for designs based on $\beta_1 \sim U(c,d)$ and $\beta_1 \sim U(ac,ad)$.