Chapter 9

MISCELLANEOUS RESPONSE SURFACE TOPICS: PREDICTION VARIANCE AND FRACTIONAL FACTORIALS

§9.1 Introduction

Designs are often selected for use based on the stability and values of prediction variance in various regions of the design space. Fractional factorials are considered the backbone of RSM in that they are used for screening designs and in designs for second order models. This chapter focuses on the development of these two concepts for the exponential model. It is shown that the prediction variance for the exponential model is a function of the individual effective concentrations of the design points. Contour plots of prediction variance are shown for the D-optimal and D_s-optimal designs for the two regressor no interaction model. A method for obtaining D-optimal fractional factorials is detailed with examples illustrating the technique. In addition, a general form for the alias structure of these designs is developed and supplemented by examples of the alias structure for several designs.

§9.2 Prediction Variance

The variance of the predicted response is a characteristic of designs that is studied in traditional response surface methodology. By comparing the prediction variance of different designs one can select the designs which predicts best in his or her region of interest or is most stable within that region. In order to compare the prediction variance of designs for the Poisson exponential model, the functional form must be found first. The general formula for the prediction variance of the mean response at x_0 in a GLM model with canonical link is

$$\mathbf{x}_0' v(\mathbf{x}_0)(\mathbf{XVX})^{-1} v(\mathbf{x}_0)\mathbf{x}_0 \tag{9.2.1}$$

where $v(\mathbf{x}_0)$ represents the variance at $\mathbf{x}_0$, $\mathbf{X}$ is the model matrix, and $\mathbf{V}$ is an $n \times n$ diagonal matrix of the variance of the responses for each observation. In the Poisson case, $v(\mathbf{x}_0) = e^{\mathbf{x}_0'\boldsymbol{\beta}} = \lambda_0$ and $\mathbf{V} = \mathrm{diag}\{\lambda_i\} = \mathrm{diag}\{e^{\mathbf{x}_0'\boldsymbol{\beta}}\}$. Thus in order to compare design based on prediction variance, it appears that it would be necessary to know the parameters. Fortunately, this is not the case. Expression (9.2.1) can be reformulated so that it is independent of the parameters for the Poisson exponential case. Recall $x_{mi} = \dfrac{\ln q_{mi}}{\beta_i}$ for any m=1,…,k and $\lambda_i = q_{1i}q_{2i}\cdots q_{ki}\lambda_c$. Define $\mathbf{X} = \mathbf{QB}^{-1}$ where

$$\mathbf{Q} = \begin{bmatrix} 1 & \ln q_{11} & \ln q_{12} & \cdots & \ln q_{1k} \\ 1 & \ln q_{21} & \ln q_{22} & \cdots & \ln q_{2k} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & \ln q_{n1} & \ln q_{n2} & \cdots & \ln q_{nk} \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \mathrm{diag}(1, \beta_1, \beta_2, \ldots, \beta_k).$$

Also re-express $\mathbf{V} = \lambda_c \mathbf{V_Q} = \lambda_c \mathrm{diag}(q_{11}q_{21}q_{31}\cdots q_{k1}, \ldots, q_{1n}q_{2n}q_{3n}\cdots q_{kn})$ and $\mathbf{x_0'} = \mathbf{q}_0'\mathbf{B}^{-1}$ where $\mathbf{q_0'} = [1 \;\ln q_{10} \;\cdots\; \ln q_{k0}]$. So,

$$\mathbf{x}_0'\mathrm{v}(\mathbf{x}_0)(\mathbf{XVX})^{-1}\mathrm{v}(\mathbf{x}_0)\mathbf{x}_0 = \mathbf{q_0'B^{-1}}(q_{10}q_{20}\cdots q_{k0})^2\lambda_c^2\left(\mathbf{B^{-1}Q}\lambda_\mathbf{c}\mathbf{V_Q QB^{-1}}\right)^{-1}\mathbf{B^{-1}q}_0 \tag{9.2.2}$$

$$= (q_{10}q_{20}\cdots q_{k0})^2\lambda_c\mathbf{q_0'}\left(\mathbf{BB^{-1}QV_Q QB^{-1}B}\right)^{-1}\mathbf{q}_0$$

$$= \lambda_c(q_{10}q_{20}\cdots q_{k0})^2\lambda_c\mathbf{q_0'}\left(\mathbf{QV_Q Q}\right)^{-1}\mathbf{q}_0$$

Thus, the final form of the prediction variance is a function of the number of organisms produced at the control and the individual ECs.

A new expression will now be defined as

$$\frac{\mathrm{var}(\hat{\lambda})}{\lambda_c} = (q_{10}q_{20}\cdots q_{k0})^2\mathbf{q_0'}\left(\mathbf{QV_Q Q}\right)^{-1}\mathbf{q}_0 \tag{9.2.3}$$

Expression (9.2.3) will be referred to as the scaled prediction variance or prediction variance apart from λ_c. Note that this is the portion of the prediction variance attributed to the design and thus under the control of the experimenter. This expression can be used to determine the best design to use in order to predict most efficiently in a particular region. To demonstrate how this is used, graphs of the contours of (9.2.3) are shown for the D and D_s-optimal design for the two variable no interaction model. Recall the D-optimal design places one-third of the design points at the control and the two extremes of the model EC contour of 13.53. The D_s design places 16.2% at the control and 41.9% at the extremes of the contour of the model $EC_{9.2}$.

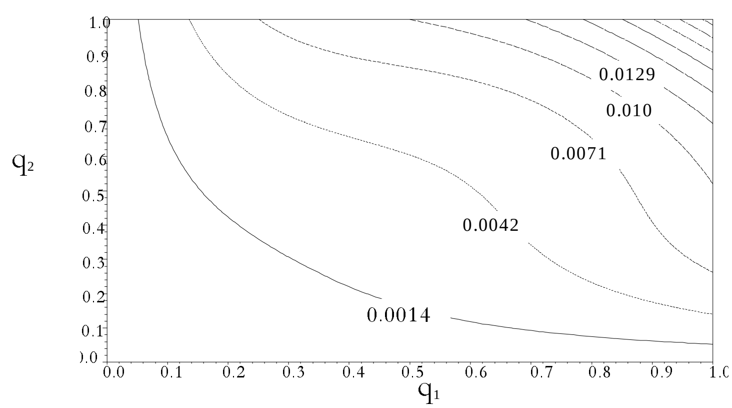

Figure 9.2.1 Scaled Prediction Variance for the D-Optimal Two Regressor No Interaction Design.

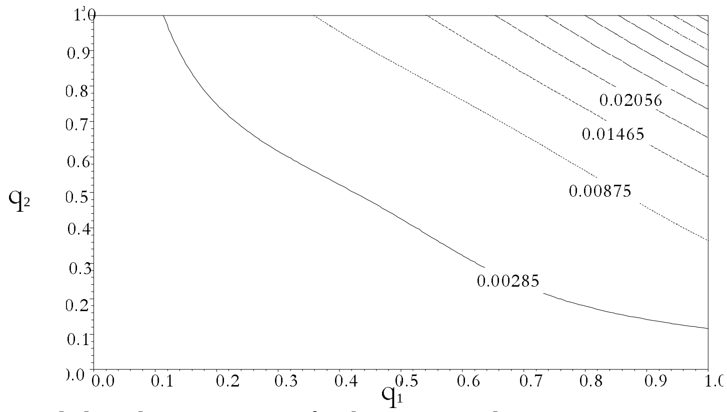

Figure 9.2.2 Scaled Prediction Variance for the D_s-Optimal Two Regressor No Interaction Design.

One may want to note that the prediction variance for the D_s-optimal design is nearly twice that of the D-design in the extremes of the region. However, it is nearly equal in the center. In both cases, the prediction variance increases as the mean increases which is what one would expect in a Poisson process. One should note that prediction variances were computed in the no interaction case. In the interaction case the form of the prediction variance would remain the same under the convenient guess that the interaction parameters are zero. As one can see from equation 9.2.2, a positive interaction would produce an increase in prediction variance whereas a negative interaction would produce a decrease in prediction variance. In conclusion, this form of the prediction variance facilitates future research with respect to prediction variance-based design optimality criteria such as G and Q optimality.

§9.3 Fractional Factorials for the Exponential Model

By its very nature, response surface methodology allows the researcher to obtain a great deal of information about a process while minimizing the cost of the experiment via minimizing the number of experimental units required. The type of experimental design that plays an instrumental role in the efficiency of response surface designs is the fractional factorial. In fact, fractional factorials are among the most widely used type of designs in industry (Myers and Montgomery, 1995). According to Myers and Montgomery, the successful use of fractional factorials is based on three key ideas:

1. **The Sparsity of Effect Principle.** When there are several variables, the system or process is likely to be driven primarily by some of the main effects and low-order interactions.

2. **The Projection Property.** Fractional factorial designs can be projected into stronger (larger) designs in the subset of significant factors.

3. **Sequential Experimentation.** It is possible to combine the runs of two (or more) fractional factorials to assemble sequentially a larger design to estimate the factor effects and interactions of interest.

Obviously, fractional factorial designs with these same properties would be extremely useful in impaired reproduction studies both in the biomedical field where experimental subjects are often limited as well as in the industrial world where experiments are very costly.

In order to highlight the features of fractional factorials for IRS models, a review of traditional fractional factorials is in order. Consider an experiment with three factors, A,B, and C, each at two levels. In order to estimate all interactions in the model, the experimenter would need eight design points, or experimental runs. However, the experimenter cannot afford to execute an experiment this large so he or she desires to reduce the number of experimental runs to four. This indicates that the experimenter wishes to use a ½ fraction of the 2^3 design or a 2^{3-1}. Recall that the defining relation for such a design is given by

$$I=ABC.$$

The design is then given by the following design points.

Table 9.3.1 One-Half Fraction of a 2^3 for a Traditional Linear Model with Defining Relation I=ABC.

Treatment	**Factorial Effect**							
Combination	I	A	B	C	AB	AC	BC	ABC
a	+	+	-	-	-	-	+	+
b	+	-	+	-	-	+	-	+
c	+	-	-	+	+	-	-	+
abc	+	+	+	+	+	+	+	+

From Table 9.3.1, one can see that columns AB through ABC are a mirror image of columns I-C. This observation indicates the alias structure of the design. Columns with identical elements indicate confounded effects. Thus, in this design, main effects are confounded with two way interactions and the mean is confounded with the three way interaction.

At this time several comments are in order to address how these types of designs satisfy the three properties of fractional factorials from Myers and Montgomery. In the previous example, a main effects model could be fit. If this model is sufficient, no further experimentation is necessary. Thus, no “cost” has been wasted estimating unnecessary effects. This illustrates the **sparsity-of-effect principle**. To demonstrate the **projection property**, assume that factor C is not significant in the model and is thus dropped from the model. The design in Table 9.3.1 then becomes a full factorial in factors A and B, fully capable of estimating AB interaction. (Note that any of the factors, A, B, or C, could have been dropped and the resulting design would be a full factorial in the remaining two.) Finally, the principle of **sequential experimentation** is addressed. If the main effects model is found to be insufficient in its fit or prediction capabilities, the second half of the full model design corresponding to I=-ABC can be used to augment the existing design in order to estimate more model terms.

This review of fractional factorials for the standard linear model leaves one wondering whether the same principles would apply to designs for the exponential model. After all, designs for the exponential model are composed of two levels, $IEC_{13.53}$ and the IEC_{100}, for each variable. To explore this possibility, consider a design for the three regressor model with interaction. In fact, this design can be coded just as designs for traditional linear models by considering the IEC_{100} as “-“ or low (no toxicant added) and the $IEC_{13.53}$ as “+” or high. With this in mind, consider the coded design for the exponential model in Table 9.3.2.

Table 9.3.2 Coded D-Optimal Design for the Full Three Regressor Exponential Model.

Treatment Combination	**Factorial Effect**							
	I	A	B	C	AB	AC	BC	ABC
a	-	+	-	-	-	-	+	+
b	-	-	+	-	-	+	-	+
c	-	-	-	+	+	-	-	+
abc	-	+	+	+	+	+	+	+
ab	-	+	+	-	+	-	-	-
ac	-	+	-	+	-	+	-	-
bc	-	-	+	+	-	-	+	-
(1)	-	-	-	-	+	+	+	-

Using the same defining relation as in the linear case, I=ABC, the first four rows correspond to the design points that would be used for the fractional factorial. However, this is not the D-optimal design for a main effects model. It was shown in Section 2 of Chapter 7 that points a, b, c and (1) comprise the D-optimal design for a main effects model. It would appear that the defining relations and alias structures associated with the traditional linear model do not apply to the exponential model. In fact, the D-optimal fractional factorial design (main effect design) is listed in Table 9.3.3 in terms of its coded units.

Table 9.3.3 Coded D-Optimal Design for the Three Regressor Main Effect Model.

Treatment Combination	**Factorial Effect**							
	I	A	B	C	AB	AC	BC	ABC
a	-	+	-	-	-	-	+	+
b	-	-	+	-	-	+	-	+
c	-	-	-	+	+	-	-	+
(1)	-	-	-	-	+	+	+	-

A glance at this table provides very little insight into the alias structure of fractional factorials for the exponential model. In fact, one must keep in mind that any study of the alias structure must account for the non-linearity of the model. The “orthogonality” or “non-orthogonality” of the columns in Table 9.3.3 do not carry the same meaning as they do in the linear model situation. The alias structure for these designs will be addressed in Section 9.6.

§9.4 General Form of D-Optimal Fractional Factorials in the Exponential Model

D-Optimal fractional factorials for impaired reproduction models can be found very easily by augmenting the main effect design with interaction points corresponding to the interactions the researcher wishes to estimate. Main effect points and interaction points must now be defined. Consider the vector $\mathbf{q_i}$ which is the vector of IECs corresponding to a particular design point $\mathbf{x_i}$**.** Main effect points are defined in terms of $\mathbf{q_i}$ as those in which the IEC for main effect of interest is set at $IEC_{13.53}$. The remaining IECs in a main effect vector are set at IEC_{100}. For example, a main effect point for X_1 in a four regressor setting would be $\mathbf{q_i}$=(0.1353,1,1,1) and a main effect point for X_2 in this same setting would be $\mathbf{q_i}$=(1,0.1353,1,1). These are analogous to the pure component points discussed in the two variable no interaction case. The vector $\mathbf{q_i}$ that corresponds to interaction points places the IECs of those regressors involved in the interaction at the $IEC_{13.53}$ and the remaining IECs at the IEC_{100}. For example, a main effect point estimating the X_1X_2 interaction is $\mathbf{q_i}$=(0.1353,0.1353,1,1) By adding the points associated with the interactions of interest to the main effect design these interactions become estimable. In this way custom fractional factorials can be built for any model defined by the researcher. Now, examples of custom fractional factorials will be shown for specific models.

§9.5 Examples of Fractional Factorial Designs

Consider an experiment with three regressors where the researcher has prior information to indicate that the model is given by (9.5.1).

$$y_{ij} = \exp\left(\beta_0 + \beta_1 x_{1i} + \beta_1 x_{2i} + \beta_1 x_{3i} + \beta_{12} x_{1i} x_{2i} + \beta_{13} x_{1i} x_{3i}\right) + \varepsilon_{ij} \tag{9.5.1}$$

Thus the optimal design would be given by Table 9.5.1

Table 9.5.1 D-Optimal Fractional Factorial Design for Model (9.5.1).

i	p_i	x_{1i}	x_{2i}	x_{3i}	***Plane***
1	$0.1\overline{6}$	IEC_{100}	IEC_{100}	IEC_{100}	MEC_{100}
2	$0.1\overline{6}$	$IEC_{13.53}$	IEC_{100}	IEC_{100}	$MEC_{13.53}$
3	$0.1\overline{6}$	IEC_{100}	$IEC_{13.53}$	IEC_{100}	$MEC_{13.53}$
4	$0.1\overline{6}$	IEC_{100}	IEC_{100}	$IEC_{13.53}$	$MEC_{13.53}$
5	$0.1\overline{6}$	$IEC_{13.53}$	$IEC_{13.53}$	IEC_{100}	$MEC_{0.0183}$
6	$0.1\overline{6}$	$IEC_{13.53}$	IEC_{100}	$IEC_{13.53}$	$MEC_{0.0183}$

This is a six point equal allocation design where point 1 estimates the control, points 2-4 estimate the main effects, and points 5 and 6 are interaction points. Equivalence theory was used to verify that this design is optimal. An evaluation over a fine grid of design points indicates that the equivalence theory function equals 6, the number of parameters, at the design points and is less than or equal to 6 at all points in the design space.

A design for a four regressor interaction model consisting of main effects and two way interactions is shown in Table 9.5.2.

Table 9.5.2 D-Optimal Fractional Factorial Design for the Four Regressor Exponential Model containing Main Effects and Two-Way Interactions.

i	p_i	x_{1i}	x_{2i}	x_{3i}	x_{4i}	***Plane***
1	$0.\overline{09}$	IEC_{100}	IEC_{100}	IEC_{100}	IEC_{100}	MEC_{100}
2	$0.\overline{09}$	$IEC_{13.53}$	IEC_{100}	IEC_{100}	IEC_{100}	$MEC_{13.53}$
3	$0.\overline{09}$	IEC_{100}	$IEC_{13.53}$	IEC_{100}	IEC_{100}	$MEC_{13.53}$
4	$0.\overline{09}$	IEC_{100}	IEC_{100}	$IEC_{13.53}$	IEC_{100}	$MEC_{13.53}$
5	$0.\overline{09}$	IEC_{100}	IEC_{100}	IEC_{100}	$IEC_{13.53}$	$MEC_{0.0183}$
6	$0.\overline{09}$	$IEC_{13.53}$	$IEC_{13.53}$	IEC_{100}	IEC_{100}	$MEC_{0.0183}$
7	$0.\overline{09}$	$IEC_{13.53}$	IEC_{100}	$IEC_{13.53}$	IEC_{100}	$MEC_{0.0183}$
8	$0.\overline{09}$	$IEC_{13.53}$	IEC_{100}	IEC_{100}	$IEC_{13.53}$	$MEC_{0.0183}$
9	$0.\overline{09}$	IEC_{100}	$IEC_{13.53}$	$IEC_{13.53}$	IEC_{100}	$MEC_{0.0183}$
10	$0.\overline{09}$	IEC_{100}	$IEC_{13.53}$	IEC_{100}	$IEC_{13.53}$	$MEC_{0.0183}$
11	$0.\overline{09}$	IEC_{100}	IEC_{100}	$IEC_{13.53}$	$IEC_{13.53}$	$MEC_{0.0183}$

Equivalence theory was used to verify the optimality of this design. Based on the examples generated in developing fractional factorials, there is strong evidence to indicate that all fractional factorials of the form described in Section 9.4 are D-optimal for their respective models.

§9.6 Alias Structure for Fractional Factorials for the Exponential Model

Prior to the design of an experiment a model is chosen consisting of a set of main effects and interactions. The purpose of using a fractional factorial is to reduce the necessary design points and thus reduce the number of estimable interactions. A successful fractional factorial leaves model terms unbiased but these same terms are confounded with higher order terms that one assumes are negligible. The purpose of this section is to explore the alias structure for the k-regressor Poisson exponential model. Prior to exploring this alias structure, which involves the GLM estimator for $\boldsymbol{\beta}$, one must be aware of the form of this estimator and the role of non-linearity in the estimation process.

Since GLM models have non-homogeneity of variance and GLM variance-covariance matrices are functions of model parameters, traditional least squares methods are not viable in this situation. Instead, a Taylor series coupled with the Gauss-Newton iterative is used in the estimation procedure (McCullagh and Nelder, 1989). First, express the exponential model as

$$y_i = e^{x_i'\boldsymbol{\beta}} + \boldsymbol{\varepsilon}_i = \mu_i + \boldsymbol{\varepsilon}_i . \tag{9.6.1}$$

Define $\boldsymbol{\beta}_0$ as the starting value for $\boldsymbol{\beta}$ and let $\mu_0 = e^{\mathbf{x}_i'\boldsymbol{\beta}_0}$. The response, y_i, can then be expressed as a first order Taylor series expanded around $\boldsymbol{\beta}_0$. Let

$$\begin{aligned} y_i &\approx \mu_0 + \left(\frac{\partial\mu}{\partial\beta_0}\right)_{\boldsymbol{\beta}_0}\left(\beta_0 - \beta_{0,0}\right) + \left(\frac{\partial\mu}{\partial\beta_1}\right)_{\boldsymbol{\beta}_0}\left(\beta_1 - \beta_{1,0}\right) + \ldots + \left(\frac{\partial\mu}{\partial\beta_k}\right)_{\boldsymbol{\beta}_0}\left(\beta_k - \beta_{k,0}\right) + e_i \\ &\approx \mu_0 + \mathbf{d}'\gamma + e_i \end{aligned} \tag{9.6.2}$$

where $\gamma = (\beta - \beta_0)$, $\mathbf{d}'$ is a vector of first derivatives, and e_i represents the error term. In matrix notation,

$$\mathbf{y} - \mu_0 \approx \mathbf{D}\gamma + \mathbf{e} \tag{9.6.3}$$

where $\mathbf{D}' = [\mathbf{d_1} \ \ \mathbf{d_2} \ \ldots \ \mathbf{d_n}]$ and $\mathbf{d_i} = \left[\frac{\partial\mu}{\partial\beta_0} \ \frac{\partial\mu}{\partial\beta_1} \ \cdots \ \frac{\partial\mu}{\partial\beta_k}\right]_{\beta_0}$. Using the Gauss-Newton procedure, the MLE is given by (McCullagh and Nelder, 1989)

$$\mathbf{b} = \beta_\mathbf{0} + \left(\mathbf{D}'\mathbf{V}^{-1}\mathbf{D}\right)^{-1}\mathbf{D}'\mathbf{V}^{-1}\left(\mathbf{y} - \mu_\mathbf{0}\right) \tag{9.6.4}$$

Assuming the model is correct, $\mathbf{b}$ is an asymptotically unbiased estimate of β as shown below. Note $\mathrm{E}(\mathbf{y} - \mu_\mathbf{0}) = \mu_\mathbf{0} + \mathbf{D}(\beta - \beta_0) - \mu_\mathbf{0} = \mathbf{D}(\beta - \beta_0)$. Thus,

$$\begin{aligned} \mathrm{E}(\mathbf{b}) &= \beta_0 + \left(\mathbf{D}'\mathbf{V}^{-1}\mathbf{D}\right)^{-1}\left(\mathbf{D}'\mathbf{V}^{-1}\right)\mathrm{E}\left(\mathbf{y} - \mu_0\right) \\ &= \beta_0 + \left(\mathbf{D}'\mathbf{V}^{-1}\mathbf{D}\right)^{-1}\left(\mathbf{D}'\mathbf{V}^{-1}\mathbf{D}\right)\left(\beta - \beta_0\right) \\ &= \beta. \end{aligned} \tag{9.6.5}$$

With this in mind, the goal is to explore the alias structure of the exponential model. Assume that the model which was fit contains p terms (k regressors and the intercept). However, the true model contains an additional r regressors as shown in (9.6.6) below.

$$\mathbf{y} = \exp\left\{\mathbf{X}_\mathrm{p}\beta_\mathrm{p} + \mathbf{X}_\mathrm{r}\beta_\mathrm{r}\right\} + \boldsymbol{\varepsilon} \tag{9.6.6}$$

The additional r regressors add corresponding terms to the Taylor series in (9.6.2). Thus, for the true model

$$\mathbf{y} - \mu_0 \approx \mathbf{D}_\mathrm{p}\gamma_\mathrm{p} + \mathbf{D}_\mathrm{r}\gamma_\mathrm{r} + \mathbf{e} \tag{9.6.7}$$

The starting value for $\boldsymbol{\beta}_r$ in the Taylor series is $\mathbf{0}$ since it was not included in the assumed model. This implies that $\boldsymbol{\gamma}_r = (\boldsymbol{\beta}_r - \mathbf{0})$. Now, the expected value of **b** must be found under the true model (9.6.6).

$$\begin{aligned} \mathrm{E}(\mathbf{b}) &\approx \boldsymbol{\beta}_0 + \left(\mathbf{D}_p'\mathbf{V}_p^{-1}\mathbf{D}_p\right)^{-1}\left(\mathbf{D}_p'\mathbf{V}_p^{-1}\right)\left(\mathbf{D}_p\boldsymbol{\gamma}_p + \mathbf{D}_r\boldsymbol{\gamma}_r\right) \\ &\approx \boldsymbol{\beta}_p + \left(\mathbf{D}_p'\mathbf{V}_p^{-1}\mathbf{D}_p\right)^{-1}\left(\mathbf{D}_p'\mathbf{V}_p^{-1}\mathbf{D}_r\right)\boldsymbol{\beta}_r \\ &\approx \boldsymbol{\beta}_p + \mathbf{A}\boldsymbol{\beta}_r \end{aligned} \tag{9.6.8}$$

In (9.6.8), $\mathbf{A} = \left(\mathbf{D}_p'\mathbf{V}_p^{-1}\mathbf{D}_p\right)^{-1}\left(\mathbf{D}_p'\mathbf{V}_p^{-1}\mathbf{D}_r\right)$ and is referred to as the alias matrix. Since the Poisson exponential model makes use of the canonical link where $\mathbf{D}_p = \mathbf{V}_p\mathbf{X}_p$ and $\mathbf{D}_r = \mathbf{V}_p\mathbf{X}_r$. (Note that $\mathbf{V}_r = \mathbf{V}_p$ since the starting value of $\boldsymbol{\beta}_r$ is $\mathbf{0}$.) This means that the alias structure in the canonical link case can be written as

$$\mathbf{A} = \left(\mathbf{X}_p'\mathbf{V}_p\mathbf{X}_p\right)^{-1}\left(\mathbf{X}_p'\mathbf{V}_p\mathbf{X}_r\right). \tag{9.6.9}$$

§9.7 Examples of Alias Structure

The way to gain a better understanding of this alias structure is through examples of this structure for specific designs. However, it would appear that parameter knowledge would be necessary for these examples since IECs which form design points must be converted to natural units for both $\mathbf{X}_p$ and $\mathbf{X}_r$. However, there is a way to eliminate this problem. The levels of the design variables are, of course, the $IEC_{13.53}$ and the IEC_{100}. Since there are two levels of these variables, they can be coded as +1 and -1 in terms of natural units. This allows the practitioner to look at the model matrices without the added complication of knowing the parameters. The same applies for the variance matrix, $\mathbf{V}_p$. The alias structure of the design is invariant to the variances and ultimately the parameter values.

Consider the main effect design for the three regressor model. This is, in fact, a special case of the fractional factorials discussed earlier in this chapter. In coded form the design matrix for the main effect design is

$$\mathbf{X_p} = \begin{bmatrix} 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}. \tag{9.7.1}$$

$$\text{Let } \mathbf{V}_\mathrm{p} = \begin{bmatrix} v_1 & 0 & 0 & 0 \\ 0 & v_2 & 0 & 0 \\ 0 & 0 & v_3 & 0 \\ 0 & 0 & 0 & v_4 \end{bmatrix}. \tag{9.7.2}$$

$$\text{So, } \left(\mathbf{X}'_\mathrm{p}\mathbf{V}_\mathrm{p}\mathbf{X}_\mathrm{p}\right)^{-1} = \frac{1}{4v_1}\begin{bmatrix} \frac{v_1v_2v_3 + v_1v_2v_4 + v_1v_3v_4 + v_2v_3v_4}{v_2v_3v_4} & \frac{v_1+v_2}{v_2} & \frac{v_1+v_3}{v_3} & \frac{v_1+v_4}{v_4} \\ \frac{v_1+v_2}{v_2} & \frac{v_1+v_2}{v_2} & 1 & 1 \\ \frac{v_1+v_3}{v_3} & 1 & \frac{v_1+v_3}{v_3} & 1 \\ \frac{v_1+v_4}{v_4} & 1 & 1 & \frac{v_1+v_4}{v_4} \end{bmatrix}. \tag{9.7.3}$$

Exploring the alias structure with regard to the two way and three way interactions not included in the model, the matrix $\mathbf{X_r}$ is

$$\mathbf{X}_\mathrm{r} = \begin{bmatrix} 1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 \end{bmatrix}. \tag{9.7.4}$$

So,

$$\left(\mathbf{X}_p'\mathbf{V}_p\mathbf{X}_r\right)=\begin{bmatrix} v_1-v_2-v_3+v_4 & v_1-v_2+v_3-v_4 & v_1+v_2-v_3-v_4 & -v_1+v_2+v_3+v_4 \\ v_1-v_2+v_3-v_4 & -v_1-v_2-v_3+v_4 & -v_1+v_2+v_3+v_4 & -v_1+v_2-v_3-v_4 \\ v_1+v_2-v_3-v_4 & -v_1+v_2+v_3+v_4 & -v_1-v_2-v_3+v_4 & -v_1-v_2+v_3-v_4 \\ -v_1+v_2+v_3+v_4 & -v_1+v_2-v_3-v_4 & -v_1-v_2+v_3-v_4 & v_1-v_2-v_3+v_4 \end{bmatrix}$$

(9.7.5)

$$\text{and} \quad \mathbf{A}=\left(\mathbf{X}_p'\mathbf{V}_p\mathbf{X}_p\right)^{-1}\left(\mathbf{X}_p'\mathbf{V}_p\mathbf{X}_r\right)=\begin{bmatrix} -1 & -1 & -1 & 2 \\ -1 & -1 & 0 & 1 \\ -1 & 0 & -1 & 1 \\ 0 & -1 & -1 & 1 \end{bmatrix}. \qquad (9.7.6)$$

This shows that the alias matrix for this particular design is invariant to the matrix $\mathbf{V}_p$. The expected value for the estimator **b** in (9.6.8) is shown in (9.7.7)

$$E(\mathbf{b})=\begin{bmatrix} \beta_0-\beta_{12}-\beta_{13}-\beta_{23}+2\beta_{123} \\ \beta_1-\beta_{12}-\beta_{13}+\beta_{123} \\ \beta_2-\beta_{12}-\beta_{23}+\beta_{123} \\ \beta_3-\beta_{13}-\beta_{23}+\beta_{123} \end{bmatrix}. \qquad (9.7.7)$$

Note that in this design the intercept is aliased with the four additional terms in the true model. The other three main effect parameters are aliased with interactions involving their respective main effect regressors.

A second example where the intercept, main effect parameters and the X_1X_2 interaction are included in the assumed model, and thus estimable through choice of design, has the following alias structure when the full model is the true model:

$$E(\mathbf{b}) = \begin{bmatrix} \beta_0 - \beta_{13} - \beta_{23} + \beta_{123} \\ \beta_1 - \beta_{13} \\ \beta_2 - \beta_{23} \\ \beta_3 - \beta_{13} - \beta_{23} + \beta_{123} \\ \beta_{12} - \beta_{123} \end{bmatrix}. \tag{9.7.8}$$

Here, the intercept as well as β_3 are aliased with all of the new terms in the true model. The main effect parameters for x_1 and x_2 are confounded with the two-way interactions and β_{12} is confounded with the three way interaction only.

A third and final example is explored where the intercept, main effect parameters and β_{12} and β_{13} are included in the model and are thus estimable. The alias structure for this design under the full model is

$$E(\mathbf{b}) = \begin{bmatrix} \beta_0 - \beta_{23} \\ \beta_1 - \beta_{123} \\ \beta_2 - \beta_{23} \\ \beta_3 - \beta_{23} \\ \beta_{12} - \beta_{123} \\ \beta_{13} - \beta_{123} \end{bmatrix}. \tag{9.7.9}$$

In this case half of the terms in the assumed model are aliased with β_{23} and half of them are aliased with β_{123}.

Some final comments are necessary regarding the alias structure of fractional factorials for the exponential model. In order for estimates to be asymptotically unbiased, the same assumption used in linear models is required. Namely, this assumption is that higher order interactions not included in the model are negligible. The alias structure for a the three regressor exponential model main effects design is much different from the one for the main effects design in the traditional linear model linear model setting. (One must also keep in mind that the designs are different in these situations.) The other examples in this section have no counterpart in the traditional linear models setting. While a general form for the alias structure has been provided in expression (9.7.8), it is obvious that more work needs to be done in this area. The most important issue to be addressed in future research is the detection of a pattern in these alias structures so that they could be specified for any design without calculating **A**.

§9.8 Conclusion

These designs fit the three characteristics needed for the successful use of fractional factorials. By only requiring the points needed for the model terms to be estimated these designs allow for the **sparsity of effect principle**. If effects are insignificant, these designs can also be projected to a design in fewer variables. This illustrates the **projection property**. Finally, the principle of **sequential experimentation** is satisfied because additional points may be added to create a larger design which estimates the factor effects and interactions of interest. It is obvious that fractional factorials for the exponential model prove an essential tool for experimentation in both the biomedical and industrial arenas where experimental runs are limited by a variety of factors.