FINITE ELEMENT FORMULATION FOR COUPLED CONSOLIDATION

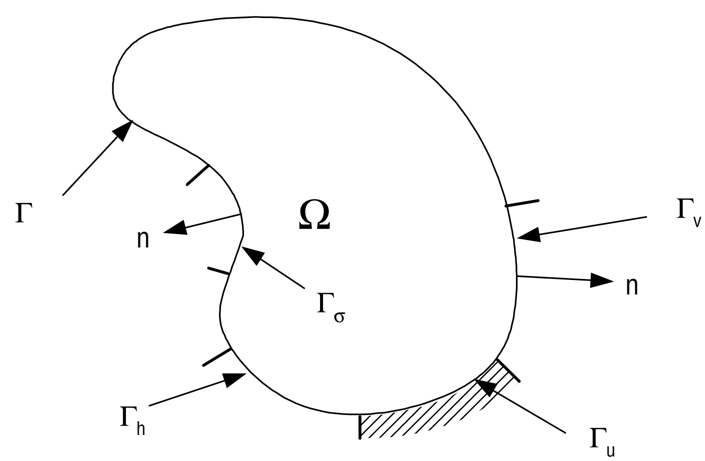
A.1. INTRODUCTION

The purpose of this appendix is to present the derivation of the finite element formulation of coupled consolidation used in the program SAGE.

A.2. DERIVATION

PROBLEM DEFINITION

A right-handed Cartesian coordinate system with principle axes x_1, x_2, and x_3 is used. The summation index notation is used here for convenience. The kronecker delta, δ_{ij}, is equal to one when $i = j$ and is equal to zero when $i \neq j$. A continuum of a soil saturated with water is considered.



Given: governing equations for saturated soil/water mixture

$$\sigma_{ij,j} = 0 \quad \text{in } \Omega \qquad \text{Equation A.1}$$

$$v_{i,i} - \dot{\varepsilon}_{ii} = 0 \quad \text{in } \Omega \qquad \text{Equation A.2}$$

with the boundary conditions:

$$\hat{f}_i = \sigma_{ij} n_j \quad \text{on } \Gamma_\sigma \qquad u = \hat{u} \quad \text{on } \Gamma_u$$
$$\hat{q} = v_i n_i \quad \text{on } \Gamma_v \qquad h = \hat{h} \quad \text{on } \Gamma_h$$

and the initial conditions:

$$h(x_1, x_2, 0) = h_0(x_1, x_2) \qquad u(x_1, x_2, 0) = u_0(x_1, x_2)$$
$$\sigma'_{ij}(x_1, x_2, 0) = \sigma'_{ij_0}(x_1, x_2)$$

where:

$$\sigma_{ij} = \sigma'_{ij} + p_w \delta_{ij} \qquad \text{Equation A.3}$$

$$\sigma'_{ij} = D_{ijkl} \varepsilon_{kl} \qquad \text{Equation A.4}$$

$$\varepsilon_{kl} = \frac{1}{2}\left(u_{k,l} + u_{l,k}\right) \qquad \text{Equation A.5}$$

$$v_i = k_{ij} h_{,j} \qquad \text{Equation A.6}$$

$$h = \frac{p_w}{\gamma_w} + h_{elev} \qquad \text{Equation A.7}$$

p_w is the pore water pressure
γ_w is the unit weight of the pore fluid (water)
h_{elev} is the elevation head ($x_2 - x_{2_{DATUM}}$; gravity acts in x_2 direction)

The given problem is an initial and boundary value problem. Equation A.1 represents the equilibrium equations in a continuum. Equation A.2 is the equation for transient seepage in a porous media. Equation A.1 is an elliptic partial differential equation and Equation A.2 is a parabolic partial differential equation.

The boundary conditions on Γ_u and Γ_h are essential boundary conditions. The boundary conditions on Γ_σ and Γ_v are natural boundary conditions. The boundary condition on Γ_σ is commonly referred to as a traction. The boundary condition on Γ_v is commonly referred to as a flux.

WEAK FORMULATION

First, the weak formulation of the problem will be derived. Next, the Galerkin formulation will be derived. The matrix formulation will be derived using the Galerkin approximation after the Galerkin formulation is obtained.

Let $\phi_i:\overline{\Omega} \rightarrow \Re$ be smooth functions for i = 1, 2, 3 such that $\phi_i = 0$ on Γ_u; that is ϕ_i vanishes on the part of the boundary where essential boundary conditions are prescribed.

Taking the inner product of Equation A.1 with ϕ_i and integrating the result over the domain Ω, we obtain:

$$\int_\Omega \phi_i \sigma_{ij,j} d\Omega = 0 \qquad \text{Equation A.8}$$

Using the chain rule of differentiation;

$$\phi_i \sigma_{ij,j} = \left(\phi_i \sigma_{ij}\right)_{,j} - \phi_{i,j} \sigma_{ij}$$

Recall the Divergence Theorem:

$$\int_\Omega A_{j,j} d\Omega = \int_{\partial\Omega} A_j n_j d\Gamma$$

Then:

$$\int_\Omega \phi_i \sigma_{ij,j} d\Omega = \int_{\partial\Omega} \left(\phi_i \sigma_{ij}\right) n_j d\Gamma - \int_\Omega \phi_{i,j} \sigma_{ij} d\Omega = 0 \qquad \text{Equation A.9}$$

Note, $\int_{\partial\Omega} \left(\phi_i \sigma_{ij}\right) n_j d\Gamma$ is only nonzero on Γ_σ

$$\int_\Omega \phi_{i,j} \sigma_{ij} d\Omega = \int_{\Gamma_\sigma} \phi_i \sigma_{ij} n_j d\Gamma$$

Recall, $\sigma_{ij} = \sigma'_{ij} + p_w \delta_{ij}$ and substitute for σ_{ij}, we obtain:

$$\int_\Omega \phi_{i,j} \sigma'_{ij} d\Omega + \int_\Omega \phi_{i,i} p_w d\Omega = \int_{\Gamma_\sigma} \phi_i \sigma_{ij} n_j d\Gamma \qquad \text{Equation A.10}$$

Now,

$$\phi_{i,j} = \phi_{(i,j)} + \phi_{[i,j]}$$

where $\phi_{(i,j)} = \frac{1}{2}\left(\phi_{i,j} + \phi_{j,i}\right)$ is symmetric

$\phi_{[i,j]} = \frac{1}{2}\left(\phi_{i,j} - \phi_{j,i}\right)$ is skew symmetric

So,

$\phi_{i,j}\sigma'_{ij} = \phi_{(i,j)}\sigma'_{ij}$ $\left(\phi_{[i,j]}\sigma'_{ij} = 0\right)$ since σ_{ij} is symmetric

Substituting this result into Equation A.10 we obtain:

$$\int_\Omega \phi_{(i,j)} \sigma'_{ij} d\Omega + \int_\Omega \phi_{i,i} p_w d\Omega = \int_{\Gamma_\sigma} \phi_i \sigma_{ij} n_j d\Gamma \qquad \text{Equation A.11}$$

Now,

$\varepsilon^{\phi}_{ij} = \phi_{(i,j)}$ where ε^{ϕ}_{ij} is the strain tensor derived from ϕ

$\sigma'_{ij} = D_{ijkl}\varepsilon^{u}_{kl}$ where ε^{u}_{kl} is the strain tensor derived from u

Then:

$$\int_\Omega \varepsilon^{\phi}_{ij} D_{ijkl} \varepsilon^{u}_{kl} d\Omega + \int_\Omega \varepsilon^{\phi}_{ii} p_w d\Omega = \int_{\Gamma_\sigma} \phi_i \hat{f}_i d\Gamma \qquad \text{Equation A.12}$$

Let $\theta: \overline{\Omega} \rightarrow \Re$ be a smooth function such that $\theta = 0$ on Γ_h; that is θ satisfies homogeneous essential boundary conditions on Γ_h.

Multiplying Equation A.2 with θ and integrating the result over the domain Ω, we obtain:

$$\int_\Omega \theta v_{i,i} d\Omega - \int_\Omega \theta \dot{\varepsilon}_{ii} d\Omega = 0 \qquad \text{Equation A.13}$$

Using the chain rule of differentiation

$$\theta v_{i,i} = (\theta v_i)_{,i} - \theta_{,i} v_i$$

Using the Divergence Theorem

$$\int_{\partial\Omega} \theta v_i n_i d\Gamma - \int_{\Omega} \theta_{,i} v_i d\Omega - \int_{\Omega} \theta \dot{\varepsilon}_{ii} d\Gamma = 0$$

Recall $v_i = k_{ij} h_{,j}$ and $\hat{q} = v_i n_i$ on Γ_v

$$v_i = \frac{k_{ij}}{\gamma_w} (\gamma_w h)_{,j}$$

Substituting, we obtain

$$\int_{\Omega} \theta_{,i} \frac{k_{ij}}{\gamma_w} (\gamma_w h)_{,j} d\Omega - \int_{\Omega} \theta \dot{\varepsilon}_{ii} d\Omega = \int_{\Gamma_v} \theta \hat{q} d\Gamma \qquad \text{Equation A.14}$$

Recall $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$, therefore $\varepsilon_{ii} = u_{i,i}$. Substituting we obtain

$$\int_{\Omega} \theta_{,i} \frac{k_{ij}}{\gamma_w} (\gamma_w h)_{,j} d\Omega - \int_{\Omega} \theta \dot{u}_{i,i} d\Omega = \int_{\Gamma_v} \theta \hat{q} d\Gamma \qquad \text{Equation A.15}$$

Equation A.12 may be rewritten as

$$a(\phi, u) + b(\phi, p_w) = l(\phi) \qquad \text{Equation A.16}$$

where

$$a(\phi, u) = \int_{\Omega} \varepsilon_{ij}^{\phi} D_{ijkl} \varepsilon_{kl}^{u} d\Omega$$

$$b(\phi, p_w) = \int_{\Omega} \varepsilon_{ii}^{\phi} p_w d\Omega$$

$$l(\phi) = \int_{\Gamma_\sigma} \phi_i \hat{f}_i d\Gamma$$

For the bilinear form $a(\cdot,\cdot)$ to be well defined, first order derivatives of functions u and ϕ should be square integrable over the domain Ω.

We define

$$H^1 = \left\{ \varphi \middle| \varphi : \overline{\Omega} \rightarrow \Re^2, \int_{\Omega} \left(\varphi_i \varphi_i + \varphi_{i,j} \varphi_{i,j} \right) d\Omega < \infty \right\}$$

$$H_0^1 = \left\{ \varphi \middle| \varphi \in H^1, \varphi = 0 \text{ on } \Gamma_u \right\}$$

Equation A.15 may be rewritten as

$$c(\theta, h) + d(\theta, \dot{u}) = m(\theta)$$ Equation A.17

where

$$c(\theta, h) = \int_{\Omega} \theta_{,i} \frac{k_{ij}}{\gamma_w} (\gamma_w h)_{,j} d\Omega$$

$$d(\theta, \dot{u}) = \int_{\Omega} \theta \dot{u}_{i,i} d\Omega$$

$$m(\theta) = \int_{\Gamma_v} \theta \hat{q} d\Gamma$$

For the bilinear form $c(\cdot,\cdot)$ to be well defined, the first order derivatives of functions θ and h should be square integrable over the domain Ω.

We define

$$\overline{H}^1 = \left\{ \varphi \middle| \varphi : \overline{\Omega} \rightarrow \Re, \int_{\Omega} \varphi_{,i} \varphi_{,i} d\Omega < \infty \right\}$$

$$\overline{H}_0^1 = \left\{ \varphi \middle| \varphi \in \overline{H}^1, \varphi = 0 \text{ on } \Gamma_h \right\}$$

Weak formulation

Find $h: \overline{\Omega} \times (0, \mathrm{T}) \rightarrow \Re^2$ and $u: \overline{\Omega} \times (0, \mathrm{T}) \rightarrow \Re^2$ such that Equation A.16 and Equation A.17, and the boundary conditions $h \in \hat{h}$ on Γ_h, $u \in H^1$, $u = \hat{u}$ on Γ_u, and $h \in \overline{H}^1$ are satisfied for every $\phi_i \in H_0^1$ and for every $\theta \in \overline{H}_0^1$.

GALERKIN FORMULATION

Now, the Galerkin formulation of the problem is sought.

Let $g \in H^1$ be such that $g(x_1, x_2, t) = \hat{u}(x_1, x_2, t)$ on $\Gamma_u \times (0, \mathrm{T})$. Let $\overline{g} \in \overline{H}^1$ be such that $\overline{g}(x_1, x_2, t) = \hat{h}(x_1, x_2, t)$ on $\Gamma_h \times (0, \mathrm{T})$. Then for every $v \in H_0^1$, $u(x_1, x_2, t) = v(x_1, x_2, t) + g(x_1, x_2, t) \in H^1$ and satisfies the essential boundary conditions involving u. For every $\overline{v} \in \overline{H}_0^1$, $h(x_1, x_2, t) = \overline{v}(x_1, x_2, t) + \overline{g}(x_1, x_2, t) \in \overline{H}^1$ and satisfies the essential boundary conditions involving h.

Recall that p_w is related to h with the relationship $h = \frac{p_w}{\gamma_w} + h_{elev}$ and $h_{elev} = x_2 - x_{2_{DATUM}}$.

$$p_w(x_1, x_2, t) = \left\{ h(x_1, x_2, t) - \left(x_2 - x_{2_{DATUM}} \right) \right\} \gamma_w \qquad \text{Equation A.18}$$

Substituting we obtain;

$$a(\phi, v) + b(\phi, \gamma_w v') = l(\phi) - a(\phi, g) - b(\phi, \gamma_w \overline{g}) + b\left(\phi, \left(x_2 - x_{2_{DATUM}}\right)\gamma_w\right) \qquad \text{Equation A.19}$$

$$c(\theta, v') + d(\theta, \dot{v}) = m(\theta) - c(\theta, \overline{g}) - d(\theta, \dot{g}) \qquad \text{Equation A.20}$$

Galerkin formulation

Find $h\colon \overline{\Omega} \times (0, \mathrm{T}) \to \Re^2$ and $u\colon \overline{\Omega} \times (0, \mathrm{T}) \to \Re^2$ such that Equation A.19 and Equation A.20, and the boundary conditions $\overline{g} = \hat{h}$ on Γ_h, $u \in H^1$, $g = \hat{u}$ on Γ_u, and $h \in \overline{H}^1$ are satisfied for each $\phi_i \in H_0^1$ and for each $\theta \in \overline{H}_0^1$.

Let $H_0^{1n} \subset H_0^1$ and $\overline{H}_0^{1m} \subset \overline{H}_0^1$ be finite dimensional sets. n is the number of nodes that will have displacement as a degree of freedom. m is the number of nodes that will have head as a degree of freedom.

GALERKIN APPROXIMATION

Find $v^n \in H_0^{1n}$ and $\overline{v}^m \in \overline{H}_0^{1m}$ such that Equation A.21 and Equation A.22 hold for each $\phi^n \in H_0^{1n}$ and for each $\theta^m \in \overline{H}_0^{1m}$.

$$a(\phi^n, v^n) + b(\phi^n, \overline{v}^m) = l(\phi^n) - a(\phi^n, g^n) - b(\phi^n, \overline{g}^m) \qquad \text{Equation A.21}$$

$$c(\theta^m, \overline{v}^m) + d(\theta^m, \dot{v}^n) = m(\theta^m) - c(\theta^m, \overline{g}^m) - d(\theta^m, \dot{g}^n) \qquad \text{Equation A.22}$$

Let $\varphi_1, \varphi_2, \ldots, \varphi_n$ be a set of basis functions in H_0^{1n}.

$$v_i^n(x_1, x_2, t) = \sum_{A=1}^{n} d_{Ai}(t)\varphi_A(x_1, x_2) \qquad \text{Equation A.23}$$

$$\phi_i^n(x_1, x_2) = \sum_{B=1}^{n} c_{Bi}\varphi_B(x_1, x_2) \qquad \text{Equation A.24}$$

Let $\overline{\varphi}_1, \overline{\varphi}_2, \ldots, \overline{\varphi}_m$ be a set of basis functions in $\overline{H}_0^{1m}$.

$$\overline{v}(x_1, x_2, t) = \sum_{C=1}^{m} f_C(t)\overline{\varphi}_C(x_1, x_2) \qquad \text{Equation A.25}$$

$$\theta^m(x_1, x_2) = \sum_{D=1}^{m} e_D\overline{\varphi}_D(x_1, x_2) \qquad \text{Equation A.26}$$

The indices A, B, C, and D refer to the basis function number (node number) and the index i indicates the x_i direction. The basis functions v_i^n and ϕ_i^n are the same.

Substituting into Equation A.21 and Equation A.22, we obtain;

$$\begin{aligned} a(c_{Bi}\varphi_B, d_{Ai}\varphi_A) + b(c_{Bi}\varphi_B, \gamma_w f_C\overline{\varphi}_C) = l(c_{Bi}\varphi_B) - \\ a(c_{Bi}\varphi_B, g) - b(c_{Bi}\varphi_B, \gamma_w\overline{g}) + \\ b\left(c_{Bi}\varphi_B, \gamma_w\left(x_2 - x_{2_{DATUM}}\right)\right) \end{aligned} \qquad \text{Equation A.27}$$

$$\begin{aligned} c(e_D\overline{\varphi}_D, f_C\overline{\varphi}_C) + d(e_D\overline{\varphi}_D, \dot{d}_{Ai}\overline{\varphi}_A) = m(e_D\overline{\varphi}_D) \\ -c(e_D\overline{\varphi}_D, \overline{g}) - d(e_D\overline{\varphi}_D, \dot{g}) \end{aligned} \qquad \text{Equation A.28}$$

MATRIX FORMULATION

Now, matrix notation will be introduced.

$$\varepsilon_{ij}^{\phi} = \frac{1}{2}\left(c_{Bi}\varphi_{B,j} + c_{Bj}\varphi_{B,i}\right) = [B]\{c\}$$
$$\varepsilon_{ii}^{\phi} = \frac{1}{2}\left(c_{Bi}\varphi_{B,i} + c_{Bi}\varphi_{B,i}\right) = [B_v]\{c\}$$

where

$$[B] = \begin{bmatrix} \varphi_{1,1} & 0 & \varphi_{2,1} & 0 & \cdots & \varphi_{n,1} & 0 \\ 0 & \varphi_{1,2} & 0 & \varphi_{2,2} & \cdots & 0 & \varphi_{n,2} \\ \varphi_{1,2} & \varphi_{1,1} & \varphi_{2,2} & \varphi_{2,1} & \cdots & \varphi_{n,2} & \varphi_{n,1} \end{bmatrix}$$

$$\{c\} = \begin{Bmatrix} c_{11} \\ c_{12} \\ c_{21} \\ c_{22} \\ \vdots \\ c_{n1} \\ c_{n2} \end{Bmatrix}$$

$$[B_v] = \begin{bmatrix} \varphi_{1,1} & \varphi_{1,2} & \varphi_{2,1} & \varphi_{2,2} & \cdots & \varphi_{n,1} & \varphi_{n,2} \end{bmatrix}$$

Similarly $\varepsilon_{ij}^{u} = [B]\{d\}$ and $\varepsilon_{ii}^{u} = [B_v]\{e\}$

$$\dot{u}_{i,i} = \dot{\varepsilon}_{ii} = [B_v]\{\dot{d}\}$$
$$\theta_{,i} = e_D \varphi'_{D,i} = [B_h]\{e\}$$

where

$$[B_h] = \begin{bmatrix} \overline{\varphi}_{1,1} & \overline{\varphi}_{2,1} & \cdots & \overline{\varphi}_{m,1} \\ \overline{\varphi}_{1,2} & \overline{\varphi}_{2,1} & \cdots & \overline{\varphi}_{m,1} \end{bmatrix}$$

$$\{e\} = \begin{Bmatrix} e_1 \\ e_2 \\ \vdots \\ e_m \end{Bmatrix}$$

Similarly, $\left(\gamma_w f\right)_{,j} = [B_h]\{\gamma_w f\}$. Note that all vectors are column vectors.

Now,

$$a\left(c_{Bi}\varphi_B, d_{Ai}\varphi_A\right) = \int_\Omega \left([B]\{c\}\right)^T [D][B]\{d\} d\Omega$$

$$= \{c\}^T \int_\Omega [B]^T [D][B]\{d\} d\Omega = \{c\}^T [K]\{d\}$$

$$b\left(c_{Bi}\varphi_B, \gamma_w f_C \overline{\varphi}_C\right) = \int_\Omega \{c\}^T \left[B_v\right]^T \{\overline{\varphi}\}\{\gamma_w f\} d\Omega$$

$$= \{c\}^T \left[K_v\right]^T \{\gamma_w f\}$$

$$l\left(c_{Bi}\varphi_B\right) = \int_\Omega \left(\{\varphi\}\{c\}\right)^T \left\{\hat{f}\right\} d\Omega = \{c\}^T \int_\Omega \{\varphi\}^T \left\{\hat{f}\right\} d\Omega = \{c\}^T \left\{F^1\right\}$$

$$a\left(c_{Bi}\varphi_B, g\right) - b\left(c_{Bi}\varphi_B, \gamma_w g'\right) = c_{Bi}\left(a\left(\varphi_B, g\right) - b\left(\varphi_B, \gamma_w g'\right)\right)$$

$$= \{c\}^T \left\{F^2\right\}$$

$$b\left(c_{Bi}\varphi_B, \gamma_w\left(x_2 - x_{2_{DATUM}}\right)\right) = c_{Bi} b\left(\varphi_B, \gamma_w\left(x_2 - x_{2_{DATUM}}\right)\right)$$

$$= \{c\}^T \left\{F^3\right\}$$

$$c\left(e_D \overline{\varphi}_D, f_C \overline{\varphi}_C\right) = \int_\Omega \left(\left[B_h\right]\{e\}\right)^T \frac{[k]}{\gamma_w} \left[B_h\right]\{\gamma_w f\} d\Omega$$

$$= \{e\}^T \int_\Omega \left[B_h\right] \frac{[k]}{\gamma_w} \left[B_h\right]\{\gamma_w f\} d\Omega = \{e\}^T \left[K_h\right]\{\gamma_w f\}$$

$$d\left(e_D \overline{\varphi}_D, \dot{d}_{Ai}\varphi_A\right) = \int_\Omega \left(\{\overline{\varphi}\}\{e\}\right)^T \left[B_v\right]\left\{\dot{d}\right\} d\Omega$$

$$= \{e\}^T \int_\Omega \{\overline{\varphi}\}^T \left[B_v\right]\left\{\dot{d}\right\} d\Omega = \{e\}^T \left[K_v\right]\left\{\dot{d}\right\}$$

$$m\left(e_D \overline{\varphi}_D\right) = \int_\Omega \left(\{\overline{\varphi}\}\{e\}\right)^T \{\hat{q}\} d\Omega = \{e\}^T \int_\Omega \{\overline{\varphi}\}^T \{\hat{q}\} d\Omega = \{e\}^T \left\{Q^1\right\}$$

$$c\left(e_D \overline{\varphi}_D, \overline{g}\right) - d\left(e_D \overline{\varphi}_D, \dot{g}\right) = e_D\left\{c\left(\overline{\varphi}_D, g\right) - d\left(\overline{\varphi}_D, \dot{g}\right)\right\}$$

$$= \{e\}^T \left\{Q^2\right\}$$

Substituting into Equation A.27 and Equation A.28, we obtain

$$\{c\}^T \left[[K]\{d\} + \left[K_v\right]^T \{\gamma_w f\} - \left\{F^1\right\} + \left\{F^2\right\} + \left\{F^3\right\}\right] = 0 \qquad \text{Equation A.29}$$

$$\{e\}^T \left[\left[K_v\right]\left\{\dot{d}\right\} + \left[K_h\right]\{\gamma_w f\} - \left\{Q^1\right\} + \left\{Q^2\right\}\right] = 0 \qquad \text{Equation A.30}$$

Equivalently we could write

$$\vec{c}\cdot\vec{E}^1=\vec{0}$$ Equation A.31

$$\vec{e}\cdot\vec{E}^2=\vec{0}$$ Equation A.32

Since Equation A.31 and Equation A.32 must hold for every $\vec{c}$ and $\vec{e}$ respectively, $\vec{E}^1=0$ and $\vec{E}^2=0$. Therefore,

$$[K]\{d\}+[K_v]^T\{\gamma_w f\}=\{F^1\}-\{F^2\}-\{F^3\}$$ Equation A.33

$$[K_v]\{\dot{d}\}+[K_h]\{\gamma_w f\}=\{Q^1\}-\{Q^2\}$$ Equation A.34

We can ignore $\{F^2\}$ and $\{Q^2\}$ because we will satisfy essential boundary conditions by modifying the finite element equations rather than selecting functions g and $\overline{g}$.

TIME DISCRETIZATION

The finite element method has been used to discretize the problem spatially. Now the finite difference method will be used to discretize the problem in the time domain.

$$\{\dot{d}\}=\frac{\{\Delta d\}}{\Delta t} \quad \text{where } \Delta d=d|_{t+\Delta t}-d|_t$$

$$\{\gamma_w\dot{f}\}=\frac{\{\Delta\gamma_w f\}}{\Delta t} \quad \text{where } \Delta\gamma_w f=\gamma_w f|_{t+\Delta t}-\gamma_w f|_t$$

$$\{\gamma_w f\}|_{t+\alpha\Delta t}=(1-\alpha)\{\gamma_w f\}|_t+\alpha\{\gamma_w f\}|_{t+\Delta t} \quad 0\le\alpha\le1$$

First, take the equations represented by Equation A.33 at time $t+\Delta t$ and subtract the equations represented by Equation A.33 at time t. We obtain,

$$[K]\{\Delta d\}+[K_v]^T\{\gamma_w f\}|_{t+\Delta t}=\{F^1\}|_{t+\Delta t}-\{F^1\}|_t+[K_v]^T\{\gamma_w f\}|_t$$

Note that F^3 is the same at times $t+\Delta t$ and t and therefore cancels out.

Next, evaluate Equation A.34 at time $t+\alpha\Delta t$ and multiply by Δt.

$$[K_v]^T\{\Delta d\} + \Delta t[K_h]\alpha\{\gamma_w f\}\big|_{t+\Delta t} =$$
$$\Delta t\left(\{Q^1\}\big|_{t+\Delta t} - \{Q^1\}\big|_t\right) - \Delta t[K_h](1-\alpha)\{\gamma_w f\}\big|_t$$

Now our problem formulation is,

$$\begin{bmatrix} [K] & [K_v]^T \\ [K_v] & \Delta t\alpha[K_h] \end{bmatrix} \begin{Bmatrix} \{\Delta d\} \\ \{\gamma_w f\}\big|_{t+\Delta t} \end{Bmatrix} = \begin{Bmatrix} \{F^1\}\big|_{t+\Delta t} - \{F^1\}\big|_t + [K_v]^T\{\gamma_w f\}\big|_t \\ \Delta t\left(\{Q^1\}\big|_{t+\Delta t} - \{Q^1\}\big|_t\right) - \Delta t(1-\alpha)[K_h]\{\gamma_w f\}\big|_t \end{Bmatrix}$$

Equation A.35

Note that Δt and α must be greater than zero for the algorithm represented by Equation A.35 to work.