Appendix A

Generalized System Equations for a Linear Predictor

The block diagram of a linear predictor is shown in Figure A1. Here $\hat{x}(n)$ is the predicted value of *x(n)* and *e(n)* is the *D* step ahead forward prediction error. The delay *D* determines how many steps ahead the predictor is designed to predict *x(n)*.

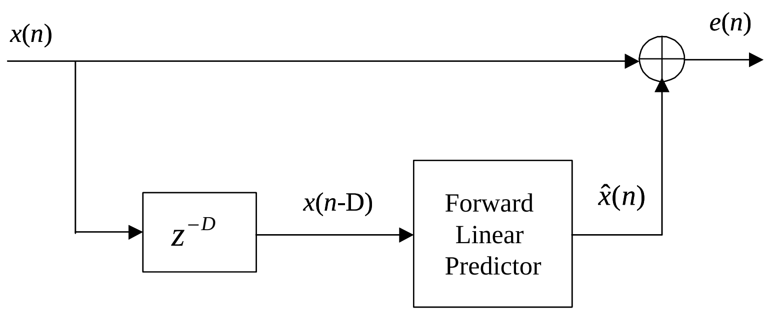

Figure A.1: Block Diagram of the General Prediction Filter.

The predicted signal:

$$\hat{x}(n) = -\sum_{k=1}^{M} a_k^M x(n - D - k + 1) \tag{A.1.1}$$

where *M* is the order of the predictor and $-a_1^M \quad -a_2^M \quad .... \quad -a_M^M$ are the predictor coefficients. The *D* step forward prediction error *e(n)* is the difference between the desired signal *d(n)* and the predicted signal, i.e.,

$$\begin{aligned} e(n) &= d(n) - \hat{x}(n) \\ &= d(n) + \sum_{k=1}^{M} a_k^M x(n - D - k + 1) \end{aligned} \tag{A.1.2}$$

The predictor coefficients are selected such that the mean square value of *e(n)*, i.e., $E\left\{|e(n)|^2\right\}$, is minimized. Therefore, for any set of predictor coefficients a_k^M $(1 \le k \le M)$,

$$\frac{d}{d\ a_k^M} E\left\{ |e(n)|^2 \right\} = 0, \quad 1 \le k \le M \tag{A.1.3}$$

Due to the linearity of the expectation and differentiation operators we can interchange these two operations and (A.1.3) can be written as:

$$\begin{aligned} \frac{d}{d\ a_k^M} E\left\{ |e(n)|^2 \right\} &= E\left\{ \frac{d}{d\ a_k^M} |e(n)|^2 \right\} \\ &= E\left\{ \left\{ d(n) + \sum_{k=1}^{M} a_k^M x(n-D-k+1) \right\} a_k^{M^*} x^*(n-D-k+1) \right. \\ &\quad \left. + \left\{ d(n) + \sum_{k=1}^{M} a_k^M x(n-D-k+1) \right\}^* a_k^M x(n-D-k+1) \right\} \end{aligned} \tag{A.1.4}$$

Since the predictor is estimating the present value of the input sample, the desired signal $d(n) = x(n)$. Therefore, from (A.1.3) and (A.1.4) we can write,

$$\left.\begin{aligned} &2\,\mathrm{Re}\left[a_k^{M^*} \left\{ r_{xx}(D+l-1) + \sum_{k=1}^{M} a_k^M r_{xx}(l-k) \right\} \right] = 0 \\ &\Rightarrow r_{xx}(D+l-1) + \sum_{k=1}^{M} a_k^M r_{xx}(k-l) = 0 \end{aligned}\right. \quad , \ l = 1, 2, \ldots, L \tag{A.1.5}$$

where $r_{xx}(k)$ is the value of the auto-correlation sequence of $x(n)$ for the k-th lag. The set of equations in (A.1.5) is known as the *normal equations*. Equation (A.1.5) indicates L distinct equations, each for a different value of l. By solving these L equations, the predictor coefficients can be determined for the minimum mean square error criterion. Now by using the identity $r_{xx}(-k) = r_{xx}^*(k)$ the L equations of (A.1.5) can be arranged in matrix format as shown below.

$$\begin{bmatrix} r_{xx}(0) & r_{xx}(1) & \cdots & r_{xx}(M-1) \\ r_{xx}^*(1) & r_{xx}(0) & \cdots & r_{xx}(M-2) \\ \vdots & \vdots & \cdots & \vdots \\ r_{xx}^*(L-1) & r_{xx}^*(L-2) & \cdots & r_{xx}^*(L-M) \end{bmatrix} \begin{bmatrix} a_1^M \\ a_2^M \\ \vdots \\ a_M^M \end{bmatrix} = - \begin{bmatrix} r_{xx}(D) \\ r_{xx}(D+1) \\ \vdots \\ r_{xx}(D+L-1) \end{bmatrix} \tag{A.1.6}$$

Equation (A.1.6) can be expressed compactly as:

$$\mathbf{Ra} = \mathbf{b} \tag{A.1.7}$$

where **R** is the $(L \times M)$ correlation matrix and **a** is the predictor coefficient vector to be determined. The vector **b** contains elements of the auto-correlation of the input sequence $x(n)$ and the particular lags of the auto-correlation values depend on how many samples ahead we want to predict. The matrix **R** has the special property that the (i,j)th element of the matrix $r(i, j) = r_{xx}(i - j)$. Again, since $r_{xx}(-k) = r_{xx}{}^{*}(k)$, $r(i, j) = r^{*}(j, i)$. A matrix with these properties is called a *Toeplitz* matrix.

Appendix B

Let us consider an estimation process that is being used to estimate a desired signal *A*. To reduce the complexity of the problem, *A* will be considered to be a stationary signal with a constant value. Let the estimated value of the desired signal be *V*. Now *V* can be written as:

$$V = A + X \tag{A2.1}$$

In (A2.1) *X* is the estimation error, which is a *random variable* (RV). For convenience, let us consider *X* to be a zero mean, white RV with variance s_x^2. Let us consider a new random variable *U* generated by transforming the RV *V* in the following way:

$$U = \frac{1}{V} = \frac{1}{A + X} \tag{A2.2}$$

If there were no estimation error in the system, *U* would be a constant with a value of 1/*A*. Therefore the desired value of *U* is 1/*A*. In the presence of the estimation error *X*, *U* will be a random variable with a constant component. To analyze the effect of the estimation error *X*, *U* will be divided into two parts: the first part is the desired value of *U* and the second part is the error signal resulting from the estimation error, i.e.,

$$U = \frac{1}{A} + e \tag{A2.3}$$

Now if we assume that the estimation error *X* is less than the desired value *A*, (A2.2) can be expanded by using the Maclaurin series as follows:

$$U = \frac{1/A}{1 + X/A} = \frac{1}{A}\sum_{n=0}^{\infty}\left(-\frac{X}{A}\right)^n, \qquad |X| < |A| \tag{A2.4}$$

Comparing (A2.3) and (A2.4) the error term of (A2.3) can be written as:

$$e = \frac{1}{A}\sum_{n=1}^{\infty}\left(-\frac{X}{A}\right)^n \tag{A2.5}$$

To analyze the effect of the estimation error on *U*, the first and the second order moments of the error term *e* (in (A2.5)) will be determined.

First Order Moment

To determine the moments of the error signal we will always start with (A2.4), the expression for *U*. The first order moment or the mean of the random variable *U*, can be expressed as

$$m_u = E\{U\} = E\left\{\frac{1}{A}\sum_{n=0}^{\infty}\left(-\frac{X}{A}\right)^n\right\} \tag{A2.6}$$

Exploiting the linearity of the expectation and summation operations, (A2.6) can be written as:

$$m_u = E\{U\} = \frac{1}{A}\sum_{n=0}^{\infty}E\left\{\left(-\frac{X}{A}\right)^n\right\} \tag{A2.7}$$

Since *X* is assumed to be zero-mean and white, and *A* is assumed to be constant, the expected value of the terms in (A2.7) with odd power *n* will be zero. Again, the expected values of $(X/A)^{2m}$, for *m*=2*n*, are negligible compared to $E\left\{(X/A)^2\right\}$, when $\frac{X}{A} << 1$. Therefore, the mean value of the error term *e* can be expressed compactly as follows:

$$m_u = E\{U\} = \frac{1}{A} + \frac{1}{A}E\left\{\left(\frac{X}{A}\right)^2\right\} = \frac{1}{A} + \frac{s_x^2}{A^3} \tag{A2.8}$$

In (A2.8), the 1/*A* term represents the desired value of *U* in the absence of the estimation error. Therefore, the mean value of the estimation error can be written as:

$$m_e = E\{e\} = \frac{s_x^2}{A^3} \tag{A2.9}$$

From (A2.9) it can be said that, even though *X* is a zero mean random variable, the mean of *e* is not zero and the mean value depends on the variance of *X* and the magnitude of *A*.

Second Order Moment

According to the assumption of a stationary desired amplitude *A*, the second order central moment of *U* should be zero in the absence of estimation error. Therefore, any non-zero value of the second order central moment of *U* will be the result of the estimation error. To find the central moment we will first determine the second order moment of *U*, which is given by:

$$\begin{aligned}
E\{U^2\} &= \frac{1}{A^2} E\left\{ \sum_{n=0}^{\infty} \left(-\frac{X}{A} \right)^n \sum_{m=0}^{\infty} \left(-\frac{X}{A} \right)^m \right\} \\
&= \frac{1}{A^2} E\left\{ \left[1 - \frac{X}{A} + \frac{X^2}{A^2} - \frac{X^3}{A^3} + \cdots \right] \times \left[1 - \frac{X}{A} + \frac{X^2}{A^2} - \frac{X^3}{A^3} + \cdots \right] \right\} \\
&= \frac{1}{A^2} E\left\{ 1 - \frac{2X}{A} + \frac{3X^2}{A^2} - \frac{4X^3}{A^3} + \frac{5X^4}{A^4} - \cdots \right\}
\end{aligned} \tag{A2.10}$$

By using the same assumptions that were used to simplify (A2.7), (A2.10) can be written as:

$$E\{U^2\} = \frac{1}{A^2} + \frac{1}{A^2}\frac{3s_x^2}{A^2} = \frac{1}{A^2} + \frac{3s_x^2}{A^4} \tag{A2.11}$$

To find the effect of the estimation error, the second order central moment of *U* will be calculated about the value of *U* in the errorless case. Therefore,

$$\begin{aligned}
s_u^2 &= E\{U^2\} - \frac{1}{A^2} = \frac{1}{A^2} + \frac{3s_x^2}{A^4} - \frac{1}{A^2} \\
&= \frac{3s_x^2}{A^4}
\end{aligned} \tag{A2.12}$$

From (A2.12) it can be concluded that the variance of U is greater than the variance of the estimation error s_x^2, if $A \le \sqrt[4]{3} = 1.3161$. , Figure B. 1 shows the effect of A and the variance of the estimation error s_x^2 on the variance of U, s_u^2. In Figure B. 1, all the values of s_u^2 above 40 were truncated to 40. X was considered a zero-mean Gaussian random process while generating Figure B. 1. It is clear from Figure B. 1 that for the smaller values of s_x^2, the values of A below which $s_u^2 > s^2$, pretty much coincide with the bound derived in (A2.12). For higher values of s^2, the higher order statistics of X are not negligible and thus the assumptions made to derive (A2.12) are no longer valid. In this situation the value of A, below which $s_u^2 > s^2$, will not be 1.3161. Figure B. 1 shows that this value of A starts to increase with an increase in s_x^2.

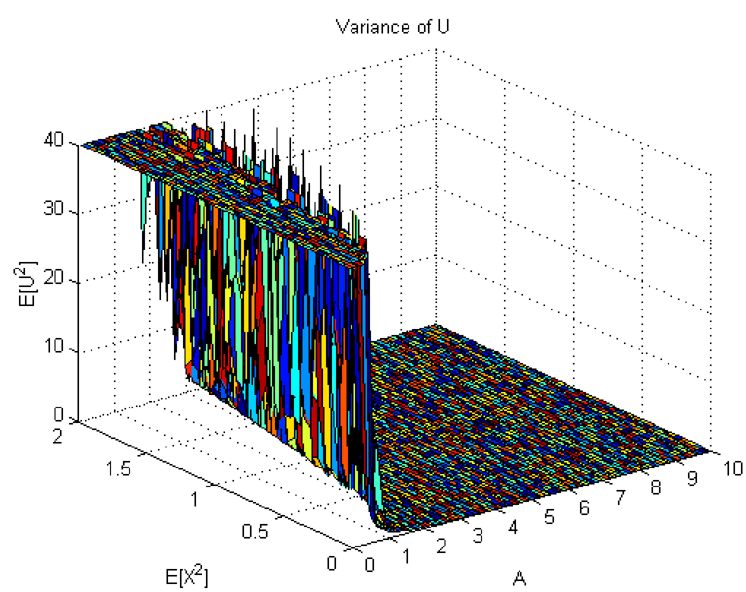

Figure B. 1: Variance of *U* as a Function of *A* and the Variance of *X*.