A geometrically exact curved beam theory and its finite element formulation/implementation

Jing Li

Thesis submitted to the faculty of the
Virginia Polytechnic Institute and State University
in partial fulfillment of the requirements for the degree of

Master of Science
in
Aerospace Engineering

Rakesh K. Kapania, Chair
Eric R. Johnson
Raymond H. Plaut

December 28, 2000
Blacksburg, Virginia

Keywords: Curved beam, geometrically exact nonlinear, finite rotations, finite element.

A geometrically exact curved beam theory and its finite element formulation/implementation

Jing Li

Rakesh K. Kapania

Aerospace and Ocean Engineering

(ABSRACT)

This thesis presents a geometrically exact curved beam theory, with the assumption that the cross-section remains rigid, and its finite element formulation/implementation. The theory provides a theoretical view and an exact and efficient means to handle a large range of nonlinear beam problems.

A geometrically exact curved/twisted beam theory, which assumes that the beam cross-section remains rigid, is re-examined and extended using orthonormal reference frames starting from a 3-D beam theory. The relevant engineering strain measures at any material point on the current beam cross-section with an initial curvature correction term, which are conjugate to the first Piola-Kirchhoff stresses, are obtained through the deformation gradient tensor of the current beam configuration relative to the initially curved beam configuration. The Green strains and Eulerian strains are explicitly represented in terms of the engineering strain measures while other stresses, such as the Cauchy stresses and second Piola-Kirchhoff stresses, are explicitly represented in terms of the first Piola-Kirchhoff stresses and engi-

neering strains. The stress resultant and couple are defined in the classical sense and the reduced strains are obtained from the three-dimensional beam model, which are the same as obtained from the reduced differential equations of motion. The reduced differential equations of motion are also re-examined for the initially curved/twisted beams. The corresponding equations of motion include additional inertia terms as compared to previous studies. The linear and linearized nonlinear constitutive relations with couplings are considered for the engineering strain and stress conjugate pair at the three-dimensional beam level. The cross-section elasticity constants corresponding to the reduced constitutive relations are obtained with the initial curvature correction term.

For the finite element formulation and implementation of the curved beam theory, some basic concepts associated with finite rotations and their parametrizations are first summarized. In terms of a generalized vector-like parametrization of finite rotations under spatial descriptions (i.e., in spatial forms), a unified formulation is given for the virtual work equations that leads to the load residual and tangent stiffness operators. With a proper explanation, the case of the non-vectorial parametrization can be recovered if the incremental rotation is parametrized using the incremental rotation vector. As an example for static problems, taking advantage of the simplicity in formulation and clear classical meanings of rotations and moments, the non-vectorial parametrization is applied to implement a four-noded 3-D curved beam element, in which the compound rotation is represented by the unit quaternion and the incremental rotation is parametrized using the incremental rotation vector. Conventional Lagrangian interpolation functions are adopted to approximate both the reference curve and incremental rotation of the deformed beam. Reduced integration is used to overcome locking problems. The finite element equations are developed for static structural analyses, including deformations, stress resultants/couples, and linearized/nonlinear bifurcation buckling, as well as post-buckling analyses of arches subjected to conservative and non-conservative loads. Several examples are used to test the formulation and the Fortran implementation of the element.

ACKNOWLEDGEMENTS

The support provided for this research by the grant DAAH04-95-1-0175 from the Army Research Office with Dr. Gary Anderson as the grant monitor is greatly appreciated. I owe a great deal of thanks to Dr. Rakesh Kapania for his kind support and patient guidance during my graduate studies. I would like to thank Dr. B. Grossman, Department Head, Aerospace and Ocean Engineering, for providing considerable computational resources. I would also like to thank Dr. Raymond Plaut (Dept. of Civil and Environmental Engineering) and Dr. Eric Johnson (Dept. of Aerospace and Ocean Engineering) for their valuable discussions and participation as members of my committee during the course of my graduate studies.

Table of Contents

List of Illustrations

List of Tables

Chapter 1: Introduction

Beams have found many applications in civil, mechanical and aerospace engineering. Exact and efficient nonlinear analysis of structures, built up from beam components, using robust numerical methods, e.g. finite element methods, should be based on proper nonlinear beam theories.

Reissner's finite strain beam theory (1972, 1973, 1981) is one of the simplest and most important ones. This theory is based on Timoshenko's plane cross-section assumption, which has been extended, given in detail, and used by many other authors (see e.g. Simo, 1985; Simo and Vu-Quoc, 1986; Cardona and Geradin, 1988; Iura and Atluri, 1988, 1989; Saje, 1991; Simo and Vu-Quoc, 1991; Jelenić and Saje, 1995; Simo *et al*, 1995; Ibrahimbegović, 1995; etc.) for 2-D and 3-D cases for both static and dynamic problems. Most authors dealt with straight beams and used orthonormal reference frames.

For curved/twisted beams, the slender beam or rod assumption is hidden in the beam theories (see e.g. Simo *et al*, 1995; Ibrahimbegović, 1995; etc.).

For truly geometrically exact curved/twisted beam theory for moderate thick beams, Iura and Atluri (1988, 1989), etc., kept the rigid cross-section assumption and used curvilinear reference frames for the formulation.

Certain authors considered shear and torsion warping by the introduction of a simple warping function (see e.g. Simo and Vu-Quoc, 1991; etc.) for straight beams.

For computationally analyzing curved beams or arches, many authors prefer using straight beam elements based on straight beam theories (see e.g. Simo and Vu-Quoc, 1986; Cardona and Geradin, 1988; Ibrahimbegović *et al*, 1995; Franchi and Montelaghi, 1996; etc.). This is a simple and good approximation for slender curved beams or flexible curved beams although more elements will be used to get a satisfactory accuracy. Others prefer using curved beam/arch elements to analyze curved beams or arches based on slender beam theories to reduce the number of elements used (see e.g. Sandhu, Stevens and Davies, 1990; Saje, 1991; Simo *et al*, 1995; Jelenić and Saje, 1995; Ibrahimbegović, 1995).

However, for thick and moderately thick curved beams, an increase in the accuracy of the finite element solution by increasing the number of straight beam elements or curved beam elements based on the slender beam theories has its limit, especially when long-term dynamic responses as well as strains and stresses in three-dimensional level are needed for design purposes. In this case, more refined curved beam theories should be used.

In this thesis, a geometrically exact finite-strain curved and twisted beam theory with large displacements/rotations is re-examined and extended using orthonormal frames and the rigid cross-section assumption. The corresponding finite element formulation/implementation is also dealt with.

In Chapter 2, we summarize the geometric assumptions and geometric descriptions of the curved/twisted beam model. A straight reference beam configuration is introduced in addition to the initially curved/twisted beam and moving beam or current configurations. In Chapter 3, we, for the reader's reference convenience, briefly review some preliminary concepts associated with finite rotations, spatial and material descriptions as well as co-rotated derivatives. In Chapter 4, the strains and stresses for initially curved/twisted beams are reduced from the 3-D to 1-D beam level. The deformation gradient tensor of the initially curved/twisted beam is obtained with the help of the straight reference beam configuration, from which the strain measures, with an initial curvature correction term, at any material point on the beam cross-section are obtained through internal power conjugate analysis. The strain measures conjugate to the first Piola-Kirchhoff stresses are addressed because of the convenience of numerical implementation. Other strains and stresses can be explicitly represented in terms of the first Piola-Kirchhoff stresses.

In Chapter 5, we summarize the Lagrangian equations of motion and the virtual work equation for initially curved/twisted beams for completeness though the final results are similar to those by Iura and Atluri (1988).

In Chapter 6, the general, but time- and rate- independent, linear and nonlinear constitutive relations are considered for the first Piola-Kirchhoff stress and engineering strain conjugate pair to reduce them from the three-dimensional beam level to the corresponding constitutive relations at the one-dimensional beam model.

The geometrically exact finite-strained curved/twisted beam theory summarized here is different from that by Iura and Atluri (1988, 1989) in that (1) Iura and Atluri (1988, 1989) used mixed curvilinear and orthonormal (i.e., Cartesian) frames while we use only orthonormal (i.e., Cartesian) reference frames; (2) Iura and Atluri (1988, 1989) used the fiber length on the beam reference curve as the reference to measure strains at any point on the beam cross-section while we use the local real fiber length to measure strains; therefore, the stress/strain conjugate pair is different; and finally, (3) the resulting constitutive relations for the same kind of materials are different. Note that in this thesis, only orthonormal reference frames or rectangular coordinate systems are used and therefore, only vectors and second-order Cartesian tensors are used.

Of the formulation and implementation of geometrically exact curved beam elements based on the geometrically exact curved and twisted beam theory as described above, the robustness of beam elements is considered in the sense that a beam element can be used to exactly and efficiently predict the linear and nonlinear static responses of the structures built up of beam components no matter if they are straight or curved, slender or thick, as long as the rigid cross-section assumption is valid in practice.

In Chapter 7, a general formulation of geometrically exact nonlinear curved beam elements is made. Three types of simple load models, self-weight, snow and pressure, are considered to illustrate the importance to identify conservative and non-conservative loads in a finite element formulation (see Section 7.1).

The concepts of finite rotations and their parametrizations are summarized in some detail in Section 7.2. The vector-like parametrization of finite rotations is generalized in terms of spatial descriptions.

In Sections 7.3 and 7.4, the admissible variations/linearizations are derived and an element formulation is performed on the continuum setting through the generalized vector-like parametrization of finite rotations, which can recover the case of non-vectorial parametrizations with a proper explanation. From the final formulae for internal/external load and tangential stiffness operators, one may make choices of which are preferred in the beam element implementation from different points of view.

In Chapter 8, a C^0 continuous four-noded 3-D curved beam element is implemented as an example for static problems. In the formulation, a non-vectorial parametrization of finite rotations is adopted taking the advantage of its simplicity and the clear and well-known classical meanings of rotations and moments. Though a non-symmetric tangential stiffness matrix is obtained, it is not a problem if a non-conservative external loading is considered.

The element has the following features:

(i) can be used for 3-D natural curved/twisted slender and thick beams;

(ii) has finite strains and finite rotations;

(iii) can handle classical coupled cross-section material constants with an initial curvature correction term; and

(iv) uses isoparametric Lagrangian interpolation with uniform reduced integration.

Several examples for slender beam cases are presented in Section 8.3 to test the element formulation and its Fortran implementation.

Chapter 2: Geometric assumptions and the geometric representation of the curved/twisted beam

In addition to the geometric assumptions in general nonlinear continuum mechanics (see e.g. Ogden, 1997: 77-83), the geometrically exact beam theory is based on some assumptions.

2.1 Rigid cross-section assumption

The beam theory is based on the *rigid cross-section assumption*: the plane cross-sections of the beam remain plane and do not undergo any shape-change during the deformation (see e.g. Iura and Atluri, 1988; Iura and Atluri, 1989; etc.). The rigid cross-section assumption is 'valid' in practice for slender beams, thin beams, or rods, and in some special cases, also 'valid' for moderately thick beams of many structural materials.

2.2 Quasi-prismatic beam assumption

The contours or shapes of different beam cross-sections are *not* necessarily exactly the *same* but *similar* to each other in the sense of local *affine mapping*[1]. They may vary smoothly (continuously) and slowly along the beam axis. This assumption is *not* really a *restriction* on, but a *removal* of, the previous *uniform* beam assumption (see e.g. Simo, 1985; Iura and Atluri, 1989; etc.) hidden in the developed beam theories, to address the fact that the beam theory can be used for varying cross-sections as long as the rigid cross-section assumption is 'valid' in practice. As will be seen later, this assumption is *not* explicitly used in the formulation except that the *local reference frames* vary continuously along the beam reference curve and the lateral surface is piece-wise smooth.

2.3 The geometries of the reference beams

The configuration of a physically *unstrained/unstressed curved/twisted beam*, simply called *curved reference beam*, is defined by a smooth and *spatial fixed* [2] *reference curve* with its

[1] Affine mapping consists of translation, rotation, stretch and shearing transformations.

[2] By *spatial fixed* we mean it is *fixed* in an arbitrarily chosen *orthonormal frame* $\mathbf{e}_i$ (i = 1, 2, 3) that has

position vector

$$\boldsymbol{\varphi}_0 = \boldsymbol{\varphi}_0(\xi_1) \in R^3 \quad ^3 \tag{1}$$

parametrized by its *real arclength* coordinate $\xi_1 \equiv s_0 \in [0, L_0]$ with $L_0 \in R^1$ as the total *real* arclength of the reference curve of the *curved reference beam*, and a family of *spatial fixed cross-sections* connected by the reference curve through the *geometry, mass, or elasticity centroids* of the cross-sections, whichever is convenient. Hence, the reference curve is also called *midcurve* in the present paper.

Consider the local *orthonormal* (i.e., Cartesian) frame $\mathbf{t}_{0i}$, which is *rigidly attached* onto the cross-section at $\xi_1 \equiv s_0 \in [0, L_0]$ with its origin at $\boldsymbol{\varphi}_0(\xi_1)$ and the orthonormal base vectors:

$$\mathbf{t}_{0i} \quad = \quad \mathbf{t}_{0i}(\xi_1) = t_{0ij}\mathbf{e}_j \in R^3; \quad \mathbf{t}_{0i} \cdot \mathbf{t}_{0j} = \delta_{ij} \tag{2}$$

referred to the arbitrarily chosen *spatial frame* $\mathbf{e}_i$ spanned by the orthonormal base vectors $\mathbf{e}_i \in R^3$ and $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$. In the above and later, the indices $i, j = 1, 2, 3$, the summation convention holds for repeated indices i and j; and δ_{ij} denotes the Kronecker delta[4].

Without losing generality, let the base vector $\mathbf{t}_{01}$ be *normal to* and base vectors $\mathbf{t}_{02}$, $\mathbf{t}_{03}$ *in* the cross-section plane. Since the curved reference beam configuration is in an *unstrained and unstressed* state, it is conventionally assumed that the cross-section plane of the curved reference beam is normal to the unit tangent vector, $\boldsymbol{\varphi}_{0,\xi_1}$, of the midcurve at $\xi_1 \equiv s_0 \in [0, L_0]$: $\boldsymbol{\varphi}_{0,\xi_1} = \mathbf{t}_{01}$, and $\boldsymbol{\varphi}_{0,\xi_1} \cdot \mathbf{t}_{0i} = \delta_{1i}$, where $(\cdot)_{,x} = \frac{\partial(\cdot)}{\partial x}$ for any variable $x \in R^1$ throughout the paper.

On the other hand, the position vector $\boldsymbol{\xi}_0 = \boldsymbol{\xi}_0(\xi_1, \xi_2, \xi_3) \in R^3$ of any *material point* (ξ_1, ξ_2, ξ_3) on the cross-section of the curved reference beam configuration can be described by

$$\boldsymbol{\xi}_0 = \boldsymbol{\varphi}_0 + \xi_2\mathbf{t}_{02} + \xi_3\mathbf{t}_{03} \tag{3}$$

no acceleration nor rotation in the 3-D *inertial physical space*. We also call $\mathbf{e}_i$ the spatial frame.

[3] R^n denotes both the Euclidean point space and Euclidean vector space of dimension n for convenience. However, one should be careful of the difference between them, especially the difference between a position vector and a free-ended vector.

[4] $\delta_{ij} = 1 \quad \text{for} \quad i = j, \quad \delta_{ij} = 0 \quad \text{for} \quad i \neq j$

where ξ_2 and ξ_3 are the coordinates along base vectors $\mathbf{t}_{02}$ and $\mathbf{t}_{03}$, respectively: $(\xi_2, \xi_3) \in \mathcal{B}$ and $\mathcal{B} = \mathcal{B}(\xi_1)$ is the compact subset of R^2 that gives the shape and size of the beam cross-section, and it may vary *smoothly* and *slowly* along the *material point* on the midcurve $\xi_1 \equiv s_0 \in [0, L_0]$ but invariant of any deformation and motion. Note that we have called (ξ_1, ξ_2, ξ_3) for $\xi_1 \equiv s_0 \in [0, L_0]$ and $(\xi_2, \xi_3) \in \mathcal{B}(\xi_1)$ the *material point* of the beam, which both *labels* a material point or particle and gives the coordinate values along the unstrained/unstressed *curved reference beam* midcurve and the unit base vectors $\mathbf{t}_{02}$ and $\mathbf{t}_{03}$. The material point (ξ_1, ξ_2, ξ_3) for $\xi_1 \equiv s_0 \in [0, L_0]$ and $(\xi_2, \xi_3) \in \mathcal{B}(\xi_1)$ is independent of any deformation and motion for a given *curved reference beam*, though its position vector may change during deformation and motion. When a material point is on the midcurve of a beam, $\xi_2 = \xi_3 = 0$.

For later convenience, we define a spatial fixed *straight* (untwisted) *reference beam* configuration whose midcurve is given by the position vector

$$\boldsymbol{\varphi}_{00} = \xi_1 \mathbf{e}_1 \tag{4}$$

with its arclength coordinate s_{00} and total arclength L_{00} exactly the same as in the unstrained and unstressed state:

$$s_{00} = \xi_1 \equiv s_0; \quad L_{00} \equiv \quad L_0 \tag{5}$$

and the local frame $\mathbf{t}_{00i}$ of the cross-section is

$$\mathbf{t}_{00i} \equiv \mathbf{E}_i = \mathbf{e}_i \tag{6}$$

Hence, the position vector $\boldsymbol{\xi}_{00} = \boldsymbol{\xi}_{00}(\xi_1, \xi_2, \xi_3)$ of any *material point* (ξ_1, ξ_2, ξ_3) for $\xi_1 \equiv s_0 \in [0, L_0]$ and $(\xi_2, \xi_3) \in \mathcal{B}(\xi_1)$ on the cross-section can be described by

$$\boldsymbol{\xi}_{00} \quad = \quad \boldsymbol{\varphi}_{00} + \xi_2 \mathbf{t}_{002} + \xi_3 \mathbf{t}_{003} = \xi_1 \mathbf{E}_1 + \xi_2 \mathbf{E}_2 + \xi_3 \mathbf{E}_3 \tag{7}$$

We call $\mathbf{E}_i$ the *material frame* in the current context, which is coincident with the *spatial frame* $\mathbf{e}_i$. A *material point* (ξ_1, ξ_2, ξ_3) referred to this *material frame* is the simplest choice because of the orthonormality condition.

Because both $\mathbf{E}_i$ and $\mathbf{t}_{0i}$ are orthonormal frames, there exists an orthogonal tensor $\boldsymbol{\Lambda}_0 = \boldsymbol{\Lambda}_0(\xi_1) \in SO(3)$[5] relating $\mathbf{t}_{0i}$ to $\mathbf{E}_i$ by

$$\mathbf{t}_{0i} \equiv \boldsymbol{\Lambda}_0 \mathbf{E}_i \quad \Leftrightarrow \quad \boldsymbol{\Lambda}_0 \;\equiv\; \mathbf{t}_{0i} \otimes \mathbf{E}_i = t_{0ij}\mathbf{e}_j \otimes \mathbf{E}_i = t_{0ji}\mathbf{e}_i \otimes \mathbf{E}_j = \Lambda_{0ij}\mathbf{e}_i \otimes \mathbf{E}_j \tag{8}$$

where $\otimes$ denotes the standard tensor product[6]. One observes that the component form of the orthogonal tensor $\boldsymbol{\Lambda}_0$, referred to base $\mathbf{e}_i \otimes \mathbf{E}_j$, is

$$[\boldsymbol{\Lambda}_0]_{\mathbf{e}_i \otimes \mathbf{E}_j} \;=\; [\Lambda_{0ij}]_{\mathbf{e}_i \otimes \mathbf{E}_j} = \begin{bmatrix} t_{011} & t_{021} & t_{031} \\ t_{012} & t_{022} & t_{032} \\ t_{013} & t_{023} & t_{033} \end{bmatrix} \tag{9}$$

Therefore, considering $t_{0ij} = \mathbf{t}_{0i} \cdot \mathbf{e}_j$, the directional cosine of $\mathbf{t}_{0i}$ with respect to $\mathbf{e}_j$, the orthogonal tensor $\boldsymbol{\Lambda}_0$ determines the *orientation* of the cross-section of the *curved reference beam* configuration. For this reason, we may call the orthogonal tensor $\boldsymbol{\Lambda}_0 = \mathbf{t}_{0i} \otimes \mathbf{E}_i$ the *orientation tensor* of the *curved reference beam* cross-section. For a straight and untwisted beam, $\boldsymbol{\Lambda}_0$ is constant along the beam midcurve, while it varies for a general curved beam, leading to the major differences between a straight beam and a curved/twisted beam.

For the *straight reference beam* configuration, the corresponding orthogonal *orientation tensor* is

$$\boldsymbol{\Lambda}_{00} \equiv \mathbf{t}_{00i} \otimes \mathbf{E}_i = \mathbf{E}_i \otimes \mathbf{E}_i = \mathbf{I}_3 = \mathbf{v}_i \otimes \mathbf{v}_i; \quad \mathbf{v}_i \in R^3 \quad \text{and} \quad \mathbf{v}_i \cdot \mathbf{v}_j = \delta_{ij} \tag{10}$$

where $\mathbf{I}_3$ is the second-order identity tensor of dimension three.

[5] $SO(3)$ denotes the special group of *proper* orthogonal tensors of order two, whose component representations, referred to a given orthonormal frame, are a set of proper three-by-three orthogonal matrices. *Proper* means that the determinant of the orthogonal tensor or matrix is 1. All the reference frames in this paper are orthonormal, whose mutually orthogonal *unit* base vectors follow the conventional *right hand rule*. Therefore, only *proper* orthogonal tensors are used.

[6] For any four vectors $\mathbf{v}_1 = v_{1j}\mathbf{e}_j, \quad \mathbf{v}_2 = v_{2j}\mathbf{e}_j, \quad \mathbf{v}_3 = v_{3j}\mathbf{e}_j, \quad \mathbf{v}_4 = v_{4j}\mathbf{e}_j \in R^3$, $\mathbf{v}_1 \otimes \mathbf{v}_2$ represents a second-order tensor, whose component representation, referred to the base $\mathbf{e}_i \otimes \mathbf{e}_j$, is $[\mathbf{v}_1 \otimes \mathbf{v}_2]_{\mathbf{e}_i \otimes \mathbf{e}_j} = [v_{1i}v_{2j}]$, a three-by-three matrix. Two important identities exist: $(\mathbf{v}_1 \otimes \mathbf{v}_2)\mathbf{v}_3 = (\mathbf{v}_2 \cdot \mathbf{v}_3)\mathbf{v}_1$ and $(\mathbf{v}_1 \otimes \mathbf{v}_2)(\mathbf{v}_3 \otimes \mathbf{v}_4) = (\mathbf{v}_2 \cdot \mathbf{v}_3)(\mathbf{v}_1 \otimes \mathbf{v}_4)$.

2.4 The geometry, elongation and shearing of the moving beam

Assume that the beam moves and deforms from the *curved reference beam configuration* at time t_0 to the *current beam* configuration at time t.

The position vector of the *material point* $\xi_1 \equiv s_0 \in [0, L_0]$ on the *beam midcurve* moves from $\boldsymbol{\varphi}_0 \in R^3$ at time t_0 to $\boldsymbol{\varphi} \in R^3$ at time t through a *translational displacement* $\mathbf{u} = \mathbf{u}(\xi_1) \in R^3$, i.e.,

$$\boldsymbol{\varphi} = \boldsymbol{\varphi}(\xi_1) = \boldsymbol{\varphi}_0 + \mathbf{u} \tag{11}$$

and during the *same* deformation and motion process, the local frame of the *moving beam* is rotated, along with the rigid cross-section, from $\mathbf{t}_{0i} = \mathbf{t}_{0i}(\xi_1) \in R^3$ at time t_0 to $\mathbf{t}_i = \mathbf{t}_i(\xi_1) \in R^3$ at time t, which stays orthogonal and unitary $(\mathbf{t}_i \cdot \mathbf{t}_j = \delta_{ij})$ through the so-called orthogonal *rotation tensor* $\boldsymbol{\Lambda}_r = \boldsymbol{\Lambda}_r(\xi_1) \in SO(3)$ as follows:

$$\begin{aligned} \mathbf{t}_i &\equiv \boldsymbol{\Lambda}_r \mathbf{t}_{0i} = \boldsymbol{\Lambda}_r \boldsymbol{\Lambda}_0 \mathbf{E}_i \equiv \boldsymbol{\Lambda} \mathbf{E}_i \\ &\Leftrightarrow \\ \boldsymbol{\Lambda}_r &\equiv \mathbf{t}_i \otimes \mathbf{t}_{0i}; \quad \boldsymbol{\Lambda} \equiv \mathbf{t}_i \otimes \mathbf{E}_i \quad \in SO(3) \end{aligned} \tag{12}$$

where

$$\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\xi_1) \equiv \boldsymbol{\Lambda}_r \boldsymbol{\Lambda}_0 \tag{13}$$

Considering $\mathbf{t}_i = t_{ij}\mathbf{e}_j$, one observes that the component representation of the orthogonal tensor $\boldsymbol{\Lambda}$, referred to $\mathbf{e}_i \otimes \mathbf{E}_j$, is

$$[\boldsymbol{\Lambda}]_{\mathbf{e}_i \otimes \mathbf{E}_j} = [\Lambda_{ij}]_{\mathbf{e}_i \otimes \mathbf{E}_j} = \begin{bmatrix} t_{11} & t_{21} & t_{31} \\ t_{12} & t_{22} & t_{32} \\ t_{13} & t_{23} & t_{33} \end{bmatrix} \tag{14}$$

Therefore, the orthogonal tensor $\boldsymbol{\Lambda}$ determines the *orientation* of the moving beam cross-section at time t. Similar to $\boldsymbol{\Lambda}_0$, we may call the orthogonal tensor $\boldsymbol{\Lambda} = \mathbf{t}_i \otimes \mathbf{E}_i$ the *orientation tensor* of the *moving or current beam* cross-section.

Note that the arc-element ds of the current beam midcurve corresponding to the material point at $\xi_1 \equiv s_0 \in [0, L_0]$ on the curved reference beam midcurve is

$$ds = \mathcal{J} d\xi_1 \tag{15}$$

where $\mathcal{J} = \mathcal{J}(\xi_1) \equiv ||\boldsymbol{\varphi}_{,\xi_1}||$ and the L_2-norm $||\mathbf{v}|| = (\mathbf{v} \cdot \mathbf{v})^{\frac{1}{2}}$ for any vector $\mathbf{v} \in R^n$. Then the *elongation* e or *extension ratio* of the midcurve of the moving beam at time t is defined by

$$e = e(\xi_1) = \frac{ds}{ds_0} - 1 = \mathcal{J} - 1 \tag{16}$$

Thus, the *unit tangent vecto*r of the midcurve of the moving beam at time t corresponding to the material point $\xi_1 \equiv s_0 \in [0, L_0]$ on the beam midcurve is calculated as

$$\boldsymbol{\varphi}_{,s} = \frac{1}{\mathcal{J}} \boldsymbol{\varphi}_{,\xi_1} = \frac{1}{1+e} \boldsymbol{\varphi}_{,\xi_1} \tag{17}$$

Generally, the unit normal vector $\mathbf{t}_1$ of the deformed beam cross-section does not coincide with the unit tangent vector $\boldsymbol{\varphi}_{,s}$ because of the shearing; the angle changes between the tangent vector of the midcurve and $\mathbf{t}_1$ and away from orthogonal to $\mathbf{t}_2$ and $\mathbf{t}_3$ are the angles of shearing, denoted by γ_{1i} and determined by (also see e.g. Iura and Atluri, 1989; etc.)

$$\begin{aligned} \boldsymbol{\varphi}_{,s} \cdot \mathbf{t}_1 &= \cos\gamma_{11} \\ \boldsymbol{\varphi}_{,s} \cdot \mathbf{t}_2 &= \cos(\frac{\pi}{2} - \gamma_{12}) = \sin\gamma_{12} \\ \boldsymbol{\varphi}_{,s} \cdot \mathbf{t}_3 &= \cos(\frac{\pi}{2} - \gamma_{13}) = \sin\gamma_{13} \end{aligned} \tag{18}$$

or equivalently,

$$\begin{aligned} \frac{1}{1+e} \boldsymbol{\varphi}_{,\xi_1} \cdot \mathbf{t}_1 &= \cos\gamma_{11} \\ \frac{1}{1+e} \boldsymbol{\varphi}_{,\xi_1} \cdot \mathbf{t}_2 &= \sin\gamma_{12} \\ \frac{1}{1+e} \boldsymbol{\varphi}_{,\xi_1} \cdot \mathbf{t}_3 &= \sin\gamma_{13} \end{aligned} \tag{19}$$

At time t_0, the moving beam coincides with the curved reference beam. Similarly, we may rewrite Eq. (19) for the curved reference beam though the corresponding elongation and shearing vanish:

$$\frac{1}{1+e_0} \boldsymbol{\varphi}_{0,\xi_1} \cdot \mathbf{t}_{01} = \cos\gamma_{011}$$

$$
\begin{aligned}
\frac{1}{1+e_0}\boldsymbol{\varphi}_{0,\xi_1} \cdot \mathbf{t}_{02} &= \sin\gamma_{012} \\
\frac{1}{1+e_0}\boldsymbol{\varphi}_{0,\xi_1} \cdot \mathbf{t}_{03} &= \sin\gamma_{013} \\
e_0 = 0 \quad &; \quad \gamma_{01i} = 0
\end{aligned} \tag{20}
$$

Also, the position vector $\boldsymbol{\xi} = \boldsymbol{\xi}(\xi_1, \xi_2, \xi_3) \in R^3$ of any material point (ξ_1, ξ_2, ξ_3) for $\xi_1 \equiv s_0 \in [0, L_0]$ and $(\xi_2, \xi_3) \in \mathcal{B}(\xi_1)$ on the moving beam cross-section at time t can be described by

$$
\boldsymbol{\xi} = \boldsymbol{\varphi} + \xi_2 \mathbf{t}_2 + \xi_3 \mathbf{t}_3 \tag{21}
$$

For the *reduced beam* or *one-dimensional beam*, the moving beam configuration at time t can be completely determined by the position vector $\boldsymbol{\varphi}$ and orthogonal orientation tensor $\boldsymbol{\Lambda}$ of the beam cross-section as (see e.g. Simo, 1985; Simo and Vu-Quoc, 1986; etc.)

$$
\boldsymbol{\Phi} = \boldsymbol{\Phi}(\xi_1) \equiv (\boldsymbol{\varphi}, \boldsymbol{\Lambda}) \quad \text{for} \quad \xi_1 \equiv s_0 \in [0, L_0] \tag{22}
$$

at any time t, whose special cases are

$$
\boldsymbol{\Phi}_0 = \boldsymbol{\Phi}_0(\xi_1) \equiv (\boldsymbol{\varphi}_0, \boldsymbol{\Lambda}_0) \quad \text{for} \quad \xi_1 \equiv s_0 \in [0, L_0] \tag{23}
$$

at time t_0 when it *coincides* with the spatial fixed *curved reference beam* configuration, and

$$
\boldsymbol{\Phi}_{00} = \boldsymbol{\Phi}_{00}(\xi_1) \equiv (\boldsymbol{\varphi}_{00}, \boldsymbol{\Lambda}_{00}) \quad \text{for} \quad \xi_1 \equiv s_0 \in [0, L_0] \tag{24}
$$

at 'time' t_{00} when it *coincides* with the spatial fixed *straight reference beam* configuration.

Note again that both the *curved* and *straight reference beam* configurations are spatially fixed and independent of time though the moving beam configuration may coincide with these by respectively taking the *pre-* subscripts '0' and '00'. Note that $\xi_1 \equiv s_0 \equiv s_{00}$.

The three configurations of a spatially curved beam are shown in Fig. 1. The *right* end cross-section is 'cut' through the midcurve at a material point $\xi_1 \equiv s_0 = [0, L_0]$ and the *left* end corresponds to $\xi_1 \equiv s_0 = 0$.

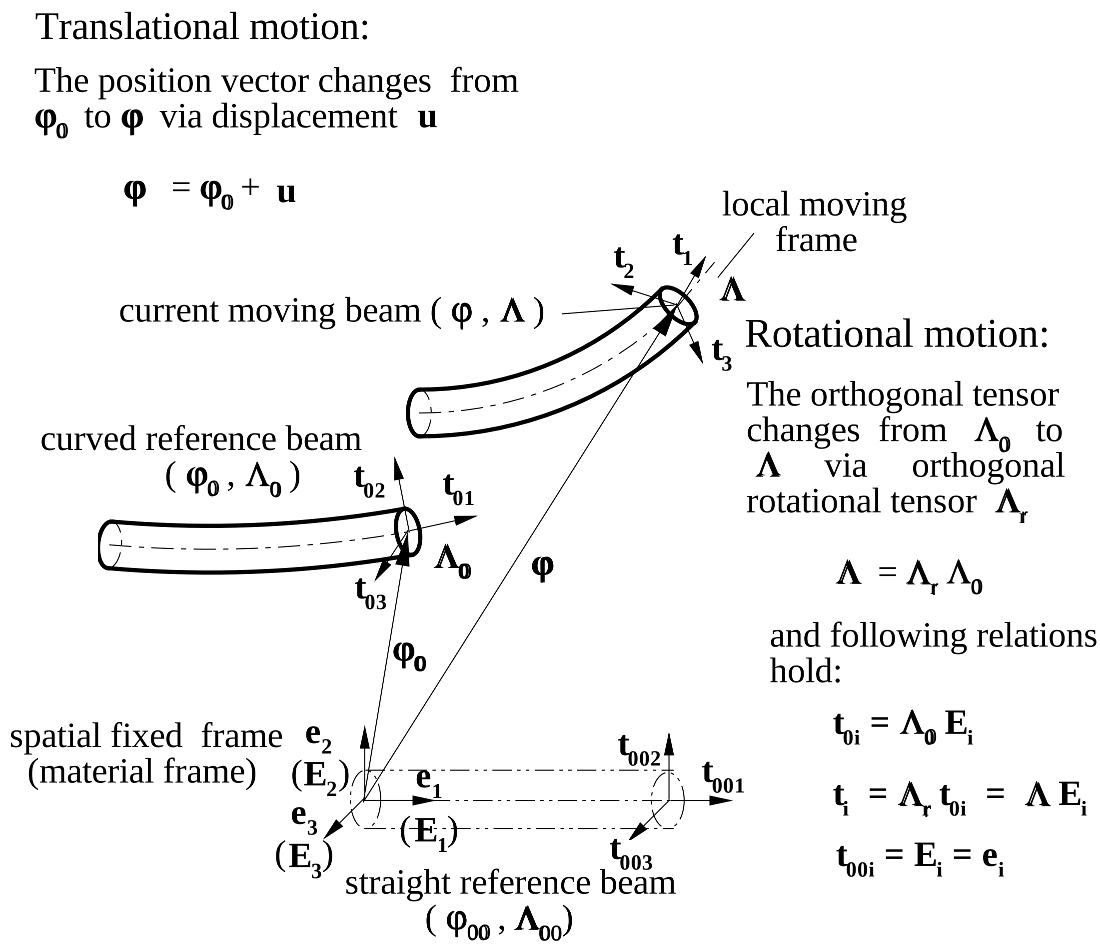

Fig. 1   A 3-D curved beam

Chapter 3: Preliminaries

In this Chapter, we review and summarize the definitions of some basic concepts associated with *finite rotations* so that the geometrically exact curved beam theory under the *rigid cross-section assumption* can be easily handled.

Mostly, we focus on the objects of the *same material point* of the moving beam, either on the midcurve which is parametrized by $\xi_1 \equiv s_0 \in [0, L_0]$, or any point on the cross-section $(\xi_1, \xi_2, \xi_3) \in R^3$ for $\xi_1 \equiv s_0 \in [0, L_0]$ and $(\xi_2, \xi_3) \in \mathcal{B}(\xi_1)$, whichever is applicable.

3.1 Finite rotations by orthogonal transformations

In Eq. (12), we have defined the orthogonal *rotation tensor* $\mathbf{\Lambda}_r$, which *rotates* the local frame of the moving beam from $\mathbf{t}_{0i}$ at time t_0 to $\mathbf{t}_i$ at time t through an *orthogonal transformation* (see e.g. Argyris, 1982; Simo, 1985; Atluri and Cazzani, 1995; etc.) $\mathbf{t}_i = \mathbf{\Lambda}_r \mathbf{t}_{0i}$.

Though the orthogonal *orientation tensors* $\mathbf{\Lambda}_0$, as defined in Eq. (8), and $\mathbf{\Lambda}$, as defined in Eqs. (12) and (13), determine the *orientations* of the local frames of the unstrained/unstressed *curved reference beam* and *current beam* configurations, respectively, we may also view them as 'rotation' tensors that '*rotate*' the local frame of the moving beam from $\mathbf{t}_{00i} = \mathbf{E}_i$ at 'time' t_{00}, to $\mathbf{t}_{0i}$ at time t_0 and to $\mathbf{t}_i$ at time t, respectively. In this sense and also from Eq. (13) , we may consider $\mathbf{\Lambda} = \mathbf{\Lambda}_r \mathbf{\Lambda}_0$ as a *compound* 'rotation', which is the result of the 'rotation' $\mathbf{\Lambda}_0$ of the moving frame from $\mathbf{t}_{00i} = \mathbf{E}_i$ at the 'time' t_{00} to $\mathbf{t}_{0i}$ at time t_0 and then a following rotation $\mathbf{\Lambda}_r$ of the moving beam from $\mathbf{t}_{0i}$ at time t_0 to $\mathbf{t}_i$ at time t.

It follows that if we consider a finite *incremental rotation* of the moving frame from $\mathbf{t}_i$ at time t by the orthogonal *incremental rotation tensor* $\mathbf{\Lambda}_{rw} \in SO(3)$ to $\mathbf{t}'_i$ at time t', then we have

$$\mathbf{t}'_i \;=\; \mathbf{\Lambda}_{rw} \mathbf{t}_i = \mathbf{\Lambda}_{rw} \mathbf{\Lambda}_r \mathbf{t}_{0i} = \mathbf{\Lambda}_{rw} \mathbf{\Lambda}_r \mathbf{\Lambda}_0 \mathbf{E}_i \equiv \mathbf{\Lambda}'_r \mathbf{t}_{0i} \equiv \mathbf{\Lambda}' \mathbf{E}_i \tag{25}$$

where

$$\mathbf{\Lambda}'_r = \mathbf{\Lambda}_{rw}\mathbf{\Lambda}_r \in SO(3) \tag{26}$$

is the *updated compound rotation* tensor that *rotates* the moving frame from $\mathbf{t}_{0i}$ at time t_0 to $\mathbf{t}'_i$ at time t' as a compound result (in fact, $\mathbf{\Lambda}_r$ is also a *compound* rotation tensor); and the updated orthogonal *orientation tensor*

$$\mathbf{\Lambda}' \;\; = \;\; \mathbf{\Lambda}_{rw}\mathbf{\Lambda} = \mathbf{\Lambda}_{rw}\mathbf{\Lambda}_r\mathbf{\Lambda}_0 \in SO(3) \tag{27}$$

represents the *orientation* of the moving frame at time t' or the 'rotation' of the moving frame from $\mathbf{t}_{00i} = \mathbf{E}_i$ at 'time' t_{00} to $\mathbf{t}'_i$ at time t', as a sequence of rotations, $\mathbf{\Lambda}_0$, $\mathbf{\Lambda}_r$ and $\mathbf{\Lambda}_{rw}$. For an *updated* rotation tensor, *the reverse-ordered multiplicative procedure* as in Eq. (26) should be used *when we observe the rotations referred to the fixed spatial frame* $\mathbf{e}_i$ (see e.g. Argyris, 1982; etc.).

3.2 Spatial and material descriptions

The objects of a beam relating finite rotations are strongly *orientation dependent* because of the geometric constraint resulting from the rigid cross-section assumption. It helps to define the *spatial and material descriptions* of an object.

An arbitrary vector $\mathbf{v} \in R^3$ of the beam induced by the 'translation', 'rotation' and 'deformation' away from the *straight reference beam* to the *moving beam* at any time t is a *spatial object* (see e.g. Simo, 1985; Simo and Vu-Quoc, 1986; and the references therein), and may be referred to either the spatial frame $\mathbf{e}_i$ or the moving frame $\mathbf{t}_i$ as

$$\mathbf{v} = v_j\mathbf{e}_j = \bar{v}_j\mathbf{t}_j \tag{28}$$

which is called the *spatial form* of the spatial vector $\mathbf{v}$. On the other hand, $\mathbf{v}$ can be further written as $\mathbf{v} = \bar{v}_j\mathbf{\Lambda}\mathbf{E}_j = \mathbf{\Lambda}\bar{v}_j\mathbf{E}_j = \mathbf{\Lambda}\bar{\mathbf{v}}$ with $\mathbf{\Lambda}$ as the corresponding *orientation tensor*; thus, we have

$$\bar{\mathbf{v}} = \bar{v}_j\mathbf{E}_j = \mathbf{\Lambda}^t\mathbf{v} \tag{29}$$

which is called the *material form* of the spatial vector $\mathbf{v}$ (see e.g. Simo, 1985; Simo and Vu-Quoc, 1986). The *superscript* t of $(\cdot)^t$ means the transpose of a vector, tensor or matrix $(\cdot)$ throughout the thesis.

The special cases of spatial vectors are the local base vectors, $\mathbf{t}_i$ at any time t, of the moving beam, and its corresponding material form is

$$\bar{\mathbf{t}}_i = \mathbf{\Lambda}^t \mathbf{t}_i = \mathbf{E}_i \tag{30}$$

For two arbitrary spatial vectors $\mathbf{v}' \in R^3$ at any time t' and $\mathbf{v}'' \in R^3$ at any time t'' of the same *material point* of the *moving beam* with $\mathbf{\Lambda}'$ and $\mathbf{\Lambda}''$ as their corresponding *orientation tensors*, respectively, if there exists a linear transformation relation given by

$$\mathbf{v}'' = \mathbf{T}\mathbf{v}' \tag{31}$$

the corresponding transformation relation in the *material form* is given by

$$\bar{\mathbf{v}}'' = \bar{\mathbf{T}}\bar{\mathbf{v}}' \tag{32}$$

where $\bar{\mathbf{v}}' = (\mathbf{\Lambda}')^t \mathbf{v}'$ and $\bar{\mathbf{v}}'' = (\mathbf{\Lambda}'')^t \mathbf{v}''$ are the *material forms* of $\mathbf{v}'$ and $\mathbf{v}''$, respectively; $\mathbf{T}$ and $\bar{\mathbf{T}}$ are the second-order tensors in the *spatial* and *material forms*, respectively. It is easy to confirm that the following relations hold:

$$\mathbf{T} = \mathbf{\Lambda}''\bar{\mathbf{T}}(\mathbf{\Lambda}')^t; \qquad \bar{\mathbf{T}} = (\mathbf{\Lambda}'')^t \mathbf{T}\mathbf{\Lambda}' \tag{33}$$

The special cases of spatial second-order tensors are the orthogonal tensors $\mathbf{\Lambda}_r$ and $\mathbf{\Lambda}$ of the moving beam at any time t which define the transformations $\mathbf{t}_i = \mathbf{\Lambda}_r \mathbf{t}_{0i}$ and $\mathbf{t}_i = \mathbf{\Lambda}\mathbf{E}_i$, respectively, from Eq. (12). Their corresponding material forms are

$$\bar{\mathbf{t}}_i = \bar{\mathbf{\Lambda}}_r \bar{\mathbf{t}}_{0i} = \mathbf{E}_i; \quad \text{and} \quad \bar{\mathbf{t}}_i = \bar{\mathbf{\Lambda}}\bar{\mathbf{E}}_i = \mathbf{E}_i \tag{34}$$

where $\bar{\mathbf{\Lambda}}_r = \mathbf{\Lambda}^t \mathbf{\Lambda}_r \mathbf{\Lambda}_0 = \mathbf{I}_3;$ and $\bar{\mathbf{\Lambda}} = \mathbf{\Lambda}^t \mathbf{\Lambda}\mathbf{\Lambda}_{00} = \mathbf{I}_3;$

The above notations (upper bar "–" for a material object and no upper bar for a spatial object) and relations are used throughout the paper.

Notice that the components of a *material object* relating the moving beam at any time referred to the *material frame* $\mathbf{E}_i$ are equal to those of its spatial object referred to its local frame (see e.g. Simo, 1985; Simo and Vu-Quoc, 1986). The same holds for second-order tensors. Every beam configuration has its own material objects. It can be seen later that the advantages of using the material descriptions are: (1) arbitrary rigid-body rotations are removed; (2) the *objectivity* is kept for any *observer transformation* (see e.g. Simo, 1985; Ogden, 1997: 74, 130-139); (3) summation and subtraction may be performed directly for certain objects.

3.3 Linearized increments and derivatives

We specify the notations for *linearized* increments and derivatives of an object.

For an infinitesimal change or increment of some parameter $x \in R^1$, denoted by Δx, the corresponding linearized or infinitesimal increment of an object $(\cdot)$ will be denoted by $\Delta_x(\cdot)$ (the variational operator $\delta_x(\cdot)$ for the same material point and the differential operator $d(\cdot)$ for a variation of the material point in space are two special examples, and the same operation rules as partial derivatives or differentiations hold); and the corresponding (partial) derivative by

$$(\cdot)_{,x} = \frac{\partial(\cdot)}{\partial x} = \frac{\Delta_x(\cdot)}{\Delta x} \quad \text{and} \quad (\cdot)_{,xy} = [(\cdot)_{,x}]_{,y} = \frac{\partial^2(\cdot)}{\partial x \partial y} = \frac{\Delta_y[\frac{\Delta_x(\cdot)}{\Delta x}]}{\Delta y}$$

for $y \in R^1$.

3.4 'Spin' and rotational 'speed' as a general term

We consider the *linearized* increment $\Delta_x \mathbf{\Lambda}$ of the orthogonal tensor $\mathbf{\Lambda}$ (which determines the orientation of the moving beam at any point of the midcurve at any time t) due to an infinitesimal increment Δx. Using the *chain rule* of partial derivatives for linearized operations, we have

$$\Delta_x(\mathbf{\Lambda}\mathbf{\Lambda}^t) \;\; = \;\; \Delta_x\mathbf{\Lambda}\mathbf{\Lambda}^t + \mathbf{\Lambda}\Delta_x\mathbf{\Lambda}^t = \Delta_x(\mathbf{I}_3) = \mathbf{0} \qquad (35)$$

and *define*

$$\Delta_x \hat{\mathbf{w}} \;\; \equiv \;\; \Delta_x \mathbf{\Lambda}\mathbf{\Lambda}^t \equiv \mathbf{\Lambda}\Delta_x \hat{\bar{\mathbf{w}}}\mathbf{\Lambda}^t; \quad (\Delta_x \hat{\bar{\mathbf{w}}} \equiv \mathbf{\Lambda}^t \Delta_x \mathbf{\Lambda}) \tag{36}$$

then

$$\Delta_x \hat{\mathbf{w}}^t \;\; = \;\; \mathbf{\Lambda}\Delta_x \mathbf{\Lambda}^t = -\Delta_x \hat{\mathbf{w}}; \quad (\Delta_x \hat{\bar{\mathbf{w}}}^t = -\Delta_x \hat{\bar{\mathbf{w}}}) \tag{37}$$

Therefore, (the '*spin*' of the *moving beam* relative to the *straight reference beam* in the linearized incremental form due to Δx) is a *skew symmetric tensor* (see e.g. Simo, 1985; Simo and Vu-Quoc, 1986; Iura and Atluri, 1989; etc.), $\Delta_x \hat{\mathbf{w}} \in so(3)$[7] ($\Delta_x \hat{\bar{\mathbf{w}}} \in so(3)$) in the spatial (material) form, associated with the spatial (material) axial vector $\Delta_x \mathbf{w} \in R^3$ ($\Delta_x \bar{\mathbf{w}} \in R^3$) (the infinitesimal or linearized incremental rotation vector of the *moving beam* relative to the *straight reference beam*). In the *derivative* form,

$$\hat{\boldsymbol{\omega}}_x \;\; \equiv \;\; \hat{\mathbf{w}}_{,x} \equiv \frac{\Delta_x \hat{\mathbf{w}}}{\Delta x} \equiv \mathbf{\Lambda}_{,x}\mathbf{\Lambda}^t = \mathbf{\Lambda}\hat{\bar{\boldsymbol{\omega}}}_x \mathbf{\Lambda}^t \in so(3); \quad (\hat{\bar{\boldsymbol{\omega}}}_x \equiv \mathbf{\Lambda}\mathbf{\Lambda}_{,x} \in so(3)) \tag{38}$$

which is called '*spin*' (see e.g. Simo, 1985; Simo and Vu-Quoc, 1986; Iura and Atluri, 1989; etc.), as x varies, of the *moving beam* relative to the *straight reference beam*, associated with the spatial (material) axial vector

$$\boldsymbol{\omega}_x \;\; = \;\; \bar{\omega}_{xj}\mathbf{t}_j \equiv \mathbf{w}_{,x} \in R^3; \quad (\bar{\boldsymbol{\omega}}_x = \bar{\omega}_{xj}\mathbf{E}_j \equiv \mathbf{\Lambda}^t \boldsymbol{\omega}_x \in R^3) \tag{39}$$

which is the *orientation change rate* (angular or rotational 'speed') vector of the *moving beam* relative to the *straight reference beam* as the parameter x varies (with x as 'time' parameter).

[7] $so(3)$ denotes the group of skew symmetric tensors. A second-order Cartesian tensor $\hat{\mathbf{v}}$ is a skew symmetric tensor if and only if $\hat{\mathbf{v}} = -\hat{\mathbf{v}}^t$, whose associated axial vector $\mathbf{v} \in R^3$ is its only eigenvector corresponding to zero eigenvalue $\hat{\mathbf{v}}\mathbf{v} = \mathbf{0}$. For another arbitrary vector $\mathbf{v}_1 \in R^3$, an identity exists: $\mathbf{v} \times \mathbf{v}_1 = \hat{\mathbf{v}}\mathbf{v}_1$. The component representation of the skew symmetric tensor $\hat{\mathbf{v}}$ corresponding to an axial vector $\mathbf{v} = v_j \mathbf{e}_j$, referred to $\mathbf{e}_i \otimes \mathbf{e}_j$, is

$$[\hat{\mathbf{v}}]_{\mathbf{e}_i \otimes \mathbf{e}_j} = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}$$

An important identity exists along with the standard tensor product: $\mathbf{v} \otimes \mathbf{v}_1 = (\mathbf{v} \cdot \mathbf{v}_1)^2 \mathbf{I}_3 + (\hat{\mathbf{v}}\hat{\mathbf{v}}_1)^t$. The above notation with an *upper head* for the skew symmetric tensor of a vector is valid throughout the paper.

Noticing $\mathbf{\Lambda} = \mathbf{\Lambda}_r \mathbf{\Lambda}_0$ according to Eq. (13) and using the chain rule of partial derivatives, we have

$$\mathbf{\Lambda}_{,x} = \mathbf{\Lambda}_{r,x}\mathbf{\Lambda}_0 + \mathbf{\Lambda}_r \mathbf{\Lambda}_{0,x} \tag{40}$$

Thus,

$$\mathbf{\Lambda}_{,x}\mathbf{\Lambda}^t \;=\; \mathbf{\Lambda}_{r,x}\mathbf{\Lambda}_0\mathbf{\Lambda}^t + \mathbf{\Lambda}_r\mathbf{\Lambda}_{0,x}\mathbf{\Lambda}^t = \mathbf{\Lambda}_{r,x}\mathbf{\Lambda}_r^t + \mathbf{\Lambda}_r\mathbf{\Lambda}_{0,x}\mathbf{\Lambda}_0^t\mathbf{\Lambda}_r^t \tag{41}$$

Therefore (for skew symmetric curvature case, see e.g. Simo, 1985; Simo and Vu-Quoc, 1986; Ibrahimbegović, 1995),

$$\hat{\boldsymbol{\omega}}_x \;=\; \hat{\boldsymbol{\omega}}_{rx} + \mathbf{\Lambda}_r \hat{\boldsymbol{\omega}}_{0x}\mathbf{\Lambda}_r^t \tag{42}$$

where

$$\hat{\boldsymbol{\omega}}_{rx} \;\equiv\; \hat{\mathbf{w}}_{r,x} = \mathbf{\Lambda}_{r,x}\mathbf{\Lambda}_r^t \in so(3); \quad (\hat{\bar{\boldsymbol{\omega}}}_{rx} = \mathbf{\Lambda}^t\hat{\boldsymbol{\omega}}_{rx}\mathbf{\Lambda} = \mathbf{\Lambda}^t\mathbf{\Lambda}_{,x} - \mathbf{\Lambda}_0^t\mathbf{\Lambda}_{0,x} \in so(3)) \tag{43}$$

is the incremental '*spin*', as x varies, in the spatial (material) form of the *moving beam* at any time t relative to the *curved reference beam* associated with the spatial (material) axial vector $\boldsymbol{\omega}_{rx} \in R^3$ ($\bar{\boldsymbol{\omega}}_{rx} \in R^3$) while

$$\hat{\boldsymbol{\omega}}_{0x} \;\equiv\; \hat{\mathbf{w}}_{0,x} = \mathbf{\Lambda}_{0,x}\mathbf{\Lambda}_0^t \in so(3); \quad (\hat{\bar{\boldsymbol{\omega}}}_{0x} = \mathbf{\Lambda}_0^t\hat{\boldsymbol{\omega}}_{0x}\mathbf{\Lambda}_0 = \mathbf{\Lambda}_0^t\mathbf{\Lambda}_{0,x} \in so(3)) \tag{44}$$

is viewed as the '*spin*', as x varies, in the spatial (material) form of the *curved reference beam* at any time t relative to the *straight reference beam* associated with the spatial (material) axial vector $\boldsymbol{\omega}_{0x} \in R^3$ ($\bar{\boldsymbol{\omega}}_{0x} \in R^3$).

Eq. (42) implies that the 'spin' of the current beam relative to the straight reference beam *cannot* be obtained by a simple addition of an incremental 'spin' relative to the previous beam orientation in their *spatial* forms; it needs to *align* the 'spin' of the previous orientation to the current orientation by applying the corresponding *relative rigid-body rotation.* The associated axial vector relation for Eq. (42) is

$$\boldsymbol{\omega}_x = \boldsymbol{\omega}_{rx} + \mathbf{\Lambda}_r\boldsymbol{\omega}_{0x} \tag{45}$$

However, one may easily confirm that a simple addition *can* be performed in their *material* forms:

$$\hat{\bar{\boldsymbol{\omega}}}_x = \hat{\bar{\boldsymbol{\omega}}}_{rx} + \hat{\bar{\boldsymbol{\omega}}}_{0x}; \quad \bar{\boldsymbol{\omega}}_x = \bar{\boldsymbol{\omega}}_{rx} + \bar{\boldsymbol{\omega}}_{0x} \tag{46}$$

Note that when $x \equiv t$ the physical *time* or *configuration transformation* due to deformation and motion,

$$\boldsymbol{\omega}_{0t} \equiv \boldsymbol{\omega}_{0x}|_{x=t} = \mathbf{0}; \quad (\bar{\boldsymbol{\omega}}_{0t} \equiv \bar{\boldsymbol{\omega}}_{0x}|_{x=t} = \mathbf{0}) \tag{47}$$

hence,

$$\boldsymbol{\omega}_t \equiv \boldsymbol{\omega}_x|_{x=t} = \boldsymbol{\omega}_{rt} \equiv \boldsymbol{\omega}_{rx}|_{x=t} \tag{48}$$

because the *curved reference beam* is *spatially fixed* and independent of the current time t! We use an *upper dot* to denote the *true* time derivative throughout the paper; thus, the time spin is

$$\dot{\hat{\mathbf{w}}} \equiv \hat{\mathbf{w}}_{,t} = \dot{\boldsymbol{\Lambda}}\boldsymbol{\Lambda}^t = \dot{\hat{\mathbf{w}}}_r \equiv \hat{\mathbf{w}}_{r,t} = \dot{\boldsymbol{\Lambda}}_r\boldsymbol{\Lambda}_r^t; \quad \dot{\hat{\mathbf{w}}}_0 \equiv \hat{\mathbf{w}}_{0,t} = \dot{\boldsymbol{\Lambda}}_0\boldsymbol{\Lambda}_0^t = \mathbf{0} \tag{49}$$

It should be mentioned that we have *no* definition for $\hat{\mathbf{w}}$ *nor* for $\mathbf{w}$; they *always* come along with the linearized operator Δ_x in front of them, and just denote an *incremental* 'rotation' as $x \in R^1$ varies.

Conventionally, we let

$$(\Delta_x\hat{\mathbf{w}})_{,y} = \frac{\Delta_y(\Delta_x\hat{\mathbf{w}})}{\Delta y}; \quad \hat{\mathbf{w}}_{,xy} = (\hat{\mathbf{w}}_{,x})_{,y} = \frac{\Delta_y[\frac{\Delta_x\hat{\mathbf{w}}}{\Delta x}]}{\Delta y} \tag{50}$$

but

$$(\Delta_x\hat{\mathbf{w}})_{,y} \neq \Delta_x(\hat{\mathbf{w}}_{,y}); \quad \hat{\mathbf{w}}_{,xy} \neq \hat{\mathbf{w}}_{,yx} \tag{51}$$

in general even though x and y are independent of each other because we have no definition for $\hat{\mathbf{w}}$ nor for $\mathbf{w}$!

3.5 Co-rotated derivatives

For any spatial vector $\mathbf{v} = \bar{v}_i \mathbf{t}_i = \mathbf{\Lambda}\bar{\mathbf{v}} \in R^3$ of the moving beam at any time t, we have, using $\hat{\boldsymbol{\omega}}_x = \mathbf{\Lambda}_{,x}\mathbf{\Lambda}^t$, (see e.g. Simo, 1985; Simo and Vu-Quoc, 1986)

$$\mathbf{v}_{,x} = \mathbf{\Lambda}_{,x}\bar{\mathbf{v}} + \mathbf{\Lambda}\bar{\mathbf{v}}_{,x} = (\mathbf{\Lambda}_{,x}\mathbf{\Lambda}^t)(\mathbf{\Lambda}\bar{\mathbf{v}}) + \mathbf{\Lambda}(\mathbf{\Lambda}^t\mathbf{v})_{,x} = \hat{\boldsymbol{\omega}}_x\mathbf{v} + \mathbf{\Lambda}(\mathbf{\Lambda}^t\mathbf{v})_{,x} \tag{52}$$

and *define* (see e.g. Simo, 1985; Simo and Vu-Quoc, 1986)

$$\tilde{\mathbf{v}}_{,x} \equiv \bar{v}_{i,x}\mathbf{t}_i \equiv \mathbf{\Lambda}(\mathbf{\Lambda}^t\mathbf{v})_{,x} \equiv \mathbf{\Lambda}\bar{\mathbf{v}}_{,x} \equiv \mathbf{v}_{,x} - \hat{\boldsymbol{\omega}}_x\mathbf{v} \equiv \mathbf{v}_{,x} - \boldsymbol{\omega}_x \times \mathbf{v} \tag{53}$$

as the *co-rotated* derivative of the spatial vector $\mathbf{v}$ with respect to x, with *upper tilde* to identify it throughout the paper.

The meaning of the *co-rotated* derivative is that the derivative of a spatial object is taken by an observer fixed in the moving frame, *only on the components* referred to the corresponding moving frame. To an observer who stays still in the fixed spatial frame, he/she needs to *pull-back* the object to the material form $\bar{\mathbf{v}} = \mathbf{\Lambda}^t\mathbf{v}$ to perform the usual derivative operation and then *push-forward* (see e.g. Simo, 1985; Simo and Vu-Quoc, 1986; and the references therein) to the spatial form $\mathbf{\Lambda}\bar{\mathbf{v}}_{,x}$; or equivalently, he/she needs to subtract the *spin effect* $\boldsymbol{\omega}_x \times \mathbf{v}$ from the usual derivative $\mathbf{v}_{,x}$ (see e.g. Simo, 1985; Simo and Vu-Quoc, 1986) to have the same *objective* observation as by the observer fixed in the moving frame.

A valuable result of application of the co-rotated derivative of a spatial vector to the current frame $\mathbf{t}_i$ is

$$\tilde{\mathbf{t}}_{i,x} = \mathbf{t}_{i,x} - \hat{\boldsymbol{\omega}}_x\mathbf{t}_i = \mathbf{\Lambda}(\mathbf{\Lambda}^t\mathbf{t}_i)_{,x} = \mathbf{\Lambda}(\mathbf{E}_i)_{,x} = \mathbf{0} \tag{54}$$

i.e. a set of important formulae:

$$\begin{aligned}
\mathbf{t}_{i,x} &= \hat{\boldsymbol{\omega}}_x\mathbf{t}_i = \boldsymbol{\omega}_x \times \mathbf{t}_i \\
\mathbf{t}_{1,x} &= \bar{\omega}_{x3}\mathbf{t}_2 - \bar{\omega}_{x2}\mathbf{t}_3 \\
\mathbf{t}_{2,x} &= -\bar{\omega}_{x3}\mathbf{t}_1 + \bar{\omega}_{x1}\mathbf{t}_3 \\
\mathbf{t}_{3,x} &= \bar{\omega}_{x2}\mathbf{t}_1 - \bar{\omega}_{x1}\mathbf{t}_2
\end{aligned} \tag{55}$$

which may be called Frenet-Serret formula in a general sense.

On the other hand, for any spatial second-order tensor $\mathbf{T} = \bar{T}_{ij}\mathbf{t}''_i \otimes \mathbf{t}'_j$ that defines a transformation, $\mathbf{v}'' = \mathbf{T}\mathbf{v}'$ between vectors $\mathbf{v}' = \bar{v}'_j\mathbf{t}'_j \in R^3$ at any time t' and $\mathbf{v}'' = \bar{v}''_i\mathbf{t}''_i \in R^3$ at any time t'' of the moving beam associated with the orientations $\mathbf{\Lambda}' = \mathbf{t}'_i \otimes \mathbf{E}_i \in SO(3)$ and $\mathbf{\Lambda}'' = \mathbf{t}''_i \otimes \mathbf{E}_i \in SO(3)$ of the same material point of the beam and the corresponding material objects as in Eqs. (31) through (33), one may *define* the corresponding *co-rotated* derivative as

$$\tilde{\mathbf{T}}_{,x} \equiv \bar{T}_{ij,x}\mathbf{t}''_i \otimes \mathbf{t}'_j \equiv \mathbf{\Lambda}''[(\mathbf{\Lambda}'')^t\mathbf{T}\mathbf{\Lambda}']_{,x}(\mathbf{\Lambda}')^t \equiv \mathbf{\Lambda}''\bar{\mathbf{T}}_{,x}(\mathbf{\Lambda}')^t \equiv \mathbf{T}_{,x} - (\hat{\boldsymbol{\omega}}''_x\mathbf{T} - \mathbf{T}\hat{\boldsymbol{\omega}}'_x) \quad (56)$$

and the chain rule holds for the *co-rotated* derivative operation:

$$\tilde{\mathbf{v}}''_{,x} = \tilde{\mathbf{T}}_{,x}\mathbf{v}' + \mathbf{T}\tilde{\mathbf{v}}'_{,x} \quad (57)$$

Chapter 4: Strains and stresses

4.1 Curvatures and curvature changes

The *elongation and shearing* on the midcurve have been taken into account *conceptually* in a 'classical' or 'engineering' sense for a moving beam in Section 3.2. That the cross-section rotates away from the orthogonality with the tangent vector of the beam midcurve is considered as shearing, while the *relative orientation change, as the material point* $\xi_1 \equiv s_0 \in [0, L_0]$ *varies along the beam midcurve, of the cross-sections at the same material point on the midcurve between the moving beam and the curved reference beam determines the curvature change of a curved beam.*

Taking the derivatives of local frames $\mathbf{t}_{0i}$ and $\mathbf{t}_i$, respectively, with respect to the arclength coordinate ξ_1 of the (unstrained and unstressed) *curved reference beam* midcurve and noticing Eq. (55) (also see e.g. Iura and Atluri, 1988; Iura and Atluri, 1989; etc.), we may obtain the Frenet-Serret formulae in their original sense

$$\begin{aligned}
\mathbf{t}_{0i,\xi_1} &= \hat{\boldsymbol{\kappa}}_0 \mathbf{t}_{0i} = \boldsymbol{\kappa}_0 \times \mathbf{t}_{0i} \\
\mathbf{t}_{01,\xi_1} &= \bar{\kappa}_{03} \mathbf{t}_{02} - \bar{\kappa}_{02} \mathbf{t}_{03} \\
\mathbf{t}_{02,\xi_1} &= -\bar{\kappa}_{03} \mathbf{t}_{01} + \bar{\kappa}_{01} \mathbf{t}_{03} \\
\mathbf{t}_{03,\xi_1} &= \bar{\kappa}_{02} \mathbf{t}_{01} - \bar{\kappa}_{01} \mathbf{t}_{02}
\end{aligned} \tag{58}$$

for the *curved reference beam* configuration, and the generalized Frenet-Serret formulae

$$\begin{aligned}
\mathbf{t}_{i,\xi_1} &= \hat{\boldsymbol{\kappa}} \mathbf{t}_i = \boldsymbol{\kappa} \times \mathbf{t}_i \\
\mathbf{t}_{1,\xi_1} &= \bar{\kappa}_3 \mathbf{t}_2 - \bar{\kappa}_2 \mathbf{t}_3 \\
\mathbf{t}_{2,\xi_1} &= -\bar{\kappa}_3 \mathbf{t}_1 + \bar{\kappa}_1 \mathbf{t}_3 \\
\mathbf{t}_{3,\xi_1} &= \bar{\kappa}_2 \mathbf{t}_1 - \bar{\kappa}_1 \mathbf{t}_2
\end{aligned} \tag{59}$$

for the *current beam*; where, noticing Eq. (43)

$$\hat{\boldsymbol{\kappa}}_0 \equiv \hat{\boldsymbol{\omega}}_{0\xi_1} = \hat{\mathbf{w}}_{0,\xi_1} = \boldsymbol{\Lambda}_{0,\xi_1} \boldsymbol{\Lambda}_0^t$$

$$\hat{\bar{\boldsymbol{\kappa}}}_0 \equiv \hat{\bar{\boldsymbol{\omega}}}_{0\xi_1} = \boldsymbol{\Lambda}_0^t \hat{\boldsymbol{\kappa}}_0 \boldsymbol{\Lambda}_0 = \boldsymbol{\Lambda}_0^t \boldsymbol{\Lambda}_{0,\xi_1} \tag{60}$$

and

$$\begin{aligned} \hat{\boldsymbol{\kappa}} &\equiv \hat{\boldsymbol{\omega}}_{\xi_1} = \hat{\mathbf{w}}_{,\xi_1} = \boldsymbol{\Lambda}_{,\xi_1} \boldsymbol{\Lambda}^t = \hat{\boldsymbol{\kappa}}_r + \boldsymbol{\Lambda}_r \hat{\boldsymbol{\kappa}}_0 \boldsymbol{\Lambda}_r^t \\ \hat{\bar{\boldsymbol{\kappa}}} &\equiv \hat{\bar{\boldsymbol{\omega}}}_{\xi_1} = \boldsymbol{\Lambda}^t \hat{\boldsymbol{\kappa}} \boldsymbol{\Lambda} = \boldsymbol{\Lambda}^t \boldsymbol{\Lambda}_{,\xi_1} = \hat{\bar{\boldsymbol{\kappa}}}_r + \hat{\bar{\boldsymbol{\kappa}}}_0 \end{aligned} \tag{61}$$

as well as

$$\begin{aligned} \hat{\boldsymbol{\kappa}}_r &\equiv \hat{\boldsymbol{\omega}}_{r\xi_1} = \hat{\mathbf{w}}_{r,\xi_1} = \boldsymbol{\Lambda}_{r,\xi_1} \boldsymbol{\Lambda}_r^t \\ \hat{\bar{\boldsymbol{\kappa}}}_r &\equiv \hat{\bar{\boldsymbol{\omega}}}_{r\xi_1} = \boldsymbol{\Lambda}^t \hat{\boldsymbol{\kappa}}_r \boldsymbol{\Lambda} = \boldsymbol{\Lambda}^t \boldsymbol{\Lambda}_{,\xi_1} - \boldsymbol{\Lambda}_0^t \boldsymbol{\Lambda}_{0,\xi_1} \end{aligned} \tag{62}$$

where, noticing Section 3.4 about 'spin' as a skew symmetric tensor and rotational 'speed' as the associated axial vector as well as the corresponding notations and operation rules, $\hat{\boldsymbol{\kappa}}_0$, $\hat{\boldsymbol{\kappa}}$ and $\hat{\boldsymbol{\kappa}}_r \in so(3)$ ($\hat{\bar{\boldsymbol{\kappa}}}_0$, $\hat{\bar{\boldsymbol{\kappa}}}$ and $\hat{\bar{\boldsymbol{\kappa}}}_r \in so(3)$) are the skew symmetric tensors in their spatial forms (material forms) associated with the axial vectors $\boldsymbol{\kappa}_0$, $\boldsymbol{\kappa}$ and $\boldsymbol{\kappa}_r \in R^3$ ($\bar{\boldsymbol{\kappa}}_0$, $\bar{\boldsymbol{\kappa}}$ and $\bar{\boldsymbol{\kappa}}_r \in R^3$) , respectively, in their spatial forms (material forms). Note that

$$\begin{aligned} \boldsymbol{\kappa}_0 &\equiv \boldsymbol{\omega}_{0\xi_1} = \mathbf{w}_{0,\xi_1} = \kappa_{0j} \mathbf{e}_j = \bar{\kappa}_{0j} \mathbf{t}_{0j} \\ \boldsymbol{\kappa} &\equiv \boldsymbol{\omega}_{\xi_1} = \mathbf{w}_{,\xi_1} = \kappa_j \mathbf{e}_j = \bar{\kappa}_j \mathbf{t}_j \\ \boldsymbol{\kappa}_r &\equiv \boldsymbol{\omega}_{r\xi_1} = \mathbf{w}_{r,\xi_1} = \kappa_{rj} \mathbf{e}_j = \bar{\kappa}_{rj} \mathbf{t}_j \end{aligned} \tag{63}$$

for the spatial forms, and

$$\begin{aligned} \bar{\boldsymbol{\kappa}}_0 &\equiv \bar{\boldsymbol{\omega}}_{0\xi_1} = \bar{\kappa}_{0j} \mathbf{E}_j = \boldsymbol{\Lambda}_0^t \boldsymbol{\kappa}_0 \\ \bar{\boldsymbol{\kappa}} &\equiv \bar{\boldsymbol{\omega}}_{\xi_1} = \bar{\kappa}_j \mathbf{E}_j = \boldsymbol{\Lambda}^t \boldsymbol{\kappa} \\ \bar{\boldsymbol{\kappa}}_r &\equiv \bar{\boldsymbol{\omega}}_{r\xi_1} = \bar{\kappa}_{rj} \mathbf{E}_j = \boldsymbol{\Lambda}^t \boldsymbol{\kappa}_r = \bar{\boldsymbol{\kappa}} - \bar{\boldsymbol{\kappa}}_0 \end{aligned} \tag{64}$$

for the material forms.

$\boldsymbol{\kappa}_0$ ($\bar{\boldsymbol{\kappa}}_0$) is the curvature (vector) of the (unstrained/unstressed) *curved reference beam* configuration in the spatial (material) form, denoting the 'rotational speed' or *orientation change rate* of the cross-section with respect to ξ_1 as the material point $\xi_1 \equiv s_0 \in [0, L_0]$ varies along the beam midcurve. The component $\bar{\kappa}_{01}$ is the *twist rate* around the tangent vector $\mathbf{t}_{01}$, and $\bar{\kappa}_{02}$ and $\bar{\kappa}_{03}$ are the corresponding *curvature* components around $\mathbf{t}_{02}$ and $\mathbf{t}_{03}$, respectively.

Similarly, we also call $\boldsymbol{\kappa}$ ($\bar{\boldsymbol{\kappa}}$) the curvature (vector) of the current beam in the spatial (material) form in a general term though it denotes the 'rotational speed' or *orientation change rate* of the cross-section of the current beam with respect to ξ_1 *not* $s \in [0, L]$ as the material point $\xi_1 \equiv s_0 \in [0, L_0]$ varies along the *curved reference beam* midcurve. The component $\bar{\kappa}_1$ is the *twist rate* around the *normal* vector $\mathbf{t}_1$, and the components $\bar{\kappa}_2$ and $\bar{\kappa}_3$ are the *curvatures* around $\mathbf{t}_2$ and $\mathbf{t}_3$ of the current beam, respectively.

Therefore, we further call $\boldsymbol{\kappa}_r$ ($\bar{\boldsymbol{\kappa}}_r$) the curvature change (vector) of the *current beam relative to the curved reference beam* in the spatial (material) form and it denotes the 'rotational speed' of the current beam cross-section relative to the curved reference beam cross-section with respect to ξ_1 as the material point $\xi_1 \equiv s_0 \in [0, L_0]$ varies along the beam reference midcurve. The component $\bar{\kappa}_{r1}$ is the *twist rate change* around the *normal* vector $\mathbf{t}_1$, and the components $\bar{\kappa}_{r2}$ and $\bar{\kappa}_{r3}$ are the *curvature changes* around $\mathbf{t}_2$ and $\mathbf{t}_3$ of the current beam relative to the curved reference beam, respectively.

On the other hand, $\hat{\boldsymbol{\kappa}}_0$, $\hat{\boldsymbol{\kappa}}$ and $\hat{\boldsymbol{\kappa}}_r$ ($\hat{\bar{\boldsymbol{\kappa}}}_0$, $\hat{\bar{\boldsymbol{\kappa}}}$ and $\hat{\bar{\boldsymbol{\kappa}}}_r$) are called the corresponding *skew symmetric curvatures or curvature changes* in their spatial (material) forms.

So far, the *elongation and shearing* on the beam midcurve as well as the *curvature change* have been specified.

4.2 Deformation gradients

The study of the *deformation gradients* helps determine the strain measures at any material point on the current beam cross-section. Simo (1985) (also see e.g. Simo and Vu-Quoc, 1986; etc.) easily obtained the deformation gradient tensor for initially straight beams simply using orthonormal reference frames by aligning the *straight beam* to coincide with the *straight reference beam configuration*. In the case of the initially curved/twisted beam, Iura and Atluri (1989) used the *covariant and contravariant* reference frames to help obtain the strain measures, in which case the real *fiber length* is not a direct consideration.

In this section, we will indirectly obtain the *deformation gradient tensor* of the *current beam* configuration relative to the (unstrained/unstressed) *curved reference beam* configuration by first obtaining the 'deformation' gradient tensors of the two curved beam configurations relative to the *straight reference beam* configuration followed by a *reference configuration change*, to avoid using the *covariant and contravariant* reference frames or *curvilinear coordinate systems*. We repeat the same procedure as Simo's (1985) for the latter two 'deformation' gradient tensors for completeness.

Noticing Eqs. (3) and (21) for the expressions of the position vectors $\boldsymbol{\xi}_0$ and $\boldsymbol{\xi}$ of any point (ξ_1, ξ_2, ξ_3) for $\xi_1 \in [0, L_0]$ and $(\xi_2, \xi_3) \in \mathcal{B}(\xi_1)$ in the curved reference beam and current beam configurations, as well as the Frenet-Serret formulae in Eqs. (58a) and (59a), respectively, we have

$$\boldsymbol{\xi}_{0,\xi_1} \;=\; \boldsymbol{\varphi}_{0,\xi_1} + \hat{\boldsymbol{\kappa}}_0(\xi_2 \mathbf{t}_{02} + \xi_3 \mathbf{t}_{03}); \quad \boldsymbol{\xi}_{0,\xi_2} = \mathbf{t}_{02}; \quad \boldsymbol{\xi}_{0,\xi_3} = \mathbf{t}_{03} \tag{65}$$

for a point in the curved reference beam, and

$$\boldsymbol{\xi}_{,\xi_1} \;=\; \boldsymbol{\varphi}_{,\xi_1} + \hat{\boldsymbol{\kappa}}(\xi_2 \mathbf{t}_2 + \xi_3 \mathbf{t}_3); \quad \boldsymbol{\xi}_{,\xi_2} = \mathbf{t}_2; \quad \boldsymbol{\xi}_{,\xi_3} = \mathbf{t}_3 \tag{66}$$

for a point in the current beam [8].

Therefore, noticing $\boldsymbol{\Lambda}_0 = \mathbf{t}_{0i} \otimes \mathbf{E}_i$ and $\boldsymbol{\Lambda} = \mathbf{t}_i \otimes \mathbf{E}_i$ as given in Eqs. (8) and (12), respectively, as well as the spatial and material descriptions for vectors and second-order tensors in Section 4.2, the 'deformation' gradient tensors, $\mathbf{F}_0$ and $\mathbf{F}$, of the *curved reference beam* and *current beam* relative to the *straight reference beam*, respectively, are defined and determined by

$$\begin{aligned} d\boldsymbol{\xi}_0 &\equiv \mathbf{F}_0 d\boldsymbol{\xi}_{00} \\ \mathbf{F}_0 &= \boldsymbol{\xi}_{0,\xi_j} \otimes \mathbf{E}_j = \boldsymbol{\epsilon}_0 \otimes \mathbf{E}_1 + \mathbf{t}_{0i} \otimes \mathbf{E}_i = \bar{\epsilon}_{0i} \mathbf{t}_{0i} \otimes \mathbf{E}_1 + \mathbf{t}_{0i} \otimes \mathbf{E}_i = \boldsymbol{\epsilon}_0 \otimes \mathbf{E}_1 + \boldsymbol{\Lambda}_0 = \boldsymbol{\Lambda}_0 \bar{\mathbf{F}}_0 \end{aligned}$$

[8] $\boldsymbol{\xi}_{0,\xi_i}$ and $\boldsymbol{\xi}_{,\xi_i}$ are used as *natural base vectors* for the curved reference beam and current beam, respectively, by some authors (see e.g. Iura and Atluri, 1988; Iura and Atluri, 1989; etc.).

$$\bar{\mathbf{F}}_0 = \bar{\boldsymbol{\epsilon}}_0 \otimes \mathbf{E}_1 + \mathbf{I}_3 = \bar{\epsilon}_{0i}\mathbf{E}_i \otimes \mathbf{E}_1 + \mathbf{E}_i \otimes \mathbf{E}_i \tag{67}$$

and

$$\begin{aligned} d\boldsymbol{\xi} &\equiv \mathbf{F} d\boldsymbol{\xi}_{00} \\ \mathbf{F} &= \boldsymbol{\xi}_{,\xi_j} \otimes \mathbf{E}_j = \boldsymbol{\epsilon} \otimes \mathbf{E}_1 + \mathbf{t}_i \otimes \mathbf{E}_i = \bar{\epsilon}_i \mathbf{t}_i \otimes \mathbf{E}_1 + \mathbf{t}_i \otimes \mathbf{E}_i = \boldsymbol{\epsilon} \otimes \mathbf{E}_1 + \boldsymbol{\Lambda} = \boldsymbol{\Lambda}\bar{\mathbf{F}} \\ \bar{\mathbf{F}} &= \bar{\boldsymbol{\epsilon}} \otimes \mathbf{E}_1 + \mathbf{I}_3 = \bar{\epsilon}_i \mathbf{E}_i \otimes \mathbf{E}_1 + \mathbf{E}_i \otimes \mathbf{E}_i \end{aligned} \tag{68}$$

respectively, where

$$\boldsymbol{\epsilon}_0 \equiv \bar{\epsilon}_{0j}\mathbf{t}_{0j} = \boldsymbol{\varepsilon}_0 + \hat{\boldsymbol{\kappa}}_0(\xi_2\mathbf{t}_{02} + \xi_3\mathbf{t}_{03}); \quad \bar{\boldsymbol{\epsilon}}_0 \equiv \bar{\epsilon}_{0j}\mathbf{E}_j = \bar{\boldsymbol{\varepsilon}}_0 + \hat{\bar{\boldsymbol{\kappa}}}_0(\xi_2\mathbf{E}_2 + \xi_3\mathbf{E}_3) \tag{69}$$

$$\boldsymbol{\epsilon} \equiv \bar{\epsilon}_j\mathbf{t}_j = \boldsymbol{\varepsilon} + \hat{\boldsymbol{\kappa}}(\xi_2\mathbf{t}_{02} + \xi_3\mathbf{t}_{03}); \quad \bar{\boldsymbol{\epsilon}} \equiv \bar{\epsilon}_j\mathbf{E}_j = \bar{\boldsymbol{\varepsilon}} + \hat{\bar{\boldsymbol{\kappa}}}(\xi_2\mathbf{E}_2 + \xi_3\mathbf{E}_3) \tag{70}$$

and

$$\begin{aligned} \boldsymbol{\varepsilon}_0 &\equiv \bar{\varepsilon}_{0j}\mathbf{t}_{0j} = \boldsymbol{\varphi}_{0,\xi_1} - \mathbf{t}_{01}; \quad \bar{\boldsymbol{\varepsilon}}_0 \equiv \bar{\varepsilon}_{0j}\mathbf{E}_j = \boldsymbol{\Lambda}_0^t\boldsymbol{\varphi}_{0,\xi_1} - \mathbf{E}_1 \\ \boldsymbol{\varepsilon} &\equiv \bar{\varepsilon}_j\mathbf{t}_{0j} = \boldsymbol{\varphi}_{,\xi_1} - \mathbf{t}_1; \quad \bar{\boldsymbol{\varepsilon}} \equiv \bar{\varepsilon}_j\mathbf{E}_j = \boldsymbol{\Lambda}^t\boldsymbol{\varphi}_{,\xi_1} - \mathbf{E}_1 \end{aligned} \tag{71}$$

Noticing Eqs. (20) and (19), the relations between the elongations and shearings for the curved beam reference beam and current beam configurations, we obtain the components of $\boldsymbol{\varepsilon}_0 = \boldsymbol{\varepsilon}_0(\xi_1)$ and $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\xi_1)$ referred to their local frames:

$$\begin{aligned} \bar{\varepsilon}_{0j} &= \boldsymbol{\varphi}_{0,\xi_1} \cdot \mathbf{t}_{0j} - \mathbf{t}_{01} \cdot \mathbf{t}_{0j} = 0 \\ \bar{\varepsilon}_{01} &= \boldsymbol{\varphi}_{0,\xi_1} \cdot \mathbf{t}_{01} - 1 = (1+e_0)\cos\gamma_{011} - 1 \\ \bar{\varepsilon}_{02} &= \boldsymbol{\varphi}_{0,\xi_1} \cdot \mathbf{t}_{02} = (1+e_0)\sin\gamma_{012} \\ \bar{\varepsilon}_{03} &= \boldsymbol{\varphi}_{0,\xi_1} \cdot \mathbf{t}_{03} = (1+e_0)\sin\gamma_{013} \\ e_0 &= 0; \quad \gamma_{01i} = 0 \end{aligned} \tag{72}$$

and

$$\begin{aligned} \bar{\varepsilon}_j &= \boldsymbol{\varphi}_{,\xi_1} \cdot \mathbf{t}_j - \mathbf{t}_1 \cdot \mathbf{t}_j \\ \bar{\varepsilon}_1 &= \boldsymbol{\varphi}_{,\xi_1} \cdot \mathbf{t}_1 - 1 = (1+e)\cos\gamma_{11} - 1 \\ \bar{\varepsilon}_2 &= \boldsymbol{\varphi}_{,\xi_1} \cdot \mathbf{t}_2 = (1+e)\sin\gamma_{12} \\ \bar{\varepsilon}_3 &= \boldsymbol{\varphi}_{,\xi_1} \cdot \mathbf{t}_3 = (1+e)\sin\gamma_{13} \end{aligned} \tag{73}$$

Then, the components of $\boldsymbol{\epsilon}_0 = \boldsymbol{\epsilon}_0(\xi_1, \xi_2, \xi_3)$ and $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}(\xi_1, \xi_2, \xi_3)$, in Eqs. (69) and (70), referred to their local frames will be, noticing Eqs. (58) and (59),

$$\begin{aligned}
\bar{\epsilon}_{01} &= \bar{\varepsilon}_{01} + (\xi_3 \bar{\kappa}_{02} - \xi_2 \bar{\kappa}_{03}) \\
\bar{\epsilon}_{02} &= \bar{\varepsilon}_{02} + (-\xi_3 \bar{\kappa}_{01}) \\
\bar{\epsilon}_{03} &= \bar{\varepsilon}_{03} + (\xi_2 \bar{\kappa}_{01})
\end{aligned} \tag{74}$$

and

$$\begin{aligned}
\bar{\epsilon}_{1} &= \bar{\varepsilon}_{1} + (\xi_3 \bar{\kappa}_{2} - \xi_2 \bar{\kappa}_{3}) \\
\bar{\epsilon}_{2} &= \bar{\varepsilon}_{2} + (-\xi_3 \bar{\kappa}_{1}) \\
\bar{\epsilon}_{3} &= \bar{\varepsilon}_{3} + (\xi_2 \bar{\kappa}_{1})
\end{aligned} \tag{75}$$

respectively.

Note that the component representations of $\mathbf{F}_0$ and $\mathbf{F}$ in the *spatial forms* as well as $\bar{\mathbf{F}}$ and $\bar{\mathbf{F}}_0$ in the *material forms* can be easily identified from Eqs. (67) and (68) as

$$\begin{aligned}
[\mathbf{F}_0]_{\mathbf{t}_{0i} \otimes \mathbf{E}_j} &= [\bar{\mathbf{F}}_0]_{\mathbf{E}_i \otimes \mathbf{E}_j} \\
&= \begin{bmatrix} 1 + \bar{\epsilon}_{01} & 0 & 0 \\ \bar{\epsilon}_{02} & 1 & 0 \\ \bar{\epsilon}_{03} & 0 & 1 \end{bmatrix}
\end{aligned} \tag{76}$$

and

$$\begin{aligned}
[\mathbf{F}]_{\mathbf{t}_{i} \otimes \mathbf{E}_j} &= [\bar{\mathbf{F}}]_{\mathbf{E}_i \otimes \mathbf{E}_j} \\
&= \begin{bmatrix} 1 + \bar{\epsilon}_{1} & 0 & 0 \\ \bar{\epsilon}_{2} & 1 & 0 \\ \bar{\epsilon}_{3} & 0 & 1 \end{bmatrix}
\end{aligned} \tag{77}$$

respectively. After some simple elementary linear algebra, the determinants of $\mathbf{F}_0$ and $\mathbf{F}$ can be easily obtained as

$$g_0 \equiv \det\mathbf{F}_0 = (\det\boldsymbol{\Lambda}_0)(\det\bar{\mathbf{F}}_0) = \det\bar{\mathbf{F}}_0 = 1 + \bar{\epsilon}_{01} = 1 + \bar{\varepsilon}_{01} + (\xi_3 \bar{\kappa}_{02} - \xi_2 \bar{\kappa}_{03}) \tag{78}$$

and

$$g \equiv \det\mathbf{F} = (\det\boldsymbol{\Lambda})(\det\bar{\mathbf{F}}) = \det\bar{\mathbf{F}} = 1 + \bar{\epsilon}_1 = 1 + \bar{\varepsilon}_1 + (\xi_3\bar{\kappa}_2 - \xi_2\bar{\kappa}_3) \tag{79}$$

respectively, and the inverses of $\mathbf{F}_0$ and $\mathbf{F}$, respectively, are

$$\begin{aligned}
\mathbf{F}_0^{-1} &= (\boldsymbol{\Lambda}_0\bar{\mathbf{F}}_0)^{-1} \\
&= \bar{\mathbf{F}}_0^{-1}\boldsymbol{\Lambda}_0^t = (-\frac{1}{g_0}\bar{\boldsymbol{\epsilon}}_0 \otimes \mathbf{E}_1 + \mathbf{I}_3)\boldsymbol{\Lambda}_0^t = -\frac{1}{g_0}\bar{\boldsymbol{\epsilon}}_0 \otimes \mathbf{t}_{01} + \boldsymbol{\Lambda}_0^t = \boldsymbol{\Lambda}_0^t(-\frac{1}{g_0}\boldsymbol{\epsilon}_0 \otimes \mathbf{t}_{01} + \mathbf{I}_3)
\end{aligned} \tag{80}$$

and, similarly,

$$\mathbf{F}^{-1} = (-\frac{1}{g}\bar{\boldsymbol{\epsilon}} \otimes \mathbf{E}_1 + \mathbf{I}_3)\boldsymbol{\Lambda}^t = -\frac{1}{g}\bar{\boldsymbol{\epsilon}} \otimes \mathbf{t}_1 + \boldsymbol{\Lambda}^t = \boldsymbol{\Lambda}^t(-\frac{1}{g}\boldsymbol{\epsilon} \otimes \mathbf{t}_1 + \mathbf{I}_3) \tag{81}$$

Noticing the definitions of $\mathbf{F}_0$ and $\mathbf{F}$ in Eqs. (67) and (68) as well as (80) and (81), one may define and obtain the *deformation gradient tensor* of the *current beam* relative to the (unstrained/unstressed) *curved reference beam* through a *change of reference configurations* (see e.g. Ogden, 1997: 120) as

$$\begin{aligned}
d\boldsymbol{\xi} &\equiv \mathbf{F}_r d\boldsymbol{\xi}_0 \\
\mathbf{F}_r &= \mathbf{F}\mathbf{F}_0^{-1} = \frac{1}{g_0}\boldsymbol{\epsilon}_r \otimes \mathbf{t}_{01} + \boldsymbol{\Lambda}_r = \frac{1}{g_0}\bar{\epsilon}_{ri}\mathbf{t}_i \otimes \mathbf{t}_{01} + \mathbf{t}_i \otimes \mathbf{t}_{0i} = \boldsymbol{\Lambda}\bar{\mathbf{F}}_r\boldsymbol{\Lambda}_0^t \\
\bar{\mathbf{F}}_r &= \frac{1}{g_0}\bar{\boldsymbol{\epsilon}}_r \otimes \mathbf{E}_1 + \mathbf{I}_3 = \frac{1}{g_0}\bar{\epsilon}_{ri}\mathbf{E}_i \otimes \mathbf{E}_1 + \mathbf{E}_i \otimes \mathbf{E}_i
\end{aligned} \tag{82}$$

where, noticing Eqs. (69) through (75), noticing $\hat{\bar{\boldsymbol{\kappa}}}_r \equiv \bar{\boldsymbol{\kappa}}_r\times$,

$$\begin{aligned}
\bar{\boldsymbol{\epsilon}}_r &\equiv \bar{\epsilon}_{rj}\mathbf{E}_j \\
&\equiv \bar{\boldsymbol{\epsilon}} - \bar{\boldsymbol{\epsilon}}_0 = \bar{\boldsymbol{\varepsilon}}_r + \hat{\bar{\boldsymbol{\kappa}}}_r(\xi_2\mathbf{E}_2 + \xi_3\mathbf{E}_3) \\
&= \bar{\boldsymbol{\varepsilon}}_r + (\xi_3\bar{\kappa}_{r2} - \xi_2\bar{\kappa}_{r3})\mathbf{E}_1 + (-\xi_3\bar{\kappa}_{r1})\mathbf{E}_2 + (\xi_2\bar{\kappa}_{r1})\mathbf{E}_3 \\
\boldsymbol{\epsilon}_r &= \boldsymbol{\Lambda}\bar{\boldsymbol{\epsilon}}_r = \boldsymbol{\epsilon} - \boldsymbol{\Lambda}_r\boldsymbol{\epsilon}_0 = \boldsymbol{\varepsilon}_r + \hat{\boldsymbol{\kappa}}_r(\xi_2\mathbf{t}_2 + \xi_3\mathbf{t}_3) \\
&= \boldsymbol{\varepsilon}_r + (\xi_3\bar{\kappa}_{r2} - \xi_2\bar{\kappa}_{r3})\mathbf{t}_1 + (-\xi_3\bar{\kappa}_{r1})\mathbf{t}_2 + (\xi_2\bar{\kappa}_{r1})\mathbf{t}_3 \\
&= \bar{\epsilon}_{rj}\mathbf{t}_j \\
\bar{\epsilon}_{rj} &\equiv \bar{\epsilon}_j - \bar{\epsilon}_{0j} = \boldsymbol{\xi}_{,\xi_1} \cdot \mathbf{t}_j - \boldsymbol{\xi}_{0,\xi_1} \cdot \mathbf{t}_{0j}
\end{aligned}$$

$$
\begin{aligned}
\bar{\epsilon}_{r1} &= \bar{\varepsilon}_{r1} + (\xi_3 \bar{\kappa}_{r2} - \xi_2 \bar{\kappa}_{r3}) \\
\bar{\epsilon}_{r2} &= \bar{\varepsilon}_{r2} + (-\xi_3 \bar{\kappa}_{r1}) \\
\bar{\epsilon}_{r3} &= \bar{\varepsilon}_{r3} + (\xi_2 \bar{\kappa}_{r1}) \\
\bar{\varepsilon}_{rj} &= \bar{\varepsilon}_j - \bar{\varepsilon}_{0j} = \boldsymbol{\varphi}_{,\xi_1} \cdot \mathbf{t}_j - \boldsymbol{\varphi}_{0,\xi_1} \cdot \mathbf{t}_{0j} = \boldsymbol{\varphi}_{,\xi_1} \cdot \mathbf{t}_j - \delta_{1j}
\end{aligned} \tag{83}
$$

and

$$
\begin{aligned}
\bar{\boldsymbol{\varepsilon}}_r &\equiv \bar{\boldsymbol{\varepsilon}} - \bar{\boldsymbol{\varepsilon}}_0 = \boldsymbol{\Lambda}^t \boldsymbol{\varphi}_{,\xi_1} - \boldsymbol{\Lambda}_0^t \boldsymbol{\varphi}_{0,\xi_1} = \bar{\varepsilon}_{rj} \mathbf{E}_j \\
\boldsymbol{\varepsilon}_r &= \boldsymbol{\Lambda} \bar{\boldsymbol{\varepsilon}}_r = \boldsymbol{\varphi}_{,\xi_1} - \boldsymbol{\Lambda}_r \boldsymbol{\varphi}_{0,\xi_1} = \boldsymbol{\varepsilon} - \boldsymbol{\Lambda}_r \boldsymbol{\varepsilon}_0 = \boldsymbol{\varepsilon} = \boldsymbol{\varphi}_{,\xi_1} - \mathbf{t}_1 = \bar{\varepsilon}_{rj} \mathbf{t}_j
\end{aligned} \tag{84}
$$

for $\varepsilon_{0j} = \bar{\varepsilon}_{0j} = 0$, where the meanings of $\bar{\boldsymbol{\varepsilon}}_r$ and $\boldsymbol{\varepsilon}_r$ are given in Table 1 at the end of this chapter.

The determinant and inverse of $\mathbf{F}_r$ can be obtained as, using Eqs. (78) and (79),

$$
g_r = \det\mathbf{F}_r = \det(\mathbf{F}\mathbf{F}_0^{-1}) = \det\mathbf{F}\det\mathbf{F}_0^{-1} = \frac{\det\mathbf{F}}{\det\mathbf{F}_0} = \frac{g}{g_0} = 1 + \frac{\bar{\epsilon}_{r1}}{g_0} \tag{85}
$$

and, using Eqs. (67) and (81),

$$
\mathbf{F}_r^{-1} = (\mathbf{F}\mathbf{F}_0^{-1})^{-1} = \mathbf{F}_0 \mathbf{F}^{-1} = \boldsymbol{\Lambda}_r^t (-\frac{1}{g} \boldsymbol{\epsilon}_r \otimes \mathbf{t}_1 + \mathbf{I}_3) \tag{86}
$$

respectively.

Note that $\boldsymbol{\epsilon}_0$ ($\bar{\boldsymbol{\epsilon}}_0$) and $\boldsymbol{\epsilon}$ ($\bar{\boldsymbol{\epsilon}}$) are, respectively, the 'strain' vectors at any point on the cross-section and $\boldsymbol{\varepsilon}_0$ ($\bar{\boldsymbol{\varepsilon}}_0$) and $\boldsymbol{\varepsilon}$ ($\bar{\boldsymbol{\varepsilon}}$) are the 'strain' vectors on the beam midcurve for the *curved reference* beam and *current beam* configurations relative to the *straight reference beam* configuration (see e.g. Simo, 1985). They determine the corresponding elongations and shearings relative to the straight reference beam as seen e.g. in Eqs. (72) and (73) for the beam midcurve.

By comparison of the similarity between Eq. (82) and Eq. (68), one may intuitively choose $\boldsymbol{\epsilon}_r/g_0$ ($\bar{\boldsymbol{\epsilon}}_r/g_0$) as the right strain vector of the *current beam* configuration relative to the *curved reference beam* configuration that is conjugate to the corresponding *first* Piola-Kirchhoff stress vector (see Section 5.4).

Now we address the geometrical meaning of g_0, which is the same as that used by Iura and Atluri (1988, 1989). First, by its definition, $g_0 = \det\mathbf{F}_0$, as given in Eq. (78), is the scalar between the differential volumes of the curved reference beam and straight reference beam configurations at any material point (ξ_1, ξ_2, ξ_3) of the beam:

$$\begin{aligned} dV_0 &= g_0 dV_{00} = g_0 d\xi_1 d\xi_2 d\xi_3 \\ dV_{00} &= d\xi_1 d\xi_2 d\xi_3 \end{aligned} \tag{87}$$

where V_0 and V_{00} are the volume domains of the *curved reference beam* and *straight reference beam* configurations, respectively.

Second, a unit-length 'fiber' parallel to $\mathbf{E}_1$ (the normal of the straight reference beam cross-section) at any material point on the cross-section is 'stretched' to be g_0 in the direction of $\mathbf{t}_{01}$ (the normal of the curved reference beam cross-section) if the moving beam moves and deforms from the straight reference beam configuration to the curved reference beam configuration. In fact, on use of Eq. (67) for the 'deformation' gradient tensor $\mathbf{F}_0$ of the curved reference beam relative to the straight reference beam and Eq. (78) for the expression of g_0, we have

$$(\mathbf{F}_0\mathbf{E}_1 d\xi_1)\cdot\mathbf{t}_{01} = (\boldsymbol{\epsilon}_0 + \mathbf{t}_{01})\cdot\mathbf{t}_{01} d\xi_1 = (\boldsymbol{\epsilon}_0\cdot\mathbf{t}_{01} + \mathbf{t}_{01}\cdot\mathbf{t}_{01})d\xi_1 = (\bar{\epsilon}_{01} + 1)d\xi_1 \equiv g_0 d\xi_1 \tag{88}$$

which is dependent on the curvatures of the curved reference beam configuration, but not on the twist for a given point on the cross-section, see Eq. (78). This is a result of the assumption that the beam cross-section remains undeformed during any motion and deformation.

Third, Eq. (88) also implies that any 'cut' slice of the *curved reference beam* through two cross-section planes with differential length $d\xi_1$, at any point on the midcurve, is *linearly tapered* and its thickness in direction $\mathbf{t}_{01}$ varies according to $g_0 d\xi_1$ as the material point varies on the curved beam cross-section. Any fiber parallel to $\mathbf{t}_{01}$ has a real length $g_0 d\xi_1$ at (ξ_1, ξ_2, ξ_3) if a fiber parallel to $\mathbf{t}_{01}$ has a real length $d\xi_1$ at ξ_1 on the midcurve.

From the above explanations, we may be clear that the factor $\frac{1}{g_0}$ in front of $\boldsymbol{\epsilon}_r$ in the deformation gradient tensor $\mathbf{F}_r$ of the current beam relative to the curved reference beam in Eq. (82) is due to the fact that it is taken with respect to the real undeformed fiber length variations, which can also be observed from the definition of $d\boldsymbol{\xi} \equiv \mathbf{F}_r d\boldsymbol{\xi}_0$ in the same set of equations, in which $d\boldsymbol{\xi}_0$ and $d\boldsymbol{\xi} \in R^3$ are the spatial vectors of an oriented differential fiber with real length *before* and *after* deformation in the physical space with the *orthonormal* reference frame $\mathbf{e}_i$ or $\mathbf{t}_i$.

Because of its importance in the truly geometrically exact curved beam theory, we may call g_0 the *initial curvature correction term*, whose effect may be significant for thick and moderately thick curved beams and small for slender beams.

Similar explanations may be made for $g = \det\mathbf{F}$ as defined and given in Eq. (79) and $g_r = \det\mathbf{F}_r$ as in Eq. (85).

4.3 Stresses, stress resultants and stress couples

There are many choices of stress measures (see e.g. Ogden, 1997: 145-168; etc. for their definitions). We only illustrate the Cauchy stress and first and second Piola-Kirchhoff stresses for the curved beam and then represent each of them in terms of the components of the first Piola-Kirchhoff stress and its conjugate strain (which will be confirmed in the next section), along with the Cauchy stress for the conjugate strain analysis in the next section. Then, we define the stress resultant and stress couple in the 'classical' or 'engineering' sense (see e.g. Reissner, 1972; Reissner, 1973; Reissner, 1981; Simo, 1985; Simo and Vu-Quoc, 1986; Iura and Atluri, 1989; etc.).

Denote by $\boldsymbol{\sigma}$ the 'true' and symmetric Cauchy stress tensor referred to a differential volume of the *current beam* at any material point (ξ_1, ξ_2, ξ_3) for $\xi_1 \equiv s_0 \in [0, L_0]$ and $(\xi_2, \xi_3) \in \mathcal{B}(\xi_1)$:

$$\begin{aligned} \boldsymbol{\sigma} &\equiv \boldsymbol{\sigma}_j \otimes \mathbf{t}_j = \bar{\sigma}_{ji} \mathbf{t}_i \otimes \mathbf{t}_j \\ \bar{\boldsymbol{\sigma}} &= \bar{\sigma}_{ji} \mathbf{E}_i \otimes \mathbf{E}_j \end{aligned}$$

$$
\begin{aligned}
\boldsymbol{\sigma}_j &\equiv \bar{\sigma}_{ji}\mathbf{t}_i \\
\bar{\boldsymbol{\sigma}}_j &= \bar{\sigma}_{ji}\mathbf{E}_i \\
\bar{\sigma}_{ij} &\equiv \bar{\sigma}_{ji}
\end{aligned} \tag{89}
$$

where $\boldsymbol{\sigma}_j$ is the associated stress vector acting on the face and referred to the *real* area of the same face of the *current beam* with $\mathbf{t}_j$ as unit normal vector.

Then, on use of Eq. (86) for the inverse of the deformation tensor, $\mathbf{F}_r^{-1} = \boldsymbol{\Lambda}_r^t(-\frac{1}{g}\boldsymbol{\epsilon}_r \otimes \mathbf{t}_1 + \mathbf{I}_3)$, and the relations $g_r = 1 + \frac{\bar{\epsilon}_{rj}}{g_0}$ as given in Eq. (85), the unsymmetric *first* Piola-Kirchhoff stress tensor $\boldsymbol{\sigma}^0$ referred to a differential volume of the *curved reference beam* can be obtained as, according to its definition (see e.g. Ogden, 1997: 85, 153; etc.),

$$
\begin{aligned}
\boldsymbol{\sigma}^0 &\equiv g_r\boldsymbol{\sigma}\mathbf{F}_r^{-t} = \boldsymbol{\sigma}_j^0 \otimes \mathbf{t}_{0j} = \bar{\sigma}_{ji}^0\mathbf{t}_i \otimes \mathbf{t}_{0j} \\
\bar{\boldsymbol{\sigma}}^0 &= \bar{\sigma}_{ji}^0\mathbf{E}_i \otimes \mathbf{E}_j \\
\boldsymbol{\sigma}_j^0 &\equiv \bar{\sigma}_{ji}^0\mathbf{t}_i = g_r\boldsymbol{\sigma}_j - \frac{\bar{\epsilon}_{rj}}{g_0}\boldsymbol{\sigma}_1 \\
\bar{\boldsymbol{\sigma}}_j^0 &= \bar{\sigma}_{ji}^0\mathbf{E}_i \\
\bar{\sigma}_{ji}^0 &= g_r\bar{\sigma}_{ji} - \frac{\bar{\epsilon}_{rj}}{g_0}\bar{\sigma}_{1i} \neq \bar{\sigma}_{ij}^0 \\
\boldsymbol{\sigma}^0\mathbf{F}_r^t &\equiv \mathbf{F}_r(\boldsymbol{\sigma}^0)^t \\
\boldsymbol{\sigma}_1^0 &= \boldsymbol{\sigma}_1
\end{aligned} \tag{90}
$$

where $\boldsymbol{\sigma}_j^0$ is the corresponding stress vector acting on the deformed face in the *current beam* corresponding to the *reference* face normal to $\mathbf{t}_{0j}$ in the *curved reference beam* and referred to the *real* area of the same reference face in the *curved reference beam.* Inversely, we have, noticing Eq. (82) for $\mathbf{F}_r$,

$$
\boldsymbol{\sigma} \equiv \frac{1}{g_r}\boldsymbol{\sigma}^0\mathbf{F}_r^t = \frac{1}{g_r}(\bar{\sigma}_{ji}^0 + \frac{\bar{\epsilon}_{rj}}{g_0}\bar{\sigma}_{1i}^0)\mathbf{t}_i \otimes \mathbf{t}_j = \frac{1}{g_r}(\bar{\sigma}_{ij}^0 + \frac{\bar{\epsilon}_{ri}}{g_0}\bar{\sigma}_{1j}^0)\mathbf{t}_i \otimes \mathbf{t}_j \tag{91}
$$

Similarly, for later reference, the first Piola-Kirchhoff stress tensor $\boldsymbol{\sigma}^{00}$ referred to a differential volume of the *straight reference beam* can be given by

$$
\boldsymbol{\sigma}^{00} \equiv g\boldsymbol{\sigma}\mathbf{F}^{-t} = g_0\boldsymbol{\sigma}_0\mathbf{F}_0^{-t} = \boldsymbol{\sigma}_i^{00} \otimes \mathbf{t}_{00i}
$$

$$
\begin{aligned}
\boldsymbol{\sigma}_1^{00} &= \boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_1^0 \\
\boldsymbol{\sigma}_2^{00} &= g\boldsymbol{\sigma}_2 - \bar{\epsilon}_2\boldsymbol{\sigma}_1 = g_0\boldsymbol{\sigma}_2^0 - \bar{\epsilon}_{02}\boldsymbol{\sigma}_1^0 \\
\boldsymbol{\sigma}_3^{00} &= g\boldsymbol{\sigma}_3 - \bar{\epsilon}_3\boldsymbol{\sigma}_1 = g_0\boldsymbol{\sigma}_3^0 - \bar{\epsilon}_{03}\boldsymbol{\sigma}_1^0 \\
\boldsymbol{\sigma}^{00}\mathbf{F}^t &\equiv \mathbf{F}(\boldsymbol{\sigma}^{00})^t
\end{aligned} \tag{92}
$$

where $\boldsymbol{\sigma}_i^{00}$ is the corresponding stress vector acting on the deformed face in the *current beam* corresponding to the *reference* face normal to $\mathbf{t}_{00i} = \mathbf{E}_i$ in the *straight reference beam* and referred to the *real* area of the same reference face in the *straight reference beam*.

Note that the *first* Piola-Kirchhoff stress vector referred to the cross-section of any beam configuration is the same as the real (Cauchy) stress vector on the cross-section because the cross-section plane remains undeformed.

Besides, the symmetric *second* Piola-Kirchhoff stress tensor (see e.g. Ogden, 1997: 159, 157; etc. for its definition), denoted by $\boldsymbol{\sigma}^{0(2)}$, is given in terms of the *first* Piola-Kirchhoff stress and Cauchy stress components as

$$
\begin{aligned}
\boldsymbol{\sigma}^{0(2)} &\equiv g_r\mathbf{F}_r^{-1}\boldsymbol{\sigma}\mathbf{F}_r^{-t} \equiv \mathbf{F}_r^{-1}\boldsymbol{\sigma}^0 = \boldsymbol{\sigma}_j^{0(2)} \otimes \mathbf{t}_{0j} = \bar{\sigma}_{ji}^{0(2)}\mathbf{t}_{0i} \otimes \mathbf{t}_{0j} \\
\bar{\boldsymbol{\sigma}}^{0(2)} &= \bar{\sigma}_{ji}^{0(2)}\mathbf{E}_i \otimes \mathbf{E}_j \\
\boldsymbol{\sigma}_j^{0(2)} &\equiv \mathbf{F}_r^{-1}\boldsymbol{\sigma}_j^0 = \mathbf{F}_r^{-1}(\bar{\sigma}_{ji}^0\mathbf{t}_i) = \bar{\sigma}_{ji}^0(\mathbf{F}_r^{-1}\mathbf{t}_i) = \bar{\sigma}_{ji}^{0(2)}\mathbf{t}_{0i} \\
\bar{\boldsymbol{\sigma}}_j^{0(2)} &= \bar{\sigma}_{ji}^{0(2)}\mathbf{E}_i \\
\bar{\sigma}_{ij}^{0(2)} &= \bar{\sigma}_{ji}^0 - \frac{\bar{\epsilon}_{ri}}{g_r g_0}\bar{\sigma}_{j1}^0 = \bar{\sigma}_{ij}^0 - \frac{\bar{\epsilon}_{rj}}{g_r g_0}\bar{\sigma}_{i1}^0 \\
&= g_r\bar{\sigma}_{ij} - (\frac{\bar{\epsilon}_{ri}}{g_0}\bar{\sigma}_{1j} + \frac{\bar{\epsilon}_{rj}}{g_0}\bar{\sigma}_{1i}) + \frac{1}{g_r}(\frac{\bar{\epsilon}_{ri}}{g_0})(\frac{\bar{\epsilon}_{rj}}{g_0})\bar{\sigma}_{11}
\end{aligned} \tag{93}
$$

where $\boldsymbol{\sigma}_j^{0(2)}$ is the stress vector acting on the deformed face in the *current beam* corresponding to the reference face normal to $\mathbf{t}_{0j}$ in the *curved reference beam* and referred to the *real* area of the same reference face in the *curved reference beam* ($\boldsymbol{\sigma}_j^0$) and then is *contracted* back to the *curved reference beam*.

It can be seen that the differences among the Cauchy stress and first and second Piola-Kirchhoff stresses are obvious for finite strain problems, though the differences tend to vanish for small strain problems.

For the *reduced beam* or *one-dimensional beam* model, it is convenient to define the stress resultant (internal *force* vector acting on the current beam cross-section) and stress couple (internal *moment* vector acting on the current beam cross-section) in the 'classical' or 'engineering' sense.

The stress resultant $\mathbf{n} = \mathbf{n}(\xi_1) \in R^3$ is defined as

$$\begin{aligned}
\mathbf{n} &\equiv \int_{\mathcal{B}} \boldsymbol{\sigma}^0 \mathbf{t}_{01} d\xi_2 d\xi_3 \equiv \int_{\mathcal{B}} \boldsymbol{\sigma}^{00} \mathbf{t}_{001} d\xi_2 d\xi_3 \equiv \int_{\mathcal{B}} \boldsymbol{\sigma} \mathbf{t}_1 d\xi_2 d\xi_3 \equiv \int_{\mathcal{B}} \mathbf{F}_r \boldsymbol{\sigma}^{0(2)} \mathbf{t}_{01} d\xi_2 d\xi_3 \\
&= \int_{\mathcal{B}} \boldsymbol{\sigma}_1^0 d\xi_2 d\xi_3 = \int_{\mathcal{B}} \boldsymbol{\sigma}_1^{00} d\xi_2 d\xi_3 = \int_{\mathcal{B}} \boldsymbol{\sigma}_1 d\xi_2 d\xi_3 = \bar{n}_i \mathbf{t}_i \\
\bar{n}_i &= \int_{\mathcal{B}} \bar{\sigma}_{1i}^0 d\xi_2 d\xi_3
\end{aligned} \tag{94}$$

and the stress couple $\mathbf{m} = \mathbf{m}(\xi_1) \in R^3$, referred to the position vector $\boldsymbol{\varphi}$ on the current beam reference curve, is defined as

$$\begin{aligned}
\mathbf{m} &\equiv \int_{\mathcal{B}} (\boldsymbol{\xi} - \boldsymbol{\varphi}) \times [\boldsymbol{\sigma}^0 \mathbf{t}_{01}] d\xi_2 d\xi_3 \equiv \int_{\mathcal{B}} (\boldsymbol{\xi} - \boldsymbol{\varphi}) \times [\boldsymbol{\sigma}^{00} \mathbf{t}_{001}] d\xi_2 d\xi_3 \\
&\equiv \int_{\mathcal{B}} (\boldsymbol{\xi} - \boldsymbol{\varphi}) \times [\boldsymbol{\sigma} \mathbf{t}_1] d\xi_2 d\xi_3 \equiv \int_{\mathcal{B}} (\boldsymbol{\xi} - \boldsymbol{\varphi}) \times [\mathbf{F}_r \boldsymbol{\sigma}^{0(2)} \mathbf{t}_{01}] d\xi_2 d\xi_3 \\
&= \int_{\mathcal{B}} (\boldsymbol{\xi} - \boldsymbol{\varphi}) \times \boldsymbol{\sigma}_1^0 d\xi_2 d\xi_3 = \int_{\mathcal{B}} (\boldsymbol{\xi} - \boldsymbol{\varphi}) \times \boldsymbol{\sigma}_1^{00} d\xi_2 d\xi_3 = \int_{\mathcal{B}} (\boldsymbol{\xi} - \boldsymbol{\varphi}) \times \boldsymbol{\sigma}_1 d\xi_2 d\xi_3 \\
&= \bar{m}_i \mathbf{t}_i \\
\bar{m}_1 &= \int_{\mathcal{B}} (\xi_2 \bar{\sigma}_{13}^0 - \xi_3 \bar{\sigma}_{12}^0) d\xi_2 d\xi_3 \\
\bar{m}_2 &= \int_{\mathcal{B}} (\xi_3 \bar{\sigma}_{11}^0) d\xi_2 d\xi_3 \\
\bar{m}_3 &= \int_{\mathcal{B}} (-\xi_2 \bar{\sigma}_{11}^0) d\xi_2 d\xi_3
\end{aligned} \tag{95}$$

in the spatial forms. For $\bar{m}_i$ in the last set of equations, use has been made of $\boldsymbol{\xi} - \boldsymbol{\varphi} = \xi_2 \mathbf{t}_2 + \xi_3 \mathbf{t}_3$ from Eq. (21).

The stress resultant and stress couple in the material forms are

$$\bar{\mathbf{n}} = \bar{\mathbf{n}}(\xi_1) = \boldsymbol{\Lambda}^t \mathbf{n} = \bar{n}_i \mathbf{E}_i \tag{96}$$

and

$$\bar{\mathbf{m}} = \bar{\mathbf{m}}(\xi_1) = \boldsymbol{\Lambda}^t \mathbf{m} = \bar{m}_i \mathbf{E}_i \tag{97}$$

respectively.

In Eqs. (94) through (97), $\bar{n}_1$ is the *normal* force component in the cross-section normal direction $\mathbf{t}_1$ while $\bar{n}_2$ and $\bar{n}_3$ are the shear force components in the directions $\mathbf{t}_2$ and $\mathbf{t}_3$, respectively. On the other hand, $\bar{m}_1$ is the torque component around the cross-section normal $\mathbf{t}_1$ while $\bar{m}_2$ and $\bar{m}_3$ are bending moment components around $\mathbf{t}_2$ and $\mathbf{t}_3$, respectively.

Note that if the *second* Piola-Kirchhoff stresses are used, one needs to *stretch* the stresses to the *first* Piola-Kirchhoff stresses to obtain the stress resultant and couple in the classical or engineering sense if the strains are not small.

4.4 Internal power: the strain measures conjugate to the first Piola-Kirchhoff stresses

There are many choices of strain measures, each of which has its own *conjugate stress* though there are some exceptions (see e.g. Ogden, 1997: 159). The general power balance condition states that *the external surface traction power plus the body force power is equal to the kinetic power plus the internal power (stress power) for a given reference volume domain of the continuum body using the Lagrangian descriptions if only the mechanical energy is concerned* (for the exact mathematical statement, see e.g. Ogden, 1997: 156). In the differential form, the internal power per unit reference volume of the continuum is $p_{int} = \text{Trace}[(\boldsymbol{\sigma}^0)^t \dot{\mathbf{F}}_r]$ in terms of the *first* Piola-Kirchhoff stress tensor $\boldsymbol{\sigma}^0$ and *material time derivative* $\dot{\mathbf{F}}_r$ of the deformation gradient tensor $\mathbf{F}_r$, which is an *objective scalar*, independent of the observer and reference frame at a given material point (see e.g. Ogden, 1997: 156).

Like Simo (1985) and Simo and Vu-Quoc (1991), we may examine the *objective internal power* to determine the strain measures that are conjugate to the first Piola-Kirchhoff stresses for the curved beam.

We first need to obtain the *material time derivative* of the deformation gradient tensor $\mathbf{F}_r$

as follows, noticing Eqs. (82), (49) and (56):

$$\dot{\mathbf{F}}_r = (\boldsymbol{\Lambda}\bar{\mathbf{F}}_r\boldsymbol{\Lambda}_0^t)_{,t} = \hat{\mathbf{w}}\mathbf{F}_r - \mathbf{F}_r\hat{\mathbf{w}}_0 + \tilde{\dot{\mathbf{F}}}_r = \hat{\mathbf{w}}\mathbf{F}_r + \tilde{\dot{\mathbf{F}}}_r \tag{98}$$

where

$$\tilde{\dot{\mathbf{F}}}_r = \boldsymbol{\Lambda}\dot{\bar{\mathbf{F}}}_r\boldsymbol{\Lambda}_0^t = \frac{1}{g_0}\tilde{\dot{\boldsymbol{\epsilon}}}_r \otimes \mathbf{t}_{01} \tag{99}$$

$$\dot{\bar{\mathbf{F}}}_r = \frac{1}{g_0}\dot{\boldsymbol{\epsilon}}_r \otimes \mathbf{E}_1 \tag{100}$$

$$\dot{\boldsymbol{\epsilon}}_r = \dot{\bar{\boldsymbol{\varepsilon}}}_r + \dot{\bar{\boldsymbol{\kappa}}}_r(\xi_2\mathbf{E}_2 + \xi_3\mathbf{E}_3) \tag{101}$$

$$\tilde{\dot{\boldsymbol{\epsilon}}}_r = \tilde{\dot{\boldsymbol{\varepsilon}}}_r + \tilde{\dot{\hat{\boldsymbol{\kappa}}}}_r(\xi_2\mathbf{t}_2 + \xi_3\mathbf{t}_3) \tag{102}$$

$$\tilde{\dot{\boldsymbol{\varepsilon}}}_r = \boldsymbol{\Lambda}\dot{\bar{\boldsymbol{\varepsilon}}}_r \tag{103}$$

$$\tilde{\dot{\hat{\boldsymbol{\kappa}}}}_r = \boldsymbol{\Lambda}\dot{\hat{\bar{\boldsymbol{\kappa}}}}_r\boldsymbol{\Lambda}^t \tag{104}$$

Note that the *upper tilde* denotes the *co-rotated* derivative as designated in Section 3.5.

The current beam *internal power* per unit volume of the *curved reference beam* at any material point (ξ_1, ξ_2, ξ_3) is, noticing that

$$\text{Trace}(\mathbf{T}_1^t\mathbf{T}_2) = \text{Trace}(\mathbf{T}_1\mathbf{T}_2^t)$$

for any two second-order Cartesian tensors $\mathbf{T}_1$ and $\mathbf{T}_2$,

$$\begin{aligned} p_{int} &= p_{int}(\xi_1, \xi_2, \xi_3) \equiv \text{Trace}[(\boldsymbol{\sigma}^0)^t\dot{\mathbf{F}}_r] = \text{Trace}(\boldsymbol{\sigma}^0\dot{\mathbf{F}}_r^t) \\ &= \text{Trace}[(\boldsymbol{\sigma}^0)^t\hat{\mathbf{w}}\mathbf{F}_r] + \text{Trace}[(\boldsymbol{\sigma}^0)^t\tilde{\dot{\mathbf{F}}}_r] \\ &= \text{Trace}[\boldsymbol{\sigma}^0(\hat{\mathbf{w}}\mathbf{F}_r)^t] + \text{Trace}[(\boldsymbol{\sigma}^0)^t\tilde{\dot{\mathbf{F}}}_r] \end{aligned} \tag{105}$$

The first term of the above equation is due to rigid-body rotation and should vanish. In fact, noticing Eq. (90) for the relation between the *first* Piola-Kirchhoff stress tensor $\boldsymbol{\sigma}^0$ and the Cauchy stress tensor $\boldsymbol{\sigma}$ as well as the symmetry of the Cauchy stress tensor $\boldsymbol{\sigma}$ and skew symmetry of $\hat{\mathbf{w}}$, we have (also see e.g. Simo and Vu-Quoc, 1991)

$$\begin{aligned} \text{Trace}[\boldsymbol{\sigma}^0(\hat{\mathbf{w}}\mathbf{F}_r)^t] &= \text{Trace}[g_r\boldsymbol{\sigma}\mathbf{F}_r^{-t}\mathbf{F}_r^t\hat{\mathbf{w}}^t] = -g_r\text{Trace}[\boldsymbol{\sigma}\hat{\mathbf{w}}] \\ &= -g_r\text{Trace}[\boldsymbol{\sigma}^t\hat{\mathbf{w}}] = -g_r\text{Trace}[\boldsymbol{\sigma}\hat{\mathbf{w}}^t] \\ &= g_r\text{Trace}[\boldsymbol{\sigma}\hat{\mathbf{w}}] = 0 \end{aligned} \tag{106}$$

The second term becomes

$$\begin{aligned}
\mathrm{Trace}[(\boldsymbol{\sigma}^0)^t\dot{\tilde{\mathbf{F}}}_r] &= \mathrm{Trace}[(\mathbf{t}_{0j}\otimes\boldsymbol{\sigma}^0_j)(\frac{1}{g_0}\dot{\tilde{\boldsymbol{\epsilon}}}_r\otimes\mathbf{t}_{01})] \\
&= \frac{1}{g_0}\mathrm{Trace}[(\boldsymbol{\sigma}^0_1\cdot\dot{\tilde{\boldsymbol{\epsilon}}}_r)(\mathbf{t}_{01}\otimes\mathbf{t}_{01})] \\
&= \frac{1}{g_0}\boldsymbol{\sigma}^0_1\cdot\dot{\tilde{\boldsymbol{\epsilon}}}_r
\end{aligned} \tag{107}$$

It follows that the current beam internal power per unit volume of the curved reference beam at any material point (ξ_1, ξ_2, ξ_3) is

$$\begin{aligned}
p_{int} &= \mathrm{Trace}[(\boldsymbol{\sigma}^0)^t\dot{\mathbf{F}}_r] = \mathrm{Trace}[(\boldsymbol{\sigma}^0)^t\dot{\tilde{\mathbf{F}}}_r] = \mathrm{Trace}[(\bar{\boldsymbol{\sigma}}^0)^t\dot{\bar{\mathbf{F}}}_r] \\
&= \boldsymbol{\sigma}^0_1\cdot(\frac{1}{g_0}\dot{\tilde{\boldsymbol{\epsilon}}}_r) = \bar{\boldsymbol{\sigma}}^0_1\cdot(\frac{1}{g_0}\dot{\bar{\boldsymbol{\epsilon}}}_r)
\end{aligned} \tag{108}$$

Therefore, $\frac{1}{g_0}\boldsymbol{\epsilon}_r$ in the spatial form (or $\frac{1}{g_0}\bar{\boldsymbol{\epsilon}}_r$ in the material form) is the current beam *strain vector* at a material point (ξ_1, ξ_2, ξ_3) on the beam cross-section *conjugate* to the *first* Piola-Kirchhoff stress vector $\boldsymbol{\sigma}^0_1$ (or $\bar{\boldsymbol{\sigma}}^0_1 = \boldsymbol{\Lambda}^t\boldsymbol{\sigma}^0_1$).

This also means that we may *define* the corresponding strain tensor $\boldsymbol{\mathcal{E}}_r$ ($\bar{\boldsymbol{\mathcal{E}}}_r$) as, noticing Eq. (82),

$$\begin{aligned}
\boldsymbol{\mathcal{E}}_r &\equiv (\mathbf{F}_r\mathbf{t}_{0i} - \boldsymbol{\Lambda}_r\mathbf{t}_{0i})\otimes\mathbf{t}_{0i} = \mathbf{F}_r - \boldsymbol{\Lambda}_r \\
&= \frac{1}{g_0}\boldsymbol{\epsilon}_r\otimes\mathbf{t}_{01} = \frac{1}{g_0}\bar{\epsilon}_{ri}\mathbf{t}_i\otimes\mathbf{t}_{01} = \bar{\mathcal{E}}_{rij}\mathbf{t}_i\otimes\mathbf{t}_{0j} \\
\bar{\boldsymbol{\mathcal{E}}}_r &\equiv (\bar{\mathbf{F}}_r\bar{\mathbf{t}}_{0i} - \bar{\boldsymbol{\Lambda}}_r\bar{\mathbf{t}}_{0i})\otimes\bar{\mathbf{t}}_{0i} = \bar{\mathbf{F}}_r - \mathbf{I}_3 \\
&= \frac{1}{g_0}\bar{\boldsymbol{\epsilon}}_r\otimes\mathbf{E}_1 = \frac{1}{g_0}\bar{\epsilon}_{ri}\mathbf{E}_i\otimes\mathbf{E}_1 = \bar{\mathcal{E}}_{rij}\mathbf{E}_i\otimes\mathbf{E}_j
\end{aligned} \tag{109}$$

which is *conjugate* to the first Piola-Kirchhoff stress tensor $\boldsymbol{\sigma}^0 = g_r\boldsymbol{\sigma}\mathbf{F}_r^{-t}$ with the component form

$$[\boldsymbol{\mathcal{E}}_r]_{\mathbf{t}_i\otimes\mathbf{t}_{0j}} = [\bar{\boldsymbol{\mathcal{E}}}_r]_{\mathbf{E}_i\otimes\mathbf{E}_j} = \begin{bmatrix} \frac{\bar{\epsilon}_{r1}}{g_0} & 0 & 0 \\ \frac{\bar{\epsilon}_{r2}}{g_0} & 0 & 0 \\ \frac{\bar{\epsilon}_{r3}}{g_0} & 0 & 0 \end{bmatrix} \tag{110}$$

In fact,

$$\dot{\tilde{\boldsymbol{\mathcal{E}}}}_r = \dot{\tilde{\mathbf{F}}}_r - \dot{\tilde{\boldsymbol{\Lambda}}}_r = \dot{\tilde{\mathbf{F}}}_r - \boldsymbol{\Lambda}(\bar{\boldsymbol{\Lambda}}_r)_{,t}\boldsymbol{\Lambda}_0^t = \dot{\tilde{\mathbf{F}}}_r - \boldsymbol{\Lambda}(\mathbf{I}_3)_{,t}\boldsymbol{\Lambda}_0^t \equiv \dot{\tilde{\mathbf{F}}}_r \tag{111}$$

and

$$\dot{\boldsymbol{\mathcal{E}}}_r \equiv \dot{\bar{\mathbf{F}}}_r \tag{112}$$

Note that the geometrical meaning of the strain vector $\frac{1}{g_0}\boldsymbol{\epsilon}_r$ can be observed from the alternative definition:

$$\frac{1}{g_0}\boldsymbol{\epsilon}_r \equiv \mathbf{F}_r\mathbf{t}_{01} - \boldsymbol{\Lambda}_r\mathbf{t}_{01} \tag{113}$$

which is the *stretching* of an oriented unit-length fiber $\mathbf{t}_{01}$ of the curved reference beam at any material point *to* $\mathbf{F}_r\mathbf{t}_{01}$ with the rigidly-rotated reference part $\mathbf{t}_1 = \boldsymbol{\Lambda}_r\mathbf{t}_{01}$ *removed.* The component $\frac{1}{g_0}\bar{\epsilon}_{r1}$ along $\mathbf{t}_1$ may be called the *extensional strain*, and the components $\frac{1}{g_0}\bar{\epsilon}_{r2}$ and $\frac{1}{g_0}\bar{\epsilon}_{r3}$ along $\mathbf{t}_2$ and $\mathbf{t}_3$, respectively, called the *shear strains*, which are much like the extensional and shear strains for small strain problems. In fact, *for small strain problems*, the three components of the strain vector $\frac{1}{g_0}\boldsymbol{\epsilon}_r$ become the extensional strain and shear strains in the 'classical' or 'engineering' sense. For example, the three components of $\frac{1}{g_0}\boldsymbol{\epsilon}_r$ on the beam midcurve become

$$\varepsilon_{r1} = e; \quad \varepsilon_{r2} = \gamma_{12}; \quad \varepsilon_{r3} = \gamma_{13}$$

if the elongation e and angles of shearing γ_{12}, γ_{13} are *small* as seen in Eq. (73). On the other hand, $\frac{1}{g_0}\bar{\epsilon}_{r1}$, $\frac{1}{g_0}\bar{\epsilon}_{r2}$ and $\frac{1}{g_0}\bar{\epsilon}_{r3}$ are really conjugate to the normal stress component $\bar{\sigma}^0_{11}$ and shear stress components $\bar{\sigma}^0_{12}$ and $\bar{\sigma}^0_{13}$, respectively. Therefore, we may call those strain components engineering strains defined by oriented fiber deformation for *finite strain* problems.

Many researchers may prefer to use the symmetric Green strain tensor

$$\begin{aligned}
\boldsymbol{\mathcal{E}}_G &\equiv \frac{1}{2}(\mathbf{F}_r^t\mathbf{F}_r - \mathbf{I}_3) = \bar{\mathcal{E}}_{Gij}\mathbf{t}_{0i}\otimes\mathbf{t}_{0j} \\
\bar{\boldsymbol{\mathcal{E}}}_G &= \bar{\mathcal{E}}_{Gij}\mathbf{E}_i\otimes\mathbf{E}_j \\
[\boldsymbol{\mathcal{E}}_G]_{\mathbf{t}_{0i}\otimes\mathbf{t}_{0j}} &= [\bar{\boldsymbol{\mathcal{E}}}_G]_{\mathbf{E}_i\otimes\mathbf{E}_j} = \begin{bmatrix} \frac{\bar{\epsilon}_{r1}}{g_0} + \frac{1}{2}\frac{\bar{\boldsymbol{\epsilon}}_r}{g_0}\cdot\frac{\bar{\boldsymbol{\epsilon}}_r}{g_0} & \frac{1}{2}\frac{\bar{\epsilon}_{r2}}{g_0} & \frac{1}{2}\frac{\bar{\epsilon}_{r3}}{g_0} \\ \frac{1}{2}\frac{\bar{\epsilon}_{r2}}{g_0} & 0 & 0 \\ \frac{1}{2}\frac{\bar{\epsilon}_{r3}}{g_0} & 0 & 0 \end{bmatrix}
\end{aligned} \tag{114}$$

which is *conjugate* to the *second* Piola-Kirchhoff stress tensor (notice Eq. (93) for its relation with the first Piola-Kirchhoff stress components) in the sense of the internal power density

$$p_{int} \equiv \mathrm{Trace}[\boldsymbol{\sigma}^{0(2)}\dot{\boldsymbol{\mathcal{E}}}_G] \equiv \mathrm{Trace}[(\boldsymbol{\sigma}^0)^t\dot{\mathbf{F}}_r] = \bar{\sigma}^{0(2)}_{ij}\dot{\mathcal{E}}_{Gij} = \bar{\sigma}^0_{ij}\dot{\mathcal{E}}_{rij} = \bar{\boldsymbol{\sigma}}^0_1 \cdot (\frac{1}{g_0}\dot{\bar{\boldsymbol{\epsilon}}}_r) \tag{115}$$

However, the symmetric Eulerian strain tensor

$$\begin{aligned}
\boldsymbol{\mathcal{E}}_E &\equiv \frac{1}{2}(\mathbf{I}_3 - \mathbf{F}_r^{-t}\mathbf{F}_r^{-1}) \equiv \mathbf{F}_r^{-t}\boldsymbol{\mathcal{E}}_G\mathbf{F}_r^{-1} = \bar{\mathcal{E}}_{Eij}\mathbf{t}_i \otimes \mathbf{t}_j \\
\bar{\boldsymbol{\mathcal{E}}}_E &= \bar{\mathcal{E}}_{Eij}\mathbf{E}_i \otimes \mathbf{E}_j \\
[\boldsymbol{\mathcal{E}}_E]_{\mathbf{t}_i\otimes\mathbf{t}_j} &= [\bar{\boldsymbol{\mathcal{E}}}_E]_{\mathbf{E}_i\otimes\mathbf{E}_j} = \frac{1}{g_r}\begin{bmatrix} \frac{\bar{\epsilon}_{r1}}{g_0} - \frac{1}{2}\frac{1}{g_r}\frac{\bar{\boldsymbol{\epsilon}}_r}{g_0}\cdot\frac{\bar{\boldsymbol{\epsilon}}_r}{g_0} & \frac{1}{2}\frac{\bar{\epsilon}_{r2}}{g_0} & \frac{1}{2}\frac{\bar{\epsilon}_{r3}}{g_0} \\ \frac{1}{2}\frac{\bar{\epsilon}_{r2}}{g_0} & 0 & 0 \\ \frac{1}{2}\frac{\bar{\epsilon}_{r3}}{g_0} & 0 & 0 \end{bmatrix}
\end{aligned} \tag{116}$$

does not have its *conjugate* stress tensor, nor does the symmetric Cauchy stress tensor have its *conjugate* strain tensor; the Green strain tensor and Cauchy stress tensor (noticing Eq. 91 for its relation with the *first* Piola-Kirchhoff stress components) relate the objective internal power density by

$$p_{int} \equiv g_r\mathrm{Trace}[\boldsymbol{\sigma}\boldsymbol{\Sigma}] \equiv \mathrm{Trace}[(\boldsymbol{\sigma}^0)^t\dot{\mathbf{F}}_r] \equiv \bar{\sigma}_{ij}\bar{\Sigma}_{ij} = \bar{\boldsymbol{\sigma}}_1 \cdot (\frac{1}{g_0}\dot{\bar{\boldsymbol{\epsilon}}}_r) = \bar{\boldsymbol{\sigma}}^0_1 \cdot (\frac{1}{g_0}\dot{\bar{\boldsymbol{\epsilon}}}_r) \tag{117}$$

where

$$\begin{aligned}
\boldsymbol{\Sigma} &\equiv \mathbf{F}_r^{-t}\dot{\boldsymbol{\mathcal{E}}}_G\mathbf{F}_r^{-1} \equiv \bar{\Sigma}_{ij}\mathbf{t}_{0i} \otimes \mathbf{t}_{0j} \\
[\boldsymbol{\Sigma}]_{\mathbf{t}_{0i}\otimes\mathbf{t}_{0j}} &= \frac{1}{g_r}\begin{bmatrix} \frac{\dot{\bar{\epsilon}}_{r1}}{g_0} & \frac{1}{2}\frac{\dot{\bar{\epsilon}}_{r2}}{g_0} & \frac{1}{2}\frac{\dot{\bar{\epsilon}}_{r3}}{g_0} \\ \frac{1}{2}\frac{\dot{\bar{\epsilon}}_{r2}}{g_0} & 0 & 0 \\ \frac{1}{2}\frac{\dot{\bar{\epsilon}}_{r3}}{g_0} & 0 & 0 \end{bmatrix}
\end{aligned} \tag{118}$$

is the Eulerian strain rate tensor (see e.g. Ogden, 1997: 155), which cannot be obtained simply taking the material time derivative on $\boldsymbol{\mathcal{E}}_E$ nor on $\bar{\boldsymbol{\mathcal{E}}}_E$ in Eq. (116).

Note that the components of both the Green strain tensor $\boldsymbol{\mathcal{E}}_G$ and Eulerian strain tensor $\boldsymbol{\mathcal{E}}_E$ consist of those of the *symmetric part* of the engineering strain tensor $\boldsymbol{\mathcal{E}}_r$. In terms of the components of $\boldsymbol{\mathcal{E}}_r$, both the Green strain tensor and Eulerian strain tensor are *nonlinear* and consist of the quadratic term

$$\frac{1}{2}\frac{\bar{\boldsymbol{\epsilon}}_r}{g_0}\cdot\frac{\bar{\boldsymbol{\epsilon}}_r}{g_0} = \frac{1}{2}\frac{\bar{\epsilon}_{rj}}{g_0}\frac{\bar{\epsilon}_{rj}}{g_0}$$

We should point out that Iura and Atluri (1989) chose

$$g_0 \mathcal{E}_{rij} = \boldsymbol{\xi}_{,\xi_i} \cdot \mathbf{t}_j - \boldsymbol{\xi}_{0,\xi_i} \cdot \mathbf{t}_{0j}$$

as their strain measures, which are conjugate to $\frac{1}{g_0}\bar{\sigma}^0_{ij}$. The only non-zero components of $g_0\mathcal{E}_{rij}$ are $g_0\mathcal{E}_{ri1} = \bar{\epsilon}_{ri}$, which are the strains that are measured with reference to the fiber length on the midcurve.

For the one-dimensional beam or *reduced* beam model, the current beam internal power per unit arclength of the *curved reference beam* midcurve is

$$\begin{aligned}
P_{int} &= P_{int}(\xi_1) = \int_{\mathcal{B}} p_{int} g_0 d\xi_2 d\xi_3 = \int_{\mathcal{B}} \boldsymbol{\sigma}_1^0 \cdot (\frac{1}{g_0}\tilde{\dot{\boldsymbol{\epsilon}}}_r) g_0 d\xi_2 d\xi_3 \\
&= \int_{\mathcal{B}} \boldsymbol{\sigma}_1^0 \cdot [\tilde{\dot{\boldsymbol{\varepsilon}}}_r + \tilde{\dot{\boldsymbol{\kappa}}}_r(\xi_2 \mathbf{t}_2 + \xi_3 \mathbf{t}_3)] d\xi_2 d\xi_3 = \int_{\mathcal{B}} \boldsymbol{\sigma}_1^0 \cdot [\tilde{\dot{\boldsymbol{\varepsilon}}}_r + \tilde{\dot{\boldsymbol{\kappa}}}_r(\boldsymbol{\xi} - \boldsymbol{\varphi})] d\xi_2 d\xi_3 \\
&= [\int_{\mathcal{B}} \boldsymbol{\sigma}_1^0 d\xi_2 d\xi_3] \cdot \tilde{\dot{\boldsymbol{\varepsilon}}}_r + [\int_{\mathcal{B}} (\boldsymbol{\xi} - \boldsymbol{\varphi}) \times \boldsymbol{\sigma}_1^0 d\xi_2 d\xi_3] \cdot \tilde{\dot{\boldsymbol{\kappa}}}_r \\
&= \mathbf{n} \cdot \tilde{\dot{\boldsymbol{\varepsilon}}}_r + \mathbf{m} \cdot \tilde{\dot{\boldsymbol{\kappa}}}_r = \bar{\mathbf{n}} \cdot \dot{\bar{\boldsymbol{\varepsilon}}}_r + \bar{\mathbf{m}} \cdot \dot{\bar{\boldsymbol{\kappa}}}_r
\end{aligned} \tag{119}$$

Therefore, $\boldsymbol{\varepsilon}_r$ and $\boldsymbol{\kappa}_r$ ($\bar{\boldsymbol{\varepsilon}}_r$ and $\bar{\boldsymbol{\kappa}}_r$) are the strain measures *conjugate* to the stress resultant $\mathbf{n}$ and stress couple $\mathbf{m}$ ($\bar{\mathbf{n}}$ and $\bar{\mathbf{m}}$), respectively.

In the literature, the one-dimensional or *reduced* strain measures for the curved beams are available in the vector/tensor forms (see e.g. Simo and Tarnow, 1995; Ibrahimbegović, 1995, etc.). In the above, we have obtained them directly from the *three-dimensional curved* beam model, and now summarize them into Table 1 for reference convenience.

Table 1 Reduced strain measures

Strain type	Spatial form	Material form
Translational strains	$\boldsymbol{\varepsilon}_r = \boldsymbol{\varphi}_{,\xi_1} - \mathbf{t}_1$	$\bar{\boldsymbol{\varepsilon}}_r = \boldsymbol{\Lambda}^t \boldsymbol{\varepsilon}_r$
Bending strains	$\boldsymbol{\kappa}_r = \boldsymbol{\omega}_{r\xi_1}$ $\hat{\boldsymbol{\omega}}_{r\xi_1} = \boldsymbol{\Lambda}_{r,\xi_1} \boldsymbol{\Lambda}_r^t$	$\bar{\boldsymbol{\kappa}}_r = \boldsymbol{\Lambda}^t \boldsymbol{\kappa}_r$

In Table 1, the translational strain vector $\boldsymbol{\varepsilon}_r = \bar{\varepsilon}_{rj} \mathbf{t}_j$ consists of the *extensional strain* component $\bar{\varepsilon}_{r1}$ along $\mathbf{t}_1$ normal to the current beam cross-section, and *shear strain* components $\bar{\varepsilon}_{r2}$ and $\bar{\varepsilon}_{r3}$ along $\mathbf{t}_2$ and $\mathbf{t}_3$, respectively. The bending strain (curvature change) vector $\boldsymbol{\kappa}_r = \bar{\kappa}_{rj} \mathbf{t}_j$ consists of the *twist rate change* component $\bar{\kappa}_{r1}$ around $\mathbf{t}_1$ normal to the current beam cross-section, and *curvature change* or *bending strain* components $\bar{\kappa}_{r2}$ and $\bar{\kappa}_{r3}$ around $\mathbf{t}_2$ and $\mathbf{t}_3$, respectively.

Once the *reduced strain* vectors are determined, the strain vector $\frac{\boldsymbol{\epsilon}_r}{g_0}$ at any material point $(\xi_1, \xi_2, \xi_3) \in R^3$ for $\xi_1 \equiv s_0 \in [0, L_0]$ and $(\xi_2, \xi_3) \in \mathcal{B}(\xi_1)$ on the current beam cross-section can be determined according to Eq. (83) for $\boldsymbol{\epsilon}_r$ and Eq. (78) for the *initial curvature correction term* $g_0 = 1 + \xi_3 \bar{\kappa}_{02} - \xi_2 \bar{\kappa}_{03}$. Other strain measures can also be obtained through their relations with $\frac{\boldsymbol{\epsilon}_r}{g_0}$, e.g. Eq. (114) for Green strains and Eq. (116) for Eulerian strains.

Chapter 5: Balance equations and principle of virtual work

5.1 Reduced balance equations

The Lagrangian differential *equations of motion* referred to the reference beam configuration are, see e.g. Simo (1985) and Ogden (1997),

$$\mathrm{Div}(\boldsymbol{\sigma}^0)^t + \rho_0 \mathbf{b} = \rho_0 \ddot{\boldsymbol{\xi}}; \quad \boldsymbol{\sigma}^0 \mathbf{F}_r^t = \mathbf{F}_r(\boldsymbol{\sigma}^0)^t \tag{120}$$

for a *material point* $\boldsymbol{\xi} = \boldsymbol{\xi}(\xi_1, \xi_2, \xi_3) \in R^3$ of the continuum body of the *current* configuration, where $\ddot{\boldsymbol{\xi}} = \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2}$ is the acceleration for *translation*; $\rho_0 = \rho_0(\xi_1, \xi_2, \xi_3)$ is the *mass* density per unit volume of the reference configuration of the continuum body, dependent on the choice of the reference configuration; $\mathbf{b} = \mathbf{b}(\xi_1, \xi_2, \xi_3)$ is the body force per unit mass of the continuum, which is *independent* of any reference configuration.

Simo (1985) started with Eq. (120a) directly on the initial beam configuration. Because the initial beam configuration is also the straight reference beam configuration, he easily obtained, using the divergence theorem for a fixed orthonormal frame and Eq. (120b), the reduced balance equations for *uniform* straight beams with the line of geometry centroids of the beam cross-sections as the beam reference curve, which may also be a good approximation for very slender curved beams.

Equivalently, Iura and Atluri (1989, 1988) started with the general principle of virtual work for the reduced balance equations of the initially curved/twisted beams.

Now we examine the case of an initially curved/twisted beam with (slowly) *varying* cross-section from the basic Lagrangian equations of motion using orthonormal frames only. In this case, it is not convenient to work directly on the initially curved/twisted beam because the divergence term in Eq. (120a) is inconvenient to expand in the *varying* local frame $\mathbf{t}_{0i}$ along the midcurve, though taking directional derivatives may be an alternative choice. Therefore, we may also work on the *straight reference beam configuration* to obtain the equations of motion of the current configuration for the initially *curved/twisted* beam, but start with the integral counterpart of Eq. (120) (according to the divergence theorem, see e.g. Ogden, 1997: 153):

$$\int_{S_{00}} \boldsymbol{\sigma}^{00}\boldsymbol{\nu}_{00} dS_{00} + \int_{V_{00}} \rho_{00}\mathbf{b} d\xi_1 d\xi_2 d\xi_3 \;=\; \int_{V_{00}} \rho_{00}\ddot{\boldsymbol{\xi}} d\xi_1 d\xi_2 d\xi_3 \tag{121}$$

for the linear momentum balance condition, and

$$\int_{S_{00}} (\boldsymbol{\xi} - \mathbf{v}) \times (\boldsymbol{\sigma}^{00}\boldsymbol{\nu}_{00}) dS_{00} + \int_{V_{00}} \rho_{00}(\boldsymbol{\xi} - \mathbf{v}) \times \mathbf{b} d\xi_1 d\xi_2 d\xi_3 \;=\; \int_{V_{00}} \rho_{00}(\boldsymbol{\xi} - \mathbf{v}) \times \ddot{\boldsymbol{\xi}} d\xi_1 d\xi_2 d\xi_3 \tag{122}$$

for the angular momentum balance condition, where S_{00} is the arbitrarily chosen surface domain, $\boldsymbol{\nu}_{00}$ the outward unit vector of the differential surface dS_{00}, V_{00} the corresponding volume domain surrounded by S_{00}, $\rho_{00} = g_0\rho_0$ the mass density because of the mass conservation condition, and $\boldsymbol{\sigma}^{00}$ the first Piola-Kirchhoff stress tensor as given in Eq. (92); and $\mathbf{v} \in R^3$ is an arbitrarily *spatially fixed* (position) vector.

Applying Eqs. (121) and (122) to a parallel 'cut' slice through the straight reference beam with the differential length $d\xi_1$ and normal to the straight reference beam midcurve (on the $\mathbf{E}_1$ axis in fact) for the surface [9] and volume integration domains, and then using variable and domain changes, we can obtain the reduced balance equations of the moving beam referred

[9]We separate dS_{00} into the lateral surface dS_{00L} and cut surface $S_{00N} = S_{00N-} \cup S_{00N+}$ for the straight reference beam configuration, and similarly, dS_0 into dS_{0L} and $S_{0N} = S_{0N-} \cup S_{0N+}$ for the corresponding tapered 'cut' slice of the curved reference beam configuration. See the Fig. 2 for the dS_{00}:

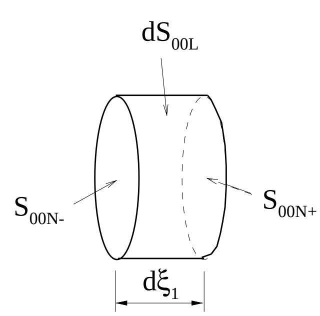

Fig. 2 A differential slice for the straight reference configuration

to the (initially unstrained/unstressed) *curved* (and twisted) *reference beam* configuration:

$$\mathbf{n}_{,\xi_1} + \boldsymbol{\mathcal{N}} \quad = \quad A_{\rho_0}\ddot{\boldsymbol{\varphi}} + \underline{\ddot{\mathbf{w}} \times \boldsymbol{\mathcal{S}}_{\rho_0} + \dot{\mathbf{w}} \times [\dot{\mathbf{w}} \times \boldsymbol{\mathcal{S}}_{\rho_0}]} \tag{123}$$

for the *translational motion* (linear momentum balance condition), and

$$\mathbf{m}_{,\xi_1} + \boldsymbol{\varphi}_{,\xi_1} \times \mathbf{n} + \boldsymbol{\mathcal{M}} \quad = \quad \underline{\boldsymbol{\mathcal{S}}_{\rho_0} \times \ddot{\boldsymbol{\varphi}}} + \boldsymbol{\mathcal{I}}_{\rho_0}\ddot{\mathbf{w}} + \dot{\mathbf{w}} \times [\boldsymbol{\mathcal{I}}_{\rho_0}\dot{\mathbf{w}}] \tag{124}$$

for *rotation* (angular momentum balance condition). In the above equations in the *spatial forms*, the stress resultant $\mathbf{n}$ and couple $\mathbf{m}$ have been given in Eqs. (94) and (95), respectively; $\ddot{\boldsymbol{\varphi}} = \frac{\partial^2 \boldsymbol{\varphi}}{\partial t^2} \in R^3$, the acceleration vector of the current beam *reference curve* for *translation*; $\dot{\mathbf{w}} = \boldsymbol{\omega}_t = \frac{\Delta_t \mathbf{w}}{\Delta t} \in R^3$ the rotational speed vector of the current beam cross-section; $\ddot{\mathbf{w}} = \frac{\partial \boldsymbol{\omega}_t}{\partial t} \in R^3$ the acceleration vector of the *current beam* cross-section for *rotation*; the (scalar) mass density A_{ρ_0} per unit arclength of the *curved reference beam* midcurve is

$$A_{\rho_0} = \int_{\mathcal{B}} g_0 \rho_0 d\xi_2 d\xi_3 \tag{125}$$

the first *mass* moment density $\boldsymbol{\mathcal{S}}_{\rho_0}$ (vector) per unit arclength of the *curved reference beam* midcurve:

$$\begin{aligned}
\boldsymbol{\mathcal{S}}_{\rho_0} &= \int_{\mathcal{B}} g_0 \rho_0 (\xi_2 \mathbf{t}_2 + \xi_3 \mathbf{t}_3) d\xi_2 d\xi_3 = \bar{\mathcal{S}}_{\rho_0 3}\mathbf{t}_2 + \bar{\mathcal{S}}_{\rho_0 2}\mathbf{t}_3 \\
\bar{\mathcal{S}}_{\rho_0 3} &= \int_{\mathcal{B}} g_0 \rho_0 \xi_2 d\xi_2 d\xi_3 \\
\bar{\mathcal{S}}_{\rho_0 2} &= \int_{\mathcal{B}} g_0 \rho_0 \xi_3 d\xi_3 d\xi_3
\end{aligned} \tag{126}$$

and the 'rotational mass' or second *mass* moment density $\boldsymbol{\mathcal{I}}_{\rho_0}$ (second-order tensor) per unit arclength of the *curved reference beam* midcurve:

$$\begin{aligned}
\boldsymbol{\mathcal{I}}_{\rho_0} &= \int_{\mathcal{B}} g_0 \rho_0 [-(\xi_2 \widehat{\mathbf{t}_2 + \xi_3} \mathbf{t}_3)^2] d\xi_2 d\xi_3 \\
&= \int_{\mathcal{B}} g_0 \rho_0 [(\|\xi_2 \mathbf{t}_2 + \xi_3 \mathbf{t}_3\|)^2 \mathbf{I}_3 - (\xi_2 \mathbf{t}_2 + \xi_3 \mathbf{t}_3) \otimes (\xi_2 \mathbf{t}_2 + \xi_3 \mathbf{t}_3)] d\xi_2 d\xi_3 \\
&= \bar{\mathcal{I}}_{\rho_0 ij} \mathbf{t}_i \otimes \mathbf{t}_j \\
\bar{\mathcal{I}}_{\rho_0 11} &= \bar{\mathcal{I}}_{\rho_0 22} + \bar{\mathcal{I}}_{\rho_0 33} \\
\bar{\mathcal{I}}_{\rho_0 12} &= \bar{\mathcal{I}}_{\rho_0 13} = \bar{\mathcal{I}}_{\rho_0 21} = \bar{\mathcal{I}}_{\rho_0 31} = 0
\end{aligned}$$

$$
\begin{aligned}
\bar{\mathcal{I}}_{\rho_0 22} &= \int_{\mathcal{B}} g_0 \rho_0 (\xi_3)^2 d\xi_2 d\xi_3 \\
\bar{\mathcal{I}}_{\rho_0 33} &= \int_{\mathcal{B}} g_0 \rho_0 (\xi_2)^2 d\xi_2 d\xi_3 \\
\bar{\mathcal{I}}_{\rho_0 23} &= \bar{\mathcal{I}}_{\rho_0 32} = -\int_{\mathcal{B}} g_0 \rho_0 \xi_2 \xi_3 d\xi_2 d\xi_3
\end{aligned}
\tag{127}
$$

while the *reduced* external *force* density per unit arclength of the *curved reference beam* midcurve is

$$
\begin{aligned}
\boldsymbol{\mathcal{N}} &= \int_{\mathcal{S}_{00L}} \boldsymbol{\sigma}^{00} (\boldsymbol{\nu}_{00} d\mathcal{S}_{00L}) + \int_{\mathcal{B}} \rho_{00} \mathbf{b} d\xi_2 d\xi_3 \\
&= \int_{\mathcal{S}_{0L}} (g_0 \boldsymbol{\sigma}^0 \mathbf{F}_0^{-t})(\frac{1}{g_0} \mathbf{F}_0^t \boldsymbol{\nu}_0 d\mathcal{S}_{0L}) + \int_{\mathcal{B}} g_0 \rho_0 \mathbf{b} d\xi_2 d\xi_3 \\
&= \int_{\mathcal{S}_{0L}} \boldsymbol{\sigma}^0 \boldsymbol{\nu}_0 d\mathcal{S}_{0L} + \int_{\mathcal{B}} g_0 \rho_0 \mathbf{b} d\xi_2 d\xi_3 \\
&= \int_{\mathcal{C}_{\mathcal{B}}} \frac{g_0}{\boldsymbol{\nu}_0 \cdot \boldsymbol{\nu}_{0\mathcal{C}_{\mathcal{B}}}} \boldsymbol{\sigma}_j^0 \bar{\nu}_{0j} d\mathcal{C}_{\mathcal{B}} + \int_{\mathcal{B}} g_0 \rho_0 \mathbf{b} d\xi_2 d\xi_3
\end{aligned}
\tag{128}
$$

and the *reduced* external *moment* density per unit arclength of the undeformed beam reference curve is

$$
\begin{aligned}
\boldsymbol{\mathcal{M}} &= \int_{\mathcal{S}_{00L}} (\boldsymbol{\xi} - \boldsymbol{\varphi}) \times [\boldsymbol{\sigma}^{00} (\boldsymbol{\nu}_{00} d\mathcal{S}_{00L})] + \int_{\mathcal{B}} \rho_{00} (\boldsymbol{\xi} - \boldsymbol{\varphi}) \times \mathbf{b} d\xi_2 d\xi_3 \\
&= \int_{\mathcal{S}_{0L}} (\boldsymbol{\xi} - \boldsymbol{\varphi}) \times (\boldsymbol{\sigma}_0 \boldsymbol{\nu}_0 d\mathcal{S}_{0L}) + \int_{\mathcal{B}} g_0 \rho_0 (\boldsymbol{\xi} - \boldsymbol{\varphi}) \times \mathbf{b} d\xi_2 d\xi_3 \\
&= \int_{\mathcal{C}_{\mathcal{B}}} \frac{g_0}{\boldsymbol{\nu}_0 \cdot \boldsymbol{\nu}_{0\mathcal{C}_{\mathcal{B}}}} (\boldsymbol{\xi} - \boldsymbol{\varphi}) \times (\boldsymbol{\sigma}_j^0 \bar{\nu}_{0j}) d\mathcal{C}_{\mathcal{B}} + \int_{\mathcal{B}} g_0 \rho_0 (\boldsymbol{\xi} - \boldsymbol{\varphi}) \times \mathbf{b} d\xi_2 d\xi_3
\end{aligned}
\tag{129}
$$

Note that the above expressions for $\boldsymbol{\mathcal{N}}$ and $\boldsymbol{\mathcal{M}}$ include the load boundary conditions for the *lateral* surface tractions with outward unit vector $\boldsymbol{\nu}_0 = \bar{\nu}_{0j} \mathbf{t}_{0j}$ of the *curved reference beam* configuration; $d\mathcal{C}_{\mathcal{B}}$ is the differential element of the contour line $\mathcal{C}_{\mathcal{B}}$ of the cross-section domain $\mathcal{B}$ and $\boldsymbol{\nu}_{0\mathcal{C}_{\mathcal{B}}} = \bar{\nu}_{2\mathcal{C}_{\mathcal{B}}} \mathbf{t}_{02} + \bar{\nu}_{3\mathcal{C}_{\mathcal{B}}} \mathbf{t}_{03}$ the unit outward normal vector of $\mathcal{C}_{\mathcal{B}}$ *in* the cross-section plane of the curved reference beam configuration; $\boldsymbol{\mathcal{N}}$ and $\boldsymbol{\mathcal{M}}$ also include the body forces.

For an untwisted straight beam of homogeneous material ($\rho_0 = \textit{constant}$) with no initial elongation of the beam midcurve ($g_0 = 1$), the first mass moment density $\boldsymbol{\mathcal{S}}_{\rho_0}$ given in Eq. (126) vanishes if the beam reference curve is chosen as the *geometry* centroid line of the beam cross-section. In this case, the underlined terms in Eqs. (123) and (124) vanish, and the balance equations reduce to the original forms given by Simo (1985) and Simo and Vu-Quoc (1986). In addition, if the beam is also *uniform*, Eqs. (128) and (129) will also reduce to those given by Simo (1985).

For an initially curved beam, $g_0 \neq 1$, and if the beam reference curve is chosen as the geometry centroid line, $\boldsymbol{\mathcal{S}}_{\rho_0}$ does *not* vanish in general though its entries are small for slender beams. On the other hand, if one chooses the *mass* centroid line as the beam reference curve, $\boldsymbol{\mathcal{S}}_{\rho_0}$ also vanishes. However, as will be seen later for linear (linearized) constitutive relations, the *elasticity* centroid line is *not* the *mass* centroid line, nor the *geometry* centroid line all because of the *initial curvature correction term* $g_0 \neq 1$.

On the other hand, in contrast to Iura and Atluri (1988), we write the inertial terms or linear and angular momentum rates on the right-hand sides of Eqs. (123) and (124), respectively, in their final forms to avoid users wrongly introducing or ignoring some terms, as we have addressed for Eq (122) that the reference position vector $\mathbf{v}$ for angular momentum is *spatially fixed* and *independent of time*. The rest, for example, the formulae for inertia constants, are the same as in Iura and Atluri (1988).

The *material forms* of the reduced equations of motion may also be given. However, it seems that the *spatial forms* are more 'natural' for dynamic problems, which is in contrast to the cases of strains and stresses as well as the constitutive relations where the *material forms* are more natural.

5.2 Reduced virtual work equations

For numerical implementations using the curved beam theory, e.g. displacement-based finite element methods, it is more convenient to use the principle of virtual work (weak formulation) to build equilibrium equations. In fact, Iura and Atluri (1989, also see the references therein) started with the general principle of virtual work to obtain the balance equations in the static case for the curved beam theory. On the other hand, the virtual work equation may also be obtained from the reduced balance equations Eq. (123) and (124) (see e.g. Ibrahimbegović, 1995; and the refences therein). For completeness, we may repeat a similar procedure to obtain the proper virtual work equation along with the end boundary conditions.

Using the 'static' method (see e.g. Iura and Atluri, 1989; and the references therein), we may let

$$\begin{aligned} \boldsymbol{\mathcal{N}}' &= \boldsymbol{\mathcal{N}} - (A_{\rho_0}\ddot{\boldsymbol{\varphi}} + \underline{\ddot{\mathbf{w}} \times \boldsymbol{\mathcal{S}}_{\rho_0} + \dot{\mathbf{w}} \times [\dot{\mathbf{w}} \times \boldsymbol{\mathcal{S}}_{\rho_0}]}) \\ \boldsymbol{\mathcal{M}}' &= \boldsymbol{\mathcal{M}} - (\underline{\boldsymbol{\mathcal{S}}_{\rho_0} \times \ddot{\boldsymbol{\varphi}}} + \boldsymbol{\mathcal{I}}_{\rho_0}\ddot{\mathbf{w}} + \dot{\mathbf{w}} \times [\boldsymbol{\mathcal{I}}_{\rho_0}\dot{\mathbf{w}}]) \end{aligned} \tag{130}$$

be the generalized external force and moment densities to include the *inertial forces*.

Taking the dot product with Eq. (123) using an arbitrary but kinematically admissible variation (virtual displacement), $\delta_a\boldsymbol{\varphi} = \delta_a\mathbf{u} \in R^3$ [10], of the position vector $\boldsymbol{\varphi}$, and taking the dot product with Eq. (124) using an arbitrary but kinematically admissible incremental rotation $\delta_a\mathbf{w} = \delta_a\mathbf{w}_r \in R^3$ associated with the skew symmetric tensor $\delta_a\hat{\mathbf{w}} = \delta_a\boldsymbol{\Lambda}\boldsymbol{\Lambda}^t = \delta_a\boldsymbol{\Lambda}_r\boldsymbol{\Lambda}_r^t \in so(3)$ (virtual incremental rotation), and summing up the two scalar virtual work equations for the tapered differential slice of the beam, and then taking integration over the whole beam, one may obtain that, after rearranging the terms, (also see e.g. Simo, 1985)

$$\int_{L_0}[\delta_a\boldsymbol{\varphi}\cdot(\mathbf{n}_{,\xi_1} + \boldsymbol{\mathcal{N}}') + \delta_a\mathbf{w}\cdot(\mathbf{m}_{,\xi_1} + \boldsymbol{\varphi}_{,\xi_1}\times\mathbf{n} + \boldsymbol{\mathcal{M}}')]d\xi_1 = 0 \tag{131}$$

or

$$\int_{L_0}[\delta_a\boldsymbol{\varphi}\cdot\mathbf{n}_{,\xi_1} + \delta_a\mathbf{w}\cdot\mathbf{m}_{,\xi_1} + \delta_a\mathbf{w}\cdot(\boldsymbol{\varphi}_{,\xi_1}\times\mathbf{n})]d\xi_1 + \int_{L_0}(\delta_a\boldsymbol{\varphi}\cdot\boldsymbol{\mathcal{N}}' + \delta_a\mathbf{w}\cdot\boldsymbol{\mathcal{M}}')d\xi_1 = 0 \tag{132}$$

For the above equation, taking integration by parts for the $\mathbf{n}_{,\xi_1}$ and $\mathbf{m}_{,\xi_1}$ terms, and noticing $\delta_a\mathbf{w}\cdot(\boldsymbol{\varphi}_{,\xi_1}\times\mathbf{n}) = (\delta_a\mathbf{w}\times\boldsymbol{\varphi}_{,\xi_1})\cdot\mathbf{n}$, one may easily obtain that

$$\int_{L_0}[(\delta_a\boldsymbol{\varphi}_{,\xi_1} - \delta_a\mathbf{w}\times\boldsymbol{\varphi}_{,\xi_1})\cdot\mathbf{n} + (\delta_a\mathbf{w})_{,\xi_1}\cdot\mathbf{m}]d\xi_1$$

[10] δ_a denotes the variation of the deformation of the current beam configuration; note that the initial beam configuration is spatially fixed and the variation of any objects associated with the initial beam configuration vanishes, e.g., $\delta_a\mathbf{t}_{0i} = \mathbf{0}$, $\delta_a\boldsymbol{\Lambda}_0 = \mathbf{0}$ and $\delta_a\mathbf{w}_0 = \mathbf{0}$

$$
\begin{aligned}
&-[(\delta_a\boldsymbol{\varphi}\cdot\mathbf{n})|_0^{L_0}+(\delta_a\mathbf{w}\cdot\mathbf{m})|_0^{L_0}]-\int_{L_0}[\delta_a\boldsymbol{\varphi}\cdot\boldsymbol{\mathcal{N}}'+\delta_a\mathbf{w}\cdot\boldsymbol{\mathcal{M}}']d\xi_1 \\
=\;& 0 \qquad (133)
\end{aligned}
$$

It follows that the virtual work equation for the beam in the current configuration can be written as (also see e.g. Simo and Vu-Quoc, 1986; Ibrahimbegović, 1995)

$$
\delta_a W_{int}-\delta_a W_{ext}=0 \qquad (134)
$$

where $\delta_a W_{ext}$ is the external virtual work [11] along with the end boundary conditions,

$$
\delta_a W_{ext} = \int_{L_0}(\delta_a\boldsymbol{\varphi}\cdot\boldsymbol{\mathcal{N}}'+\delta_a\mathbf{w}\cdot\boldsymbol{\mathcal{M}}')d\xi_1+(\delta_a\boldsymbol{\varphi}\cdot\mathbf{n})|_0^{L_0}+(\delta_a\mathbf{w}\cdot\mathbf{m})|_0^{L_0} \qquad (135)
$$

$\delta_a W_{int}$ is the internal virtual work, the variational form of the reduced internal power [12] as given in Eq. (119):

$$
\delta_a W_{int} = \int_{L_0}(\tilde{\delta}_a\boldsymbol{\varepsilon}_r\cdot\mathbf{n}+\tilde{\delta}_a\boldsymbol{\kappa}_r\cdot\mathbf{m})d\xi_1=\int_{L_0}(\delta_a\bar{\boldsymbol{\varepsilon}}_r\cdot\bar{\mathbf{n}}+\delta_a\bar{\boldsymbol{\kappa}}_r\cdot\bar{\mathbf{m}})d\xi_1 \qquad (136)
$$

in which, noticing the *co−rotated* variation notation $\tilde{\delta}_a$, the variational form of the *co-rotated* derivatives as summarized in Section 3.5,

$$
\begin{aligned}
\tilde{\delta}_a\boldsymbol{\varepsilon}_r &= \delta_a\boldsymbol{\varphi}_{,\xi_1}-\delta_a\mathbf{w}\times\boldsymbol{\varphi}_{,\xi_1}=\tilde{\delta}_a\boldsymbol{\varphi}_{,\xi_1}=\tilde{\delta}_a\boldsymbol{\varphi}_{,\xi_1}-\tilde{\mathbf{t}}_1=\tilde{\delta}_a(\boldsymbol{\varphi}_{,\xi_1}-\mathbf{t}_1) \\
\delta_a\bar{\boldsymbol{\varepsilon}}_r &= \boldsymbol{\Lambda}^t\tilde{\delta}_a\boldsymbol{\varepsilon}_r \\
\tilde{\delta}_a\boldsymbol{\kappa}_r &= (\delta_a\mathbf{w})_{,\xi_1}=(\delta_a\mathbf{w}_r)_{,\xi_1} \\
\delta_a\bar{\boldsymbol{\kappa}}_r &= \boldsymbol{\Lambda}^t\tilde{\delta}_a\boldsymbol{\kappa}_r \qquad (137)
\end{aligned}
$$

To this end, it should be mentioned that Ibrahimbegović (1995) used the reduced strain vector (co-rotated) variations to obtain the reduced strain measures for the curved beam

[11] In the 'static' method, the external virtual work includes the virtual kinetic-energy-lost because the generalized external loads include the inertia terms as in Eq. (130). We may separate the virtual kinetic energy from the external virtual work in Eq. (134) if necessary for the dynamic problems.

[12] In the more general term, the power (energy) balance equation in its incremental or variational form becomes the virtual work equation while the internal power term becomes the internal virtual work. Therefore, the time derivative terms in Section 5.4 may be replaced by the corresponding variational terms with the operator δ_a.

theory as demonstrated for $\boldsymbol{\varepsilon}_r$ in Eq. (137) with the consideration of the vanishing $\boldsymbol{\varphi}_{,\xi_1} - \mathbf{t}_1$ as the pure rigid-body motion.

In order to use the solution procedures of Newton type, one needs the linearized equilibrium or state equation, which can be achieved through the linearization of the principle of virtual work in its continuum form. The linearized virtual work equation can be written as[13] (also see e.g. Ibrahimbegović, 1995)

$$\delta_a W_{int} - \delta_a W_{ext} + \Delta_a \delta_a W_{int} - \Delta_a \delta_a W_{ext} + \Delta_\lambda \delta_a W_{int} - \Delta_\lambda \delta_a W_{ext} \quad = \quad 0 \qquad (138)$$

[13]Note again that the symbol $\Delta_a(\cdot)$ denotes the linearized increment of the quantity $(\cdot)$ and $\delta_a(\cdot)$ the admissible variation of the quantity $(\cdot)$ at the same point of the beam midcurve (the same arc-length coordinate ξ_1), due to deformation only while both the loading factor λ and initial configuration of the structure remain constant; the subscript a is used to identify deformation. Also, $\Delta_\lambda(\cdot)$ denotes the linearized increment of the quantity $(\cdot)$ due to a change in the loading factor λ while both the initial and deformed configurations of the structure remain constant. For dynamic problems, the loading factor λ may be replaced by the time parameter t.

Chapter 6: Constitutive relations

For the finite strain beam theories, a hyperelastic, isotropic and homogeneous material was usually assumed (for straight beams, see e.g. Simo and Vu-Quoc, 1986; etc.; for curved beams, see e.g. Iura and Atluri, 1989; etc.). Therefore, the reduced constitutive relations are very simple.

In fact, it is easy to extend the constitutive relations to more general cases (see e.g. Simo, 1985). In what follows, we consider the general linear and nonlinear constitutive equations in more explicit forms for the convenience of performing computations.

6.1 Linear constitutive relations

Let us first assume that the beam material is hyperelastic but not necessarily isotropic nor necessarily homogeneous. The linearity is kept between the first Piola-Kirchhoff stress vector and its conjugate strain vector at any material point $(\xi_1, \xi_2, \xi_3) \in R^3$ for $\xi_1 \equiv s_0 \in [0, L_0]$ and $(\xi_2, \xi_3) \in \mathcal{B}(\xi_1)$ on the current beam cross-section and given in the component forms as

$$\bar{\sigma}^0_{1i} = \bar{\mathcal{C}}_{ij} \frac{\bar{\epsilon}_{rj}}{g_0}; \quad \bar{\mathcal{C}}_{ij} = \bar{\alpha} \bar{\mathcal{C}}^0_{ij}; \quad \bar{\alpha} = \bar{\alpha}(\xi_1, \xi_2, \xi_3) \tag{139}$$

where $\bar{\mathcal{C}}_{ij} = \bar{\mathcal{C}}_{ji}$ are the general material elasticity constants for a given material point but may vary over different material points; $\bar{\mathcal{C}}^0_{ij} = \bar{\mathcal{C}}^0_{ji}$ are arbitrarily chosen reference material constants and do not vary over different material points; $\bar{\alpha}$ is the corresponding scalar between $\bar{\mathcal{C}}_{ij}$ and $\bar{\mathcal{C}}^0_{ij}$, and may vary over material points according to $\bar{\mathcal{C}}_{ij}$.

Then, the linear constitutive relation for a material point on the current beam cross-section may be described in terms of vectors and tensors in the *spatial* and *material form* as

$$\begin{aligned} \boldsymbol{\sigma}^0_1 &= \frac{1}{g_0} \boldsymbol{\mathcal{C}} \boldsymbol{\epsilon}_r; \quad \bar{\boldsymbol{\sigma}}^0_1 = \frac{1}{g_0} \bar{\boldsymbol{\mathcal{C}}} \bar{\boldsymbol{\epsilon}}_r \\ \boldsymbol{\mathcal{C}} &= \bar{\mathcal{C}}_{ij} \mathbf{t}_i \otimes \mathbf{t}_j; \quad \bar{\boldsymbol{\mathcal{C}}} = \bar{\mathcal{C}}_{ij} \mathbf{E}_i \otimes \mathbf{E}_j \end{aligned} \tag{140}$$

Substituting Eq. (139) into the formulae for the components $\bar{n}_i$ and $\bar{m}_i$ of the stress resultant ($\mathbf{n}$) and couple ($\mathbf{m}$) vectors in Eqs. (94) and (95), respectively, also using the formulae for the components $\bar{\epsilon}_{rj}$ of the strain vector $\boldsymbol{\epsilon}_r$ without the initial curvature correction term in Eq. (83), we may finally obtain the general reduced linear constitutive relations for the curved/twisted beam:

$$\begin{aligned}
\mathbf{n} &= \mathbf{C}_{nn}\boldsymbol{\varepsilon}_r + \mathbf{C}_{nm}\boldsymbol{\kappa}_r \\
\mathbf{m} &= \mathbf{C}_{mn}\boldsymbol{\varepsilon}_r + \mathbf{C}_{mm}\boldsymbol{\kappa}_r \\
\bar{\mathbf{n}} &= \bar{\mathbf{C}}_{nn}\bar{\boldsymbol{\varepsilon}}_r + \bar{\mathbf{C}}_{nm}\bar{\boldsymbol{\kappa}}_r \\
\bar{\mathbf{m}} &= \bar{\mathbf{C}}_{mn}\bar{\boldsymbol{\varepsilon}}_r + \bar{\mathbf{C}}_{mm}\bar{\boldsymbol{\kappa}}_r
\end{aligned}$$

$$\begin{aligned}
\mathbf{C}_{pq} &= \bar{C}_{pqij}\mathbf{t}_i \otimes \mathbf{t}_j \\
\bar{\mathbf{C}}_{pq} &= \bar{C}_{pqij}\mathbf{E}_i \otimes \mathbf{E}_j \\
\mathbf{C}_{pq} &= \boldsymbol{\Lambda}\bar{\mathbf{C}}_{pq}\boldsymbol{\Lambda}^t \\
\text{for} & \quad p, q = m, n
\end{aligned}$$

$$\begin{aligned}
\bar{C}_{nnij} &= \bar{\mathcal{C}}^0_{ij}\bar{A}^* \\
\bar{C}_{nmi1} &= \bar{\mathcal{C}}^0_{i3}\bar{S}^*_3 - \bar{\mathcal{C}}^0_{i2}\bar{S}^*_2 \\
\bar{C}_{mn1j} &= \bar{\mathcal{C}}^0_{3j}\bar{S}^*_3 - \bar{\mathcal{C}}^0_{2j}\bar{S}^*_2 \\
\bar{C}_{nmi2} &= \bar{\mathcal{C}}^0_{i1}\bar{S}^*_2 \\
\bar{C}_{mn2j} &= \bar{\mathcal{C}}^0_{1j}\bar{S}^*_2 \\
\bar{C}_{nmi3} &= -\bar{\mathcal{C}}^0_{i1}\bar{S}^*_3 \\
\bar{C}_{mn3j} &= -\bar{\mathcal{C}}^0_{1j}\bar{S}^*_3 \\
\bar{C}_{mm11} &= \bar{\mathcal{C}}^0_{22}\bar{I}^*_{22} + \bar{\mathcal{C}}^0_{33}\bar{I}^*_{33} - (\bar{\mathcal{C}}^0_{23}\bar{I}^*_{23} + \bar{\mathcal{C}}^0_{32}\bar{I}^*_{32}) \\
\bar{C}_{mm22} &= \bar{\mathcal{C}}^0_{11}\bar{I}^*_{22} \\
\bar{C}_{mm33} &= \bar{\mathcal{C}}^0_{11}\bar{I}^*_{33} \\
\bar{C}_{mm12} &= \bar{\mathcal{C}}^0_{31}\bar{I}^*_{32} - \bar{\mathcal{C}}^0_{21}\bar{I}^*_{22} \\
\bar{C}_{mm21} &= \bar{\mathcal{C}}^0_{13}\bar{I}^*_{32} - \bar{\mathcal{C}}^0_{12}\bar{I}^*_{22}
\end{aligned}$$

$$
\begin{aligned}
\bar{C}_{mm13} &= \bar{\mathcal{C}}^0_{21}\bar{I}^*_{23} - \bar{\mathcal{C}}^0_{31}\bar{I}^*_{33} \\
\bar{C}_{mm31} &= \bar{\mathcal{C}}^0_{12}\bar{I}^*_{23} - \bar{\mathcal{C}}^0_{13}\bar{I}^*_{33} \\
\bar{C}_{mm23} &= -\bar{\mathcal{C}}^0_{11}\bar{I}^*_{23} \\
\bar{C}_{mm32} &= -\bar{\mathcal{C}}^0_{11}\bar{I}^*_{32}
\end{aligned}
$$

$$
\begin{aligned}
\bar{A}^* &= \int_{\mathcal{B}} \frac{\bar{\alpha}}{g_0} d\xi_2 d\xi_3 \\
\bar{S}^*_2 &= \int_{\mathcal{B}} \frac{\bar{\alpha}}{g_0} \xi_3 d\xi_2 d\xi_3 \\
\bar{S}^*_3 &= \int_{\mathcal{B}} \frac{\bar{\alpha}}{g_0} \xi_2 d\xi_2 d\xi_3 \\
\bar{I}^*_{22} &= \int_{\mathcal{B}} \frac{\bar{\alpha}}{g_0} (\xi_3)^2 d\xi_2 d\xi_3 \\
\bar{I}^*_{33} &= \int_{\mathcal{B}} \frac{\bar{\alpha}}{g_0} (\xi_2)^2 d\xi_2 d\xi_3 \\
\bar{I}^*_{23} &= \bar{I}^*_{32} = \int_{\mathcal{B}} \frac{\bar{\alpha}}{g_0} \xi_2 \xi_3 d\xi_2 d\xi_3
\end{aligned} \tag{141}
$$

Note that the reduced elasticity constants have an overall symmetry for any hyperelastic material because of $\bar{\mathcal{C}}_{ij} = \bar{\mathcal{C}}_{ji}$. For other materials, the symmetry may not hold because of non-existence of the strain energy functional. In general, the couplings exist, such as stretch-bending coupling, stretch-torsion coupling and torsion-bending coupling, etc.

On the other hand, we may align the beam reference curve so that $\bar{S}^*_2 = \bar{S}^*_3 = 0$. This means that the cross-section elasticity centroid line is chosen as the beam reference curve. However, the elasticity centroid line does *not* coincide with the mass centroid line for a general curved beam even though the beam material is homogeneous because the *initial curvature correction term* g_0 appears as the numerator in the integrals for the inertia constants while it appears as the denominator for the elasticity constants.

The simplest case of the cross-section elasticity constants is when the beam material is isotropic and homogeneous,

$$
\bar{\sigma}^0_{11} = E\frac{\bar{\epsilon}_{r1}}{g_0}
$$

$$
\begin{aligned}
\bar{\sigma}^0_{12} &= G\frac{\bar{\epsilon}_{r2}}{g_0} \\
\bar{\sigma}^0_{13} &= G\frac{\bar{\epsilon}_{r3}}{g_0} \\
&\text{i.e.} \\
\bar{\mathcal{C}}_{11} &= E \\
\bar{\mathcal{C}}_{22} &= \bar{\mathcal{C}}_{33} = G \\
\bar{\mathcal{C}}_{ij} &= 0, \quad \text{otherwise} \\
&\text{and} \\
\bar{\alpha} &= 1 \qquad (142)
\end{aligned}
$$

where E is the Young's modulus and G the shearing modulus in the conventional sense. Then, the cross-section elasticity constants become

$$
\begin{aligned}
[\mathbf{C}_{nn}]_{\mathbf{t}_i\otimes\mathbf{t}_j} &= [\bar{\mathbf{C}}_{nn}]_{\mathbf{E}_i\otimes\mathbf{E}_j} = \begin{bmatrix} E\bar{A} & 0 & 0 \\ 0 & Gk_s\bar{A} & 0 \\ 0 & 0 & Gk_s\bar{A} \end{bmatrix} \\
[\mathbf{C}_{nm}]_{\mathbf{t}_i\otimes\mathbf{t}_j} &= [\mathbf{C}_{nm}]_{\mathbf{E}_i\otimes\mathbf{E}_j} = \begin{bmatrix} 0 & E\bar{S}_2 & -E\bar{S}_3 \\ -G\bar{S}_2 & 0 & 0 \\ G\bar{S}_3 & 0 & 0 \end{bmatrix} \\
[\mathbf{C}_{mm}]_{\mathbf{t}_i\otimes\mathbf{t}_j} &= [\bar{\mathbf{C}}_{mm}]_{\mathbf{E}_i\otimes\mathbf{E}_j} = \begin{bmatrix} Gk_t\bar{I}_{11} & 0 & 0 \\ 0 & E\bar{I}_{22} & -E\bar{I}_{23} \\ 0 & -E\bar{I}_{32} & E\bar{I}_{33} \end{bmatrix} \\
[\mathbf{C}_{mn}]_{\mathbf{t}_i\otimes\mathbf{t}_j} &= [\bar{\mathbf{C}}_{mn}]_{\mathbf{E}_i\otimes\mathbf{E}_j} = \begin{bmatrix} 0 & -G\bar{S}_2 & G\bar{S}_3 \\ E\bar{S}_2 & 0 & 0 \\ -E\bar{S}_3 & 0 & 0 \end{bmatrix} \\
\bar{A} &= \int_{\mathcal{B}} \frac{1}{g_0} d\xi_2 d\xi_3 \\
\bar{S}_2 &= \int_{\mathcal{B}} \frac{1}{g_0} \xi_3 d\xi_2 d\xi_3 \\
\bar{S}_3 &= \int_{\mathcal{B}} \frac{1}{g_0} \xi_2 d\xi_2 d\xi_3 \\
\bar{I}_{22} &= \int_{\mathcal{B}} \frac{1}{g_0} (\xi_3)^2 d\xi_2 d\xi_3
\end{aligned}
$$

$$
\begin{aligned}
\bar{I}_{33} &= \int_{\mathcal{B}} \frac{1}{g_0}(\xi_2)^2 d\xi_2 d\xi_3 \\
\bar{I}_{11} &= \bar{I}_{22} + \bar{I}_{33} \\
\bar{I}_{23} &= \bar{I}_{32} = \int_{\mathcal{B}} \frac{1}{g_0}\xi_2\xi_3 d\xi_2 d\xi_3
\end{aligned} \tag{143}
$$

which would be the same as given by Iura and Atluri (1989) if we replaced $\frac{1}{g_0}$ by g_0, and k_s and k_t are the correction factors for shearing and torsion, respectively.

For slender curved beams or straight beams, we may let $g_0 = 1$. If the beams are further assumed to be built of isotropic, homogeneous and linear elastic material, we may take the beam cross-section geometry centroid line as the beam reference curve and align $\mathbf{t}_2$ and $\mathbf{t}_3$ to coincide with the cross-section principal axes. Then, $[\mathbf{C}_{nn}]$ and $[\mathbf{C}_{mm}]$ become diagonal, and $\mathbf{C}_{nm}$ and $\mathbf{C}_{mn}$ vanish. The simple constitutive form, used by Simo (1985) and Simo and Vu-Quoc (1986), Simo *et al* (1995), etc., is recovered.

6.2 Material nonlinearity and its linearization

For finite strain problems, the linearity for the constitutive relations may not be applicable. The problems also involve material nonlinearities. Assume that the nonlinear stress and strain relations exist, which are, for example, time-and-rate independent,

$$
\bar{\sigma}^0_{1i} = \bar{\sigma}^0_{1i}\left(\frac{\bar{\epsilon}_{rj}}{g_0}\right) \tag{144}
$$

then, we may obtain the tangential coefficients as

$$
\bar{\mathcal{C}}_{ij} = g_0(\bar{\sigma}^0_{1i})_{,\bar{\epsilon}_{rj}} \tag{145}
$$

which are dependent on the strain state $\frac{\bar{\epsilon}_{rj}}{g_0}$ and may vary over different material points.

In this case, the linearized constitutive relations may be used for the solution procedure of Newton type, i.e.,

$$
\begin{aligned}
\tilde{\Delta}_a \boldsymbol{\sigma}^0_1 &= \frac{1}{g_0}\boldsymbol{\mathcal{C}}\tilde{\Delta}_a \boldsymbol{\epsilon}_r \\
\Delta_a \bar{\boldsymbol{\sigma}}^0_1 &= \frac{1}{g_0}\bar{\boldsymbol{\mathcal{C}}}\Delta_a \bar{\boldsymbol{\epsilon}}_r
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\Delta}_a \boldsymbol{\sigma}_1^0 &= \boldsymbol{\Lambda} \Delta_a \bar{\boldsymbol{\sigma}}_1^0 \\
\tilde{\Delta}_a \boldsymbol{\epsilon}_r &= \boldsymbol{\Lambda} \Delta_a \bar{\boldsymbol{\epsilon}}_r
\end{aligned} \tag{146}
$$

in the three-dimensional beam level, and (also see e.g. Simo, 1985)

$$
\begin{aligned}
\tilde{\Delta}_a \mathbf{n} &= \mathbf{C}_{nn} \tilde{\Delta}_a \boldsymbol{\varepsilon}_r + \mathbf{C}_{nm} \tilde{\Delta}_a \boldsymbol{\kappa}_r \\
\tilde{\Delta}_a \mathbf{m} &= \mathbf{C}_{mn} \tilde{\Delta}_a \boldsymbol{\varepsilon}_r + \mathbf{C}_{mm} \tilde{\Delta}_a \boldsymbol{\kappa}_r \\
\Delta_a \bar{\mathbf{n}} &= \bar{\mathbf{C}}_{nn} \Delta_a \bar{\boldsymbol{\varepsilon}}_r + \bar{\mathbf{C}}_{nm} \Delta_a \bar{\boldsymbol{\kappa}}_r \\
\Delta_a \bar{\mathbf{m}} &= \bar{\mathbf{C}}_{mn} \Delta_a \bar{\boldsymbol{\varepsilon}}_r + \bar{\mathbf{C}}_{mm} \Delta_a \bar{\boldsymbol{\kappa}}_r \\
\tilde{\Delta}_a \mathbf{n} &= \boldsymbol{\Lambda} \Delta_a \bar{\mathbf{n}} = \Delta_a \mathbf{n} - \Delta_a \mathbf{w} \times \mathbf{n} \\
\tilde{\Delta}_a \mathbf{m} &= \boldsymbol{\Lambda} \Delta_a \bar{\mathbf{m}} = \Delta_a \mathbf{m} - \Delta_a \mathbf{w} \times \mathbf{m} \\
\tilde{\Delta}_a \boldsymbol{\varepsilon}_r &= \boldsymbol{\Lambda} \Delta_a \bar{\boldsymbol{\varepsilon}}_r = \Delta_a \boldsymbol{\varphi}_{,\xi_1} - \Delta_a \mathbf{w} \times \boldsymbol{\varphi}_{,\xi_1} \\
\tilde{\Delta}_a \boldsymbol{\kappa}_r &= \boldsymbol{\Lambda} \Delta_a \bar{\boldsymbol{\kappa}}_r = (\Delta_a \mathbf{w})_{,\xi_1}
\end{aligned} \tag{147}
$$

in the one-dimensional beam level, where $\tilde{\Delta}_a(\cdot)$ is the co-rotated derivative in the linearized incremental form (see Section 4.5). $\boldsymbol{\mathcal{C}}$ ($\bar{\boldsymbol{\mathcal{C}}}$) and $\mathbf{C}_{pg}$ ($\bar{\mathbf{C}}_{pg}$) will be understood as the tangential material coefficient tensors, dependent on deformation. The rest is the same as the linear case. However, attention should be paid that the stresses in the three-dimensional level should be updated according to the nonlinear relations between the stresses and strains. Then, Eqs. (94) and (95) should be used to update the stress resultants and couples.

Note that Eq. (147) is used to linearize the internal virtual work in Eq. (138) for both linear and nonlinear constitutive relations. For dynamic problems, the (co-rotated) linearized increment operators in Eq. (147) may be replaced by the corresponding (co-rotated) time rate operators.

Chapter 7: A general formulation of geometrically exact nonlinear curved beam elements

7.1 External loads

External loads applied to a structure are very complex in practice. The interaction between the structure and its environment must be considered. In the present study, only simplified distributed loads are considered in the formulation, while the concentrated loads can be treated with no additional effort but will not appear in the formulation.

Here, three types of simplified distributed loads are considered to model self-weight, snow, and pressure loads, each of which is given in the form of some load density. The relevant definitions are in Table 2 and the descriptions following it.

Table 2 Applied external load densities

Load type	Force density	Moment density
I: Self-weight	$\boldsymbol{\mathcal{N}}_g(\xi_1) = \mathcal{N}_{gj}(\xi_1)\mathbf{e}_j$	$\boldsymbol{\mathcal{M}}_g(\xi_1) = \mathcal{M}_{gj}(\xi_1)\mathbf{e}_j$
II: Snow	$\boldsymbol{\mathcal{N}}_d(s_d) = \mathcal{N}_{dj}(s_d)\mathbf{e}_j$	$\boldsymbol{\mathcal{M}}_d(s_d) = \mathcal{M}_{dj}(s_d)\mathbf{e}_j$
III: Pressure	$\boldsymbol{\mathcal{N}}_p(\xi_1) = \bar{\mathcal{N}}_{pj}(\xi_1)\mathbf{t}_j$	$\boldsymbol{\mathcal{M}}_p(\xi_1) = \bar{\mathcal{M}}_{pj}(\xi_1)\mathbf{t}_j$

Note that all the components of the applied load densities in Table 2 are constant for given ξ_1 in load types I and III and s_d in load type II.

Type I: The applied load density is given by per unit *unstressed arc-length* of the beam midcurve and referred to the global fixed frame $\mathbf{e}_i$, invariant with respect to the deformation of the beam. Self-weight may be defined by this type of load. Therefore, this type of load may be called self-weight type of load.

The differential force, $d\mathbf{f}_g$, and moment, $d\mathbf{m}_g$, exerted on the arc-element $d\xi_1$ are calculated as follows:

$$d\mathbf{f}_g = \lambda \boldsymbol{\mathcal{N}}_g(\xi_1)d\xi_1 \tag{148}$$

$$d\mathbf{m}_g = \lambda \boldsymbol{\mathcal{M}}_g(\xi_1)d\xi_1 \tag{149}$$

where λ is the proportional loading factor. This type of load is deformation invariant and usually conservative.

Type II: The applied load density is given as constant in space in the sense that the load acting on unit projection length ds_d of the deformed arc-element ds corresponding to the undeformed arc-element $d\xi_1$ at a material point ξ_1 on the midcurve onto any plane with normal $\mathbf{d}_N = \frac{\boldsymbol{\mathcal{N}}_d}{||\boldsymbol{\mathcal{N}}_d||} \in R^3$ is constant, given by

$$d\mathbf{f}_d = \int_0^\lambda \boldsymbol{\mathcal{N}}_d ds_d d\lambda \tag{150}$$

where $\boldsymbol{\mathcal{N}}_d$ (therefore, $\mathbf{d}_N$) is constant with respect to both space and the beam itself; however, ds_d depends on the deformation and motion of the beam.

ds_d relates the deformation and undeformed arc-element $d\xi_1$ as well as the direction of $\boldsymbol{\mathcal{N}}_d$ by

$$\begin{aligned} ds_d &= [(\mathbf{d}_N \times \boldsymbol{\varphi}_{,s}) \times \mathbf{d}_N] \cdot \boldsymbol{\varphi}_{,\xi_1} d\xi_1 \\ &= -[(\hat{\mathbf{d}}_N)^2 \boldsymbol{\varphi}_{,s}] \cdot \boldsymbol{\varphi}_{,\xi_1} d\xi_1 \end{aligned} \tag{151}$$

Assuming that the extension or elongation of the beam midcurve is small and can be ignored, i.e. $ds = d\xi_1$, the above equation for ds_d can be simplified as

$$ds_d = -[(\hat{\mathbf{d}}_N)^2 \boldsymbol{\varphi}_{,\xi_1}] \cdot \boldsymbol{\varphi}_{,\xi_1} d\xi_1 \tag{152}$$

Therefore, Eq. (150) can be rewritten as

$$\begin{aligned} d\mathbf{f}_d &= \int_0^\lambda \boldsymbol{\mathcal{N}}_d\{-[(\hat{\mathbf{d}}_N)^2\boldsymbol{\varphi}_{,\xi_1}]\cdot\boldsymbol{\varphi}_{,\xi_1}d\xi_1\}d\lambda \\ &= \boldsymbol{\mathcal{N}}_d\int_0^\lambda\{-[(\hat{\mathbf{d}}_N)^2\boldsymbol{\varphi}_{,\xi_1}]\cdot\boldsymbol{\varphi}_{,\xi_1}\}d\lambda d\xi_1 \\ &= \lambda c_N\mathbf{N}_d d\xi_1 \end{aligned} \tag{153}$$

where

$$c_N = -\frac{1}{\lambda}\int_0^\lambda[(\hat{\mathbf{d}}_N)^2\boldsymbol{\varphi}_{,\xi_1}]\cdot\boldsymbol{\varphi}_{,\xi_1}d\lambda \tag{154}$$

Similarly, we may define the differential moment $d\mathbf{m}_d$, exerted on the arc-element $d\xi_1$ of the beam midcurve, as

$$d\mathbf{m}_d = \lambda c_M\boldsymbol{\mathcal{M}}_d d\xi_1 \tag{155}$$

where

$$\begin{aligned} c_M &= -\frac{1}{\lambda}\int_0^\lambda[(\hat{\mathbf{d}}_M)^2\boldsymbol{\varphi}_{,\xi_1}]\cdot\boldsymbol{\varphi}_{,\xi_1}d\lambda \\ \mathbf{d}_M &= \frac{\boldsymbol{\mathcal{M}}_d}{||\boldsymbol{\mathcal{M}}_d||}\in R^3 \end{aligned} \tag{156}$$

Note that both $d\mathbf{f}_d$ and $d\mathbf{m}_d$ are *dependent* on deformation. The force density $\boldsymbol{\mathcal{N}}_d$ may be a good approximation for snow loads acting on a uniform beam when the loading process of snowing is addressed and the snow itself may cause relatively large displacements/rotations if the slipping of snow is not considered. Therefore, when large displacements/rotations are considered, the snow loads can be non-conservative.

On the other hand, when displacements/rotations due to snow loading are small and other loads dominate, all the snow can be considered to add onto the structure before deformation and then move along with the structure if the slipping of snow is not considered. In this case, the coefficients for the differential force $d\mathbf{f}_d$, and moment, $d\mathbf{m}_d$, exerted on the arc-element $d\xi_1$ may be calculated as

$$c_{N0} = -[(\hat{\mathbf{d}}_N)^2\boldsymbol{\varphi}_{0,\xi_1}]\cdot\boldsymbol{\varphi}_{0,\xi_1} \tag{157}$$

$$c_{M0} = -[(\hat{\mathbf{d}}_M)^2 \boldsymbol{\varphi}_{0,\xi_1}] \cdot \boldsymbol{\varphi}_{0,\xi_1} \tag{158}$$

which are really *independent* of deformation and may be conservative.

Type III: The applied load density is given by per unit *unstressed arc-length* of the beam midcurve referred to the local moving frame $\mathbf{t}_i$, invariant with respect to the deformation of the beam. Pressure or hydrostatic (wind) loads belong to this type if both the shear deformation and cross-section variation along the beam axis are small and may be ignored.

The differential force, $d\mathbf{f}_p$, and moment, $d\mathbf{m}_p$, exerted on the arc-length $d\xi_1$ are calculated as follows:

$$d\mathbf{f}_p = \lambda \boldsymbol{\mathcal{N}}_p d\xi_1 = \lambda \boldsymbol{\Lambda} \bar{\boldsymbol{\mathcal{N}}}_p d\xi_1 \tag{159}$$

$$d\mathbf{m}_p = \lambda \boldsymbol{\mathcal{M}}_p d\xi_1 = \lambda \boldsymbol{\Lambda} \bar{\boldsymbol{\mathcal{M}}}_p d\xi_1 \tag{160}$$

where $\bar{\boldsymbol{\mathcal{N}}}_p = \bar{\mathcal{N}}_{pj} \mathbf{e}_j$ and $\bar{\boldsymbol{\mathcal{M}}}_p = \bar{\mathcal{M}}_{pj} \mathbf{e}_j$ are in the material form. This type of load depends on rotational displacements and therefore is non-conservative.

See Fig. 3 for the demonstration of the three types of loads.

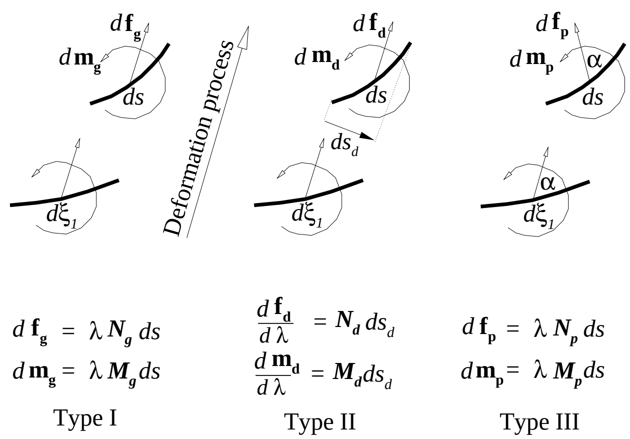

Fig. 3 Different load models considered

Now define

$$\boldsymbol{\mathcal{N}} = \lambda[\boldsymbol{\mathcal{N}}_g + c_N \boldsymbol{\mathcal{N}}_d + \boldsymbol{\mathcal{N}}_p + \cdots] \tag{161}$$

as the force density, and

$$\boldsymbol{\mathcal{M}} = \lambda[\boldsymbol{\mathcal{M}}_g + c_M \boldsymbol{\mathcal{M}}_d + \boldsymbol{\mathcal{M}}_p + \cdots] \tag{162}$$

as the moment density along the beam midcurve at current loading in general.

7.2 Parametrization of finite rotations

In the previous context, the so-called rotation tensor $\mathbf{\Lambda}_r \in SO(3)$ has been frequently encountered, e.g., Eqs. (12) and (13), Table 1, and wherever the orthogonal tensor $\mathbf{\Lambda} \in SO(3)$ or the base vectors $\mathbf{t}_i$ of the local moving frame are involved, which is in contrast to small rotation problems. The proper orthogonal rotation tensor $\mathbf{\Lambda}_r$ consists of nine parameters, of which only three are independent because of $\mathbf{\Lambda}_r \mathbf{\Lambda}_r^t = \mathbf{I}_3$, the orthogonality condition. Therefore, it seems necessary to parametrize the rotation tensor using three *independent* parameters (see e.g. Iura and Atluri, 1989; etc.), which represent the proper three degrees of freedom of the rotation, so that the three independent rotation parameters and the associated rotation tensor as well as their linearized increments and admissible variations are well defined (one-to-one mapping) and updated, which is the key to represent in detail the virtual work equation (equilibrium equation) and its linearization (to obtain the tangential stiffness matrix for the solution procedures of Newton type). In what follows, therefore, a brief summary of the parametrizations of finite rotations is given and discussed.

In the available literature on finite rotations, the dominant methods to parametrize the rotation tensor $\mathbf{\Lambda}_r$ are to use Euler angles (the three angles rotated in a certain sequence about three reference axes attached to the rotated rigid body), the rotation vectors (to be defined soon), as well as quaternion parameters (e.g. see Argyris, 1982; Simo and Vu-Quoc, 1986; Cardona and Geradin, 1988; Ibrahimbegović *et al*, 1995; Betsch *et al*, 1998; and references therein), of which there are total parametrizations and incremental parametrizations as well as the combinations, leading to multiplicative or additive updating procedures and symmetric or non-symmetric tangential stiffness matrices. Efforts have been made to overcome the singularities (rank deficient matrices) among those parametrizations. In what follows, only the rotation vectors (including the incremental rotation vector in the spatial form) and quaternion parametrizations are addressed.

• **Rotation tensor, rotation vector and compound rotation**

When a rigid body rotates from one orientation to another with a *finite* angle $||\boldsymbol{\theta}||$ about a *spatially fixed axis* (e.g. see Argyris, 1982) represented by a unit vector $\mathbf{i}_\theta \in R^3$, the rotation will carry a vector $\mathbf{v} \in R^3$, which is rigidly attached to the rigid body, to be $\mathbf{v}' \in R^3$ on a *cone* around axis $\mathbf{i}_\theta$. This rotation can be described mathematically using an *orthogonal* rotation tensor $\boldsymbol{\Lambda}_r \in SO(3)$, via the (exact) linear transformation (mapping) (e.g. see Argyris, 1982)

$$\mathbf{v}' = \boldsymbol{\Lambda}_r \mathbf{v} \in R^3 \tag{163}$$

while the so-called rotation vector is defined as (e.g. see Argyris, 1982)

$$\boldsymbol{\theta} = \|\boldsymbol{\theta}\|\, \mathbf{i}_\theta \in R^3 \tag{164}$$

Argyris (1982) used a simple geometric approach to confirm the famous Rodrigues formula:

$$\boldsymbol{\Lambda}_r(\boldsymbol{\theta}) = \mathbf{I}_3 + \frac{\sin\|\boldsymbol{\theta}\|}{\|\boldsymbol{\theta}\|}\hat{\boldsymbol{\theta}} + \frac{1-\cos\|\boldsymbol{\theta}\|}{(\|\boldsymbol{\theta}\|)^2}(\hat{\boldsymbol{\theta}})^2 \tag{165}$$

where $\hat{\boldsymbol{\theta}} \in so(3)$ is the skew symmetric tensor associated with the axial vector $\boldsymbol{\theta}$.

Using the Taylor series for $\sin\|\boldsymbol{\theta}\|$ and $\cos\|\boldsymbol{\theta}\|$ and the relations $(\hat{\mathbf{i}}_\theta)^{2k-1} = (-1)^{k-1}\hat{\mathbf{i}}_\theta$, $(\hat{\mathbf{i}}_\theta)^{2k} = (-1)^{k-1}(\hat{\mathbf{i}}_\theta)^2$ for $k = 1, 2, \cdots$, and $\mathbf{i}_\theta = \boldsymbol{\theta}/\|\boldsymbol{\theta}\|$, it can be easily verified that (e.g. see Argyris, 1982):

$$\boldsymbol{\Lambda}_r(\boldsymbol{\theta}) = \mathbf{I}_3 + \frac{\hat{\boldsymbol{\theta}}}{1!} + \frac{(\hat{\boldsymbol{\theta}})^2}{2!} + \cdots = \exp[\hat{\boldsymbol{\theta}}] \tag{166}$$

Therefore, the rotation tensor $\boldsymbol{\Lambda}_r$ is an exponential mapping (e.g. see Simo and Vu-Quoc, 1986; etc.).

According to Euler's theorem, when a rigid body rotates from one orientation to another, which may be the result of a series of rotations with one rotation superposed onto the previous one, the rotation will be equivalent to a single *compound rotation* (e.g. see Argyris, 1982) around a *spatial fixed axis*. In fact, one can easily confirm, by post-multiplication of both sides of Eq. (165) with $\boldsymbol{\theta}$ and noticing the identity $\hat{\boldsymbol{\theta}}\boldsymbol{\theta} = \mathbf{0}$ for a skew symmetric tensor, that the rotation vector $\boldsymbol{\theta}$ is the eigenvector of the rotation tensor $\boldsymbol{\Lambda}_r$ with 1 as the eigenvalue (see e.g. Argyris, 1982):

$$(\boldsymbol{\Lambda}_r - \mathbf{I}_3)\boldsymbol{\theta} = \mathbf{0} \tag{167}$$

Hence, the rotation tensor $\boldsymbol{\Lambda}_r$ carries its corresponding rotation vector $\boldsymbol{\theta}$ to itself or does not affect its corresponding rotation vector. Therefore, in the above context and later, the rotation vector $\boldsymbol{\theta}$ can and will be understood as a compound rotation, which globally or *totally* parametrizes the compound rotation tensor $\boldsymbol{\Lambda}_r$ via the Rodrigues formula (Eq. 165).

In general, *if the compound rotation tensor $\boldsymbol{\Lambda}_r$ can be uniquely represented by some real parameters $p^k \in R^1$, $(k = 1, 2, 3, \cdots)$, i.e. $\boldsymbol{\Lambda}_r = \boldsymbol{\Lambda}_r(p^1, p^2, p^3, \cdots)$, then this kind of parametrization of the finite rotations can be called total parametrization.* Therefore, the use of the rotation vector $\boldsymbol{\theta} = \theta_j \mathbf{e}_j$ to represent the rotation tensor as in Eq. (165) is a total parametrization. The use of the nine components Λ_{ij} of the rotation tensor $\boldsymbol{\Lambda}_r = \Lambda_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$ is also a total parametrization. In the case of the use of the compound rotation vector $\boldsymbol{\theta}$ to parametrize the compound rotation tensor $\boldsymbol{\Lambda}_r$ as in Eq. (165), for given $\boldsymbol{\theta}$, $\boldsymbol{\Lambda}_r$ can be uniquely determined, but for given $\boldsymbol{\Lambda}_r$, $\boldsymbol{\theta}$ cannot be uniquely determined without knowledge of the number of full cycles because of the periodical nature of the sine and cosine functions.

• Incremental rotation and multiplicative updating rule

Following the previous compound rotation, e.g. parametrized by the rotation vector as $\boldsymbol{\Lambda}_r = \boldsymbol{\Lambda}_r(\boldsymbol{\theta}) = \exp[\hat{\boldsymbol{\theta}}]$, if there is an *incremental* rotation with an angle $\|\Delta_a \mathbf{w}\|$ about a *spatially fixed* axis represented by a unit vector $\mathbf{i}_w \in R^3$, the rotation will carry the previously rotated vector (on the rigid body) $\mathbf{v}'$ to be $\mathbf{v}'' \in R^3$ on a *cone* around axis $\mathbf{i}_w$. If, similar to defining the rotation vector $\boldsymbol{\theta}$ previously, the incremental rotation is (directly) described by the *incremental rotation vector* $\Delta_a \mathbf{w}$ defined as

$$\Delta_a \mathbf{w} = \|\Delta_a \mathbf{w}\|\, \mathbf{i}_w \tag{168}$$

whose corresponding rotation tensor can be determined using the Rodrigues formula (Eq. 165) or its equivalent exponential mapping form

$$\mathbf{\Lambda}_{rw} = \mathbf{\Lambda}_r(\Delta_a \mathbf{w}) = \exp[\Delta_a \hat{\mathbf{w}}] \tag{169}$$

then

$$\mathbf{v}'' = \mathbf{\Lambda}_{rw}\mathbf{v}' = \mathbf{\Lambda}_{rw}\mathbf{\Lambda}_r\mathbf{v} = \mathbf{\Lambda}'_r\mathbf{v} \in R^3 \tag{170}$$

where

$$\mathbf{\Lambda}'_r = \mathbf{\Lambda}_{rw}\mathbf{\Lambda}_r \tag{171}$$

is the *updated compound rotation* tensor corresponding to the *previous compound rotation* $\mathbf{\Lambda}_r$ followed by the *superposed incremental rotation* $\mathbf{\Lambda}_{rw}$ – that is, the *reverse-ordered multiplicative rule* applies to obtaining the compound rotation tensor of a series of rotations about a series of *spatially fixed axes* with one rotation following the previous compound one.

It should be noted that Eq. (171) is, in fact, independent of any particular parametrizations of the compound rotation tensor $\mathbf{\Lambda}_r$ and incremental rotation tensor $\mathbf{\Lambda}_{rw}$. However, similar to the use of the compound rotation vector $\boldsymbol{\theta}$ to parametrize the compound rotation, the use of the incremental rotation vector $\Delta_a \mathbf{w}$ to parametrize the incremental rotational tensor $\mathbf{\Lambda}_{rw}$ is a more *natural and convenient* choice.

Consider the new compound rotation vector

$$\boldsymbol{\theta}' = \boldsymbol{\theta} + \Delta_a \boldsymbol{\theta} \tag{172}$$

which parametrizes the new compound rotation tensor $\mathbf{\Lambda}'_r$ as in Eq. (171), with $\Delta_a \boldsymbol{\theta}$ the *increment of the rotation vector* $\boldsymbol{\theta}$ due to the incremental rotation described by the incremental rotation vector $\Delta_a \mathbf{w}$.

Now rewrite Eq. (171) in two parts in the form with a scalar parameter ε as follows:

$$\mathbf{\Lambda}'_{r\varepsilon} = \mathbf{\Lambda}_r(\varepsilon \Delta_a \mathbf{w})\mathbf{\Lambda}_r(\boldsymbol{\theta}) \tag{173}$$

and

$$\mathbf{\Lambda}_r(\varepsilon\Delta_a\mathbf{w}) = \mathbf{\Lambda}_r(-\boldsymbol{\theta})\mathbf{\Lambda}'_r(\boldsymbol{\theta}+\varepsilon\Delta_a\boldsymbol{\theta}) \tag{174}$$

Taking the derivative of Eq. (173) with respective to ε and then setting $\varepsilon = 0$, one can, using the Rodrigues formula Eq. (165), obtain the formula for the linearized increment of the rotation tensor:

$$\Delta_a\mathbf{\Lambda}_r = \frac{\partial}{\partial\varepsilon}[\mathbf{\Lambda}_r(\varepsilon\Delta_a\mathbf{w})\mathbf{\Lambda}_r(\boldsymbol{\theta})]_{\varepsilon=0} = \Delta_a\hat{\mathbf{w}}\mathbf{\Lambda}_r \tag{175}$$

or the admissible variation of the rotation tensor (see e.g. Simo, 1985; Simo and Vu-Quoc, 1986; Ibrahimbegović, 1995; etc.):

$$\delta_a\mathbf{\Lambda}_r = \delta_a\hat{\mathbf{w}}\mathbf{\Lambda}_r \tag{176}$$

One can observe that, because of $\delta_a\hat{\mathbf{w}}$ being skew symmetric, the linearized increment or admissible variation of the rotation tensor, $\delta_a\mathbf{\Lambda}_r$, is no longer orthogonal. In fact, $\delta_a\hat{\mathbf{w}}$ belongs to the tangential space of the rotation tensor $\mathbf{\Lambda}_r \in SO(3)$ (see. e.g. Simo and Vu-Quoc, 1986; Ibrahimbegović and Frey, 1995; etc.). Therefore, Eq. (171) is used for updating the rotation tensor $\mathbf{\Lambda}_r$ while Eqs. (175) and (176) are used for the linearized increment and admissible variation operations, respectively, if the incremental rotation is parametrized using the incremental rotation vector.

Similar to Eq. (175), taking the derivative of both sides of Eq. (174) with respect to the scalar parameter ε and setting $\varepsilon = 0$, one can, repeatedly using the Rodrigues formula (Eq. 165), obtain the linearized relation between the incremental rotation vector, $\Delta_a\mathbf{w}$, and the increment of the rotation vector, $\Delta_a\boldsymbol{\theta}$, as (see e.g. Ibrahimbegović, 1995; Ibrahimbegović *et al*, 1995;)

$$\Delta_a\mathbf{w} = \mathbf{T}_{w\theta}\Delta_a\boldsymbol{\theta} \tag{177}$$

where

$$\mathbf{T}_{w\theta} = \mathbf{T}_{w\theta}(\boldsymbol{\theta}) = \frac{\sin\|\boldsymbol{\theta}\|}{\|\boldsymbol{\theta}\|}\mathbf{I}_3 + \frac{1-\cos\|\boldsymbol{\theta}\|}{\|\boldsymbol{\theta}\|^2}\hat{\boldsymbol{\theta}} + \frac{\|\boldsymbol{\theta}\|-\sin\|\boldsymbol{\theta}\|}{\|\boldsymbol{\theta}\|^3}\boldsymbol{\theta}\otimes\boldsymbol{\theta} \tag{178}$$

and the determinant of $\mathbf{T}_{w\theta}$ is (e.g. see Ibrahimbegović, 1995; Ibrahimbegović *et al*, 1995; etc.)

$$\det\mathbf{T}_{w\theta} = \frac{2(1-\cos\|\boldsymbol{\theta}\|)}{(\|\boldsymbol{\theta}\|)^2} \tag{179}$$

Therefore, $\Delta_a \boldsymbol{\theta} \neq \Delta_a \mathbf{w}$ for general 3-D rotations unless $\Delta_a \mathbf{w}$ and $\boldsymbol{\theta}$ are coaxial. Besides, when $\|\boldsymbol{\theta}\| = 2k\pi$ for $k = 1, 2, 3, \cdots$, $\mathbf{T}_{w\theta}(\boldsymbol{\theta})$ becomes rank deficient, which may lead the tangential stiffness tensor to be rank deficient too. In this case, iterative solution procedures of Newton type do not converge and *give spurious bifurcation information because the determinant of the tangential stiffness matrix vanishes.* Apart from the above singular points, the following inverse transformation is well defined (e.g. see Ibrahimbegović, 1995; Ibrahimbegović *et al*, 1995; etc.):

$$\Delta_a \boldsymbol{\theta} = \mathbf{T}_{\theta w} \Delta_a \mathbf{w} \tag{180}$$

where

$$\mathbf{T}_{\theta w} = \mathbf{T}_{\theta w}(\boldsymbol{\theta}) = \mathbf{T}_{w\theta}^{-1} = \frac{\|\boldsymbol{\theta}\|/2}{\tan(\|\boldsymbol{\theta}\|/2)} \mathbf{I}_3 - \frac{1}{2}\hat{\boldsymbol{\theta}} + \frac{1}{(\|\boldsymbol{\theta}\|)^2}\left[1 - \frac{\|\boldsymbol{\theta}\|/2}{\tan(\|\boldsymbol{\theta}\|/2)}\right] \boldsymbol{\theta} \otimes \boldsymbol{\theta} \tag{181}$$

To avoid the singularity due to the use of the rotation vector to parametrize the rotation tensor, a rescaling remedy is available (e.g. see Ibrahimbegović *et al.*) as follows when $\|\boldsymbol{\theta}\| > \pi$ is identified:

$$\boldsymbol{\theta}^* = \boldsymbol{\theta} - 2n\pi \mathbf{i}_\theta \tag{182}$$

where $n = \mathrm{int}[(\|\boldsymbol{\theta}\| + \pi)/2\pi]$, the number of full-cycle rotations. This remedy makes sure $\|\boldsymbol{\theta}^*\| \in [-\pi, \pi]$, and therefore overcomes the singularity.

It should be addressed that either Eq. (177) or (180) is a *linearized* relation between $\Delta_a \boldsymbol{\theta}$ and $\Delta_a \mathbf{w}$ in the general 3–D case. For a 2–D problem, however, one can confirm that $\Delta_a \mathbf{w} = \Delta_a \boldsymbol{\theta}$ as mentioned previously though $\mathbf{T}_{\theta w} \neq \mathbf{I}_3$ or $\mathbf{T}_{w\theta} \neq \mathbf{I}_3$ in general.

• Quaternion parametrization and quaternion multiplicative updating rule

Another way to avoid a singularity is to use Euler parameters or quaternion parameters to parametrize the rotation tensor (see e.g. Argyris, 1982; Simo and Vu-Quoc, 1986; Ibrahimbegović, 1995; Betsch *et al*, 1998; and the relevant references therein). The *unit* quaternion parameters $\{q_0, \mathbf{q}\}$ associated with the rotation vector $\boldsymbol{\theta}$ are defined as (see e.g. Argyris, 1982; Simo and Vu-Quoc, 1986; Betsch *et al*, 1998; and the relevant references therein)

$$\{q_0, \mathbf{q}\} = \{\cos\frac{\|\boldsymbol{\theta}\|}{2}, \frac{\boldsymbol{\theta}}{\|\boldsymbol{\theta}\|}\sin\frac{\|\boldsymbol{\theta}\|}{2}\} \tag{183}$$

and similarly, the unit quaternion representing the rotation tensor that corresponds to the incremental rotation, $\Delta_a\mathbf{w}$, is given as (e.g. see Simo and Vu-Quoc, 1986; Ibrahimbegović, 1995)

$$\{q_{0w}, \mathbf{q}_w\} = \{\cos\frac{\|\Delta_a\mathbf{w}\|}{2}, \frac{\Delta_a\mathbf{w}}{\|\Delta_a\mathbf{w}\|}\sin\frac{\|\Delta_a\mathbf{w}\|}{2}\} \tag{184}$$

Note that the normality conditions $(q_0)^2 + \mathbf{q}\cdot\mathbf{q} = 1$ and $(q_{0w})^2 + \mathbf{q}_w\cdot\mathbf{q}_w = 1$ have to be satisfied.

In order to establish the relations between the unit quaternions and the associated rotation tensors, one can obtain the alternative form of the Rodrigues formula (Eq. 165) using $\boldsymbol{\theta} = \|\boldsymbol{\theta}\|\,\mathbf{i}_\theta$ and the standard relation $(\hat{\boldsymbol{\theta}})^2 = \boldsymbol{\theta}\otimes\boldsymbol{\theta} - (\|\boldsymbol{\theta}\|)^2\mathbf{I}_3$ as follows (see e.g. Betsch *et al*, 1998):

$$\boldsymbol{\Lambda}_r(\boldsymbol{\theta}) = \boldsymbol{\Lambda}_r(\|\boldsymbol{\theta}\|\,\mathbf{i}_\theta) = \cos\|\boldsymbol{\theta}\|\mathbf{I}_3 + (1-\cos\|\boldsymbol{\theta}\|)\mathbf{i}_\theta\otimes\mathbf{i}_\theta + \sin\|\boldsymbol{\theta}\|\hat{\mathbf{i}}_\theta \tag{185}$$

Using the standard relations

$$\cos\|\boldsymbol{\theta}\| = 2(\cos\frac{\|\boldsymbol{\theta}\|}{2})^2 - 1$$

$$1-\cos\|\boldsymbol{\theta}\| = 2(\sin\frac{\|\boldsymbol{\theta}\|}{2})^2$$

and

$$\sin\|\boldsymbol{\theta}\| = 2\sin\frac{\|\boldsymbol{\theta}\|}{2}\cos\frac{\|\boldsymbol{\theta}\|}{2}$$

as well as the definition of the unit quaternion in Eq. (183), it can be easily confirmed that (see e.g. Argyris, 1982; Ibrahimbegović, 1995; Betsch *et al*, 1998; and the relevant references therein)

$$\boldsymbol{\Lambda}_r = \boldsymbol{\Lambda}_r(q_0, \mathbf{q}) = [2(q_0)^2 - 1]\mathbf{I}_3 + 2q_0\hat{\mathbf{q}} + 2\mathbf{q}\otimes\mathbf{q} \tag{186}$$

and therefore,

$$\boldsymbol{\Lambda}_{rw} = \boldsymbol{\Lambda}_r(q_{0w}, \mathbf{q}_w) = [2(q_{0w})^2 - 1]\mathbf{I}_3 + 2q_{0w}\hat{\mathbf{q}}_w + 2\mathbf{q}_w\otimes\mathbf{q}_w \tag{187}$$

Substituting the above two equations into Eq. (171), one can verify that the new compound rotation tensor can be written as

$$\mathbf{\Lambda}'_r = \mathbf{\Lambda}_r(q'_0, \mathbf{q}') = \mathbf{\Lambda}_r(q_{0w}, \mathbf{q}_w)\mathbf{\Lambda}_r(q_0, \mathbf{q}) = [2(q'_0)^2 - 1]\mathbf{I}_3 + 2q'_0\hat{\mathbf{q}}' + 2\mathbf{q}' \otimes \mathbf{q}' \tag{188}$$

where

$$q'_0 = q_0 q_{0w} - \mathbf{q} \cdot \mathbf{q}_w \tag{189}$$

and

$$\mathbf{q}' = q_0\mathbf{q}_w + q_{0w}\mathbf{q} - \hat{\mathbf{q}}\mathbf{q}_w \tag{190}$$

are the unit quaternion parameters for the new compound rotation following the *quaternion multiplicative rule* (see e.g. Argyris, 1982; Ibrahimbegović, 1995; Betsch *et al*, 1998; etc.)

• A general form of vector-like parametrizations of finite rotations

Assume that there exists a vector $\mathbf{p} = p_j\mathbf{e}_j \in R^3$ with three independent components p_i, which can uniquely determine the rotation tensor, i.e.

$$\mathbf{\Lambda}_r = \mathbf{\Lambda}_r(\mathbf{p}) \tag{191}$$

One expects to establish the additive update rule

$$\mathbf{p}' = \mathbf{p} + \Delta_a\mathbf{p} \tag{192}$$

but has to determine the linearized transformation relation between $\Delta_a\mathbf{p}$ and incremental rotation vector $\Delta_a\mathbf{w}$

$$\Delta_a\mathbf{p} = \mathbf{T}_{pw}(\mathbf{p})\Delta_a\mathbf{w} \tag{193}$$

and its inverse linearized transformation

$$\Delta_a\mathbf{w} = \mathbf{T}_{wp}(\mathbf{p})\Delta_a\mathbf{p} \tag{194}$$

because the incremental rotation vector $\Delta_a\mathbf{w}$ has a clear geometrical and physical meaning. Usually, $\Delta_a\mathbf{p} \neq \Delta_a\mathbf{w}$ or $\mathbf{T}_{wp} \neq \mathbf{I}_3$, and the relation between $\mathbf{p}$ and $\Delta_a\mathbf{w}$ is a nonlinear differential manifold (Simo and Vu-Quoc, 1986; etc.). Equation (193) serves as projecting

$\Delta_a \mathbf{w}$ onto the linear vector space to which $\mathbf{p}$ belongs, so that the simple additive update rule Eq. (192) can be used. The parametrizations of the compound rotation using $\mathbf{p}$ through Eq. (191) and incremental rotation vector $\Delta_a \mathbf{w}$ using $\Delta_a \mathbf{p}$ through Eq. (194) can be called vector-like parametrizations (see e.g. Ibrahimbegović *et al*, 1995) or consistent linear parametrization (see e.g. Pacoste and Eriksson, 1997). These, in fact, belong to the class of total Lagrangian descriptions (see e.g. Iura and Atluri, 1989).

Within the class of vector-like parametrizations, the bending strain vector has an alternative vector form

$$\boldsymbol{\kappa}_r = \boldsymbol{\kappa}_r(\mathbf{p}) = \mathbf{T}_{wp}(\mathbf{p})\mathbf{p}_{,\xi_1} \tag{195}$$

for the compound bending strains, and

$$\boldsymbol{\kappa}_{rw} = \boldsymbol{\kappa}_r(\mathbf{p}_w) = \mathbf{T}_{wp}(\mathbf{p}_w)\mathbf{p}_{w,\xi_1} \tag{196}$$

for the bending strains due to the incremental rotation vector $\Delta_a \mathbf{w}$, in which $\mathbf{p}_w$ is the parameter $\mathbf{p}$ due to incremental rotation superposed onto the configuration with vanishing rotation.

For example, when $\mathbf{p}$ is chosen to be the compound rotation vector $\boldsymbol{\theta}$, therefore, $\mathbf{p}_w = \Delta_a \mathbf{w}$, then

$$\boldsymbol{\kappa}_r = \boldsymbol{\kappa}_r(\boldsymbol{\theta}) = \mathbf{T}_{w\theta}(\boldsymbol{\theta})\boldsymbol{\theta}_{,\xi_1} \tag{197}$$

and

$$\boldsymbol{\kappa}_{rw} = \boldsymbol{\kappa}_r(\Delta_a \mathbf{w}) = \mathbf{T}_{w\theta}(\Delta_a \mathbf{w})\Delta_a \mathbf{w}_{,\xi_1} \tag{198}$$

which can be found in the papers by Ibrahimbegović (1995), Ibrahimbegović *et al*, 1995, etc.

7.3 Admissible variations/linearizations with respect to deformation

• Variations of translational displacement and position vector

It is well known that a position vector and its displacement vector are in the same vector space. The vector additive rule applies to them. From Eq. (11), one can readily obtain that

$$\delta_a \mathbf{u} = \delta_a \boldsymbol{\varphi} \tag{199}$$

• Variations of orthogonal tensors

One may repeat the procedures for Eq. (175) to obtain the admissible variation of the rotation tensor with respect to deformation:

$$\delta_a \boldsymbol{\Lambda}_r = \frac{\partial}{\partial \varepsilon}[\boldsymbol{\Lambda}_r(\varepsilon \delta_a \mathbf{w})\boldsymbol{\Lambda}_r(\boldsymbol{\theta})]_{\varepsilon=0} = \delta_a \hat{\mathbf{w}} \boldsymbol{\Lambda}_r \tag{200}$$

Therefore, the variation of the orientation tensor is

$$\delta_a \boldsymbol{\Lambda} = \left[\frac{\partial \boldsymbol{\Lambda}_\varepsilon}{\partial \varepsilon}\right]_{\varepsilon=0} = \left[\frac{\partial(\boldsymbol{\Lambda}_{r\varepsilon}\boldsymbol{\Lambda}_0)}{\partial \varepsilon}\right]_{\varepsilon=0} = \delta_a \hat{\mathbf{w}} \boldsymbol{\Lambda}_r \boldsymbol{\Lambda}_0 = \delta_a \hat{\mathbf{w}} \boldsymbol{\Lambda} \tag{201}$$

• Variations of strains

Since the variations of the orthogonal tensors have been determined, the other variations can be obtained using the chain rule of partial derivatives.

In what follows, one needs to recall the strain measures given in Table 1.

Considering $\boldsymbol{\varepsilon}_r = \boldsymbol{\varphi}_{,\xi_1} - \mathbf{t}_1$ and noticing Eq. (200) and $\mathbf{t}_1 = \boldsymbol{\Lambda}_r \mathbf{t}_{01}$ as from Eq. (12), one has the usual variation operation:

$$\begin{aligned}
\delta_a \boldsymbol{\varepsilon}_r &= \delta_a(\boldsymbol{\varphi}_{,\xi_1} - \mathbf{t}_1) \\
&= \delta_a(\boldsymbol{\varphi}_{,\xi_1} - \boldsymbol{\Lambda}_r \mathbf{t}_{01}) \\
&= \delta_a \boldsymbol{\varphi}_{,\xi_1} - \delta_a(\boldsymbol{\Lambda}_r \mathbf{t}_{01}) \\
&= \delta_a \boldsymbol{\varphi}_{,\xi_1} - \delta_a \boldsymbol{\Lambda}_r \mathbf{t}_{01} \\
&= \delta_a \boldsymbol{\varphi}_{,\xi_1} - \delta_a \hat{\mathbf{w}} \boldsymbol{\Lambda}_r \mathbf{t}_{01} \\
&= \delta_a \boldsymbol{\varphi}_{,\xi_1} - \delta_a \hat{\mathbf{w}} \mathbf{t}_1
\end{aligned} \tag{202}$$

From $\bar{\boldsymbol{\varepsilon}}_r = \boldsymbol{\Lambda}^t \boldsymbol{\varepsilon}_r$, noticing $\delta_a \boldsymbol{\Lambda}^t = -\boldsymbol{\Lambda}^t \delta_a \hat{\mathbf{w}}$ according to Eq. (201), one obtains that

$$\delta_a \bar{\boldsymbol{\varepsilon}}_r = \delta_a(\boldsymbol{\Lambda}^t \boldsymbol{\varepsilon}_r)$$

$$
\begin{aligned}
&= \delta_a \boldsymbol{\Lambda}^t \boldsymbol{\varepsilon}_r + \boldsymbol{\Lambda}^t \delta_a \boldsymbol{\varepsilon}_r \\
&= -\boldsymbol{\Lambda}^t \delta_a \hat{\mathbf{w}}(\boldsymbol{\varphi}_{,\xi_1} - \mathbf{t}_1) + \boldsymbol{\Lambda}^t(\delta_a \boldsymbol{\varphi}_{,\xi_1} - \delta_a \hat{\mathbf{w}} \mathbf{t}_1) \\
&= \boldsymbol{\Lambda}^t[-\delta_a \hat{\mathbf{w}}(\boldsymbol{\varphi}_{,\xi_1} - \mathbf{t}_1) + \delta_a \boldsymbol{\varphi}_{,\xi_1} - \delta_a \hat{\mathbf{w}} \mathbf{t}_1] \\
&= \boldsymbol{\Lambda}^t(\delta_a \boldsymbol{\varphi}_{,\xi_1} - \delta_a \hat{\mathbf{w}} \boldsymbol{\varphi}_{,\xi_1}) \\
&= \boldsymbol{\Lambda}^t(\delta_a \boldsymbol{\varphi}_{,\xi_1} + \hat{\boldsymbol{\varphi}}_{,\xi_1} \delta_a \mathbf{w})
\end{aligned} \tag{203}
$$

Therefore, one has the *co-rotated* variation:

$$
\tilde{\delta}_a \boldsymbol{\varepsilon}_r = \boldsymbol{\Lambda} \delta_a \bar{\boldsymbol{\varepsilon}}_r = \delta_a \boldsymbol{\varphi}_{,\xi_1} + \hat{\boldsymbol{\varphi}}_{,\xi_1} \delta_a \mathbf{w} \tag{204}
$$

which is equivalent to that given in Eq. (147). Similarly, considering $\hat{\boldsymbol{\kappa}}_r = \hat{\boldsymbol{\omega}}_{r\xi_1} = \boldsymbol{\Lambda}_{r,\xi_1} \boldsymbol{\Lambda}_r^t$ from Table 1 and $\delta_a \boldsymbol{\Lambda}_r^t = -\boldsymbol{\Lambda}_r^t \delta_a \hat{\mathbf{w}}$ from Eq. (200), one has

$$
\begin{aligned}
\delta_a \hat{\boldsymbol{\kappa}}_r &= \delta_a(\boldsymbol{\Lambda}_{r,\xi_1} \boldsymbol{\Lambda}_r^t) \\
&= \delta_a \boldsymbol{\Lambda}_{r,\xi_1} \boldsymbol{\Lambda}_r^t + \boldsymbol{\Lambda}_{r,\xi_1} \delta_a \boldsymbol{\Lambda}_r^t \\
&= (\delta_a \boldsymbol{\Lambda}_r)_{,\xi_1} \boldsymbol{\Lambda}_r^t + \boldsymbol{\Lambda}_{r,\xi_1}(-\boldsymbol{\Lambda}_r^t \delta_a \hat{\mathbf{w}}) \\
&= (\delta_a \hat{\mathbf{w}} \boldsymbol{\Lambda}_r)_{,\xi_1} \boldsymbol{\Lambda}_r^t - \boldsymbol{\Lambda}_{r,\xi_1} \boldsymbol{\Lambda}_r^t \delta_a \hat{\mathbf{w}} \\
&= (\delta_a \hat{\mathbf{w}}_{,\xi_1} \boldsymbol{\Lambda}_r + \delta_a \hat{\mathbf{w}} \boldsymbol{\Lambda}_{r,\xi_1}) \boldsymbol{\Lambda}_r^t - \boldsymbol{\Lambda}_{r,\xi_1} \boldsymbol{\Lambda}_r^t \delta_a \hat{\mathbf{w}} \\
&= \delta_a \hat{\mathbf{w}}_{,\xi_1} \boldsymbol{\Lambda}_r \boldsymbol{\Lambda}_r^t + \delta_a \hat{\mathbf{w}} \boldsymbol{\Lambda}_{r,\xi_1} \boldsymbol{\Lambda}_r^t - \boldsymbol{\Lambda}_{r,\xi_1} \boldsymbol{\Lambda}_r^t \delta_a \hat{\mathbf{w}} \\
&= \delta_a \hat{\mathbf{w}}_{,\xi_1} + \delta_a \hat{\mathbf{w}} \hat{\boldsymbol{\kappa}}_r - \hat{\boldsymbol{\kappa}}_r \delta_a \hat{\mathbf{w}}
\end{aligned} \tag{205}
$$

After some direct operations on the component form, one can show that for any two vectors $\mathbf{v}_1, \mathbf{v}_2 \in R^3$ and $\mathbf{v} = \mathbf{v}_1 \times \mathbf{v}_2$,

$$
\hat{\mathbf{v}} \equiv \hat{\mathbf{v}}_1 \hat{\mathbf{v}}_2 - \hat{\mathbf{v}}_2 \hat{\mathbf{v}}_1 \tag{206}
$$

which leads Eq. (205) to its axial vector form:

$$
\delta_a \boldsymbol{\kappa}_r = \delta_a \mathbf{w}_{,\xi_1} + \delta_a \mathbf{w} \times \boldsymbol{\kappa}_r = \delta_a \mathbf{w}_{,\xi_1} - \hat{\boldsymbol{\kappa}}_r \delta_a \mathbf{w} \tag{207}
$$

From $\bar{\boldsymbol{\kappa}}_r = \boldsymbol{\Lambda}^t \boldsymbol{\kappa}_r$, one obtains that

$$
\delta_a \bar{\boldsymbol{\kappa}}_r = \delta_a(\boldsymbol{\Lambda}^t \boldsymbol{\kappa}_r)
$$

$$
\begin{aligned}
&= \delta_a \boldsymbol{\Lambda}^t \boldsymbol{\kappa}_r + \boldsymbol{\Lambda}^t \delta_a \boldsymbol{\kappa}_r \\
&= -\boldsymbol{\Lambda}^t \delta_a \hat{\mathbf{w}} \boldsymbol{\kappa}_r + \boldsymbol{\Lambda}^t (\delta_a \mathbf{w}_{,\xi_1} - \hat{\boldsymbol{\kappa}}_r \delta_a \mathbf{w}) \\
&= \boldsymbol{\Lambda}^t \delta_a \mathbf{w}_{,\xi_1} \qquad (208)
\end{aligned}
$$

and therefore, one obtains the *co-rotated* variation:

$$
\tilde{\delta}_a \boldsymbol{\kappa}_r = \boldsymbol{\Lambda} \delta_a \bar{\boldsymbol{\kappa}}_r = \delta_a \mathbf{w}_{,\xi_1} \qquad (209)
$$

which is also equivalent to that given in Eq. (147).

If one chooses a vector-like parametrization for finite rotations, it is convenient, noticing the identity $\delta_a \mathbf{w}_{,\xi_1} = \delta_a \boldsymbol{\kappa}_r + \hat{\boldsymbol{\kappa}}_r \delta_a \mathbf{w}$ from Eq. (207), to rewrite Eqs. (208) and (209) as

$$
\delta_a \bar{\boldsymbol{\kappa}}_r = \boldsymbol{\Lambda}^t (\delta_a \boldsymbol{\kappa}_r + \hat{\boldsymbol{\kappa}}_r \delta_a \mathbf{w}) \qquad (210)
$$

and

$$
\tilde{\delta}_a \boldsymbol{\kappa}_r = \delta_a \mathbf{w}_{,\xi_1} = \delta_a \boldsymbol{\kappa}_r + \hat{\boldsymbol{\kappa}}_r \delta_a \mathbf{w} \qquad (211)
$$

respectively.

Considering the transformation relation as given in Eq. (194) and the bending strain vector as given in Eq. (195) for a general case of the vector-like parametrizations, Eqs. (203), (204), (210), and (211) can be further written as

$$
\delta_a \bar{\boldsymbol{\varepsilon}}_r = \boldsymbol{\Lambda}^t (\delta_a \boldsymbol{\varphi}_{,\xi_1} + \hat{\boldsymbol{\varphi}}_{,\xi_1} \mathbf{T}_{wp} \delta_a \mathbf{p}) \qquad (212)
$$

$$
\tilde{\delta}_a \boldsymbol{\varepsilon}_r = \delta_a \boldsymbol{\varphi}_{,\xi_1} + \hat{\boldsymbol{\varphi}}_{,\xi_1} \mathbf{T}_{wp} \delta_a \mathbf{p} \qquad (213)
$$

$$
\begin{aligned}
\delta_a \bar{\boldsymbol{\kappa}}_r &= \boldsymbol{\Lambda}^t [\delta_a (\mathbf{T}_{wp} \mathbf{p}_{,\xi_1}) + \hat{\boldsymbol{\kappa}}_r \mathbf{T}_{wp} \delta_a \mathbf{p}] \\
&= \boldsymbol{\Lambda}^t [\delta_a \mathbf{T}_{wp} \mathbf{p}_{,\xi_1} + \mathbf{T}_{wp} \delta_a \mathbf{p}_{,\xi_1} + \hat{\boldsymbol{\kappa}}_r \mathbf{T}_{wp} \delta_a \mathbf{p}] \\
&= \boldsymbol{\Lambda}^t [(\mathbf{T}_{wp,\mathbf{p}} \delta_a \mathbf{p}) \mathbf{p}_{,\xi_1} + \mathbf{T}_{wp} \delta_a \mathbf{p}_{,\xi_1} + \hat{\boldsymbol{\kappa}}_r \mathbf{T}_{wp} \delta_a \mathbf{p}] \\
&= \boldsymbol{\Lambda}^t [(\mathbf{T}_{wp} \underline{\mathbf{p}_{,\xi_1}})_{,\mathbf{p}} \delta_a \mathbf{p} + \mathbf{T}_{wp} \delta_a \mathbf{p}_{,\xi_1} + \hat{\boldsymbol{\kappa}}_r \mathbf{T}_{wp} \delta_a \mathbf{p}] \\
&= \boldsymbol{\Lambda}^t \{[(\mathbf{T}_{wp} \underline{\mathbf{p}_{,\xi_1}})_{,\mathbf{p}} + \hat{\boldsymbol{\kappa}}_r \mathbf{T}_{wp}] \delta_a \mathbf{p} + \mathbf{T}_{wp} \delta_a \mathbf{p}_{,\xi_1}\} \\
&= \boldsymbol{\Lambda}^t [\mathbf{A} \delta_a \mathbf{p} + \mathbf{T}_{wp} \delta_a \mathbf{p}_{,\xi_1}]
\end{aligned}
\qquad (214)
$$

and

$$\begin{aligned}\tilde{\delta}_a\boldsymbol{\kappa}_r &= \delta_a\mathbf{w}_{,\xi_1}\\ &= \delta_a(\mathbf{T}_{wp}\mathbf{p}_{,\xi_1}) + \hat{\boldsymbol{\kappa}}\mathbf{T}_{wp}\delta_a\mathbf{p}\\ &= \mathbf{A}\delta_a\mathbf{p} + \mathbf{T}_{wp}\delta_a\mathbf{p}_{,\xi_1}\end{aligned} \tag{215}$$

respectively, where the second-order tensor $\mathbf{A}$ and the two directional derivatives, $\mathbf{A}_1$ and $\mathbf{A}_2$ of $\mathbf{A}^t$ along $\mathbf{m}$ with respect to $\mathbf{p}$ and $\mathbf{p}_{,\xi_1}$, respectively, are calculated as follows:

$$\mathbf{A} = \mathbf{R} + \hat{\boldsymbol{\kappa}}_r\mathbf{T}_{wp} \tag{216}$$

$$\begin{aligned}\mathbf{A}_1 &= (\mathbf{A}^t\underline{\mathbf{m}})_{,\mathbf{p}}\\ &= [\mathbf{R}^t\underline{\hat{\mathbf{m}}} + \mathbf{T}^t_{wp}\underline{\hat{\mathbf{m}}}\mathbf{T}_{wp}\mathbf{p}_{,\xi_1}]_{,\mathbf{p}}\\ &= (\mathbf{R}^t\underline{\mathbf{m}})_{,\mathbf{p}} + (\mathbf{T}^t_{wp}\underline{\hat{\mathbf{m}}\mathbf{T}_{wp}\mathbf{p}_{,\xi_1}})_{,\mathbf{p}} + \mathbf{T}^t_{wp}\hat{\mathbf{m}}(\mathbf{T}_{wp}\underline{\mathbf{p}_{,\xi_1}})_{,\mathbf{p}}\\ &= \mathbf{H} + \boldsymbol{\Xi}(\hat{\mathbf{m}}\boldsymbol{\kappa}) + \mathbf{T}^t_{wp}\hat{\mathbf{m}}\mathbf{R}\end{aligned} \tag{217}$$

$$\begin{aligned}\mathbf{A}_2 &= (\mathbf{A}^t\underline{\mathbf{m}})_{,\mathbf{p}_{,\xi_1}}\\ &= (\mathbf{R}^t\underline{\mathbf{m}})_{,\mathbf{p}_{,\xi_1}} + \mathbf{T}^t_{wp}\hat{\mathbf{m}}\mathbf{T}_{wp}\\ &= \boldsymbol{\Xi}'(\mathbf{m}) + \mathbf{T}^t_{wp}\hat{\mathbf{m}}\mathbf{T}_{wp}\end{aligned} \tag{218}$$

where

$$\mathbf{R} = (\mathbf{T}_{wp}\underline{\mathbf{p}_{,\xi_1}})_{,\mathbf{p}} \tag{219}$$

$$\mathbf{H} = (\mathbf{R}^t\underline{\mathbf{m}})_{,\mathbf{p}} \tag{220}$$

$$\boldsymbol{\Xi}(\mathbf{v}) = (\mathbf{T}^t_{wp}\underline{\mathbf{v}})_{,\mathbf{p}} \tag{221}$$

and

$$\boldsymbol{\Xi}'(\mathbf{m}) = (\mathbf{R}^t\underline{\mathbf{m}})_{,\mathbf{p},\xi_1} \quad (= \boldsymbol{\Xi}^t(\mathbf{m})) \tag{222}$$

In Eqs. (214) through (222), brief notations have been used: for any second-order tensor $\mathbf{T} = \mathbf{T}(\mathbf{x})$, the third-order tensor $\mathbf{T}_{,\mathbf{x}}$ is the derivative of $\mathbf{T}$ with respect to the vector $\mathbf{x}$; the second-order tensor $(\mathbf{T}\underline{\mathbf{v}})_{,\mathbf{x}} = \mathbf{T}_{,\mathbf{x}} \cdot \mathbf{v}$ is the derivative of the vector $\mathbf{T}\mathbf{v}$ with respect to the vector $\mathbf{x}$ while the vector $\mathbf{v}$ remains constant, which is a convenient way to calculate directional derivatives. The closed-form representations for those tensors can be obtained after some *elaborate* operations for certain vector-like parametrizations. For the vector-like parametrization using the rotation vector in the same spatial form, those tensors have been obtained (see e.g. Ibrahimbegović *et al*, 1995). Here, clear definitions of them have been presented for readers' convenience to reproduce the same results. Cardona and Geradin (1988), and then Pacoste and Eriksson (1997), gave different versions of similar tensors (the material form for incremental rotation vector was used) but the second-order derivatives of $\mathbf{T}_{wp}$ were neglected.

• Variations of stress resultants/couples

Recalling the linearized relations between stress resultants/couples and strain measures in Eq. (147) and considering the variations of strains in Eqs. (202) through (209), and denoting $\mathbf{C}_n = \mathbf{C}_{nn}$ and $\mathbf{C}_m = \mathbf{C}_{mm}$, one obtains that

$$\begin{aligned} \delta_a\bar{\mathbf{n}} &= \bar{\mathbf{C}}_n\delta_a\bar{\boldsymbol{\varepsilon}}_r + \bar{\mathbf{C}}_{nm}\delta_a\bar{\boldsymbol{\kappa}}_r \\ &= \bar{\mathbf{C}}_n\boldsymbol{\Lambda}^t(\delta_a\boldsymbol{\varphi}_{,\xi_1} + \hat{\boldsymbol{\varphi}}_{,\xi_1}\delta_a\mathbf{w}) + \bar{\mathbf{C}}_{nm}\boldsymbol{\Lambda}^t\delta_a\mathbf{w}_{,\xi_1} \end{aligned} \tag{223}$$

and hence (by pull-back/push-forward operations)

$$\begin{aligned} \tilde{\delta}_a\mathbf{n} &= \boldsymbol{\Lambda}\delta_a\boldsymbol{\Lambda}^t\mathbf{n} \\ &= \boldsymbol{\Lambda}\delta_a\bar{\mathbf{n}} \\ &= \boldsymbol{\Lambda}\bar{\mathbf{C}}_n\boldsymbol{\Lambda}^t(\delta_a\boldsymbol{\varphi}_{,\xi_1} + \hat{\boldsymbol{\varphi}}_{,\xi_1}\delta_a\mathbf{w}) + \boldsymbol{\Lambda}\bar{\mathbf{C}}_{nm}\boldsymbol{\Lambda}^t\delta_a\mathbf{w}_{,\xi_1} \end{aligned}$$

$$= \mathbf{C}_n(\delta_a \boldsymbol{\varphi}_{,\xi_1} + \hat{\boldsymbol{\varphi}}_{,\xi_1} \delta_a \mathbf{w}) + \mathbf{C}_{nm} \delta_a \mathbf{w}_{,\xi_1} \tag{224}$$

for stress resultants, and similarly,

$$\delta_a \bar{\mathbf{m}} = \bar{\mathbf{C}}_{mn} \boldsymbol{\Lambda}^t (\delta_a \boldsymbol{\varphi}_{,\xi_1} + \hat{\boldsymbol{\varphi}}_{,\xi_1} \delta_a \mathbf{w}) + \bar{\mathbf{C}}_m \boldsymbol{\Lambda}^t \delta_a \mathbf{w}_{,\xi_1} \tag{225}$$

and hence,

$$\tilde{\delta}_a \mathbf{m} = \mathbf{C}_{mn}(\delta_a \boldsymbol{\varphi}_{,\xi_1} + \hat{\boldsymbol{\varphi}}_{,\xi_1} \delta_a \mathbf{w}) + \mathbf{C}_m \delta_a \mathbf{w}_{,\xi_1} \tag{226}$$

for stress couples.

For the conventional variation of spatial stress resultant, noticing the other form of the relation for the *co-rotated* variation $\tilde{\delta}_a \mathbf{n} = \delta_a \mathbf{n} - \delta_a \hat{\mathbf{w}} \mathbf{n}$, one may easily obtain that

$$\begin{aligned} \delta_a \mathbf{n} &= \tilde{\delta}_a \mathbf{n} + \delta_a \hat{\mathbf{w}} \mathbf{n} \\ &= \tilde{\delta}_a \mathbf{n} - \hat{\mathbf{n}} \delta_a \mathbf{w} \\ &= \mathbf{C}_n(\delta_a \boldsymbol{\varphi}_{,\xi_1} + \hat{\boldsymbol{\varphi}}_{,\xi_1} \delta_a \mathbf{w}) + \mathbf{C}_{nm} \delta_a \mathbf{w}_{,\xi_1} - \hat{\mathbf{n}} \delta_a \mathbf{w} \\ &= \mathbf{C}_n \delta_a \boldsymbol{\varphi}_{,\xi_1} + (\mathbf{C}_n \hat{\boldsymbol{\varphi}}_{,\xi_1} - \hat{\mathbf{n}}) \delta_a \mathbf{w} + \mathbf{C}_{nm} \delta_a \mathbf{w}_{,\xi_1} \end{aligned} \tag{227}$$

Similarly,

$$\delta_a \mathbf{m} = \mathbf{C}_{mn} \delta_a \boldsymbol{\varphi}_{,\xi_1} + (\mathbf{C}_{mn} \hat{\boldsymbol{\varphi}}_{,\xi_1} - \hat{\mathbf{m}}) \delta_a \mathbf{w} + \mathbf{C}_m \delta_a \mathbf{w}_{,\xi_1} \tag{228}$$

for the conventional variation of stress couple, which will be used in the derivation of Eq. (231).

For a vector-like parametrization, one can substitute $\delta_a \mathbf{w} = \mathbf{T}_{wp} \delta_a \mathbf{p}$ and $\delta_a \mathbf{w}_{,\xi_1} = \mathbf{A} \delta_a \mathbf{p} + \mathbf{T}_{wp} \delta_a \mathbf{p}_{,\xi_1}$ into Eqs. (223) through (228) for further representations.

7.4 Virtual work equation and its linearization

The virtual work equation and its linearization, which produces the equilibrium or state equation, have been introduced in Eqs. (134) through (138). Now one may expand them as follows in the form more convenient to a C^1 continuous finite element formulation, though a C^0 continuous curved beam element will be implemented in the next chapter.

Substituting Eqs. (203), (208), (204), and (209) into Eq. (136), one has the internal virtual work as (noticing the equivalence between the material and spatial representations)

$$\begin{aligned}
\delta_a W_{int} &= \int_L (\delta_a \bar{\boldsymbol{\varepsilon}}_r \cdot \bar{\mathbf{n}} + \delta_a \bar{\boldsymbol{\kappa}}_r \cdot \bar{\mathbf{m}}) d\xi_1 \\
&= \int_L \{[\boldsymbol{\Lambda}^t(\delta_a \boldsymbol{\varphi}_{,\xi_1} + \hat{\boldsymbol{\varphi}}_{,\xi_1}\delta_a \mathbf{w})] \cdot \bar{\mathbf{n}} + (\boldsymbol{\Lambda}^t \delta_a \mathbf{w}_{,\xi_1}) \cdot \bar{\mathbf{m}}\} d\xi_1 \\
&= \int_L [(\delta_a \boldsymbol{\varphi}_{,\xi_1} + \hat{\boldsymbol{\varphi}}_{,\xi_1}\delta_a \mathbf{w}) \cdot (\boldsymbol{\Lambda}\bar{\mathbf{n}}) + \delta_a \mathbf{w}_{,\xi_1} \cdot (\boldsymbol{\Lambda}\bar{\mathbf{m}})] d\xi_1 \\
&= \int_L [(\delta_a \boldsymbol{\varphi}_{,\xi_1} + \hat{\boldsymbol{\varphi}}_{,\xi_1}\delta_a \mathbf{w}) \cdot \mathbf{n} + \delta_a \mathbf{w}_{,\xi_1} \cdot \mathbf{m}] d\xi_1 \\
&= \int_L (\tilde{\delta}_a \boldsymbol{\varepsilon}_r \cdot \mathbf{n} + \tilde{\delta}_a \boldsymbol{\kappa}_r \cdot \mathbf{m}) d\xi_1 \\
&= \int_L \begin{bmatrix} \delta_a \boldsymbol{\varphi} \\ \delta_a \boldsymbol{\varphi}_{,\xi_1} \\ \delta_a \mathbf{w} \\ \delta_a \mathbf{w}_{,\xi_1} \end{bmatrix}^t \begin{bmatrix} \mathbf{0} \\ \mathbf{n} \\ -\hat{\boldsymbol{\varphi}}_{,\xi_1}\mathbf{n} \\ \mathbf{m} \end{bmatrix} d\xi_1
\end{aligned} \tag{229}$$

which gives rise to internal loading.

The external virtual work, recalling Eqs. (148) through (160), can be further written as

$$\begin{aligned}
\delta_a W_{ext} &= \int_L [\delta_a \boldsymbol{\varphi} \cdot (d\mathbf{f}_g + d\mathbf{f}_d + d\mathbf{f}_p) + \delta_a \mathbf{w} \cdot (d\mathbf{m}_g + d\mathbf{m}_d + d\mathbf{m}_p)] \\
&= \lambda \int_L \{\delta_a \boldsymbol{\varphi} \cdot [\boldsymbol{\mathcal{N}}_g + c_N \boldsymbol{\mathcal{N}}_d + \boldsymbol{\Lambda}\bar{\boldsymbol{\mathcal{N}}}_p] \\
&\quad + \delta_a \mathbf{w} \cdot [\boldsymbol{\mathcal{M}}_g + c_M \boldsymbol{\mathcal{M}}_d + \boldsymbol{\Lambda}\bar{\boldsymbol{\mathcal{M}}}_p]\} d\xi_1 \\
&= \lambda \int_L \begin{bmatrix} \delta_a \boldsymbol{\varphi} \\ \delta_a \boldsymbol{\varphi}_{,\xi_1} \\ \delta_a \mathbf{w} \\ \delta_a \mathbf{w}_{,\xi_1} \end{bmatrix}^t \begin{bmatrix} \boldsymbol{\mathcal{N}}_g + c_N \boldsymbol{\mathcal{N}}_d + \boldsymbol{\Lambda}\bar{\boldsymbol{\mathcal{N}}}_p \\ \mathbf{0} \\ \boldsymbol{\mathcal{M}}_g + c_M \boldsymbol{\mathcal{M}}_d + \boldsymbol{\Lambda}\bar{\boldsymbol{\mathcal{M}}}_p \\ \mathbf{0} \end{bmatrix} d\xi_1
\end{aligned} \tag{230}$$

which gives rise to external loading.

Using Eqs. (227) and (228) in their linearized incremental forms, one can obtain the linearized increment of the internal virtual work:

$$
\begin{aligned}
\Delta_a \delta_a W_{int} &= \int_L \begin{bmatrix} \delta_a \boldsymbol{\varphi} \\ \delta_a \boldsymbol{\varphi}_{,\xi_1} \\ \delta_a \mathbf{w} \\ \delta_a \mathbf{w}_{,\xi_1} \end{bmatrix}^t \Delta_a \begin{bmatrix} \mathbf{0} \\ \mathbf{n} \\ -\hat{\boldsymbol{\varphi}}_{,\xi_1} \mathbf{n} \\ \mathbf{m} \end{bmatrix} d\xi_1 \\
&= \int_L \begin{bmatrix} \delta_a \boldsymbol{\varphi} \\ \delta_a \boldsymbol{\varphi}_{,\xi_1} \\ \delta_a \mathbf{w} \\ \delta_a \mathbf{w}_{,\xi_1} \end{bmatrix}^t \begin{bmatrix} \mathbf{0} \\ \Delta_a \mathbf{n} \\ -\Delta_a \hat{\boldsymbol{\varphi}}_{,\xi_1} \mathbf{n} - \hat{\boldsymbol{\varphi}}_{,\xi_1} \Delta_a \mathbf{n} \\ \Delta_a \mathbf{m} \end{bmatrix} d\xi_1 \\
&= \int_L \begin{bmatrix} \delta_a \boldsymbol{\varphi} \\ \delta_a \boldsymbol{\varphi}_{,\xi_1} \\ \delta_a \mathbf{w} \\ \delta_a \mathbf{w}_{,\xi_1} \end{bmatrix}^t \begin{bmatrix} \mathbf{0} \\ \mathbf{C}_n \Delta_a \boldsymbol{\varphi}_{,\xi_1} + (\mathbf{C}_n \hat{\boldsymbol{\varphi}}_{,\xi_1} - \hat{\mathbf{n}}) \Delta_a \mathbf{w} + \mathbf{C}_{nm} \Delta_a \mathbf{w}_{,\xi_1} \\ -\Delta_a \hat{\boldsymbol{\varphi}}_{,\xi_1} \mathbf{n} - \hat{\boldsymbol{\varphi}}_{,\xi_1} [\mathbf{C}_n \Delta_a \boldsymbol{\varphi}_{,\xi_1} + (\mathbf{C}_n \hat{\boldsymbol{\varphi}}_{,\xi_1} - \hat{\mathbf{n}}) \Delta_a \mathbf{w} + \mathbf{C}_{nm} \Delta_a \mathbf{w}_{,\xi_1}] \\ \mathbf{C}_{mn} \delta_a \boldsymbol{\varphi}_{,\xi_1} + (\mathbf{C}_{mn} \hat{\boldsymbol{\varphi}}_{,\xi_1} - \hat{\mathbf{m}}) \Delta_a \mathbf{w} + \mathbf{C}_m \Delta_a \mathbf{w}_{,\xi_1} \end{bmatrix} d\xi_1 \\
&= \int_L \begin{bmatrix} \delta_a \boldsymbol{\varphi} \\ \delta_a \boldsymbol{\varphi}_{,\xi_1} \\ \delta_a \mathbf{w} \\ \delta_a \mathbf{w}_{,\xi_1} \end{bmatrix}^t \left(\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_n & \mathbf{C}_n \hat{\boldsymbol{\varphi}}_{,\xi_1} & \mathbf{C}_{nm} \\ \mathbf{0} & -\hat{\boldsymbol{\varphi}}_{,\xi_1} \mathbf{C}_n & -\hat{\boldsymbol{\varphi}}_{,\xi_1} \mathbf{C}_n \hat{\boldsymbol{\varphi}}_{,\xi_1} & -\hat{\boldsymbol{\varphi}}_{,\xi_1} \mathbf{C}_{nm} \\ \mathbf{0} & \mathbf{C}_{mn} & \mathbf{C}_{mn} \hat{\boldsymbol{\varphi}}_{,\xi_1} & \mathbf{C}_m \end{bmatrix}_M \right. \\
&\quad \left. + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\hat{\mathbf{n}} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{n}} & \hat{\boldsymbol{\varphi}}_{,\xi_1} \hat{\mathbf{n}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\hat{\mathbf{m}} & \mathbf{0} \end{bmatrix}_G \right) \begin{bmatrix} \Delta_a \boldsymbol{\varphi} \\ \Delta_a \boldsymbol{\varphi}_{,\xi_1} \\ \Delta_a \mathbf{w} \\ \Delta_a \mathbf{w}_{,\xi_1} \end{bmatrix} d\xi_1
\end{aligned} \tag{231}
$$

in which the matrix with subscript M corresponds to the material part of the tangential stiffness operator and the matrix with subscript G corresponds to the geometric part of the tangential stiffness operator, and both of the matrices are due to the internal loading. The symmetry of the material part of the tangential stiffness is obvious, while the geometric part is not symmetric in general. The geometric part of the tangential operator can recover symmetry at the equilibrium state if the external loads are conservative; if external loads are nonconservative, it will be *non-symmetric* (see e.g. Simo and Vu-Quoc, 1986).

One can also obtain the linearized increment of the external virtual work from Eq. (230) as follows:

$$
\begin{aligned}
\Delta_a \delta_a W_{ext} &= \lambda \int_L \begin{bmatrix} \delta_a \boldsymbol{\varphi} \\ \delta_a \boldsymbol{\varphi}_{,\xi_1} \\ \delta_a \mathbf{w} \\ \delta_a \mathbf{w}_{,\xi_1} \end{bmatrix}^t \Delta_a \begin{bmatrix} \boldsymbol{\mathcal{N}}_g + c_N \boldsymbol{\mathcal{N}}_d + \boldsymbol{\Lambda} \bar{\boldsymbol{\mathcal{N}}}_p \\ \mathbf{0} \\ \boldsymbol{\mathcal{M}}_g + c_M \boldsymbol{\mathcal{M}}_d + \boldsymbol{\Lambda} \bar{\boldsymbol{\mathcal{M}}}_p \\ \mathbf{0} \end{bmatrix} d\xi_1 \\
&= \lambda \int_L \begin{bmatrix} \delta_a \boldsymbol{\varphi} \\ \delta_a \boldsymbol{\varphi}_{,\xi_1} \\ \delta_a \mathbf{w} \\ \delta_a \mathbf{w}_{,\xi_1} \end{bmatrix}^t \begin{bmatrix} \boldsymbol{\mathcal{N}}_d \Delta_a c_N + \Delta_a \boldsymbol{\Lambda} \bar{\boldsymbol{\mathcal{N}}}_p \\ \mathbf{0} \\ \boldsymbol{\mathcal{M}}_d \Delta_a c_M + \Delta_a \boldsymbol{\Lambda} \bar{\boldsymbol{\mathcal{M}}}_p \\ \mathbf{0} \end{bmatrix} d\xi_1 \\
&= \lambda \int_L \begin{bmatrix} \delta_a \boldsymbol{\varphi} \\ \delta_a \boldsymbol{\varphi}_{,\xi_1} \\ \delta_a \mathbf{w} \\ \delta_a \mathbf{w}_{,\xi_1} \end{bmatrix}^t \begin{bmatrix} -\boldsymbol{\mathcal{N}}_d (\mathbf{c}_N \cdot \Delta_a \boldsymbol{\varphi}_{,\xi_1}) + \Delta_a \hat{\mathbf{w}} \boldsymbol{\Lambda} \bar{\boldsymbol{\mathcal{N}}}_p \\ \mathbf{0} \\ -\boldsymbol{\mathcal{M}}_d (\mathbf{c}_M \cdot \Delta_a \boldsymbol{\varphi}_{,\xi_1}) + \Delta_a \hat{\mathbf{w}} \boldsymbol{\Lambda} \bar{\boldsymbol{\mathcal{M}}}_p \\ \mathbf{0} \end{bmatrix} d\xi_1 \\
&= \lambda \int_L \begin{bmatrix} \delta_a \boldsymbol{\varphi} \\ \delta_a \boldsymbol{\varphi}_{,\xi_1} \\ \delta_a \mathbf{w} \\ \delta_a \mathbf{w}_{,\xi_1} \end{bmatrix}^t \begin{bmatrix} -(\boldsymbol{\mathcal{N}}_d \otimes \mathbf{c}_N) \Delta_a \boldsymbol{\varphi}_{,\xi_1} + \Delta_a \hat{\mathbf{w}} \boldsymbol{\mathcal{N}}_p \\ \mathbf{0} \\ -(\boldsymbol{\mathcal{M}}_d \otimes \mathbf{c}_M) \Delta_a \boldsymbol{\varphi}_{,\xi_1} + \Delta_a \hat{\mathbf{w}} \boldsymbol{\mathcal{M}}_p \\ \mathbf{0} \end{bmatrix} d\xi_1 \\
&= \lambda \int_L \begin{bmatrix} \delta_a \boldsymbol{\varphi} \\ \delta_a \boldsymbol{\varphi}_{,\xi_1} \\ \delta_a \mathbf{w} \\ \delta_a \mathbf{w}_{,\xi_1} \end{bmatrix}^t \begin{bmatrix} -(\boldsymbol{\mathcal{N}}_d \otimes \mathbf{c}_N) \Delta_a \boldsymbol{\varphi}_{,\xi_1} - \hat{\boldsymbol{\mathcal{N}}}_p \Delta_a \mathbf{w} \\ \mathbf{0} \\ -(\boldsymbol{\mathcal{M}}_d \otimes \mathbf{c}_M) \Delta_a \boldsymbol{\varphi}_{,\xi_1} - \hat{\boldsymbol{\mathcal{M}}}_p \Delta_a \mathbf{w} \\ \mathbf{0} \end{bmatrix} d\xi_1 \\
&= -\lambda \int_L \begin{bmatrix} \delta_a \boldsymbol{\varphi} \\ \delta_a \boldsymbol{\varphi}_{,\xi_1} \\ \delta_a \mathbf{w} \\ \delta_a \mathbf{w}_{,\xi_1} \end{bmatrix}^t \begin{bmatrix} \mathbf{0} & \boldsymbol{\mathcal{N}}_d \otimes \mathbf{c}_N & \hat{\boldsymbol{\mathcal{N}}}_p & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\mathcal{M}}_d \otimes \mathbf{c}_M & \hat{\boldsymbol{\mathcal{M}}}_p & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Delta_a \boldsymbol{\varphi} \\ \Delta_a \boldsymbol{\varphi}_{,\xi_1} \\ \Delta_a \mathbf{w} \\ \Delta_a \mathbf{w}_{,\xi_1} \end{bmatrix} d\xi_1
\end{aligned}
$$

$$= \quad -\lambda \int_L \begin{bmatrix} \delta_a \boldsymbol{\varphi} \\ \delta_a \boldsymbol{\varphi}_{,\xi_1} \\ \delta_a \mathbf{w} \\ \delta_a \mathbf{w}_{,\xi_1} \end{bmatrix}^t \left(\begin{bmatrix} \mathbf{0} & \boldsymbol{\mathcal{N}}_d \otimes \mathbf{c}_N & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\mathcal{M}}_d \otimes \mathbf{c}_M & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}_d \right.$$

$$\left. + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \hat{\boldsymbol{\mathcal{N}}}_p & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \hat{\boldsymbol{\mathcal{M}}}_p & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}_p \right) \begin{bmatrix} \Delta_a \boldsymbol{\varphi} \\ \Delta_a \boldsymbol{\varphi}_{,\xi_1} \\ \Delta_a \mathbf{w} \\ \Delta_a \mathbf{w}_{,\xi_1} \end{bmatrix} d\xi_1 \tag{232}$$

where the vectors $\mathbf{c}_N$ and $\mathbf{c}_M$ are defined as

$$\Delta_a c_N = -\mathbf{c}_N \cdot \Delta_a \boldsymbol{\varphi}_{,\xi_1} \tag{233}$$

and

$$\Delta_a c_M = -\mathbf{c}_M \cdot \Delta_a \boldsymbol{\varphi}_{,\xi_1} \tag{234}$$

respectively, and given as

$$\mathbf{c}_N = \frac{2}{\lambda} \int_0^\lambda (\hat{\mathbf{d}}_N)^2 \boldsymbol{\varphi}_{,\xi_1} d\lambda \tag{235}$$

and

$$\mathbf{c}_M = \frac{2}{\lambda} \int_0^\lambda (\hat{\mathbf{d}}_M)^2 \boldsymbol{\varphi}_{,\xi_1} d\lambda \tag{236}$$

for the *deformation-dependent* loading of type II, or

$$\mathbf{c}_N = \mathbf{0} \tag{237}$$

and

$$\mathbf{c}_M = \mathbf{0} \tag{238}$$

for the *deformation-independent* loading of type II. In Eq. (232), the matrix with subscript d corresponds to the part of the tangential stiffness matrix operator contributed by the external loading of type II, and the matrix with subscript p by the external loading of type III - the pressure type of loading, both of which are *non-symmetric*.

The above formulation has been performed for parametrizing incremental rotations with the incremental rotation vector $\Delta_a \mathbf{w}$ independent of any particular parametrizations of the compound rotation tensor.

One may also formulate with a vector-like parametrization as follows.

Referred to Eqs. (213) through (222) , the internal virtual work can be written as

$$\begin{aligned}
\delta_a W_{int} &= \int_L \{(\tilde{\delta}\boldsymbol{\varepsilon}_r \cdot \mathbf{n} + \tilde{\delta}_a \boldsymbol{\kappa}_r \cdot \mathbf{m}) d\xi_1 \\
&= \int_L \{[\delta_a \boldsymbol{\varphi}_{,\xi_1} + \hat{\boldsymbol{\varphi}}_{,\xi_1} \mathbf{T}_{wp} \delta_a \mathbf{p}] \cdot \mathbf{n} \\
&\quad + [\mathbf{A} \delta_a \mathbf{p} + \mathbf{T}_{wp} \delta_a \mathbf{p}_{,\xi_1}] \cdot \mathbf{m}\} d\xi_1 \\
&= \int_L [\delta_a \boldsymbol{\varphi}_{,\xi_1} \cdot \mathbf{n} + (\hat{\boldsymbol{\varphi}}_{,\xi_1} \mathbf{T}_{wp} \delta_a \mathbf{p}) \cdot \mathbf{n} \\
&\quad + (\mathbf{A} \delta_a \mathbf{p}) \cdot \mathbf{m} + (\mathbf{T}_{wp} \delta_a \mathbf{p}_{,\xi_1}) \cdot \mathbf{m}] d\xi_1 \\
&= \int_L \{\delta_a \boldsymbol{\varphi}_{,\xi_1} \cdot \mathbf{n} - \delta_a \mathbf{p} \cdot [\mathbf{T}^t_{wp} \hat{\boldsymbol{\varphi}}_{,\xi_1} \mathbf{n}] \\
&\quad + \delta_a \mathbf{p} \cdot [\mathbf{A}^t \mathbf{m}] + \delta_a \mathbf{p}_{,\xi_1} \cdot [\mathbf{T}^t_{wp} \mathbf{m}]\} d\xi_1 \\
&= \int_L \begin{bmatrix} \delta_a \boldsymbol{\varphi} \\ \delta_a \boldsymbol{\varphi}_{,\xi_1} \\ \delta_a \mathbf{p} \\ \delta_a \mathbf{p}_{,\xi_1} \end{bmatrix}^t \begin{bmatrix} \mathbf{0} \\ \mathbf{n} \\ \mathbf{A}^t \mathbf{m} - \mathbf{T}^t_{wp} \hat{\boldsymbol{\varphi}}_{,\xi_1} \mathbf{n} \\ \mathbf{T}^t_{wp} \mathbf{m} \end{bmatrix} d\xi_1
\end{aligned} \tag{239}$$

which gives rise to internal loading, and the external virtual work as

$$\begin{aligned}
\delta_a W_{ext} &= \lambda \int_L \{\delta_a \boldsymbol{\varphi} \cdot [\boldsymbol{\mathcal{N}}_g + c_N \boldsymbol{\mathcal{N}}_d + \boldsymbol{\Lambda} \bar{\boldsymbol{\mathcal{N}}}_p] \\
&\quad + \delta_a \mathbf{w} \cdot [\boldsymbol{\mathcal{M}}_g + c_M \boldsymbol{\mathcal{M}}_d + \boldsymbol{\Lambda} \bar{\boldsymbol{\mathcal{M}}}_p]\} d\xi_1 \\
&= \lambda \int_L \{\delta_a \boldsymbol{\varphi} \cdot [\boldsymbol{\mathcal{N}}_g + c_N \boldsymbol{\mathcal{N}}_d + \boldsymbol{\Lambda} \bar{\boldsymbol{\mathcal{N}}}_p] \\
&\quad + (\mathbf{T}_{wp} \delta_a \mathbf{p}) \cdot [\boldsymbol{\mathcal{M}}_g + c_M \boldsymbol{\mathcal{M}}_d + \boldsymbol{\Lambda} \bar{\boldsymbol{\mathcal{M}}}_p]\} d\xi_1 \\
&= \lambda \int_L \{\delta_a \boldsymbol{\varphi} \cdot [\boldsymbol{\mathcal{N}}_g + c_N \boldsymbol{\mathcal{N}}_d + \boldsymbol{\Lambda} \bar{\boldsymbol{\mathcal{N}}}_p] \\
&\quad + \delta_a \mathbf{p} \cdot [\mathbf{T}^t_{wp} \boldsymbol{\mathcal{M}}_g + \mathbf{T}^t_{wp} c_M \boldsymbol{\mathcal{M}}_d + \mathbf{T}^t_{wp} \boldsymbol{\Lambda} \bar{\boldsymbol{\mathcal{M}}}_p]\} d\xi_1
\end{aligned}$$

$$
= \lambda \int_L \begin{bmatrix} \delta_a \boldsymbol{\varphi} \\ \delta_a \boldsymbol{\varphi}_{,\xi_1} \\ \delta_a \mathbf{p} \\ \delta_a \mathbf{p}_{,\xi_1} \end{bmatrix}^t \begin{bmatrix} \boldsymbol{\mathcal{N}}_g + c_N \boldsymbol{\mathcal{N}}_d + \boldsymbol{\Lambda} \bar{\boldsymbol{\mathcal{N}}}_p \\ \mathbf{0} \\ \mathbf{T}^t_{wp} [\boldsymbol{\mathcal{M}}_g + c_M \boldsymbol{\mathcal{M}}_d + \boldsymbol{\Lambda} \bar{\boldsymbol{\mathcal{M}}}_p] \\ \mathbf{0} \end{bmatrix} d\xi_1 \tag{240}
$$

which gives rise to external loading.

Using Eqs. (227) and (228) in their linearized incremental forms, and considering $\delta_a \mathbf{w} = \mathbf{T}_{wp} \delta_a \mathbf{p}$ and $\delta_a \mathbf{w}_{,\xi_1} = \mathbf{A} \delta_a \mathbf{p} + \mathbf{T}_{wp} \delta_a \mathbf{p}_{,\xi_1}$, as well as the derivative tensor notations in Eqs. (216) through (222), one can obtain the linearized increment of the internal virtual work as

$$
\begin{aligned}
\Delta_a \delta_a W_{int} &= \int_L \begin{bmatrix} \delta_a \boldsymbol{\varphi} \\ \delta_a \boldsymbol{\varphi}_{,\xi_1} \\ \delta_a \mathbf{p} \\ \delta_a \mathbf{p}_{,\xi_1} \end{bmatrix}^t \Delta_a \begin{bmatrix} \mathbf{0} \\ \mathbf{n} \\ \mathbf{A}^t \mathbf{m} - \mathbf{T}^t_{wp} \hat{\boldsymbol{\varphi}}_{,\xi_1} \mathbf{n} \\ \mathbf{T}^t_{wp} \mathbf{m} \end{bmatrix} d\xi_1 \\
&= \int_L \begin{bmatrix} \delta_a \boldsymbol{\varphi} \\ \delta_a \boldsymbol{\varphi}_{,\xi_1} \\ \delta_a \mathbf{p} \\ \delta_a \mathbf{p}_{,\xi_1} \end{bmatrix}^t \begin{bmatrix} \mathbf{0} \\ \Delta_a \mathbf{n} \\ \Delta_a (\mathbf{A}^t \mathbf{m} - \mathbf{T}^t_{wp} \hat{\boldsymbol{\varphi}}_{,\xi_1} \mathbf{n}) \\ \Delta_a (\mathbf{T}^t_{wp} \mathbf{m}) \end{bmatrix} d\xi_1 \\
&= \int_L \begin{bmatrix} \delta_a \boldsymbol{\varphi} \\ \delta_a \boldsymbol{\varphi}_{,\xi_1} \\ \delta_a \mathbf{p} \\ \delta_a \mathbf{p}_{,\xi_1} \end{bmatrix}^t \begin{bmatrix} \mathbf{0} \\ \\ \Delta_a \mathbf{n} \\ \\ \Delta_a \mathbf{A}^t \mathbf{m} + \mathbf{A}^t \Delta_a \mathbf{m} \\ -\Delta_a \mathbf{T}^t_{wp} \hat{\boldsymbol{\varphi}}_{,\xi_1} \mathbf{n} - \mathbf{T}^t_{wp} \Delta_a \hat{\boldsymbol{\varphi}}_{,\xi_1} \mathbf{n} \\ -\mathbf{T}^t_{wp} \hat{\boldsymbol{\varphi}}_{,\xi_1} \Delta_a \mathbf{n} \\ \\ \Delta_a \mathbf{T}^t_{wp} \mathbf{m} + \mathbf{T}^t_{wp} \Delta_a \mathbf{m} \end{bmatrix} d\xi_1
\end{aligned}
$$

$$
= \int_L \begin{bmatrix} \delta_a \boldsymbol{\varphi} \\ \delta_a \boldsymbol{\varphi}_{,\xi_1} \\ \delta_a \mathbf{p} \\ \delta_a \mathbf{p}_{,\xi_1} \end{bmatrix}^t \begin{bmatrix} \mathbf{0} \\ \\ \Delta_a \mathbf{n} \\ \\ (\mathbf{A}^t \underline{\mathbf{m}})_{,\mathbf{p}} \Delta_a \mathbf{p} + (\mathbf{A}^t \underline{\mathbf{m}})_{,\mathbf{p}_{,\xi_1}} \Delta_a \mathbf{p}_{,\xi_1} \\ + \mathbf{A}^t \Delta_a \mathbf{m} - (\mathbf{T}_{wp}^t \underline{\hat{\boldsymbol{\varphi}}_{,\xi_1} \mathbf{n}})_{,\mathbf{p}} \Delta_a \mathbf{p} \\ - \mathbf{T}_{wp}^t \Delta_a \hat{\boldsymbol{\varphi}}_{,\xi_1} \mathbf{n} - \mathbf{T}_{wp}^t \hat{\boldsymbol{\varphi}}_{,\xi_1} \Delta_a \mathbf{n} \\ \\ (\mathbf{T}_{wp}^t \underline{\mathbf{m}})_{,\mathbf{p}} \Delta_a \mathbf{p} + \mathbf{T}_{wp}^t \Delta_a \mathbf{m} \end{bmatrix} d\xi_1
$$

$$
= \int_L \begin{bmatrix} \delta_a \boldsymbol{\varphi} \\ \delta_a \boldsymbol{\varphi}_{,\xi_1} \\ \delta_a \mathbf{p} \\ \delta_a \mathbf{p}_{,\xi_1} \end{bmatrix}^t \begin{bmatrix} \mathbf{0} \\ \\ \Delta_a \mathbf{n} \\ \\ \mathbf{A}_1 \Delta_a \mathbf{p} + \mathbf{A}_2 \Delta_a \mathbf{p}_{,\xi_1} + \mathbf{A}^t \Delta_a \mathbf{m} \\ - \boldsymbol{\Xi}(\hat{\boldsymbol{\varphi}}_{,\xi_1} \mathbf{n}) \Delta_a \mathbf{p} + \mathbf{T}_{wp}^t \hat{\mathbf{n}} \Delta_a \boldsymbol{\varphi}_{,\xi_1} \\ - \mathbf{T}_{wp}^t \hat{\boldsymbol{\varphi}}_{,\xi_1} \Delta_a \mathbf{n} \\ \\ \boldsymbol{\Xi}(\mathbf{m}) \Delta_a \mathbf{p} + \mathbf{T}_{wp}^t \Delta_a \mathbf{m} \end{bmatrix} d\xi_1
$$

$$
= \int_L \begin{bmatrix} \delta_a \boldsymbol{\varphi} \\ \delta_a \boldsymbol{\varphi}_{,\xi_1} \\ \delta_a \mathbf{p} \\ \delta_a \mathbf{p}_{,\xi_1} \end{bmatrix}^t \begin{bmatrix} \mathbf{0} \\ \\ \Delta_a \mathbf{n} \\ \\ \mathbf{T}_{wp}^t \hat{\mathbf{n}} \Delta_a \boldsymbol{\varphi}_{,\xi_1} \\ + [\mathbf{A}_1 - \boldsymbol{\Xi}(\hat{\boldsymbol{\varphi}}_{,\xi_1} \mathbf{n})] \Delta_a \mathbf{p} \\ + \mathbf{A}_2 \Delta_a \mathbf{p}_{,\xi_1} \\ + \mathbf{A}^t \Delta_a \mathbf{m} - \mathbf{T}_{wp}^t \hat{\boldsymbol{\varphi}}_{,\xi_1} \Delta_a \mathbf{n} \\ \\ \boldsymbol{\Xi}(\mathbf{m}) \Delta_a \mathbf{p} + \mathbf{T}_{wp}^t \Delta_a \mathbf{m} \end{bmatrix} d\xi_1
$$

$$
= \int_L \begin{bmatrix} \delta_a \boldsymbol{\varphi} \\ \delta_a \boldsymbol{\varphi}_{,\xi_1} \\ \delta_a \mathbf{p} \\ \delta_a \mathbf{p}_{,\xi_1} \end{bmatrix}^t \left(\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_n & \begin{matrix} \mathbf{C}_n \hat{\boldsymbol{\varphi}}_{,\xi_1} \mathbf{T}_{wp} \\ +\mathbf{C}_{nm} \mathbf{A} \end{matrix} & \mathbf{C}_{nm} \mathbf{T}_{wp} \\ \mathbf{0} & \begin{matrix} -\mathbf{T}^t_{wp} \hat{\boldsymbol{\varphi}}_{,\xi_1} \mathbf{C}_n \\ +\mathbf{A}^t \mathbf{C}_{mn} \end{matrix} & \begin{matrix} -\mathbf{T}^t_{wp} \hat{\boldsymbol{\varphi}}_{,\xi_1} \mathbf{C}_n \hat{\boldsymbol{\varphi}}_{,\xi_1} \mathbf{T}_{wp} \\ +\mathbf{A}^t \mathbf{C}_m \mathbf{A} \\ +\mathbf{A}^t \mathbf{C}_{mn} \hat{\boldsymbol{\varphi}}_{,\xi_1} \mathbf{T}_{wp} \\ -\mathbf{T}^t_{wp} \hat{\boldsymbol{\varphi}}_{,\xi_1} \mathbf{C}_{nm} \mathbf{A} \end{matrix} & \begin{matrix} -\mathbf{T}^t_{wp} \hat{\boldsymbol{\varphi}}_{,\xi_1} \mathbf{C}_{nm} \mathbf{T}_{wp} \\ +\mathbf{A}^t \mathbf{C}_m \mathbf{T}_{wp} \end{matrix} \\ \mathbf{0} & \mathbf{T}^t_{wp} \mathbf{C}_{mn} & \begin{matrix} \mathbf{T}^t_{wp} \mathbf{C}_{mn} \hat{\boldsymbol{\varphi}}_{,\xi_1} \mathbf{T}_{wp} \\ +\mathbf{T}^t_{wp} \mathbf{C}_m \mathbf{A} \end{matrix} & \mathbf{T}^t_{wp} \mathbf{C}_m \mathbf{T}_{wp} \end{bmatrix}_M \right.
$$

$$
\left. + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\hat{\mathbf{n}} \mathbf{T}_{wp} & \mathbf{0} \\ \mathbf{0} & \mathbf{T}^t_{wp} \hat{\mathbf{n}} & \begin{matrix} \mathbf{T}^t_{wp} \hat{\boldsymbol{\varphi}}_{,\xi_1} \hat{\mathbf{n}} \mathbf{T}_{wp} \\ +\mathbf{A}_1 - \boldsymbol{\Xi}(\hat{\boldsymbol{\varphi}}_{,\xi_1} \mathbf{n}) - \mathbf{A}^t \hat{\mathbf{m}} \mathbf{T}_{wp} \end{matrix} & \mathbf{A}_2 \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\Xi}(\mathbf{m}) - \mathbf{T}^t_{wp} \hat{\mathbf{m}} \mathbf{T}_{wp} & \mathbf{0} \end{bmatrix}_G \right) \begin{bmatrix} \Delta_a \boldsymbol{\varphi} \\ \Delta_a \boldsymbol{\varphi}_{,\xi_1} \\ \Delta_a \mathbf{p} \\ \Delta_a \mathbf{p}_{,\xi_1} \end{bmatrix} d\xi_1 \tag{241}
$$

in which the matrix with subscript M corresponds to the material part of the tangential stiffness operator and the matrix with subscript G corresponds to the geometric part of the tangential stiffness operator, both of which are due to the internal loading. The symmetry of the material part of the tangential stiffness operator is obvious, while the symmetry of the geometric part is not transparent. However, one may reason that the geometric part is always symmetric as long as the beam material works in the elastic range and the second-order derivatives of the transformation tensors are continuous for a vector-like parametrization. Therefore, in Eq. (222), the last term is enforced.

For the tangential stiffness operator due to the extential loading, one can obtain the linearized increment of the external virtual work, using the results in Eq. (232), as follows:

$$\begin{aligned}
\Delta_a\delta_a W_{ext} &= \lambda\int_L \begin{bmatrix} \delta_a\boldsymbol{\varphi} \\ \delta_a\boldsymbol{\varphi}_{,\xi_1} \\ \delta_a\mathbf{p} \\ \delta_a\mathbf{p}_{,\xi_1} \end{bmatrix}^t \Delta_a \begin{bmatrix} \boldsymbol{\mathcal{N}}_g + c_N\boldsymbol{\mathcal{N}}_d + \boldsymbol{\Lambda}\bar{\boldsymbol{\mathcal{N}}}_p \\ \mathbf{0} \\ \mathbf{T}^t_{wp}[\boldsymbol{\mathcal{M}}_g + c_M\boldsymbol{\mathcal{M}}_d + \boldsymbol{\Lambda}\bar{\boldsymbol{\mathcal{M}}}_p] \\ \mathbf{0} \end{bmatrix} d\xi_1 \\
&= \lambda\int_L \begin{bmatrix} \delta_a\boldsymbol{\varphi} \\ \delta_a\boldsymbol{\varphi}_{,\xi_1} \\ \delta_a\mathbf{p} \\ \delta_a\mathbf{p}_{,\xi_1} \end{bmatrix}^t \begin{bmatrix} -(\boldsymbol{\mathcal{N}}_d\otimes\mathbf{c}_N)\Delta_a\boldsymbol{\varphi}_{,\xi_1} - \hat{\boldsymbol{\mathcal{N}}}_p\Delta_a\mathbf{w} \\ \mathbf{0} \\ \mathbf{T}^t_{wp}[-(\boldsymbol{\mathcal{M}}_d\otimes\mathbf{c}_M)\Delta_a\boldsymbol{\varphi}_{,\xi_1} - \hat{\boldsymbol{\mathcal{M}}}_p\Delta_a\mathbf{w}] \\ +\Delta_a\mathbf{T}^t_{wp}[\boldsymbol{\mathcal{M}}_g + c_M\boldsymbol{\mathcal{M}}_d + \boldsymbol{\Lambda}\bar{\boldsymbol{\mathcal{M}}}_p] \\ \mathbf{0} \end{bmatrix} d\xi_1 \\
&= \lambda\int_L \begin{bmatrix} \delta_a\boldsymbol{\varphi} \\ \delta_a\boldsymbol{\varphi}_{,\xi_1} \\ \delta_a\mathbf{p} \\ \delta_a\mathbf{p}_{,\xi_1} \end{bmatrix}^t \begin{bmatrix} -(\boldsymbol{\mathcal{N}}_d\otimes\mathbf{c}_N)\Delta_a\boldsymbol{\varphi}_{,\xi_1} - \hat{\boldsymbol{\mathcal{N}}}_p\mathbf{T}_{wp}\Delta_a\mathbf{p} \\ \mathbf{0} \\ \mathbf{T}^t_{wp}[-(\boldsymbol{\mathcal{M}}_d\otimes\mathbf{c}_M)\Delta_a\boldsymbol{\varphi}_{,\xi_1} - \hat{\boldsymbol{\mathcal{M}}}_p\mathbf{T}_{wp}\Delta_a\mathbf{p}] \\ +\boldsymbol{\Xi}(\boldsymbol{\mathcal{M}}_g + c_M\boldsymbol{\mathcal{M}}_d + \boldsymbol{\Lambda}\bar{\boldsymbol{\mathcal{M}}}_p)\Delta_a\mathbf{p} \\ \mathbf{0} \end{bmatrix} d\xi_1
\end{aligned}$$

$$
\begin{aligned}
&= -\lambda \int_L \begin{bmatrix} \delta_a \boldsymbol{\varphi} \\ \delta_a \boldsymbol{\varphi}_{,\xi_1} \\ \delta_a \mathbf{p} \\ \delta_a \mathbf{p}_{,\xi_1} \end{bmatrix}^t \begin{bmatrix} (\boldsymbol{\mathcal{N}}_d \otimes \mathbf{c}_N)\Delta_a \boldsymbol{\varphi}_{,\xi_1} + \hat{\boldsymbol{\mathcal{N}}}_p \mathbf{T}_{wp} \Delta_a \mathbf{p} \\ \mathbf{0} \\ -\boldsymbol{\Xi}(\boldsymbol{\mathcal{M}}_g)\Delta_a \mathbf{p} \\ +\mathbf{T}^t_{wp}(\boldsymbol{\mathcal{M}}_d \otimes \mathbf{c}_M)\Delta_a \boldsymbol{\varphi}_{,\xi_1} \\ -\boldsymbol{\Xi}(c_M \boldsymbol{\mathcal{M}}_d)\Delta_a \mathbf{p} \\ +[\mathbf{T}^t_{wp} \hat{\boldsymbol{\mathcal{M}}}_p \mathbf{T}_{wp} - \boldsymbol{\Xi}(\boldsymbol{\Lambda} \bar{\boldsymbol{\mathcal{M}}}_p)]\Delta_a \mathbf{p} \\ \mathbf{0} \end{bmatrix} d\xi_1 \\
&= -\lambda \int_L \begin{bmatrix} \delta_a \boldsymbol{\varphi} \\ \delta_a \boldsymbol{\varphi}_{,\xi_1} \\ \delta_a \mathbf{p} \\ \delta_a \mathbf{p}_{,\xi_1} \end{bmatrix}^t \left(\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\boldsymbol{\Xi}(\boldsymbol{\mathcal{M}}_g) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}_g \right. \\
&\quad + \begin{bmatrix} \mathbf{0} & \boldsymbol{\mathcal{N}}_d \otimes \mathbf{c}_N & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{T}^t_{wp}(\boldsymbol{\mathcal{M}}_d \otimes \mathbf{c}_M) & -c_M \boldsymbol{\Xi}(\boldsymbol{\mathcal{M}}_d) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}_d \\
&\quad \left. + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \hat{\boldsymbol{\mathcal{N}}}_p \mathbf{T}_{wp} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{T}^t_{wp} \hat{\boldsymbol{\mathcal{M}}}_p \mathbf{T}_{wp} - \boldsymbol{\Xi}(\boldsymbol{\mathcal{M}}_p) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}_p \right) \begin{bmatrix} \Delta_a \boldsymbol{\varphi} \\ \Delta_a \boldsymbol{\varphi}_{,\xi_1} \\ \Delta_a \mathbf{p} \\ \Delta_a \mathbf{p}_{,\xi_1} \end{bmatrix} d\xi_1
\end{aligned}
\tag{242}
$$

where the matrices with subscripts g, d, and p correspond to the parts of the tangential stiffness operator contributed by the external loadings of type I, II, and III, respectively–all of which are *non-symmetric*. Therefore, when the deformation-dependent external loading is involved, the overall tangential stiffness matrix is *non-symmetric*.

It is interesting to note that if one defines

$$\Delta_a \mathbf{p} \equiv \Delta_a \mathbf{w} \quad \text{and} \quad \mathbf{T}_{ww} \equiv \mathbf{T}_{wp} \tag{243}$$

which means using the incremental rotation vector $\Delta_a \mathbf{w}$ to parametrize incremental rotation, while $\mathbf{p}$ has no meaning (noticing $\Delta_a \mathbf{w} = \mathbf{T}_{wp} \Delta_a \mathbf{p}$),

$$\Delta_a \mathbf{w} \equiv \mathbf{T}_{ww} \Delta_a \mathbf{w} \tag{244}$$

therefore,

$$\mathbf{T}_{wp} = \mathbf{T}_{ww} = \mathbf{I}_3 \tag{245}$$

and let the directional derivative tensors $\boldsymbol{\Xi} = \mathbf{A} = \mathbf{A}_1 = \mathbf{A}_2 = \mathbf{0}$, then the formulae for a vector-like parametrization of finite rotations as obtained in Eqs. (239) through (242) recover those for the parametrization of the incremental rotation using the incremental vector $\Delta_a \mathbf{w}$, as given in Eqs. (229) through (232). Now, therefore, one has a unified formulation for both vector-like parametrizations and pure incremental parametrization without considering how the compound rotation is parametrized.

Also note that the moment contributed by $\boldsymbol{\mathcal{M}}_g$, as in load type I, is deformation invariant. If the incremental rotation is parametrized using the incremental rotation vector $\Delta_a \mathbf{w}$, and $\boldsymbol{\mathcal{M}}_g$ does not contribute any stiffness to the beam, as seen in Eq. (232), then is conservative; however, if a vector-like parametrization of finite rotations is used, $\boldsymbol{\mathcal{M}}_g$ does contribute stiffness to the beam, as seen in Eq. (242)!

It should be addressed that the benefits of vector-like parametrizations of finite rotations have been said to be: (i) less secondary storage is needed; (ii) a simple additive update rule is applied; (iii) a symmetric element tangential stiffness matrix may be deduced under conservative loading. However, much effort of vector-like parametrizations is on pure mathematical treatments; therefore, clear physical meanings may be lost. If the external loading is non-conservative, the symmetry of tangential stiffness is destroyed, and because of some extra calculations needed to obtain the second-order tensors $\mathbf{T}_{wp}$, $\mathbf{A}$, $\mathbf{A}_1$, $\mathbf{A}_2$, and $\boldsymbol{\Xi}$ the advantage of vector-like parametrizations of finite rotations is *not* obvious. Therefore, in general, a vector-like parametrization is not necessarily a better choice.

On the other hand, parametrizing the incremental rotation using the incremental rotation vector $\Delta_a \mathbf{w}$ results in the simplest and cleanest formulation. With the help of quaternion representations for compound rotations and some results of vector-like parametrizations, it can still be a good choice, especially when the external loading is not conservative and a precise bifurcation analysis is needed.

Chapter 8: A 3-D four-noded curved beam element implementation

8.1 Formulation

Assume that the whole curved beam is discretized into N_{elm} finite elements, and the e-th element has N^e_{node} nodes (the total number of nodes in the curved beam is N_{node}). As previously implied, the non-vectorial parametrization of finite rotations is used (because at least the pressure type of loading will be considered): the incremental rotation is parametrized using the incremental rotation vector $\Delta_a \mathbf{w}$, while the unit quaternion parameters are used to help update the compound rotations and reduce secondary storage.

Here, the conventional Lagrangian interpolation is used for the displaced position vector of the e-th element, $\boldsymbol{\varphi}^e$ (therefore, $\boldsymbol{\varphi}^e_0$, $\delta_a \boldsymbol{\varphi}^e$, and $\Delta_a \boldsymbol{\varphi}^e$, etc.), and the incremental rotation vector $\delta_a \mathbf{w}$ (therefore, $\Delta_a \mathbf{w}$). That is,

$$\boldsymbol{\varphi}^e = N_I \boldsymbol{\varphi}^e_I \tag{246}$$

$$\delta_a \boldsymbol{\varphi}^e = N_I \delta_a \boldsymbol{\varphi}^e_I \tag{247}$$

$$\delta_a \mathbf{w}^e = N_I \delta_a \mathbf{w}^e_I \tag{248}$$

for example, where $N_I = N_I(\xi)$ is the Lagrangian interpolation function of the node I and $\boldsymbol{\varphi}^e_I$, $\delta_a \boldsymbol{\varphi}^e_I$, $\delta_a \boldsymbol{\varphi}^e_I$, etc. are the corresponding nodal values. Note that in the present paper, the subscripts I, $J = 1, 2, \cdots, N^e_{node}$, and the summation convention holds.

Substituting Eqs. (247) and (248) into Eq. (229), one obtains the internal virtual work in the discretized form

$$\delta_a W^e_{int} = \mathbf{P}^e_{intI} \delta_a \boldsymbol{\varphi}^e_I + \mathbf{M}^e_{intI} \delta_a \mathbf{w}^e_I = \mathbf{q}^e_{intI} \cdot \delta_a \mathbf{a}^e_I \tag{249}$$

where

$$\mathbf{P}^e_{intI} = \int_{L^e} N_{I,\xi_1} \mathbf{n}^e d\xi_1 \tag{250}$$

$$\mathbf{M}^e_{intI} = \int_{L^e} (N_{I,\xi_1} \mathbf{m}^e - N_I \hat{\boldsymbol{\varphi}}^e_{,\xi_1} \mathbf{n}^e) d\xi_1 \tag{251}$$

$$\mathbf{q}^e_{intI} = \begin{bmatrix} \mathbf{P}^e_{intI} \\ \mathbf{M}^e_{intI} \end{bmatrix} \tag{252}$$

and

$$\delta_a \mathbf{a}_I^e = \begin{bmatrix} \delta_a \boldsymbol{\varphi}_I^e \\ \\ \delta_a \mathbf{w}_I^e \end{bmatrix} \tag{253}$$

The $\mathbf{q}_{intI}^e$ is interpreted as the nodal internal load vector at node I contributed by element e, and $\mathbf{a}^e$ the nodal variational dispacement vector.

Similarly, the external virtual work becomes

$$\delta_a W_{ext}^e = \lambda \mathbf{P}_{ext0I}^e \delta_a \boldsymbol{\varphi}_I^e + \lambda \mathbf{M}_{ext0I}^e \delta_a \mathbf{w}_I^e = \lambda \mathbf{q}_{ext0I}^e \cdot \delta_a \mathbf{a}_I^e \tag{254}$$

where

$$\begin{aligned} \mathbf{P}_{ext0I}^e &= \int_{L^e} N_I [\boldsymbol{\mathcal{N}}_g^e + c_N^e \boldsymbol{\mathcal{N}}_d^e + \boldsymbol{\Lambda}^e \bar{\boldsymbol{\mathcal{N}}}_p^e] d\xi_1 \\ &= \int_{L^e} N_I [\boldsymbol{\mathcal{N}}_g^e + c_N^e \boldsymbol{\mathcal{N}}_d^e + \boldsymbol{\mathcal{N}}_p^e] d\xi_1 \end{aligned} \tag{255}$$

$$\begin{aligned} \mathbf{M}_{ext0I}^e &= \int_{L^e} N_I [\boldsymbol{\mathcal{M}}_g^e + c_M^e \boldsymbol{\mathcal{M}}_d^e + \boldsymbol{\Lambda}^e \bar{\boldsymbol{\mathcal{M}}}_p^e] d\xi_1 \\ &= \int_{L^e} N_I [\boldsymbol{\mathcal{M}}_g^e + c_M^e \boldsymbol{\mathcal{M}}_d^e + \boldsymbol{\mathcal{M}}_p^e] d\xi_1 \end{aligned} \tag{256}$$

and

$$\mathbf{q}_{ext0I}^e = \begin{bmatrix} \mathbf{P}_{ext0I}^e \\ \mathbf{M}_{ext0I}^e \end{bmatrix} \tag{257}$$

The $\mathbf{q}_{ext0I}^e$ is interpreted as the nodal external load vector at node I contributed by the distributed loads acting on element e for unit loading factor.

The element tangential stiffness matrix is obtained by the linearization of the nodal internal and external loads of the corresponding element, which gives the same result as from Eqs. (231) and (232), the continuum form.

By the linearization of the nodal internal load vector (Eq. 252), one obtains that

$$\Delta_a \mathbf{q}_{intI}^e = \mathbf{K}_{IJ}^{ce} \Delta_a \mathbf{a}_J^e \tag{258}$$

The tangential stiffness matrix $\mathbf{K}_{IJ}^{ce}$ (corresponding to internal loads) consists of two parts:

$$\mathbf{K}_{IJ}^{ce} = \mathbf{K}_{mIJ}^{ce} + \mathbf{K}_{gIJ}^{ce} \tag{259}$$

The material part is

$$\mathbf{K}^{ce}_{mIJ} = \int_{L^e} \begin{bmatrix} N_{I,\xi_1} N_{J,\xi_1} \mathbf{C}^e_n & N_{I,\xi_1} N_J \mathbf{C}^e_n \hat{\boldsymbol{\varphi}}^e_{,\xi_1} + N_{I,\xi_1} N_{J,\xi_1} \mathbf{C}^e_{nm} \\ \\ -N_I N_{J,\xi_1} \hat{\boldsymbol{\varphi}}^e_{,\xi_1} \mathbf{C}^e_n + N_{I,\xi_1} N_{J,\xi_1} \mathbf{C}^e_{mn} & N_{I,\xi_1} N_{J,\xi_1} \mathbf{C}^e_m - N_I N_J \hat{\boldsymbol{\varphi}}^e_{,\xi_1} \mathbf{C}^e_n \hat{\boldsymbol{\varphi}}^e_{,\xi_1} \\ & +N_{I,\xi_1} N_J \mathbf{C}^e_{mn} \hat{\boldsymbol{\varphi}}^e_{,\xi_1} - N_I N_{J,\xi_1} \hat{\boldsymbol{\varphi}}^e_{,\xi_1} \mathbf{C}^e_{nm} \end{bmatrix} d\xi_1 \tag{260}$$

which is always symmetric. The geometric part is

$$\mathbf{K}^{ce}_{gIJ} = \int_{L^e} \begin{bmatrix} \mathbf{0} & -N_{I,\xi_1} N_J \hat{\mathbf{n}}^e \\ N_I N_{J,\xi_1} \hat{\mathbf{n}}^e & N_I N_J \hat{\boldsymbol{\varphi}}^e_{,\xi_1} \hat{\mathbf{n}}^e - N_{I,\xi_1} N_J \hat{\mathbf{m}}^e \end{bmatrix} d\xi_1 \tag{261}$$

which is not necessarily symmetric (see Simo and Vu-Quoc, 1986).

By the linearization of the nodal external load vector (Eq. 254), one obtains that

$$\Delta_a \mathbf{q}^e_{extI} = -(\mathbf{K}^{de}_{IJ} + \mathbf{K}^{pe}_{IJ}) \Delta_a \mathbf{a}^e_J \tag{262}$$

The tangential stiffness $\mathbf{K}^{de}_{IJ}$ turns out to be

$$\mathbf{K}^{de}_{IJ} = \lambda \int_{L^e} \begin{bmatrix} N_I N_{J,\xi_1} \boldsymbol{\mathcal{N}}^e_d \mathbf{c}^{et}_N & \mathbf{0} \\ N_I N_{J,\xi_1} \boldsymbol{\mathcal{M}}^e_d \mathbf{c}^{et}_M & \mathbf{0} \end{bmatrix} d\xi_1 \tag{263}$$

The tangential stiffness matrix $\mathbf{K}^{pe}_{IJ}$ (called the pressure stiffness matrix) is

$$\mathbf{K}^{pe}_{IJ} = \lambda \int_{L^e} \begin{bmatrix} \mathbf{0} & N_I N_J \hat{\boldsymbol{\mathcal{N}}}^e_p \\ \mathbf{0} & N_I N_J \hat{\boldsymbol{\mathcal{M}}}^e_p \end{bmatrix} d\xi_1 \tag{264}$$

Note that both $\mathbf{K}^{de}_{IJ}$ and $\mathbf{K}^{pe}_{IJ}$ can be neglected for small displacements/rotations but *not* for large displacements/rotations, especially when an exact bifurcation analysis is needed.

The nodal displacement vector in the incremental form for element e is

$$\Delta_a \mathbf{a}^e = \begin{bmatrix} \Delta_a \mathbf{a}^e_1 \\ \Delta_a \mathbf{a}^e_2 \\ \vdots \\ \Delta_a \mathbf{a}^e_{N_{node}} \end{bmatrix} \tag{265}$$

The global displacement vector in the incremental form is grouped to be $\mathbf{\Delta a}$ with consideration of the compatibility conditions at nodes.

The nodal internal and external load vectors for element e are

$$\mathbf{q}^e_{int} = \begin{bmatrix} \mathbf{q}^e_{int1} \\ \mathbf{q}^e_{int2} \\ \vdots \\ \mathbf{q}^e_{intN_{node}} \end{bmatrix} \text{ and } \mathbf{q}^e_{ext0} = \begin{bmatrix} \mathbf{q}^e_{ext01} \\ \mathbf{q}^e_{ext02} \\ \vdots \\ \mathbf{q}^e_{ext0N_{node}} \end{bmatrix} \tag{266}$$

respectively.

The global nodal internal and external load vectors are summed up at the corresponding nodes to be $\mathbf{q}_{int}$ and $\mathbf{q}_{ext0}$, respectively.

The virtual work equation in the discretized form becomes

$$\delta\mathbf{a}^t\mathbf{R} = 0 \tag{267}$$

or, equivalently, one obtains the equilibrium or state equation at nodes:

$$\mathbf{R} = \mathbf{q}_{int} - \lambda\mathbf{q}_{ext0} = 0 \tag{268}$$

where $\mathbf{R}$ is the nodal load vector residual.

The tangential stiffness matrix of element e is

$$\mathbf{K}^e_T = \mathbf{K}^{ce}_m + \mathbf{K}^{ce}_g + \mathbf{K}^{pe} + \mathbf{K}^{de} \tag{269}$$

where the material part is

$$\mathbf{K}^{ce}_m = [\,\mathbf{K}^{ce}_{mIJ}\,] \tag{270}$$

and similarly for the geometric part $\mathbf{K}^{ce}_g$ and the parts contributed by external loads, $\mathbf{K}^{pe}$ and $\mathbf{K}^{de}$.

Assemblying $\mathbf{K}^e_T$, one obtains the global tangential stiffness matrix $\mathbf{K}_T = \mathbf{R}_{,\mathbf{a}}$. Therefore, the state equation in linearized incremental form is

$$\mathbf{R} + \mathbf{R}_{,\mathbf{a}}\Delta\mathbf{a} + \mathbf{R}_{,\lambda}\Delta\lambda = \mathbf{0} \tag{271}$$

or equivalently,

$$\mathbf{K}_T \Delta \mathbf{a} = \Delta\lambda \mathbf{q}_{ext0} - \mathbf{R} \tag{272}$$

It should be pointed out that the formulae for both the nodal internal load and tangential stiffness matrix for an element are, in fact, identical to those by Simo and Vu-Quoc (1986) and Ibrahimbegović (1995) except the parts related to the different treatments of locking phenomena and external loads in addition to the different arrangements of the order of the nodal translational and rotational displacements. Simo and Vu-Quoc (1986) and the present study apply the *uniform* reduced integration technique to the nodal load vectors and tangential stiffness matrix to remove the lockings, while Ibrahimbegović (1995) uses the so-called hierarchical terms for the same purpose. The reason for using reduced integration in the present study is based on the paper by Min and Kim (1996), which shows that *for C^0-continuous beam elements under (even) the non-uniform mapping, using $(N^e_{node} - 1)$-pointed Gaussian quadrature can produce (enough) $(N^e_{node} - 1)$ constraints with no spurious constraints (energy) when N^e_{node}-noded (curved) beam element is used.* This conclusion makes it possible to elevate the degree of the curved beam element with C^0-continuity to any order following the same procedure. Therefore, the current formulation of the curved beam element with large displacements/rotations is *uniformly* valid for any number of nodes in an element, though a four-noded element may be a wise choice for a general spatially curved beam to have a better representation of the initial geometry of the beam and keep a relative small band-width of the tangential stiffness matrix. It should be addressed again that the use of the incremental rotational vector $\Delta_a \mathbf{w}$ to parametrize the incremental rotation makes the expressions for the nodal internal/external load vectors and tangential stiffness matrix concise and explicit, as opposed to using the increment of the compound rotation vector $\Delta_a \boldsymbol{\theta}$ in the vector-like parametrization (see e.g. Ibrahimbegović, 1995). This seems more efficient and robust for computations and more convenient to programming.

8.2 Solution procedures

The update procedures for large displacements/rotations are the exact same as suggested by Ibrahimbegović (1995) for the same class of parametrizations, and therefore are omitted here.

The full Newton-Raphson iterative method (see e.g. Simo and Vu-Quoc, 1986) and Riks' arc-length control technique (see Crisfield, 1981) are used to find the nonlinear solutions (deformations, stress resultants/couples, nonlinear bifurcation buckling point, limit point, and post-buckling analyses of arches) subjected to the three types of distributed loads as defined in Table 2. For the details of those, one can refer to relevant references.

8.3 Numerical examples

While the formulation in the present study is suitable for any number (three or more) of nodes in an element, only a four-noded curved beam element is implemented in the Fortran code. Several geometrically nonlinear test problems are solved using the four-noded 3-D curved beam element. Some notations are first given here. E represents Young's modulus; G shear modulus; ν Poisson's ratio; A and A' cross-sectional areas; I area moment of the cross-section corresponding to bending; M bending moment; P point load; p distributed load; U, V, and W displacement components in X, Y, and Z directions, respectively; R radius; L length; and N_{elm} total number of the four-noded curved beam elements used. In all the test examples, the grid points of the finite element mesh are uniformly distributed along the beam midcurve for convenience. Whether or not a non-uniform mesh would induce significant mesh-distortion problems will be tested in the 45-degree bend example. Note also that all examples are run with the slender beam assumption, in which the initial curvature correction term has trivial effects on the numerical solutions.

Example 1. Unrolling and Re-rolling of a Circular Cantilever Beam

This example represents the geometrically-nonlinear analysis of a circular cantilever beam with a bending moment applied at the free end. See Fig. 4 for the relevant data (Ibrahimbegović, 1995:).

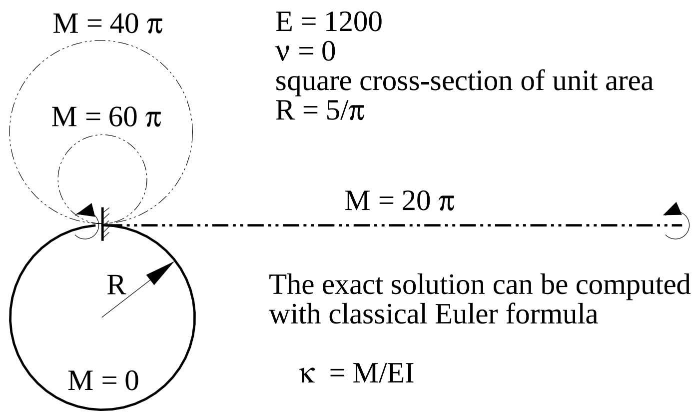

Fig. 4 Unrolling and re-rolling a circular cantilever beam

For the chosen data, the values of the free-end bending moments that turn the reference circular shape into the corresponding semi-circle and straight line are $M = 10\pi$ and $M = 20\pi$, respectively. If the moment is increased after the straight line is reached, for example, $M = 30\pi$, $M = 40\pi$, the semi-circle and complete circle can be formed in the other side. Moreover, another smaller circle is formed at $M = 60\pi$. Eight elements are used to unroll and re-roll the beam as shown in the figure, which shows that the deformed shapes of the beam fit the expected results every well.

Different numbers of the 4-noded elements were tried. It turns out that four elements are enough to get good results for the unrolling case. In this case, two load steps are needed to unroll the beam to a straight line. Eighteen iterations are needed within each load step.

Table 3 shows the displacement comparison.

Table 3 Displacement comparison of loaded point of unrolling a beam

Model	N_{elm}	Horiz. displ	Vert. displ
$M = 10\pi$			
Present	4	0.009	6.374
(4-noded, curved)	6	0.002	6.368
	8	0.001	6.367
Ibrahimbegović (1995)	10	-0.004	6.364
(3-noded, curved)			
Reference solution		0.000	6.366
$M = 20\pi$			
Present	4	10.009	0.026
(4-noded, curved)	6	10.002	0.007
	8	10.001	0.002
Ibrahimbegović (1995)	10	9.998	0.017
(3-noded, curved)			
Reference solution		10.000	0.000

Example 2. Post-buckling analysis of a clamped-hinged circular arch

This well-known example presents the nonlinear analysis of the pre- and post-buckling deformation of a circular arch, hinged at one end and clamped at the other end, under a vertical force applied at the apex. Material and geometric data for the arch are shown in Fig. 5(a).

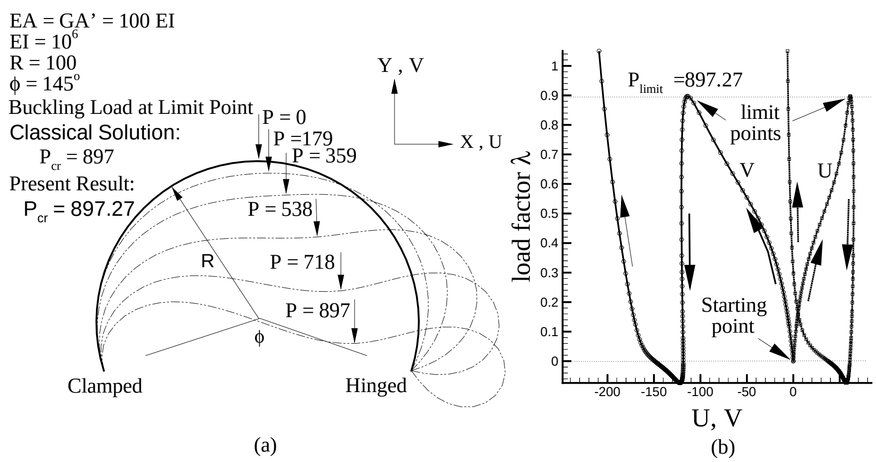

Fig. 5 Clamped-hinged circular arch

When twelve 4-noded elements are used, the limit point with value 897.30 is found automatically using the full Newton-Raphson iterative method. One load step with 44 iterations is enough to converge to the limit point value if it is prescribed as a load increment. Different numbers of the 4-noded elements were tried, and only six elements were enough to obtain good results: see Table 4 (the classical solution was contributed by DaDeppo and Schmidt, 1975).

Table 4 Buckling load comparison of clamped-hinged circular arch

Model	N_{elm}	Limit Buckling Load
Present	6	897.27
(4-noded, curved)	12	897.30
Ibrahimbegović (1995)	20	897.3
(3-noded, curved)		
Simo and Vu-Quoc (1986)	40	905.28
(2-noded, straight)		
Classical solution		897

Fig. 5(a) shows, using five equal-load increments, the deformation process of the arch (six elements) up to the load when the limit point is reached. Finally, the pre- and post-buckling analysis was done using six elements. The relations between the displacements of the loaded point and the load are shown in Fig. 5(b). Note that the second limit point is reached at $P_{limit} = -73.60$ here while it is -77.07 by Simo and Vu-Quoc (1986). A total of 198 load steps are used automatically for Fig. 5(b). Obviously, the arc-length control technique used here is quite robust.

Example 3. 45-degree bend

This example serves to test the performance of the curved beam element in a 3-D deformation problem. The curved cantilever beam with the reference configuration given as a 45-degree circular segment of radius $R = 100$ in the horizontal plane is shown in Fig. 6(a), from which one sees that the cantilever beam is undergoing a 3-D deformation.

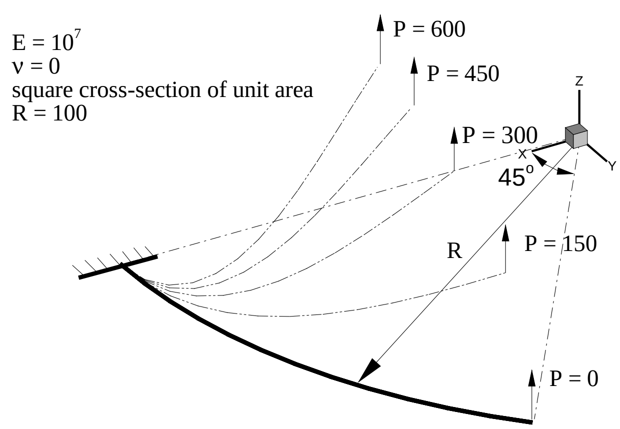

Fig. 6(a) 45-degree bend

Different numbers of 4-noded elements were tried. The tip displacement components are given in Table 5(a) for comparison. It can be seen that one element is enough to obtain converged results[14].

[14] The present results in Table 5(a) were obtained using torsion constant $J = 0.1406 \times 1^4$, while $J = 1/3 \times 1^4$ was used in by Kapania and Li (1998)

Table 5(a) Displacement comparison of loaded point of 45-degree bend

Load level	Model	N_{elm}	U	V	W
	Present	1	7.18	-12.17	40.46
	(4-noded, curved)	2	7.18	-12.18	40.48
$P = 300$		8	7.18	-12.17	40.48
	Bathe and Bolourchi (1979)		6.97	-11.87	40.08
	(2-noded, straight)				
	Simo and Vu-Quoc (1986)		6.8	-11.51	39.50
	(2-noded, straight)				
	Cardona and Geradin (1988)		7.16	-12.07	40.35
	(2-noded, straight)				
	Present(4-noded)	1	13.72	-23.76	53.51
		2	13.73	-23.82	53.61
$P = 600$		8	13.73	-23.82	53.61
	Ibrahimbegović (1995)	8	13.73	-23.81	53.61
	(3-noded, curved)				
	Bathe and Bolourchi (1988)	8	13.51	-23.48	53.37
	Simo and Vu-Quoc (1986)		13.40	-23.51	53.40
	(2-noded, straight)				
	Cardona and Geradin (1988)		13.75	-23.67	53.50
	(2-noded, straight)				

In case of a non-uniform mesh, we tested whether or not the four-noded curved beam has any significant *mesh-distortion* problems through the 45-degree bend example under the point load $P = 600$. A non-uniform mesh was created using geometric progression: the *p-th* arc-length, numbered from the clamped end, of two adjacent nodes of the mesh is given by $s_p = r^{p-1} s_1$ and $s_1 = L(1-r)/(1-r^{N_{node}})$. Figure 6(b) shows how the mesh was distorted for the case of one element.

The results are given in Table 5(b) with different values of ratio r. The results show that the four-noded curved beam element is, for all practical purposes, free from mesh-distortion errors.

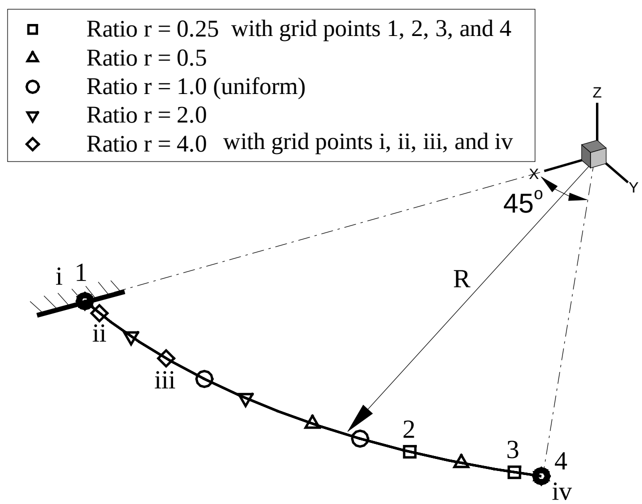

Fig. 6(b) Non-uniform mesh of 45-degree bend

Table 5(b) Distortion test of non-uniform meshes

N_{elm}	Ratio r	U	V	W
1	0.25	11.14	-21.81	51.34
	0.5	12.69	-23.04	52.43
	0.75	13.44	-23.55	53.16
	1.0	13.72	-23.76	53.51
	1.5	13.67	-23.82	53.55
	2.0	13.41	-23.76	53.38
	4.0	12.54	-23.59	53.34
2	0.25	10.87	-22.14	51.86
	0.5	12.94	-23.22	52.69
	0.75	13.65	-23.55	53.46
	1.0	13.73	-23.82	53.61
	1.5	13.70	-23.82	53.59
	2.0	13.53	-23.80	53.49
	4.0	12.59	-23.59	53.34
8	0.25			
	0.5	12.97	-23.25	52.72
	0.75	13.69	-23.78	53.53
	1.0	13.73	-23.82	53.61
	1.5	13.71	-23.82	53.59
	2.0	13.55	-23.81	53.50
	4.0	12.60	-23.59	53.34

Example 4. Cantilever beam subjected to conservative and non-conservative distributed loads

This example is used to illustrate the three types of distributed loads as defined previously (Table 2) through a uniform elastic cantilever beam: Type I – self weight type of load, whose density is given per unit arc-length; this type of load is conservative, being identified as I–c. Type II – the density is given along the direction **d**, being coincident with the **X** axis in this example; for the case that the load is carried by the beam, it is conservative, identified as II–c; for the case that the load is fixed in space, it is non-conservative, identified as II–n . Type III – pressure type of load, which is non-conservative, identified as III–n. Only force densities are involved in this example. Because the beam is straight, cases I–c and II–c are identical. See Fig. 7(a) for the illustration. Except for case II–n, the other two have been used by Kondoh and Atluri (1987) and Jiang and Olson (1994).

Figure 7(a) also shows the deformed shapes for the four cases of loads at $PL^2/EI = 25$. Figure 7(b) shows the tip deflections corresponding to the four cases of loads. The continuous curves are obtained by the present study. The scattered data in Fig. 7(b) were scanned from the paper by Kondoh and Atluri (1987). Results by Jiang and Olson (1994) are also very close to the current ones, though they are not shown here.

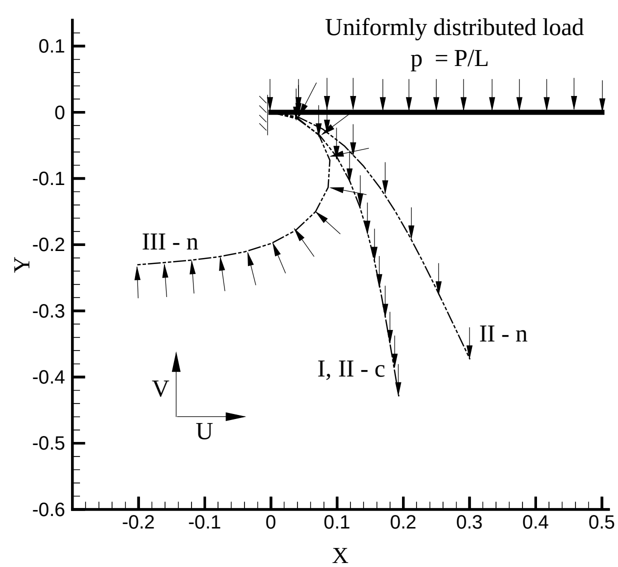

Fig. 7(a) Deformed shapes of a cantilever beam at $PL^2/EI = 25$

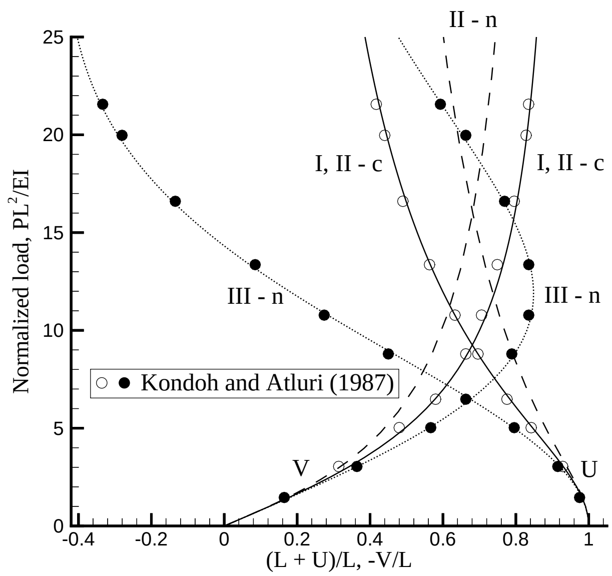

Fig. 7(b) Tip deflections of a cantilever beam

Four 4-noded curved beam elements are used for the data in Fig. 7(a-b), though two such elements are enough to obtain the results close to exact ones. The full Newton-Raphson method is used for all the cases except for case II-n, in which Riks' arc-length control technique is used because of the *over-hardening* phenomena.

Note that for a small loading factor (therefore, small displacements/rotations), the deflections are almost the same for both the conservative and non-conservative loadings, which can be seen in Fig. 7(b). However, the deflections are very different from each other for a large loading factor or large displacements/rotations. Therefore, it is necessary to identify non-conservative loads for flexible structures so that a better prediction of the structural responses can be achieved.

Example 5. Bifurcation buckling of a circular arch with two hinges

This example, as shown in Fig. 8, is used to test the performance of the 4-noded curved beam element in a bifurcation buckling problem under pressure load.

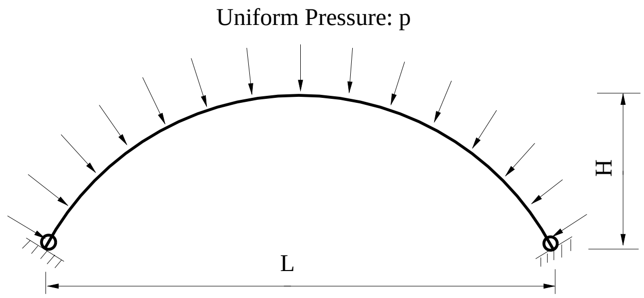

Fig. 8 Circular arch with two hinges

Both linearized and nonlinear analyses were used. The results using 4 elements are shown in Table 6 (the buckling load is given in terms of the factor γ in the equation $p_{cr} = \gamma EI/L^3$), as a comparison with the classical solution available in the book by Timoshenko and Gere (1961).

Table 6 Buckling load comparison of a circular arch with two hinges

H/L	Classical solution	Linearized analysis	Nonlinear analysis
0.1	28.4	28.4	28.2
0.2	39.3	42.1	42.2
0.3	40.9	40.9	40.9
0.4	32.8	32.8	32.8
0.5	24.0	24.0	24.0

It turns out that the results from the linearized analysis are very close to the classical solution. The same holds for the nonlinear analysis except that when the arch tends to be shallower, the difference increases.

Chapter 9: Summary and Conclusions

In the present study, we deal with a truly geometrically exact curved and twisted beam theory without placing emphasis on the computational aspects.

Except for the rigid cross-section assumption, no further approximations are made in the formulation. All reference frames are orthonormal and therefore the geometric aspects may become more direct.

The formulation starts with the three-dimensional theory and ends up with the one-dimensional beam model. The deformation gradient tensor of the current beam configuration relative to the initially curved/twisted beam configuration is explicitly obtained through the introduction of a straight reference beam.

The first Piola-Kirchhoff stress and its conjugate strain with an initial curvature correction term at any material point on the current beam cross-section are addressed, in terms of which other stresses and strains can be explicitly expressed.

The stress resultant and couple are defined in a classical sense and the reduced strains are obtained from the three-dimensional beam model, which are the same as obtained from the reduced differential equations of motion.

The reduced differential equations of motion are also re-examined for the initially curved/twisted beams. The corresponding equations of motion in general include additional inertial terms.

The linear and linearized non-linear constitutive relations with couplings are considered for the engineering strain and stress conjugate pair in the three-dimensional beam level. The cross-section elasticity constants corresponding to the reduced constitutive relations are obtained with the initial curvature correction term.

The geometry, mass and elasticity centroid lines do not coincide with each other in general.

It is believed that the initial curvature correction term may be significant for thick and moderately thick curved beams, especially when long-term dynamic responses are concerned as well as when the strains and stresses in three-dimensional level are needed for strength checking.

The current formulation method by the introduction of a straight reference beam can be used to extend the straight beam theory that considers simple shear-torsion warping effect (see e.g. Simo and Vu-Quoc, 1991; etc.) to the curved beam theory.

Some effort has been made to discuss finite rotations using rotation vectors and unit quaternions, including total and incremental parametrizations. The so-called vector-like parametrizations, which may lead to an additive update rule and symmetric element tangential stiffness matrix, are addressed and generalized in contrast to the non-vectorial parametrizations, which leads to a multiplicative update rule and non-symmetric tangential stiffness matrix, when external loading is not a consideration. In terms of a general vector-like parametrization, a unified formulation is given for the linearized virtual work equations, which will lead to the nodal residual vector and tangential stiffness matrix in the finite element formulation, which can recover the case of the non-vectorial parametrization if the incremental rotation is parametrized using the incremental rotation vector. It is addressed that when complex external loading is considered, the overall tangential stiffness is non-symmetric in general. Therefore, the vector-like parametrization of finite rotations is not necessarily a better choice in general.

As an example, taking advantage of the simplicity in formulation and clear classical meanings of rotations and moments, the previously suggested non-vectorial parametrization is applied to implement a four-noded 3-D curved beam element using Reissner's slender beam theory, in which the compound rotation is parametrized using the unit quaternion and the incremental rotation is parametrized using the incremental rotation vector. The element has three displacement and three rotational degrees of freedom at each node. Conventional Lagrangian interpolation functions are adopted to independently model the displacements and rotations. Reduced integration is used in order to overcome locking problems. The finite element equations are developed for static structural analyses, including deformations, stress resultants/couples, and linearized/nonlinear bifurcation buckling, as well as post-buckling analyses of arches subjected to conservative and non-conservative loads. Illustrative examples show that the consideration of the initial curvature of a curved beam or arch and the elevation of the degree of interpolation polynomials to be cubic can increase the precision and efficiency in the analysis of arches. Besides, the element is mesh-distortion-free from the practical view-point if a non-uniform mesh is used, which is especially important for shape optimization of arches.

Appendix

Fortran programs for a 3-D four-noded curved beam element: ARCHCODE version 1.0

(See the separate PDF file: 2appendix.pdf)

References

Argyris, J. 1982: An excursion into large rotations. *Computer Methods in Applied Mechanics and Engineering* 49, 85-155

Atluri, S.N.; Cazzani, A. 1995: Rotations in computational solid mechanics. *Archives Comput. Methods Engrg*, 2(1), 49-138

Cardona, A.; Geradin, M. 1988: A beam finite element non-linear theory with finite rotations. *International Journal for Numerical Methods in Engineering* 26, 2403-2438

Franchi, C.G.; Montelaghi, F. 1996: A weak-weak formulation for large displacements beam statics: a finite volumes approximation. *International Journal for Numerical Methods in Engineering* 39, 585-604

Ibrahimbegović, A. 1995: On finite element implementation of geometrically nonlinear Reissner's beam theory: three dimensional curved beam elements. *Computer Methods in Applied Mechanics and Engineering* 122, 11-26

Ibrahimbegović, A.; Frey, F. 1994: Stress resultant geometrically nonlinear shell theory with drilling rotations – Part II. Computational aspects. *Computer Methods in Applied Mechanics and Engineering* 118, 285-308

Ibrahimbegović, A.; Frey, F.; Kožar, I. 1995: Computational aspects of vector-like parametrization of three-dimensional finite rotations. *International Journal for Numerical Methods in Engineering* 38, 3653-3673

Iura, M.; Atluri, S.N. 1988: Dynamic analysis of finitely stretched and rotated three-dimensional space-curved beams. *Computers & Structures* 29, 875-889

Iura, M.; Atluri, S.N. 1989: On a consistent theory, and variational formulation of finitely stretched and rotated 3-D space-curved beams. *Computational Mechanics* 4, 73-88

Jelenić, G.; Saje, M. 1995: A kinematically exact space finite strain beam model – finite element formulation by generalized virtual work principle, *Computer Methods in Applied Mechanics and Engineering* 120, 131-161

Kapania, R.K. and Li, J. 1998: A Four-Noded 3-D geometrically nonlinear curved beam element with large displacements/rotations, in *Modeling and Simulation Based Engineering*, S.N. Atluri and P.E. O'Donoghue (ed.), Technology Science Press, 679-684

Ogden, R.W. 1997: Non-linear Elastic Deformations, Dover Publications, Inc., New York

Reissner, E. 1972: On one-dimensional finite-strain beam theory: the plane problem. *Journal of Applied Mathematics and Physics* 23, 795-804

Reissner, E. 1973: On one-dimensional large displacement finite beam theory. *Studies in Applied Mechanics* 52, 87-95

Reissner, E. 1981: On finite deformations of space curved beams. *Journal of Applied Mathematics and Physics* 32, 734-744

Saje, M. 1991: Finite element formulation of finite planar deformation of curved elastic beams. *Computers & Structures* 19, 327-337

Sandhu, J.S.; Stevens, K.A.; Davies, G.A.O. 1990: A 3-D, co-rotational, curved and twisted beam element. *Computers & Structures* 35, 69-79

Simo, J.C. 1985: A finite strain beam formulation: the three-dimensional dynamics. Part I. *Computer Methods in Applied Mechanics and Engineering* 49, 55-70

Simo, J.C.; Tarnow, N.; and Doblare, M. 1995: Non-linear dynamics of three-dimensional rods: exact energy and momentum conserving algorithms. *International Journal for Numerical Methods in Engineering* 38, 1431-1473

Simo, J.C.; Vu-Quoc, L. 1986: A three dimensional finite-strain rod model. Part II: computational aspects, *Computer Methods in Applied Mechanics and Engineering* 58, 79-116

Simo, J.C.; Vu-Quoc, L. 1991: A geometrically-exact rod model incorporating shear and torsion-warping deformation, *International Journal of Solids and Structures* 27, 371-393

Vita

Jing Li was born in Dalian, Liaoning, P.R. China in 1961. He received the B.S. and M.S. degrees in Aircraft Engineering from Northwestern Polytechnical University at Xi'an, China, in 1984, and 1987, respectively.

From 1987 to 1995, he worked as a mechanical/structural engineer in the Department of Structures and Strength at Shenyang Aircraft Research Institute, Shenyang, China.

In 1995, he joined in the Department of Aerospace and Ocean Engineering, Virginia Polytechnical Institute and State University. As a graduate research assistant, he is mainly involved in researches in the areas of geometrically nonlinear curved beam theory and its finite element implementation/simulation.