Appendix A

This appendix explains the derivation of Stirling's formula. Consider the function $y = Ln(x)$ as shown in Figure A.1 below.

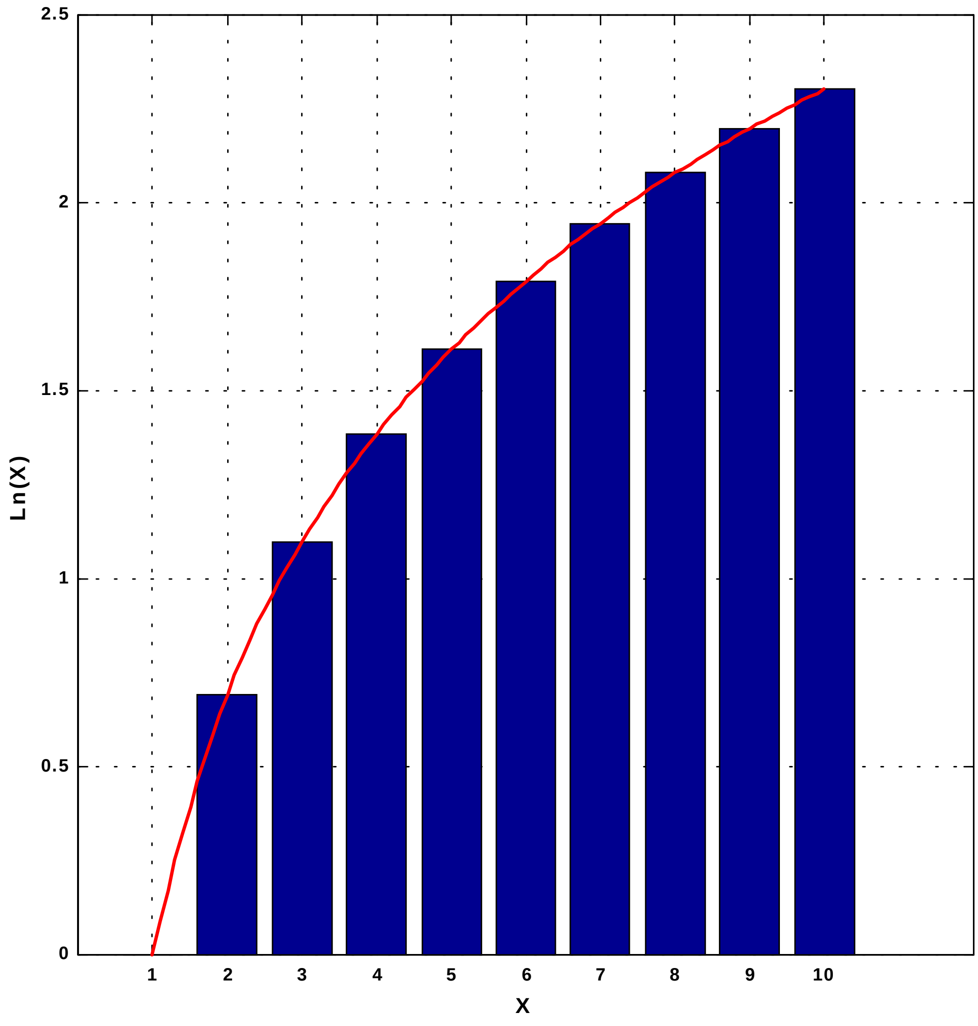

Figure A.1: X vs. Ln(X)

By approximating the area under the curve with the rectangle method, we find that;

Method 1: Area under the curve between x = 1 and x = N is

$$\approx (1)Ln(1) + (1)Ln(1) + (1)Ln(3) + (1)Ln(4) + \ldots(1)Ln(N) = Ln(N!) \qquad \text{(A.1)}$$

If we evaluate the exact area under the curve from x = 1 to x = N, we find that it can be expressed in the form of an integral.

Method 2: Area under the curve between x = 1 and x = N is equal to

$$\int_1^N Ln(x)dx \qquad \text{(A.2)}$$

We know

$$\int Ln(x)dx = xLn(x) - x \qquad \text{(A.3)}$$

Therefore

$$\int_1^N Ln(x)dx = NLn(N) - N - 0 + 1 \qquad \text{(A.4)}$$

If N is large, then the '1' in this expression has negligible effect and can be dropped. As method 1 and 2 compute the area under the curve equating Equation A.1 and A.4 we get;

$$Ln(N!) = NLn(N) - N \qquad \text{(A.5)}$$

A careful treatment using Euler-Maclaurin series gives the asymptotic series

$$x! = \sqrt{2\pi x}\, x^x e^{-x}\left(1 + \frac{1}{12x} + \frac{1}{288x^2} - \frac{139}{51840x^3} - .....\right) \tag{A.6}$$

Ignoring the terms with x in the asymptotic series we get

$$x! = \sqrt{2\pi x}\, x^x e^{-x} \tag{A.7}$$

Taking natural logarithms on both sides of equation A.7, we get

$$Ln(x!) = \left(x + \frac{1}{2}\right)Ln(x) - x + \frac{1}{2}Ln(2\pi) \tag{A.8}$$

which is more accurate than equation A.5. For large values the difference between Equation A.5 and A.8 is practically zero.

Appendix B

This appendix contains the various formulations in an expanded form for the 5 link and 2 link networks discussed in Chapter 6.

5 LINK NETWORK

Equation 6.1

Maximize: $$Z\left(T_{ij}, t_{ij}\right) = \frac{T!}{\prod_{ij}\left(T_{ij}!\right)} \prod_{ij}\left(\frac{t_{ij}}{\sum_{ij} t_{ij}}\right)^{T_{ij}}$$

Maximize: $$Z = \frac{T!}{T_{ac}!T_{ad}!T_{bc}!T_{bd}!}\left(\frac{t_{ac}}{t}\right)^{T_{ac}}\left(\frac{t_{ad}}{t}\right)^{T_{ad}}\left(\frac{t_{bc}}{t}\right)^{T_{bac}}\left(\frac{t_{bd}}{t}\right)^{T_{bd}}$$

Equation 6.2

Maximize: $$Z\left(T_{ij}, t_{ij}\right) = TLn\left(\frac{T}{t}\right) - T - \sum_{ij}\left(T_{ij}Ln\left(\frac{T_{ij}}{t_{ij}}\right) - T_{ij}\right)$$

Maximize: $$\begin{aligned} Z = TLn\left(\frac{T}{t}\right) - T &- \left(T_{ac}Ln\left(\frac{T_{ac}}{t_{ac}}\right) - T_{ac}\right) - \left(T_{ad}Ln\left(\frac{T_{ad}}{t_{ad}}\right) - T_{ad}\right) \\ &- \left(T_{bc}Ln\left(\frac{T_{bc}}{t_{bc}}\right) - T_{bc}\right) - \left(T_{bd}Ln\left(\frac{T_{bd}}{t_{bd}}\right) - T_{bd}\right) \end{aligned}$$

Equation 6.3

$$\text{Maximize: } Z(T_{ij}, t_{ij}) = -\sum_{ij}\left(T_{ij} Ln\left(\frac{T_{ij}}{t_{ij}}\right) - T_{ij} \right)$$

$$\text{Maximize: } \begin{aligned} Z = &-\left(T_{ac} Ln\left(\frac{T_{ac}}{t_{ac}}\right) - T_{ac} \right) - \left(T_{ad} Ln\left(\frac{T_{ad}}{t_{ad}}\right) - T_{ad} \right) \\ &-\left(T_{bc} Ln\left(\frac{T_{bc}}{t_{bc}}\right) - T_{bc} \right) - \left(T_{bd} Ln\left(\frac{T_{bd}}{t_{bd}}\right) - T_{bd} \right) \end{aligned}$$

Equation 6.4

$$\text{Maximize: } Z(T_{ij}, t_{ij}) = -\sum_{ij}\left(T_{ij} Ln\left(\frac{T_{ij}}{t_{ij}}\right) - T_{ij} + t_{ij} \right)$$

$$\text{Maximize: } \begin{aligned} Z = &-\left(T_{ac} Ln\left(\frac{T_{ac}}{t_{ac}}\right) - T_{ac} + t_{ac} \right) - \left(T_{ad} Ln\left(\frac{T_{ad}}{t_{ad}}\right) - T_{ad} + t_{ad} \right) \\ &-\left(T_{bc} Ln\left(\frac{T_{bc}}{t_{bc}}\right) - T_{bc} + t_{bc} \right) - \left(T_{bd} Ln\left(\frac{T_{bd}}{t_{bd}}\right) - T_{bd} + t_{bd} \right) \end{aligned}$$

Equation 6.5

$$\text{Minimize: } Z(T_{ij}, t_{ij}) = \sum_{ij}\left(\frac{1}{2T_{ij}} (T_{ij} - t_{ij})^2 \right)$$

$$\text{Maximize: } Z = \frac{1}{2T_{ac}}(T_{ac} - t_{ac})^2 + \frac{1}{2T_{ad}}(T_{ad} - t_{ad})^2 + \frac{1}{2T_{bc}}(T_{bc} - t_{bc})^2 + \frac{1}{2T_{bd}}(T_{bd} - t_{bd})^2$$

Equation 6.6

$$\text{Minimize: } Z(T_{ij}, t_{ij}) = \sum_{ij} \left(\frac{1}{2t_{ij}} (T_{ij} - t_{ij})^2 \right)$$

$$\text{Minimize: } Z = \frac{1}{2t_{ac}}(T_{ac} - t_{ac})^2 + \frac{1}{2t_{ad}}(T_{ad} - t_{ad})^2 + \frac{1}{2t_{bc}}(T_{bc} - t_{bc})^2 + \frac{1}{2t_{bd}}(T_{bd} - t_{bd})^2$$

Equation 6.7

$$\text{Maximize: } Z(T_{ij}, t_{ij}) = \prod_a \frac{V_a!}{\prod_{ij} (T_{ij} p_{ij}^a)!} \prod_{ij} \left(\frac{t_{ij} p_{ij}^a}{v_a} \right)^{T_{ij} p_{ij}^a}$$

$$Z_1 = \frac{V_1!}{T_{ac}!T_{ad}!} \times \left(\frac{t_{ac}}{t_{ac} + t_{ad}} \right)^{T_{ac}} \times \left(\frac{t_{ad}}{t_{ac} + t_{ad}} \right)^{T_{ad}}$$

$$Z_2 = \frac{V_2!}{T_{bc}!T_{bd}!} \times \left(\frac{t_{bc}}{t_{bc} + t_{bd}} \right)^{T_{bc}} \times \left(\frac{t_{bd}}{t_{bc} + t_{bd}} \right)^{T_{bd}}$$

$$Z_3 = \frac{V_3!}{T_{ac}!T_{ad}!T_{bc}!T_{bd}!} \times \left(\frac{t_{ac}}{t_{ac} + t_{ad} + t_{ac} + t_{ad}} \right)^{T_{ac}} \times \left(\frac{t_{ad}}{t_{ac} + t_{ad} + t_{ac} + t_{ad}} \right)^{T_{ad}} \times$$
$$\left(\frac{t_{bc}}{t_{ac} + t_{ad} + t_{ac} + t_{ad}} \right)^{T_{bc}} \times \left(\frac{t_{bd}}{t_{ac} + t_{ad} + t_{ac} + t_{ad}} \right)^{T_{bd}}$$

$$Z_4 = \frac{V_4!}{T_{ac}!T_{bc}!} \times \left(\frac{t_{ac}}{t_{ac} + t_{bc}} \right)^{T_{ac}} \times \left(\frac{t_{bc}}{t_{ac} + t_{bc}} \right)^{T_{bc}}$$

$$Z_5 = \frac{V_5!}{T_{ad}!T_{bd}!} \times \left(\frac{t_{ad}}{t_{ad} + t_{bd}} \right)^{T_{ad}} \times \left(\frac{t_{bd}}{t_{ad} + t_{bd}} \right)^{T_{bd}}$$

$$\text{Maximize: } Z = Z_1 \times Z_2 \times Z_3 \times Z_4 \times Z_5$$

Equation 6.8

$$\text{Minimize: } Z(T_{ij}, t_{ij}) = \sum_a \sum_{ij} T_{ij}\, p_{ij}^a Ln\left(\frac{T_{ij} v_a}{V_a t_{ij}}\right)$$

$$Z_1 = T_{ac} Ln\left(\frac{T_{ac} \times v_1}{V_1 \times t_{ac}}\right) + T_{ad} Ln\left(\frac{T_{ad} \times v_1}{V_1 \times t_{ad}}\right)$$

$$Z_2 = T_{bc} Ln\left(\frac{T_{bc} \times v_2}{V_2 \times t_{bc}}\right) + T_{bd} Ln\left(\frac{T_{bd} \times v_2}{V_2 \times t_{bd}}\right)$$

$$Z_3 = T_{ac} Ln\left(\frac{T_{ac} \times v_3}{V_3 \times t_{ac}}\right) + T_{ad} Ln\left(\frac{T_{ad} \times v_3}{V_3 \times t_{ad}}\right) + T_{bc} Ln\left(\frac{T_{bc} \times v_3}{V_3 \times t_{bc}}\right) + T_{bd} Ln\left(\frac{T_{bd} \times v_3}{V_3 \times t_{bd}}\right)$$

$$Z_4 = T_{ac} Ln\left(\frac{T_{ac} \times v_4}{V_4 \times t_{ac}}\right) + T_{bc} Ln\left(\frac{T_{bc} \times v_4}{V_4 \times t_{bc}}\right)$$

$$Z_5 = T_{ad} Ln\left(\frac{T_{ad} \times v_5}{V_5 \times t_{ad}}\right) + T_{bd} Ln\left(\frac{T_{bd} \times v_5}{V_5 \times t_{bd}}\right)$$

$$Z = Z_1 + Z_2 + Z_3 + Z_4 + Z_5$$

$$\text{Minimize: } Z = Z_1 + Z_2 + Z_3 + Z_4 + Z_5$$

Equation 6.9

$$L \equiv TLn\left(\frac{T}{t}\right) - T - \sum_{ij}\left(T_{ij} Ln\left(\frac{T_{ij}}{t_{ij}}\right) - T_{ij}\right) + \sum_{ij}\left(\lambda_{ij} \cdot 2\left(\sum_a \left(V_a \cdot p_{ij}^a\right) - \left(p_{ij}^a \sum_a \left(\sum_{xy} T_{xy}\, p_{xy}^a\right)\right)\right)\right) \qquad \forall\, i, j$$

$$\left.\begin{aligned}
&Ln(T)-Ln(t)-Ln(T_{ac})+Ln(t_{ac})-3\lambda_1-2\lambda_2-2\lambda_3-1\lambda_4=0\\
&Ln(T)-Ln(t)-Ln(T_{ad})+Ln(t_{ad})-2\lambda_1-3\lambda_2-1\lambda_3-2\lambda_4=0\\
&Ln(T)-Ln(t)-Ln(T_{bc})+Ln(t_{bc})-2\lambda_1-1\lambda_2-3\lambda_3-2\lambda_4=0\\
&Ln(T)-Ln(t)-Ln(T_{bd})+Ln(t_{bd})-1\lambda_1-2\lambda_2-2\lambda_3-3\lambda_4=0\\
&(V_1+V_3+V_4)-3T_{ac}-2T_{ad}-2T_{bc}-1T_{bd}=0\\
&(V_1+V_3+V_5)-2T_{ac}-3T_{ad}-1T_{bc}-2T_{bd}=0\\
&(V_2+V_3+V_4)-2T_{ac}-1T_{ad}-3T_{bc}-2T_{bd}=0\\
&(V_2+V_3+V_5)-1T_{ac}-2T_{ad}-2T_{bc}-3T_{bd}=0
\end{aligned}\right\}$$

2 LINK NETWORK

Equation 6.1

$$\text{Maximize: } Z(T_{ij},t_{ij})=\frac{T!}{\prod_{ij}(T_{ij}!)}\prod_{ij}\left(\frac{t_{ij}}{\sum_{ij}t_{ij}}\right)^{T_{ij}}$$

$$\text{Maximize: } Z=\frac{T!}{T_{ab}!T_{ac}!T_{bc}!}\left(\frac{t_{ab}}{t}\right)^{T_{ab}}\left(\frac{t_{ac}}{t}\right)^{T_{ac}}\left(\frac{t_{bc}}{t}\right)^{T_{bac}}$$

Equation 6.2

$$\text{Maximize: } Z(T_{ij},t_{ij})=TLn\left(\frac{T}{t}\right)-T-\sum_{ij}\left(T_{ij}Ln\left(\frac{T_{ij}}{t_{ij}}\right)-T_{ij}\right)$$

Maximize: $$Z = TLn\left(\frac{T}{t}\right) - T - \left(T_{ab}Ln\left(\frac{T_{ab}}{t_{ab}}\right) - T_{ab}\right) - \left(T_{ac}Ln\left(\frac{T_{ac}}{t_{ac}}\right) - T_{ac}\right) - \left(T_{bc}Ln\left(\frac{T_{bc}}{t_{bc}}\right) - T_{bc}\right)$$

Equation 6.3

Maximize: $$Z(T_{ij}, t_{ij}) = -\sum_{ij}\left(T_{ij}Ln\left(\frac{T_{ij}}{t_{ij}}\right) - T_{ij}\right)$$

Maximize: $$Z = -\left(T_{ab}Ln\left(\frac{T_{ab}}{t_{ab}}\right) - T_{ab}\right) - \left(T_{ac}Ln\left(\frac{T_{ac}}{t_{ac}}\right) - T_{ac}\right) - \left(T_{bc}Ln\left(\frac{T_{bc}}{t_{bc}}\right) - T_{bc}\right)$$

Equation 6.7

Maximize: $$Z(T_{ij}, t_{ij}) = \prod_{a}\frac{V_a!}{\prod_{ij}(T_{ij}\,p_{ij}^{a})!}\prod_{ij}\left(\frac{t_{ij}\,p_{ij}^{a}}{v_a}\right)^{T_{ij}\,p_{ij}^{a}}$$

$$Z_1 = \frac{V_1!}{T_{ab}!\,T_{ac}!} \times \left(\frac{t_{ab}}{t_{ab} + t_{ac}}\right)^{T_{ab}} \times \left(\frac{t_{ac}}{t_{ab} + t_{ac}}\right)^{T_{ac}}$$

$$Z_2 = \frac{V_2!}{T_{ac}!\,T_{bc}!} \times \left(\frac{t_{ac}}{t_{ac} + t_{bc}}\right)^{T_{ac}} \times \left(\frac{t_{bc}}{t_{ac} + t_{bc}}\right)^{T_{bc}}$$

Maximize: $$Z = Z_1 \times Z_2$$

Equation 6.8

$$\text{Minimize: } Z(T_{ij}, t_{ij}) = \sum_a \sum_{ij} T_{ij}\, p_{ij}^a Ln\left(\frac{T_{ij} v_a}{V_a t_{ij}}\right)$$

$$\begin{aligned}
Z_1 &= T_{ab} Ln\left(\frac{T_{ab} \times v_1}{V_1 \times t_{ab}}\right) + T_{ac} Ln\left(\frac{T_{ac} \times v_1}{V_1 \times t_{ac}}\right)\\
Z_2 &= T_{ac} Ln\left(\frac{T_{ac} \times v_2}{V_2 \times t_{ac}}\right) + T_{bd} Ln\left(\frac{T_{bd} \times v_2}{V_2 \times t_{bd}}\right)\\
Z &= Z_1 + Z_2
\end{aligned}$$

$$\text{Minimize: } \quad Z = Z_1 + Z_2$$

Equation 6.9

$$L \equiv TLn\left(\frac{T}{t}\right) - T - \sum_{ij}\left(T_{ij} Ln\left(\frac{T_{ij}}{t_{ij}}\right) - T_{ij}\right) + \sum_{ij}\left(\lambda_{ij} \cdot 2\left(\sum_a \left(V_a \cdot p_{ij}^a\right) - \left(p_{ij}^a \sum_a \left(\sum_{xy} T_{xy}\, p_{xy}^a\right)\right)\right)\right) \qquad \forall\, i, j$$

$$\left.\begin{aligned}
&Ln(T) - Ln(t) - Ln(T_{ab}) + Ln(t_{ab}) - 1\lambda_1 - 1\lambda_2 - 0\lambda_3 = 0\\
&Ln(T) - Ln(t) - Ln(T_{ac}) + Ln(t_{ac}) - 1\lambda_1 - 2\lambda_2 - 1\lambda_3 = 0\\
&Ln(T) - Ln(t) - Ln(T_{bc}) + Ln(t_{bc}) - 0\lambda_1 - 1\lambda_2 - 1\lambda_3 = 0\\
&(V_1) - 1T_{ab} - 1T_{ac} - 0T_{bc} = 0\\
&(V_1 + V_2) - 1T_{ab} - 2T_{ac} - 1T_{bc} = 0\\
&(V_2) - 0T_{ab} - 1T_{ac} - 1T_{bc} = 0
\end{aligned}\right\}$$