CHAPTER 2. BACKGROUND

This chapter covers the basic concepts of Technical efficiency, Data Envelopment Analysis (*DEA*) and Goal Programming.

2.1 TECHNICAL EFFICIENCY

Fundamentally, efficiency can be defined as the ratio of outputs to inputs. For many production scenarios, it is imperative to consider multiple inputs and outputs. Moreover, the computation of efficiency for the more realistic scenario of multiple inputs and outputs is difficult. This computation requires that weights be given to the different outputs and inputs. Given these weights, technical efficiency can be defined as

$$\text{Technical Efficiency} = \frac{\text{Weighted Sum of Outputs}}{\text{Weighted Sum of Inputs}} \tag{2.1}$$

Technical Efficiency refers to the ability to:

1. Produce the maximum amount of outputs for a specific quantity of inputs (output increasing notion), and/or
2. Use the minimum amount of inputs to produce a specific quantity of outputs (input reducing notion).

2.1.1 BASIC CONCEPTS OF TECHNICAL EFFICIENCY

2.1.1.1 Production Functions

A production function is an abstract mathematical relationship that describes the quantity of output as a function of the quantity of inputs. The production function assumes technical efficiency meaning that it represents the maximum output possible for every feasible combination of inputs. For a single output q and variable inputs x_1 and x_2, the production relates the quantity of q to the quantities x_1 and x_2 respectively, i.,e

$$q = f(x_1 , x_2) \tag{2.2}$$

2.1.1.2 Isoquant

An isoquant is the locus of all possible combinations of inputs from which a specific quantity of output can be produced. Each point on the isoquant represents a different technique to get that specific quantity of output. An isoquant with constant output q^0 quantity is expressed as:

$$q^0 = f(x_1, x_2) \tag{2.3}$$

The further away an isoquant lies from the origin, the greater is the output quantity that it represents. Consider the isoquant for output q^0 as shown in the figure. A, B, C, D, E, F and G are production units, which produce the same output q^0, using a different combination of inputs x_1 and x_2. Units B, C, E and F define and lie on the isoquant q^0. A and G are enveloped by the isoquant q^0, use more of x_1 and x_2 as compared to the other units, and are therefore inefficient. The efficiency of any unit can be obtained by comparing the performance of that unit to the specific units of the isoquant.

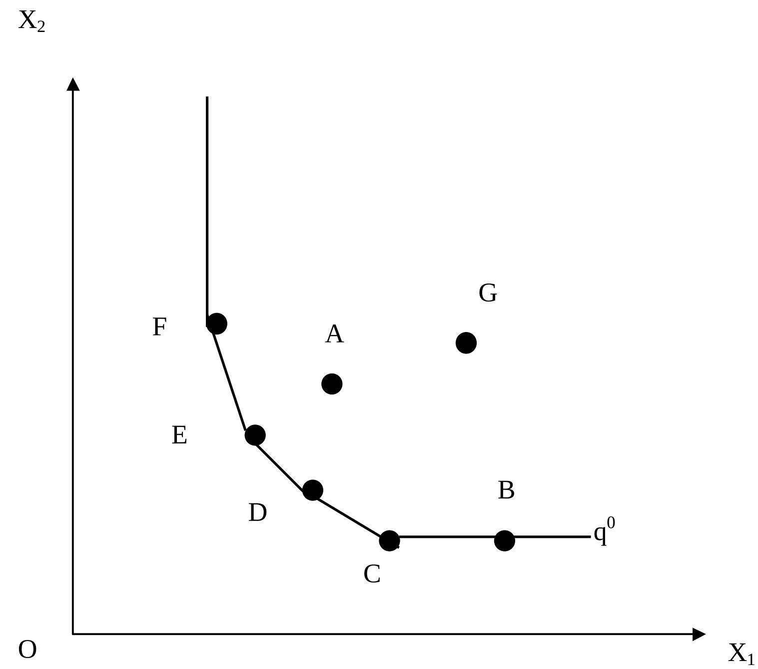

Figure 2.1 Isoquant

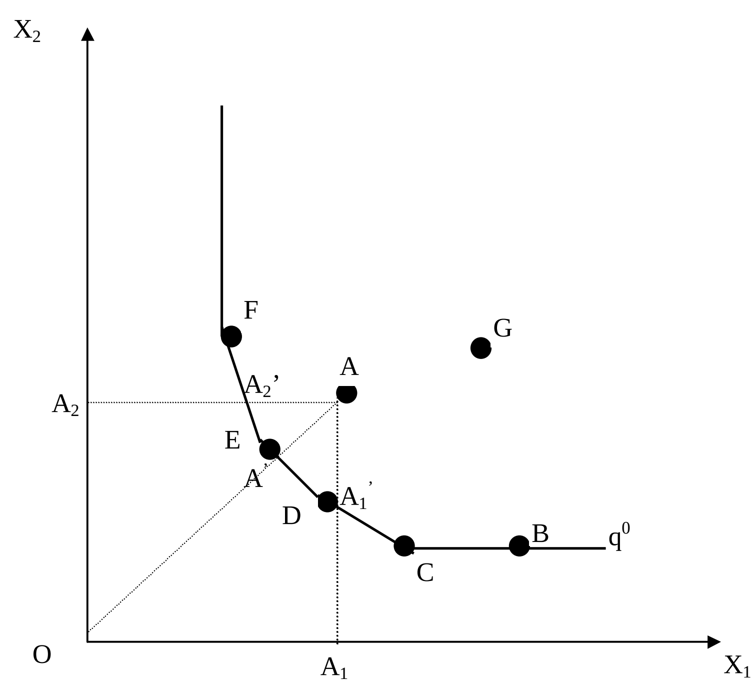

Figure 2.2 Radial and Non-Radial Measures of Efficiency

From Figure 2.2 the radial measure of technical efficiency for unit A is given by:

$$TE_R = \frac{OA^{'}}{OA} \tag{2.4}$$

$A^{'}$ is hypothetical unit which can be obtained as a weighted average of the actual units E and D. Sometimes it may not be practical to reduce both (all) inputs equiproportionately and for such cases the notion of non-radial measure of technical efficiency is useful. From figure 2.2, the non-radial measures of technical efficiency for unit A for the two inputs x_1 and x_2 are as follows:

$$TE_{X1} = \frac{A\,A_1^{'}}{A\,A} \tag{2.5a}$$

$$TE_{X2} = \frac{A\,A_2^{'}}{A\,A} \tag{2.5b}$$

2.2 DATA ENVELOPMENT ANALYSIS (*DEA*)[2]

In contrast to the parametric approaches whose object is to optimize a single regression plane through the data, *DEA* optimizes on each individual observation with an objective of calculating a discrete piecewise frontier determined by a set of pareto-efficient decision making units (*DMUs*). A feasible allocation is pareto-efficient, if there does not exist another feasible allocation of inputs and outputs that makes either of the variables achieve a better solution. Pareto-efficient *DMUs* are those which lie on the frontier. The decision-making units for this research are represented by time periods for which the analyses are conducted. The time period in this case is a month. The efficiency of the processes for each month are compared and an efficient frontier is determined by the months for which the processes are most efficient. Both the parametric and the non-

[2] Charnes, A., Cooper, W.W., Lewin, A., and Seiford., L *Data Envelopment Analysis: Theory, Methodology and Applications*, Kluwer Academic Publisher, 1994.

parametric (mathematical programming) approaches use all the information contained in the data. In the parametric approach, the single optimized regression equation is assumed to apply to all *DMUs*. *DEA*, in contrast, optimizes the performance measure of each *DMU*. This results in a revealed understanding about each *DMU* instead of the depiction of an "average" *DMU*. In other words, the focus of *DEA* is on the individual observations as represented by the *n* optimizations (one for each observation) required in *DEA* analysis. In contrast, regression analysis focuses on finding a plane that passes through an average for all inputs and outputs.

The parametric approach requires the imposition of a specific functional form (e.g., linear, quadratic etc.) when relating the independent variables to the dependent variable(s). The parametric approach also requires specific assumptions about the distributions of error terms (e.g., independently and identically normally distributed). In contrast, *DEA* does not require any assumption about the production function. *DEA* calculates a maximal performance measure for each *DMU* relative to all other *DMUs* in the observed population. Each *DMU* not on the frontier is compared against a convex combination of the *DMUs* on the frontier facet closest to it.

Charnes, Cooper and Rhodes (1978) extended Farrell's (1957) idea of linking the computation of technical efficiency and production frontiers. Their *CCR* model generalized the single-output/input ratio of efficiency for a single *DMU* to a fractional linear-programm7ing formulation transforming the multiple output/input characterization of each *DMU* to that of a single "virtual" output and "virtual" input. The relative technical efficiency of any *DMU* is calculated by forming the ratio of a weighted sum of outputs to a weighted sum of inputs, where the weights (multipliers) for both outputs and inputs are to be selected in a manner that calculates the Pareto efficiency measure of each *DMU* subject to the constraint that no *DMU* can have a relative efficiency score greater than unity.

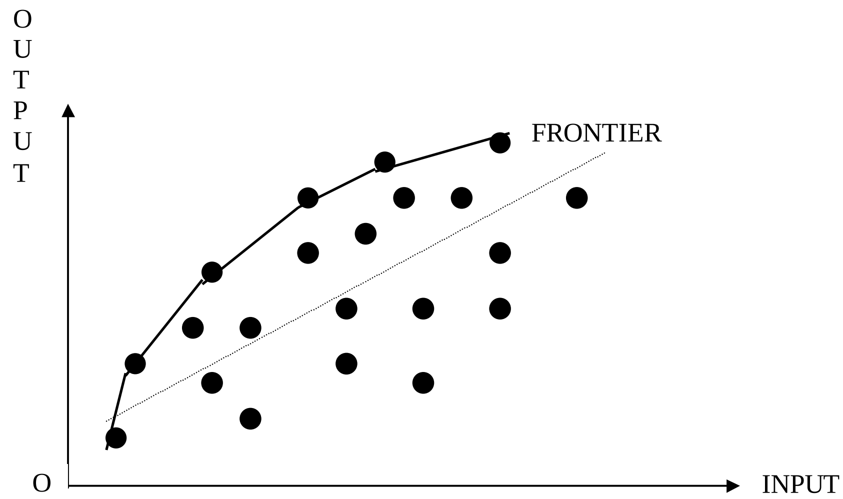

Figure 2.3 Comparison of DEA and Regression

The solid line in Figure 2.3 represents a frontier derived by *DEA* from data on a population of *DMUs* each utilizing different amounts of a single input to produce various amounts of a single output. It is important to note that *DEA* calculations, because they are generated from actual observed data for each *DMU*, produce only relative efficiency measures. The relative efficiency of each *DMU* is calculated in relation to all the other *DMUs*, using the actual observed values for the outputs and inputs of each *DMU*. The *DEA* calculations are designed to maximize the relative efficiency score of each *DMU*, subject to the condition that the set of weights obtained in this manner for each *DMU* must also be feasible for all the other *DMUs* included in the calculation. *DEA* produces a piecewise empirical extremal production surface, which in economic terms reveals the best practice production frontier - the maximum output empirically obtainable from any *DMU* in the observed population, given its level of inputs.

For each inefficient *DMU* (one that lies below the frontier), *DEA* identifies the sources and level of inefficiency for each of the inputs and outputs. The level of inefficiency is determined by comparing a single referent *DMU* to a convex combination of other referent *DMUs* located on the efficient frontier that utilize the same level of inputs and produce the same or higher level of outputs. This is achieved by obtaining solutions to mathematical programming formulations that satisfy inequality constraints.

These inequality constraints designate that a *DMU* can increase some outputs (or decrease some inputs) without worsening the other inputs or outputs. The calculation of potential improvement for each inefficient *DMU* does not necessarily correspond to the observed performance of any actual *DMU*. The calculated improvements (in each of the inputs and outputs) for inefficient *DMUs* are indicative of potential improvements obtainable because the projections are based on revealed best-practice performance of "comparable" *DMUs* that are located on the efficient frontier.

DEA is of interest to operations analysts, management scientists, and industrial engineers because of three features of the method.

1. Characterization of each *DMU* by a single summary relative efficiency score.
2. *DMU* - specific projections for improvements based on observable referent revealed best-practice *DMUs*; and
3. *DEA* does not require specifying abstract statistical models (This is also a weakness because only recently has the statistical properties of the approach been examined).

2.2.1 BASIC DEA MODELS

DEA is a body of concepts and methodologies that have now been incorporated in a collection of models with accompanying interpretive possibilities as follows:

1. the *CCR* ratio model (1978)
   (i) yields an objective evaluation of overall efficiency and
   (ii) identifies the sources and estimates the amounts of the identified inefficiencies;
2. the *BCC* model (1984) distinguishes between technical and scale efficiencies by
   (i) estimating pure technical efficiency at the given scale of operations and
   (ii) identifying whether increasing, decreasing, or constant returns to scale are present for further exploitation;
3. the multiplicative models (Charnes et al., 1982, 1983) provide
   (i) a log-linear envelopment or
   (ii) a piecewise Cobb-Douglas interpretation of the production process
4. the additive model and the extended additive model
   (i) relate *DEA* to the earlier Charnes-Cooper inefficiency analysis and in the process
   (ii) relate the efficiency results to the economic concept of Pareto optimality .

While each of these models address managerial and economic issues and provide useful results, their orientations are different. Thus models may focus on increasing, decreasing, or constant returns to scale. They may determine an efficient frontier that may be piecewise linear, piecewise loglinear, or piecewise Cobb-Douglas.

Essentially, the various models for *DEA* each seek to establish which subsets of *n* *DMUs* determine parts of an envelopment surface. The geometry of the envelopment surface is prescribed by the specified model employed. To be efficient, the point P_j corresponding to DMU_j must lie on this surface. Units that do not lie on this surface are termed inefficient, and the *DEA* analysis identifies the sources and the amounts of inefficiency and/or provides a summary measure of relative efficiency.

2.2.1.1 The BCC Model

The inefficient *DMU* can be made fully efficient by projection onto a point on the envelopment surface. The particular point of projection selected is dependent upon the type of the model selected. In the input-reducing model, the focus is on the maximal movement towards the frontier through proportional reduction of inputs. The output-increasing model focuses on the maximal movement towards the frontier through proportional augmentation of outputs.

2.2.1.1.1 Input Reducing Model (IRM)

The linear programs for the BCC (Banker, et al. 1984) input-reducing model are given below:

$$Min\ \theta - \varepsilon\left(\sum_{i=1}^{m} e_i + \sum_{r=1}^{s} s_r\right) \qquad (2.6)$$

subject to

$$\sum_{j=1}^{n} z_j x_{ij} + e_i = \theta x_{ijo} \qquad i = 1, 2, \ldots, m$$

$$\sum_{j=1}^{n} z_j x_{ij} - s_r = y_{rjo} \qquad r = 1, 2, \ldots, s$$

$$\sum_{j=1}^{n} z_j = 1$$

$\theta, z_j, e_i, s_r \geq 0, \ \forall j, i \text{ and } r$

where

y_{rj} is the amount of output r produced by DMU_j

x_{ij} is the amount of input i used by DMU_j

θ is the radial (input reducing) measure of technical efficiency

e_i is the excess or surplus of input i used by DMU_j

s_r is the slack in output r produced by DMU_j

n is the total number of DMUs

s is the total number of output variables

m is the total number of input variables

z_j is the vector of intensity factors that defines the hypothetical DMU to which DMU_{jo} is compared

2.2.1.1.2 Output Increasing Model (OIM)

The essential difference between the previous input reducing model and the output increasing model is that the linear programming maximizes on θ to achieve proportional output augmentation.

$$Max\ \theta - \varepsilon \left(\sum_{r=1}^{s} s_r - \sum_{i=1}^{m} e_i \right) \qquad (2.7)$$

Subject to

$$\sum_{j=1}^{n} z_j x_{ij} + e_i = x_{ijo} \qquad i = 1, 2, \ldots, m$$

$$\sum_{j=1}^{n} z_j y_{rj} - s_r = \theta\ y_{rjo} \qquad r = 1, 2, \ldots, s$$

$$\sum_{j=1}^{n} z_j = 1$$

$\theta, z_j, e_i, s_r \geq 0,\ \forall j,\ i \text{ and } r$

2.2.2 FURTHER INSIGHTS INTO TECHNICAL EFFICIENCY

There are two interpretations of technical efficiency. Debreu (1951) and Farrel (1957) proposed one of these. This is a radial interpretation which defines input-based technical efficiency as one minus the maximum equiproportionate reduction in all inputs that still allows for specific levels of output to be realized. Similarly, output-based technical efficiency is defined as one minus the maximum equiproportionate expansion of outputs that can be produced by using the same level of inputs.

The other interpretation by Koopmans (1951) proposes that a producer is technically efficient if and only if an increase in an output requires a decrease in at least one of the other outputs or if a decrease in any input requires an increase in at least one of the other outputs.

At this juncture, some notations developed by Färe and Lovell (1978) need to be introduced. If producers use inputs $x = (x_1, \ldots, x_m) \in R_+^m$ to produce $y = (y_1, \ldots, y_s) \in R_+^s$, then the production technology can be represented with an input set L(y),

$$L(y) = \{x: (y,x) \text{ is feasible}\} \tag{2.8}$$

which has an Isoquant IsoL(y) for every $y \in R_+^s$,

$$\mathrm{IsoL}(y) = \{x : x \in L(y),\ \lambda x \notin L(y),\ \lambda \in [0,1)\} \tag{2.9}$$

and an efficient subset EffL(y),

$$\mathrm{EffL}(y) = \{x : x \in L(y),\ x' \notin L(y),\ x' \leq x\} \tag{2.10}$$

The discrepancy between the two definitions comes to light for units which are on the isoquant but are consuming surplus (or excess) inputs as compared to other efficient units which are also on the isoquant. In such cases the non-radial measures are useful alternative indicators since they do not depict units with excess input usage that are on the isoquant as being efficient. The Debreu-Farrell and Koopmans definitions of technical efficiency give conflicting interpretations of the technical efficiency status for DMUs lying on the isoquant that use surplus inputs, i.e., whether such DMUs should be characterized as efficient as not.

2.2.2.1 RADIAL MEASURES

Radial measures of technical efficiency such as Debreu-Farrell allow equiproportional reduction in all the inputs used to produce a specific output. If θ is the radial measure of technical input reducing efficiency, then:

$$DF_i(x,y) = Min\{\ \theta\ :\ \theta \geq 0,\ \theta\ x \in\ L(y)\ \} \tag{2.11}$$

If a reduction in inputs is possible then the optimal θ value is less than one. If an equiproportional reduction in inputs is not possible then the optimal value of θ equals one. However this alone is not sufficient to measure technical efficiency. The excess in inputs and slack in outputs associated with a DMU should also be zero. Thus for a DMU to be technically efficient there are two necessary and sufficient conditions:

1. $\theta = 1$
2. Residual excess in inputs and slack in outputs is also zero.

Thus, the Debreu-Farrell radial efficiency measure (θ) alone is not sufficient to declare a DMU efficient although it is the first necessary condition. Consequently, a two stage model is required because resulting equiproportional reductions from the first stage alone do not guarantee a place in the efficient subset. The second stage model helps to calculate the excess in inputs or slack in outputs. Reporting the slack and excess along with the radial efficiency measure θ in the final analysis is necessary in defining the efficient *DMUs.*

2.2.2.1.1 Input Reducing Model (IRM)

The input reducing orientation of the *BCC* model is presented here as a two stage approach. In the second stage θ is not an unknown variable but is the radial measure from the solution of the first stage.

Stage 1

$$Min\ \theta \tag{2.12}$$

subject to

$$\sum_{j=1}^{n} z_j x_{ij} + e_i = \theta x_{ijo} \qquad i = 1, 2, \ldots, m$$

$$\sum_{j=1}^{n} z_j x_{ij} - s_r = y_{rjo} \qquad r = 1, 2, \ldots, s$$

$$\sum_{j=1}^{n} z_j = 1$$

$$\theta, z_j, e_i, s_r \geq 0,\ \forall j, i \text{ and } r$$

Stage 2

$$Min\ - \varepsilon (\sum_{i=1}^{m} e_i + \sum_{r=1}^{s} s_r) \tag{2.13}$$

subject to

$$\sum_{j=1}^{n} z_j x_{ij} + e_i = \theta^* x_{ijo} \qquad i = 1, 2, \ldots, m$$

$$\sum_{j=1}^{n} z_j x_{ij} - s_r = y_{rjo} \qquad r = 1, 2, \ldots, s$$

$$\sum_{j=1}^{n} z_j = 1$$

$$z_j, e_i, s_r \geq 0,\ \forall j, i \text{ and } r$$

2.2.2.1.2 Output Increasing Model (*OIM*)

The output increasing orientation of the *BCC* model is presented here as a two stage approach. In the second stage θ is not an unknown variable but is the radial measure from the solution of the first stage.

Stage 1

$$Max\ \theta \tag{2.14}$$

subject to

$$\sum_{j=1}^{n} z_j x_{ij} + e_i = x_{ijo} \qquad i = 1, 2, \ldots, m$$

$$\sum_{j=1}^{n} z_j x_{ij} - s_r = \theta y_{rjo} \qquad r = 1, 2, \ldots, s$$

$$\sum_{j=1}^{n} z_j = 1$$

$$\theta, z_j, e_i, s_r \geq 0,\ \forall j, i \text{ and } r$$

Stage 2

$$Max\ -\varepsilon\left(\sum_{i=1}^{m} e_i + \sum_{r=1}^{s} s_r\right) \tag{2.15}$$

subject to

$$\sum_{j=1}^{n} z_j x_{ij} + e_i = x_{ijo} \qquad i = 1, 2, \ldots, m$$

$$\sum_{j=1}^{n} z_j x_{ij} - s_r = \theta^* y_{rjo} \qquad r = 1, 2, \ldots, s$$

$$\sum_{j=1}^{n} z_j = 1$$

$$z_j, e_i, s_r \geq 0,\ \forall j, i \text{ and } r$$

2.2.2.2 NON-RADIAL MEASURES

The underlying notion of non-radial measures is that at least the DMUs which are efficient should belong to the efficient subset. Non-radial measures scale each input individually by different proportions so that they get projected on to the efficient frontier.

2.2.2.2.1 THE FÄRE-LOVELL NON-RADIAL MEASURE

Färe and Lovell (1978) proposed non-radial measures of relative technical efficiency. The input based non-radial measure is defined as follows:

$$FL_i(x,y) = \mathrm{Min} \sum_{i=1}^{m} \lambda_i / m : \{(\lambda_1 x_1, \ldots, \lambda_m x_m) \in L(y), \lambda_i \in (0,1] \ \forall \ i \} \qquad (2.16)$$

As seen from the above model, this measure minimizes the mean of the reduction for each input which is the scalar λ_i. The reduction for each input is carried out separately. The mathematical programming formulation for the Färe-Lovell (1978) non-radial efficiency measure for inputs, $FL_i(x, y)$, is given as follows:

$$Min\ 1/m \sum_{i=1}^{m} \lambda_i \qquad (2.17)$$

Subject to

$$\sum_{j=1}^{n} z_j x_{ij} \le \lambda_i x_{ijo} \qquad i = 1, 2, \ldots, m$$

$$\sum_{j=1}^{n} z_j y_{rj} \ge y_{rjo} \qquad r = 1, 2, \ldots, s$$

$$\sum_{j=1}^{n} z_j = 1$$

$$\lambda_i \le 1 \qquad i = 1, 2, \ldots, m$$

$$\lambda_i, z_j \ge 0, \ \forall\, i, j$$

The mathematical programming formulation for the Färe-Lovell non-radial efficiency measure of outputs, $FL_r(x, y)$, is given as follows:

$$Max\ 1/m \sum_{r=1}^{s} \lambda_r \tag{2.18}$$

Subject to

$$\sum_{j=1}^{n} z_j x_{ij} \leq x_{ijo} \qquad i = 1, 2, \ldots, m$$

$$\sum_{j=1}^{n} z_j y_{rj} \geq \lambda_r\, y_{rjo} \qquad r = 1, 2, \ldots, s$$

$$\sum_{j=1}^{n} z_j = 1$$

$$\lambda_r \leq 1 \qquad r = 1, 2, \ldots, s$$

$$\lambda_{i,} z_j \geq 0, \forall\, i, j$$

2.3 LINEAR GOAL PROGRAMMING[3]

The concept of efficiency plays an important role in Data Envelopment Analysis and Multiple Objective Linear Programming (*MOLP)*. Joro and Korhonen (1996) have shown that structurally the formulation of *DEA* that identifies the efficient units is similar to the *MOLP* model based on the reference point approach that generate efficient solutions. *DEA* and *MOLP* have been shown to complement each other. In *DEA*, the projection is performed by letting a mathematical program determine weights that associate the analyzed point with the best possible efficiency score. In *MOLP*, the direction of the projection is based on the use of weights which the decision maker can directly or indirectly influence through his/her preference structure. Mathematically, the

CCR model by Charnes etal. (1978) and the reference point approach proposed by Wierzbicki (1980) for solving *MOLP* problems use similar formulations.

There are a number of approaches to use for the multiple objective linear model. There are three basic approaches that form the basis for nearly all the multiple objective techniques. These are:

1. Weighting or utility methods
2. Ranking or prioritizing methods
3. Efficient solution (or generating) methods

Goal programming has been selected as the multiple objective linear model for the current research. This is for the following reasons:

- Goal Programming is a methodology for modeling, solving, and analyzing problems for which we wish to consider the impact of multiple, conflicting objectives.
- The model development is relatively simple and straightforward.
- The objective functions can be prioritized and weighted by a decision maker
- The method of solution is quite simple.
- The model and its assumptions are consistent with typical real - world problems.

2.3.1 TERMINOLOGY AND CONCEPTS

2.3.1.1 Objective: An objective is a relatively general statement that reflects the desires of the decision-maker.

2.3.1.2 Aspiration Level: An aspiration level is a specific value associated with a desired or acceptable level of achievement of an objective. Thus, an aspiration level is used to measure the achievement of an objective and generally serves to "anchor" the objective to reality (The objective may not be completely achievable).

2.3.1.3 Goal: An objective in conjunction with an aspiration level is termed a goal. For example, we may wish to "achieve at least Y units of output" or "reduce the cost of inputs by at least X percent".

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[3] James P Igniozio: *Linear Programming in Single and Multiple Objective Systems*, 1982, Prentice-Hall, Englewood Cliffs, New Jersey.

2.3.1.4 Goal Deviation: The difference between what is accomplished and what is aspired is termed as a goal deviation. A deviation can represent *over*- as well as *under*-achievement of a goal.

2.3.1.5 Goal Formulation: Consider the objective function expressed in general terms as $f_i(x)$ (a linear form of the objective is assumed).

$f_i(x)$ = mathematical representation of objective *i* as a function of the decision variables $x = (x_1, x_2, x_{3,} \ldots x_n)$

b_i = value of the aspiration level associated with the objective *i*.

Three possible forms of goals may then result:

1. $f_i(x) \leq b_i$; a value of $f_i(x)$ that is less than equal to b_i is desired. (2.19a)
2. $f_i(x) \geq b_i$; a value of $f_i(x)$ that is greater than or equal to b_i is desired. (2.19b)
3. $f_i(x) = b_i$; $f_i(x)$ must exactly equal b_i . (2.19c)

Regardless of the form, these relations can be transformed into the goal programming format by adding a negative deviation variable ($\eta_i \geq 0$) and subtracting a positive deviation variable ($\rho_i \geq 0$). Table 2.1 summarizes the statement.

Goal Type	Goal Programming Form	Deviation Variables to be Minimized
$f_i(x) \leq b_i$	$f_i(x) + \eta_i - \rho_i = b_i$	ρ_i
$f_i(x) \geq b_i$	$f_i(x) + \eta_i - \rho_i = b_i$	η_i
$f_i(x) = b_i$	$f_i(x) + \eta_i - \rho_i = b_i$	$\rho_i + \eta_i$

Table 2.1 Goal Formulations

From the above table it is clear that:

1. To satisfy $f_i(x) \leq b_i$, the positive deviation (ρ_i) should be minimized.
2. To satisfy $f_i(x) \geq b_i$, the negative deviation (η_i) should be minimized.

3. To satisfy $f_i(x) = b_i$, both the deviations should be minimized ($\rho_i + \eta_i$).

2.3.1.6 The Achievement Function: After getting a solution, x, to a multiple-objective model, the next step is to determine how good the solution is. Some of the measures used to evaluate the "goodness" of a solution are:

1. How well does it minimize the sum of weighted goal deviations?
2. How well does it minimize some polynomial form of the goal deviations?
3. How well does it minimize the maximum deviation?
4. How well does it lexicographically minimize an ordered (ranked or prioritized) set of goal deviations?
5. Various combinations of the above.

2.3.1.7 Lexicographic Minimum: Given an ordered array **a** of nonnegative elements a_k's, the solution given by $a^{(1)}$ is preferred to $a^{(2)}$ if $a_k^{(1)} < a_k^{(2)}$ and all higher-order elements $(a_1, a_2, \ldots a_{k-1})$ are equal. If no other solution is preferred to **a** , then **a** is the lexicographic minimum.

The achievement function, or vector is $\mathbf{a} = (a_1, a_2, \ldots, a_k, \ldots, a_K)$ (2.20)

Where **a** = achievement vector for which the lexicographic minimum is desired

k = ranking or priority, where

$a_k = g_k(\eta,\rho) \quad k = 1, 2, \ldots, K$ (2.21)

$a_k = g_k(\eta,\rho)$ = linear function of the goal or constraint deviation variables that are to be minimized at rank or priority k.

2.3.2 STEPS IN MODEL CONSTRUCTION

The initial phase in the model construction of the goal programming model is the development of the baseline model. Once the baseline model has been constructed, the next stage is the conversion of the baseline model into the specific linear goal programming model. The following assumptions are necessary in this conversion:

1. Aspiration levels may be associated with all objectives so as to transform them into goals.

2. Any rigid constraints (i.e., absolute goals) are ranked at priority 1. All remaining goals may be ranked according to importance.
3. All the goals within a given priority, except priority 1 must be commensurable (i.e., measured in the same units) or made commensurable by means of weights.

The steps in the formulation of a linear goal program are then:

Step 1: Develop the baseline model.

Step 2: Specify aspiration levels for each and every objective.

Step 3: Include negative and positive deviation variables for each and every goal constraint.

Step 4: Rank the goals in the order of preference. Priority 1 is always reserved for rigid constraints.

Step 5: Establish the achievement function.

The linear goal programming then has the general form:

Find $x = (x_1, x_2, \dots, x_n)$ *so as to* (2.22)

Lexicographically minimize $a = \{g_1(\eta,\rho), \dots, g_k(\eta,\rho)\}$

Subject to

$f_i(x) + \eta_i - \rho_i = b_i \qquad \text{for } i = 1, 2, \dots, m$

$x, \eta, \rho \geq 0$

Since a linear form of the model is assumed, the form of $f_i(x)$ is given as

$$f_i(x) = \sum_{j=1}^{n} c_{ij}\, x_j$$

where c_{ij} is the coefficient associated with variable j in goal or constraint i.

2.3.3 METHODS OF SOLUTIONS

The method of solution proposed for the lexicographic minimum form of the linear goal programming is the sequential linear goal programming. The underlying basis for this method, the *SGLP* algorithm, is a sequential solution to a series of conventional linear programming models. This is accomplished by partitioning the goal programming models according to priority levels.

Given the linear goal programming model, first consider just the portion of the achievement vector and the goals associated with priority 1. This results in the establishment of a single-objective linear programming model given as:

$$\textit{Minimize } a_1 = g_1(\eta,\rho) \tag{2.23}$$

Subject to

$$\sum_{j=1}^{n} c_{i,j}\, x_j + \eta_i - \rho_i = \mathrm{b_i} \qquad \forall\ i \in P_1$$

$$x, \eta, \rho \geq 0$$

That is, the first term in the achievement function is minimized subject only to those goals in priority level i ($i \in P_1$). Once this is done, we have the best solution to a_1, designated as a_1^*.

Then the second term in the achievement function a_2 is minimized. However this is done subject to:

1. All goals at priority 1.
2. All goals at priority 2.
3. Plus an extra goal (or rigid constraint) that assures that any solution to priority 2 cannot degrade the achievement level previously obtained in priority one. That is,

$$g_1(\eta,\rho) = a_1^*.$$

This procedure is continued until all priorities have been considered. The solution to the final linear programming model is then also the solution to the equivalent linear goal program.

In modeling efficiency performance using goal programming, both input and output orientations can simultaneously be considered. Since goal programming has the ability to handle multiple objective functions, there is no need to model the two orientations separately. In proposed model for the research, the objective function minimizes the deviations from the goals for both inputs and outputs at the individual process level and the plant level and also considers line-balance among consecutive production processes. This type of a model also gives the flexibility of achieving the most prioritized objective.