

Appendix A – Finite Element Analysis

Due to the complexity of the geometry of the test cavity, no analytical solution is available. The finite element method was used to numerically approximate the acoustic behavior of the test cavity. The analysis was performed using the commercial software package Sysnoise [53]. This appendix gives a brief overview of the method and equations used in the analysis as described in [53, 54].

The objective of the method is to find an approximate solution of a given boundary-value problem. The basis of the method is the representation of the acoustic region as a collection of finite sub-volumes (Finite Elements). The finite elements can have different shapes and a different number of nodes. These nodes are the locations where the solution is computed as it will be described in section A.2. The type of elements used during the analysis was a 'linear brick' with one node per vertices as illustrated in Figure A.1. This type of element was chosen because the union of these elements matches exactly the automobile cabin domain.

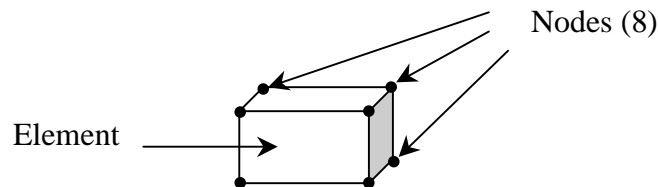


Figure A.1 **Finite element**

Section A.1 describes the equations necessary to the finite element analysis. First, the concept of velocity potential is introduced and is later applied to the Hamilton's principle. Typical boundary conditions are also discussed briefly. Section A.2 give details of the discretization process, as a result, a solution equation is presented in section A.3.

A.1 Governing equations

Velocity Potential

Recalling the linearized form of Euler's equation [55]:

$$\rho \frac{\partial v}{\partial t} + \nabla p = 0, \quad (\text{A.1})$$

after computing the curl of equation (A.1), equation (A.2) is obtained [55]:

$$\frac{\partial}{\partial t} (\nabla \times v) = 0, \quad (\text{A.2})$$

where v is the particle velocity. A first approach is to consider the particle velocity as the summation of an irrotational component v_a ($\nabla \times v_a = 0$) and a rotational component v_i ($\nabla \times v_i \neq 0$).

After substitution in equation (A.2), it can be shown that $\frac{\partial v_i}{\partial t} = 0$. The component v_i does not represent an acoustic velocity field, since it does not fluctuate. Hence it can be concluded that the particle velocity is irrotational, and therefore can be expressed as the gradient of a scalar variable, also known as the velocity potential Ψ , as in equation (A.3):

$$v = \nabla \Psi. \quad (\text{A.3})$$

The pressure can be expressed in terms of the velocity potential as in equation (A.4)

$$p = -\rho \frac{\partial \Psi}{\partial t} = k \text{div}(u), \quad (\text{A.4})$$

with u the particle displacement. The classical hyperbolic wave equation can be expressed in terms of the velocity potential as in equation (A.5):

$$\nabla^2 \Psi - \frac{1}{c^2} \frac{\partial^2 \Psi}{\partial t^2} = 0, \quad (\text{A.5})$$

where c is the speed of sound. Assuming an harmonic behavior of frequency ω : $\Psi = \Phi(x, y, z).e^{i\omega t}$, the wave equation (A.5) becomes the elliptic Helmholtz equation, as expressed in equation (A.6) in terms of the velocity potential, or as in equation (A.7) in terms of the pressure:

$$\nabla^2 \Phi + k^2 \Phi = 0, \quad (\text{A.6})$$

$$\nabla^2 p + k^2 p = 0, \quad (\text{A.7})$$

with $p = -\rho i \omega \Phi$.

Hamilton's principle

In the case of the vibration of a fluid, the Hamilton's principle is used for the finite element analysis. It can be seen as a variational approach, where the condition is expressed as in (A.8):

$$\delta \int_{t_1}^{t_2} L dt = 0. \quad (\text{A.8})$$

In which δ denotes the first variation of the integral, and $L = T - U_i - V_e$, with T the kinetic energy, U_i the internal potential energy, and V_e the external potential energy. Assuming a fixed pressure on the boundary S_1 and fixed particle velocity on the boundary S_2 , the different terms can be expressed as follows in terms of the velocity potential ψ :

$$T = \frac{\rho}{2} \int_V \nabla \Psi^T \nabla \Psi dV, \quad (\text{A.9})$$

$$U_i = \frac{\rho}{2c^2} \int_V (\dot{\Psi})^2 dV, \quad (\text{A.10})$$

$$V_e = \int_{S_2} \rho \dot{\Psi} u_n dS, \quad (\text{A.11})$$

where u_n is the boundary displacement in the normal direction.

Boundary conditions

Several types of boundary conditions can be assigned to the numerical problem. Typical conditions are presented below:

- Velocity constraints $v_n = \bar{v}_n$,
- Pressure constraints $p = \bar{p}$,
- Absorption Condition: $\frac{p}{v_n} = Z_n$,
- A permeable membrane inside the domain is described by transfer impedance conditions, which relates the normal velocity to the pressure jump: $\frac{\Delta p}{v_n} = Z_t$.

A.2 Discretization

Continuous functions as the pressure or velocity potential can be represented over the finite element as the linear combination of polynomials called interpolation functions as in equation (A.12):

$$\Psi = [N_e](\Psi_e), \quad (\text{A.12})$$

where $[N_e]$ is a set of suitable interpolation functions and (Ψ_e) is the set of velocity potentials evaluated at the nodes of the elements. From the expression described in equation (A.12), the gradient can be computed as in equation (A.13):

$$\nabla \Psi = [B_e](\Psi_e), \quad (\text{A.13})$$

where $[B_e]$ contains the spatial derivatives of the interpolation functions.

The cost function L of equation (A.8) is formed as the summation of L_e evaluated on each element:

$$L = \sum_e L_e. \quad (\text{A.14})$$

After substitution of equation (A.12) in equations (A.10) and (A.11) and substitution of equation (A.13) in equation (A.9), an expression for L_e is found:

$$\frac{L_e}{\rho} = \frac{1}{2}(\Psi_e)^T [K_e](\Psi_e) - \frac{1}{2}(\dot{\Psi}_e)^T [M_e](\dot{\Psi}_e) + (\dot{\Psi}_e)[C_e](u_{ens}), \quad (\text{A.15})$$

where expressions for the stiffness, mass and loss matrices are respectively as follows:

$$[K_e] = \int_{V_e} [B_e]^T [B_e] dV_e, \quad (\text{A.16})$$

$$[M_e] = \int_{V_e} \frac{1}{c^2} [N_e]^T [N_e] dV_e, \quad (\text{A.17})$$

$$[C_e] = \int_{S_{2e}} [N_{es}]^T [N_{es}] dS_e, \quad (\text{A.18})$$

where V_e is the volume of the element and S_{2e} is the surface of the element on the boundary S_2 .

A.3 Solution

After substitution of equation (A.15) into equation (A.14), and then into Hamilton's equation (A.8), integration by part leads to the relation:

$$\partial J = \int_{t_0}^{t_1} \sum_e (\partial \Psi_e)^T \left[[K_e](\Psi_e) + [M_e](\ddot{\Psi}_e) - [C_e](\dot{u}_{ens}) \right] dt = 0. \quad (\text{A.19})$$

After assembling the local matrices K_e , M_e and C_e into global matrices K , M and C , equation (A.19) can be solved, and the solution leads to equation (A.20):

$$[K](\Psi) + [M](\ddot{\Psi}) = [C](\dot{u}_n). \quad (\text{A.20})$$

Equation (A.20) is a typical eigen value problem. Classic techniques can be used to solve for the normal modes of the cavity. The response can be determined by modal superposition.