Hardy-space Function Theory on Finitely Connected Planar Domains

Moises Daniel Guerra Huaman

Thesis submitted to the Faculty of the
Virginia Polytechnic Institute and State University
in partial fulfillment of the requirements for the degree of

Master of Science
in
Mathematics

Joseph A. Ball, Chair
George A. Hagedorn
Michael Renardy

April 17, 2008
Blacksburg, Virginia

Keywords: Hardy space, planar domain, unit disk.

Hardy-space Function Theory on Finitely Connected Planar Domains

Moises Daniel Guerra Huaman

(ABSTRACT)

Hardy space scalar theory on the disk is now classical. Some extensions have been done, one of them is the approach done by Donald Sarason using Laurent series. We present the more complicated function theory, without the use of either power series or Laurent series, for finitely-connected planar domains.

Dedication

To my family, without them I would not either be who I am or to be where I am.

Acknowledgements

I am very thankful to Dr. Joseph A. Ball for his important clarifying ideas we provided while advising during the development of this thesis. Thanks to Dr. George Hagedorn and Dr. Michael Renardy for reading this paper and for their important corrections to improve my thesis. Special thanks to Edgar Saenz Maldonado for his friendship and his important remarks done in our informal discussions of several topics of mathematics. Thanks to David Murrugarra, Bart Ordonez, Maminiaina Rasamoelina and Collin Fox for their support. Special thanks to my brothers David and Miguel and my sister Susana because they allowed me to have all the free time I had to focus totally in my studies.

Contents

Chapter 1

Introduction

The classical Hardy space over the unit disk, denoted as $H^p(\Delta)$, consists of those analytic functions f on the unit disk Δ satisfying the growth condition

$$\sup_{r<1}\left\{\int_{-\pi}^{\pi}|f(re^{i\theta})|^p \mathtt{d}\theta\right\}<\infty.$$

Hardy space scalar function theory on the disk is now classical, this goes back to Hardy, Riesz (who introduced it with the name of G. H. Hardy in 1923) and Fejér. All this theory was capped off by Beurling's theorem. There resulted a nice setting for the study of the interplay of function theory and operator theory. The shift operator $S : l_2 \to l_2$, is defined as follows

$$S(a_0, a_1, a_2, \dots) = (0, a_0, a_1, a_2, \dots),$$

has a function-theoretic representation $f(z) \mapsto zf(z)$. Beurling's theorem describes the invariant subspaces for S (giving a explicit of them description, making use of no eigenvectors) with a concrete example radically different from the finite-dimensional case (which describes the invariant subspaces determined by eigenvectors and generalized eigenvectors).

The study of Hardy spaces has been extended in many directions, one of them is the work done by Donald Sarason in 1965, he worked with $H^p(A)$ functions where $A = \{r_0 < |z| < 1\}$. He also introduces the concept of a modulus automorphic function, which is simply a function F that is analytic on the slit disk $\{re^{i\theta} \colon r_0 < r < 1, 0 < \theta < 2\pi\}$ and $F(z + 2\pi i) = F(z)$, such that

$$\lim_{\theta \uparrow 2\pi} F(re^{i\theta}) = \alpha \lim_{\theta \downarrow 0} F(re^{i\theta})$$

where $|\alpha| = 1$ (so $|F(e^{i\theta})|$ is single valued subharmonic on the annulus $r < |z| < 1$). With this he finds analogues of the canonical factorization of an H^p function into a Blaschke product, singular inner function, and outer function.

For the case of a general domain one cannot make use of either power series or Laurent series. The involved function theory is more complicated due to the presence of the space N, which is described as follows. Take Ω a finitely-connected planar domain $\Gamma = \partial\Omega = \Gamma_0 \cup \Gamma_1 \cup \cdots \cup \Gamma_m$ and $R(\Omega)$ the set of rational functions whose poles are off $\Omega \cup \Gamma$, then there are m linear independent measures $\nu_1, \ldots, \nu_m$ on Γ orthogonal to $ReR(\Omega)$ and of the form

$$\mathtt{d}\nu_j = Q_j \mathtt{d}\omega_q, \ 1 \leq j \leq m$$

where Q_j is C^∞ on Γ, Q_j is nonnegative on Γ_j and nonpostive on Γ_k, $k \neq j$. Take N to be the complex span of such Q_j. When $N = 0$ we are in the disk case.

The main goal of this thesis is to develop all the preliminary results needed leading to a self-contained explanation of the Main Result: *there is an isometric isomorphism between the Hardy space* $H^p(\Omega)$ *and a closed subspace of* $L^p(\Gamma)$ *(*$\Gamma =$ *the boundary of* Ω*).* This in turn is one of the main prerequisites required for the understanding of the analog of Beurling's theorem for the multiply connected domain case. After a preliminary chapter reviewing needed basic material concerning measure and integration and the theory of Banach spaces, Chapter 3 introduces the notion of subharmonic functions that will help to give a proof of the Dirichlet problem. The fourth chapter introduces the notion of harmonic measures and presents the main results of Hardy spaces for the unit disk, namely Theorem 4.2.3. Finally the last chapter presents the generalization of the results given in Chapter 4 for the case of a finitely-connected planar domain providing us our Main Result. Solving the Dirichlet problem is one of the tools for understanding this. This Main Result establishes a fertile interplay between measure theory and complex analysis as in Rudin's "Real and Complex Analysis".

Chapter 2

Preliminaries

In this chapter we introduce the basic facts that will be taken for granted through the development of this thesis.

2.1 Measure and Integration

If X is a set, then the collection of all subsets of X forms a ring, using the operations

$A + B = (A \cup B) - (A \cap B)$.

$AB = A \cap B$.

A σ-ring of subsets of X is a subring of the ring of all subsets of X which is closed under the formation of countable unions.

Suppose that X is a locally compact Hausdorff topological space. Take the smallest σ-ring of subsets of X which contains every compact G_δ, where a G_δ set is a set which is the intersection of a countable number of open sets. The members of this σ-ring are called **Baire subsets**. Also the **Borel subsets** of X are the members of the smallest σ-ring of subsets of X which contains all compact subsets. It is important to note that in Euclidian space, every compact (closed and bounded) set is a G_δ; hence, if X is a closed subset of Euclidian space, the Baire and Borel subsets of X coincide. When X is the real line or a closed interval in the real line, the ring of Baire (Borel) subsets of X may also be described as the σ-ring generated by the half-open intervals [a,b).

Consider a locally compact Hausdorff space X. A **positive Baire (Borel) measure** on X is a function μ whose domain consist of Baire (Borel) subsets of X and whose range is [0,∞] and has the following property: if A_i is a disjoint countable collection of Baire (Borel) sets in X, then

$$\mu\left(\bigcup_{i=0}^{\infty} A_i\right) = \sum_{i=0}^{\infty} \mu(A_i).$$

A positive Baire measure is called **finite** if $\mu(X) < \infty$ is finite.

Now suppose X is the real line or a closed interval. Consider F a monotone increasing function on X which is continuous from the left:

$$F(x) = \sup_{t<x} F(t).$$

Define a function μ on semi-closed intervals $[a,b)$ by

$$\mu([a,b)) = F(b) - F(a).$$

Then μ has a unique extension to a positive Baire measure on X. The measure is finite if and only if F is bounded. If X is the real line, then every positive Baire measure on X arises in this way from a left-continuous increasing function F. If X is a closed interval, a monotone function on X is necessarily bounded. Thus, every finite positive Baire measure on X comes from such a increasing function. If X is the real line or an interval, the measure induced by $F(x) = x$ is called **Lebesgue measure**.

Given a locally compact Hausdorff space X, a **Baire (Borel) function** on X is a complex-valued function f on X such that $f^{-1}(S)$ is a Baire set for each Baire (Borel) set S in the plane. A **simple Baire function** for μ is a complex-valued function f on X of the form

$$f(x) = \sum_{i=1}^{n} \alpha_i \chi_{E_i}(x)$$

where

1. $\alpha_1, \ldots, \alpha_n$ are complex numbers;
2. $E_1, \ldots, E_n$ are disjoint Baire sets of finite μ-measure;
3. χ_E denotes the characteristic function of the set E.

The simple Baire functions form a vector space over the field of complex numbers. For a simple Baire function f we define

$$\int f d\mu = \sum_{i=1}^{n} \alpha_i \mu(E_i).$$

If f is a simple function, so is $|f|$ and

$$\left|\int f \mathtt{d}\mu\right| \leq \int |f| \mathtt{d}\mu$$

A Baire function f is called integrable with respect to μ if there exists a sequence of functions f_n such that

1. each function f_n is a simple Baire function for μ;
2. $$\lim_{m,n\to\infty} \int |f_m - f_n| \mathtt{d}\mu = 0;$$
3. f_n converges to f in measure; i.e., given $\epsilon > 0$,
$$\lim_{n\to\infty} \mu(\{x\colon |f(x) - f_n(x)| \geq \epsilon\}) = 0.$$

If f is integrable, then for any such sequence f_n the sequence $\int f_n \mathtt{d}\mu$ converges and the limit of this sequence (which is independent of f_n) is denoted by $\int f \mathtt{d}\mu$. Denote the class of μ integrable functions by $L^1(\mathtt{d}\mu)$. Then $L^1(\mathtt{d}\mu)$ is a vector space and $f \mapsto \int f \mathtt{d}\mu$ is a linear functional on L^1.

The Baire function f is in $L^1(\mathtt{d}\mu)$ if and only if its real part and imaginary part are in $L^1(\mathtt{d}\mu)$, or equivalently if and only if $|f|$ is in $L^1(\mathtt{d}\mu)$. When f is in $L^1(\mathtt{d}\mu)$,

$$\left|\int f \mathtt{d}\mu\right| \leq \int |f| \mathtt{d}\mu.$$

A subset S of X has **μ-measure zero** if for each $\epsilon > 0$ there is a Baire set containing S with $\mu(A) < \epsilon$. Any phenomenon which occurs except on a set of μ-measure zero is said to happen **almost everywhere** (relative to μ).

If f_n is a sequence of integrable functions such that the $\lim_{n\to\infty} f_n(x) = f(x)$, and if there is a fixed integrable function g such that $|f_n| \leq |g|$ for each n, then f is integrable and

$$\int f \mathtt{d}\mu = \lim_{n\to\infty} \int f_n \mathtt{d}\mu$$

This is called the **Lebesgue Dominated Convergence Theorem**.

Another important result is the following. Suppose μ is finite and f is a non-negative Baire function on the product space $X \times X$. If $f(x, y)$ is integrable in x for each fixed y and in y for each fixed x, then

$$\int \left[\int f(x,y) \mathtt{d}\mu(x) \right] \mathtt{d}\mu(y) = \int \left[\int f(x,y) \mathtt{d}\mu(y) \right] \mathtt{d}\mu(x).$$

This is a weak version of **Fubini's Theorem**.

If p is a positive number, the space $L^p(\mathtt{d}\mu)$ consists of all Baire functions f such that $|f|^p$ is in $L^1(\mathtt{d}\mu)$. If

$$f \in L^p(\mathtt{d}\mu), g \in L^q(\mathtt{d}\mu), \text{and } 1/p + 1/q = 1$$

then $(fg) \in L^1(\mathtt{d}\mu)$ and (**Hölder's inequality**)

$$\left|\int fg\mathtt{d}\mu\right| \leq \left(\int |f|^p\mathtt{d}\mu\right)^{1/p}\left(\int |g|^q\mathtt{d}\mu\right)^{1/q}.$$

Let us note something about $L^p(\mathtt{d}\mu)$ when X is compact and μ is a finite measure. In this case, every continuous function on X is integrable and the space of continuous functions is dense in $L^1(\mathtt{d}\mu)$; i.e., for all $f \in L^1(\mathtt{d}\mu)$ and $\epsilon > 0$, there is a continuous function g such that

$$\int |f-g|\mathtt{d}\mu < \epsilon.$$

Also if $p \geq 1$, then $L^p(\mathtt{d}\mu)$ is contained in $L^1(\mathtt{d}\mu)$, and the continuous functions are a dense subspace of $L^p(\mathtt{d}\mu)$:

$$\int |f-g|^p\mathtt{d}\mu < \epsilon$$

where $f \in L^p(\mathtt{d}\mu)$ and g is a continuous function.

Let μ_1 and μ_2 be positive Baire measures on X. We say that μ_1 is absolutely continuous with respect μ_2 if every set of measure zero for μ_2 is a set of measure zero for μ_1. The **Radon-Nikodym Theorem** states the following about finite measures: if μ_1 and μ_2 are finite, then μ_1 is absolutely continuous with respect to μ_2 if and only if

$$\mathtt{d}\mu_1 = f\mathtt{d}\mu_2$$

where f is some non-negative function in $L^1(\mathtt{d}\mu_2)$. We say μ_1 and μ_2 are **mutually singular** if there are disjoint Baire sets B_1 and B_2 such that

$$\mu_j(A) = \mu_j(A \cap B_j),\ j = 1, 2,$$

for every Baire set A. The generalized **Lebesgue Decomposition Theorem** states the following: if μ_1 and μ_2 are any two finite positive Baire measures, then μ_1 is uniquely expressible in the form

$$\mu_1 = \mu_a + \mu_s$$

where μ_a is absolutely continuous with respect to μ_2, and μ_s and μ_2 are mutually singular. That is,

$$\mathtt{d}\mu_1 = f\mathtt{d}\mu_2 + \mathtt{d}\mu_s$$

where $f \in L^1(\mathtt{d}\mu_2)$, and μ_s and μ_2 are mutually singular. One usually calls f the derivative of μ_1 with respect to μ_2.

Let X be a closed interval, and μ_2 Lebesgue measure. Suppose μ is the positive measure determined by the increasing function F. Then F is differentiable except on a set of Lebesgue measure zero, and if $f = \mathtt{d}F/\mathtt{d}x$, then f is Lebesgue integrable and

$$\mathtt{d}\mu = f\mathtt{d}x + \mathtt{d}\mu_s$$

where μ_s is mutually singular with Lebesgue measure. This means that μ_s is determined by an increasing function F_s such that $\mathtt{d}F_s/dx = 0$ almost everywhere with respect to Lebesgue measure.

A finite real Baire measure on X is a countably additive and real-valued function μ on the class of Baire sets. One way to construct such a measure is by forming the difference of two finite positive Baire measures

$$\mu = \mu_1 - \mu_2.$$

The **Jordan decomposition theorem** states that this is the only kind there is. In fact, given such a real measure μ there are disjoint Baire sets B_1 and B_2 and finite positive measures μ_1 and μ_2 carried on B_1 and B_2, respectively, such that $\mu = \mu_1 - \mu_2$. This splitting (with B_1 and B_2) is unique up to sets of measure zero. The positive measure $\mu = \mu_1 + \mu_2$ is called **total variation** of μ, denoted by $|\mu|$. The notions of absolutely continuous and singular can be extended for real measures as follows. We say that the real measure μ_1 is absolutely continuous with respect to μ_2 if $|\mu_1|$ is absolutely continuous with respect to $|\mu_2|$; similarly we say that μ_1 and μ_2 are singular if $|\mu_1|$ and $|\mu_2|$ are singular. In the case where X is a closed interval on the real line, the finite real Baire measures on X are the ones induced by real-valued functions of bounded variation which are continuous from the left. The Jordan decomposition for such a measure corresponds to the canonical expression for a function of bounded variation as the difference of increasing functions.

Finite complex Baire measures are defined similarly. We can write such a measure μ as a function of the form $\mu_1 + \mu_2$, where μ_1 and μ_2 are finite real Baire measures.

2.2 Banach Spaces

Let X be a real or complex vector space. A **norm** on X is a non-negative real valued function $\| \cdot \|$ on X such that:

1. $\|x\| \geq 0$ if and only if $x = 0$;

2. $\|x+y\| \leq \|x\| + \|y\|$;

3. $\|\lambda x\| = |\lambda| \|x\|$.

A real (complex) **normed linear space** is a real (complex) vector space X together with a specified norm on X. On such a space one has a metric ρ defined by:

$\rho(x, y) = \|x - y\|$.

If X is complete in this metric we call X a **Banach Space**. Completeness, then, means that if $\{x_n\}$ is a sequence of elements of X such that:

$$\lim_{m,n \to \infty} \|x_m - x_n\| = 0$$

there exists an element x in X such that:

$$\lim_{n \to \infty} \|x - x_n\| = 0.$$

Now, consider a locally compact Hausdorff space S and let us fix a positive Baire measure μ on S. Take a number $p \geq 1$ and let $X = L^p(\mathtt{d}\mu)$.

Define the L^p-norm of an f in L^p to be

$$\|f\|_p = (\int |f|^p \mathtt{d}\mu)^{1/p}.$$

This is not a norm, since we may have $\|f\|_p = 0$ without $f = 0$. We will agree to identify two functions in $L^p(\mathtt{d}\mu)$ which agree almost everywhere with respect to μ. So strictly speaking elements of $L^p(\mathtt{d}\mu)$ will be equivalence classes but we will continue with the same notation. Therefore with this convention $L^p(\mathtt{d}\mu)$ is a Banach space using the L^p-norm ($p \geq 1$).

We write $L^{\infty}(\mathtt{d}\mu)$ for the space of bounded Baire functions with μ-essential sup norm:

$$\|f\|_{\infty} = \mu_{\text{ess}} \sup_{x} |f(x)|$$

which means the infimum of

$$\sup_{x} \|g(x)\|$$

as g ranges over all bounded Baire function which agree with f almost everywhere with respect to μ.

Let X be a Banach space. Then X^* stands for the set of all linear functionals F on X which are continuous:

The set X^* forms a vector space with the usual sum of function and product of a scalar and a function. It is known that the linear functional F is continuous if and only if it is bounded, i.e., if and only if there is a constant $K \geq 0$ such that

$$|F(x)| \leq K\|x\|$$

for every x in X. The smallest such K is called the norm of F, i.e.,

$$\|F\| = \sup_{\|x\| \leq 1} |F(x)|.$$

The set X^* together with this norm becomes a Banach space. and is called the **dual space** of X.

If we take S to be a locally compact space, μ a positive Baire measure on S and $1 \leq p < \infty$, then

$$(L^p(\mathtt{d}\mu))^* = L^q(\mathtt{d}\mu)$$

where $1/p + 1/q = 1$, if $p > 1$, and $q = \infty$ if $p = 1$.

It is also true that for any continuous linear functional F on L^p there exists a $g \in L^q$ such that

$$F(f) = \int fg\,\mathtt{d}\mu, \quad \text{for } f \in L^p$$

and in that case

$$\|F\| = \|g\|_q.$$

Let us consider the special case when S is a compact Hausdorff space and $X = C(S)$, the space of all continuous real (or complex) valued functions on S. By defining the norm as

$$\|f\|_\infty = \sup_{x \in S} |f(x)|,$$

$C(S)$ is a Banach space and for $F \in (C(S))^*$ we have:

$$\lim_{n\to\infty} \|x - x_n\|_\infty = 0 \text{ implies } |F(x_n) - F(x)| \to 0.$$

The dual space of $C(S)$ can be identified (isomorphically and isometrically) with the space of real (complex) Baire measures on S. This is the statement of the **Riesz representation theorem** which can be formulated as follows. If S is a compact Hausdorff space, then every bounded linear functional ϕ on $C(S)$ is represented by a unique complex Borel measure μ, in the sense that

$$\phi(f) = \int f\mathtt{d}\mu \text{ for } f \in C(S).$$

The norm of $\|\phi\|$ equals to the **total variation** of μ on S. If μ is complex, the total variation of μ on S is best thought of as the norm of the corresponding functional on $C(S)$, since the relation of this number to the total variations of the real and imaginary parts of μ is rather involved. Of course in case μ is a positive measure, the norm of ϕ is $\mu(S)$. It is also true, in the result above, that for such a measure μ there is a complex Borel function h such that $|h| = 1$ and

$$\mathtt{d}\mu = h\mathtt{d}|\mu|.$$

Now, suppose X is a Banach space. The following result is very important. If F is a bounded linear functional on a subspace Y of X, then F can be extended to a linear functional on X which has the same norm as F. This result is called the **Hahn-Banach extension theorem.**

Over the conjugate space X^* we can consider the **weak-star topology** which is defined as follows. For $F_0 \in X^*$, let

$$x_1, x_2, \ldots, x_n \in X \text{ and } \epsilon > 0.$$

Define

$$U = \{F \in X^* \colon |F(x_k) - F_0(x_k)| < \epsilon, k = 1, \ldots, n\}.$$

Such a set U is a basic weak-star neighborhood of F_0 and the union of such neighborhoods U is an weak-star open set. Then we have a topology on X^* such that for each $x \in X$ the function $F \mapsto F(x)$ is continuous on X^*. In this topology a sequence $\{F_n\}$ converges to F

in the weak-star topology if and only if

$$\lim_{n\to\infty} F_n(x) = F(x)$$

for each x in X.

The following result is also very important. If B is the closed unit ball in X^*

$$B = \{F \in X^*; \|F\| \leq 1\},$$

then B is compact in the weak-star topology. This result is called **Banach-Alaoglu theorem**. We will use this result as follows. If $\{F_n\}$ is a sequence of linear functionals on X with $\|F_n\| \leq 1$, then this sequence has a weak-star cluster point, i.e., there exists an $F \in X^*$ with $\|F\| \leq 1$ such that $F(x)$ is a cluster point of the sequence $\{F_n(x)\}$ for every $x \in X$. As an example, if we have $\{\mu_n\}$ is a sequence of positive Baire measures on the compact space V and if $\mu_n(V) \leq 1$ for each n, then there exists a finite measure μ such that $\int f \mathtt{d}\mu$ is a cluster point of $\{\int f \mathtt{d}\mu_n\}$ for every $f \in C(V)$.

Chapter 3

The Dirichlet problem on a domain Ω

In this chapter our main purpose will be to solve the Dirichlet problem for a domain whose boundary components are nontrivial. For such purpose I will follow the approach described in [1], i.e., we will use a limiting procedure involving subharmonic functions to solve our problem.

3.1 Some results about the Poisson Formula for the disk case $\Omega = \Delta$

Given a domain Ω on the Riemann sphere, and given u a continuous real-valued function on $\Gamma = \partial\Omega$, the Dirichlet problem consists in finding a function f which is continuous on $CL(\Omega) = \Omega \cup \Gamma$ such that f satisfies the following conditions:

1. The function f is harmonic on Ω.

2. The function f equals u on Γ.

I will follow the approach described in [1] in order to give reasonable conditions that are sufficient to solve the Dirichlet problem.

We consider first the case where Ω is the unit disk Δ.

Let us recall that the **Poisson kernel** P for the unit disk is the function given by

$$P(r,\theta) = \frac{1-r^2}{1-2r\cos(\theta)+r^2} \tag{3.1}$$

where $0 < r < 1$ and $0 \leq \theta \leq 2\pi$. The Poisson kernel has the following properties:

$$P(r,\theta) = Re\left(\frac{1+z}{1-z}\right),\ z = re^{i\theta}. \tag{3.2}$$

$$P(r,\theta) > 0. \tag{3.3}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P(r,\theta)\mathtt{d}\theta = 1,\ 0 < r < 1. \tag{3.4}$$

$$\text{for } \delta > 0,\ \lim_{r\to 1}\max\{P(r,\theta)\colon \delta \leq |\theta| \leq \pi\} = 0. \tag{3.5}$$

If we consider u a real-valued continuous function on the unit circle $\mathbb{T}$, where

$$\mathbb{T} = \{e^{i\theta} : -\pi \leq \theta \leq \pi\}$$

and we set

$$P_u(re^{it}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P(r, t-\theta)u(e^{i\theta})\mathtt{d}\theta$$

then the function P_u is a harmonic function of $z = re^{it}$. In fact, because of (3.2) we have

$$P_u(re^{it}) = Re\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{1+z}{1-z}\right)u(e^{i\theta})\mathtt{d}\theta\right]$$

and so P_u is the real part of an analytic function which implies P_u is harmonic. What it is important is to find out the behavior of $P_u(z)$ as z tends to a point in $\mathbb{T}$. For that purpose we have the following theorem.

Theorem 3.1.1. *Given* $\lambda \in \mathbb{T}$*, then*

$$\lim_{z\longrightarrow\lambda} P_u(z) = u(\lambda)$$

that is, P_u *is continuous on* $\Delta \cup \mathbb{T}$ *and agrees with* u *on* $\mathbb{T}$*, where* $\Delta = \{z\colon |z| < 1\}$.

Proof. Let $\lambda = e^{is}$. By continuity of u, given $\epsilon > 0$, choose $0 < \delta < \pi$ such that if $|\theta - s| < \delta$ implies $|u(e^{i\theta}) - u(e^{is})| < \epsilon/2$. Let t be such that $|t-s| < \delta/2$. Because of (3.5), for this δ there exists r_1 such that, if $r_1 \leq r < 1$, then

$$A = \max\{P(r,\theta)\colon \delta/2 \leq |s| \leq \pi\} < \frac{\epsilon}{4m} \tag{3.6}$$

where

$$m = \max_{|\theta|\leq\pi}\{|u(\theta)|\}.$$

Then, because of (3.3), (3.4) and (3.6)

$$\begin{aligned}
|P_u(re^{it}) - u(e^{is})| &= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} P(r,t-\theta)u(e^{i\theta})\mathtt{d}\theta - \frac{1}{2\pi}\int_{-\pi}^{\pi} P(r,t-\theta)u(e^{is})\mathtt{d}\theta\right| \\
&= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} P(r,t-\theta)\left[u(e^{i\theta}) - u(e^{is})\right]\mathtt{d}\theta\right| \\
&\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} P(r,t-\theta)|u(e^{i\theta}) - u(e^{is})|\mathtt{d}\theta \\
&= \left(\frac{1}{2\pi}\int_{|\theta-s|<\delta} + \frac{1}{2\pi}\int_{\delta\leq|\theta-s|<\pi}\right) P(r,t-\theta)|u(e^{i\theta}) - u(e^{is})|\mathtt{d}\theta \\
&\leq \epsilon/2 + 2mA \\
&< \epsilon/2 + \epsilon/2 = \epsilon.
\end{aligned}$$

when $|s-t| < \delta/2$ and $r \geq r_1$, which is what we wanted. □

Definition 3.1.1. Let μ be a measure on $\mathbb{T}$ and set

$$P_\mu(re^{it}) = \int_{\mathbb{T}} P(r,t-\theta)\mathtt{d}\mu(\theta). \tag{3.7}$$

Note that because of (3.4) and Fubini's Theorem we have

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} P_\mu(re^{it})\mathtt{d}t &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\int_{\mathbb{T}} P(r,t-\theta)\mathtt{d}\mu(\theta)\right]\mathtt{d}t \\
&= \int_{\mathbb{T}}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} P(r,t-\theta)\mathtt{d}t\right]\mathtt{d}\mu(\theta) \\
&= \int_{\mathbb{T}}\mathtt{d}\mu(\theta) = P_\mu(0).
\end{aligned}$$

Thus P_μ is continuous and satisfies the mean value property at 0, so P_μ is harmonic.

Theorem 3.1.2. *Let $\mathtt{d}\mu = v\mathtt{d}\theta + \mathtt{d}\alpha$ the Lebesgue decomposition of μ where $v \in L^1(\mathbb{T}, \mathtt{d}\theta)$ and $\mathtt{d}\alpha$ is singular with respect to $\mathtt{d}\theta$. Then*

$$\lim_{r\to 1} P_\mu(re^{it}) = 2\pi v(t) \ a.e. \mathtt{d}t \tag{3.8}$$

Proof. This proof basically follows the same ideas as the proof of Theorem 3.1.1. ☐

Theorem 3.1.3. *A harmonic function u in Δ can be written as*

$$u(re^{i\theta}) = \int_{\mathbb{T}} P(r, \theta - t)\, d\mu(t) \tag{3.9}$$

for some measure μ on $\mathbb{T}$, if and only if

$$\sup_{r<1} \left\{ \int_{-\pi}^{\pi} |u(re^{i\theta})|\, d\theta \right\} \text{ is finite.} \tag{3.10}$$

If (3.9) holds, then μ is uniquely determined. Moreover, if u is also positive then μ is a non-negative measure.

Proof. Assume (3.9) holds, then because (3.3) and (3.4)

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} |u(re^{i\theta})| \mathtt{d}\theta &\leq \frac{1}{2\pi} \int_{\mathbb{T}} P(r, \theta - t) \mathtt{d}\theta \mathtt{d}|\mu(t)| \\ &= \int_{\mathbb{T}} \mathtt{d}|\mu(t)| = \|\mu\|, \text{ the total variation of } \mu. \end{aligned}$$

Conversely, assume (3.10) is true. Define μ_ρ on $\mathbb{T}$ given by

$$\mathtt{d}\mu_\rho(t) = \frac{1}{2\pi} u(\rho e^{it}) \mathtt{d}t,\ 0 < \rho < 1.$$

These μ_ρ are measures on $\mathbb{T}$ and by (3.10) we have

$$\|\mu_\rho\| \leq c, 0 < \rho < 1$$

for some constant c that without loss of generality we can assume is 1. Then, by the example given at the end of Chapter 2, there is a measure μ on $\mathbb{T}$ which is a weak-star cluster point of $\{\mu_\rho\}$. Also since u is harmonic (and the unicity of the harmonic extension) then $u(r\rho e^{i\theta}) = P_u(r\rho e^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P(r,\theta-t)u(\rho e^{it})\mathtt{d}t$.

Thus, by the fact that u is continuous and the last observation and considering the definition of μ_ρ, we have

$$\begin{aligned} u(re^{i\theta}) &= \lim_{\rho\to 1} u(\rho r e^{i\theta}) \\ &= \lim_{\rho\to 1} \frac{1}{2\pi}\int_{-\pi}^{\pi} P(r,\theta-t)u(\rho e^{it})\mathtt{d}t \\ &= \lim_{\rho\to 1} \int_{\mathbb{T}} P(r,\theta-t)\mathtt{d}\mu_\rho(t) \\ &= \int_{\mathbb{T}} P(r,\theta-t)\mathtt{d}\mu(t). \end{aligned}$$

For the unicity of μ, if there is μ_0 that satisfies (3.9), then

$$\int_{\mathbb{T}} P(r,\theta-t)(\mathtt{d}\mu - \mathtt{d}\mu_0) = 0$$

if we write $\mu = Re(\mu) + iIm(\mu)$ and $\mu_0 = Re(\mu_0) + iIm(\mu_0)$, then

$$\int_{\mathbb{T}} P(r,\theta-t)(\mathtt{d}Re(\mu) - \mathtt{d}Re(\mu_0)) + i\int_{\mathbb{T}} P(r,\theta-t)(\mathtt{d}Im(\mu) - \mathtt{d}Im(\mu_0)) = 0$$

implies

$$\int_{\mathbb{T}} P(r,\theta-t)(\mathtt{d}Re(\mu) - \mathtt{d}Re(\mu_0)) = 0, \text{ and, } \int_{\mathbb{T}} P(r,\theta-t)(\mathtt{d}Im(\mu) - \mathtt{d}Im(\mu_0)) = 0. \quad (3.11)$$

Let $\nu = Re(\mu-\mu_0)$ and $\tau = Im(\mu-\mu_0)$. By (3.11)

$$0 = Re\int_{\mathbb{T}} \frac{e^{it}+z}{e^{it}-z}\mathtt{d}\nu(t), \ |z| < 1$$

and so is its harmonic conjugate (chosen to be zero at the origen). So the analytic function

$$h(z) = \int_{\mathbb{T}} \frac{e^{it}+z}{e^{it}-z}\mathtt{d}\nu(t)$$

is identically constant and therefore 0 since $h(0) = 0$. But we know

$$h(z) = \int_{\mathbb{T}} \mathtt{d}\nu + 2\sum_{n=1}^{\infty} z^n \left\{ \int_{\mathbb{T}} e^{-int} \mathtt{d}\nu(t) \right\}$$

so,

$$\int_{\mathbb{T}} e^{-int} \mathtt{d}\nu(t),\ n = 0, 1, 2, \ldots$$

since ν is real then ν is the zero measure. Similarly τ is the zero measure, therefore $\mu - \mu_0 = 0$.

Finally we know that a measure μ is positive if and only if $\int f \mathtt{d}\mu \geq 0$ for all nonnegative continuous function f. Now if u is positive then, because of the way μ_ρ is been defined, μ_ρ is non-negative measure for each ρ and so

$$\int_{\mathbb{T}} f \mathtt{d}\mu = \lim_{\rho \to 1} \int_{\mathbb{T}} f \mathtt{d}\mu_\rho \geq 0$$

for any nonnegative continuous function f, therefore μ is a non-negative measure. ☐

3.2 Subharmonic Functions

We now return to the case of a general domain Ω contained in the Riemann sphere.

Definition 3.2.1. Consider Ω a domain on the sphere. A function $u(z)$ defined for z in Ω is **subharmonic** on Ω if it satisfies the following conditions:

$$-\infty \leq u(z) < \infty,\ z \in \Omega \tag{3.12}$$

(1) u is upper semicontinuous on Ω, i.e.,

$$u(a) \geq \limsup\{u(z) \colon z \to a\} \text{ for all } a \in \Omega, \text{ and} \tag{3.13}$$

(2) if the closed disc $\{z\colon |z-p| \leq r\}$ lies in Ω, then

$$u(p) \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} u(p+re^{it})\mathtt{d}t. \tag{3.14}$$

It is clear that every real-valued harmonic function on Ω is subharmonic and if u and $-u$ are subharmonic then u is harmonic. It is also clear that the sum as well as the maximum of two subharmonic functions are also subharmonic. A positive multiple of a subharmonic function will be a subharmonic function as well. I will list some facts about subharmonic functions in the following propositions that will be needed later on. A detailed explanation of them can be found in [1].

Proposition 3.2.1. *Let u be a subharmonic on Ω an let ϕ be a monotonically increasing convex function on $\mathbb{R}$. Then $\phi(u(z))$ is subharmonic on Ω.*

As an application of the previous proposition we have: if f is an holomorphic function on Ω, then both $\log|f|$ and $|f|^q$, $0 < q < \infty$, are subharmonic on Ω.

Lemma 3.2.2. *Let $\mathbf{K}$ be a compact set and let u be a function on $\mathbf{K}$ with values on $[-\infty, \infty)$. Then u is upper semicontinuous if and only if there is a sequence $\{f_n\}$ of continuous function on $\mathbf{K}$ with*

$$f_1 \geq f_2 \geq \dots \text{ and } \lim_{n\to\infty} f_n(z) = u(z),\ z \in \mathbf{K}.$$

Proposition 3.2.3. *Suppose there is a number $M < \infty$ such that*

$$\limsup\{u(z)\colon z \to \zeta\} \leq M \text{ for all } \zeta \in \partial\Omega.$$

Then $u(z) \leq M$ for all $z \in \Omega$. If $u(z_0) = M$ for some $z_0 \in \Omega$, then $u \equiv M$ in Ω.

3.3 Solution of the Dirichlet Problem

In order to solve the Dirichlet problem we will need the following fundamental result.

Proposition 3.3.1. *Let $\mathfrak{F}$ be a family of subharmonic functions satisfying the following conditions:*

$$\text{for } u, v \in \mathfrak{F}, \text{ then } \max(u,v) \in \mathfrak{F} \tag{3.15}$$

if $\{z\colon |z-p| \leq r\} \subset \Omega$ and if $u \in \mathfrak{F}$, then the function

$$s(u,z) = \begin{cases} u(z) \text{ if } |z-p| \geq r \\ P_u(z) \text{ if } |z-p| < r \end{cases} \tag{3.16}$$

is in $\mathfrak{F}$. Set

$$v(z) = \sup_{u \in \mathfrak{F}} u(z) \tag{3.17}$$

Then either $v \equiv \infty$ in Ω or v is harmonic in Ω.

Proof. **First case**: there exists $z_0 \in \Omega$ such that $v(z_0) = \infty$. Then there is a sequence $\{u_i\}$ in $\mathfrak{F}$ such that $\{u_i(z_0)\}$ increases to ∞ as $i \to \infty$. Let $v_n = max\{u_1, u_2, \ldots, u_n\}$, then by (3.15) $v_n \in \mathfrak{F}$ for all $n = 1, 2, \ldots$. So $v_1 \leq v_2 \leq \ldots$, on all Ω and $v_n(z_0) \to \infty$ as $n \to \infty$. Considering the disc $D = \{|z-z_0| \leq r\} \subset \Omega$ we have $s(v_n, z) \in \mathfrak{F}$ by (3.16). We also know

$$P_{v_n}(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{r^2-s^2}{r^2-2rs\cos(\theta-t)+s^2} v_n(z_0+re^{it})\mathtt{d}t,\ z = z_0+se^{i\theta},\ s<r$$

and

$$a = \frac{r^2-s^2}{r^2-2rs\cos(\theta-t)+s^2} \geq \frac{r-s}{r+s} = b$$

then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} a(v_n - v_1)(z_0 + re^{it})\mathtt{d}t \geq \frac{1}{2\pi}\int_{-\pi}^{\pi} b(v_n - v_1)(z_0 + re^{it})\mathtt{d}t.$$

Let $L = \frac{1}{2\pi}\int_{-\pi}^{\pi} av_1(z_0 + re^{it})\mathtt{d}t - \frac{1}{2\pi}\int_{-\pi}^{\pi} bv_1(z_0 + re^{it})\mathtt{d}t$, then, by Theorem 19.4.11 of [5], L is finite, hence

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} av_n(z_0 + re^{it})\mathtt{d}t \geq \frac{1}{2\pi}\int_{-\pi}^{\pi} bv_n(z_0 + re^{it})\mathtt{d}t + L$$

using the fact that v_n is subharmonic, for $z = z_0 + se^{i\theta}$, with $s < r$, we have

$$s(v_n, z) = P_{v_n}(z) \geq bv_n(z_0) + L$$

thus, since $v_n(z_0) \to \infty$ then $s(v_n, z) \to \infty$, for all $z = z_0 + se^{i\theta},\ s < r$; also, since $s(v_n, z) \in \mathfrak{F}$, then

$$v(z) \geq s(v_n, z),\ \text{for all } n,\ |z - z_0| < r$$

which implies $v(z) = \infty$, if $|z - z_0| < r$. Thus, this implies that the set

$$\Omega_1 = \{z \in \Omega \colon v(z) = \infty\}$$

is open. Also if we take a sequence $\{z_n\}$ of elements in Ω_1 such that $z_n \to \beta$ as $n \to \infty$, then

$$v(\beta) \geq \limsup\{v(z) \colon z \to \beta\} \geq \limsup\{v(z_n) \colon z_n \to \beta\} = \infty$$

This implies that Ω_1 is closed. Since $z_0 \in \Omega_1$ and Ω is connected, then $\Omega_1 = \Omega$ and, therefore, $v = \infty$ in Ω.

Second case: v is finite at all points of Ω. Let a be a point of Ω and let D be a disc centered at a whose closure lies in Ω. As in the first case, we can get $u_n \in \mathfrak{F}$ such that $u_1 \leq u_2 \leq \ldots$, on Ω, and $u_n(a) \to v(a)$ as $n \to \infty$.

Claim. Using the disc D, we may assume that each u_n is harmonic in D. In fact, for fixed n, by Lemma 3.2.2, there is a sequence of continuous functions $\{f_l\}$ such that f_l decreases to u_n on ∂D. Because the Dirichlet problem in the disc D is solvable then for each l there exists a harmonic extension F_l of f_l such that $F_l = f_l$ on ∂D. By Harnack's theorem there is a harmonic function F such that $F_l \longrightarrow F$ as $l \longrightarrow \infty$ on D and $F = u_n$ on ∂D. Then in the disk D, for $z = a + se^{is},\ s < r$:

$$\begin{aligned} s(u_n, z) = P_{u_n}(z) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{r^2 - s^2}{r^2 - 2rs\cos(\theta - t) + s^2} u_n(a + re^{it})\mathtt{d}t \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{r^2 - s^2}{r^2 - 2rs\cos(\theta - t) + s^2} F(a + re^{it})\mathtt{d}t = F(z) \end{aligned}$$

so $s(u_n, z)$ is harmonic in the disc D. This concludes the proof of our claim.

Then, by Harnack's theorem, $\{u_n\}$ converges to a function U which is harmonic in D and $U(a) = v(a)$. Taking any $b \in D, b \neq a$, we can do the same as before and get $w_n \in \mathfrak{F}$ with $w_1 \leq w_2 \leq \ldots$ on Ω, w_n harmonic, and $w_n(b) \longrightarrow v(b)$ as $n \longrightarrow \infty$. Since the Dirichlet problem is solvable in the disc, then there exists $r_n(z)$ such that $r_n(z)$ is harmonic in D and equal $\max\{u_n, w_n\}$ on ∂D. But, for $z = a + se^{is},\ s < r$

$$\begin{aligned} s(t_n, z) = P_{t_n}(z) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{r^2 - s^2}{r^2 - 2rs\cos(\theta - t) + s^2} t_n(a + re^{it})\mathtt{d}t \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{r^2 - s^2}{r^2 - 2rs\cos(\theta - t) + s^2} r_n(a + re^{it})\mathtt{d}t = r_n(z) \end{aligned}$$

where $t_n = \max\{u_n, w_n\}$. So $r_n(z) = s(t_n, z)$ in D and $s(t_n, z) \in \mathfrak{F}$ so we may assume $r_n \in \mathfrak{F}$. Now, using Theorem 19.4.5 in [5], we can conclude that $r_n(b) \geq w_n(b)$ and

$$v(a) \geq r_n(a) \geq u_n(a).$$

Again, by Harnack's theorem, $\{r_n\}$ increases to a function R that is harmonic in D, and since $R \geq r_n \geq u_n$ in ∂D then $R \geq U$ in ∂D and so $R \geq U$ in D, but $U(a) = v(a) = R(a)$, $U = R$ in D. Since $U(b) = v(b) \leq R(b)$ then $v = R$ in D with R harmonic, so v is harmonic. $\square$

Definition 3.3.1. Given $x \in \partial\Omega$. We will say that there is a **barrier** at x if for given $\delta > 0$ it is possible to find a function $b(z)$ satisfying the following conditions:

$$-b \text{ is subharmonic in } \Omega \tag{3.18}$$

$$b \geq 0 \tag{3.19}$$

$$b(z) \geq 1 \text{ if } z \in \Omega \text{ and } |z - x| \geq \delta \tag{3.20}$$

$$b(z) \to 0 \text{ if } z \in \Omega \text{ and } z \to x. \tag{3.21}$$

Definition 3.3.2. We will say that the set $V \subset \mathbb{C}$ is a **continuum** if it is closed and connected consisting of more than one point.

Theorem 3.3.2. *Let Ω be a domain an let $x \in \partial\Omega$. If there is a continuum V in the complement of Ω which contains x, then there is a barrier at x.*

Proof. Let x' be another point in V, then there is a linear fractional transformation, which sends x to ∞ and x' to 0. So without loss of generality we will work the case when $x = \infty$ and the continuum V in the complement of Ω contains both 0 and ∞. Set $\mathfrak{D} = \mathbb{C} \backslash V$, then $\Omega \subset \mathfrak{D}$; also because of $\mathbb{C} \backslash \mathfrak{D} = V$ and V is connected then, by Theorem 8.2.2 of [4], $\mathfrak{D}$ is simply connected and there is a single-valued branch of $\log(z)$ in the domain $\mathfrak{D}$. Set $\mathfrak{R} = \log(\mathfrak{D})$, then $\mathfrak{R}$ is a domain, since $\log(z)$ is an open mapping and continuous. We can assume that if $\mathfrak{R}$ meets the imaginary axis, then this intersection in an open set in the imaginary line and so is the disjoint union of open intervals; moreover the sum of the length of such intervals is at most 2π since the $\log(z)$ is analytic in the branch we have chosen and $\mathfrak{R}$ is in the domain where the $\exp(z)$ (the inverse of $\log(z)$) is single valued for such selection of the branch. So we can write:

$$\mathfrak{R} \cap \{it \colon t \in \mathbb{R}\} = \bigcup_{j=1}^{\infty} (i\alpha_j, i\beta_j)$$

where

$$\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \dots \text{ and } \sum_{j=1}^{\infty} (\beta_j - \alpha_j) \leq 2\pi$$

Define

$$h_j(z) = \arg\left(\frac{z - i\alpha_j}{z - i\beta_j}\right), \ Re\, z > 0, \ j = 1, 2, \dots.$$

These h_j are well defined; in fact, if

$$z - i\alpha_j = \tau(z - i\beta j), \ \tau < 0$$

then

$$0 < Re(z - i\alpha_j) = Re(\tau(z - i\beta j)) < 0$$

which is a contradiction. Then $\frac{z-i\alpha_j}{z-i\beta j}$ never meets the negative real axis, so $\arg\left(\frac{z-i\alpha_j}{z-i\beta_j}\right)$ makes sense. Also, because of how each h_j is defined then h_j is positive and harmonic on $Re\, z > 0$ and

$$0 < \sum_{j=1}^{\infty} h_j(z) < \pi.$$

Now, define

$$h(z) = -\frac{1}{\pi} \sum_{j=1}^{\infty} h_j(z), \ Re\, z > 0.$$

Let us see why the function $h(z)$ is well defined. Note that

$$\sum_{j=1}^{\infty} h_j(z) = \sum_{j=1}^{\infty} \arg\left(\frac{z-i\alpha_j}{z-i\beta_j}\right) = Im\left(\sum_{j=1}^{\infty} \log\left(\frac{z-i\alpha_j}{z-i\beta_j}\right)\right).$$

Take any compact K not meeting the imaginary axis. Since the imaginary axis is closed then

$$c = \min_{z\in K,\, j\geq 1}\{|z-i\beta_j|\} > 0$$

and so

$$\begin{aligned} \left|\sum_{j=1}^{\infty}\left[1-\frac{z-i\alpha_j}{z-i\beta_j}\right]\right| &= \left|\sum_{j=1}^{\infty}\left[\frac{i\alpha_j-i\beta_j}{z-i\beta_j}\right]\right| \\ &\leq \sum_{j=1}^{\infty}\left|\frac{\alpha_j-\beta_j}{z-i\beta_j}\right| \leq 2\pi/c. \end{aligned}$$

So

$$\sum_{j=1}^{\infty}\left[\frac{z-i\alpha_j}{z-i\beta_j}-1\right]$$

converges absolutely and uniformly then, so for j large enough $|\frac{z-i\alpha_j}{z-i\beta_j}-1| < 1$, then by Theorem 7.1.2 of [5],

$$\sum_{j=1}^{\infty}\log\left[\frac{z-i\alpha_j}{z-i\beta_j}\right]$$

converges. Thus the definition of h makes sense. We also have $-1 < h(z) < 0$.

We also have to notice that h is harmonic on $Re\ z > 0$. In fact, h is increasing limit of the partial sums of its series and each h_j is harmonic, and

$$\left|-\frac{1}{\pi}\sum_{j=1}^{N} h_j(z)\right| \leq 1.$$

Consider $p \in \mathbb{C}$ and $Re\ p > 0$ and$\{z \colon |z-p| < r\}$ for $Re\ z > 0$. Then, by the Lebesgue dominated convergence theorem,

$$\begin{aligned} \frac{1}{2\pi}\int_{-\pi}^{\pi} h(p+re^{i\theta})\mathtt{d}\theta &= -\frac{1}{\pi}\sum_{j=1}^{\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi} h_j(p+re^{i\theta})\mathtt{d}\theta \\ &= -\frac{1}{\pi}\sum_{j=1}^{\infty} h_j(p) = h(p). \end{aligned}$$

So h satisfies the mean value property, and therefore h is harmonic.

If $x \in (\alpha_i, \beta_j)$ for some j and if $\{z_m\}$ is a sequence in $Re\ z > 0$ such that $z_m \to ix$, then $\frac{ix-i\alpha_j}{ix-i\beta_j} = \frac{x-\alpha_j}{x-\beta_j} < 0$ which implies

$$\lim_{m\to\infty} h_j(z_m) = \pi$$

so h is continuous with $h(ix) = -1$. Finally, if $Re\ z > 0$ and $|z_m| \to \infty$ then $h(z_m) \to 0$. Define

$$g(z) = \begin{cases} -1 \text{ if } Re\ z \leq 0\ z \in \mathfrak{R} \\ h(z) \text{ if } Re\ z > 0\ z \in \mathfrak{R} \end{cases}$$

Then it is clear that g is continuous in $\mathfrak{R}$, subharmonic in $\mathfrak{R}$, $-1 \leq g(z) \leq 0$ and $g(z) \to 0$ if $Re\ z > 0$ and $|z| \to \infty$. Now set

$$G(z) = g(\log(z)),\ z \in \mathfrak{D}.$$

Then G is subharmonic in $\mathfrak{D}$, $-1 \leq G \leq 0$, and $G(z) \to 0$ as $|z| \to \infty$. It can happen that $G \to 0$ at some finite boundary point. To compensate for this, take $\{s_n\}$ real numbers increasing to ∞ such that all the lines $Re\ z = t_n$ meet $\mathfrak{R}$. Let g_n be the function corresponding to $Re\ z = t_n$ constructed as above, set

$$H(z) = \sum_{n=1}^{\infty}\frac{1}{2^n} g_n(\log(z)),\ z \in \mathfrak{D}$$

then we have

$$\left|\sum_{n=1}^{\infty} \frac{1}{2^n} g_n(\log(z))\right| \leq \sum_{n=1}^{\infty} \left|\frac{1}{2^n} g_n(\log(z))\right| \leq \sum_{n=1}^{\infty} \frac{1}{2^n} < \infty$$

so the series converges uniformly which implies that H is continuous on $\mathfrak{D}$, subharmonic on $\mathfrak{D}$, $-1 \leq H \leq 0$ and $H(z) \to 0$ if $z \in \mathfrak{D}$ and $|z| \to \infty$. Given y a finite point in $\partial\mathfrak{D}$, them $\log(y)$ is a finite point in $\partial\mathfrak{R}$ and so $g_n(\log(y)) = -1$, for all $n \geq 0$. Thus

$$\limsup\{H(z)\colon z \in \mathfrak{D},\ z \to y\} < 0$$

hence for M large enough

$$\rho = \sup_{|y| \leq M} \{\limsup\{H(z)\colon z \in \mathfrak{D},\ z \to y\}\} < 0.$$

Then $b(z) = \rho^{-1}H(z)$ is the desired function for the barrier at $x = \infty$. □

Now, consider h a bounded function on $\partial\Omega$. Let us consider the following family $\mathfrak{F}(h)$ of subharmonic functions satisfying

$$\limsup\{u(z)\colon z \in \Omega, z \to \zeta\} \leq h(\zeta), \forall \zeta \in \partial\Omega. \tag{3.22}$$

Set

$$v(z) = v_h(z) = \sup\{u(z)\colon u \in \mathfrak{F}(h)\}. \tag{3.23}$$

Then we have the following theorem.

Theorem 3.3.3. *The function v given by (3.23) is harmonic on Ω. Moreover, if h is continuous at $x \in \partial\Omega$ and if there is a barrier at x then*

$$\lim_{z \to x} v(z) = h(x) \tag{3.24}$$

Proof. We have

$$\limsup\{u(z)\colon z \in \Omega, z \to \zeta\} \quad \leq \quad h(\zeta) \leq M = \sup_{\zeta \in \partial\Omega}\{h(\zeta)\} < \infty$$

then by Proposition 3.2.3

$$u(z) \leq M < \infty \,, z \in \Omega$$

and so, by Proposition 3.3.1, v given by (3.24) is harmonic on Ω. Notice that without loss of generality we can assume $M > 0$.

Using the continuity of h, given $\epsilon > 0$, choose $\delta > 0$ so that if $y \in \partial\Omega$ and $|x - y| < \delta$ implies $|h(x) - h(y)| < \epsilon/2$. Since there is a barrier at x then, for this δ, there is a barrier b. Now, set

$$s(z) = h(z) - \epsilon - 2Mb(z), \; z \in \Omega.$$

Suppose $y \in \partial\Omega$ and $|y - x| < \delta$, then (3.22) and continuity of h implies

$$\limsup\{s(z)\colon z \to y\} \leq h(x) - \epsilon < h(y)$$

And, if $y \in \partial\Omega$ and $|y - x| \geq \delta$ then by (3.21)

$$s(z) \leq h(x) - 2M - \epsilon < h(x) - 2M$$

therefore

$$\limsup\{s(z)\colon z \to y\} \leq h(x) - 2M \leq h(y).$$

Thus, $s \in \mathfrak{F}(h)$ and so $(v(z) \geq s(z)$ for all $z \in \Omega$. We then have

$$\begin{aligned} \liminf\{v(z)\colon z \to x\} &\geq \liminf\{s(z)\colon z \to x\} \\ &\geq h(x) - \epsilon. \end{aligned}$$

Because ϵ was chosen arbitrarily, we have

$$\liminf\{v(z)\colon z \to x\} \geq h(x).$$

Similarly, if we consider the family $\mathfrak{F}(-h)$ and set

$$w(z) = - \sup_{u \in \mathfrak{F}(-h)} \{u(z)\}$$

then w is harmonic inΩ and

$$\liminf\{-w(z)\colon z \to x\} \geq -h(x)$$

in other words,

$$\limsup\{w(z)\colon z \to x\} \leq h(x).$$

Finally, if $u_1 \in \mathfrak{F}(h)$ and $u_2 \in \mathfrak{F}(-h)$, then $u_1 + u_2$ is subharmonic in Ω and

$$\begin{aligned} \limsup\{u_1(z) + u_2(z)\colon z \to \zeta\} &\leq \limsup u_1 + \limsup u_2 \\ &\leq h(\zeta) + (-h(\zeta) = 0 \end{aligned}$$

so, by Proposition 3.2.3, $u_1 + u_2 \leq 0$ in Ω, and therefore $v - w \leq 0$ in Ω. Thus

$$\begin{aligned} h(x) &\geq \limsup\{w(z)\colon z \to x\} \\ &\geq \limsup\{v(z)\colon z \to x\} \\ &\geq \liminf\{v(z)\colon z \to x\} \\ &\geq h(x) \end{aligned}$$

which implies

$$\lim_{z\to x} v(z) = h(x).$$

$\square$

Corollary 3.3.4. *If there is a barrier at each point of $\partial\Omega$, then the Dirichlet problem is solvable for Ω.*

Corollary 3.3.5. *If each component of $\partial\Omega$ is nontrivial, then the Dirichlet problem is solvable in Ω.*

Now let us talk a little bit about Green's function and some of its principal properties.

Suppose that Ω is a domain on the extended plane and that $p \in \Omega$. A function $g(z; p)$ is a **Green's function** for Ω with pole (or singularity) at p, $p \neq \infty$, if

1. $g(z; p)$ is harmonic on $\Omega - \{p\}$
2. $g(z; p) + \log|z - p|$ is harmonic near p
3. $\lim\{g(z; p) \colon z \to \zeta\} = 0$ for all $\zeta \in \partial\Omega$.

If $p = \infty$, then (2) is modified to

$$g(z, \infty) - \log|z|, \text{ is harmonic near } \infty$$

Proposition 3.3.6. *Let Ω be a domain for which the Dirichlet problem is solvable and let $p \in \Omega$. Then Ω has a Green's function with pole at p.*

Proposition 3.3.7. *Let* g *be the Green's function for* Ω*. Then for all pairs of points* p, q *in* Ω *with* $p \neq q$ *in* Ω*, we have*

$$g(p, q) = g(q, p).$$

Chapter 4

Harmonic measure and Hardy spaces on a domain Ω

In this chapter we introduce some additional concepts we need in order to resolve our main problem. We will solve our main problem in the case our domain Ω is Δ the unit disc. This will be the crucial result to solve our main problem for the more general case where Ω is a finitely connected planar domains.

4.1 Harmonic Measure

Let Ω be a domain on the extended plane for which the Dirichlet problem is solvable and let p be a point in Ω. Given u a real-valued continuous function on $\Gamma = \partial\Omega$, let U be the harmonic extension to Ω of u. Then we can define

$$\Lambda : \{u \colon u : \Gamma \to \mathbb{R}, \text{continuous}\} \to \mathbb{R}$$

by

$$\Lambda(u) = U(p)$$

then Λ is linear and applying the maximum principal to U we have

$$|U(p)| \leq \|u\|_\Gamma = \sup\{|u(z)| \colon z \in \Gamma\}$$

so

$$\|\Lambda\| \leq 1$$

then by the Riesz representation theorem there is a unique Borel real measure ω_p on Γ such that

$$\Lambda(u) = U(p) = \int_\Gamma u \mathtt{d}\omega_p, \ u \in C(\Gamma).$$

This measure will be called the **harmonic measure** on Γ for p. Let us remark that if $u \geq 0$ then $U \geq 0$ (in fact if $U < 0$ then, by continuity of U, $u = 0$ and so $U = 0$ on $\Omega \cup \Gamma$, giving us a contradiction), thus $U(p) \geq 0$ and so ω_p is a non-negative Borel measure. Also

$$\|\omega_p\| = \int_\Gamma 1 \mathtt{d}\omega_p = 1(p) = 1.$$

We notice that ω_p depends of the point p, but it can be shown that for p and q in Ω, ω_p and ω_q are boundedly mutually absolutely continuous. Further, if K is a compact set in Ω, then there is a constant M such that

$$\omega_q(E) \leq M\omega_p(E), \text{ for all } q \in K \text{ and for all measurable set } E \text{ in } \Gamma.$$

For a proof of this fact see Theorem 1.6.1 of [1].

Now, let us assume that $\Gamma = \partial\Omega$ consist of $m+1$ disjoint analytic simple connected curves. Let $p \in \Omega$ and $g(z;p)$ its Green's function for Ω at p, set $h(z) = h(z;p)$ the harmonic conjugate of $g(z;p)$ (of course this h is multivalued). Then we have that locally $Q = g + ih$ is analytic and its derivative is single-valued on Ω. Then we have the following three results (whose proofs can be seen in Chapter 1, Section 6 of [1]).

Theorem 4.1.1. *Suppose Ω is bounded by a finite number of disjoint analytic simple closed curves. Then for each $p \in \Omega$ we have*

$$d\omega_p = -\frac{1}{2\pi}\frac{\partial}{\partial n} g(\cdot;p)\, ds$$

where $g(\cdot;p)$ is the Green's function for Ω with pole at p, $\frac{\partial}{\partial n}$ is the derivative in the direction of outwards normal at Γ, and ds is arc length.

Theorem 4.1.2.

$$d\omega_p(\zeta) = \frac{i}{2\pi} Q^{'}(\zeta)\, d\zeta.$$

Theorem 4.1.3. *Let $\Gamma = \partial\Omega$ consist of $m+1$ disjoint analytic simple closed curves, let $p \in \Omega$, and Q as before. Then*

- *$Q^{'}$ does not vanish on Γ.*
- *$Q^{'}$ has precisely m zeros in Ω, counting multiplicity.*

4.2 Some properties of $H^p(\Omega)$

Definition 4.2.1. Let $0 < p < \infty$; a holomorphic function f on a domain Ω is in $H^p(\Omega)$ if the subharmonic function $|f(z)|^p$ has a harmonic majorant on Ω, i.e, there is a harmonic function $v(z)$ such that

$$|f(z)|^p \leq v(z),\ z \in \Omega$$

The function f is in $H^\infty(\Omega)$ if it is both holomorphic and bounded on Ω.

It is easy to see that $H^\infty(\Omega) \subset H^p(\Omega)$. It can be proved that there is a unique harmonic function u_f such that

$$|f(z)|^p \leq u_f(z),\ z \in \Omega$$

and

$$u_f(z) \leq v(z),\ z \in \Omega$$

if v is any harmonic majorant of $u = |f|^p$. This u_f will be called the **least harmonic majorant** of f. In fact, consider $\{\Omega_n\}$ a regular exhaustion of Ω. Set $v_n = (|f|^p)|_{\partial\Omega_n}$, and V_m the corresponding harmonic extension to Ω_n, for $n = 1, 2, \ldots$. Now is $n > m$, then $\partial\Omega_m \subset \Omega_n$ and so on $\partial\Omega_m$ it holds $V_m = u \leq V_n$ since V_n is also harmonic on Ω_m and so $V_n = V_m$, on Ω_m. Hence, by Theorem 19.4.5 in [5],

$$V_m \leq V_n \text{ on } \Omega_m.$$

We have, therefore, $\{V_n\}$ is an increasing sequence on Ω that tends to ∞ or to a harmonic function W on Ω (by Harnack's theorem). But since $f \in H^p(\Omega)$, for any g harmonic majorant of u, and therefore, if $a \in \Omega_m,\ m \geq 1$:

$$\begin{aligned} V_m(a) &= \int_{\partial\Omega_n} v_m \mathtt{d}\omega_{m,a},\ \text{by definition of } \mathtt{d}\omega_{m,a} \\ &= \int_{\partial\Omega_n} u \mathtt{d}\omega_{m,a},\ \text{by definition of } v_m \\ &\leq \int_{\partial\Omega_n} g \mathtt{d}\omega_{m,a},\ \text{because } u \leq g \\ &= g(a) < \infty,\ \text{by definition of } \mathtt{d}\omega_{m,a} \end{aligned}$$

so $\{V_n\}$ tends to a harmonic function W and $W \leq g$ on Ω. This W is the function that we have denoted above by u_f.

Remark 4.2.1. It is important to note the following: if $\Omega = \Delta = \{z\colon |z| < 1\}$,

$$\Delta_r = \{z\colon |z| < r\} \text{ for } r < 1,$$

and $f \in H^p(\Delta)$, then the Green's function for Δ_r with pole at 0 is

$$g(z;0) = \log(r) - \log|z|$$

and Q, in Theorem 4.1.2 above, is $Q(z) = \log(r) - \log(z)$ and therefore:

$$\int_{\partial\Delta_r} |f(\zeta)|^p \mathtt{d}\omega_0(\zeta) = -\frac{i}{2\pi}\int_{-\pi}^{\pi} |f(re^{it})|^p \left[\frac{ire^{it}}{re^{it}}\right] \mathtt{d}t = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{it})|^p \mathtt{d}t$$

and

$$\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{it})|^p \mathtt{d}t\right\} \text{ tends increasingly to } u_f(0) \text{ as } r \to 1.$$

Definition 4.2.2. Fixed $z_0 \in \Omega$, set

$$\|f\| = \begin{cases} (u_f(z_0))^{1/p},\ 0 < p < \infty \\ \sup\{|f(z)|\colon z \in \Omega\},\ p = \infty \end{cases} \tag{4.1}$$

It can be shown that the function in (4.1) is a norm on $H^p(\Omega)$, $1 \leq p \leq \infty$, and the resulting topology does not depend on the choice of $z_0 \in \Omega$. Furthermore $H^p(\Omega)$ together with this norm is a Banach space for $1 \leq p \leq \infty$, i.e., the norm defined (4.1) is complete. (For a detailed proof of the independence of the choice of z_0 and the completeness of $H^p(\Omega)$ see Chapter 3, Section 2 of [1]).

Now we focus our attention to $H^p(\Delta)$. We start with the following elementary facts concerning this conformally invariant definition of $H^p(\Omega)$. A function f holomorphic in Δ is in $H^p(\Delta)$, $0 < p < \infty$ if and only if

$$\sup_{0<r<1} \left\{ \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{it})|^p \mathrm{d}t \right)^{1/p} \right\}$$

is bounded. This follows in a straightforward way from the Remark 4.2.1. We will set $M_p(f;r) = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{it})|^p \mathrm{d}t \right)^{1/p}$. It is also important to note that if $\phi : \Omega \to \Omega'$ is conformal with $\phi(z_0) = z_0'$, then

$$\|f\|_{H^p(\Omega', z_0')} = \|f \circ \phi\|_{H^p(\Omega, z_0)}.$$

We have the following theorem that will help us to analyze the zeros of an $f \in H^p(\Delta)$.

Theorem 4.2.1. *Let $f \in H^p(\Delta)$, $0 < p \leq \infty$, f not identically zero. Let $z_1, z_2, \ldots$ be the zeros of f in Δ repeated according to their respective multiplicities. If f has infinitely many zeros, then they satisfy*

$$\sum_{1}^{\infty} (1 - |z_j|) < \infty. \tag{4.2}$$

If the points $z_1, z_2, \ldots$ satisfy (4.2) then

$$B(z) = \prod_{j=1}^{\infty} \left(\frac{-\bar{z}_j}{|z_j|} \right) \left(\frac{z - z_j}{1 - \bar{z}_j z} \right)$$

is holomorphic in Δ bounded by 1 which vanishes precisely at the points $\{z_j\}$. Furthermore

$$f = BF$$

where $F \in H^p(\Delta)$, $\|F\|_p = \|f\|_p$, and F has no zeros in Δ.

The proof of this theorem can be seen in Chapter 3, Section 3 of [1].

Proposition 4.2.2. *A holomorphic function*

$$f(z) = \sum_{i=0}^{\infty} a_n z^n$$

on Δ is in $H^2(\Delta)$ if and only if

$$\sum_{i=0}^{\infty} |a_n|^2 < \infty,$$

and $\|f\|_{H^2(\Delta)} = \left(\sum_{n=0}^{\infty} |a_n|^2\right)^{1/2}$

Proof. For $r < 1$ we have

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{it})|^2 \mathrm{d}t = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}$$

which is just an straightforward calculation. Then the result follows immediately from this equality and the remark above. □

Theorem 4.2.3. *Let $f \in H^p(\Delta)$, f not identically zero, $0 < p < \infty$. Then*

1. $\lim_{r \to 1} f(re^{it}) = f^*(e^{it})$ *exists a.e.* $\mathrm{d}t$

2. $f^* \in L^p(\partial\Delta, \mathrm{d}t)$

3. $\int_{-\pi}^{\pi} |f(re^{it}) - f^*(e^{it})|^p \, dt \to 0$ *as* $r \to 1$

4. $\log|f(re^{i\theta})| \le \frac{1}{2\pi}\int_{-\pi}^{\pi} P(r,\theta - t)\log|f^*(e^{it})|\, dt.$

Proof. First, let us take $f \in H^2(\Delta)$, and $f(z) = \sum_{i=0}^{\infty} a_n z^n$. Set

$$g(e^{i\theta}) = \sum_{n=0}^{\infty} a_n e^{in\theta} \in L^2(\partial\Delta, \mathrm{d}t)$$

and $f_r(e^{i\theta}) = \sum_{n=0}^{\infty} a_n r^n e^{in\theta}$, $0 < r < 1$. After a calculation we get

$$\|f_r - g\|_2^2 = \sum_{n=1}^{\infty} |a_n|^2 (1 - r^{2n})$$

now making $r \to 1$ implies $f_r \to g$ in $L^2(\partial\Delta, \frac{1}{2\pi}\mathrm{d}t)$, and therefore a subsequence of f_r converges almost everywhere to g. We also know that $Re(f)$ and $Im(f)$ are harmonic functions, and because $f \in H^2(\Delta)$, we can deduce:

$$\sup_{0<r<1}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} |Re(f)(re^{it})|\mathrm{d}t\right\} < M$$

and

$$\sup_{0<r<1}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} |Im(f)(re^{it})|\mathrm{d}t\right\} < M$$

for some fixed $M > 0$, then applying Theorem 3.1.3 we get

$$Re(f)(re^{i\theta}) = \int_{\mathbb{T}} P(r, \theta - t)\mathrm{d}\mu(t) \text{ and } Im(f)(re^{i\theta}) = \int_{\mathbb{T}} P(r, \theta - t)\mathrm{d}\nu(t)$$

for measures μ and ν on $\mathbb{T}$ respectively. Applying Theorem 3.1.2, gives

$$\lim_{r\to 1} Re(f)(re^{i\theta}) = m(\theta) \text{ a.e. } \mathrm{d}\theta$$

and

$$\lim_{r\to 1} Im(f)(re^{i\theta}) = n(\theta) \text{ a.e. } \mathtt{d}\theta$$

and therefore

$$\lim_{r\to 1} f(re^{i\theta}) = m(\theta) + in(\theta) \text{ a.e. } \mathtt{d}\theta$$

but $f_r(e^{i\theta}) = f(re^{i\theta})$, and because a subsequence converges to g, then $g = m + in$, so

$$\lim_{r\to 1} f(re^{i\theta}) = g(\theta) \text{ a.e } \mathtt{d}\theta.$$

Next, suppose that $f \in H^p(\Delta)$ and let us write as $f = BF$ where B and F are as in Theorem 4.2.1, then $F^{p/2} \in H^2(\Delta)$ and, because of what was done at the beginning of the proof for $f \in H^2(\Delta)$, has radial limits a.e. $\mathtt{d}\theta$ and this define a function in $L^2(\partial\Delta, \mathtt{d}t)$, denoted as $(F^{p/2})^*$.

Claim: B has radial limits a.e. $\mathtt{d}\theta$. In fact, setting

$$B_N = \prod_{j=1}^{N} \left(\frac{-\bar{z}_j}{|z_j|}\right)\left(\frac{z - z_j}{1 - \bar{z}_j z}\right),$$

then for $|z| = 1$,

$$\begin{aligned}
\left|\left(\frac{-\bar{z}_j}{|z_j|}\right)\left(\frac{z - z_j}{1 - \bar{z}_j z}\right)\right| &= \left|\frac{z - z_j}{1 - \bar{z}_j z}\right| \\
&= \left|\frac{1}{\bar{z}}\left(\frac{\bar{z}z - \bar{z}z_j}{1 - \bar{z}_j z}\right)\right| \\
&= \left|\frac{1 - \bar{z}z_j}{1 - \bar{z}_j z}\right| = 1
\end{aligned}$$

Thus $|B_N| = 1$ on $\partial\Delta$, $N \geq 1$. Also

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} |B_M - B_N|^2 \mathtt{d}\theta &= \frac{1}{2\pi}\int_{-\pi}^{\pi} [|B_M|^2 + |B_N|^2 + 2Re(B_N \bar{B}_M)\mathtt{d}\theta \\
&= 2\left[1 - Re\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{B_N}{B_M}\right)\right] \mathtt{d}\theta
\end{aligned}$$

and for $N > M$, $\frac{B_N}{B_M}$ is analytic, then it satisfies the mean value property, i.e.,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{B_N}{B_M}\mathtt{d}\theta = \left(\frac{B_N}{B_M}\right)(0) = \frac{\prod_{j=1}^{N}\left(\frac{-\bar{z_j}}{|z_j|}\right)\left(\frac{0-z_j}{1-\bar{z_j}0}\right)}{\prod_{j=1}^{M}\left(\frac{-\bar{z_j}}{|z_j|}\right)\left(\frac{0-z_j}{1-\bar{z_j}0}\right)} = \prod_{k=M+1}^{N}|z_k|.$$

Thus

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|B_M - B_N|^2\mathtt{d}\theta = 2\left(1 - \prod_{k=M+1}^{N}|z_k|\right)$$

since $\prod_{k=1}^{\infty}|z_k|$ converges then $B_N \to B$ in $L^2(\partial\Delta, \mathtt{d}\theta)$ and therefore a subsequence of the B_N converges a.e. $\mathtt{d}\theta$ to B on the circle, and this implies $|B| = 1$ and so our claim is proven.

These radial limits of B define a function B^* is in $L^\infty(\partial\Delta, \mathtt{d}\theta)$ and $|B^*| = 1$ a.e. $\mathtt{d}\theta$. Thus, $f \in H^p(\Delta)$ has radial limits a.e. $\mathtt{d}\theta$ and the limits define a function f^* which is in $L^p(\partial\Delta, \mathtt{d}\theta)$. This concludes the proof of items 1 and 2.

If we repeat what we did at beginning of the proof, we get $F_r^{p/2} \to (F^*)^{p/2}$ in $L^2(\partial\Delta, \mathtt{d}\theta)$, with F^* the radial limit of F. Thus,

$$\begin{aligned}
\limsup_{r\to 1}\{M_p(f;r)\} &\leq \limsup_{r\to 1}\{M_p(F;r)\} \\
&= \|F^*\|_p = \left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}|F^*(e^{it})|^p\mathtt{d}t\right\}^{1/p} \\
&= \left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(|F^*(e^{it})||B^*(e^{it})|\right)^p\mathtt{d}t\right\}^{1/p} \\
&= \left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}|f^*(e^{it})|^p\mathtt{d}t\right\}^{1/p} = \|f^*\|_p.
\end{aligned}$$

On the other hand, $|f_r|^p \to |f^*|^p$ a.e. $\mathtt{d}\theta$, so by Fatou's lemma

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi}\liminf\{|f(re^{it})|^p\colon r\to 1\}\mathtt{d}t &= \|f^*\|_p^p \\
&\leq \left(\liminf\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(re^{it})|^p\mathtt{d}t\colon r\to 1\right\}\right)^p \\
&= \left(\liminf\{M_p(f;r)\colon r\to 1\}\right)^p
\end{aligned}$$

therefore

$$\lim_{r\to 1} M_p(f_r; r) = \|f^*\|_p.$$

Now, define $g_r = |f_r - f^*|^p$ and $h_r = 2^p\left(|f^*|^p + |f_r|^p\right)$ then we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} h_r(e^{it})\mathtt{d}t \to 2^p\left(\frac{2}{2\pi}\int_{-\pi}^{\pi}|f^*(e^{it})|^p\mathtt{d}t\right) \text{ as } r\to 1$$

and $g_r \to 0$ as $r \to 1$. Then, by Theorem 4.17 of [8],

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} g_r(e^{it})\mathtt{d}t &= \frac{1}{2\pi}\int_{-\pi}^{\pi}|f_r(e^{it}) - f^*(e^{it})|^p\mathtt{d}t\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(re^{it}) - f^*(e^{it})|^p\mathtt{d}t \to 0 \text{ as } r\to 1.\end{aligned}$$

This conclude the proof of item 3.

To prove item 4, we do the following. Assume $f(re^{it}) \neq 0$, and take $\rho < 1$, using the fact that $\log|F(\rho z)|$ is harmonic on the closed disc we conclude

$$\begin{aligned}\log|f(\rho re^{i\theta})| &\leq \log|F(\rho re^{i\theta})|\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} P(r,\theta - t)\log|F(\rho e^{i\theta})|\mathtt{d}t\\ &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} P(r,\theta - t)\log\left(|F(\rho e^{i\theta})| + \epsilon\right)\mathtt{d}t\end{aligned}$$

Since $\log(|F(\rho e^{i\theta})| + \epsilon)$ is bounded below, Fatou's lemma can be used to justify

$$\limsup_{\rho\to 1}\left\{\frac{1}{2\pi}\int_{\pi}^{\pi} P(r,\theta - t)\log(|F(\rho e^{i\theta})| + \epsilon)\mathtt{d}\theta\right\} \leq \frac{1}{2\pi}\int_{\pi}^{\pi} P(r,\theta - t)\log(|F^*(e^{i\theta})| + \epsilon)\mathtt{d}\theta$$

Therefore

$$\begin{aligned}
\log|f(re^{i\theta})| &= \limsup\{\log|f(\rho re^{i\theta})|\colon \rho\to 1\}\\
&\le \limsup\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} P(r,\theta-t)\log\left(|F(\rho e^{i\theta})|+\epsilon\right)\mathtt{d}t\colon \rho\to 1\right\}\\
&\le \frac{1}{2\pi}\int_{-\pi}^{\pi}\limsup\{P(r,\theta-t)\log\left(|F(\rho e^{i\theta})|+\epsilon\right)\,:\,\rho\to 1\}\mathtt{d}t\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} P(r,\theta-t)\log\left(|F^*(e^{i\theta})|+\epsilon\right)\mathtt{d}t\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} P(r,\theta-t)\log\left(|f^*(e^{i\theta})|+\epsilon\right)\mathtt{d}t,\ \text{(since } |f^*|=|F^*|).
\end{aligned}$$

Now making $\epsilon\to 0$, we get the desired inequality. □

Chapter 5

Finitely Connected Planar Domains

In this chapter we will present the main results for a finitely connected planar domain analogous to those presented at the end of chapter 4 for the disk case.

5.1 Preliminaries for the main result

Let us take Ω a domain on the sphere whose complement relative to the sphere consists of exactly $m+1$ (closed) components, each of which is non-trivial. Then $m+1$ applications of the Riemann mapping theorem produces a one-to-one holomorphic map of Ω onto a bounded domain whose boundary consists of $m+1$ disjoint analytic simple closed curves. This holomorphic map induces an isometry of the corresponding H^p spaces. So we may assume that Ω is a bounded domain. Thus,

$$\Gamma = \Delta\Omega = \Gamma_0 \cup \cdots \cup \Gamma_m$$

where Γ_j is an analytic simple closed curve and $\Gamma_j \cap \Gamma_k = \phi$ if $j \neq k$. Let us set Γ_0 equal to the boundary of the unbounded component of the complement of Ω. Let

$$\mathcal{U}_0 = \text{ bounded component of } \mathbf{S}^2 \setminus \Gamma_0 \tag{5.1}$$

and

$$\mathcal{U}_j = \text{ unbounded component of } \mathbf{S}^2 \setminus \Gamma_j,\ j = 1, \ldots, m. \tag{5.2}$$

We notice if $m = 0$ then we are in the case of the unit disc Δ; from now on, we focus on the case $m \geq 1$.

Let us denote the set of rational functions whose poles are off $\Omega \cup \Gamma$ by $R(\Omega)$, and $A(\Omega)$ the set of functions which are continuous on $\Omega \cup \Gamma$ and analytic in Ω.

Proposition 5.1.1. *Let* $\mathcal{U}_0, \ldots, \mathcal{U}_m$ *be the domains defined by* (5.1) *and* (5.2). *If* $f \in H^p(\Omega)$, *then*

$$f = f_0 + \cdots + f_m \text{ on } \Omega \tag{5.3}$$

where $f_j \in H^p(\mathcal{U}_j)$ *for* $0 \leq j \leq m$.

Proof. Following the same ideas of Lemma A9 in page 224 in [11], given $\epsilon > 0$, and considering

$$t_i : [-\pi, \pi] \to \mathbb{C},\ \text{parametrization of } \Gamma_i,\ i = 0, \ldots, m,$$

we can get

$$s_i : [-\pi, \pi] \to \mathbb{C},\ i = 0, \dots, m$$

a smooth map such that

$$\|t_i - s_i\|_\infty \leq \epsilon.$$

If we let $C_i = t_i([-\pi, \pi])$, then C_i is a smooth simple closed curve, $i = 0, \dots, m$. Now take $z \in \Omega$ exterior to $C_1, \dots, C_m$ and interior to C_0. Let

$$f_k(z) = \frac{1}{2\pi} \int_{C_k} \frac{f(w)}{w - z} \mathtt{d}w,\ k = 0, \dots, m.$$

If we take any simple closed curve homotopic to C_k then f_k takes the same value. Therefore f_k is independent of the choice of C_k, and f_k is holomorphic in $\mathcal{U}_k$ for $k = 0, \dots, m$, and $f_k(\infty) = 0$ for $k = 1, \dots, m$. Moreover an application of Cauchy's formula shows that (5.3) is satisfied. Now fix k and take $j \neq k$ and $\mathcal{O}$ a neighborhood of Γ_k. For $z \in \mathcal{O}$

$$\inf_{w \in C_j, z \in \mathcal{O}} \{|z - w|\} = a > 0$$

so

$$|f_j(z)| \leq \frac{1}{2\pi} \frac{1}{a}\ l(C_k) \max_{w \in C_k} \{|f(w)|\} = M_j,\ z \in \mathcal{O}$$

(where $l(C_k)$ is the arc length of C_k), therefore f_j is bounded in $\mathcal{O}$. From (5.3) we deduce

$$|f_k(z)| \leq |f(z)| + \sum_{j=1, j \neq k}^{m} |f_j(z)| \leq |f(z)| + mM$$

where

$$M = \max_{1 \leq j \leq m;\, j \neq k} \{M_j\}.$$

Thus

$$\begin{aligned} |f(z)|^p &\leq (|f(z)| + mM)^p \\ &\leq 2^p\,[|f(z)|^p + (mM)^p] \\ &\leq 2^p\,[|u_f(z)| + (mM)^p] = h(z),\ z \in \mathcal{O} \end{aligned}$$

where h is harmonic, therefore $f_k \in H^p(\mathcal{O})$.

Now, for z not in $\mathcal{O}$, we have

$$\inf_{w \in C_k, z \notin \mathcal{O}} \{|z-w|\} = b > 0$$

thus

$$|f_k(z)| \leq \frac{1}{2\pi}\frac{1}{b}\, l(C_k) \max_{w \in\ C_k} \{|f(w)|\} = N,\ z \notin \mathcal{O}$$

which implies

$$|f_k(z)|^p \leq h(z)\chi_{\mathcal{O}}(z) + N^p \chi_{\mathcal{U}_k \setminus \mathcal{O}}(z) = r(z)$$

with r is harmonic, so $f_k \in H^p(\mathcal{U}_k)$. ☐

Proposition 5.1.2. *If $1 \leq p < \infty$, $R(\Omega)$ is dense in $H^p(\Omega)$ and boundedly pointwise dense in $H^\infty(\Omega)$; $R(\Omega)$ is uniformly dense in $A(\Omega)$.*

Proof. Fix j, then $\Omega \subset \mathcal{U}_j$, and if $h \in H^p(\mathcal{U}_j)$, we have

$$h(z) \leq \widehat{u_h}(z),\ \text{for } z \in \mathcal{U}_j$$

where $\widehat{u_h}$ is the least harmonic majorant of h in $\mathcal{U}_j$. This implies

$$h(z) \leq \widehat{u_h}(z),\ \text{for } z \in \Omega$$

with $\widehat{u_h}$ harmonic on $\mathcal{U}_j$ (and therefore on Ω), so $h \in H^p(\Omega)$ and

$$u_h(z) \leq \widehat{u_h}(z), \text{ for } z \in \Omega$$

which implies that the $H^p(\mathcal{U}_j)$ norm is larger than the $H^p(\Omega)$ norm.

Now by (5.3), we have

$$f = f_0 + \cdots + f_m \text{ on } \Omega.$$

By the analysis at the beginning of the proof we see that it is sufficient to show that for each $j = 1, \ldots, m$, f_j is the limit in $H^p(\mathcal{U}_j)$ of a sequence of functions holomorphic in a neighborhood of $\mathcal{U}_j \cup \Gamma_j$. For this purpose, let ϕ be the Riemann mapping of $\mathcal{U}_j$ onto Δ. Since Γ_j is analytic, this mapping can be extended continuously to the boundary of $\mathcal{U}_j$ by Theorem 14.19 in [9]. Also $\phi(z) \to 1$ as z tend to Γ_j, for $z \in \mathcal{U}_j$. Therefore by Theorem in page 286 of [10], we can extend ϕ analytically and one-to-one in a neighborhood of $\mathcal{U}_j \cup \Gamma_j$. Moreover, $g_j = f_j \circ \phi^{-1}$ is in $H^p(\Delta)$ and therefore, by Runge's theorem, there is a function G analytic on a neighborhood of $\Delta \cup \mathbb{T}$, with $\|G - g_j\| < \epsilon$, in $H^p(\Delta)$. Thus

$$\|f_j - G \circ \phi\| < \epsilon, \text{ in } H^p(\mathcal{U}_j)$$

and $G \circ \phi$ is analytic in a neighborhood of $\mathcal{U}_j \cup \Gamma_j$. Now applying one more time Runge's theorem we can approximate $G \circ \phi$ uniformly on $\mathcal{U}_j \cup \Gamma_j$ by elements of $R(\Omega)$ (and therefore approximate f_j).

For $p = \infty$, we follows the same ideas and we get $g_j \in H^\infty(\Delta)$, and, applying Runge's theorem, there are rational functions G_{jn}, $n = 1, 2, \ldots$ with no poles in $\Delta \cup \mathbb{T}$ such that

$$|G_{jn}(z) - g_j(z)| < 1/n, \text{ for } z \in \mathbb{T}$$

which implies

$$\|G_{jn}\|_{\mathbb{T}} \leq \|g_j\|_{\mathbb{T}}, \text{ and } \lim_{n\to\infty} G_{jn}(z) = g_j(z),\ z \in \Delta$$

and so the functions $F_{jn} = G_{jn} \circ \phi$ are holomorphic in a neighborhood of $\mathcal{U}_j \cup \Gamma_j$ and

$$\sup_{w\in\Gamma_j} |(G_{jn} \circ \phi)(z)| = \|G_{jn} \circ \phi\|_{\mathbb{T}} \leq \|f_j\|_{\mathbb{T}}$$

which tell us

$$\lim_{n\to\infty} F_j(z) = \lim_{n\to\infty} (G_{jn} \circ \phi)(z) = f_j(z),\ z \in \mathcal{U}_j$$

then, applying Runge's theorem, we can get a sequence of functions $R_{nj} \in R(\Omega)$ such that

$$\lim_{n\to\infty} R_{nj}(z) = f_j(z),\ z \in \mathcal{U}_j.$$

Finally, we notice that if $f \in A(\Omega)$ then $f_j \in A(\mathcal{U}_j \cup \Gamma_j)$, $j = 0, \ldots, m$. So, doing a process like the one above we can get a sequence of polynomials $\{p_{jn}\}$ with $p_{jn}\left(\frac{1}{z-a_j}\right) \to f_j$ uniformly on Γ_j and hence uniformly on Γ, where a_j is in the bounded component of the complement of Γ_j, $j = 1, \ldots, m$. Also we can get a sequence $\{p_{0n}\}$ of polynomial such that

$$p_{0n} \to f_0 \text{ uniformly on } \Gamma_0$$

and therefore uniformly on Γ. Set

$$\sum_{j=1}^{m} p_{jn}\left(\frac{1}{z-a_j}\right) + p_{0n}(z) = q_n(z)$$

then $q_n \in R(\Omega)$ and $q_n \to f_0 + \cdots + f_m = f$ uniformly on Γ. □

Proposition 5.1.3. *If $u \in L^1(\Gamma, ds)$ and*

$$\int_\Gamma \frac{u(\zeta)}{\zeta - z} d\zeta = 0,\ z \notin \Gamma$$

then $u = 0$ *a.e.* $\mathtt{d}s$.

Proof. Fixed j, let

$$g_j(z) = \int_{\Gamma_j} \frac{u(\zeta)}{\zeta - z} \mathtt{d}\zeta, \; z \notin \Gamma_j, \; j = 0, \dots, m \tag{5.4}$$

Then g_j is holomorphic off Γ_j and $g_j(\infty) = 0$. Also from (5.4) we have

$$g_0 + \cdots + g_m = 0, \text{ off } \Gamma.$$

Also

$$g_j = g_0 + \cdots + g_{j-1} + g_{j+1} + \cdots + g_m$$

and each g_k, with $k \neq j$, is holomorphic on Γ_j, therefore g_j is holomorphic on Γ_j and so, by Liouville's theorem, it has to be constant. But $g_j(\infty) = 0$, and hence $g_j \equiv 0$. Now that we have

$$0 = g_j(z) = \int_{\Gamma_j} \frac{u(\zeta)}{\zeta - z} \mathtt{d}\zeta$$

then

$$0 = g_j^{(n)}(z) = \int_{\Gamma_j} \frac{u(\zeta)}{(\zeta - z)^n} \mathtt{d}\zeta.$$

Fixed $z_0 \notin \Gamma_j$, considering h analytic function in some neighborhood of Γ_j containing z_0, we can get a sequence of polynomial P_{jn} such that

$$P_{jn}\left(\frac{1}{z - z_0}\right) \to h, \text{ uniformly on } \Gamma_j$$

and

$$P_{jn}(z) = \lim_{q \to \infty} \sum_{n=0}^{q} a_{qjn} z^n.$$

So

$$\int_{\Gamma_j} u(\zeta) P_{jn}\left(\frac{1}{\zeta - z_0}\right) \mathtt{d}\zeta = \lim_{q\to\infty} \sum_{n=0}^{q} a_{qjn} \int_{\Gamma_j} \frac{u(\zeta)}{(\zeta - z_0)^n} \mathtt{d}\zeta = 0$$

and therefore

$$\int_{\Gamma_j} u(\zeta) h(\zeta) \mathtt{d}\zeta = 0.$$

This implies

$$\int_{\mathbb{T}} u(\varphi(e^{it})) H(e^{it}) \mathtt{d}t = 0$$

where φ is holomorphic and one-to-one in some neighborhood of $\mathbb{T}$, mapping $\mathbb{T}$ onto Γ_j, and H is analytic in some neighborhood of $\mathbb{T}$. Now taking $H(e^{it}) = e^{it}$ for $n = \pm 1, \pm 2, \ldots$, we get $u \circ \varphi$ a.e. $\mathtt{d}t$ on $\mathbb{T}$ and so $u = 0$ a.e. $\mathtt{d}s$ on Γ_j. Since we started the proof with any fixed j then $u = 0$ a.e. $\mathtt{d}s$ on Γ. $\square$

5.2 Main Result

Let $z \in \Omega$ at which the $H^p(\Omega)$ norm is determined and let ω the harmonic measure on Γ for z. Now we are ready for our main result.

Theorem 5.2.1. ***(Main Result)***

Each $f \in H^p(\Omega)$ has boundary values f^* almost everywhere $(d\omega)$ on Γ and $f^* \in L^p(\Gamma, \omega)$. Moreover

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f^*(w)}{w - z} dw, \; z \in \Omega, \tag{5.5}$$

$$\int_{\Gamma} \frac{f^*(w)}{w - z} dw = 0, \; z \notin \Omega \cup \Gamma, \tag{5.6}$$

$$f(z)=\int_{\Gamma} f^*(\zeta)\,d\omega_z(\zeta),\ z\in\Omega, \tag{5.7}$$

and the mapping $f\mapsto f^*$ *is an isometry of* $H^p(\Omega)$ *on to a closed subspace of* $L^p(\Gamma,\omega)$*.*

Proof. By the decomposition (5.3), we see that it is enough to prove that f_j has boundary values a.e. $\mathtt{d}s$ on Γ and that this boundary-value functions lies in $L^p(\Gamma)$.

Fixed j, for $k\neq j$, f_j is actually analytic on Γ_k because of the way f_j was defined in Proposition 5.1.1, so (5.5), (5.6) and (5.7) hold immediately. Let us focus on Γ_j. Let ϕ be the Riemann mapping of $\mathcal{U}_j$ onto Δ. So, for the same reasons given in Proposition 5.1.2, ϕ can be extended to be analytic on a neighborhood of $\mathcal{U}_j\cup\Gamma_j$ and $g_j=f_j\circ\phi^{-1}\in H^p(\Delta)$ and so, by Theorem 4.2.3, g_j has boundary values g_j^* a.e. $\mathtt{d}\theta$ on $\mathbb{T}$, and $g_j^*\in L^p(\mathbb{T},\mathtt{d}\theta)$. Therefore $f_j=g_j\circ\phi$ has boundary values $f_j^*=g_j^*\circ\phi^*=g_j^*\circ\phi$ a.e. because $\phi=\phi^*$ on Γ_j. So $f_j^*\in L^p(\Gamma_j,\mathtt{d}s)$, and, because of our observation at the beginning of the paragraph $f_j^*\in L^p(\Gamma_k,\mathtt{d}s)$ for $k\neq j$. So $f_j^*\in L^p(\Gamma,\omega)$.

If $z\in\Omega$, then

$$f_j(z)=g_j(\phi(z))=\frac{1}{2\pi i}\int_{|\xi|=1}\frac{g_j^*(\xi)}{\xi-\phi(z)}\mathtt{d}\xi.$$

Making $\xi=\phi(\zeta)$ we have

$$\begin{aligned} f_j(z) &= g_j(\phi(z)) \\ &= \frac{1}{2\pi i}\int_{\Gamma_j}\frac{g_j^*(\phi(\zeta))}{\phi(\zeta)-\phi(z)}\phi^{'}(\zeta)\mathtt{d}\zeta \\ &= \frac{1}{2\pi i}\int_{\Gamma_j}\frac{f_j^*(\zeta)}{\phi(\zeta)-\phi(z)}\phi^{'}(\zeta)\mathtt{d}\zeta \end{aligned}$$

we notice that

$$\frac{\phi'(\zeta)}{\phi(\zeta)-\phi(z)} = \frac{1}{\zeta - z} + S(\zeta)$$

where S (depends on the choice of z) is analytic in a neighborhood of $\Omega \cup \Gamma$, since the function in the left-hand side of the equality has a simple pole at z with residue equal to 1. Then

$$f_j(z) = \frac{1}{2\pi i}\int_{\Gamma_j} f_j^*(\zeta)\left(\frac{1}{\zeta - z} + S(\zeta)\right)\mathtt{d}\zeta.$$

Let $f_{j,r}(z) = g_j(r\phi(z))$ for $r < 1$, then, by Cauchy's theorem, we have

$$\int_{\Gamma_j} f_{j,r}(\zeta)S(\zeta)\mathtt{d}\zeta = 0.$$

Because of Theorem 4.2.3, we also have

$$\lim_{r\to 1} g_j(r\cdot) \to g_j^*(\cdot), \text{ in } L^p(\mathbb{T}, \mathtt{d}s)$$

which implies

$$\lim_{r\to 1} f_{j,r} = f_j^* \text{ in } L^p(\Gamma_j, \mathtt{d}s)$$

(and also in $L^1(\Gamma_j, \mathtt{d}s)$, since $p \geq 1$, and Γ_j is compact). Therefore

$$\int_{\Gamma_j} f_j^*(\zeta)S(\zeta)\mathtt{d}\zeta = \lim_{r\to 1}\int_{\Gamma_j} f_{j,r}(\zeta)S(\zeta)\mathtt{d}\zeta = 0.$$

Hence

$$f_j(z) = \frac{1}{2\pi i}\int_{\Gamma_j}\frac{f_j^*(\zeta)}{\zeta - z}\mathtt{d}\zeta, \ z \in \Omega.$$

Also

$$\int_{\Gamma_k}\frac{f_j^*(\zeta)}{\zeta - z}\mathtt{d}\zeta = \int_{\Gamma_k}\frac{f_j(\zeta)}{\zeta - z}\mathtt{d}\zeta = 0, \ k \neq j$$

since f_j is analytic on Γ_k. So for $z \notin \Omega \cup \Gamma$

$$\int_\Gamma \frac{f_j^*(\zeta)}{\zeta - z}\mathtt{d}\zeta = \int_{\Gamma_j} \frac{f_j^*(\zeta)}{\zeta - z}\mathtt{d}\zeta + \sum_{k=0,k\neq j}^{m} \int_{\Gamma_k} \frac{f_j^*(\zeta)}{\zeta - z}\mathtt{d}\zeta = 0.$$

To prove (5.7), remember from Theorem 4.1.2 that

$$\mathtt{d}\omega_z(\zeta) = \frac{i}{2\pi} Q_z^{'}(\zeta)\mathtt{d}\zeta$$

where $Q_z(\zeta) = g(\zeta; z) + ih(\zeta; z)$ ($g(\zeta; z)$ is the Green's function for Ω with pole at z, and $h(\zeta; z)$ is its harmonic conjugate). Then

$$Q_z^{'}(\zeta) = \frac{1}{z - \zeta} + R(\zeta) \tag{5.8}$$

where R is holomorphic in a neighborhood of $\Omega \cup \Gamma$ (and, because of the same reasoning for S above, $\int_\Gamma f_j^*(\zeta) R(\zeta)\mathtt{d}\zeta = 0$). So

$$\begin{aligned} \int_\Gamma f^*(\zeta)\mathtt{d}\omega_z(\zeta) &= \int_\Gamma f^*(\zeta)\frac{i}{2\pi}Q_z^{'}(\zeta)\mathtt{d}\zeta \\ &= -\frac{i}{2\pi}\int_\Gamma \frac{f^*(\zeta)}{\zeta - z}\mathtt{d}\zeta + \frac{i}{2\pi}\int_\Gamma f^*(\zeta)R(\zeta)\mathtt{d}\zeta \\ &= f(z) \end{aligned}$$

because (5.5) already holds.

Finally, for $1 \le p < \infty$, in order to show that the mapping $f \mapsto f^*$ is an isometry, let $q \in R(\Omega)$ and let $u(z)$ the harmonic extension of the continuous function $|q(z)|^p$ in Ω. Then

$$u(z) = \int_\Gamma |q(\zeta)|^p \mathtt{d}\omega_z(\zeta) \tag{5.9}$$

then q satisfies (5.7), and, by Hölder's inequality, we get

$$|q(z)|^p \le u(z).$$

Moreover if v is any harmonic majorant of $|q|^p$ we have

$$u(x) = |q(x)|^p \leq \liminf\{v(z)\colon z \to x\},\ x \in \Gamma$$

so the harmonic function $v - u$ is non-negative on Γ and hence on all Ω, therefore u given by (5.9) is the least harmonic majorant of $|q|^p$ if $q \in R(\Omega)$ and clearly in this case

$$\|q\|_{L^p(\Gamma,\omega)} = \|q\|_{H^p(\Omega)}.$$

Now take $f \in H^p(\Omega)$. Since $R(\Omega)$ is dense in $H^p(\Omega)$ by Proposition 5.1.2, we can take $\{q_n\}$ to be a sequence in $R(\Omega)$ converging to f (in $H^p(\Omega)$). Then $\{q_n\}$ converges to f uniformly on compact subsets of Ω. Even more,

$$\|q_n - q_m\|_{H^p(\Omega)} = \|q_n - q_m\|_{L^p(\Gamma,\omega)}$$

by the foregoing, thus $\{q_n\}$ is a Cauchy sequence in $L^p(\Gamma, \omega)$, and therefore convergent. Let

$$g = \lim_{n\to\infty} q_n,\ \text{in } L^p(\Gamma, \omega)$$

then, since all harmonic measures are boundedly mutually absolutely continuous, we may take the limit as $n \to \infty$ in the formula

$$q_n(z) = \int_\Gamma q_n(\zeta)\mathrm{d}\omega_z(\zeta)$$

to set

$$f(z) = \int_\Gamma g(\zeta)\mathrm{d}\omega_z(\zeta)$$

from which we also have

$$f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{g(\zeta)}{\zeta - z}\mathrm{d}\zeta,\ z \in \Omega$$

and

$$\int_\Gamma \frac{g(\zeta)}{\zeta - z}\mathsf{d}\zeta = \int_\Gamma \frac{f^*(\zeta)}{\zeta - z}\mathsf{d}\zeta = 0,\ z \notin \Omega \cup \Gamma$$

thus

$$\int_\Gamma \left(\frac{g(\zeta) - f^*(\zeta)}{\zeta - z}\right)\mathsf{d}\zeta = 0,\ \text{for } z \notin \Gamma,\ \text{with } g - f^* \in L^p(\Gamma, \omega)$$

and $L^p(\Gamma, \omega) \subset L^1(\Gamma, \omega)$, since $p \geq 1$ and Γ is compact, then by Proposition 5.1.3, $g = f^*$ a.e. $\mathsf{d}\omega$. Consequently $f_n \to f^*$ in $L^p(\Gamma, \omega)$ and, therefore

$$\|f^*\|_{L^p(\Gamma,\omega)} = \lim_{n\to\infty} \|f_n\|_{L^p(\Gamma,\omega)} = \lim_{n\to\infty} \|f_n\|_{H^p(\Omega)} = \|f\|_{H^p(\Omega)}.$$

For $p = \infty$, if $f \in H^\infty(\Omega)$, then, because of (5.7), $|f(z)| \leq \|f^*\|_{L^\infty(\Gamma,\omega)}$ which implies

$$\|f\|_{H^\infty(\Omega)} \leq \|f^*\|_{L^\infty(\Gamma,\omega)}.$$

Also

$$\|f\|_{H^\infty(\Omega)} \geq \limsup_{p\to\infty} \|f\|_{H^p(\Omega)} = \limsup_{p\to\infty} \|f^*\|_{L^p(\Gamma,\omega)} = \|f^*\|_{L^\infty(\Gamma,\omega)}.$$

So

$$\|f\|_{H^\infty(\Omega)} = \|f^*\|_{L^\infty(\Gamma,\omega)}.$$

□

An immediate consequence of this theorem is the following.

Corollary 5.2.2. *If $f \in H^p(\Omega)$, $1 \leq p < \infty$, then*

$$u_f(z) = \int_\Gamma |f^*(\zeta)|^p \mathsf{d}\omega_z(\zeta),\ z \in \Omega.$$

Corollary 5.2.3. *If $f \in H^1(\Omega)$, f not identically zero, then* $\log|f^*(\zeta)|$ *is in* $L^1(\Gamma, \omega)$ *and*

$$\log|f(z)| \leq \int_\Gamma \log|f^*(\zeta)| d\omega_z(\zeta),\ z \in \Omega.$$

Proof. The first part of theorem can be obtained by realizing that $f^* \in L^1(\Gamma, \mathrm{d}\omega)$ (because of Theorem 5.2.1), and therefore using Jensen Inequality we get what we want. The second part can be obtained by getting the result for a function $f \in R(\Omega)$ and then use Theorem 5.1.2 to get the inequality when $f \in H^1(\Omega)$. ☐

Theorem 5.2.1 tells that $H^p(\Omega)$ is isometrically isomorphic to a closed subspace of L^p. The next result will tell us which L^p functions are boundary values of $H^p(\Omega)$ functions.

Theorem 5.2.4. *Let $f \in L^p(\Gamma, \omega)$, $1 \leq p \leq \infty$. There is an $F \in H^p(\Omega)$ with $F^* = f$ a.e. ω if and only if*

$$0 = \int_\Gamma \frac{f(\zeta)}{\zeta - w} d\zeta,\ \textit{for all}\ w \notin \Omega \cup \Gamma. \tag{5.10}$$

Proof. It can be proved that

$$F(z) = \int_\Gamma f(\zeta) \mathrm{d}\omega_z(\zeta),\ z \in \Omega$$

is harmonic, since f can be approximated by continuous functions f_n and the corresponding function f_n^*. Also by Hölder's inequality

$$|F(z)|^p \leq \int_\Gamma |f(\zeta)|^p \mathrm{d}\omega_z(\zeta),\ 1 \leq p < \infty$$

and

$$|F(z)| \leq \|f\|_\infty,\ p = \infty$$

thus F has a harmonic majorant. Moreover, by (5.8)

$$\mathtt{d}\omega_z(\zeta) = \frac{1}{2\pi i}\frac{\mathtt{d}\zeta}{\zeta - z} + R(\zeta)\mathtt{d}\zeta$$

where R is holomorphic in a neighborhood of $\Omega \cup \Gamma$. This implies

$$F(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z}\mathtt{d}\zeta,\ z \in \Omega$$

and therefore F is analytic in Ω and hence $F \in H^p(\Omega)$. By Theorem 5.2.1 F^* exists a.e., using (5.10) and Proposition 5.1.3, again implies $F^* = f$ a.e. $\mathtt{d}s$.

The converse is just (5.7). □

Finally we can characterize F^* (for $F \in H^p(\Omega)$) in terms of the measure ω. For this purpose we start by fixing a point $q \in \Omega$, then consider $g(z;q)$ the Green's function for q in Ω and $h(z;q)$ its corresponding harmonic conjugate, then Theorem 4.1.3 says that Q' ($Q = g + ih$) has precisely m zeros in Ω (counting multiplicity). Let $z_1^*, \ldots, z_m^*$ be such zeros and set

$$P(z) = \prod_{j=1}^{m}(z - z_j^*).$$

Theorem 5.2.5. *Let $f \in L^p(\Gamma, \omega)$. Then*

$$\int_\Gamma f(\zeta)h^*(\zeta)\,\mathtt{d}\omega(\zeta) = 0,\ \textit{all}\ h \in H^\infty(\Omega)\ \textit{with}\ h(q) = 0 \tag{5.11}$$

if and only if there is $F \in H^p(\Omega)$ such that

$$F^* = fP\ \textit{a.e.}\ \mathtt{d}\omega\ \textit{on}\ \Gamma. \tag{5.12}$$

Proof. Suppose (5.12) is satisfied. Take $h \in H^p(\Omega)$ with $h(q) = 0$, then using Theorem 4.1.2. we have

$$-2\pi i \int_\Gamma f h^* \mathtt{d}\omega = \int_\Gamma \frac{F^*(\zeta)h^*(\zeta)}{P(\zeta)} Q^{'}(\zeta)(\zeta - q)\frac{\mathtt{d}\zeta}{\zeta - q}.$$

But

$$K(z) = \frac{Q^{'}(z)(z-q)}{P(z)}$$

is analytic and single-valued in a neighborhood of $\Omega \cup \Gamma$ since P and $Q^{'}$ has the same zeros. Moreover, because (5.8), K is zero-free in $\Omega \cup \Gamma$. Thus

$$-\int_\Gamma f h^* \mathtt{d}\omega = \frac{1}{2\pi i}\int_\Gamma F^*(\zeta)h^*(\zeta)K(\zeta)\frac{\mathtt{d}\zeta}{\zeta - q} = F(q)K(q)h(q) = 0.$$

Conversely, if (5.11), take any $\hat{h} \in H^\infty(\Omega)$, with $\hat{(h)}(q) = 0$, then we can write

$$\hat{h}(\zeta) = (\zeta - q)\hat{\hat{h}}(\zeta),\ \hat{\hat{h}} \in H^\infty(\Omega).$$

Using Theorem 4.1.2. one more time, we have

$$0 = \int_\Gamma f(\zeta)(\hat{\hat{h}})^*(\zeta)(\zeta - q)Q^{'}(\zeta)\mathtt{d}\zeta = \int_\Gamma f(\zeta)(\hat{\hat{h}})^*(\zeta)K(\zeta)P(\zeta)\mathtt{d}\zeta$$

In particular taking $\hat{\hat{h}}(\zeta) = \frac{1}{\zeta - w}$, $w \notin \Omega \cup \Gamma$, we have

$$0 = \int_\Gamma \frac{f(\zeta)K(\zeta)P(\zeta)}{\zeta - w}\mathtt{d}\zeta,\ w \notin \Omega \cup \Gamma.$$

Because of Theorem 5.2.4, there is an $V \in H^p(\Omega)$ with $V^*(\zeta) = f(\zeta)K(\zeta)P(\zeta)$ a.e. ω. Then take $F = \frac{V}{K}$, we still have $F \in H^p(\Omega)$ since K is zero-free. With this selection of F we have the desired conclusion. □

5.3 Final Comments

Let us remember how N was defined in the introduction. Take Ω a finitely-connected planar domain $\Gamma = \partial\Omega = \Gamma_0 \cup \Gamma_1 \cup \cdots \cup \Gamma_m$ and $R(\Omega)$ the set of rational functions whose poles are off $\Omega \cup \Gamma$, then there are m linear independent measures $\nu_1, \ldots, \nu_m$ on Γ orthogonal to $ReR(\Omega)$ and of the form

$$\mathtt{d}\nu_j = Q_j \mathtt{d}\omega_q, \ 1 \leq j \leq m$$

where Q_j is C^∞ on Γ, Q_j is nonnegative on Γ_j and nonpostive on Γ_k, $k \neq j$. It can be proven that

$$Q_j \mathtt{d}\omega_q = \left(\frac{\partial h_j}{\partial n}\right) \mathtt{d}s \ \left(\frac{\partial}{\partial n} \text{ is the derivative in the direction of the outward normal at } \Gamma\right),$$

where h_j is the solution of the Dirichlet problem with boundary value 1 on Γ_j and 0 on Γ_k, $k \neq j$. If $N = 0$, there do not exist nonzero measures orthogonal to $ReR(\Omega)$, and therefore $ReR(\Omega)$ is uniformly dense on $A(\Omega)$. Then $A(\Omega)$ is called a Dirichlet algebra. Also P, in Theorem 5.2.5, equals 1, and so Theorem 5.2.5 can be restated in the following way. Let $f \in L^p(\Gamma, \omega)$. Then

$$\int_\Gamma f(\zeta) h^*(\zeta) \mathtt{d}\omega(\zeta) = 0, \text{ all } h \in H^\infty(\Omega) \text{ with } h(q) = 0$$

if and only if there is $F \in H^p(\Omega)$ such that

$$F^* = f \text{ a.e. } \mathtt{d}\omega \text{ on } \Gamma.$$

For $1 \leq p \leq \infty$. Let us denote by $H^p(\Gamma)$ the closed subspace of $L^p(\Gamma, \omega)$ consisting of boundary values of $H^p(\Omega)$ functions and let $H_0^p(\Gamma)$ be the space of functions f of $H^p(\Gamma)$ with

$f(q) = 0 = \int_\Gamma f \mathtt{d}\omega$. Also let $\bar{H}_0^p(\Gamma)$ denote the complex conjugates of the elements of $H_0^p(\Gamma)$. It can be proven

$$H^p(\Gamma) + \bar{H}_0^p(\Gamma) + N \text{ is dense in } L^p(\Gamma, \omega),\ 1 \leq p < \infty,$$

when $p = 2$ it can be proven more, namely

$$H^2(\Gamma) \bigoplus \bar{H}_0^2(\Gamma) \bigoplus N = L^2(\Gamma, \omega),$$

and for $p = \infty$

$$H^\infty(\Gamma) + \bar{H}_0^\infty(\Gamma) + N \text{ is weak-star dense in } L^\infty(\Gamma, \omega).$$

Finally P in Theorem 5.2.5 can be related to N in the following way:

$$P(N + H^\infty(\Gamma)) = H^\infty(\Gamma).$$

Bibliography

[1] Stephen D. Fisher, Function theory on planar domain: a second course in complex analysis, Dover, 1983.

[2] Kenneth Hoffman, Banach spaces of analytic functions, Dover, 1990.

[3] Edwin Hewitt and Karl Stromberg, Real and abstract analysis, Spring-Verlag New York,Inc. 1965.

[4] John B. Conway, Function of one complex variable, Second edtion, Spring-Verlag New York, Inc. 1978.

[5] John B. Conway, Function of one complex variable II, Spring-Verlag New York, Inc. 1995.

[6] Jerrold E. Marsden and Michael J. Hoffman, Basic complex analysis, Third edition, W. H. Freeman New York, 1999.

[7] Steven G. Krantz, Function theory of several complex variables, Second edition, American Mathematical Society, 1992.

[8] H.L. Royden, Real Analysis, Third Edition, Prentice-Hall, Inc. 1988.

[9] Walter Rudin, Real and Complex Analysis, Third edition, McGraw-Hill Companies Inc. 1987.

[10] Theodore W. Gamelin, Complex Analysis, Springer-Verlag New York, Inc. 2001.

[11] Ib Madsen and Jorgen Tornehave, From Calculus to Cohomology: de Rham cohomology and characteristic classes, Cambridge University Press, 1997.

[12] Lars V. Ahlfors, Complex Analysis, Second edition, International student edtion, McGraw-Hill Book Company, 1966.

[13] I.M. Singer and J.A. Thorpe, Lectures Notes on Elementary Topology and Geometry, Springer-Verlag New York-Heidelberg-Berlin, 1967.

[14] George F. Simmons, Introduction to Topology and Modern Analysis, McGraw-Hill New York, 1963.

[15] Peter A. Fillmore, Notes on Operator Theory, Litton Educational Publishing, Inc. 1970.

[16] Michael Reed and Barry Simon, Methods of Modern Mathematical Physics Vol. 1: Functional Analysis, Revised and Enlarged Edition, Academic Press, 1980.