Chapter 3

METHODOLOGY

The present research incorporates data envelopment analysis, goal programming and fuzzy set theory. The literature reports extensive work in the three fields. However, the framework developed in this thesis draws upon the strengths of the three approaches to provide a tool to facilitate decision-making in a fuzzy environment.

In most real life situations the decision-maker generally seeks to *satisfy* aspiration levels of goals within a certain tolerance limit rather than *optimize* crisp goals. Sengupta (1992) first developed a fuzzy DEA framework for efficiency measurement where he introduced fuzziness in the objective function and the resource (right hand side) vector and applied it to the Charnes, Cooper and Rhodes (1978) model. However, Sengupta’s (1992) theoretical concepts were not implemented with real data. Furthermore, his formulation only extends conventional DEA to a fuzzy environment. Girod (1996) developed fuzzy models to measure efficiency in a fuzzy environment employing fuzzy inputs and outputs. However, limited research has been done in the area of achieving multiple organizational goals with multiple decision-making levels in a fuzzy environment. In such scenarios optimal achievement of all goals is rarely possible. The decision-maker usually chooses to achieve goals as close to optimal as possible or in some order of priority.

Under this scenario the decision-maker can avail of the fuzzy model developed in this research which allows fuzziness in the aspiration levels. This fuzzy model is formulated using Athanassopoulos' (1995) framework. First, a reformulation of Athanassopoulos' (1995) model with slight modifications is presented. Second, the membership functions employed in this research are defined. Third, the reformulated model derived from Athannasopoulos (1995) is then fuzzified using techniques from fuzzy goal programming. The Fuzzy GoDEA model is followed by variations that capture different scenarios related to decision-making. Finally, an interpretation of the membership functions related to a production environment is described.

3.1 REFORMULATION OF THE ATHANASSOPOULOS (1995) GODEA MODEL

Athanassopoulos (1995) developed a framework for multi-objective multi-level planning where a central coordinating entity controls and/or allocates global resources between relatively homogenous decision-making units. This framework is used to reformulate the GoDEA model for the current research. The central coordinating entity desires the maximization of pre-specified input/output global targets. To achieve this goal, the individual DMUs are expected to maximize their contribution toward achievement of the global organizational targets. Global organizational targets are reflected in the objectives of efficiency (contribution of individual DMUs to individual targets), effectiveness (achievement of global organizational targets), and operational viability (maintaining minimal resource allocation of individual DMUs). This framework gives rise to issues regarding the assessment of global and individual DMU targets and the reallocation of resources among the DMUs. The challenge for the decision-maker is to decide the relative importance of achievement of the objectives of efficiency, effectiveness and operational viability depending on the level of decision-making. In reality, the optimal achievement of any one of the three objectives cannot guarantee the best use of available resources. Therefore, the decision-maker is likely to choose a compromise between pre-specified satisfaction levels of the objectives. At a given decision-making level in the organizational hierarchy the decision-maker can prioritize the achievement of objectives according to relative importance. A crisp formulation does not allow linguistic specifications such as "essentially satisfied" or "approximately satisfied". The need for such *imprecise* specification of multiple organizational goals with varying relative importance in a hierarchical system motivates the fuzzy model formulation in this research. The model in this research is developed for a two-level hierarchy where the global and individual DMU targets are known *a priori* (*e.g.*, historical process knowledge).

Athanassopoulos' (1995) GoDEA model is reformulated as:

Model 3.1

$$\underset{p_i, p_j, n_i, n_j}{Min} \quad \sum_{k=1}^{N}\sum_{i\in I}(P_i^- \frac{n_i^k}{x_{ik}} + P_i^+ \frac{p_i^k}{x_{ik}}) + \sum_{k=1}^{N}\sum_{j\in J}(P_j^- \frac{n_j^k}{y_{jk}} + P_j^+ \frac{p_j^k}{y_{jk}}),$$
$$\underset{d_i, d_j,}{Max} \quad (\sum_{i\in I} P_i^g \frac{d_i^-}{TX_i} + \sum_{j\in J} P_j^g \frac{d_j^+}{TY_j}) \qquad 3.1$$

Subject to:

DMU representation:

$$\sum_{k=1}^{N} \lambda_k^c y_{jk} + n_j^c - p_j^c = y_j^c, \quad j \in J, \ \forall c \qquad 3.2$$

$$\sum_{k=1}^{N} \lambda_k^c x_{ik} + n_i^c - p_i^c = x_i^c, \quad i \in I, \ \forall c \qquad 3.3$$

Effectiveness through Achievement of Global Targets:

$$\sum_{k=1}^{N} \lambda_k^1 x_{ik} + \ldots + \sum_{k=1}^{N} \lambda_k^N x_{ik} + d_i^- = TX_i, \quad \forall i \in I \qquad 3.4$$

$$\sum_{k=1}^{N} \lambda_k^1 y_{jk} + \ldots + \sum_{k=1}^{N} \lambda_k^N y_{jk} - d_j^+ = TY_j, \quad \forall j \in J \qquad 3.5$$

Minimal Resource Share Allocation:

$$\sum_{k=1}^{N} \lambda_k^h x_{ik} \geq r_i^h \cdot (TX_i), \forall i \in I_s \subseteq I, h \in H \subseteq N = \{1,2,\ldots,N\} \qquad 3.6$$

$$\sum_k \lambda_k^c = 1, \qquad \forall c = \{1,2,\ldots,N\} \qquad 3.7$$

$$\lambda_k^c \geq 0, \qquad \forall k = \{1,2,\ldots,N\}, \quad \forall c \qquad 3.8$$

where:

N:	number of DMUs
I:	set of inputs
J:	set of outputs
x_{ik}:	level of input i for DMU k
y_{jk}:	level of output j for DMU k
x_i^c, y_j^c:	level of input i and output j for DMU c when assessing DMU c
λ_k^c:	activity level of DMU k when assessing DMU c
n_i^k, p_i^k:	negative and positive deviation variables for input i of DMU k
n_j^k, p_j^k:	negative and positive deviation variables for output j of DMU k
d_i^+, d_i^-:	positive and negative deviation variables from global targets of input i and output j
P_i^-, P_i^+:	user defined preferences over the minimization of positive and negative goal deviations of input i
P_j^-, P_j^+:	user defined preferences over the minimization of positive and negative goal deviations of output j
P_i^g, P_j^g:	user-defined preferences related to global targets of input i and output j
TX_i, TY_j:	global target levels known *a priori* for input i and output j
I_s:	subset of inputs chosen to be maintained at least at pre-specified levels
H:	subset of DMUs chosen to be maintained at least at pre-specified levels for inputs $i \in I_s$
r_i^h:	pre-specified proportion of global target TX_i for input $i \in I_s$ and $h \in H$

Model 3.1 is a goal programming formulation. The model has an objective function and three sets of constraints. The first set of constraints (equations 3.2-3.3) provides the individual DMU representations and reflects the objective of efficiency. These DEA-like constraints compare the inputs and outputs of the assessed DMU c with the composite units $\Sigma_k \lambda_k^c x_{ik}$ and $\Sigma_k \lambda_k^c y_{jk}$ respectively. Each composite unit is basically a convex combination[3] (as in the BCC Model) of all DMUs in the system under study with a set of activity levels λ_k^c when assessing DMU c. These constraints differ from the conventional DEA constraints due to the introduction of positive and negative goal deviation variables, p_i^c and n_i^c for inputs and p_j^c and n_j^c for outputs, instead of the contraction and expansion factors respectively. The two-way deviation variables allow under- and over-achievement of the input and output factors and also impact the construction of the efficiency frontier. In conventional DEA the objective function seeks to minimize (maximize) the contraction (expansion) factor for the inputs (outputs). The objective function thus drives the solution to the problem. The efficient DMUs are evaluated when the contraction (expansion) variable is equal to unity and the distance between the efficient facet and the DMU is minimized via the input excess (output slack) variables. The efficient DMUs then represent points on the efficient frontier. In the goal programming model presented above the objective function seeks to minimize the two-way deviation variables. The two-way deviation variables represent possible contraction and expansion of both inputs and outputs. The minimization of these variables thus drives the solution to the problem. The specific cases of input and output orientations can be obtained by appropriately modifying the objective function. Thus, the efficient frontier constructed in the goal programming formulation may differ from the frontier constructed by conventional DEA

The second set of constraints (equations 3.4-3.5) reflects the objective of effectiveness through the achievement of global input and output targets. In his

[3] Equation 3.7 is the convexity constraint that ensures variable returns to scale. The proposed model and its variations maintain the convexity property to model variable returns to scale.

formulation Athanassopoulos (1995) makes an assumption that individual DMUs will tend to use more inputs than made available by and produce less outputs than desired by the central coordinating entity. In the current reformulation the aim is to restrict global consumption of each input to less than or equal to the global target and to enable global production of output that is more than or equal to the global target. That is, the decision-maker desires to maximize the negative deviation from the input target and the positive deviation from the output target. Therefore, only deviation variables corresponding to reduction in input usage (d_i^-) and the augmentation of output production (d_j^+) are present in these constraints. Fuzzification of the global target constraints would allow positive input deviation and negative output deviation within pre-specified tolerance limits. Therefore, the fuzzy global target constraints correspond to the assumption made by Athanassopoulos (1995).

The third constraint set (equation 3.6) reflects the objective of operational viability through the concept of minimal resource share allocation. These constraints differ from those proposed by Athanassopoulos (1995). While Athanassopoulos (1995) seeks to balance commensurate (measured in the same units) inputs and outputs (*e.g.,* income-expenses relationship), the constraint set in the reformulated model seeks to maintain a minimal resource allocation for all inputs from a subset of the input set I for a subset of the DMUs. These constraints, thus, address the concept of maintaining minimal operating levels of certain inputs and DMUs to remain viable in the generation of outputs. This constraint can be extended to apply to a pre-specified subset of the outputs. In the current formulation only a subset of the input set is employed, as the case considered is that of input reduction.

Equation 3.7 restricts the sum of the activity parameters λ_k^c's to one and enables variable returns to scale in the formulation. This convexity constraint is used in the same manner as in the BCC model for conventional DEA. Equation 3.8 imposes the non-negativity condition on the λ_k^c's.

The objective function of the model (equation 3.1) has two parts. The deviation variables are standardized to achieve a standard evaluation system (see Chapter 2, Section 2.6, page 54). The first part contains the positive and negative deviation variables associated with the inputs and outputs of individual DMUs. This allows for over- and under-achievement of individual input/output targets for each DMU. The priorities associated with these deviation variables can be interpreted as the extent to which individual DMUs contribute toward achievement of global organizational targets. This feature differentiates Athanassopoulos' (1995) model from conventional DEA, which always assumes input contraction and output expansion for the assessed DMU. Also, by appropriately modifying the signs and magnitudes of the preferences P_i^-, P_i^+, P_j^+, P_j^- different planning scenarios can be implemented (*e.g.*, input contraction and output expansion (conventional DEA), input contraction and output contraction, input expansion and output expansion, etc.). The second part of the objective function contains the deviation variables associated with the global input and output targets. The priorities associated with these deviation variables represent the reward per unit deviation from the global targets.

This reformulated version of Athanassopoulos' (1995) model provides the base model that is fuzzified and presented in this research. The following section presents a discussion of the membership function employed in this research. The concepts of membership functions are discussed related to a production environment and presented for the input and output spaces. The Fuzzy GoDEA model is presented subsequently.

3.2 MEMBERSHIP FUNCTIONS FOR THE INPUT AND OUTPUT SPACES

A membership function is constructed for both the input and output spaces based on the interpretation of the bounds or tolerance limits specified for the satisfaction of the fuzzy constraints. The assumptions for the membership functions are outlined next. Figures 3.1 and 3.2 show the membership functions for the input and output spaces respectively.

3.2.1 Assumptions

The following assumptions are made regarding the membership functions associated with the achievement of the fuzzy goals (or constraints). The membership functions are assumed to be linear and monotonically increasing. The membership function associated with each constraint is evaluated as a linear expression (Zimmermann (1978)) when the constraint is satisfied *within* the specified tolerance limits *i.e., essentially satisfied*. The value of the membership function is equal to zero when the constraint is evaluated at or beyond the tolerance limits *i.e., completely dissatisfied* and is equal to one when the constraint is satisfied crisply.

3.2.2 The Input Space

Consider the membership function shown in Figure 3.1 for the input space. The observed input i for DMU c is represented by x_i^c. Using the structural efficiency concept of DEA, the input efficiency of DMU c is assessed by comparing input x_i^c with the composite unit or convex combination in the BCC case (linear combination in the CCR case) of all the DMUs for input i in the system. The objective is to find a composite unit that is less than or equal to x_i^c *i.e.*, to find a composite unit that utilizes less than or as much of input i as DMU c. This mathematical representation when fuzzified allows the inequality to be satisfied up to an upper bound u_i^c where $u_i^c \geq x_i^c$. That is, the decision-maker is satisfied to varying degrees when the inequality is satisfied within the interval (x_i^c, u_i^c). When the composite unit is greater than or equal to u_i^c then the constraint is dissatisfied completely. Therefore, an input realization greater than or equal to u_i^c is undesirable. Hence, the membership function takes the value zero at u_i^c and all values greater than u_i^c. The membership function increases monotonically from zero to one in the interval (x_i^c, u_i^c) as the input realization moves from u_i^c to x_i^c. This is consistent with crisp constraint satisfaction at x_i^c. For the decision-maker, the membership function values for the efficiency constraints represent the degree of satisfaction of the DEA representation of the constraints. Therefore, when the membership function for the DEA representation of an input is equal to one it implies crisp satisfaction of the DEA structure for that input and when the membership function value is less than one it implies a relaxation of the DEA structure. Accordingly, it follows that when the membership function for an input is equal to zero the DEA structure fails to hold for that input.

A priori there is no knowledge that x_i^c is an (in)efficient observation. Therefore, prior to the efficiency evaluation of the DMUs the input space membership function can be considered to have a *hypothetical* nature. In other words, the membership function in the input case is assumed to be one at all values equal to and less than the observed input realization x_i^c. Since DEA is based on "best observed practices" this assumption is justified in the sense that it may be possible to further reduce inputs as the best observed may not be the best possible. Therefore, if the observed input realization is evaluated as inefficient then the efficient frontier can be considered to lie at x_i^{c*} (efficient input usage)at a distance "a" units from x_i^c (observed input) . When a = 0 then x_i^c (DMU *c*) is an efficient observation and lies on the efficient frontier ($x_i^c = x_i^{c*}$). Alternately, "a" can be interpreted as the projection of DMU *c* on to the efficient frontier that would make DMU *c* an efficient observation.

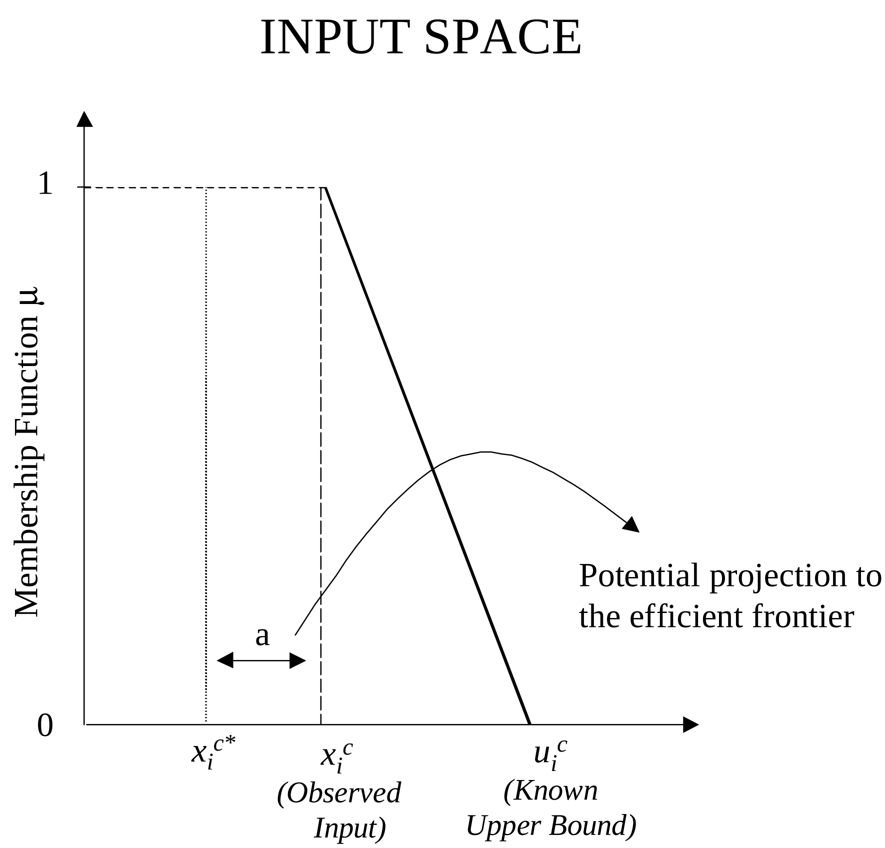

Figure 3.1 Membership Function μ for the Input Space

3.2.3 The Output Space

The membership function for the output space is constructed in an analogous manner. Consider the membership function shown in Figure 3.2 for the output space. The observed output j for DMU c is represented by y_j^c and is compared with the composite unit to assess DMU c's output efficiency. The objective is to find a composite unit that is greater than or equal to y_j^c *i.e.*, to find a composite unit that produces more than or as much of output j as DMU c. This mathematical representation when fuzzified allows the inequality to be satisfied up to a lower bound l_j^c where $l_j^c \leq y_j^c$ and is interpreted as the decision-maker's varying degrees of satisfaction when the inequality is satisfied within the interval (l_j^c, y_j^c). When the composite unit is less than or equal to l_j^c then the constraint is dissatisfied completely. Therefore, an output realization less than or equal to l_j^c is undesirable. Hence, the membership function takes the value zero at l_j^c and all values less than l_j^c. The membership function increases monotonically from zero to one in the interval (l_j^c, y_j^c) as the input realization moves from l_j^c to y_j^c. This is consistent with crisp constraint satisfaction at y_j^c. As in the input case, the membership function values for the efficiency constraints represent the degree of satisfaction of the DEA representation of the constraints. Therefore, when the membership function for the DEA representation of an output is equal to one it implies crisp satisfaction of the DEA structure for that output and when the membership function value is less than one it implies a relaxation of the DEA structure. Accordingly, it follows that when the membership function for an output is equal to zero the DEA structure fails to hold for that output.

In the absence of *a priori* knowledge whether y_j^c is an (in)efficient observation, the output space membership function is assumed to be equal to one at all values equal to and greater than the observed output realization y_j^c. Since DEA is based on "best observed practices" it may be possible to increase outputs as the best observed may not be the best possible. Therefore, if the observed output realization is evaluated as inefficient then the efficient frontier can be considered to lie at y_j^{c*} (efficient output production) at a distance "b" units from y_j^c (observed output). When b = 0 then y_j^c (DMU *c*) is an efficient observation and lies on the efficient frontier ($y_j^c = y_j^{c*}$). Alternately, "b" can be interpreted as the projection of DMU *c* on to the efficient frontier that would make DMU *c* an efficient observation.

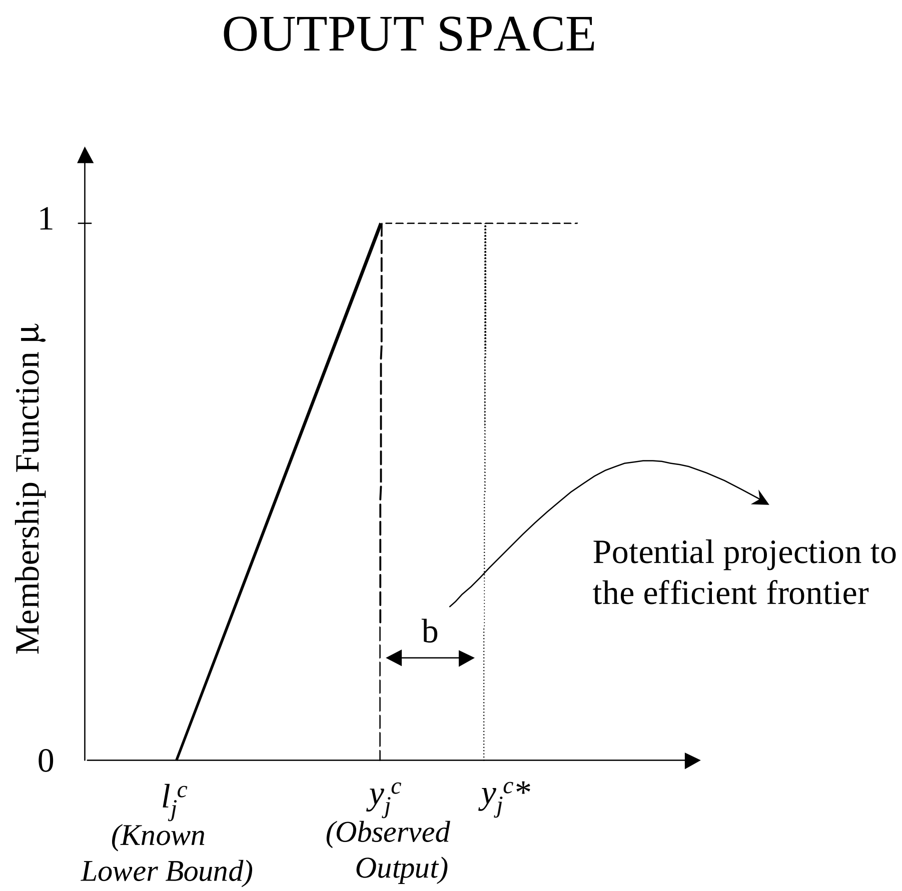

Figure 3.2 Membership Function μ for the Output Space

The membership functions developed in the preceding sections can be evaluated deterministically only after a solution to Model 3.3 is obtained. Based on the solution to Model 3.3 it is possible to calculate the values for the distances "a" and "b" for the inputs and outputs for each DMU. This provides a new graphical representation of the DMUs and their location with respect to the efficient frontier. Conventional DEA enables graphical representation of the frontier only in the two dimensional case *i.e.*, two inputs – one output or two outputs – one input cases. The construction of the membership function proposed above is not limited to two dimensions as each input/output for each DMU can be graphically portrayed as described. The Fuzzy GoDEA model is presented in the next section.

3.3 FUZZY GOAL PROGRAMMING AND DEA (Fuzzy GODEA) MODEL

The reformulated GoDEA model presented in section 3.1 is fuzzified in this section. The fuzzy model is transformed into an equivalent crisp formulation using fuzzy set theory. This crisp formulation is then solved with suitably developed computer programs in CPLEX. Variations of the base Fuzzy GoDEA model are presented subsequently to capture different decision-making scenarios. This research aims to present various decision-making scenarios in a fuzzy environment using the proposed Fuzzy GoDEA model and its variations.

3.3.1 The Conceptual Model

The proposed fuzzy model structure includes goal programming, data envelopment analysis and fuzzy set theory. The model is developed for a hierarchical environment with two levels of decision-making. At the higher- or super-level is the central decision making unit (CDMU) and at the lower- or sub-level are the individual decision making units (DMUs). The DMUs are under the control of the CDMU insofar as allocation of resources and setting global targets are concerned. The CDMU has a given amount of resources that it wishes to allocate among the DMUs while trying to achieve its global objectives of effectiveness, efficiency and minimal operational viability. The CDMU specifies global input and output targets for the DMUs based on historical process knowledge and statistical analyses. To achieve these objectives the CDMU could possibly choose to give most importance to meeting global input and output targets through maximal contribution of the DMUs. The CDMU could consider the objectives of efficiency and minimal viable operating levels for the DMUs to bear secondary importance. The DMUs could assign primary importance to the objectives of efficiency and minimal viable operating levels, and assign secondary importance to the objective of meeting global targets. The model can be solved with different priorities for the fuzzy goals depending on the level of decision-making. The variations to the model follow the formulation of the base model.

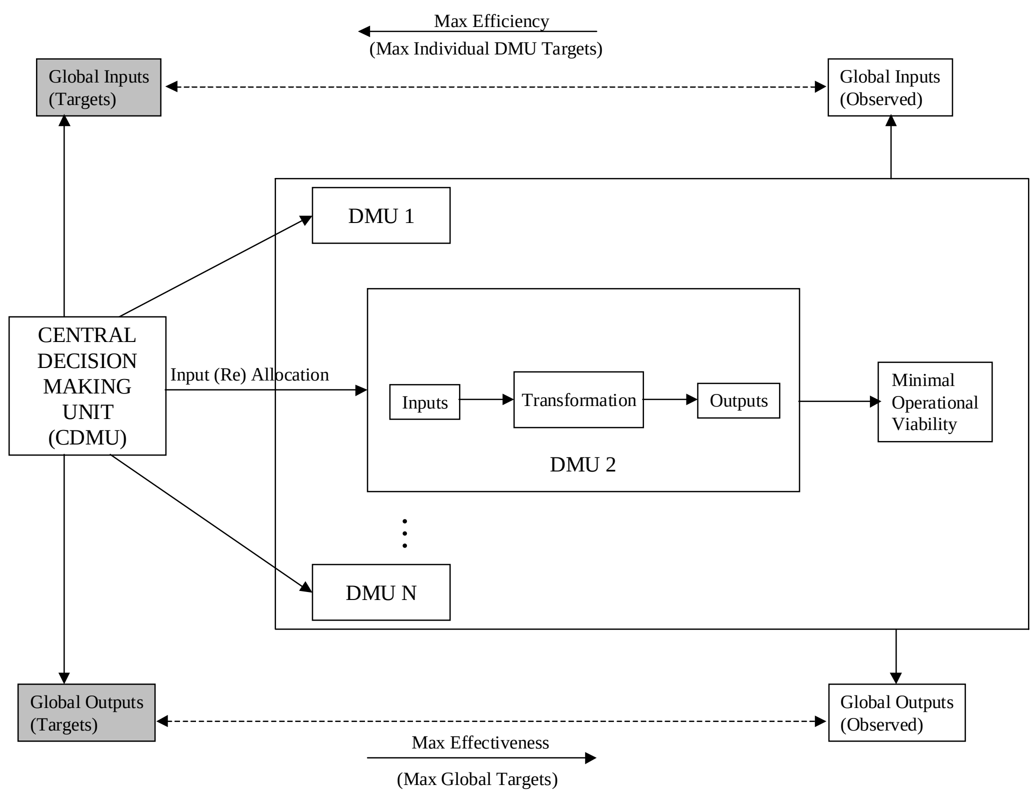

Figure 3.3 Conceptual Model

3.3.2 The Mathematical Formulation

Problem Statement: A hierarchical system consists of N DMUs and a central coordinating DMU. The CDMU provides global input and output targets and pre-specifies tolerance limits for the global targets. The individual DMUs specify the tolerance limits for the individual DMU inputs and outputs. Then, the problem is to determine the activity levels that maximally achieve the fuzzy goals of effectiveness (meeting global targets) and efficiency (meeting individual DEA targets) and the crisp goal of operational viability (minimal resource allocation share).

The Fuzzy GoDEA model can thus be written as:

Model 3.2

Find λ_k^c 3.9

Subject to:

DMU Representation:

$$\left.\begin{array}{ll} \sum_{k=1}^{N} \lambda_k^c y_{jk} \underset{\approx}{\geq} y_j^c, \quad \forall j \in J & (3.10) \\ \sum_{k=1}^{N} \lambda_k^c x_{ik} \underset{\approx}{\leq} x_i^c, \quad \forall i \in I & (3.11) \end{array}\right] \forall\, c = \{1,2,\ldots,N\}$$

Achievement of Global Targets (Effectiveness):

$$\sum_{k=1}^{N} \lambda_k^1 y_{jk} + \ldots + \sum_{k=1}^{N} \lambda_k^N y_{jk} \underset{\approx}{\geq} TY_j, \qquad \forall j \in J \qquad 3.12$$

$$\sum_{k=1}^{N} \lambda_k^1 x_{ik} + \ldots + \sum_{k=1}^{N} \lambda_k^N x_{ik} \underset{\approx}{\leq} TX_i, \qquad \forall i \in I \qquad 3.13$$

Minimal Resource Share Allocation:

$$\sum_{k=1}^{N} \lambda_k^h x_{ik} \geq r_i^h \cdot (TX_i), \qquad \forall i \in I_s \subseteq I, h \in H \subseteq N = \{1,2,\ldots,N\} \tag{3.14}$$

$$\sum_{k=1}^{N} \lambda_k^c = 1, \qquad \forall c = \{1,2,\ldots,N\} \tag{3.15}$$

$$\lambda_k^c \geq 0, \qquad \forall k = \{1,2,\ldots,N\}, \forall c \tag{3.16}$$

where:

N:	number of DMUS
I:	set of inputs
J:	set of outputs
x_{ik}:	level of input i of DMU k
y_{jk}:	level of output j for DMU k
x_i^c, y_j^c:	level of input i and output j for DMU c when assessing DMU c
λ_k^c:	activity level of DMU k when assessing DMU c
TX_i, TY_j:	global target levels known *a priori* for input i and output j
I_s:	subset of inputs chosen to be maintained at least at pre-specified levels
H:	subset of DMUs chosen to be maintained at least at pre-specified levels for inputs $i \in I_s$
r_i^h:	pre-specified proportion of global target TX_i for input $i \in I_s$ and $h \in$ H
$"\underset{\approx}{\leq}", "\underset{\approx}{\geq}"$:	denote fuzzification of the goal or constraint.

The objective function and the constraints of the model are related through the activity levels λ_k^c's. A set of activity levels is obtained when each DMU is assessed. These activity levels are DMU specific. In other words each DMU when assessed has its own set of activity levels for each input and output for all the DMUs in the data set. The activity levels are free to take on any non-negative value. The convexity constraint models variable returns to scale. Restrictions can be relaxed on the activity levels to incorporate constant returns to scale or weighting schemes imposed so as to avoid all the weight to be directed toward a few inputs or outputs.

The fuzzy constraints can be treated as fuzzy goals. The fuzzy goals imply that the goals have to be *essentially* met within the specified tolerance limits or bounds. These bounds are pre-specified by the decision-maker based on historical knowledge.

Consider the r^{th} fuzzy goal $G_r \underset{\approx}{\geq} g_r$, which signifies that the decision-maker accepts the constraint satisfaction up to a certain tolerance limit less than g_r. Consequently, a membership function μ_r for the r^{th} goal $G_r \underset{\approx}{\geq} g_r$ is defined by Zimmermann (1978) as:

$$\mu_r = \begin{cases} 1, & \text{if } \quad G_r \geq g_r \qquad (3.17) \\ \dfrac{G_r - L_r}{g_r - L_r}, & \text{if } L_r < G_r < g_r \qquad (3.18) \\ 0, & \text{if } \quad G_r \leq L_r \qquad (3.19) \end{cases}$$

where L_r is the lower bound or lower tolerance limit for fuzzy goal $G_r \geq g_r$.

Analogously, for the s^{th} fuzzy goal $G_s \underset{\approx}{\leq} g_s$, which signifies that the decision-maker accepts the constraint satisfaction up to a certain tolerance limit greater than g_s, the membership function μ_s is defined as:

$$\mu_s = \begin{cases} 1, & \text{if } \quad G_s \leq g_s \qquad (3.20) \\ \dfrac{U_s - G_s}{U_s - g_s}, & \text{if } g_s < G_s < U_s \qquad (3.21) \\ 0, & \text{if } \quad U_s \leq G_s \qquad (3.22) \end{cases}$$

where U_s is the upper bound or upper tolerance limit for fuzzy goal $G_s \leq g_s$.

The membership functions associated with the fuzzy goals in Model 3.2 can be expressed based on Zimmermann's (1978) definition for linear membership functions. However, the fuzzy model outlined above cannot be solved in the present form. Therefore, a linear crisp translation is required. A membership function μ_q is associated with each fuzzy goal G_q. There are $(i+j)$ input/output factors and therefore $(i+j)$ DMU representation constraints (DEA type constraints) for every DMU $k = 1,2,\ldots,N$. There are $(i+j)$ global target constraints. Therefore, in total there are $N(i+j)$ $+(i+j)$ fuzzy constraints and consequently, $N(i+j)$ $+(i+j)$ membership functions. Let $(i+j) = m$. Then there are $m(N+1)$ membership functions. Note that the minimal resource share allocation constraints are maintained as crisp inequalities. These constraints serve as implicit weight restrictions on the activity levels λ_k^h for the chosen subset of DMUs. Additionally, these constraints seek to secure at least a pre-specified proportion of the global input target for certain DMUs and inputs. Physically, this translates into the decision-maker deeming certain DMUs to be maintained at minimal operating levels for inputs determined to be critical for operational viability to generate outputs. The structure of these constraints also allows for sensitivity analysis on the proportion r_i^h of the global input target TX_i for the chosen input subset I_s. The model results can be compared by varying the specification of r_i^h as well as the subsets H and I_s.

The crisp equivalent linear program for the Fuzzy GoDEA model (Model 3.2) is written as:

Model 3.3

$$Max \sum_{x_i^c, y_j^c, x_i, y_j} (\mu_{y_j^c} + \mu_{x_i^c} + \mu_{y_j} + \mu_{x_i}) \qquad 3.23$$

Subject to:

For the DMU representations:

$$\left.\begin{array}{ll} \mu_{y_j^c} = \dfrac{\sum_{k=1}^{N} \lambda_k^c y_{jk} - l_j^c}{y_j^c - l_j^c}, & \forall j \in J \qquad 3.24 \\[2ex] \mu_{x_i^c} = \dfrac{u_i^c - \sum_{k=1}^{N} \lambda_k^c x_{ik}}{u_i^c - x_i^c}, & \forall i \in I \qquad 3.25 \end{array}\right] \forall \quad c \in \{1,2,\ldots,N\}$$

For the achievement of Global Targets:

$$\mu_{y_j} = \frac{(\sum_{k=1}^{N} \lambda_k^1 y_{jk} + \ldots + \sum_{k=1}^{N} \lambda_k^N y_{jk}) - L_j}{TY_j - L_j}, \qquad \forall j \in J \qquad 3.26$$

$$\mu_{x_i} = \frac{U_i - (\sum_{k=1}^{N} \lambda_k^1 x_{ik} + \ldots + \sum_{k=1}^{N} \lambda_k^N x_{ik})}{U_j - TX_i}, \qquad \forall i \in I \qquad 3.27$$

For the Minimal Resource Allocation Share:

$$\sum_{k=1}^{N} \lambda_k^h x_{ik} \geq r_i^h \cdot (TX_i), \qquad \forall i \in I_s \subseteq I, h \in H \subseteq N = \{1,2,\ldots,N\} \qquad 3.28$$

$$\sum_{k=1}^{N} \lambda_k^c = 1, \qquad \forall c = \{1,2,\ldots,N\} \qquad 3.29$$

$$0 \leq \mu_{y_j^c}, \mu_{x_i^c}, \mu_{y_j}, \mu_{x_i} \leq 1, \qquad \lambda_k^c \geq 0 \qquad 3.30$$

where:

l_j^c: lower bound on DMU output target y_j^c

u_i^c: upper bound on DMU input target x_i^c and

L_j: lower bound on DMU global output target TY_j and $L_j = \Sigma_k\, l_j^k$

U_i: upper bound on global input target TX_i and $U_i = \Sigma_k\, u_i^k$

In this research U_i and L_j are assumed to be the sum of the individual input/output bounds. The μ_q's represent the degree of satisfaction of the decision-maker. The interpretation of these membership functions is explained in the next section.

3.4 INTERPRETATION OF THE MEMBERSHIP FUNCTIONS

This section provides the interpretation for the membership functions associated with the fuzzy constraints or goals in the Fuzzy GoDEA Model.

3.4.1 Membership Functions for the Constraints

Each fuzzy constraint of Model 3.2 has a membership function μ associated with it. Each membership function ranges from zero to one and expresses the degree of satisfaction of the constraint. In other words, the membership function denotes the degree of satisfaction of the decision-maker in achieving the aspiration levels of his/her fuzzy goals. If the membership function takes the value one then the associated goal is achieved crisply and if it takes the value zero then the associated goal is dissatisfied completely. The range between zero and one can be viewed as the percentage satisfaction of the decision-maker in satisfying the goal. The graphical representations of the membership function for the "≤" and "≥" constraints are given in Figure 3.4 and Figure 3.5 respectively. The quantities g_s^b and g_r^b represent the production limits. This means that the membership function virtually retains its value as one beyond these bounds ("≤" and "≥" respectively) and is represented by a dashed line. In production terms though it may not be possible to decrease/increase the factors to levels beyond these bounds.

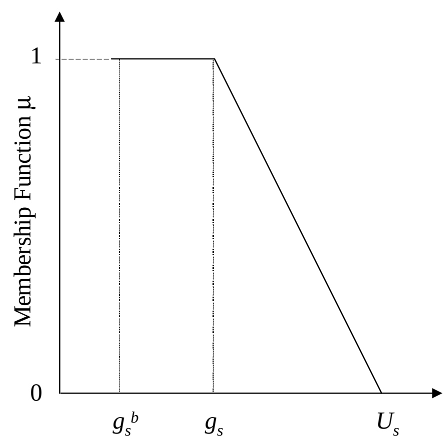

Figure 3.4 Membership Function μ for the "≤" Type Fuzzy Goal

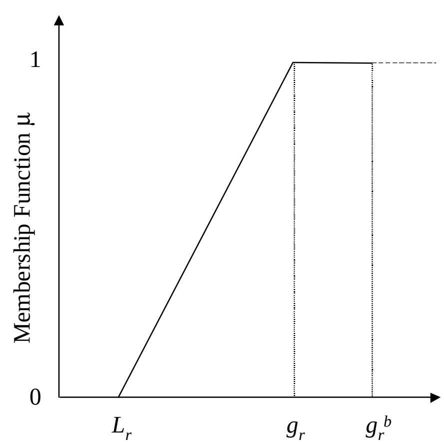

Figure 3.5 Membership Function μ for the "≥" Type Fuzzy Goal

3.5 ILLUSTRATION

Consider a system with N = 3 DMUs, I = 2 inputs, J = 1 output *i.e.*, we have

$$N = 1,2,3 \tag{3.31}$$

$$I = x_1, x_2 \tag{3.32}$$

$$J = y_1 \tag{3.33}$$

Say, $H = \{1,3\}$ and $I_s = x_1$. (3.34)

Then the Fuzzy GoDEA formulation as given in Model 3.2 is given as :

$$\text{Find } \lambda_k^{\,c} \tag{3.35}$$

Subject to:

DMU Representation (*Efficiency*):

(DMU 1)

$$\lambda_1^1 y_{11} + \lambda_2^1 y_{12} + \lambda_3^1 y_{13} \underset{\approx}{\geq} y_1^1 \tag{3.36}$$

$$\lambda_1^1 x_{11} + \lambda_2^1 x_{12} + \lambda_3^1 x_{13} \underset{\approx}{\leq} x_1^1 \tag{3.37}$$

$$\lambda_1^1 x_{21} + \lambda_2^1 x_{22} + \lambda_3^1 x_{23} \underset{\approx}{\leq} x_2^1 \tag{3.38}$$

(DMU 2)

$$\lambda_1^2 y_{11} + \lambda_2^2 y_{12} + \lambda_3^2 y_{13} \underset{\approx}{\geq} y_1^2 \tag{3.39}$$

$$\lambda_1^2 x_{11} + \lambda_2^2 x_{12} + \lambda_3^2 x_{13} \underset{\approx}{\leq} x_1^2 \tag{3.40}$$

$$\lambda_1^2 x_{21} + \lambda_2^2 x_{22} + \lambda_3^2 x_{23} \underset{\approx}{\leq} x_2^2 \tag{3.41}$$

$$\lambda_1^3 y_{11} + \lambda_2^3 y_{12} + \lambda_3^3 y_{13} \underset{\approx}{\geq} y_1^3 \qquad \text{(DMU 3)} \qquad 3.42$$

$$\lambda_1^3 x_{11} + \lambda_2^3 x_{12} + \lambda_3^3 x_{13} \underset{\approx}{\leq} x_1^3 \qquad 3.43$$

$$\lambda_1^3 x_{21} + \lambda_2^3 x_{22} + \lambda_3^3 x_{23} \underset{\approx}{\leq} x_2^3 \qquad 3.44$$

Achievement of Global Targets: (*Effectiveness*):

$$(\lambda_1^1 y_{11} + \lambda_2^1 y_{12} + \lambda_3^1 y_{13}) + (\lambda_1^2 y_{11} + \lambda_2^2 y_{12} + \lambda_3^2 y_{13}) + (\lambda_1^3 y_{11} + \lambda_2^3 y_{12} + \lambda_3^3 y_{13}) \underset{\approx}{\geq} TY_1 \qquad 3.45$$

$$(\lambda_1^1 x_{11} + \lambda_2^1 x_{12} + \lambda_3^1 x_{13}) + (\lambda_1^2 x_{11} + \lambda_2^2 x_{12} + \lambda_3^2 x_{13}) + (\lambda_1^3 x_{11} + \lambda_2^3 x_{12} + \lambda_3^3 x_{13}) \underset{\approx}{\leq} TX_1 \qquad 3.46$$

$$(\lambda_1^1 x_{21} + \lambda_2^1 x_{22} + \lambda_3^1 x_{23}) + (\lambda_1^2 x_{21} + \lambda_2^2 x_{22} + \lambda_3^2 x_{23}) + (\lambda_1^3 x_{21} + \lambda_2^3 x_{22} + \lambda_3^3 x_{23}) \underset{\approx}{\leq} TX_2 \qquad 3.47$$

Minimal Resource Share Allocation: (*Operational Viability*):

$$\lambda_1^1 x_{11} + \lambda_2^1 x_{12} + \lambda_3^1 x_{13} \geq r_1^1 \cdot (TX_1) \qquad 3.48$$

$$\lambda_1^3 x_{11} + \lambda_2^3 x_{12} + \lambda_3^3 x_{13} \geq r_1^3 \cdot (TX_1) \qquad 3.49$$

$$\Sigma_k \lambda_k^c = 1, \qquad \forall\ c = \{1,2,\ldots,N\} \qquad 3.50$$

$$\lambda_k^c \geq 0, \qquad 0 < r_i^h < 1 \qquad 3.51$$

3.6 VARIATIONS OF THE FUZZY GoDEA MODEL

The Fuzzy GoDEA model (Model 3.2) (and its crisp equivalent (Model 3.3)) presented in section 3.3 is developed for the scenario when the decision-maker wishes to specify imprecise aspiration levels for *all* goals. However, real life data often offers a combination of complete and incomplete information with respect to achievement of goals. Further, in the case of two or more levels of decision-making, multiple goals carry different degrees of importance depending on the level of decision-making. In such circumstances the decision-maker may wish to partition the goals into fuzzy and crisp. For example, in one case the decision-maker may choose to specify the goals related to efficiency (DEA representation) in a crisp sense and maintain the goals related to effectiveness (achievement of global targets) as fuzzy. Further, the relative importance of goals can also be modeled by setting priority levels for the goals.

The different models that capture these variations are outlined in this section. All the variations outlined carry only the goals of efficiency and effectiveness and represent a two-level hierarchy. The variations are presented as the final crisp equivalents obtained after crisp transformation of any fuzzy goals. Due to unavailability of information and the need to maintain consistency for comparison the third constraint of operational viability is not modeled for the application, and henceforth omitted. Thus the variations of the Fuzzy GoDEA Base Model presented next contain only two goals, that of efficiency and effectiveness. It should be noted, however, that the operational viability goal maybe relevant for certain situations and can be modeled with suitably available information.

3.6.1 *Variation 1:*

Model 3.4

$$Max \sum_{x_i^c, y_j^c, x_i, y_j} (w_{y_j^c} \mu_{y_j^c} + w_{x_i^c} \mu_{x_i^c} + w_{y_j} \mu_{y_j} + w_{x_i} \mu_{x_i}) \qquad 3.52$$

Subject to:

For the DMU representations:

$$\left.\begin{array}{ll} \mu_{y_j^c} = \dfrac{\sum_{k=1}^{N} \lambda_k^c y_{jk} - l_j^c}{y_j^c - l_j^c}, & \forall j \in J \qquad 3.53 \\[2ex] \mu_{x_i^c} = \dfrac{u_i^c - \sum_{k=1}^{N} \lambda_k^c x_{ik}}{u_i^c - x_i^c}, & \forall i \in I \qquad 3.54 \end{array}\right] \forall \quad c \in \{1,2,\ldots,N\}$$

For the achievement of Global Targets:

$$\mu_{y_j} = \frac{(\sum_{k=1}^{N} \lambda_k^1 y_{jk} + \ldots + \sum_{k=1}^{N} \lambda_k^N y_{jk}) - L_j}{TY_j - L_j}, \quad \forall j \in J \qquad 3.55$$

$$\mu_{x_i} = \frac{U_i - (\sum_{k=1}^{N} \lambda_k^1 x_{ik} + \ldots + \sum_{k=1}^{N} \lambda_k^N x_{ik})}{U_i - TX_i}, \quad \forall i \in I \qquad 3.56$$

$$\sum_{k=1}^{N} \lambda_k^c = 1, \qquad \forall c = \{1,2,\ldots,N\} \qquad 3.57$$

$$0 \le \mu_{y_j^c}, \mu_{x_i^c}, \mu_{y_j}, \mu_{x_i} \le 1, \quad \lambda_k^c \ge 0, \quad \text{and} \quad w_{y_j^c}, w_{x_i^c}, w_{y_j}, w_{x_i} \ge 0 \qquad 3.58$$

In this case the objective function is of the weighted additive type. Here the μ's are weighted according to the importance that the decision-maker wishes to assign each goal.

3.6.2 *Variation 2:*

Model 3.5

Stage 1:

$$Max \sum_{x_i, y_j} (\mu_{y_j} + \mu_{x_i}) \qquad 3.59$$

Subject to:

For the achievement of Global Targets:

$$\mu_{y_j} = \frac{(\sum_{k=1}^{N} \lambda_k^1 y_{jk} + \ldots + \sum_{k=1}^{N} \lambda_k^N y_{jk}) - L_j}{TY_j - L_j}, \quad \forall j \in J \qquad 3.60$$

$$\mu_{x_i} = \frac{U_i - (\sum_{k=1}^{N} \lambda_k^1 x_{ik} + \ldots + \sum_{k=1}^{N} \lambda_k^N x_{ik})}{U_i - TX_i}, \quad \forall i \in I \qquad 3.61$$

$$\sum_{k=1}^{N} \lambda_k^c = 1, \qquad \forall c = \{1,2,\ldots,N\} \qquad 3.62$$

$$0 \le \mu_{y_j}, \mu_{x_i} \le 1, \quad \lambda_k^c \ge 0, \qquad 3.63$$

Stage 2:

$$Max \sum_{x_i^c, y_j^c} (\mu_{y_j^c} + \mu_{x_i^c}) \qquad 3.64$$

Subject to:

For the DMU representations:

$$\mu_{y_j^c} = \frac{\sum_{k=1}^{N} \lambda_k^c y_{jk} - l_j^c}{y_j^c - l_j^c}, \qquad \forall j \in J \qquad 3.65$$

$$\mu_{x_i^c} = \frac{u_i^c - \sum_{k=1}^{N} \lambda_k^c x_{ik}}{u_i^c - x_i^c}, \qquad \forall i \in I \qquad 3.66$$

$$\forall \quad c \in \{1,2,\ldots,N\}$$

For the achievement of Global Targets:

$$\mu_{y_j} = \frac{(\sum_{k=1}^{N} \lambda_k^1 y_{jk} + \ldots + \sum_{k=1}^{N} \lambda_k^N y_{jk}) - L_j}{TY_j - L_j}, \quad \forall j \in J \qquad 3.67$$

$$\mu_{x_i} = \frac{U_i - (\sum_{k=1}^{N} \lambda_k^1 x_{ik} + \ldots + \sum_{k=1}^{N} \lambda_k^N x_{ik})}{U_i - TX_i}, \quad \forall i \in I \qquad 3.68$$

$$\sum_{k=1}^{N} \lambda_k^c = 1, \qquad \forall c = \{1,2,\ldots,N\} \qquad 3.69$$

$$0 \le \mu_{y_j^c}, \mu_{x_i^c} \le 1, \quad \lambda_k^c \ge 0, \quad \text{and} \quad \mu_{y_j} = \mu_{y_j}^*, \quad \mu_{x_i} = \mu_{x_i}^* \qquad 3.70$$

Here the achievement of global targets is considered as the more important goal in stage 1 and solved for optimality. The optimal solution μ*'s obtained from stage 1 are passed as constraints for the stage 2 problem. The objective of efficiency is then evaluated in stage 2. The solution obtained for stage 2 will thus maintain the solution to stage 1.

3.6.3 *Variation 3:*

Model 3.6

Stage 1:

$$Max \sum_{x_i^c, y_j^c} (\mu_{y_j^c} + \mu_{x_i^c}) \qquad 3.71$$

Subject to:

For the DMU representations:

$$\left.\begin{aligned} &\mu_{y_j^c} = \frac{\sum_{k=1}^{N} \lambda_k^c y_{jk} - l_j^c}{y_j^c - l_j^c}, \qquad \forall j \in J && 3.72 \\ &\mu_{x_i^c} = \frac{u_i^c - \sum_{k=1}^{N} \lambda_k^c x_{ik}}{u_i^c - x_i^c}, \qquad \forall i \in I && 3.73 \end{aligned}\right\} \forall \quad c \in \{1,2,\ldots,N\}$$

$$\sum_{k=1}^{N} \lambda_k^c = 1, \qquad \forall c = \{1,2,\ldots,N\} \qquad 3.74$$

$$0 \le \mu_{y_j^c}, \mu_{x_i^c} \le 1, \quad \lambda_k^c \ge 0 \qquad 3.75$$

Stage 2:

$$Max \sum_{x_i, y_j} (\mu_{y_j} + \mu_{x_i}) \qquad 3.76$$

Subject to:

For the DMU representations:

$$\left.\begin{array}{ll}\mu_{y_j^c} = \dfrac{\sum_{k=1}^{N} \lambda_k^c y_{jk} - l_j^c}{y_j^c - l_j^c}, & \forall j \in J \quad (3.77) \\[2ex] \mu_{x_i^c} = \dfrac{u_i^c - \sum_{k=1}^{N} \lambda_k^c x_{ik}}{u_i^c - x_i^c}, & \forall i \in I \quad (3.78)\end{array}\right] \forall \quad c \in \{1,2,\ldots,N\}$$

For the achievement of Global Targets:

$$\mu_{y_j} = \frac{(\sum_{k=1}^{N} \lambda_k^1 y_{jk} + \ldots + \sum_{k=1}^{N} \lambda_k^N y_{jk}) - L_j}{TY_j - L_j}, \quad \forall j \in J \tag{3.79}$$

$$\mu_{x_i} = \frac{U_i - (\sum_{k=1}^{N} \lambda_k^1 x_{ik} + \ldots + \sum_{k=1}^{N} \lambda_k^N x_{ik})}{U_i - TX_i}, \quad \forall i \in I \tag{3.80}$$

$$\sum_{k=1}^{N} \lambda_k^c = 1, \qquad \forall c = \{1,2,\ldots,N\} \tag{3.81}$$

$$0 \le \mu_{y_j} \mu_{x_i} \le 1, \quad \lambda_k^c \ge 0, \quad \text{and} \quad \mu_{y_j^c} = \mu_{y_j^c}^*, \quad \mu_{x_i} = \mu_{x_i^c}^* \tag{3.82}$$

Here the achievement of individual DMU targets *i.e.*, the goal of efficiency is considered as the more important goal in stage 1 and solved for optimality. The optimal solution μ*'s obtained from stage 1 are passed as constraints for the stage 2 problem. The objective of effectiveness *i.e.*, achievement of global targets is then evaluated in stage 2. The solution obtained for stage 2 will thus maintain the solution to stage 1.

3.6.4 *Variation 4:*

Model 3.7

$$Max \sum_{x_i^c, y_j^c} (\mu_{y_j^c} + \mu_{x_i^c}) \qquad 3.83$$

Subject to:

For the DMU representations:

$$\left.\begin{aligned} \mu_{y_j^c} &= \frac{\sum_{k=1}^{N} \lambda_k^c y_{jk} - l_j^c}{y_j^c - l_j^c}, \qquad \forall j \in J \qquad 3.84 \\ \mu_{x_i^c} &= \frac{u_i^c - \sum_{k=1}^{N} \lambda_k^c x_{ik}}{u_i^c - x_i^c}, \qquad \forall i \in I \qquad 3.85 \end{aligned}\right] \forall \quad c \in \{1,2,\ldots,N\}$$

For the achievement of Global Targets:

$$(\sum_{k=1}^{N} \lambda_k^1 y_{jk} + \ldots + \sum_{k=1}^{N} \lambda_k^N y_{jk}) \geq TY_j, \quad \forall j \in J \qquad 3.86$$

$$(\sum_{k=1}^{N} \lambda_k^1 x_{ik} + \ldots + \sum_{k=1}^{N} \lambda_k^N x_{ik}) \leq TX_i, \quad \forall i \in I \qquad 3.87$$

$$\sum_{k=1}^{N} \lambda_k^c = 1, \qquad \forall c = \{1,2,\ldots,N\} \qquad 3.88$$

$$0 \leq \mu_{y_j^c}, \mu_{x_i^c} \leq 1, \quad \lambda_k^c \geq 0 \qquad 3.89$$

In this case the constraints related to efficiency (DMU representation) are fuzzy while the constraints related to achievement of global targets are crisp. The objective function seeks to maximize the sum of the μ's associated with the efficiency constraints. The decision-maker would typically desire that the global consumption of inputs is maintained below the global target and the global production of output exceeds the global target. Accordingly, the right hand side of each of these constraints represents the sum of the individual bounds of the inputs and outputs. This is consistent with the fact that the individual bounds represent the risk-free scenarios and, therefore, the goal would be to improve upon the sum of these bounds at the global level.

3.6.5 *Variation 5:*

Model 3.8

$$Max \sum_{x_i, y_j} (\mu_{y_j} + \mu_{x_i}) \tag{3.90}$$

Subject to:

For the DMU representations:

$$\left.\begin{aligned} &\sum_{k=1}^{N} \lambda_k^c y_{jk} \geq y_j^c, \qquad \forall j \in J \qquad (3.91)\\ &\sum_{k=1}^{N} \lambda_k^c x_{ik} \leq x_i^c, \qquad \forall i \in I \qquad (3.92) \end{aligned}\right] \quad \forall \quad c \in \{1,2,\ldots,N\}$$

For the achievement of Global Targets:

$$\mu_{y_j} = \frac{(\sum_{k=1}^{N} \lambda_k^1 y_{jk} + \ldots + \sum_{k=1}^{N} \lambda_k^N y_{jk}) - L_j}{TY_j - L_j}, \quad \forall j \in J \tag{3.93}$$

$$\mu_{x_i} = \frac{U_i - (\sum_{k=1}^{N} \lambda_k^1 x_{ik} + \ldots + \sum_{k=1}^{N} \lambda_k^N x_{ik})}{U_i - TX_i}, \quad \forall i \in I \tag{3.94}$$

$$\sum_{k=1}^{N} \lambda_k^c = 1, \qquad \forall c = \{1,2,\ldots,N\} \tag{3.95}$$

$$0 \le \mu_{y_j}\, \mu_{x_i} \le 1, \quad \lambda_k^c \ge 0 \tag{3.96}$$

In this case the constraints related to efficiency (DMU representation) are crisp while the constraints related to achievement of global targets are fuzzy. The objective function seeks to maximize the sum of the μ's associated with the effectiveness constraints. The crisp DMU representation constraints measure each unit relative to a composite unit as in conventional DEA to measure efficiency.

3.6.6 *Variation 6:*

Model 3.9

Stage 1:

$$Max \sum_{x_i^c, y_j^c} (\mu_{y_j^c} + \mu_{x_i^c}) \tag{3.97}$$

Subject to:

For the DMU representations:

$$\left.\begin{aligned} \mu_{y_j^c} &= \frac{\sum_{k=1}^{N} \lambda_k^c y_{jk} - l_j^c}{y_j^c - l_j^c}, && \forall j \in J && (3.98) \\ \mu_{x_i^c} &= \frac{u_i^c - \sum_{k=1}^{N} \lambda_k^c x_{ik}}{u_i^c - x_i^c}, && \forall i \in I && (3.99) \end{aligned}\right] \forall \quad c \in \{1,2,\ldots,N\}$$

$$\sum_{k=1}^{N} \lambda_k^c = 1, \qquad \forall c = \{1,2,\ldots,N\} \qquad 3.100$$

$$0 \le \mu_{y_j^c}, \mu_{x_i^c} \le 1, \quad \lambda_k^c \ge 0 \qquad 3.101$$

Stage 2:

$$Min \sum_{x_i, y_j} (d_{y_j}^{-} + d_{x_i}^{+}) \qquad 3.102$$

Subject to:

For the DMU representations:

$$\left.\begin{array}{lll}
\mu_{y_j^c} = \dfrac{\sum_{k=1}^{N} \lambda_k^c y_{jk} - l_j^c}{y_j^c - l_j^c}, & \forall j \in J & 3.103 \\[2ex]
\mu_{x_i^c} = \dfrac{u_i^c - \sum_{k=1}^{N} \lambda_k^c x_{ik}}{u_i^c - x_i^c}, & \forall i \in I & 3.104
\end{array}\right] \forall \quad c \in \{1,2,\ldots,N\}$$

For the achievement of Global Targets:

$$(\sum_{k=1}^{N} \lambda_k^1 y_{jk} + \ldots + \sum_{k=1}^{N} \lambda_k^N y_{jk}) + d_{y_j}^{-} - d_{y_j}^{-} = L_j, \qquad \forall j \in J \qquad 3.105$$

$$(\sum_{k=1}^{N} \lambda_k^1 x_{ik} + \ldots + \sum_{k=1}^{N} \lambda_k^N x_{ik}) + d_{x_i}^{-} - d_{x_i}^{+} = U_i, \qquad \forall i \in I \qquad 3.106$$

$$\sum_{k=1}^{N} \lambda_k^c = 1, \qquad \forall c = \{1,2,\ldots,N\} \qquad 3.107$$

$$\lambda_k^c \ge 0, \quad d_{y_j}^{-}, d_{y_j}^{-}, d_{x_i}^{-}, d_{x_i}^{+} \ge 0 \qquad \text{and} \qquad \mu_{y_j^c} = \mu_{y_j^c}^{*}, \quad \mu_{x_i} = \mu_{x_i^c}^{*} \qquad 3.108$$

Here the stage 1 problem tries to maximize satisfaction of the fuzzy DMU representations or efficiency constraints. The stage 2 problem consists of fuzzy efficiency constraints and crisp effectiveness constraints. The optimal solution μ^*'s is passed as a constraint to the stage 2 problem. The objective function in stage 2 tries to minimize the deviations from the global targets. Only negative deviation from the output target and positive deviation from the input target are minimized since positive output deviation and negative input deviation are considered acceptable. Thus the solution to the stage 2 problem will maintain the optimal solution to the stage 1 problem. Consequently, the solution to the deviation variables in stage 2 reveal the extent of satisfaction of the effectiveness constraints given a certain acceptable level of satisfaction of the efficiency constraints.

3.6.7 *Variation 7:*

Model 3.10

Stage 1:

$$Min \sum_{x_i, y_j} (d^-_{y_j} + d^+_{x_i}) \qquad 3.109$$

Subject to:

For the achievement of Global Targets:

$$(\sum_{k=1}^{N} \lambda_k^1 y_{jk} + \ldots + \sum_{k=1}^{N} \lambda_k^N y_{jk}) + d^-_{y_j} - d^-_{y_j} = L_j, \qquad \forall j \in J \qquad 3.110$$

$$(\sum_{k=1}^{N} \lambda_k^1 x_{ik} + \ldots + \sum_{k=1}^{N} \lambda_k^N x_{ik}) + d^-_{x_i} - d^+_{x_i} = U_i, \qquad \forall i \in I \qquad 3.111$$

$$\sum_{k=1}^{N} \lambda_k^c = 1, \qquad \forall c = \{1,2,\ldots,N\} \qquad 3.112$$

$$\lambda_k^c \geq 0, \quad \text{and} \quad d_{y_j}^-, d_{y_j}^-, d_{x_i}^-, d_{x_i}^+ \geq 0 \tag{3.113}$$

Stage 2:

$$Max \sum_{x_i^c, y_j^c} (\mu_{y_j^c} + \mu_{x_i^c}) \tag{3.114}$$

Subject to:

For the DMU representations:

$$\left.\begin{aligned} \mu_{y_j^c} &= \frac{\sum_{k=1}^{N} \lambda_k^c y_{jk} - l_j^c}{y_j^c - l_j^c}, \qquad \forall j \in J \quad (3.115) \\ \mu_{x_i^c} &= \frac{u_i^c - \sum_{k=1}^{N} \lambda_k^c x_{ik}}{u_i^c - x_i^c}, \qquad \forall i \in I \quad (3.116) \end{aligned}\right] \forall \quad c \in \{1, 2, \ldots, N\}$$

For the achievement of Global Targets:

$$(\sum_{k=1}^{N} \lambda_k^1 y_{jk} + \ldots + \sum_{k=1}^{N} \lambda_k^N y_{jk}) + d_{y_j}^- - d_{y_j}^- = L_j, \qquad \forall j \in J \tag{3.117}$$

$$(\sum_{k=1}^{N} \lambda_k^1 x_{ik} + \ldots + \sum_{k=1}^{N} \lambda_k^N x_{ik}) + d_{x_i}^- - d_{x_i}^+ = U_i, \qquad \forall i \in I \tag{3.118}$$

$$\sum_{k=1}^{N} \lambda_k^c = 1, \qquad \forall c = \{1, 2, \ldots, N\} \tag{3.119}$$

$$0 \leq \mu_{y_j^c}, \mu_{x_i^c} \leq 1, \quad \lambda_k^c \geq 0, \quad \text{and} \tag{3.120}$$

$$d_{y_j}^- = d_{y_j}^{-*}, d_{y_j}^- = d_{y_j}^{-*}, d_{x_i}^- = d_{x_i}^{-*}, d_{x_i}^+ = d_{x_i}^{+*}$$

This variation reverses the priority attached to the goals in variation 6. In stage 1 the objective function minimizes the deviations from the global targets. The solution to the deviation variables is then passed as a constrain to the stage 2 problem. In stage 2 the objective is to maximize satisfaction of the fuzzy DMU representations or efficiency constraints. The stage 2 problem consists of fuzzy efficiency constraints and crisp effectiveness constraints. The solution to the stage 2 problem reveals the extent of satisfaction of the efficiency constraints given the least deviation from the effectiveness constraints.

3.6.8 *Variation 8:*

Model 3.11

Stage 1:

$$Min \sum_{x_{ii}^{c}, y_j^c} (d^{+}_{y_j^c} + d^{-}_{y_j^c} + d^{+}_{x_i^c} + d^{-}_{x_i^c}) \quad 3.121$$

Subject to:

For the DMU representations:

$$\left.\begin{array}{ll} \sum_{k=1}^{N} \lambda_k^c y_{jk} + d^{-}_{y_j^c} - d^{+}_{y_j^c} = y_j^c, & \forall j \in J \quad 3.122 \\ \sum_{k=1}^{N} \lambda_k^c x_{ik} + d^{-}_{x_i^c} - d^{+}_{x_i^c} = x_i^c, & \forall i \in I \quad 3.123 \end{array}\right] \forall \quad c \in \{1,2,\ldots,N\}$$

$$\sum_{k=1}^{N} \lambda_k^c = 1, \qquad \forall c = \{1,2,\ldots,N\} \quad 3.124$$

$$\lambda_k^c \geq 0, \qquad \text{and} \qquad d^{-}_{y_j^c}, d^{+}_{y_j^c}, d^{+}_{x_i^c}, d^{-}_{x_i^c} \geq 0 \quad 3.125$$

Stage 2:

$$Max \sum_{x_i, y_j} (\mu_{y_j} + \mu_{x_i}) \tag{3.126}$$

Subject to:

For the DMU representations:

$$\left.\begin{aligned} &\sum_{k=1}^{N} \lambda_k^c y_{jk} + d_{y_j^c}^- - d_{y_j^c}^+ = y_j^c, \qquad \forall j \in J \quad (3.127) \\ &\sum_{k=1}^{N} \lambda_k^c x_{ik} + d_{x_i^c}^- - d_{x_i^c}^+ = x_i^c, \qquad \forall i \in I \quad (3.128) \end{aligned}\right] \forall \quad c \in \{1,2,\ldots,N\}$$

For the achievement of Global Targets:

$$\mu_{y_j} = \frac{(\sum_{k=1}^{N} \lambda_k^1 y_{jk} + \ldots + \sum_{k=1}^{N} \lambda_k^N y_{jk}) - L_j}{TY_j - L_j}, \quad \forall j \in J \tag{3.129}$$

$$\mu_{x_i} = \frac{U_i - (\sum_{k=1}^{N} \lambda_k^1 x_{ik} + \ldots + \sum_{k=1}^{N} \lambda_k^N x_{ik})}{U_i - TX_i}, \quad \forall i \in I \tag{3.130}$$

$$\sum_{k=1}^{N} \lambda_k^c = 1, \qquad \forall c = \{1,2,\ldots,N\} \tag{3.131}$$

$$0 \le \mu_{y_j}, \mu_{x_i} \le 1, \quad \lambda_k^c \ge 0, \quad \text{and} \tag{3.132}$$

$$d_{y_j^c}^- = d_{y_j^c}^{-*}, d_{y_j^c}^+ = d_{y_j^c}^{+*}, d_{x_i^c}^- = d_{x_i^c}^{-*}, d_{x_i^c}^+ = d_{x_i^c}^{+*}$$

In this variation the stage 1 problem is to minimize the deviations from the crisp efficiency targets for each DMU. These crisp DMU representations measure relative efficiency in the conventional DEA sense. The DMUs for which the deviations reach zero are evaluated as efficient. The solutions for the deviations are passed as constraints to the stage 2 problem. The stage 2 objective is to maximize the satisfaction of the fuzzy effectiveness constraints while maintaining the efficiency goal achieved in stage 1.

3.6.9 *Variation 9:*

Model 3.12

Stage 1:

$$Max \sum_{x_i, y_j} (\mu_{y_j} + \mu_{x_i}) \tag{3.133}$$

Subject to:

For the achievement of Global Targets:

$$\mu_{y_j} = \frac{(\sum_{k=1}^{N} \lambda_k^1 y_{jk} + \ldots + \sum_{k=1}^{N} \lambda_k^N y_{jk}) - L_j}{TY_j - L_j}, \quad \forall j \in J \tag{3.134}$$

$$\mu_{x_i} = \frac{U_i - (\sum_{k=1}^{N} \lambda_k^1 x_{ik} + \ldots + \sum_{k=1}^{N} \lambda_k^N x_{ik})}{U_i - TX_i}, \quad \forall i \in I \tag{3.135}$$

$$\sum_{k=1}^{N} \lambda_k^c = 1, \qquad \forall c = \{1,2,\ldots,N\} \tag{3.136}$$

$$0 \le \mu_{y_j}, \mu_{x_i} \le 1, \quad \lambda_k^c \ge 0 \tag{3.137}$$

Stage 2:

$$Min \sum_{x_{ii}^c, y_j^c} (d_{y_j^c}^+ + d_{y_j^c}^- + d_{x_i^c}^+ + d_{x_i^c}^-) \tag{3.138}$$

Subject to:

For the DMU representations:

$$\left.\begin{array}{ll} \sum_{k=1}^{N} \lambda_k^c y_{jk} + d^-_{y_j^c} - d^+_{y_j^c} = y_j^c, & \forall j \in J \qquad (3.139) \\ \sum_{k=1}^{N} \lambda_k^c x_{ik} + d^-_{x_i^c} - d^+_{x_i^c} = x_i^c, & \forall i \in I \qquad (3.140) \end{array}\right] \forall \quad c \in \{1,2,\ldots,N\}$$

For the achievement of Global Targets:

$$\mu_{y_j} = \frac{(\sum_{k=1}^{N} \lambda_k^1 y_{jk} + \ldots + \sum_{k=1}^{N} \lambda_k^N y_{jk}) - L_j}{TY_j - L_j}, \quad \forall j \in J \qquad 3.141$$

$$\mu_{x_i} = \frac{U_i - (\sum_{k=1}^{N} \lambda_k^1 x_{ik} + \ldots + \sum_{k=1}^{N} \lambda_k^N x_{ik})}{U_i - TX_i}, \quad \forall i \in I \qquad 3.142$$

$$\sum_{k=1}^{N} \lambda_k^c = 1, \qquad \forall c = \{1,2,\ldots,N\} \qquad 3.143$$

$$\mu_{y_j} = \mu_{y_j}^*, \mu_{x_i} = \mu_{x_i}^*, \quad \lambda_k^c \geq 0 \qquad 3.144$$

In this variation the priorities associated with the efficiency and effectiveness constraints are reversed. The stage 1 problem is to maximize the satisfaction of the fuzzy effectiveness constraints. The solution μ^*s are passed to the stage 2 problem where the objective is to minimize the deviations from the efficiency targets for the individual DMUs. The solution to the stage 2 problem maintains the satisfaction of the fuzzy effectiveness goals achieved in stage 1.

Chapter 4 presents a real-life application of the current research.