Analysis of Decision Postponement Strategies for Aircraft Assignment under Uncertainty

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ABSTRACT

The ability to effectively match supply and demand can lead to significant revenue benefits in the airline industry. Airline supply management deals with assigning the right resources (i.e., aircraft and crew) to the right routes in the flight network. Due to certain crew regulations, operating characteristics, and constraints of the airline companies, these supply management decisions need to be made well in advance of departures, at a time when demand is highly uncertain. However, demand forecasts improve markedly over time, as more information on demand patterns is gathered. Thus, exploiting the flexibilities in the system that allows the partial postponement of supply decisions to a later time, when more accurate demand information is obtained, can significantly improve the airline's revenue. In this thesis, we propose and analyze the Demand Driven Swapping (DDS) approach that aims at improving the airline's revenue by reducing the supply-demand mismatches through dynamically swapping aircraft as departures approach. This research has been done in collaboration with our industrial partner, the United Airlines Research and Development Division.

Due to the proximity to departures, the DDS problem is restricted by two main constraints: 1) the initial crew schedule needs to be kept intact (due to certain union contracts); and 2) airport services and operations need to be preserved to the greatest extent possible. As a result, only a limited number of simple swaps can be performed between aircraft types of the same family (i.e. crew-compatible aircraft types). However, the swaps can be potentially performed on a daily basis given the initial fleet assignments. Clearly, the swapping criteria, frequency, and timing will highly impact the revenue benefits of the DDS approach. When the swapping decisions are made several weeks prior to departures (i.e., 4-6 weeks before departures), they will not cause much disturbance to the operations, but will be performed under highly uncertain demand information. On the other hand, swapping decisions that are delayed to a time later (i.e., 1-3 weeks before departures) will decrease the possibility of bad swaps, but will result in larger costs due to the higher disruptions to airport services and operations. Thus our research objective is to provide guidelines and principles on how the flexible capacity should be managed in the system. For this purpose, we study the effectiveness of different swapping strategies, characterized in terms of their frequency and timing, for hedging against the demand uncertainty. We first study stylized analytical models to gain insights into the critical parameters that affect these benefits. Simulation models are then conducted to test the validity of our analytical findings as well as to analyze more complex strategies and assess the dynamic performance of these strategies.

The analytical results indicate that strategies that make the swapping decision early in time (in order to minimize disturbances to the operations) perform very well on routes, where the demand uncertainty is low and the expected demands on the legs are well-balanced. Otherwise, a swapping strategy, which revises the swapping decision over time, should be implemented. Our simulation results, based on real data obtained from United Airlines, confirm the analytical findings.

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Chapter 1

Introduction

Today's highly uncertain and competitive market place is forcing the airline industry to look for new approaches and strategies to stay profitable. The ability to closely match supply and demand, which has always been a determining factor of financial success in any industry, is attracting renewed attention in the airline industry. Airline "supply management" deals with assigning the right *resources* (i.e., aircraft, each having different capacity; and crew, each having different qualifications) to the right flight legs in the network (i.e., flight legs at specific departure times between origin-destination pairs), whereas airline "demand management" focuses on capturing the right mix of *passengers* (i.e., passengers with different itineraries, flexibilities, and price utilities).

Certain regulations, operating characteristics, and constraints of the airline industry dictate that most tactical scheduling and assignment decisions, which affect the airline's supply, be made well in advance of flight departures, at a time when demand is highly uncertain. As a result, the focus of the airline industry has, so far, been on demand management techniques (also referred to as revenue or yield management), and their use of supply management has been very limited: With capacity utilizations around 70% (based on personal communications with United Airlines managers), airlines still experience a significant passenger loss due to insufficient capacity. Thus, there is much room for improvement, because no matter how sophisticated their demand management systems are, airlines are still working under sub-optimal solutions when they need to *fix* their supply well in advance, under limited demand information. In fact, more effective and robust supply decisions are still possible by exploiting the inherent *flexibilities* in the system, which allow the partial *postponement* (or modification) of these decisions to a time closer to departures, when more information on demand patterns is gathered and demand uncertainty is greatly reduced. Thus, this flexibility needs to be considered when managing supply and demand in the airline industry. However, research that focuses on the benefits of decision postponement strategies for airline supply management, and studies their value, is extremely limited.

Motivated by these observations, Bish and Sherali (2001) propose a three-stage airline supply management framework, which *systematically* revises the supply decisions over time, taking advantage of the more accurate demand forecasts obtained as departures approach. The focus is on the airline's *fleeting* decisions (i.e., assignment of aircraft capacity to different flight legs in the network); see Bish and Sherali, 2001, for details on this framework. This supply management framework is devised so as to postpone certain parts of the initial fleeting decision to a time closer to departures, by utilizing the flexibility in the system. This is accomplished through an initial fleet assignment model that is paired with downstream models, demand driven re-fleeting and swapping. Our focus in this thesis is on the "demand-driven swapping" model. This thesis is a sequel to the Demand Driven Re-fleeting stage (see Sherali, Bish, and Zhu, 2001), and has been conceived in collaboration with the *United Airlines Research and Development Division* as our industry partner.

This chapter is organized as follows. In Section 1.1, we provide more details on the problem. Section 1.2 then presents the research questions that we propose to study in this thesis.

1.1 Background and Motivation

Two major components of airline supply management include the "fleet assignment" process, which refers to the assignment of aircraft to the flight legs in the network, and the "crew scheduling" process, which refers to the assignment of each crew to those flight legs that it is qualified to fly. Throughout this thesis, we will refer to an "aircraft type" as the set of aircraft, each having the same cockpit configuration, crew rating, and capacity. An example is the B737-300 aircraft type. Similarly, we refer to an "aircraft family" as the set of aircraft types having the same cockpit configuration and crew rating. Thus, the same crew can fly any aircraft type of the same family, even when they are of different capacities. An example is the Boeing 737 family, which consists of multiple aircraft types, including the B737-300, B737-400, and B737-500, having capacity ranges between 110-168 passengers.

In order to manage their supply, most major airlines utilize a fleet comprised of multiple aircraft types so as to be able to match the aircraft capacity with the forecasted flight demand. This is important, because if the demand observed on a flight leg exceeds the capacity of the aircraft assigned to that leg, then unsatisfied passengers will be *spilled* (lost). On the other hand, if demand turns out to be lower than the capacity assigned, then excess seats will be *spoiled*, incurring higher operational costs. However, dependencies between several airline processes dictate that the fleeting decision is made quite early in time, under limited demand information. For example, the fleeting decision is an essential input to the crew scheduling process, which, by typical union contracts and government regulations, needs to be completed well in advance of departures (i.e., 8-12 weeks out at United Airlines). Thus, a fleeting decision needs to be made even earlier, under high demand uncertainty (demand standard deviations of 20-50% of the mean for any flight leg are typical at this time; see Berge and Hopperstad, 1993; Kniker, 1998). In addition, since most crew members are certified to fly only one aircraft family due to the cockpit configuration, any revision to fleet assignments that requires changes in the current crew assignments will be very difficult and expensive to implement. As a result, the main focus of airlines so far has been on demand management (see, for instance, the many references in Mcgill and van Ryzin, 1999). In contrast, their use of *systematic supply management* strategies has been very limited.

Thus, the fleet assignment problem needs to be solved quite early in time for the entire fleet over a planning period. Solving such a large scale problem early in time hinders the consideration of a more accurate and detailed demand information in the fleeting process. Indeed, the demand information, utilized by the traditional fleet assignment models proposed in the Operations Research literature and in use by major airlines, is very limited. Specifically, the demand information used in the traditional fleet assignment process suffers from the following major drawbacks.

- Most fleeting models generate the "same-every-day" fleeting solution (i.e., the same fleet assignments are used every day of the planning period), considering only an *aggregated* demand information over the different days of the week (or month). This is due to the need to limit the size of the fleeting model to a manageable level. However, demand patterns on the same leg may vary *significantly* over different days of the week ("**day-of-week (DOW) variation**"); for example, Mondays and Fridays are usually higher demand days for business markets. In addition, such an aggregated demand information fails to consider the uncertainty that demand on any given day will have around its mean ("**variation**

around the mean"). In addition, the parameters of the demand distribution need to be estimated well in advance of departures. However, demand parameters will probably not be known with certainty *a priori* at that time ("**forecast error**").

Although the need to address these limitations in the fleeting decision has been acknowledged by many researchers in the airline industry, the related work has been extremely limited due to the underlying difficulties. Specifically, the large size of the fleeting problem, which can easily involve thousands of flight legs and several aircraft types, and the time-frame involved (i.e., demand forecasts at the time of the fleeting are not very reliable, but they improve markedly over time) hinder the consideration of all the foregoing enhancements in the initial fleeting stage. Thus, updates to the initial fleet assignment, as departures approach and more accurate demand forecasts are obtained, become inevitable to implement in order to manage the day-of-week variation, demand variation around its mean on any given day, and the forecast error. However, there are two main restrictions on these re-assignments:

1. The initial crew schedule needs to be kept intact.

2. Due to the close proximity to departures, scheduled airport services and operations need to be preserved to the greatest extent.

As a result, only a limited number of swaps (exchanges) between aircraft types of the same family are possible at this later stage, but the swaps can be potentially performed on a daily basis, given the initial fleet assignments. This added flexibility provides an increased ability to hedge against the two types of demand variation: (i) DOW variation ("variation in the mean") and (ii) demand variation around its mean on any particular day ("variation around the mean"), as well as (iii) demand forecast errors (i.e., when demand distribution parameters are not perfectly known *a priori*). Thus, revenue can still be improved significantly via the utilization of the flexible capacity in the system, due to the ability to incorporate the more accurate and detailed demand forecasts available in this stage. In this research, we refer to the *limited swapping of aircraft within one family based on updated demand forecasts* as the *Demand Driven Swapping* (*DDS*) problem. This problem is the focus of this thesis.

The next section presents our proposed research directions.

1.2 Research Questions

To summarize, the value of the DDS (Demand Driven Swapping) approach lies in the ability to postpone certain parts of the initial fleeting decision, through the utilization of dynamic swaps within each family, to a later point in time, when more accurate demand information is available. This added flexibility provides an increased ability to hedge against the DOW variation, demand variation around its mean on any particular day, and demand forecast errors. In this research, we focus on one of these effects, demand variation around its mean on a particular day.

Clearly, the swapping criteria, frequency, and timing will have a significant impact on its benefits. Swapping decisions that are made several weeks prior to departure (i.e., 4-6 weeks out) will not cause much disturbance to operations, but will be based on more uncertain demand. On the other hand, delaying (or revising) the swapping decision until demand uncertainty is greatly resolved (i.e., 1-3 weeks out), will benefit from having improved demand information, thus reducing the possibility of "bad" swaps (swaps with loss), but at a possibly larger cost of disrupting operations. In addition, if the swapping decision is made too late, some customers may have already been rejected due to capacity restrictions. Thus, in the case of *delayed* swaps, it is crucial to perform the revenue management study while considering the possibility of subsequent swaps, instead of simply using fixed capacities. As a result of the foregoing trade-off, delayed swapping decisions will, probably, be made only for a set of "critical" legs. In addition, since each swap is limited within aircraft types of the same family, there will be high dependency between the initial fleeting solution and the DDS problem.

In this research, our focus will be on the following research questions: How should we manage the *flexible capacity* in the system (obtained by the later swapping capability) to hedge against the demand variation around its mean? When should these swaps be performed? What is the value of more information on the swapping decision?

In our late DDS implementation stage, swaps that are most likely to be performed on a daily basis will be *simple* swaps, assigned to aircraft types within the same family. Thus, our research plan will be to consider simple "swappable loops", which are associated with a set of round-trips originating and ending at a common hub airport with similar departure and arrival times so that they can be swapped without violating the flow balance in the flight network. Our objective is to study the impact of demand characteristics (mean and variance of demand on a particular day) on the benefits of swapping. Thus, we will also consider that initial demand forecasts are accurate (i.e., mean and variance of flight demands are known with certainty at the outset). Although this assumption is not realistic, it is necessary to extract the pure benefits of demand characteristics on the benefits of swapping. Specifically, we will study various swapping strategies, considering loops having different demand characteristics. To achieve our goals, we will (1) develop stylized analytical models to gain insights into the effectiveness of different swapping strategies, and (2) create simulation environments to test the validity of the insights gained through the mathematical models as well as analyze more complex strategies.

This thesis is organized as follows. In Chapter 2, we present a very brief overview of the related literature on decision postponement strategies in manufacturing and service industries. Chapter 3 introduces our model, assumptions, and the notation that will be used throughout the thesis as well as some preliminary analysis and derivations that will be utilized when studying more complex models in Chapter 4. Chapters 4 and 5 then present a comprehensive analysis of different swapping strategies under different assumptions. Finally, we conclude, in Chapter 6, with a discussion of our results and future research directions. All derivations are included in the Appendix.

Chapter 2

Literature Review

The Demand Driven Swapping (DDS) approach presented in this thesis represents a *systematic decision postponement strategy* that benefits from the utilization of the partial flexible capacity in the system, which refers to the ability to revise the fleeting decision as departures approach and more information on demand is gathered. In our late DDS implementation stage, swaps that are most likely to be performed on a daily basis will be simple loop swaps, assigned to aircraft types of the same family. Recall that the latter restriction is due to the need to preserve the initial crew schedule. Thus, the potential DDS swaps in this stage will be entirely determined by how the *flexible capacity* (i.e., aircraft types within each family) in the system is assigned to *swappable loops* in the flight network. Thus, there will be high dependency between the initial fleeting assignment and the later swapping capability. As a result, we limit our review of literature to two main areas: (1) Decision postponement strategies and the value of information in manufacturing and service industries; and (2) airline fleeting, re-fleeting, and swapping problems. Sections 2.1 and 2.2 provide a very brief overview of research in each of these areas.

2.1 An Overview of Decision Postponement Strategies in Manufacturing and Service Industries

The main value of *decision postponement* strategies lies in the ability to delay some portion of the tactical and operational decisions to a time when more information on demand patterns is gathered and demand uncertainty is reduced. The literature on decision postponement strategies, and especially on the value of information in managing the supply chain, has grown significantly in the last decade. However, most of this research has focused on manufacturing and other service industries, but not the airline industry. Thus, much research is still needed to study the value of a *systematic decision postponement approach*, along with the strategies that can be used to effectively manage supply, in the airline industry. In what follows, we present a very brief overview of research in this area and give some examples to illustrate the different decision postponement strategies possible in different environments. We refer the interested reader to Tayur, Ganeshan, and Magazine (2000) for an extensive review.

Delayed product differentiation is one of the most commonly used decision postponement strategies in manufacturing environments (see, for instance, Aviv and Federgruen, 2000, for an extensive review of research on these strategies). The underlying idea is to delay the point a semi-finished product is differentiated into its end-products via a re-design of the manufacturing process so as to manage the risks associated with product variety and demand uncertainty. The success of this strategy is due to statistical pooling effects and improved demand forecasts. Clearly, this strategy is most beneficial to implement for a product family, whose end-products share a high level of commonality, while having highly variable demand patterns; see, for instance Aviv and Federgruen (2001a, 2001b); Gavirneni and Tayur (2000); Swamanithan and Tayur

(2000); and the references cited therein, as well as several researchers in the marketing literature, including Alderson (1950), who first introduced the idea; and Cox and Goodman (1956). A similar idea can be used in the transshipment of goods from manufacturers to the retailers. A commonly used strategy that makes use of this idea is the *delayed geographic differentiation* strategy, which involves coordinating several outlets through a regional distribution center, thus delaying the time shipments are made to the outlets (Aviv and Federgruen, 2000).

Similar ideas can be beneficial when implemented in manufacturing industries, where capacity investment decisions are capital intensive and need to be made long before production starts, with limited information on future demand patterns. Although it might be possible to update the initial capacity investment levels in the production stage in such environments, investment costs will generally be much higher and lead times of installing additional capacity will be very long. Consequently, decision postponement strategies that can reduce the sensitivity of the initial capacity investment decisions to demand uncertainty will incur large benefits. The decision postponement strategies considered in van Mieghem and Dada (1999) include price and production postponement strategies in a single product environment. They show, through an analytical model, that price postponement strategies make the capacity investment and production (inventory) decisions more insensitive to uncertainty. Hence, such postponement strategies can be valuable options to the firm. A related area is investments in *flexible manufacturing capacity*, which provides the ability to delay the allocation of the total production capacity to different product demands to a later period in time, when demand uncertainty is greatly resolved (see, for instance, Biller, Bish, and Muriel, 2000; Bish, Muriel, and Biller, 2001; Netessine, Dobson, and Shumsky, 2000; van Mieghem, 1999; and the references therein). In the same vein, several researchers have studied the impact of demand information on inventory levels (see

Anand, 2000, for an extensive review), and on variability in the supply chain (see Chen et al., 2000; Gavirneni and Tayur, 2000, for extensive reviews). The value of information is also studied in the context of supply contracts in a supply chain, where a buyer and a supplier need to specify different levels of commitments for financial, material, and information flows under uncertainty (see Tsay, Nahmias, and Agrawal, 2000, for an extensive review on supply contracts).

To our knowledge, research that focuses on decision postponement strategies and studies their benefits considering the airline industry is extremely limited. Next, we present a brief overview of research on airline fleeting decisions.

2.2 Fleeting, Re-fleeting, and Swapping Models in Airline Supply Management

In this section, we summarize approaches used for airline fleeting, re-fleeting, and swapping. Please refer to Zhu (2001) for an extensive literature review. Although the focus of this research is not on the fleeting problem, our DDS implementation will be highly impacted by the initial fleeting solution. In the following, we start with a very brief overview of the fleet assignment process, discuss its major short-comings, and then present the literature on airline re-fleeting and swapping strategies.

Since the fleet assignment problem (FAP) is a major component of the airline scheduling process, it has been extensively studied by researchers and practitioners (see Gopalan and Talluri, 1998; Yu and Yang, 1998; and Zhu, 2001, for extensive reviews). Most researchers have used the “same-every-day model” (i.e., the same fleet assignments are used every day of the planning period) for the FAP. This is mainly due to the need to limit the size of the resulting problem, and thus the computational requirements, to a manageable level. Formulating the FAP as an integer program, researchers have developed a variety of solution approaches to obtain good fleeting solutions in reasonable computing times (see, for instance, Abara, 1989; Daskin and Panayotopoulos, 1989; Gu et al., 1994; Hane et al., 1995; Rushmeier and Kontogiorgis, 1997; Yan and Young, 1996; and the references cited therein). All these fleeting models are based on integer programming formulations, which consider three main sets of basic constraints: (1) *cover* constraints, which require each flight leg to be assigned to exactly one aircraft type; (2) *balance* constraints, which ensure that flow in and out of each airport in the network is balanced; and (3) *count* constraints, which ensure that the number of each aircraft type used in the fleeting solution does not exceed the number of that type available in the airline's fleet. In addition, most of these formulations incorporate leg demand versus aircraft capacity information only through the use of suitable cost terms in the objective function. Several extensions to the FAP have also been studied, such as including aggregate aircraft maintenance considerations (Clarke et al., 1996; Subramanian et al., 1994); combining aircraft fleeting and routing (Barnhart et al., 1998; Desaulniers et al., 1997); and allowing for flight departure times to vary within a small time-window so that more choices of assigning aircraft to legs are possible (Rexing et al., 2000). Not surprisingly, the problem size grows considerably with each enhancement, and the problem becomes more difficult to solve to optimality.

As discussed in the previous chapter, most of these fleet assignment models consider the demand distribution information only via suitable cost terms in the objective function, determined several months prior to departures, and the demand distributions considered are *aggregated* over the planning horizon. However, demand forecasts at the time the fleet assignment problem is solved (usually 8-12 weeks in advance of departures) are not very reliable, but the forecast accuracy improves markedly over time. Consequently, a re-fleeting or a swapping model that makes use of a more accurate and detailed demand information that these fleeting models lack can provide significant revenue impact to airline companies. We note here that a "systematic supply management strategy" (i.e., systematically revising fleeting decisions over time so as to incorporate the improved demand forecasts into the fleeting decisions), as in Sherali, Bish, and Zhu, 2001, requires solving the initial fleeting problem considering the dependencies between the initial fleeting and the later re-fleeting and/or swapping problems. However, such an approach has not been addressed at all in these fleeting models and in the re-fleeting literature cited below. All fleeting models are solved to obtain a "fixed" fleeting solution, whereas the re-fleeting models are solved rather sporadically. In fact, to our knowledge, research that focuses on these interactions to propose an overall supply management scheme for airlines is nonexistent.

Research in the re-fleeting area is relatively new and limited, and has mostly focused on recovery from irregular operations (see, for instance, Jarrah et al., 1993; Thengwall et al., 2000; Yu and Luo, 1997). Berge and Hopperstad (1993) are one of the first researchers to study the benefits of *systematically* revising fleet assignments over time, due to demand-capacity mismatches, as flight departures approach and forecasts improve. Their proposed model consists of solving the fleet assignment model at the beginning of each period, based on updated demand information. Heuristics are proposed to solve the fleet assignment problem, which is formulated as a multi-commodity network problem. After aircraft types are assigned to legs in the flight network, simulation is performed to generate leg demands based on the demand forecasts, and to estimate the resulting loads, spill (customers rejected due to capacity restrictions), and the revenue. At the beginning of the next period, the assignment costs for each (aircraft type-demand leg) pair are updated and this process is repeated. Their computational study, based on real airline data, suggests that the proposed approach can improve the profit by 1-5%.

More recently, Jarrah et al. (2000) present a re-fleeting model, having side constraints added to model the maintenance opportunities, crew staffing levels, and noise restrictions, together with a user specified parameter on the maximum number of changes that can be made to the current schedule. The focus of Jarrah et al. is on generating several near-optimal solutions so that the user can select the most "appropriate" solution based on an operational perspective.

Talluri (1996) develops algorithms to modify the initial fleet assignment solution, when two aircraft types need to be swapped between a pair of airports. The algorithms proposed by Talluri are based on a series of shortest-path algorithms, which minimize the number of changes to the initial fleeting solution, but are limited to two aircraft types. Ahuja (2000) also focuses on algorithmic developments to swap aircraft, but in his approach the swapping problem needs to be solved just after the initial fleet assignment phase so as to perform swaps such that a set of flight legs can be assigned to the same aircraft type (i.e., *through flights* are formed), if possible. While both of these papers and our proposed research focus on the aircraft swapping stage, the research objectives are totally different. Talluri and Ahuja focus on algorithmic developments to update the initial fleet assignment solution so as to swap aircraft. On the contrary, while focusing on simple loop swaps, where the fleet assignment solution after the swap can be easily obtained, our objective is to understand how to manage the swappable capacity in the system.

Although some airlines have been using re-fleeting models in the earlier stages (i.e., to react to changes in assumptions and to manage the demand versus capacity discrepancies), to our knowledge there is no major US airline that is managing its supply through a systematic manipulation of the flexible capacity in the system to its full extent. However, several airlines are in the process of evaluating the benefits of such an approach. Two examples include United Airlines and Continental Airlines. A preliminary simulation study performed by United Airlines exhibited high benefits as a result of managing supply by taking advantage of this flexible capacity. Similarly, Continental Airlines has been implementing a pilot study to test the benefits of a demand driven swapping approach (see Pastor, 1999). In their pilot study, Continental Airlines makes use of two types of swaps: 60 DTD (Days to Departure) swaps and 14 DTD swaps. The former causes less disruptions to operations, because the swap is performed before most airport services and operations are scheduled. At this time, changes to the crew schedule are allowed at Continental Airlines. Thus, the swap might involve aircraft types of different families. On the other hand, the 14 DTD swaps are restricted within each family due to the need to preserve the crew schedule. However, the swaps are based on more accurate demand forecasts, since the swap decision is made closer to departures. In a simulation study, they first use the expectation of revenue gain in the swapping decision. The simulation study shows that while most swaps performed under this rule turn out to be successful, there might be some bad swaps resulting in loss. In order to remedy this situation, information on the probability distribution of the revenue gain, including its standard deviation, minimum, maximum, and average, as well as the percentage of simulation trials that achieve positive revenue gain and the percentage that meet the minimum revenue goal, is used in the swapping decision. For a given swap possibility, all these performance measures are estimated via simulation. For example, when the decision rule is such that the swap is performed only when at least 80% of the simulation trials meet the minimum revenue goal, the revenue is improved by 8%, while the number of unprofitable swaps are reduced over the policy using only the expected revenue gain in the swapping decision.

Other examples of airline companies that make use of systematic swapping approaches include Austrian Airlines and KLM. However, both of these airline companies make use of limited swaps in a very constrained way (swaps are performed 6-8 weeks out in Austrian Airlines and 2-4 weeks out in KLM; the process is manual in both airlines; see Barocio-Cots, 1999; Berge and Hopperstad, 1993).

Chapter 3

Model, Notation, and Preliminaries

3.1 Research Objectives

As mentioned in Chapter 1, the *frequency* and *timing* of the swaps in the DDS stage will highly impact their revenue benefits. Swapping decisions that are made several weeks prior to departures (i.e., 4-6 weeks out) will not cause much disturbance to operations, but will be based on *limited* demand information. On the other hand, delaying the swapping decision to a time when the demand uncertainty is greatly resolved (i.e., 1-3 weeks out) or revising the swapping decision later will benefit from the consideration of the most up-to-date demand information, thus resulting in more profitable and less risky swaps, but at a larger cost of disrupting operations. In addition, if the swapping decision is made too late, some customers may have already been rejected due to capacity restrictions. Thus, in the case of a delayed swapping strategy, the swap potential needs to be considered when passengers, arriving over time, are accepted or rejected from their requested flights. This, however, requires the consideration of the swap potential in the revenue management process.

Airline revenue management is based on segmenting the market by offering multiple "fare-classes" on each flight leg (an average of 5 fare-classes per leg is offered by United Airlines): each fare-class corresponds to a different (fare, restriction) combination. For example, a Y fare-class does not impose any restrictions (i.e., it can be cancelled anytime after purchase with full refund), is offered at a higher price, and is aimed towards capturing the *business* passengers, whereas a W fare-class is more restricted (i.e., it should be purchased usually two weeks in advance of departures), and is offered at lower prices, so as to attract *vacation* passengers. Thus, a delayed swapping strategy needs to be integrated within the revenue management process. In addition, as mentioned above, a delayed swapping strategy might require the re-scheduling of airport services and operations. Consequently, a delayed swapping strategy is more difficult to implement in practice, and thus, such strategies need to be limited to a set of "critical" legs, which would benefit most from the potential to revise/modify the swapping decision. On the other hand, swaps carried out under limited demand information will be "risky" swaps, having the possibility of leading to losses, due to the inability to react to updates on demand information. In addition, since each swap is limited within aircraft types of the same family, there will be high dependency between the initial fleeting solution and the DDS problem.

Consequently, our objective in this thesis is to study important research questions How should we manage the *flexible capacity* in the system, obtained by utilizing the capability to swap aircraft types, within each family, as departures approach? What is the value of more information on the swapping decision? Can we identify a set of "critical" legs that would benefit most from a delayed swapping strategy?

Thus, these research questions attempt to explore the benefits of different swapping mechanisms. Our objectives are thus (1) to analyze the trade-off between better information (i.e., a delayed swapping versus an early swapping) and higher disruption costs to operations and the possibility of customers being rejected prior to the swapping decision; and (2) to analyze how demand characteristics affect the benefits of swapping. Our research methodology will consist of studying stylized analytical models to gain insights into these effects; and creating simulation environments to test the validity of the insights gained through the mathematical models as well as analyze more complex strategies and assess the dynamic performance of these strategies when several model assumptions are relaxed.

First, we develop simple analytical models to study the effectiveness of different swapping strategies, characterized in terms of their timing and frequency, while considering legs with different demand parameters. Our focus in this thesis is limited to the benefits of swapping to hedge against the variation in demand around its mean on a particular day, and the capacity-demand discrepancies on each leg. Then, we extend our analysis and understanding to more general cases through the use of simulation models that relax several assumptions used in the analytical study.

This chapter is organized as follows. In Sections 3.2 and 3.3, we present the model and the notation that will be used throughout this thesis. Then, in Sections 3.4 and 3.5, we present some preliminary analysis, which will be used as the building blocks when we analyze more complicated models in the subsequent chapters.

3.2 Model and Assumptions

To gain insights into our research questions, we first study a simple analytical model that considers a pair of routes, *swappable* with each other (routes originating and terminating at a common airport within similar time frames so that the aircraft assigned to those routes can be swapped with each other, if needed), and having been assigned two *swappable* aircraft types (of the same family), with respective capacities of C_1 and C_2 passengers ($C_1 < C_2$), in the initial fleeting solution. We study the effectiveness of several swapping strategies, characterized by their *timing* (i.e., when the swapping decision is made), and *frequency* (i.e., how often the swapping decision is revised). Two types of swapping strategies are considered: (1) *Incomplete information* swapping strategies that are allowed to revise the swapping decision only a *limited* number of times until departures, under *incomplete* information on demand; and (2) *Perfect information* swapping strategies that make the swapping decision under *perfect* information on demand. Although perfect information swapping policies are not realistic, they provide an *upper bound* on the possible benefits of swapping, and are thus useful to analyze and compare with the former ones. Thus, the former are used to hedge against both demand uncertainty and demand-capacity discrepancies, whereas the latter allows us to single out the effectiveness of flexibility to hedge against demand-capacity discrepancies in the absence of demand uncertainty, since the swapping decision is made under perfect information on demand. Comparing the two, we can understand the additional benefits that can be realized by having the ability to revise the swapping decision later on, under reduced uncertainty. In the following, we let t denote the number of periods until departures (i.e., the number of periods "to go"). That is, departures occur at time 0.

Although many swapping strategies are possible, we focus on the following set of policies, since they are easy to implement in practice, and also these policies represent extreme cases that can help develop insights. Specifically, we analyze the following swapping strategies in our DDS model.

Incomplete Information Swapping Strategies:

- **Limited Swapping Strategy** ("*Limited*"): Under the limited swapping strategy, swapping decision is made T periods before departures, under *limited* information on demand, and *is not* revised later, in order not to disrupt operations. Consequently, swaps performed under this strategy can be "risky"; that is, they may lead to loss.

- **Delayed Swapping Strategy** ("*Delayed*"): Under the delayed swapping strategy, swapping decision is made T periods before departures, under *limited* information on demand, but can be *revised* later, at the beginning of each period t, $t = T-1, T-2, \cdots, 2, 1$, until departures, at the cost of disruptions to operations.

Under incomplete information, swapping decisions can be made based on various measures of merit, including the expected revenue gain of the swap, the probability that the revenue gain is positive, and/or incorporating the variance of gain into the decision. In our models, we consider that under both the limited and delayed strategies, the swapping decision is made based only on the expected revenue gain of the swap. It is a future research direction to incorporate other merits into our analysis.

Perfect Information Swapping Strategies:

- **Perfect Information Swapping Strategy** ("*Perfect*"): Under the perfect information swapping strategy, we assume that swapping decision is made under perfect information on demands.

Although the perfect information strategy is not realistic, it provides an *upper bound* on the revenue benefits possible under any swapping policy, and is needed to extract the *value of information* in our swapping decision.

As stated above, each of these policies is attractive in practice for different reasons: Limited swapping strategies do not cause much disturbance to operations, but carry a higher risk, since the decision is made under high demand uncertainty. On the other hand, delayed information strategies might lead to the re-scheduling of several airport and service operations, but will yield higher revenue gains. Thus, this *trade-off* needs to be considered when determining a swapping policy. Of course many other swapping strategies can be devised, each with different timing, frequency, and decision criteria. However, since the limited, delayed, and perfect information swapping strategies represent extreme cases (in the first one, swapping decision is made only once, under high demand uncertainty, and is not revised later; in the second one, swapping decision is revised every period; and in the last one, the decision is made under no demand uncertainty), insights developed for these strategies should be useful for developing other strategies that are combinations of these.

As stated above, although delayed swapping strategies are attractive in reducing the riskiness of the swaps, they face several problems in practice. First, it is costly and undesirable to reschedule airport and service opertations, including cargo and food catering. As an example, consider the cargo planning process. If passenger aircraft is used for cargo transportation, which is the case for our industry partner, then changes in aircraft capacity (and hence, its cargo capacity) would require changes in the cargo schedule as well. Secondly, it may not be possible to swap aircraft at a time closer to departures, if the number of accepted ticketed customers has already exceeded the capacity of the smaller aircraft. This is due to the constraint that does not allow spilling of any ticketed customers. Indeed, in our analysis,(see Chapter 4), we found that delayed strategy cannot perform many swaps when the time gets closer to departures.

Next, we describe the *sequence of events* in our model.

1. We are given the initial capacity assignments, of C_1 and C_2, for the two swappable loops;

2. T time periods before departures, we make our swapping decision (under both limited and delayed strategies) based on the expected revenue gain of the swap, determined using the current demand forecasts; and swap aircraft, if beneficial (i.e., if the expected revenue gain is positive);

3. At the beginning of each period t, $t = T-1, \cdots, 2, 1$, we *update* our demand forecasts based on the demand realizations in periods $t+1, \cdots, T$. Under the delayed strategy, we then *re-evaluate* our swapping decision based on the expected revenue gain corresponding to the updated demand forecasts; and revise our swapping decision, if needed. Demand is realized in period t;

4. Demand in the last period (period 1) is realized and the resulting revenue gain or loss (over the "base case", which does not involve any DDS swapping) is determined under the limited, delayed, and perfect information swapping strategies.

Recall that under the perfect information swapping strategy, swapping decision will simply be made based on the overall demand realized on each leg at departures, and revenue will be determined assuming that no demand will be lost due to the delayed timing of the swapping decision. This will be detailed in Section 3.4.

We assume that the total demand observed on any leg initially assigned to the larger capacity, C_2, up to time T is **not** larger than the smaller aircraft capacity, C_1. Thus, a swap at the beginning of time T is still possible, since it will not spill any passengers already ticketed. This is a very reasonable assumption, especially when considering domestic flights 4-6 weeks prior to departures, which is the time period the swapping decision needs to be made under the limited swapping strategy. This assumption will simplify the expressions for the limited swapping policy, but as will be explained in Chapter 4, we will still consider that a revision to the swapping decision under the delayed swapping strategy will not be made in time t, $t = T-1, \cdots, 1$ if the swap spills any ticketed passengers. Since the delayed swapping strategy makes use of a number of revisions to the swapping decision until departures, we still need to consider this possibility.

Recall that our research objective is to study the effectiveness of different swapping strategies to hedge against the variation around mean demand and demand-capacity discrepancies in the swappable routes, so that we can devise effective swapping strategies based on demand characteristics.Consequently, we make certain assumptions that allow us to isolate the pure effect of these demand characteristics on the benefits of swapping. Specifically, we assume that there is no *demand forecast error* (i.e., all parameters of the demand distributions are known with certainty at the outset). It is an interesting future research direction to incorporate the effect of forecast error in our analysis. We first consider a single fare-class (class) on each leg in our analytical models. Then, we will extend our model to multiple fare-classes through a simulation model in Chapter 5.

In the following section, we present the notation that will be used throughout this thesis.

3.3 Notation

We let L_i denote the set of legs in the swappable routes, assigned to aircraft having capacity C_i in the initial fleeting solution, for $i = 1, 2$, where $L_1 = \{l_1, l_2, \cdots, l_{n_1}\}$ and $L_2 = \{k_1, k_2, \cdots, k_{n_2}\}$, and let $L = L_1 \cup L_2$. Thus, if a swap is made, then aircraft having capacity C_2 will be assigned to legs in set L_1, and aircraft having capacity C_1 to legs in set L_2. In what follows, we consider, without loss of generality, that $n_1 = n_2 = n$. However, all our expressions can be extended to the case with $n_1 \neq n_2$. We let $C_{mid} = \frac{C_1+C_2}{2}$.

As stated in the previous section, we first study a model, considering a single fare-class (also equivalent to a multiple fare-class model under some restrictive assumptions). We let D_i denote the demand on flight leg i, $i \in L$, which is a random variable. In this thesis, we consider that each demand D_i is independently, normally distributed with mean μ_i, standard deviation σ_i, probability density function (pdf) $f_i(.)$ and cumulative distribution function (CDF) $F_i(.)$. Observe that the normal demand distribution considered represents a continuous approximation of the discrete demand. A normal distribution approximation has been widely used in the literature to represent airline demand as well as demand in more general settings (see, for instance, Barocio-Cots, 1999; Kniker, 1999; and the references cited therein). In addition, our airline partner has been modeling leg demands using the normal distribution. This is because it is possible to specify the first two moments of the random variable under the normal distribution, and the normal distribution lends itself to analytical tractability. In addition, the probability of negative values in this normal distribution is assumed to be negligible. In our analysis, this is justified for demand coefficient of variations (c.v.), $\frac{\sigma_i}{\mu_i}$, of at most 0.30, which are typical for the variability in demand on a particular day observed in the airline industry in this time period (4-6 weeks out).

We let D_{it} denote the demand on leg i in period t, which is also assumed to be normally distributed with mean μ_{it} and standard deviation σ_{it}, for $i \in L$; $t = 1, 2, \cdots, T$. We assume that demand on each leg is independent across periods, and also between legs. These assumptions, also used in the demand forecasting system of our industrial partner, allow us to express D_i as $\sum_{t=1}^{T} D_{it}$, where $\mu_i = \sum_{t=1}^{T} \mu_{it}$ and $\sigma_i^2 = \sum_{t=1}^{T} \sigma_{it}^2$, for $i \in L$. Similarly, D_{it}^c denotes the cumulative demand (demand to go) on leg i in periods $t, t-1, \cdots, 1$; that is, $D_{it}^c = \sum_{j=1}^{t} D_{ij}$, which then is also normally distributed with parameters $\mu_{it}^c = \sum_{j=1}^{t} \mu_{ij}$ and $(\sigma_{it}^c)^2 = \sum_{j=1}^{t} \sigma_{ij}^2$, for $i \in L,\ t = 1, \cdots, T$. We let d_{it} represent the demand realized (observed) on flight leg i in period t, for $i \in L, t = 1, \cdots, T$. In the delayed swapping strategy, we update demand forecasts at the beginning of each period as demands are realized. This is detailed below.

We let $\tilde{D}_{it}^c$ denote the cumulative demand forecast on leg i, updated at the beginning of period t, based on demand realizations in periods $t+1, \cdots, T$. The mean, $\tilde{\mu}_{it}^c$, and standard deviation, $\tilde{\sigma}_{it}^c$, of the random variable $\tilde{D}_{it}^c$ are updated as follows:

$$\tilde{\mu}_{it}^c = \sum_{j=t+1}^{T} d_{ij} + \sum_{j=1}^{t} \mu_{ij}$$

$$(\tilde{\sigma}_{it}^c)^2 = \sum_{j=1}^{t} \sigma_{ij}^2$$

Thus, $\tilde{D}_{it}^c$ is normally distributed with $\tilde{\mu}_{it}^c$ and $(\tilde{\sigma}_{it}^c)^2$ for $i \in L$, $t = 1, \cdots, T-1$.

Finally, for each $i \in L$, we define the following disjoint events: $s_i = \{D_i < C_1\}$; $\quad b_i = \{C_1 < D_i < C_2\}$; $\quad g_i = \{D_i > C_2\}$. Thus, we can write:

$$Pr(s_i) = F_i(C_1), \quad Pr(b_i) = F_i(C_2) - F_i(C_1), \quad Pr(g_i) = 1 - F_i(C_2).$$

where $F_i(C_j)$ is an accumulative demand distribution function of leg i up to capacity C_j.

For random variable X having mean μ_X and standard deviation σ_X, we define $e_X(x) \equiv exp(\frac{-(x-\mu_X)^2}{2\sigma_X^2})$; $x^+ \equiv \max\{x, 0\}$ and $x^- \equiv \min\{x, 0\}$. We let $E(X)$ and $Var(X)$ denote the expectation and variance of X that is, $E(X) = \mu_X$ and $Var(X) = \sigma_X^2$. Throughout, lower-case letters represent realizations of the random variables, whereas capital letters represent the random variables.

In the following, we use superscripts P, D, and L to denote the perfect information ("*perfect*"), delayed ("*delayed*"), and limited ("*limited*") swapping strategies, respectively. We let G^P, G^D and G^L respectively denote the revenue gains under swapping strategies *perfect, delayed*, and *limited*. Clearly, $E[G^L] \leq E[G^D] \leq E[G^P]$. As mentioned above, in order to extract the pure impact of demand characteristics (variation around mean on a particular day and demand-capacity discrepancies) on the revenue gains of the different swapping policies, we consider the same fare on each leg in the swappable routes, and focus on the following performance measures: (1) how the expected gain and the variability in gain differ under these three strategies; (2) the proportion of time the gain of *perfect* is higher than those of *delayed* and *limited*, and (3) the proportion of time the gain of *delayed* is higher than that of *limited*. As the following, we derive analytical expressions for the revenue gains under the perfect information and limited swapping strategies, and study the delayed swapping strategy through a simulation model in Section 4.2.2.

3.4 Deriving Expressions for the Perfect Information and Limited Swapping Strategies

As defined above, L_1 and L_2 respectively denote the set of legs assigned to capacities C_1 and C_2 in the initial fleeting solution; and $L = L_1 \cup L_2$. For each $i \in L$, we define the random variable $\Delta_i \equiv \min\{D_i, C_2\} - \min\{D_i, C_1\}$. Thus, for each $i \in L_1$, revenue contribution of leg i to the swap is $+\Delta_i$, and for each $j \in L_2$, its revenue contribution is $-\Delta_j$. Hence, we can write,

$$
\begin{aligned}
G^P &= \max\{\sum_{i \in L_1} \Delta_i - \sum_{j \in L_2} \Delta_j, 0\} \\
G^L &= \begin{cases} \sum_{i \in L_1} \Delta_i - \sum_{j \in L_2} \Delta_j, & \text{if swapped in period } T, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned} \tag{3.1}
$$

As previously stated, the revenue gain under the perfect information strategy provides an *upper bound* on the gain possible under any swapping strategy, since it increases revenue by eliminating all bad swaps by utilizing the perfect information on demands.

We let $G \equiv \sum_{i \in L_1} \Delta_i - \sum_{j \in L_2} \Delta_j$ and let Π denote a binary *decision variable*, which is 1, if a swap decision is made under the limited swapping strategy in period T; and 0, otherwise. Thus, due to our decision criteria used in the limited swapping strategy, we can write:

$$\Pi = \begin{cases} 1, & \text{if } E[G] > 0 \text{ in period } T, \\ 0, & \text{otherwise.} \end{cases}$$

Next we define random variables $\omega_{ij} = \Delta_i - \Delta_j$, for each $i \in L_1, j \in L_2$. Recall that $|L_1| = |L_2| = n$ Thus, $L_1 = \{l_1, \cdots, l_n\}$ and $L_2 = \{k_1, \cdots, k_n\}$. Hence, we write:

$$\begin{aligned} E[G^L] = \max\ \{E[\sum_{i=1}^{n} \omega_{l_i k_i}],\ 0\} \leq E[G^P] &= E[\ \max\ \{\sum_{i=1}^{n} \omega_{l_i k_i},\ 0\}] \\ &\leq \sum_{i=1}^{n} E[\ \max\{\omega_{l_i k_i},\ 0\}] \\ &= \sum_{i=1}^{n} E[\omega_{l_i k_i}^{+}] \end{aligned} \tag{3.2}$$

In the following, we will determine analytical expressions for the expectation and variance of random variable G^L. However, determining the expectation and variance of random variable G^P gets analytically messy. Therefore, in our analysis we will make use of the upper bound on $E[G^P]$, derived in Equation (3.2). Our computational study indicates that this is a strong upper bound; see Chapter 4.

Consider the case when a swap **is** made under the limited strategy. Then, only when the realized gain is negative (i.e., the swap actually leads to a loss), the gain under the limited and perfect information strategies will be different, since perfect information strategy will eliminate all swaps with loss. On the other hand, if a swap **is not** made under the limited strategy, then whenever the realized gain is positive, this gain will be realized only under the perfect information strategy, but not under the limited strategy. Thus, we can write:

$$E[G^P - G^L] = \begin{cases} -E[\min\{\sum_{i=1}^n \omega_{l_i k_i},\ 0\}\ ] = -E[\sum_{i=1}^n \omega^-_{l_i k_i}], & \text{if } \Pi = 1 \\ E[\max\{\sum_{i=1}^n \omega_{l_i k_i},\ 0\}\ ] = E[\sum_{i=1}^n \omega^+_{l_i k_i}], & \text{otherwise} \end{cases} \tag{3.3}$$

$$Pr[G^P > G^L] = \begin{cases} Pr(\sum_{i=1}^n \omega_{l_i k_i} < 0), & \text{if } \Pi = 1 \\ Pr(\sum_{i=1}^n \omega_{l_i k_i} > 0), & \text{otherwise.} \end{cases} \tag{3.4}$$

Next, we derive exact expressions as well as bounds on the expectation and variance of random variables G^L and G^P as well as their difference. The remainder of this chapter focuses on fundamentals that will be used in these derivations in the subsequent chapters.

3.5 Preliminaries

3.5.1 Derivations of Some Basic Functions

We first derive expressions for random variables ω_{ij}, $i \in L_1, j \in L_2$. Please refer to the Appendix for all derivations in this section.

$$\omega_{ij} = \Delta_i - \Delta_j = \begin{cases} 0, & \text{if } s_i, s_j \ or \ g_i, g_j, \\ C_1 - D_j, & \text{if } s_i, b_j \quad (\omega_{ij} < 0), \\ C_1 - C_2, & \text{if } s_i, g_j \quad (\omega_{ij} < 0), \\ D_i - C_1, & \text{if } b_i, s_j \quad (\omega_{ij} > 0), \\ D_i - D_j, & \text{if } b_i, b_j, D_i > D_j \quad (\omega_{ij} > 0), \\ D_i - D_j, & \text{if } b_i, b_j, D_i < D_j \quad (\omega_{ij} < 0), \\ D_i - C_2, & \text{if } b_i, g_j \quad (\omega_{ij} < 0), \\ C_2 - C_1, & \text{if } g_i, s_j \quad (\omega_{ij} > 0), \\ C_2 - D_j, & \text{if } g_i, b_j \quad (\omega_{ij} > 0). \end{cases} \tag{3.5}$$

We can write:

$$\begin{aligned} p_{ij}^{+} &\equiv Pr(\omega_{ij} > 0) = Pr\{(b_i, s_j) \ or \ (g_i, s_j) \ or \ (b_i, b_j, D_i > D_j) \ or \ (g_i, b_j)\} \\ &= [F_j(C_2) - F_j(C_1)][1 - F_i(C_2)] + F_j(C_1)[1 - F_i(C_2)] + F_j(C_1)[F_i(C_2) - F_i(C_1)] \\ &\quad + \int_{C_1}^{C_2} \int_{C_1}^{d_i} f_j(d_j) f_i(d_i) dd_j \ dd_i \\ &= F_j(C_2) \ [1 - F_i(C_2)] + \int_{C_1}^{C_2} f_i(d) F_j(d) \ dd \end{aligned} \tag{3.6}$$

$$\begin{aligned} p_{ij}^{-} &\equiv Pr(\omega_{ij} < 0) = Pr\{(s_i, b_j) \ or \ (b_i, b_j, D_i < D_j) \ or \ (s_i, g_j) \ or \ (b_i, g_j)\} \\ &= F_i(C_1)[F_j(C_2) - F_j(C_1)] + F_i(C_2)[1 - F_j(C_2)] \\ &\quad + \int_{C_1}^{C_2} \int_{C_1}^{d_j} f_i(d_i) f_j(d_j) dd_i \ dd_j \\ &= F_i(C_2)[1 - F_j(C_2)] + \int_{C_1}^{C_2} f_j(d) F_i(d) \ dd \end{aligned} \tag{3.7}$$

The expressions for the expectation and variance of random variables $\omega_{ij}^{+}, i \in$

$L_1, j \in L_2$, can be derived as follows.

$$\omega_{ij}^{+} = \max\{\omega_{ij}, 0\} = \begin{cases} D_i - C_1, & \text{if } b_i, s_j \\ D_i - D_j, & \text{if } b_i, b_j, D_i > D_j \\ C_2 - C_1, & \text{if } g_i, s_j \\ C_2 - D_j, & \text{if } g_i, b_j \\ 0, & \text{otherwise.} \end{cases} \tag{3.8}$$

Hence,

$$\begin{aligned} E[\omega_{ij}^{+}] &= E[\omega_{ij}|\omega_{ij} > 0]\ Pr[\omega_{ij} > 0] \\ &= E[D_i - C_1|b_i, s_j]\ Pr(bi, sj) + E[D_i - D_j|b_i, b_j, D_i > D_j]\ Pr(b_i, b_j, D_i > D_j) \\ &\quad + E[C_2 - C_1|g_i, s_j]\ Pr(g_i, s_j) + E[C_2 - D_j|g_i, b_j]\ Pr(g_i, b_j) \\ &= C_2 F_j(C_2)[1 - F_i(C_2)] - C_1 F_j(C_1)[1 - F_i(C_1)] \\ &\quad - \mu_j[F_j(C_2) - F_j(C_1)][1 - F_i(C_2)] - \sigma_j^2[f_j(C_1) - f_j(C_2)][1 - F_i(C_2)] \\ &\quad + \mu_i F_j(C_1)[F_i(C_2) - F_i(C_1)] + \sigma_i^2[f_i(C_1) - f_i(C_2)]F_j(C_1) \\ &\quad + \int_{C_1}^{C_2} \int_{d_j}^{C_2} (d_i - d_j) f_j(d_j) f_i(d_i)\ dd_i\ dd_j \\ &= C_2 F_j(C_2)[1 - F_i(C_2)] - C_1 F_j(C_1)[1 - F_i(C_1)] \\ &\quad - \mu_j[F_j(C_2) - F_j(C_1)] - \sigma_j^2[f_j(C_1) - f_j(C_2)] \\ &\quad + \mu_i[F_i(C_2)F_j(C_2) - F_i(C_1)F_j(C_1)] + \sigma_i^2[f_i(C_1)F_j(C_1) - f_i(C_2)F_j(C_2)] \\ &\quad + \sigma_i^2 \int_{C_1}^{C_2} f_i(d) f_j(d)\ dd - \mu_i \int_{C_1}^{C_2} f_j(d) F_i(d)\ dd \\ &\quad + \int_{C_1}^{C_2} d\ f_j(d) F_i(d)\ dd \end{aligned} \tag{3.9}$$

Similarly, we derive:

$$\begin{aligned}
E[(\omega_{ij}^{+})^2] &= E[\omega_{ij}^2|\omega_{ij} > 0]\ Pr[\omega_{ij} > 0] \\
&= E[(D_i - C_1)^2|b_i, s_j]\ Pr(bi, sj) + E[(D_i - D_j)^2|b_i, b_j, D_i > D_j]\ Pr(b_i, b_j, D_i > D_j) \\
&\quad + E[(C_2 - C_1)^2|g_i, s_j]\ Pr(g_i, s_j) + E[(C_2 - D_j)^2|g_i, b_j]\ Pr(g_i, b_j) \\
&= F_j(C_1)\ [F_i(C_2) - F_i(C_1)]\ (\mu_i^2 + \sigma_i^2) \\
&\quad + F_j(C_1)\sigma_i^2\ [\ (C_1 + \mu_i)\ f_i(C_1)\ -\ (C_2 + \mu_i)\ f_i(C_2)\ ] \\
&\quad - 2F_j(C_1)\ [F_i(C_2) - F_i(C_1)]\ C_1\ \mu_i - 2F_j(C_1)\ C_1\ \sigma_i^2\ [f_i(C_1) - f_i(C_2)] \\
&\quad + [1 - F_i(C_2)]\ [F_j(C_2) - F_j(C_1)]\ (\mu_j^2 + \sigma_j^2) \\
&\quad + [1 - F_i(C_2)]\ \sigma_j^2\ [(C_1 + \mu_j)\ f_j(C_1)\ -\ (C_2 + \mu_j)\ f_j(C_2)] \\
&\quad - 2[1 - F_i(C_2)]\ [F_j(C_2) - F_j(C_1)]\ C_2\ \mu_j - 2[1 - F_i(C_2)]\ C_2\ \sigma_j^2\ [f_j(C_1) - f_j(C_2)] \\
&\quad + C_1^2\ F_j(C_1)\ [1 - F_i(C_1)] + C_2^2\ F_j(C_2)\ [1 - F_i(C_2)] - 2\ C_1\ C_2\ F_j(C_1)\ [1 - F_i(C_2)] \\
&\quad + \int_{C_1}^{C_2} \int_{d_j}^{C_2} (d_i^2 - 2d_i d_j + d_j^2)\ f_j(d_j)\ f_i(d_i)\ dd_i\ dd_j
\end{aligned}$$

We can now determine:

$$Var(\omega_{ij}^{+}) = E[(\omega_{ij}^{+})^2] - (\ E[\omega_{ij}^{+}]\ )^2 \tag{3.10}$$

Similarly, we derive the expectation for $\omega_{ij}^- = \min\{\omega_{ij}, 0\}$ as:

$$
\begin{aligned}
E[\omega_{ij}^-] &= E[\omega_{ij}|\omega_{ij} < 0]\ Pr[\omega_{ij} < 0] \\
&= F_i(C_1)\ \int_{C_1}^{C_2} (C_1 - d) f_j(d)\ dd \\
&\quad + (C_1 - C_2) F_i(C_1)[1 - F_j(C_2)] + \int_{C_1}^{C_2} \int_{C_1}^{d_j} (d_i - d_j) f_j(d_j) f_i(d_i)\ dd_i\ dd_j \\
&\quad + [1 - F_j(C_2)] \int_{C_1}^{C_2} (d - C_2) f_i(d)\ dd \\
&= C_1\ F_i(C_1)[1 - F_j(C_1)] - C_2\ F_i(C_2)[1 - F_j(C_2)] \\
&\quad + \mu_i[F_i(C_2)(1 - F_j(C_2)) - F_i(C_1)(1 - F_j(C_1)] + \sigma_i^2 f_i(C_1)[1 - F_j(C_1)] \\
&\quad - \sigma_i^2 f_i(C_2)[1 - F_j(C_2)] + \mu_i \int_{C_1}^{C_2} f_j(d) F_i(d)\ dd \\
&\quad - \sigma_i^2 \int_{C_1}^{C_2} f_i(d) f_j(d)\ dd - \int_{C_1}^{C_2} d\ f_j(d) F_i(d)\ dd
\end{aligned} \tag{3.11}
$$

Finally, observe that $E[\omega_{ij}] = E[\omega_{ij}^+] + E[\omega_{ij}^-]$.

Next, we derive expressions for expectation and variance of random variables Δ_i, $i \in L$. Letting $a_{ki} = \min\{C_k, D_i\}$, for $k = 1, 2$, we can write $E[a_{ki}] = C_k + F_i(C_k)(\mu_i - C_k) - \sigma_i^2 f_i(C_k)$. Thus:

$$
\begin{aligned}
E[\Delta_i] &= E[a_{2i} - a_{1i}] \\
&= (C_2 - C_1) + F_i(C_2)(\mu_i - C_2) + F_i(C_1)(C_1 - \mu_i) + \sigma_i^2[f_i(C_1) - f_i(C_2)]
\end{aligned} \tag{3.12}
$$

Observe that

$$
\begin{aligned}
E[\omega_{ij}] &= E[\Delta_i - \Delta_j] \\
&= F_i(C_2)(\mu_i - C_2) + F_i(C_1)(C_1 - \mu_i) + \sigma_i^2[f_i(C_1) - f_i(C_2)] \\
&\quad - F_j(C_2)(\mu_j - C_2) - F_j(C_1)(C_1 - \mu_j) - \sigma_j^2[f_j(C_1) - f_j(C_2)]
\end{aligned} \tag{3.13}
$$

Similarly, we write,

$$\begin{aligned} Var(\Delta_i) &= Var(a_{2i} - a_{1i}) \\ &= E[(a_{2i} - a_{1i})^2] - (E[a_{2i} - a_{1i}])^2 \end{aligned} \tag{3.14}$$

We start with the first term. We can write $E[(a_{2i} - a_{1i})^2] = E[a_{2i}^2] + E[a_{1i}^2] - 2E[a_{1i} \times a_{2i}]$, where:

$$\begin{aligned} E[a_{1i} \times a_{2i}] =& (\mu_i^2 + \sigma_i^2)\ F_i(C_1) - \sigma_i^2\ (C_1 + \mu_i) f_i(C_1) + C_1 \mu_i\ [F_i(C_2) - F_i(C_1)] \\ & + C_1 \sigma_i^2\ [f_i(C_1) - f_i(C_2)]\ + C_1\ C_2\ [1 - F_i(C_2)] \end{aligned}$$

Similarly, for $k = 1, 2$, we obtain:

$$E[a_{ki}^2] = (\mu_i^2 + \sigma_i^2)\ F_i(C_k) - \sigma_i^2\ (C_k + \mu_i)\ f_i(C_k) + C_k^2\ [1 - F_i(C_k)]$$

Thus,

$$\begin{aligned} Var(\Delta_i) = & (\mu_i^2 + \sigma_i^2)[F_i(C_2) - F_i(C_1)] + \sigma_i^2[(C_1 + \mu_i) f_i(C_1) - (C_2 + \mu_i) f_i(C_2)] \\ & - 2C_1 \mu_i [F_i(C_2) - F_i(C_1)] - 2C_1 \sigma_i^2 [f_i(C_1) - f_i(C_2)] \\ & + (C_2^2 - 2C_1 C_2)[1 - F_i(C_2)] + C_1^2 [1 - F_i(C_1)] \\ & - \{(C_2 - C_1) + F_i(C_2)(\mu_i - C_2) + F_i(C_1)(C_1 - \mu_i) + \sigma_i^2 [f_i(C_1) - f_i(C_2)]\}^2 \end{aligned} \tag{3.15}$$

Thus, we can also obtain:

$$Var(\omega_{ij}) = Var(\Delta_i) + Var(\Delta_j) \tag{3.16}$$

We will make use of these expressions in our analysis in the subsequent sections.

3.5.2 Analysis of Some Basic Functions

We start by considering only two legs, $i \in L_1$ and $j \in L_2$, in the swappable routes. Thus, leg i is initially assigned to the smaller capacity C_1, and leg j to the larger capacity C_2. We assume, in this section, that swapping decisions under both perfect information and limited swapping strategies are made based *only* on the expected revenue gain of these two legs. We will relax this assumption in the next chapter, when we study a more realistic model. If we let G^P_{ij} and G^L_{ij} respectively denote the *revenue gain component* corresponding *only* to legs i and j under the perfect information and limited swapping strategies, then we can write:

$$E[G^P_{ij}] = E[max\{\omega_{ij}, 0\}] = E[\omega^+_{ij}], \quad E[G^L_{ij}] = \max\{E[\omega_{ij}], 0\}$$

Thus, the expected difference in gain between the *perfect* and *limited* strategies corresponding to leg pair (i, j), $E[G^P_{ij} - G^L_{ij}]$, the probability that the gain corresponding to leg pair (i, j) under the *perfect* strategy is larger than that under the *limited* strategy, $Pr(G^P_{ij} > G^L_{ij})$, and the variances of random variables G^P_{ij} and G^L_{ij} can be written as follows:

$$E[G^P_{ij} - G^L_{ij}] = \begin{cases} -E[\omega^-_{ij}], & \text{if II} = 1 \\ E[\omega^+_{ij}], & \text{otherwise} \end{cases} \tag{3.17}$$

$$Pr(G^P_{ij} > G^L_{ij}) = \begin{cases} Pr(\omega_{ij} < 0) = p^-_{ij}, & \text{if II} = 1 \\ Pr(\omega_{ij} > 0) = p^+_{ij}, & \text{otherwise} \end{cases} \tag{3.18}$$

$$Var(G^P_{ij}) = Var(\omega^+_{ij}) \tag{3.19}$$

$$Var(G^L_{ij}) = \begin{cases} Var(\omega_{ij}), & \text{if II} = 1 \\ 0, & \text{otherwise,} \\ i \end{cases} \tag{3.20}$$

where

$$\text{II} = \begin{cases} 1, & \text{if } E[\omega_{ij}] > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Keeping all demand and capacity parameters, except for the mean demand on leg i, constant, we can show that there exists a *threshold* value μ_i^{th} such that if $\mu_i \geq \mu_i^{th}$, then we swap under the limited strategy (i.e., $\amalg = 1$); and otherwise, we do not swap (i.e., $\amalg = 0$). This result is intuitive and directly follows from Result 3.5.1 below, which shows that the function $E[\omega_{ij}]$ is non-decreasing in μ_i. Thus, we can write:

$$E[G_{ij}^P - G_{ij}^L] = \begin{cases} E[\omega_{ij}^+], & \text{if } \mu_i < \mu_i^{th} \\ -E[\omega_{ij}^-], & \text{if } \mu_i \geq \mu_i^{th} \end{cases} \tag{3.21}$$

$$Pr(G_{ij}^P > G_{ij}^L) = \begin{cases} p_{ij}^+, & \text{if } \mu_i < \mu_i^{th} \\ p_{ij}^-, & \text{if } \mu_i \geq \mu_i^{th}. \end{cases} \tag{3.22}$$

In a similar way, we can show (see Result 3.5.2 in the next section) that keeping all demand and capacity parameters, except for μ_j, constant, there exists a *threshold* value μ_j^{th} such that if $\mu_j \leq \mu_j^{th}$, then we swap under the limited strategy ($\amalg = 1$); and otherwise, we do not swap ($\amalg = 0$).

In Section 3.5.3, we study how functions $E[\omega_{ij}^+], E[\omega_{ij}^-], E[\omega_{ij}], Var(\omega_{ij}^+)$, and $Var(\omega_{ij})$ behave as demand parameters change so as to derive some insights. Then, in Section 3.5.4, we extend this analysis to probability functions p_{ij}^+ and p_{ij}^-. We will use these results in the next chapter, when we compare the *total* gain functions corresponding to all legs of the swappable routes, under the perfect information and limited swapping strategies.

3.5.3 Analysis of the Expectation and Variance Components Corresponding to One Leg Pair

We first study how functions $E[\omega_{ij}^+]$, $E[\omega_{ij}^-]$, and $E[\omega_{ij}]$ behave as the mean demand on the leg initially assigned to the smaller capacity, μ_i, increases. The following result characterizes their behaviors.

Result 3.5.1 *Consider any leg $i \in L_1$ and $j \in L_2$. We have:*

$$\frac{\delta E[\omega_{ij}^+]}{\delta \mu_i} = F_i(C_2)F_j(C_2) - F_i(C_1)F_j(C_1) - \int_{C_1}^{C_2} f_j(d)F_i(d)\ dd \geq 0$$
$$\frac{\delta E[\omega_{ij}^-]}{\delta \mu_i} = F_i(C_2)[1 - F_j(C_2)] - F_i(C_1)[1 - F_j(C_1)] + \int_{C_1}^{C_2} f_j(d)F_i(d)\ dd \geq 0$$
$$\frac{\delta E[\omega_{ij}]}{\delta \mu_i} = F_i(C_2) - F_i(C_1) \geq 0; \qquad \frac{\delta^2 E[\omega_{ij}]}{\delta \mu_i^2} = \sigma_i^2\ [f_i(C_1) - f_i(C_2)]$$

Thus, $E[\omega_{ij}^+]$, $E[\omega_{ij}^-]$, and $E[\omega_{ij}]$ are non-decreasing in μ_i.

Proof: We can write the following lower and upper bounds on $\int_{C_1}^{C_2} f_j(d)F_i(d)\ dd$.

$$F_i(C_1)\int_{C_1}^{C_2} f_j(d)\ dd \leq \int_{C_1}^{C_2} f_j(d)F_i(d)\ dd \leq F_i(C_2)\int_{C_1}^{C_2} f_j(d)\ dd$$
$$\Rightarrow F_i(C_1)[F_j(C_2) - F_j(C_1)] \leq \int_{C_1}^{C_2} f_j(d)F_i(d)\ dd \leq F_i(C_2)[F_j(C_2) - F_j(C_1) \quad (3.23)$$

Using the upper bound in Equation (3.23), we can thus write:

$$\frac{\delta E[\omega_{ij}^+]}{\delta \mu_i} = F_i(C_2)F_j(C_2) - F_i(C_1)F_j(C_1) - \int_{C_1}^{C_2} f_j(d)F_i(d)\ dd$$
$$\geq F_j(C_1)[F_i(C_2) - F_i(C_1)] \geq 0$$

Similarly, using the lower bound in Equation (3.23), we can write:

$$\frac{\delta E[\omega_{ij}^-]}{\delta \mu_i} = F_i(C_2)[1 - F_j(C_2)] - F_i(C_1)[1 - F_j(C_1)] + \int_{C_1}^{C_2} f_j(d)F_i(d)\ dd$$
$$\geq [F_i(C_2) - F_i(C_1)]\ [1 - F_j(C_2)] \geq 0$$

The last two derivations are given in the Appendix. This completes the proof. ∎

In addition, since $C_1 < C_2$, we have that for $\mu_i \leq C_{mid}$, $f_i(C_1) \geq f_i(C_2)$, and thus, $E[\omega_{ij}]$ is convex non-decreasing in μ_i, whereas for $\mu_i > C_{mid}$, it is concave non-decreasing in μ_i. Observe that this makes sense: As the mean demand on leg i increases, the expected gain of swapping the capacity assigned to that leg (C_1) with a larger capacity (C_2) also increases. However, as the mean demand on the leg gets larger, the benefits of assigning it a larger capacity will increase at a slower rate, since even the larger capacity will not be able to capture all demand. Similarly, for small values of μ_i, the rate of increase of $E[\omega_{ij}]$ is very small, since almost all leg demand can be captured by the smaller capacity C_1. Thus, for very small or very large μ_i (i.e., $\mu_i << C_1$ or $\mu_i >> C_2$), the rate of increase of function $E[\omega_{ij}]$ approaches to 0.

Next, we evaluate these functions using *numerical integration*, considering parameters typical in practice. In all the following numerical integration analysis, we consider two aircraft types of the Boeing 737 family, having capacities of $C_1 = 112$ and $C_2 = 126$ passengers. Thus, $C_{mid} = 119$. In addition, unless otherwise noted, we consider σ_i values of $\{15, 25\}$ and σ_j value of 15.

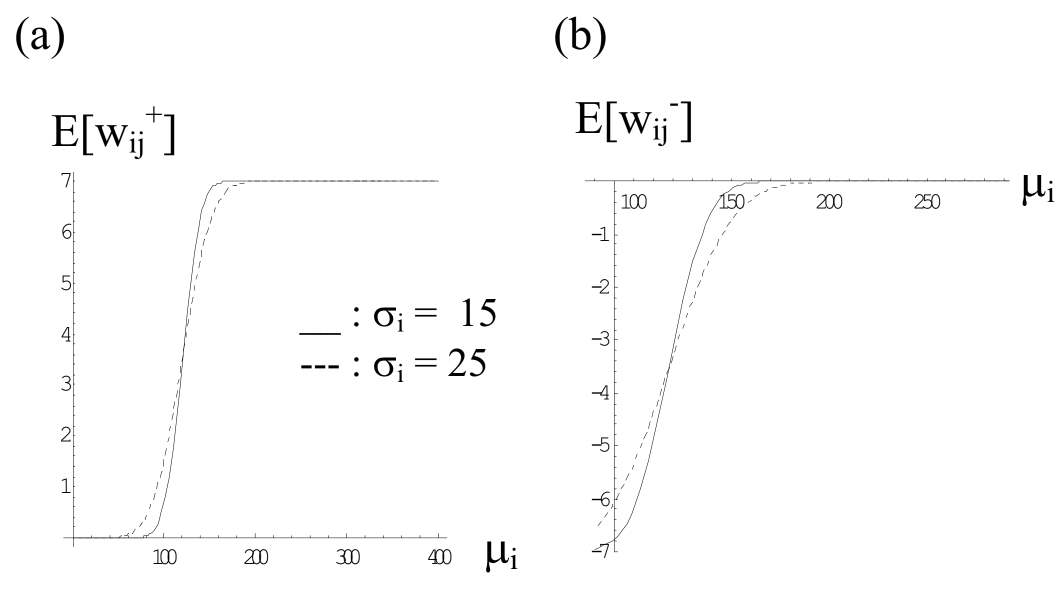

Figure 3.1: a)$E[\omega_{ij}^{+}]$ versus μ_i ; b) $E[\omega_{ij}^{-}]$ versus μ_i.

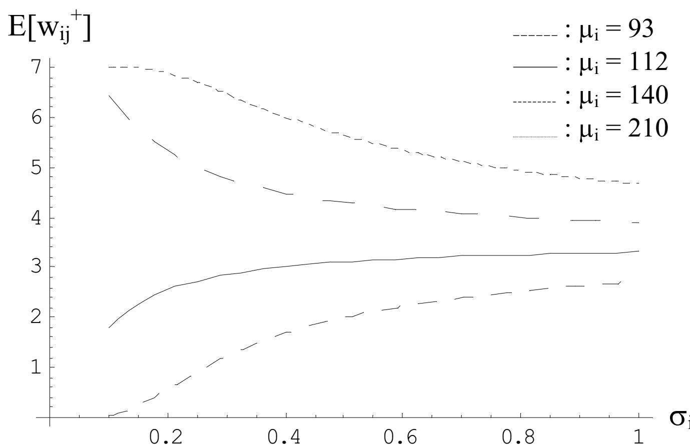

Figure 3.2: $E[\omega_{ij}^{+}]$ versus σ_i.

Figure 3.1 depicts how functions $E[\omega_{ij}^{+}]$ and $E[\omega_{ij}^{-}]$ behave as the mean demand on leg i, μ_i, increases, when considering $\mu_j = 119$. We observe the same behavior for other μ_j values. As Result 3.5.1 indicates, both $E[\omega_{ij}^{+}]$ and $E[\omega_{ij}^{-}]$ are non-decreasing in μ_i. Observing both graphs for $\sigma_i = 15$ and 25, we see that for lower values of μ_i ($\mu_i \leq C_{mid} = 119$), higher demand variability is more desirable (i.e., both $E[\omega_{ij}^{+}]$ and $E[\omega_{ij}^{-}]$ are larger in the $\sigma_i = 25$ case than the $\sigma_i = 15$ case for μ_i values of up to C_{mid}). On the other hand, for higher values of μ_i ($\mu_i \geq C_{mid}$), lower variability leads to higher values for both functions. This pattern can also be observed in Figure 3.2, which plots $E[\omega_{ij}^{+}]$ versus σ_i for different μ_i values, considering again, $\mu_j = 119$ and σ_j = 15. For μ_i values of up to C_{mid}, $E[\omega_{ij}^{+}]$ is increasing in σ_i; otherwise, it is decreasing in σ_i. Finally, the rates of increase of both $E[\omega_{ij}^{+}]$ and $E[\omega_{ij}^{-}]$ decrease to zero as $\mu_i \to \infty$. Also observe that both functions are initially convex and become concave at point $\mu_i = C_{mid}$.

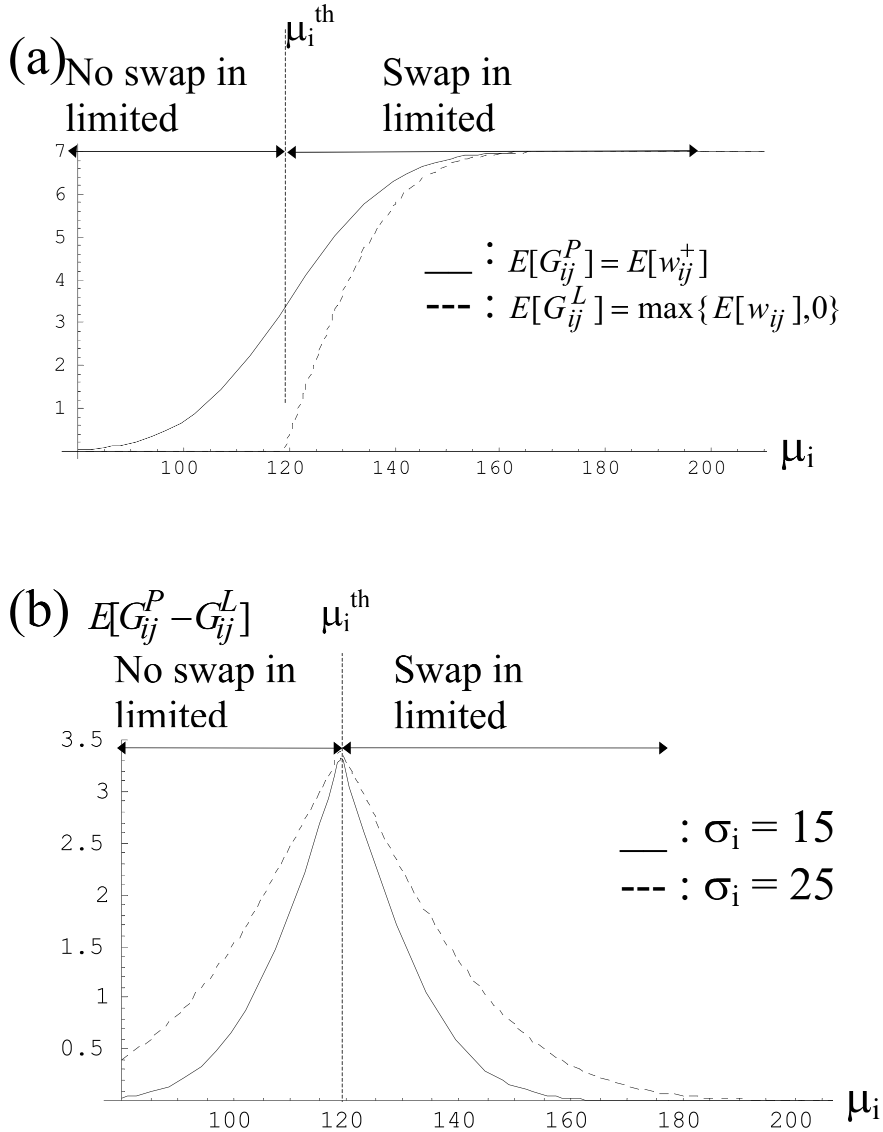

Figure 3.3: a)$E[G_{ij}^P]$ and $E[G_{ij}^L]$ versus μ_i; b) $E[G_{ij}^P - G_{ij}^L]$ versus μ_i.

Next we study how the expected difference in gain under the perfect information and limited swapping strategies changes as μ_i increases. For this purpose, in Figure 3.3(a), we plot functions $E[G_{ij}^P]$ and $E[G_{ij}^L]$, whereas in Figure 3.3(b), we plot the expected difference, $E[G_{ij}^P - G_{ij}^L]$, as μ_i increases, considering $\sigma_i = 15$ and 25, and $\mu_j = 119$. In order to interpret the results, we introduce the concept of *demand imbalance (ν_{ij})*, which we define as $\nu_{ij} \equiv \mu_i - \mu_j$. We will extend this term to consider the multiple legs in the swappable routes, when we study the *total* gain functions under the limited and perfect information swapping strategies in Chapter 4. Clearly, the threshold point, μ_i^{th}, will be different for cases with different σ_i values. However, for the $\sigma_i = 15$ and 25 cases considered in Figure 3.3(b), both threshold values turn out to be equal. Observe that the region up to point μ_i^{th} (i.e., $\nu_{ij} \leq 0$) represents cases where no swaps are made under the limited strategy; thus, its expected gain is zero. Similarly, the region after point μ_i^{th} (i.e., $\nu_{ij} > 0$) is the region where swaps are made under the limited swapping strategy. Recall that the expected difference, $E[G_{ij}^P - G_{ij}^L]$, is given by $E[\omega_{ij}^+]$ up to the swapping point μ_i^{th}, and is given by $-E[\omega_{ij}^-]$ after the swapping point; see Equation (3.21). Thus, Result 3.5.1 indicates that the expected difference, $E[G_{ij}^P - G_{ij}^L]$, is non-decreasing in μ_i up to the swapping point, and is non-increasing in μ_i after the swapping point. This pattern can be observed in Figure 3.3(b) for both $\sigma_i = 15$ and 25 cases. In addition, we have observed, in Figures 3.1 and 3.2, that $E[\omega_{ij}^+]$ is increasing in σ_i up to the swapping point, and $-E[\omega_{ij}^-]$ is also increasing in σ_i after the swapping point. Thus, the expected difference, $E[G_{ij}^P - G_{ij}^L]$ is increasing in σ_i, over the values evaluated in Figure 3.3(b).

In summary, *as the absolute value of the demand imbalance ($|\nu_{ij}|$) decreases and/or demand standard deviation, σ_i, increases, the additional expected gain that can be achieved under the perfect information strategy over the limited swapping strategy increases.* We will observe the same behavior when we study the total gain functions corresponding to multiple legs under the limited and perfect information strategies in Chapter 4.

Next we study how functions $E[\omega_{ij}^+]$, $E[\omega_{ij}^-]$, and $E[\omega_{ij}]$ behave as μ_j, the mean demand on the leg initially assigned to the larger capacity, increases.

Result 3.5.2 *Consider any leg $i \in L_1$ and $j \in L_2$. We have:*

$$\begin{aligned}
\frac{\delta E[\omega_{ij}^+]}{\delta \mu_j} &= \sigma_i^2[f_i(C_2) - f_i(C_1)]f_j(C_1) + C_2 F_i(C_2) f_j(C_2) - C_1 F_i(C_1) f_j(C_1) \\
&\quad + \mu_i[F_i(C_1)f_j(C_1) - F_i(C_2)f_j(C_2)] - F_j(C_1) - F_j(C_2) \\
&\quad + \int_{C_1}^{C_2} \frac{d}{\sigma_j^2} F_i(d) f_j(d)\ dd + \frac{\sigma_i^2}{\sigma_j^2}\int_{C1}^{C_2}(d-\mu_j) f_i(d) f_j(d)\ dd \\
&\quad + \frac{\mu_j}{\sigma_j^2}\int_{C_1}^{C_2}(\mu_i - d)F_i(d)f_j(d)\ dd + \frac{1}{\sigma_j^2}\int_{C_1}^{C_2} d^2 F_i(d) f_j(d)\ dd \\
\frac{\delta E[\omega_{ij}^-]}{\delta \mu_j} &= C_1 F_i(C_1) f_j(C_1) - C_2 F_i(C_2) f_j(C_2) - \mu_i[F_i(C_1)f_j(C_1) - F_i(C_2)f_j(C_2)] \\
&\quad + \sigma_i^2[f_i(C_1)f_j(C_1) - f_i(C_2)f_j(C_2)] + \frac{\mu_i}{\sigma_j^2}\int_{C_1}^{C_2} d\ F_i(d) f_j(d)\ dd \\
&\quad - \frac{\sigma_i^2}{\sigma_j^2}\int_{C_1}^{C_2} d\ F_i(d) f_j(d)\ dd - \frac{1}{\sigma_j^2}\int_{C_1}^{C_2} d^2 F_i(d) f_j(d)\ dd - \frac{\mu_i \mu_j}{\sigma_j^2}\int_{C_1}^{C_2} F_i(d) f_j(d)\ dd \\
\frac{\delta E[\omega_{ij}]}{\delta \mu_j} &= F_j(C_1) - F_j(C_2) \le 0; \qquad \frac{\delta^2 E[\omega_{ij}]}{\delta \mu_j^2} = -\sigma_j^2\ [f_j(C_1) - f_j(C_2)]
\end{aligned}$$

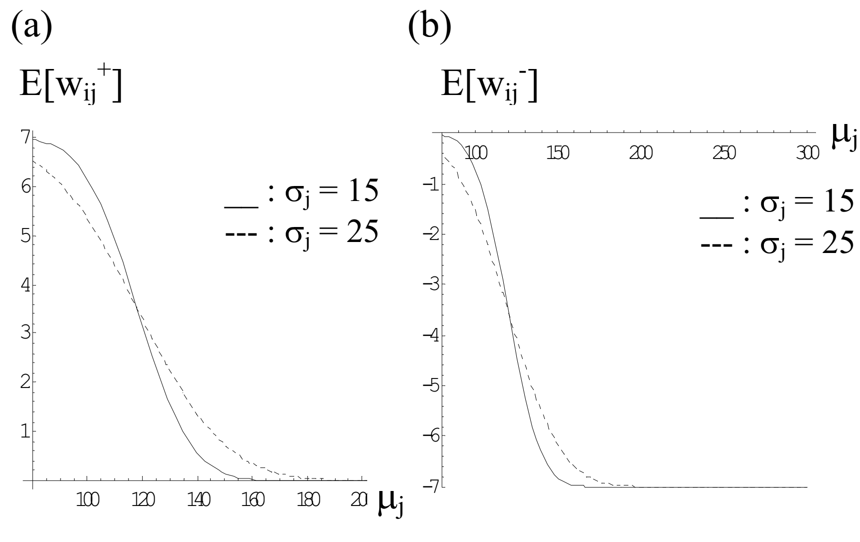

Figure 3.4: a)$E[\omega_{ij}^{+}]$ versus μ_j; b) $E[\omega_{ij}^{-}]$ versus μ_j.

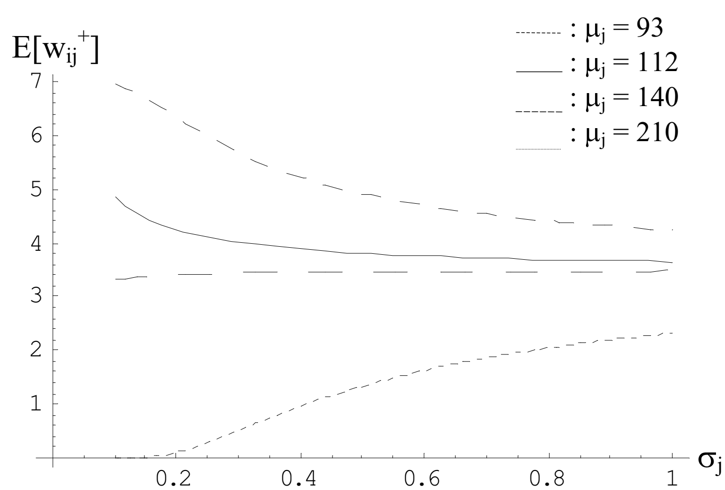

Figure 3.5: $E[\omega_{ij}^{+}]$ versus σ_j.

Figure 3.4 illustrates how functions $E[\omega_{ij}^+]$ and $E[\omega_{ij}^-]$ change as μ_j increases, considering $\mu_i = 119$, $\sigma_i = 15$, and $\sigma_j = 15$ and 25. We observe the same behavior for other values of μ_i. In case of a swap, the initial capacity (C_2) on leg j, $j \in L_2$, will be exchanged with the smaller capacity (C_1). Thus, $E[\omega_{ij}^+]$ and $E[\omega_{ij}^-]$ are both non-increasing in μ_j, over the values evaluated in these graphs. Moreover, we observe that for lower values of μ_j, smaller demand standard deviation, σ_j, leads to higher gain (or lower loss), whereas for larger values of μ_j, higher demand standard deviations are preferable. This pattern can also be observed in Figure 3.5, which plots $E[\omega_{ij}^+]$ versus σ_j for different values of μ_j, considering again $\mu_i = 119$ and $\sigma_i = 15$. For μ_j values of up to C_{mid}, $E[\omega_{ij}^+]$ is decreasing in σ_j; otherwise, it is increasing in σ_j. As μ_j gets very large (i.e., $\mu_j >> C_2$), the rates of decrease of both $E[\omega_{ij}^+]$ and $E[\omega_{ij}^-]$ reduce to 0.

Result 3.5.2 also shows that function $E[\omega_{ij}]$ is non-increasing in μ_j. Thus, there exists a *threshold* value, μ_j^{th}, such that if $\mu_j \leq \mu_j^{th}$, then we swap under the limited strategy; and otherwise, we do not swap. Our numerical integration results, plotted in Figure 3.6, illustrate how the expected difference, $E[G_{ij}^P - G_{ij}^L]$, changes as μ_j increases, considering $\sigma_i = 15$, $\sigma_j = 15$ and 25, and $\mu_j = 119$. Observe that function $E[G_{ij}^P - G_{ij}^L]$ behaves similarly as μ_i increases or μ_j increases; see Figures 3.3(b) and 3.6(b).

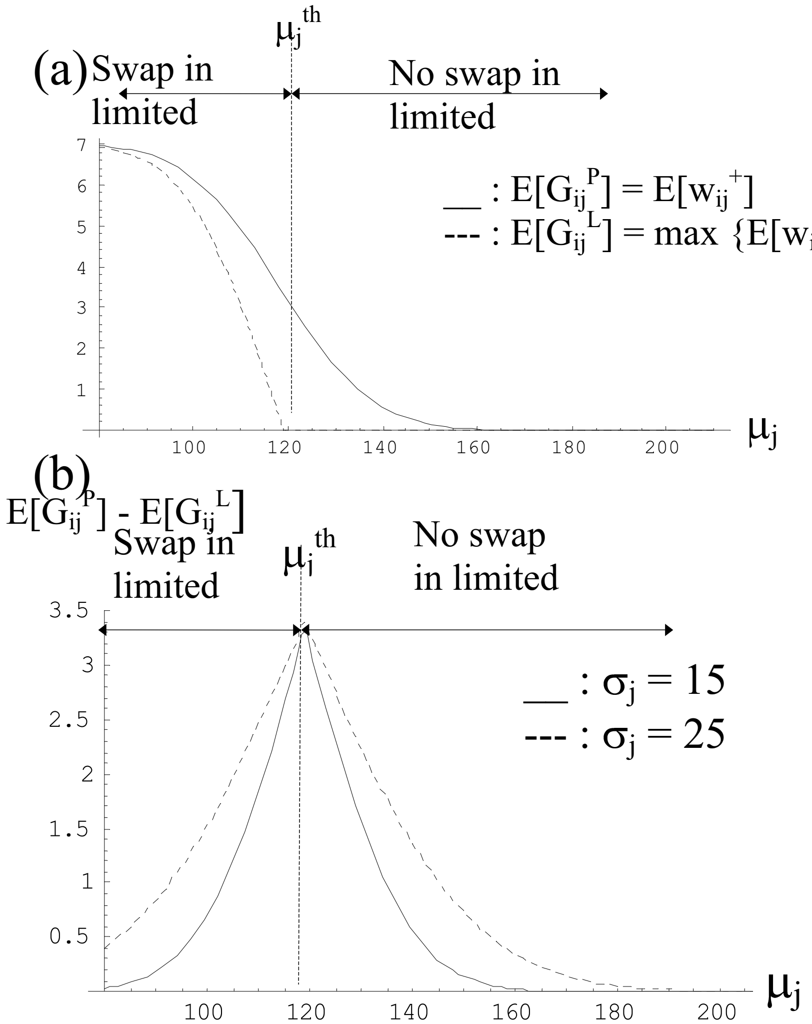

Figure 3.6: a)$E[G_{ij}^P]$ and $E[G_{ij}^L]$ versus μ_j; b) $E[G_{ij}^P] - E[G_{ij}^L]$ versus μ_j.

Next we characterize the behaviors of functions $E[\omega_{ij}^+]$ and $E[\omega_{ij}]$ as the difference between aircraft capacities increases (i.e., C_2 increases as C_1 stays constant). We have the following result.

Result 3.5.3 *Consider any leg $i \in L_1$ and $j \in L_2$. We have:*

$$\frac{\delta E[\omega_{ij}^+]}{\delta C_2} = [1 - F_i(C_2)]\ F_j(C_2) \geq 0$$

$$\frac{\delta E[\omega_{ij}]}{\delta C_2} = F_j(C_2) - F_i(C_2)$$

Since $E[G_{ij}^P] = E[\omega_{ij}^+]$, the expected gain component on leg pair (i, j) under the perfect information swapping strategy is non-decreasing in C_2. However, as C_2 becomes very large, its rate of change approaches zero. On the other hand, recall that $E[G_{ij}^L] = \max\{E[\omega_{ij}], 0\}$. Thus, the expected gain on leg pair (i, j) under the limited swapping strategy is **not** necessarily non-decreasing in C_2. Figure 3.7 depicts such a situation with $\mu_i = 112$, $\sigma_i = 5$, $\mu_j = 119$, $\sigma_j = 15$; see Figure 3.7(a), where $E[\omega_{ij}]$ is first non-decreasing and then non-increasing as C_2 increases. On the other hand, when $\mu_i = 128$, $\sigma_i = 25$, $\mu_j = 119$, $\sigma_j = 15$, $E[\omega_{ij}]$ is non-decreasing in C_2; see Figure 3.7(b).

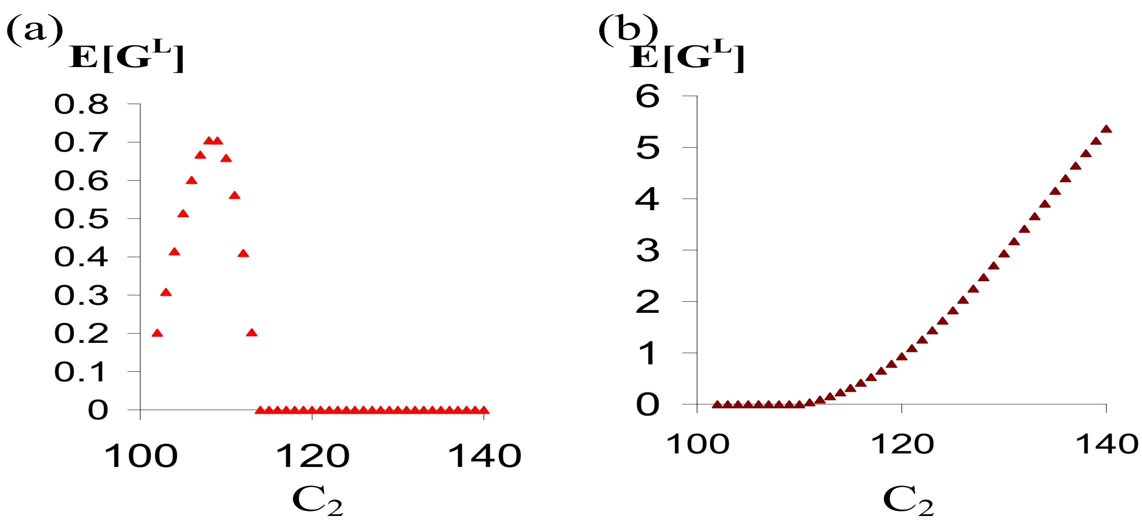

Figure 3.7: $E[\omega_{ij}]$ versus C_2 for a) $\mu_i = 112$ and $\sigma_i = 5$; b) $\mu_i = 128$ and $\sigma_i = 25$.

Next we study the impact of demand parameters on the variances of random variables G_{ij}^P and G_{ij}^L, where $Var(G_{ij}^P) = Var(\omega_{ij}^+)$ and

$$Var(G_{ij}^P) = \begin{cases} Var(\omega_{ij}), & \text{if II} = 1 \\ 0, & \text{otherwise,} \end{cases} \tag{3.24}$$

as given in Equations (3.19) and (3.20). Recall also that $Var(\omega_{ij}) = Var(\Delta_i) + Var(\Delta_j)$. Thus, we make use of Equations (3.10), (3.15), and (3.16), to evaluate these functions, considering different demand parameters.

Figures 3.8(a) and (b) suggest that $Var(\Delta_i)$ is unimodal in μ_i: First it is non-decreasing and then non-increasing in μ_i, whereas it is non-decreasing in σ_i over the values evaluated. When we evaluate $Var(G_{ij}^L)$, we observe the same pattern: It is non-decreasing in σ_i; see Figure 3.9(a). In addition, we observe, in Figure 3.9(a), that as σ_i increases, $Var(G_{ij}^L)$ converges to some value, independent of μ_i. On the contrary, the behavior of $Var(G_{ij}^P)$ can be both non-decreasing and non-increasing in σ_i, and this depends on the value of μ_i; see Figure 3.9(b). Both graphs in Figure 3.9 consider $\mu_j = 119$ and $\sigma_j = 15$.

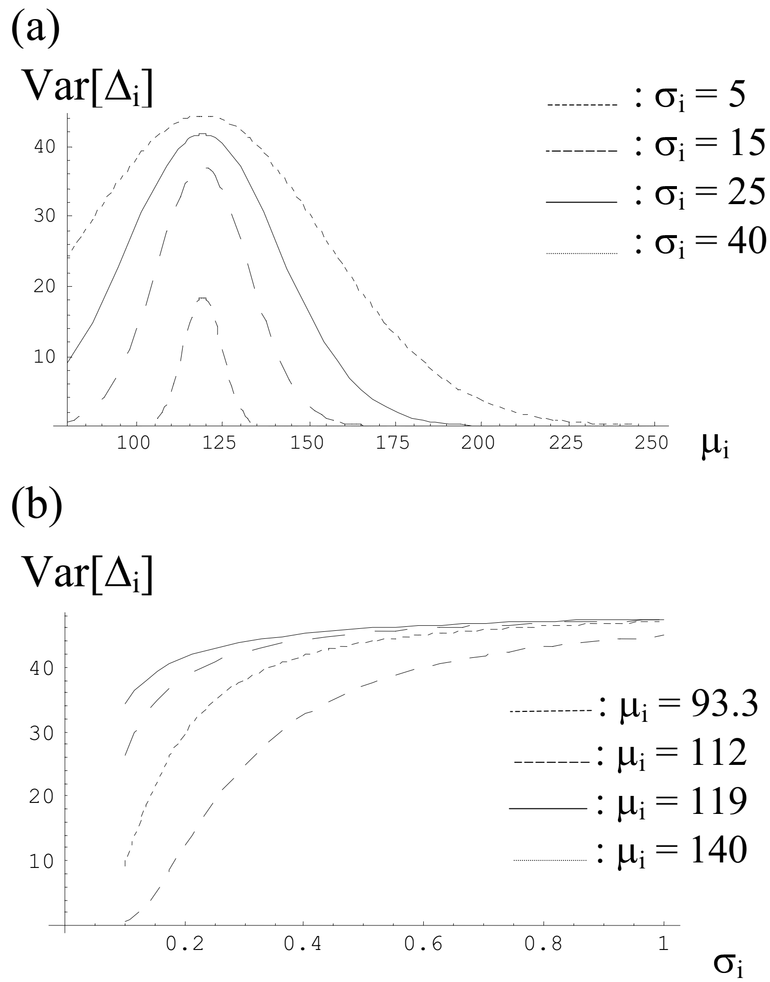

Figure 3.8: a) $Var(\Delta_i)$ versus μ_i; b) $Var(\Delta_i)$ versus σ_i.

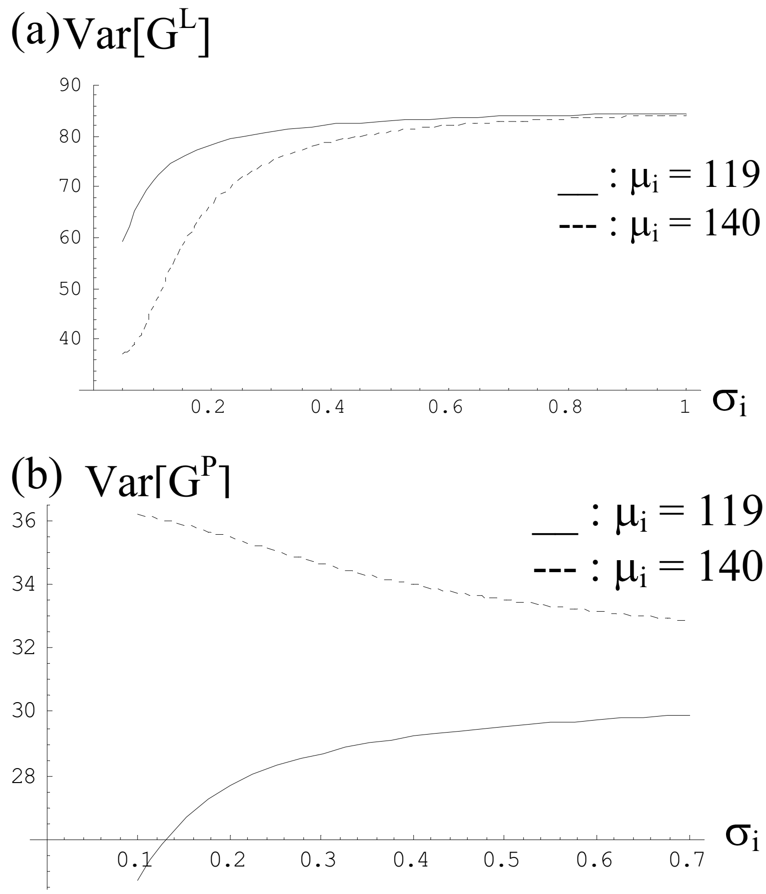

Figure 3.9: a)$Var(G_{ij}^L)$ versus σ_i; b) $Var(G_{ij}^P)$ versus σ_i.

Next we compare the variances of the gain component corresponding to leg pair (i, j) under the limited and perfect information swapping strategies. We find that the gain component under the limited strategy, given by G_{ij}^L, is more variable than that under the perfect information strategy, given by G_{ij}^P, when a swap is performed under the limited strategy. Obviously, when a swap is not performed under the limited strategy, the corresponding variance will be zero. Please refer to Figure 3.10, which plots $Var(G_{ij}^P)$ and $Var(G_{ij}^L)$ versus μ_i, considering $\sigma_i = \sigma_j = 15$ and $\mu_j = 112$ in (a) and $\mu_j = 119$ in (b). In both graphs, both $Var(G_{ij}^P)$ and $Var(G_{ij}^L)$ converge to the same value as μ_i gets large. We also observe that $Var(G_{ij}^L)$ is non-increasing in μ_i in its swapping region, whereas $Var(G_{ij}^P)$ can be both non-increasing or non-decreasing in μ_i. Finally, Figure 3.11 plots $Var(G_{ij}^P)$ and $Var(G_{ij}^L)$ versus μ_j, considering $\mu_i = 119$ and $\sigma_i = \sigma_j = 15$.

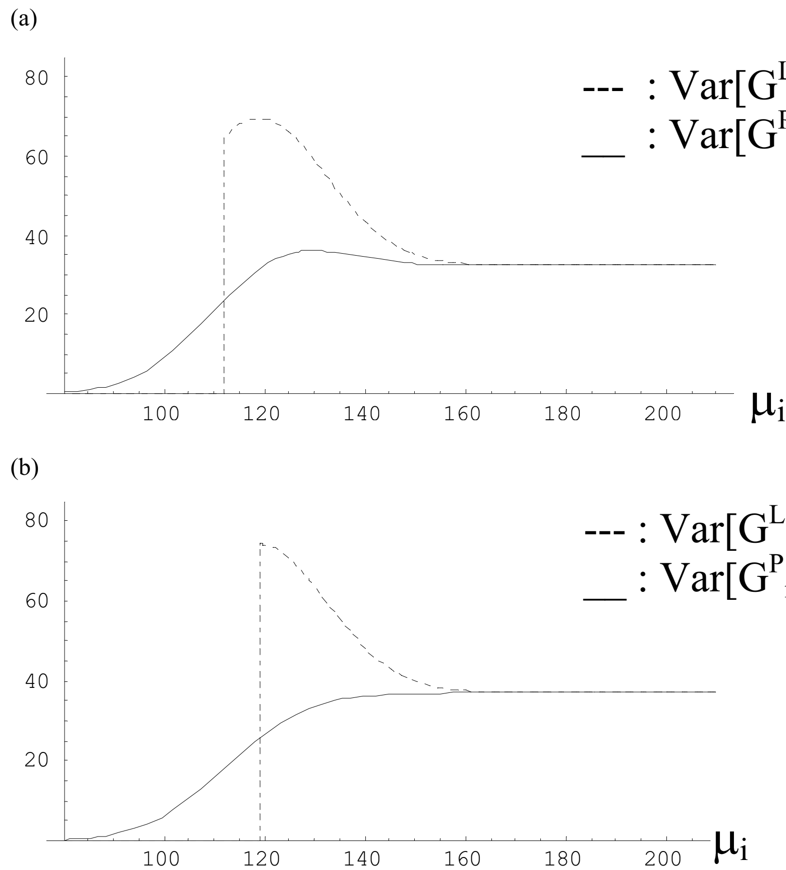

Figure 3.10: $Var(G_{ij}^L)$ and $Var(G_{ij}^P)$ versus μ_i for (a) $\mu_j = 112$; (b) $\mu_j = 119$.

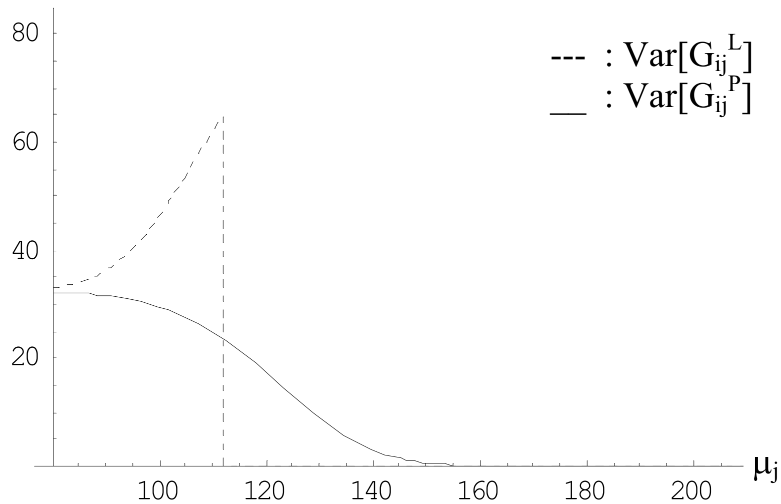

Figure 3.11: $Var(G_{ij}^L)$ and $Var(G_{ij}^P)$ versus μ_j.

3.5.4 Analysis of the Probability Functions Corresponding to One Leg Pair

In this section, we analyze how probability functions p_{ij}^+ and p_{ij}^- behave as demand characteristics change.

Result 3.5.4 *Consider any leg $i \in L_1$ and $j \in L_2$. We have:*

$$\frac{\delta p_{ij}^+}{\delta \mu_i} = f_i(C_2)F_j(C_2) + \int_{C_1}^{C_2} \frac{(d-\mu_i)}{\sigma_i^2} f_i(d)F_j(d)\ dd \quad \geq 0$$
$$\frac{\delta p_{ij}^-}{\delta \mu_i} = -f_i(C_2)[1 - F_j(C_2)] - \int_{C_1}^{C_2} f_i(d)f_j(d)\ dd \quad \leq 0$$

Thus, p_{ij}^+ is non-decreasing in μ_i, whereas p_{ij}^- is non-increasing in μ_i.

Proof: We can write the following lower bound on $\int_{C_1}^{C_2} \frac{(d-\mu_i)}{\sigma_i^2} f_i(d)F_j(d)\ dd$.

$$\begin{aligned}
\int_{C_1}^{C_2} \frac{(d-\mu_i)}{\sigma_i^2} f_i(d) F_j(d)\ dd \ &\geq F_j(C_1) \int_{C_1}^{C_2} \frac{(d-\mu_i)}{\sigma_i^2} f_i(d)\ dd \\
&= F_j(C_1)\ \frac{1}{\sigma_i^2}\ [\ \int_{C_1}^{C_2} d\ f_i(d)\ dd - \mu_i \int_{C_1}^{C_2} f_i(d)\ dd\ ] \\
&= F_j(C_1)\ \frac{1}{\sigma_i^2}\ [\ \mu_i F_i(C_2) - \sigma_i^2 f_i(C_2) - \mu_i F_i(C_1) + \sigma_i^2 f_i(C_1) \\
&\quad - \mu_i(F_i(C_2) - F_i(C_1))\ ] \\
&= F_j(C_1)\ [\ f_i(C_1) - f_i(C_2)\ ] \qquad (3.25)
\end{aligned}$$

Using the lower bound derived in Equation (3.25), we can write:

$$\begin{aligned}
\frac{\delta p_{ij}^+}{\delta \mu_i} &= f_i(C_2) F_j(C_2) + \int_{C_1}^{C_2} \frac{(d-\mu_i)}{\sigma_i^2} f_i(d) F_j(d)\ dd \\
&\geq f_i(C_2) F_j(C_2) + F_j(C_1)[f_i(C_1) - f_i(C_2)] \\
&= F_j(C_1) f_i(C_1) + f_i(C_2)[F_j(C_2) - F_j(C_1)]\ \geq 0.
\end{aligned}$$

The second result is obvious. ∎

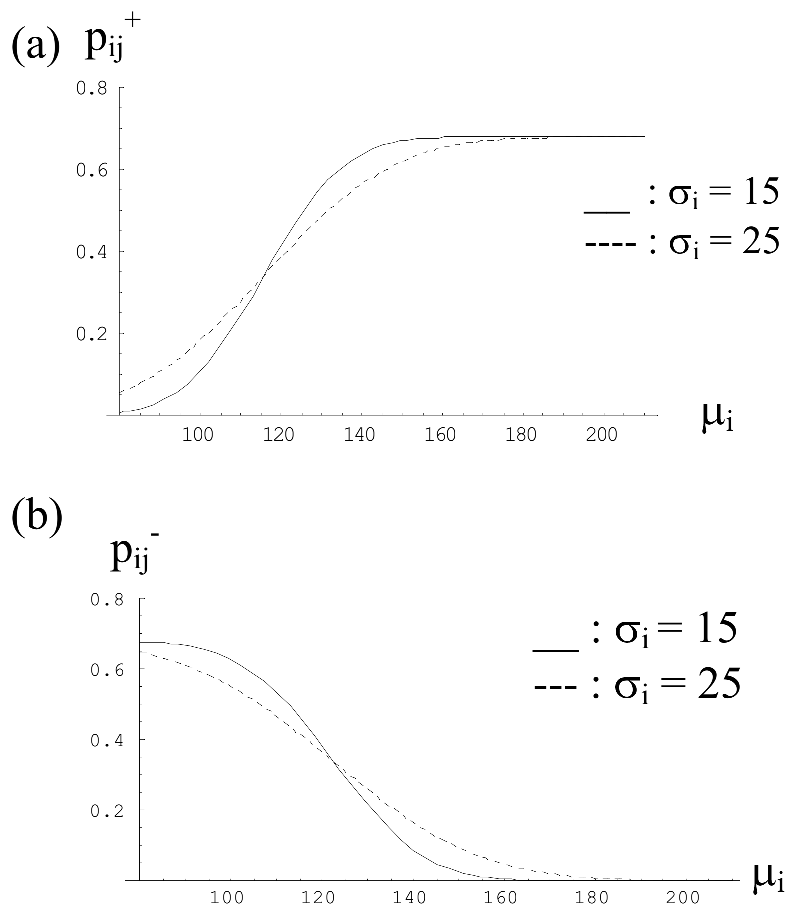

Figure 3.12: a) p_{ij}^{+} versus μ_i; b) p_{ij}^{-} versus μ_i.

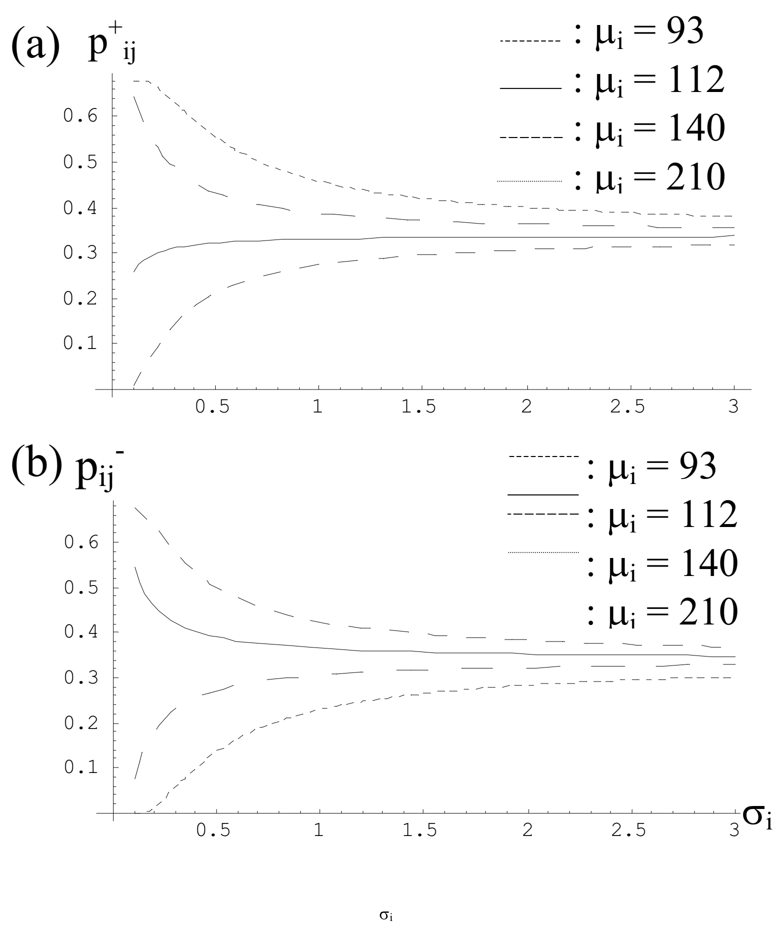

Figure 3.13: a) p_{ij}^+ versus σ_i; b) p_{ij}^- versus σ_i.

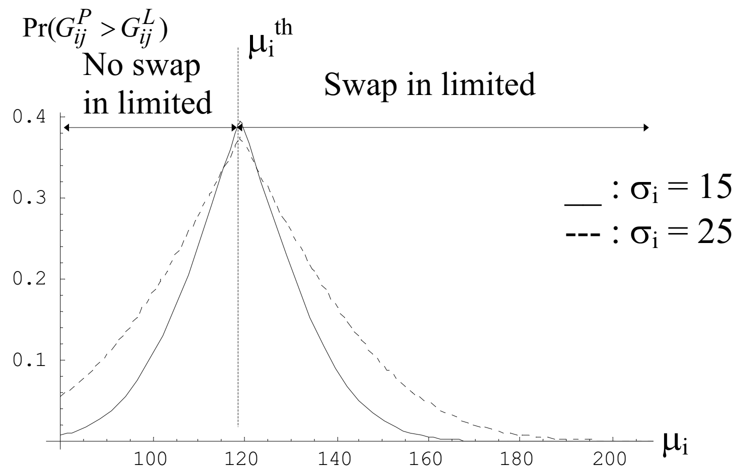

Figure 3.14: $Pr(G_{ij}^P > G_{ij}^L)$ versus μ_i.

These results are intuitive: The swap assigns leg i to the aircraft with the larger capacity (of C_2). Therefore, we expect that the probability of having a positive swap gain under the limited strategy (p_{ij}^+) will increase as μ_i increases. Similarly, we expect the probability of having a swap loss under the limited strategy (p_{ij}^-) to decrease as μ_i increases.

Through numerical integration, we evaluate these probability functions for various values of μ_i. Figure 3.12 illustrates how functions p_{ij}^+ and p_{ij}^- change as μ_i increases, considering $\mu_j = 119$, $\sigma_i = 15$ and 25, and $\sigma_j = 15$. We observe the same behavior for other values of μ_j. As Result 3.5.4 implies, function p_{ij}^+ is non-decreasing in μ_i and function p_{ij}^- is non-increasing in μ_i. For lower values of μ_i, higher demand standard deviation, σ_i, is better, since it leads to higher values of p_{ij}^+ and lower values of p_{ij}^-. On the other hand, for higher values of μ_i, Figure 3.12 suggests that lower variability is more desirable. This pattern can also be observed in Figure 3.13, which plots p_{ij}^+ and p_{ij}^- versus σ_i for various values of μ_i, considering again $\mu_j = 119$ and $\sigma_j = 15$. For μ_i

values of up to C_{mid} $(= 119)$, p_{ij}^{+} is non-decreasing in σ_i, whereas p_{ij}^{-} is non-increasing in σ_i; thus, higher variability is preferable. On the other hand, for μ_i values larger than C_{mid}, p_{ij}^{+} is non-increasing in σ_i and p_{ij}^{-} is non-decreasing in σ_i, and thus, lower variability is more desirable. Finally, the rates of change of both p_{ij}^{+} and p_{ij}^{-} reduce to zero as $\mu_i \to \infty$. These results are expected: As μ_i increases beyond C_2, the benefits of the higher demand on leg i reduces, since we cannot satisfy all demand due to our capacity constraint.

Next we analyze the probability that the gain under the perfect information swapping strategy is higher than that under the limited swapping strategy, that is, $Pr(G_{ij}^{P} > G_{ij}^{L})$, which is given by probability p_{ij}^{+} up to the swapping point, and by probability p_{ij}^{-} after the swapping point; see Equation (3.22). Thus, by Result 3.5.4, this probability is non-decreasing in μ_i up to the swapping point μ_i^{th}, and is non-increasing in μ_i after the swapping point. This is illustrated in Figure 3.14, considering $\mu_j = 119$, $\sigma_j = 15$, $\sigma_i = 15$ and 25.

In addition, as we have observed in Figures 3.12 and 3.13, p_{ij}^{+} is non-decreasing in σ_i up to the swapping point, and p_{ij}^{-} is non-decreasing in σ_i after the swapping point. Thus, $Pr(G_{ij}^{P} > G_{ij}^{L})$ is non-decreasing in σ_i over the values evaluated. Observe also that both functions p_{ij}^{+} and p_{ij}^{-} converge to some value as σ_i gets large, and this value is independent of μ_i; see again Figures 3.13(a) and (b). We observe the same behavior as σ_j gets large; see Figures 6.1(a) and (b) in the Appendix.

In summary, *as the absolute magnitude of the demand imbalance* $(|\nu_{ij}|)$ *decreases and/or demand standard deviation,* σ_i*, increases, the probability that the gain of the perfect information swapping strategy is larger than that of the limited strategy,* $Pr(G_{ij}^{P} > G_{ij}^{L})$*, increases. Recall that we have observed the same behavior for the expected difference,* $E[G_{ij}^{P} - G_{ij}^{L}]$*, between the perfect information and the limited swapping strategies; please refer to Figures 3.3(b), 3.6(b), and 3.14.*

Next, we analyze the behavior of these probability functions as μ_j changes.

Result 3.5.5 *Consider any leg $i \in L_1$ and $j \in L_2$. Then, we have:*

$$\frac{\delta p_{ij}^{+}}{\delta \mu_j} = -f_j(C_2)[1 - F_i(C_2)] - \int_{C_1}^{C_2} f_i(d) f_j(d) dd \ \leq 0$$
$$\frac{\delta p_{ij}^{-}}{\delta \mu_j} = F_i(C_2) f_j(C_2) + \int_{C_1}^{C_2} \frac{(d - \mu_j)}{\sigma_j^2} f_j(d) F_i(d) dd \ \geq 0$$

Thus, p_{ij}^{+} is non-increasing in μ_j, whereas p_{ij}^{-} is non-decreasing in μ_j.

The proof is very similar to the previous proof, and thus, will be omitted.

The behavior of p_{ij}^{+} versus μ_j is similar to the behavior of p_{ij}^{-} versus μ_i, and the behavior of p_{ij}^{-} versus μ_j is similar to that of p_{ij}^{+} versus μ_i; please see Figures 6.4(a) and (b) in the Appendix.

Finally, Result 3.5.5 indicates that the probability, $Pr(G_{ij}^{P} > G_{ij}^{L})$, is non-decreasing in μ_j up to the swapping point μ_j^{th}, and is non-increasing in μ_j after the swapping point. This is depicted in Figure 3.15, considering $\mu_i = 119$, $\sigma_i = 15$, and $\sigma_j = 15$ and 25. Observe that the behavior of $Pr(G_{ij}^{P} > G_{ij}^{L})$ is similar as μ_i increases or μ_j increases; see Figures 3.14 and 3.15.

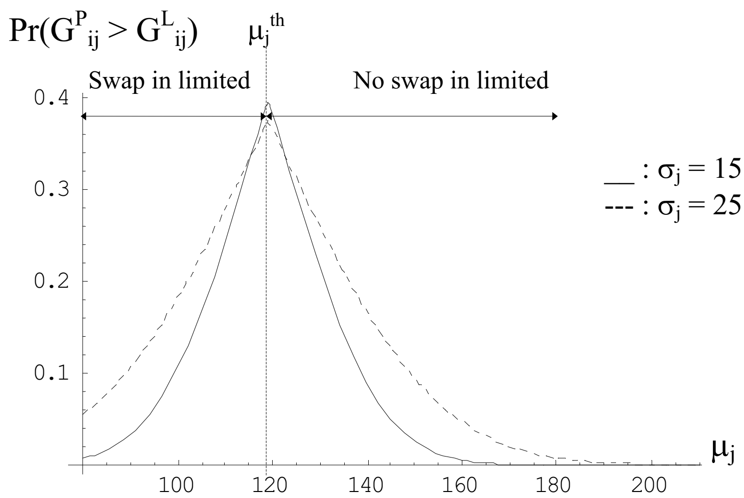

Figure 3.15: $Pr(G_{ij}^P > G_{ij}^L)$ versus μ_j.

We will use these results when we analyze the *total* gain functions of the different swapping strategies in the next chapter, considering a model that consists of two swappable loops.

Chapter 4

Comparison of Limited, Delayed, and Perfect Information Swapping Strategies

4.1 Outline

In this chapter, we present a comprehensive study on the benefits (i.e., the revenue gain) of the various swapping strategies introduced in the previous chapter, as well as on the impact of demand characteristics on the benefits of these strategies. As stated previously, the **trade-off** between a *delayed swapping* decision, which utilizes a more accurate demand information, but causes high disturbance to operations, and an *early swapping* decision, which needs to be made under highly uncertain demand forecasts, but will possibly require little re-scheduling of airport services and operations, should be considered when designing a swapping strategy. In this chapter, we study various swapping strategies that represent different levels of this trade-off.

Specifically, we focus on the three strategies described in Chapter 3: the perfect information swapping strategy (*perfect*), the limited swapping strategy (*limited*), and the delayed swapping strategy (*delayed*). Recall that under the limited swapping strategy, swapping decision is made 4-6 weeks ahead of departures, at a time when demand is highly uncertain, and will not be revised later, whereas the delayed swapping strategy allows the revision of the swapping decision at the beginning of each period until departures, based on updated demand forecasts. Finally, a third strategy, the perfect information swapping strategy, is also included in our analysis in order to obtain an *upper bound* on the benefits of any swapping strategy, since this last strategy considers that perfect demand information is available at the time of making the swapping decision, while assuming that no passengers will be lost due to the delayed timing of the swapping decision. As stated previously, we limit our attention to these three strategies, because each of these policies represent extreme cases. Therefore, insights developed for these strategies can help when developing other strategies that are combinations of these.

In this chapter, we extend the two-leg analysis of Chapter 3 to a more realistic model, by considering two “swappable loops”, each composed of a round-trip (of 2 legs) starting and ending at a common airport at similar times so that the aircraft originally assigned to these loops can be swapped with each other, if needed. Such loop swaps are highly desirable in practice, especially in the DDS stage, at close proximity to departures, due to their ease of implementation and small disturbance to operations.

Next, we use *numerical integration* to compare the limited and perfect information swapping strategies, building on the results derived in Chapter 3. This is presented in Section 4.2. Then, we extend our study to also include the delayed swapping strategy, and compare all three strategies via a simulation model, as detailed in Section 4.2.2.

4.2 Analysis

In this chapter we extend the two-leg analysis of Chapter 3 to a more general analytical model, which consists of two "swappable loops", L_1 and L_2, where $L_1 = \{1, 3\}$, $L_2 = \{2, 4\}$, and $L = L_1 \cup L_2$. As before, C_k represents the capacity of the aircraft type assigned to the legs in set L_k in the initial fleeting solution, for $k = 1, 2$; and D_i represents the independently, normally distributed demand on leg i, for $i \in L$. We consider that both aircraft types are in the same family so that they *can* be swapped with each other in the DDS stage.

Recall that we have defined and used the term "demand imbalance" ($\nu_{ij} = \mu_i - \mu_j$), when we analyzed the two-leg model, consisting of leg pair $(i, j), i \in L_1, j \in L_2$, in Chapter 3. In this chapter, we extend this term so as to include the multiple legs considered in our swappable loops. Thus, in our loop model, we now let $\nu \equiv \sum_{i \in L_1} \mu_i - \sum_{j \in L_2} \mu_j = (\mu_1 + \mu_3) - (\mu_2 + \mu_4)$. Since we consider $C_1 < C_2$, a swap will assign the aircraft having the larger capacity (of C_2) to legs 1 and 3, and the aircraft having the smaller capacity (of C_1) to legs 2 and 4. Therefore, we would expect the swap benefits to increase as the demand imbalance increases. Indeed, for the simple two-leg model studied in Chapter 3, we find that both functions $E[G^L_{ij}]$ and $E[G^P_{ij}]$ are non-decreasing in $\mu_i, i \in L_1$, while converging to a constant for large values of μ_i; see Result 3.5.1 in Chapter 3. Similarly, we find that both $E[G^L_{ij}]$ and $E[G^P_{ij}]$ are non-increasing in $\mu_j, j \in L_2$, while converging to zero as μ_j becomes large; see Result 3.5.2 in Chapter 3. However, as we shall see in Section 3.4, demand imbalance is **not** the only factor that affects the benefits of a swap; we will introduce the other important factors that impact the revenue gain of a swap subsequently in our analysis.

In all our analysis, we consider two aircraft types of the Boeing 737 family, having capacities of $C_1 = 112$ and $C_2 = 126$ passengers, as was done in the previous chapter.

We consider independent normal distributions for each of the four leg demands in our loop model, having means in the values of $\mu_i = \{93, 112, 126, 140, 210\}$, for $i \in L$; these values cover different demand patterns ranging from demand being below capacity of the smaller aircraft type to being above capacity of the larger aircraft type. We plan to analyze four different scenarios, each with different demand standard deviations: (i) $\sigma_i = 15$, $\forall i \in L$; (ii) $\sigma_i = 25$, $\forall i \in L$; (iii) $\frac{\sigma_i}{\mu_i} = 0.15$, $\forall i \in L$; and (iv) $\frac{\sigma_i}{\mu_i} = 0.25$, $\forall i \in L$. These values are typical of demand standard deviations experienced 4-6 weeks before departures, the stage considered in our DDS model, by our industry partner; please refer to the Appendix for more details on our numerical experiments. Thus, scenarios (i) and (ii) represent systems having similar demand variability on each leg, whereas scenarios (iii) and (iv) represent systems having similar demand coefficients of variation ($\frac{\sigma_i}{\mu_i}$, $i \in L$) on each leg in the swappable loops. Consequently, in scenarios (i) and (ii), we can observe the pure effect of demand imbalance on the gain functions by varying the mean leg demands, while keeping the demand variability constant. Then, in scenarios (iii) and (iv), we can observe the additional effect of demand variability on the gain functions, since now demand variability will also increase as mean demands (demand imbalance) increase. When the same σ_i is considered on each leg $i, i \in L$, we denote the common demand standard deviation as σ, and refer to it as the *system standard deviation*. Similarly, when the demand coefficient of variation is the same on each leg considered, we refer to it as the *system coefficient of variation*.

We first study the pure effect of demand imbalance on the gain functions under the limited and perfect information swapping strategies. As mentioned above, for this purpose we study systems having the same level of demand standard deviation on each leg (i.e., scenarios (i) and (ii)), which we denote by σ. Thus, we vary levels of demand imbalance in the system, while keeping demand standard deviations constant. Then we extend this analysis to systems having a common demand coefficient of variation by considering scenarios (iii) and (iv).

In Section 4.2.1, we first study the effectiveness of the limited swapping strategy via our analytical model. Then, in Section 4.2.2, we extend this analysis to also include the delayed swapping strategy in our comparison. This is done by a simulation model.

4.2.1 Comparison of the Limited and Perfect Information Swapping Strategies

The limited swapping strategy is attractive in practice, especially when the re-scheduling of operations is highly undesirable. However, since it requires the swapping decision to be made early in time, under high demand uncertainty, it can lead to risky swaps that might result in loss. Thus, our objective in this section is to characterize the effectiveness of the limited swapping strategy. For this purpose, we compare the revenue gain of the limited swapping strategy, G^L, with G^P, the revenue gain of the perfect information strategy, which represents an *upper bound* on the benefits of any swapping policy. Our objective is to characterize how this difference depends on demand characteristics (i.e., mean and variance of demand on each leg in the swappable routes).

Recall that in the limited swapping strategy, we make our swapping decision based on the expected revenue gain of the swap (i.e., we swap if the expected revenue gain of the swap in period T is positive; and do not swap otherwise). Thus, the expectations of random variables G^P and G^L can be expressed as follows:

$$\begin{aligned} E[G^P] &= E[\max\{\sum_{i \in L_1} \Delta_i - \sum_{j \in L_2} \Delta_j, 0\}] = E[\max\{\omega_{12} + \omega_{34}, 0\}] \\ E[G^L] &= \max\{E[\sum_{i \in L_1} \Delta_i - \sum_{j \in L_2} \Delta_j], 0\} = \max\{E[\omega_{12}] + E[\omega_{34}], 0\} \end{aligned} \tag{4.1}$$

Thus, the expected gain under the limited strategy can be obtained using Equation (3.13), derived in Chapter 3. However, as stated in Chapter 3, it becomes analytically messy to derive a closed-form expression for the expected gain under the perfect information swapping strategy. Observe that:

$$\begin{aligned} E[G^P] &= E[\max\{\omega_{12} + \omega_{34}, 0\}] \\ &\leq E[\max\{\omega_{12}, 0\}] + E[\max\{\omega_{34}, 0\}] \\ &= E[\omega_{12}^+] + E[\omega_{34}^+] \end{aligned}$$

Similarly,

$$\begin{aligned} E[G^P] &= E[\max\{\omega_{14} + \omega_{32}, 0\}] \\ &\leq E[\omega_{14}^+] + E[\omega_{32}^+] \end{aligned}$$

Consequently, we make use of the following *upper bound* on $E[G^P]$:

$$E[G^P] \leq \min\{\ E[\ \omega_{12}^+] + E[\omega_{34}^+],\quad E[\omega_{14}^+] + E[\omega_{32}^+\ ]\ \} \equiv \overline{E[G^P]} \tag{4.2}$$

Thus, we can determine our upper bound using the closed-form expressions for $E[\omega_{ij}^+]$, $i \in L_1, j \in L_2$, derived in Chapter 3 (see Equation (3.9)).

As below, we first analyze the quality of the upper bound, $\overline{E[G^P]}$, given in Equation (4.2), which in fact provides an upper bound on the expected revenue gain possible in any swapping strategy. We simulate the loop model under the perfect information strategy, considering various values of mean and standard deviations for demands on the four legs; and determine the expected gain under the perfect information swapping strategy over 1000 replications. Figure 4.1 shows how $\overline{E[G^P]}$, obtained analytically using Equation (4.2), compares with $E[G^P]$, the expected revenue of the perfect information strategy determined in the simulation, when $\sigma_i = 15$, $\forall i \in L$. We observe similar patterns for different values of demand standard deviations. Thus, our simulation results show that this upper bound is quite close to the expectations obtained by simulation (for example, an average deviation of 7.5% for the case where $\sigma_i = 15$, $\forall i \in L$). Observe also that the larger the demand imbalance is, the smaller the difference between our upper bound and the simulation result; our upper bound gets stronger as demand imbalance increases. The reason is very intuitive: As the demand imbalance increases, random variables ω_{ij}, $i \in L_1, j \in L_2$ generally take on negative values with lower probabilities. Thus, the upper bound better approximates $E[G^P]$.

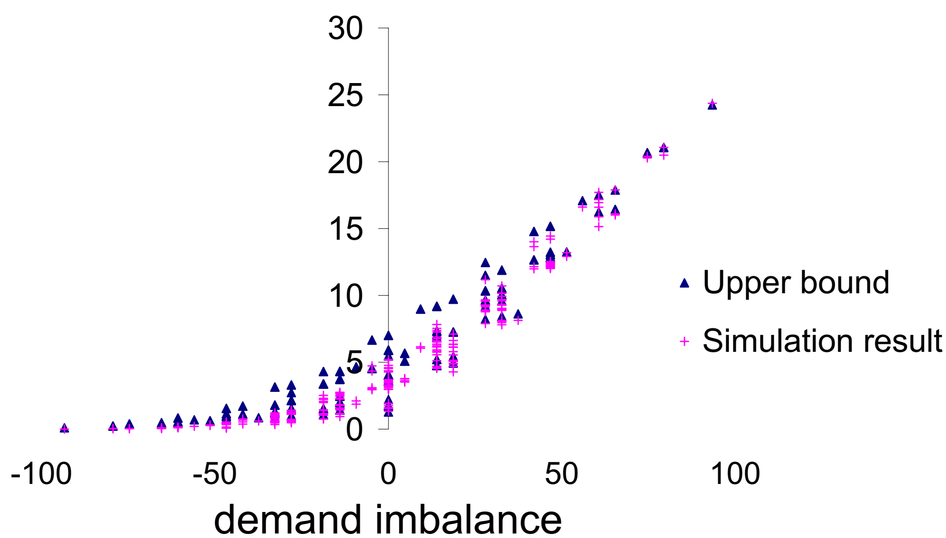

Figure 4.1: $\overline{E[G^P]}$ (obtained analytically) and $E[G^P]$ (obtained via simulation) versus demand imbalance when $\sigma_i = 15$, $\forall i \in L$.

In what follows, we use *numerical integration* to determine the expected revenue under the limited swapping policy, given in Equation (4.1), using Equation (3.13) derived in Chapter 3; and the upper bound on the expected revenue under the perfect information swapping policy, given in Equation (4.2), using Equation (3.9) derived in Chapter 3, considering various values of demand parameters on the four legs and scenarios (i) and (ii). Our objective is to determine how far the gain of the limited strategy is from the potential gain possible, and how this difference is affected by demand characteristics such as demand imbalance and variability so that we can obtain insights on how to manage the flexible capacity in the system (i.e., when the limited strategy should be used in practice).

We first study how the upper bound on the expected gain of swapping changes as demand imbalance and/or demand variability increase; see Figure 4.2, which plots $\overline{E[G^P]}$ versus demand imbalance for system demand standard deviations 15 and 25 (scenarios (i) and (ii)). We observe that the upper bound is generally non-decreasing in demand imbalance for both system standard deviations. In addition, we observe that for non-positive levels of demand imbalance ($\nu \leq 0$), higher system standard deviation leads to a higher expected gain, whereas for positive levels of demand imbalance ($\nu > 0$), lower system standard deviation is preferable. Observe that these findings are consistent with the results obtained in Chapter 3 for the two-leg model; see Results 3.5.1 and 3.5.2; and Figures 3.1(a), 3.2, 3.4(a) and 3.5 in Chapter 3. We observed a similar behavior in scenarios (iii) and (iv).

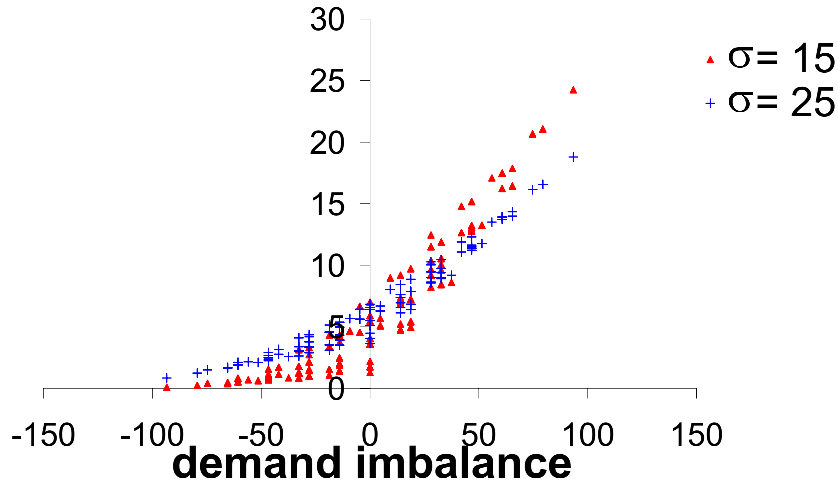

Figure 4.2: $\overline{E[G^P]}$ versus demand imbalance.

Similarly, Figure 4.3 shows how $E[G^L]$, the expected gain under the limited swapping strategy, changes as (i) demand imbalance varies, and (ii) system demand standard deviation increases from 15 to 25.

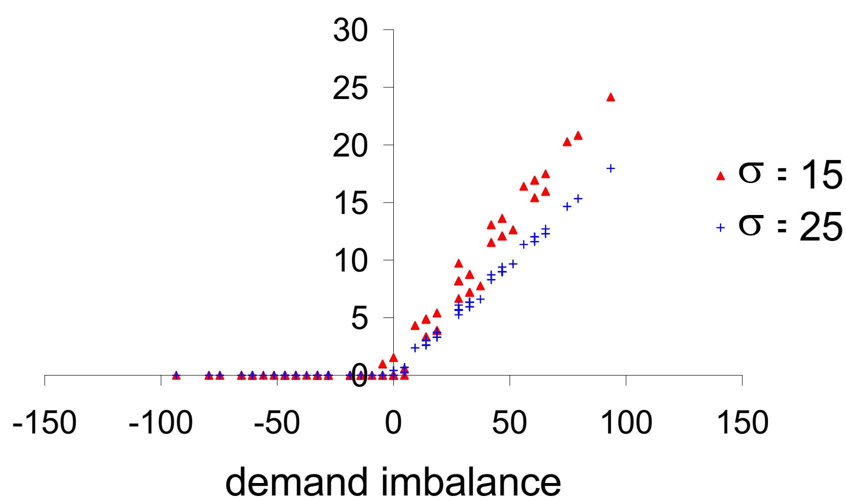

Figure 4.3: $E[G^L]$ versus demand imbalance.

Observe that when demand imbalance is non-positive (i.e., $\nu \leq 0$), a swap is *generally* not made under the limited swapping strategy; thus, its expected gain is zero. The cases where a swap is made under the limited strategy usually correspond to those with positive demand imbalance (i.e., $\nu > 0$): We observe that in this region, $E[G^L]$ is non-decreasing in demand imbalance. In addition, when a swap is made, then lower demand variability leads to higher expected gain under the limited swapping strategy, as was the case with $\overline{E[G^P]}$. These findings are consistent with the results obtained in Chapter 3 for the two-leg model; see Results 3.5.1 and 3.5.2; and Figures 3.3(a), 3.6(a), 6.2, and 6.3. Thus, the limited strategy performs better under high demand imbalance and low demand variability.

Next we compare the expected gain under the limited strategy with the upper bound, $\overline{E[G^P]}$, and analyze their difference. In Figure 4.4(a), we plot the expected gain under the limited swapping strategy, $E[G^L]$, and the upper bound, $\overline{E[G^P]}$, versus demand imbalance when the system demand standard deviation is 15 (scenario (i)); and in Figure 4.4(b), we plot their difference, given by $\overline{E[G^P]} - E[G^L]$, versus demand imbalance for system standard deviations of 15 and 25 (scenarios (i) and (ii)). We can analyze both graphs in two regions: The first region corresponds to non-positive levels of demand imbalance (i.e., $\nu \leq 0$). In this region, generally no swaps are made under the limited strategy, and therefore, $E[G^L]$ is zero. Then, $\overline{E[G^P]} - E[G^L]$ is simply given by $\overline{E[G^P]}$, which is increasing both in demand imbalance and also in σ in this region as was observed in Figure 4.2. The second region corresponds to positive levels of demand imbalance ($\nu > 0$). In this region, a swap is usually made under the limited strategy. We observe that the difference, $\overline{E[G^P]} - E[G^L]$, is now decreasing in demand imbalance, but is still increasing in σ. Observe that these results are very similar to the findings of Chapter 3; see Results 3.5.1 and 3.5.2; and Figures 3.3 and 3.6. These results are very intuitive. *When the absolute magnitude of the demand imbalance in* *the system, $|\nu|$, is large, the swapping decision under the limited swapping strategy will not be very risky; that is, the decision of to swap in the limited swapping strategy will likely result in revenue gain, whereas the decision of not to swap will likely avoid loss. Thus, the swapping decisions under the limited and perfect information strategies will likely be similar in such cases, and therefore, their expected difference small. However, when the absolute magnitude of the demand imbalance, $|\nu|$, is small, the swapping decision in the limited swapping strategy will be more risky, and hence, the swapping decisions under the limited and perfect information strategies will not always be the same.*

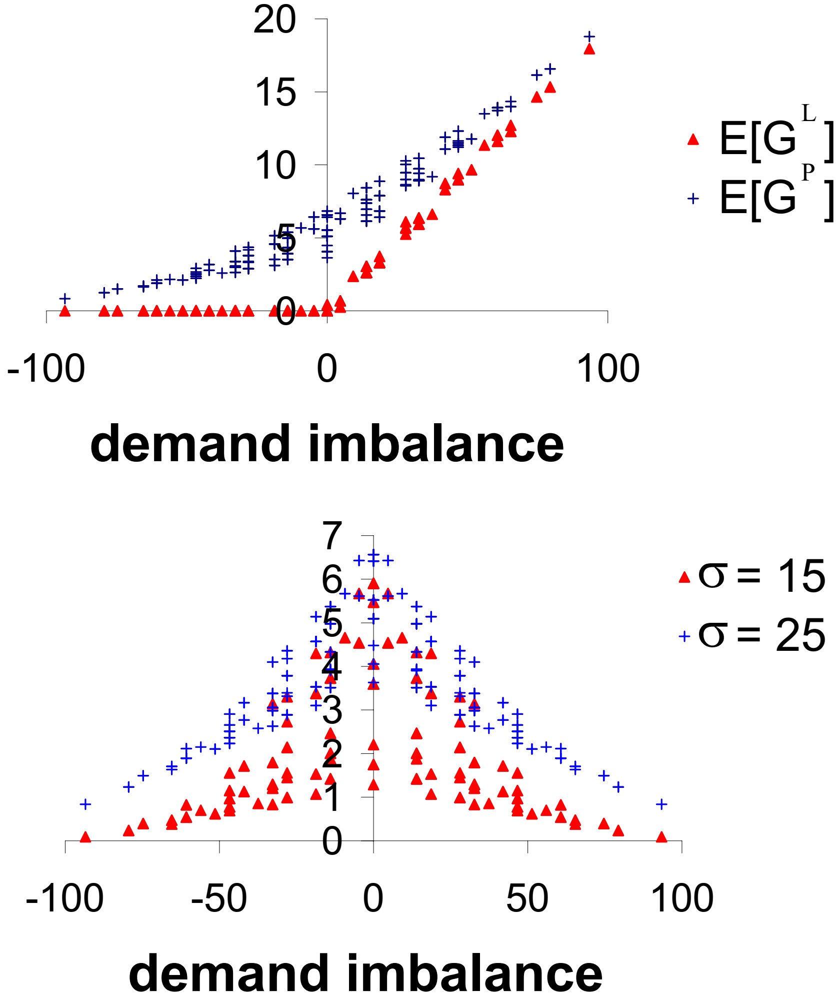

Figure 4.4: a) $\overline{E[G^P]}$ and $E[G^L]$ versus demand imbalance; b)$\overline{E[G^P]} - E[G^L]$ versus demand imbalance.

In fact, as can be observed in Figure 4.4, we find that although the concept of demand imbalance explains most of the variations in the expected gain functions in the loop model under the perfect information and limited swapping strategies, there still exists some noise, especially when the absolute magnitude of the demand imbalance, $|\nu|$, is small, and hence, the swapping decision under the limited swapping strategy is highly risky. This is because the risk factor in such cases is not uniquely determined by the demand imbalance; similar levels of demand imbalance can correspond to different combinations of the mean demands, which would affect the benefits of swapping. In order to demonstrate this behavior, we now graph $\overline{E[G^P]}$ and $E[G^L]$ versus the sum of the mean demands on legs assigned to the smaller capacity of C_1 ($\sum_{i\in L_1}\mu_i = \mu_1+\mu_3$), while holding all mean demands on legs assigned to the larger capacity of C_2 constant (i.e., μ_2 and μ_4 are held constant at values of $\mu_2 = \mu_4 = 112$, while $\sigma_2 = \sigma_4 = 15$); see Figures 4.5 and 4.6. For each value of $\mu_1+\mu_3$, we plot different combinations of μ_1 and μ_3, thus generating different curves with same levels of demand imbalance. Similarly, in Figures 4.7 and 4.8, we graph $\overline{E[G^P]}$ and $E[G^L]$ versus $\sum_{j\in L_2}\mu_j = \mu_2+\mu_4$, while holding all mean demands on legs in set L_1 constant (i.e., μ_1 and μ_3 are constant). Thus, we observe that although each curve exhibits a similar behavior (i.e., functions $\overline{E[G^P]}$ and $E[G^L]$ are both non-decreasing in $\mu_1+\mu_3$, and are non-increasing in $\mu_2+\mu_4$), their values can be different for the same level of demand imbalance. In addition, we observe that usually the larger the absolute difference between μ_1 and μ_3 (i.e., $|\mu_1-\mu_3|$), the lower the values of $E[G^L]$ and $\overline{E[G^P]}$ are.

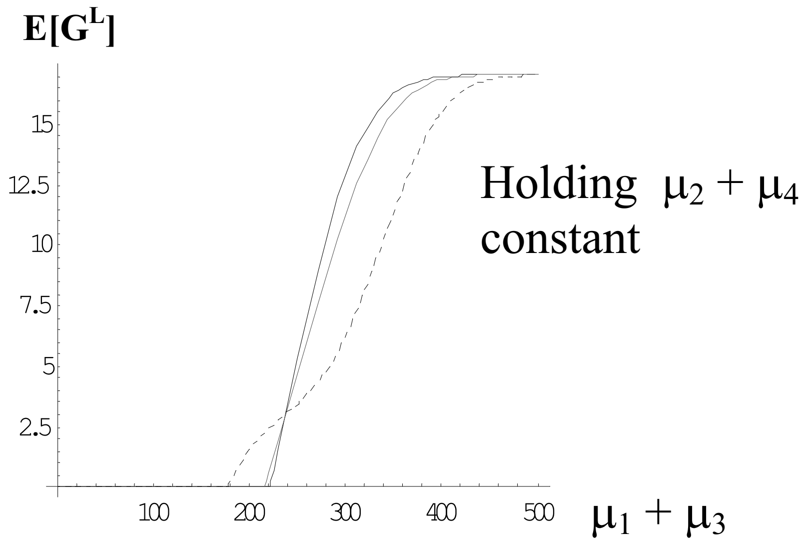

Figure 4.5: $E[G^L]$ versus $\mu_1 + \mu_3$

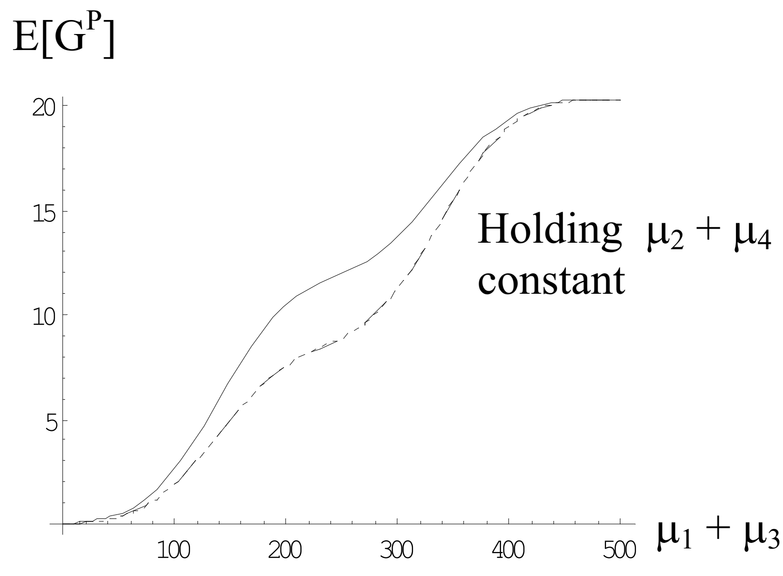

Figure 4.6: $E[G^P]$ versus $\mu_1 + \mu_3$

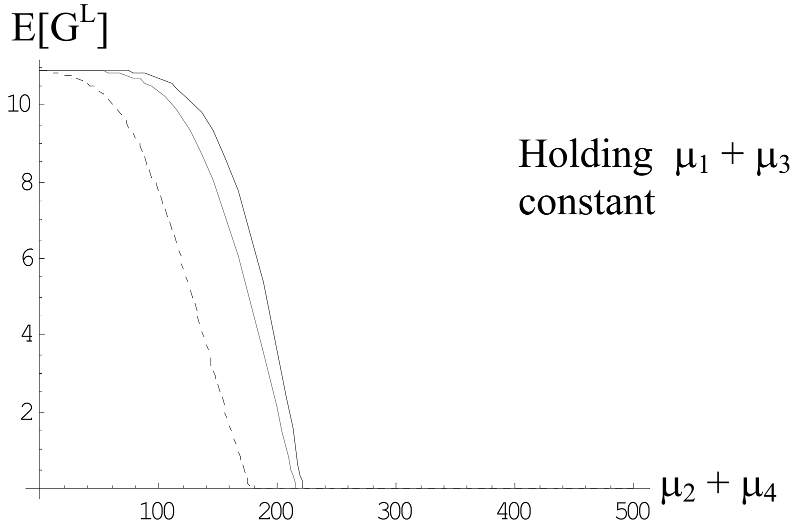

Figure 4.7: $E[G^L]$ versus $\mu_2 + \mu_4$

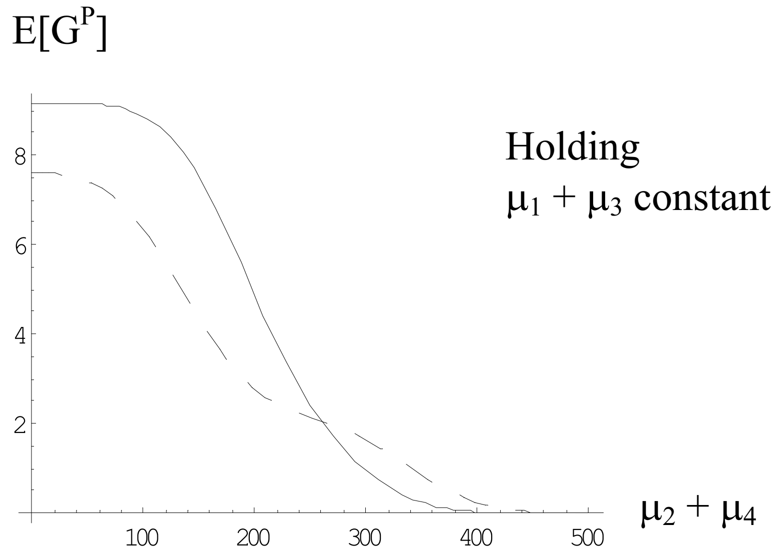

Figure 4.8: $E[G^P]$ versus $\mu_2 + \mu_4$

In order to give the reader a better idea on behavior of the expected gain functions under the limited and perfect information swapping strategies, we also plot the 3-dimensional graphs of these functions, where the x-axis represents $\sum_{i \in L_1} \mu_i = \mu_1 + \mu_3$, y-axis represents $\sum_{i \in L_2} \mu_i = \mu_2 + \mu_4$, and z-axis represents the functions $E[G^L]$ or $\overline{E[G^P]}$. These graphs are given in Figure 4.10.

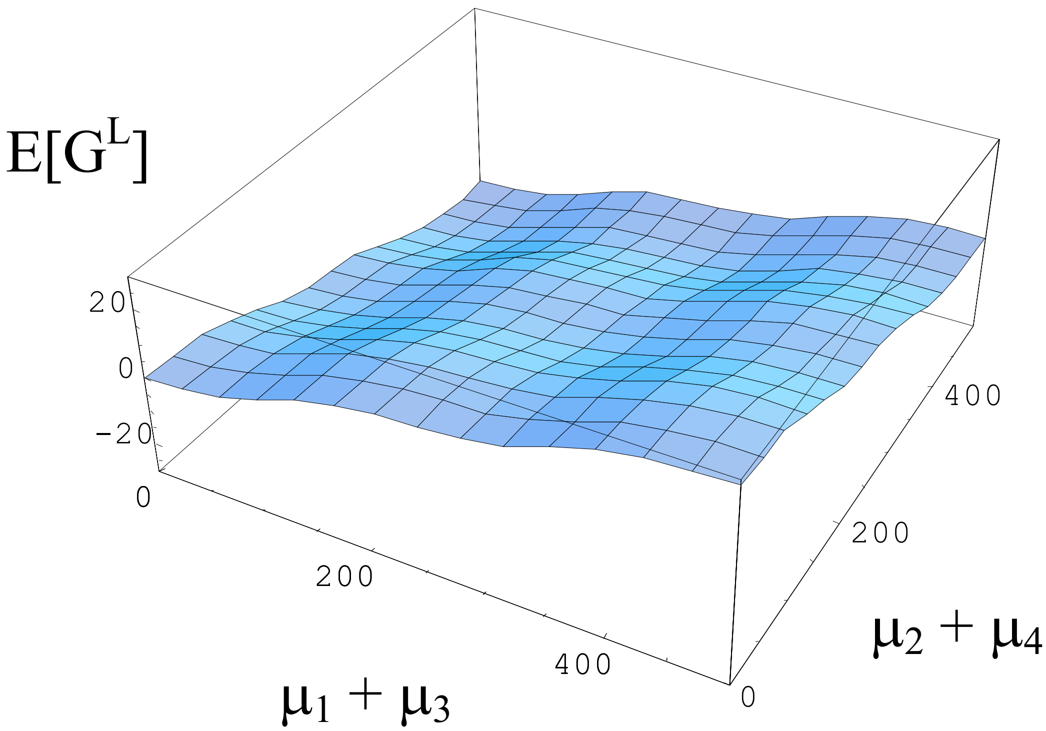

Figure 4.9: $E[G^L]$ versus $\mu_1 + \mu_3$ and $\mu_2 + \mu_4$.

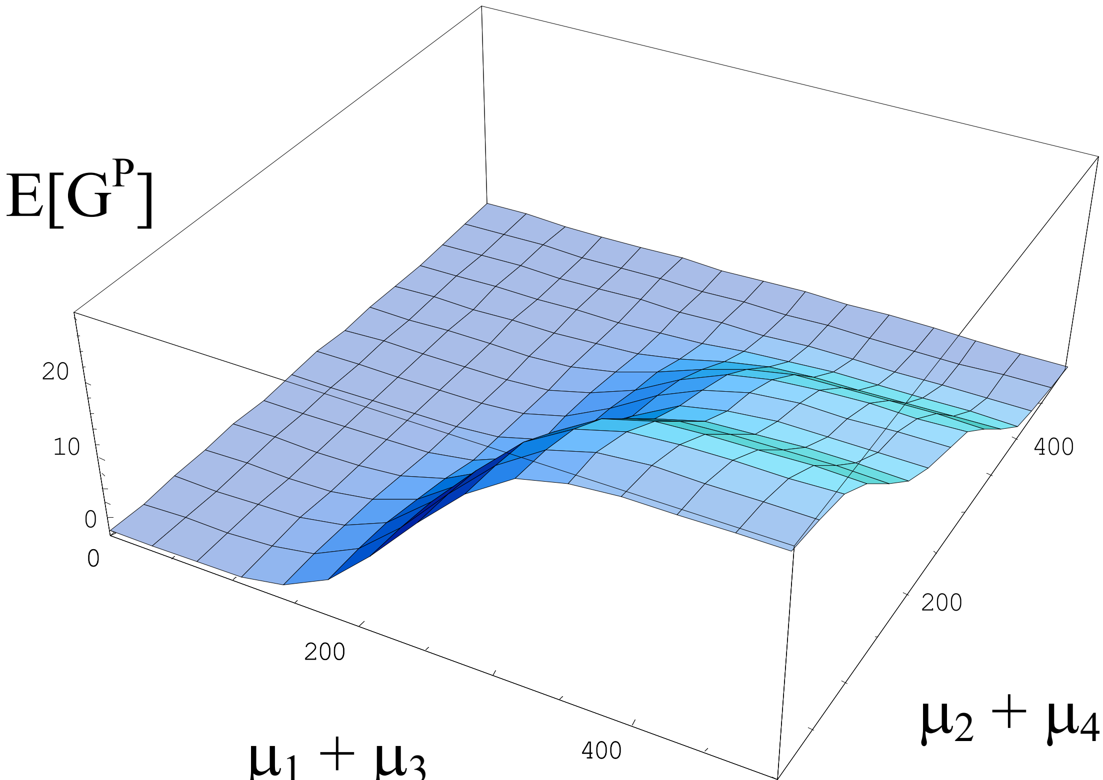

Figure 4.10: $\overline{E[G^P]}$ versus $\mu_1 + \mu_3$ and $\mu_2 + \mu_4$.

Thus, in addition to demand imbalance, we need other factors to explain the variation in the difference between the expected gain under the limited swapping strategy, $E[G^L]$, and our upper bound, $\overline{E[G^P]}$, for cases where the absolute magnitude of the demand imbalance is small. Consider first the case with small and positive demand imbalance; that is, the sum of the mean demands initially assigned to the aircraft having the smaller capacity (of C_1), $\mu_1 + \mu_3$, is larger, but still close, in magnitude, to $\mu_2 + \mu_4$, the sum of the mean demands initially assigned to the aircraft having the larger capacity (of C_2). In this case, a swap will generally be performed under the limited swapping strategy, but the swap will be highly risky. The same level of demand imbalance can be realized for different combinations of the means on the four legs. In fact, the smaller μ_1 and μ_3 are compared to aircraft capacity C_2 (or, alternatively, the larger μ_2 and μ_4 compared to C_1), the more risky the swapping decision under the limited swapping strategy, and hence the larger the difference will be between the gain under the limited swapping strategy and the upper bound. As an example of this case, consider the following two scenarios, under each of which a swap is performed under the limited swapping strategy:

$$(1)(\mu_1, \mu_2, \mu_3, \mu_4) = (140, 126, 140, 126); \quad \sigma_i = 15, \forall i \in L$$
$$(2)(\mu_1, \mu_2, \mu_3, \mu_4) = (126, 112, 126, 112); \quad \sigma_i = 15, \forall i \in L.$$

Thus, we have the same level of demand imbalance of $\nu = 28$ for both scenarios. Recall that $C_1 = 112$ and $C_2 = 126$ in our analysis. Observe that μ_1 and μ_3 in scenario (2) are at the value of C_2 (and μ_2 and μ_4 are at the value of C_1), whereas μ_1 and μ_3 in scenario (1) are larger than C_2 (and μ_2 and μ_4 are larger than C_1). Thus, the swap performed under the limited swapping strategy in scenario (2) will be more risky than that in scenario (1), and hence, the perfect information swapping strategy should be more beneficial for scenario (2). Indeed, $\overline{E[G^P]} - E[G^L]$ in the first scenario is 1.6, while that in the second scenario is 2.7; please refer to the Appendix for all results. Consider now the following two scenarios, under each of which a swap is performed under the limited swapping strategy:

$$(3)(\mu_1, \mu_2, \mu_3, \mu_4) = (126, 112, 126, 126); \quad \sigma_i = 15, \forall i \in L$$
$$(4)(\mu_1, \mu_2, \mu_3, \mu_4) = (140, 112, 112, 126); \quad \sigma_i = 15, \forall i \in L.$$

Thus, we have the same level of demand imbalance of $\nu = 14$ for both scenarios. Observe that the sum of $\mu_1 + \mu_3$, and the values of μ_2, and μ_4 are the same in both scenarios. However, both μ_1 and μ_3 are at the values of C_2 in scenario (3), whereas μ_1 is above C_2, and μ_3 is below C_2 in scenario (4). We have that $\overline{E[G^P]} - E[G^L]$ in the third scenario is 4.3, while that in the fourth scenario is 3.7. In fact, analysis several other scenarios, where demand imbalance is small but positive such that a swap is performed under the limited swapping strategy. We observe that keeping $\mu_2 + \mu_4$ constant, $\overline{E[G^P]} - E[G^L]$ increases as one of μ_1 or μ_3 increases above C_2 (and thus, the other one reaches a value below C_2). Similarly, for scenarios with small but negative demand imbalance such that a a swap is not performed under the limited swapping strategy, we observe that keeping $\mu_1 + \mu_3$ constant, the expected difference, $\overline{E[G^P]} - E[G^L]$, increases as one of μ_2 or μ_4 increases above C_2 (and thus, the other one reaches a value below C_2). Hence, we include another factor, the *absolute deviation from capacity*, in our analysis, which is defined as:

$$DevCap \equiv \begin{cases} \sum_{i \in L_1} |\mu_i - C_2| + \sum_{i \in L_2} |\mu_i - C_1|, & \text{if } \mathrm{II} = 1, \\ \sum_{i \in L_1} |\mu_i - C_1| + \sum_{i \in L_2} |\mu_i - C_2|, & \text{otherwise.} \end{cases}$$

Thus, for a given level of demand imbalance, we observe that the expected difference is non-increasing as the absolute deviation from capacity increases; see Figures 4.11 and 4.12, which plot $\overline{E[G^P]} - E[G^L]$ versus absolute deviation from capacity for cases having demand imbalance levels of 0 and 14, respectively.

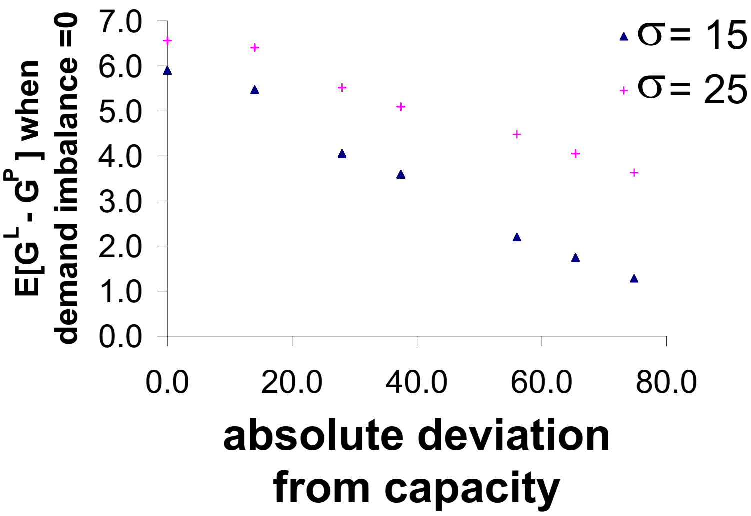

Figure 4.11: $\overline{E[G^P]} - E[G^L]$ versus absolute deviation from capacity for scenarios having demand imbalance $= 0$.

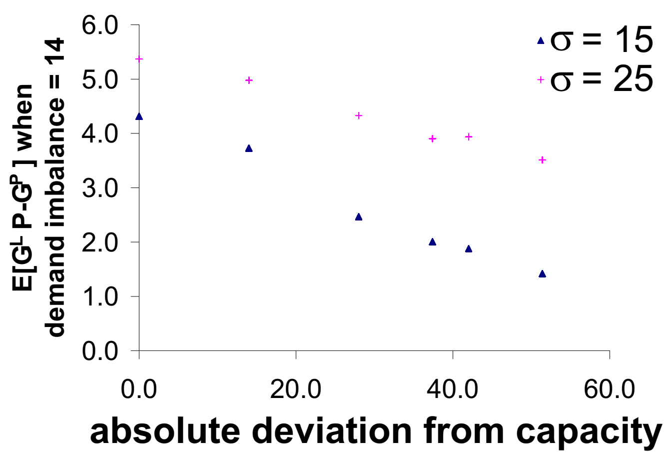

Figure 4.12: $\overline{E[G^P]} - E[G^L]$ versus absolute deviation from capacity for scenarios having demand imbalance $= 14$.

These results suggest that the limited swapping strategy is an attractive strategy in practice, when the demand uncertainty in the system is low, and demand imbalance is positive and large, or demand imbalance is negative and large (i.e., when $|\nu|$ is large). However, its performance degrades as demand uncertainty increases and demand imbalance is medium and absolute deviation from capacity is low. It performs worst for systems with zero demand imbalance.

The next section presents a comprehensive simulation study carried out to compare the benefits of the limited swapping strategy with the delayed swapping strategy when demand variability is the same on all legs in the swappable loops.

4.2.2 Comparison of the Delayed Swapping Strategy with the Limited and Perfect Information Swapping Strategies

As mentioned before, the delayed swapping strategy reduces the number of risky swaps by allowing revisions to the swapping decision later on, based on updated demand forecasts, but at the expense of larger disruptions to operations. Thus, in this section, we study the effectiveness of the delayed swapping strategy by comparing it with the limited and perfect information swapping strategies through a simulation model. In order to study the pure effect of demand characteristics on the swapping gain, we consider two swappable loops, each having the same level of demand standard deviation of 15, and mean demands in the range of $\mu_i = \{93, 112, 126, 140, 210\}$, for $i \in L$, as was done in the previous section. We plan to extend this analysis to systems having a common demand standard deviation of 25 as well as a common demand coefficient of variation of 0.15 and 0.25 in our subsequent research.

Specifically, in our simulation model we consider a time horizon of $T = 6$ weeks in the DDS stage. The swapping decision under the limited and delayed swapping strategies is made 6 weeks prior to departures, based on the expected revenue gain of the swap. Then, under the delayed swapping strategy, revisions are allowed to the swapping decision at the beginning of every week t, $t = T-1, \cdots, 1$, where t represents the number of periods until departures, and these revisions are based on the demand forecasts updated at the beginning of each week. However, no revisions to the swapping devision are allowed under the delayed swapping strategy, if the number of tickets already sold on legs that will be swapped to the smaller capacity C_1 is already above the capacity of this aircraft; that is, no revisions are allowed at the beginning of period t if $\sum_{j=t+1}^{T} d_{2j} > C_1$ or $\sum_{j=t+1}^{T} d_{4j} > C_1$; please see Chapter 3, Section 3.2 for a detailed description of the delayed swapping strategy and our demand forecast updating mechanism. As described in Section 3.2, we consider a single class on each leg and assume that the demand, D_{it}, on leg i in period t is normally distributed, and is independent among legs and across time; this assumption is also used in the demand forecasting system of our industry partner. Thus, D_i, the total demand on leg $i, i \in L$, is still normally distributed with mean μ_i and standard deviation σ_i, as has been considered in all the previous analysis. Observing the real data on several flight legs provided to us by our industry partner, we have derived some factors to obtain the mean and variance of the demand on leg i in week t, $D_{it}, i \in L, t = T, \cdots, 1$, as well as the cumulative demand observed on the leg prior to time T. Of course the factors selected for this model will affect the performance of the delayed swapping strategy. We have used the same set of factors in all the following analysis. In our simulation, we use the greatest integer less than or equal to the demand value generated to obtain integral demands. Each simulation scenario is replicated $10,000$ times and performance measures have been averaged over all replications; please see the Appendix for details on the scenarios. Observe that in our simulation model we can obtain the exact expression for the expected gain under the perfect information swapping strategy, $E[G^P]$, which, as mentioned above, represents an upper bound on the expected revenue gain of any swapping strategy. Hence, we make use of this expression in our comparisons instead of $\overline{E[G^P]}$, the upper bound on it.

Our simulation results for a system demand standard deviation of 15 are summarized in Figures 4.13–4.16. Specifically, Figure 4.13 plots the expected differences between the perfect information and the delayed swapping strategies, $E[G^P - G^D]$, and between the perfect and the limited swapping strategies, $E[G^P - G^L]$, versus demand imbalance, whereas Figure 4.14 plots $Pr(G^P > G^L)$, the probability that the gain under the limited swapping strategy is strictly less than the upper bound, versus demand imbalance. Finally, Figures 4.15 and 4.16 plot the average number of swaps (or revisions to the swapping decision) under the delayed and limited swapping strategies, respectively. Observe that under the limited swapping strategy, the number of swaps in each scenario will be either 0 or 1, whereas under the delayed swapping strategy, the number of swaps (including the revisions) can be between 0 and 6, since there is a potential to revise the swap at the beginning of every week during the horizon of 6 weeks considered in the simulation. Note also that simulation results will have more noise than the analytical results.

These figures suggest the following conclusions:

- We observe that the behavior of the expected difference between the perfect and limited information swapping strategies in the simulation versus demand imbalance is very similar to the analytical results; please see Figure 4.4(b): As the absolute magnitude of the demand imbalance increases, the expected gain of the limiting swapping strategy approaches to the upper bound. We observe the same behavior on the expected difference between the delayed and limited swapping strategies, $E[G^D - G^L]$, versus demand imbalance. In addition, the expected difference between the perfect information and delayed swapping strategies is much smaller than that between the delayed and limited swapping strategies. Thus, the delayed swapping strategy achieves most benefits of a perfect information strategy with a finite number (and as we shall see, in fact a small number) of swaps. We also found that in addition to the demand imbalance, the absolute deviation from capacity factor, discussed in the previous section, is successful in explaining the variations in these expected differences.

- Recall that we have studied the behavior of the probability function, $Pr(G_{ij}^P > G_{ij}^L)$, only for the simple two-leg model in Chapter 3 (please see Results 3.5.4 and 3.5.5 and Figures 3.14 and 3.15), but **not** for our two-loop model, since determining this probability, considering the four legs, involves a large number of integrations. However, our simulation results in Figure 4.14 show that this probability function behaves similar to those analytical results for the two-leg model: The larger the absolute magnitude of the demand imbalance is, the smaller this probability.

- Interestingly, we observe that the average number of swaps performed under the delayed swapping strategy in the DDS stage is surprisingly low (a maximum of 1.33 swaps on the average under the delayed strategy); please see Figure 4.15. However, as discussed above, the delayed swapping strategy achieves most benefits of the perfect information strategy (the upper bound). Thus, these benefits are achieved by a very small number of additional swaps performed under the delayed strategy! Analyzing the simulation results in detail, we find that most swaps under the delayed swapping strategy are, in fact, performed $4 - 6$ weeks away from departures, thus not causing much disturbance to operations. This is partly due to our swapping rule, which prohibits swaps when more passengers than the smaller capacity are already ticketed on legs that will be swapped to

> the smaller capacity. However, the results suggest that swaps performed early in time can be highly effective. Also observe that when the absolute demand imbalance in the system is large, then the number of swaps under the delayed swapping strategy is very similar to those under the limited swapping strategy; please see Figures 4.15 and 4.16. This is because swaps made under the limited strategy in these regions are generally not risky. Only when the absolute demand imbalance in the system is small, the delayed strategy needs to perform revisions to the initial swapping decision. However, as mentioned, the number of these additional swaps is very low (0.32 more swaps on the averages).

Thus, this analysis suggests that for swappable loops having similar demand variability on each legs: (1) the limited swapping strategy provides an excellent way to hedge against the demand uncertainty, when the absolute demand imbalance in the system is high and demand uncertainty is low; (2) otherwise, a strategy that allows the revision of the swapping decision later on, such as the delayed swapping strategy studied here, can be utilized. Such a strategy can achieve most potential benefits of swapping, and with only a very small number of additional swaps, mostly performed 1-2 weeks after the initial swapping decision.

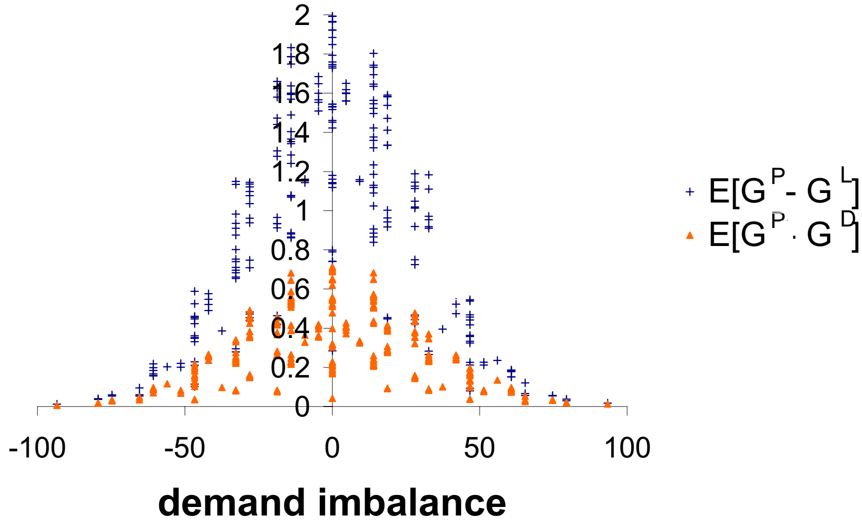

Figure 4.13: $E[G^P - G^D]$ and $E[G^D - G^L]$ versus demand imbalance.

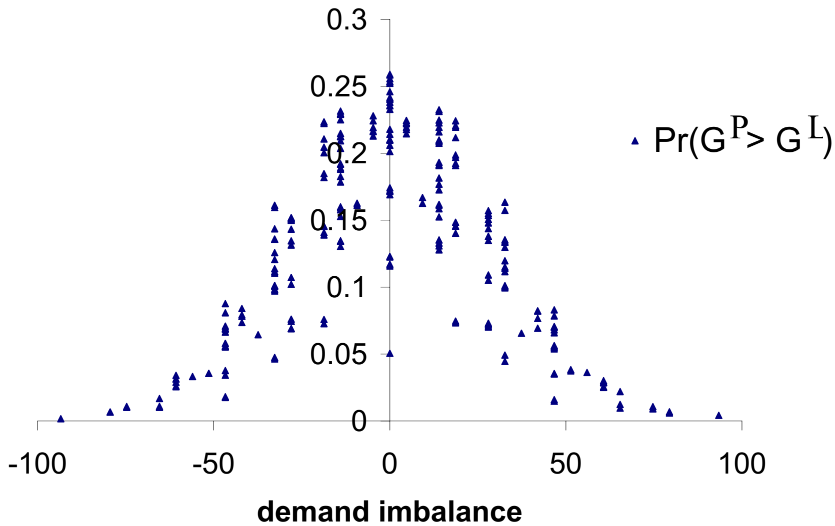

Figure 4.14: $Pr(G^P > G^L)$ versus demand imbalance.

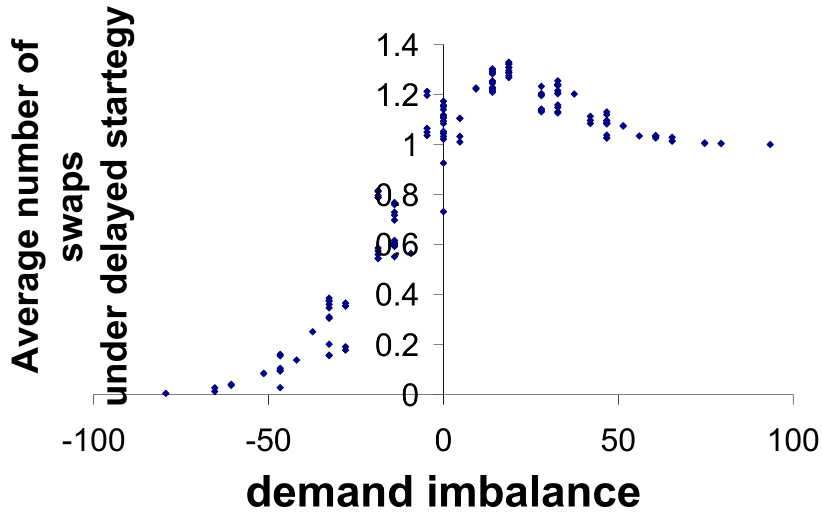

Figure 4.15: Average number of swaps under the delayed swapping strategy in the DDS stage.

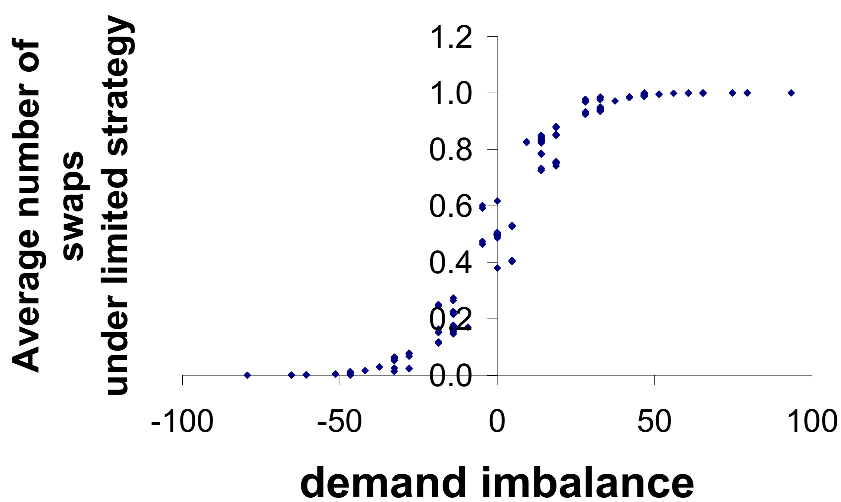

Figure 4.16: Average number of swaps under the limited swapping strategy in the DDS stage.

Chapter 5

A Simulation Study Considering Multiple Fare-Classes

5.1 Simulation Model with Multiple Fare Classes

In this section, we extend our simulation model to consider two fare-classes on each flight leg, where fare-class 1 corresponds to the class having the more expensive fare. Thus, we would like to fill our capacities with as many class 1 customers as possible. However, since the higher fare customers might arrive after lower fare customers, it is common practice in the airline industry to set aside a certain number of seats for anticipated future arrivals of the higher fare customers and protect them from the lower fare customers. The number of these reserved seats is referred to as the *protection level* in the airline revenue management literature. One of the most commonly used models to determine protection levels is called the EMSRb model (Barocio-Cots, 1999). This model determines the optimal protection levels under the following assumptions. The model considers a single leg having only two fare-classes and assumes that demands on each fare-class are independent and all lower fare customers arrive before the higher fare customers. Let f_i denote the fare of class $i, i = 1, 2$, and F_1 denote the CDF of class 1 demand. Under these assumptions, the protection level P can be calculated as:

$$P = F_1^{-1}(1 - \frac{f_2}{f_1}) \tag{5.1}$$

Observe that the protection level set aside for class 1 is not dependent on the capacity of the aircraft (as long as $P \leq C_1$, which is a very reasonable assumption). Thus, in the case of two fare-classes only, swapping aircraft is not going to affect the protection level for class 1 (under the above assumption).

Next we describe our simulation model in detail. In order to study the pure effect of demand variation on the benefits of swapping, we will assume that the fare on each class $i, i = 1, 2$, is the same on each leg in the swappable loops, as was done for the single fare-class models. Let D^c_{ilt} denote the cumulative demand on leg $l \in L$ and fare-class $i, i = 1, 2$, based on the forecast updated in period t, and assume that each D^c_{ilt} is normally distributed with CDF F_{ilt}. As in the single fare-class model, each demand can have different parameters, but we assume that demands are independent across fare-classes, across legs, and over time.

In our simulation model, we use Equation (5.1) to determine the protection levels for the higher fare customers on each leg $l \in L$. We must note, however, that these protection levels are not necessarily optimal in our model due to the way demands are generated. Specifically, our simulation model generates demands in each period such that all class 2 demand is generated before all class 1 demand. However, we still choose to determine the protection levels in the simulation using the EMSRb equation simply because our industry partner uses the same model to determine their protection levels.

Next we extend our simulation model to consider two fare-classes on each leg. Let P_{lt} denote the protection level on leg $l \in L$ updated in period t and d^a_{ilt} denotes the total realized demand of class i customers on leg i up to time t.

Recall that t represents the number of periods until departures (i.e., number of periods "to go"). The initial capacity assignment is such that capacity of C_i is assigned to legs in $L_i, i = 1, 2$, where $C_1 < C_2$ and $L_1 \cup L_2$ correspond to the set of legs in the swappable loops.

The flow-chart of the simulation model is given below.

Set $t = T$

1. Update the demand forecast for class i and leg l, $i = 1, 2, l \in L$.

2. Update the protection level, P_{lt}, $l \in L$. (Note that if the updated protection level is less than the number of the realized class 1 customers, then we adjust the protection level to be the number of realized class 1 customers.

3. Determine the expected revenue gain, $E[G]$, for the swappable loops. If $E[G] > 0$ and $\sum_{i=1}^{2} d^a_{ilt} \leq C_1$, for $\forall l \in L_2$, then swap capacities C_1 and C_2.

4. Generate d_{2lt}, $\forall l \in L$. Update the available capacities.

5. Generate d_{1lt}, $\forall l \in L$. Update the available capacities.

6. If $t = 1$, then stop. Otherwise, let $t = t - 1$ and go to step 1.

Under the delayed swapping strategy, all the above steps are performed every period, whereas the limited swapping strategy requires this process to be performed only at time T, which, as mentioned previously, is typically $4 - 6$ weeks prior to departures when partial demand information is available. Also recall that under the perfect swapping strategy, the swapping decision is made under perfect demand information.

In addition, forecast updating is performed in a way similar to the single fare-class simulation model (see Section 4.2.2).

In the next section we describe our simulation experiments and discuss the results.

5.2 Simulation Experiments and Discussion of the Results

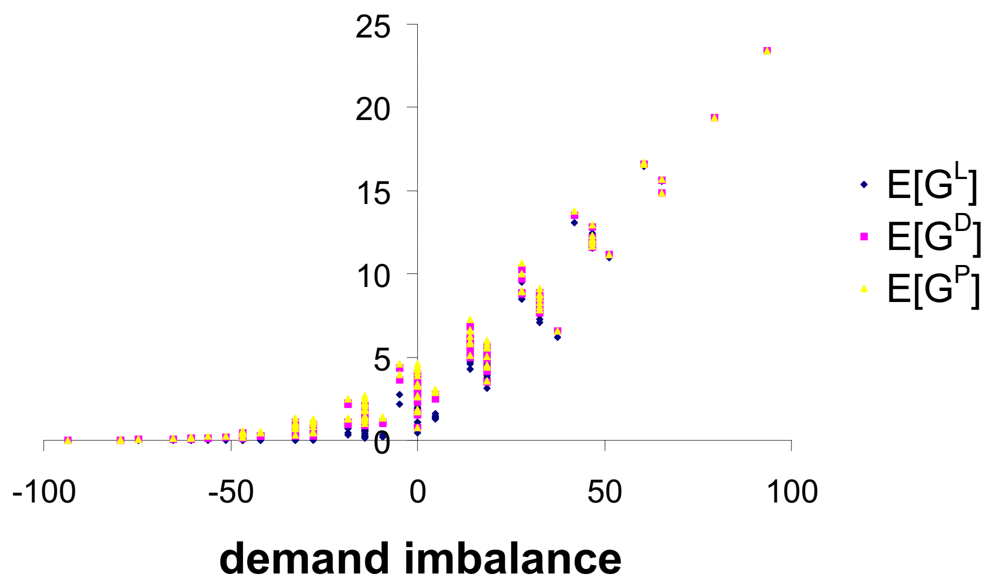

Figure 5.1: Expected gain under the limited, delayed, and perfect swapping strategies.

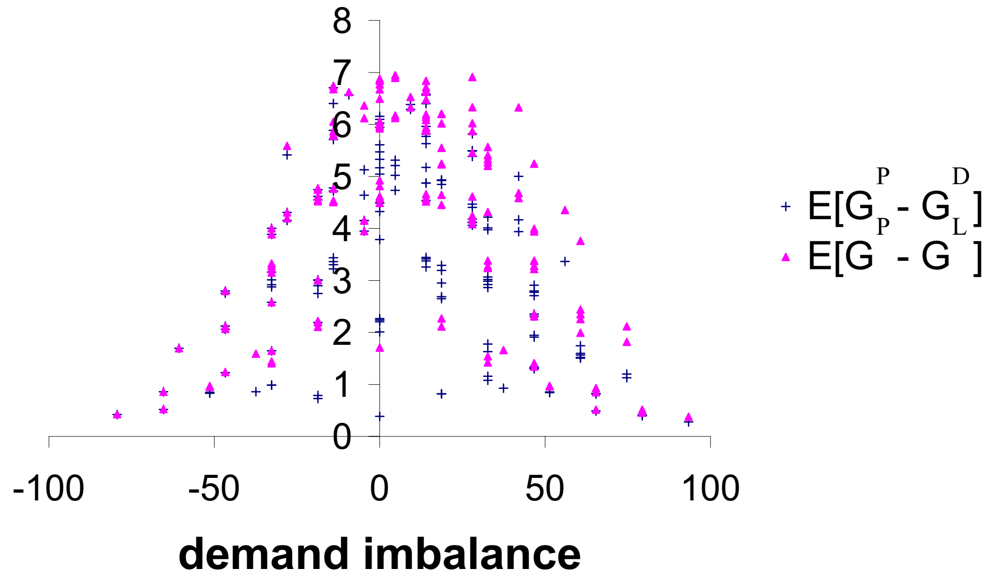

Figure 5.2: Difference in gain between the perfect and limited swapping strategies and between the perfect and delayed swapping strategies.

Recall that the demand for each class on each leg is considered to be normally distributed and independent among classes, across legs and over time. Clearly demand patterns for the different class customers over time will significantly affect the effectiveness of our swapping strategies. We analyzed some data provided to us by our industry partner, United Airlines, to model the demand patterns for each customer class over time. We found that demand patterns for different fare classes vary depending on the markets served and the time of the day. However, we also found out that demand patterns, where the mean demand for higher fare class customers is increasing over time and the mean demand for lower fare class customers is decreasing over time is common for most markets. Since the data obtained from United Airlines was confidential, we decided to use the following factors to approximate demands for the two fare-classes in our simulation model: Considering a seven period model to cover a six week period (the length of each period is not necessarily equal) our factors for the higher fare-class were taken as $\{0.03, 0.05, 0.08, 0.12, 0.16, 0.22, 0.34\}$, whereas those for the lower fare-class were taken as $\{0.37, 0.19, 0.14, 0.12, 0.1, 0.07, 0.01\}$. These factors are used to split a total mean demand for each class and leg over the seven periods to the individual periods. Then we used demand standard deviations of 15 and 25 respectively for each demand distribution. We must note here, however, that an extensive simulation study, considering different demand patterns for the different fare classes over time, needs to be carried out to assess a more reliable performance of the different swapping strategies. Our preliminary results are intended to give an idea on this effect.

As in the single fare-class model, we consider two aircraft of the same family having capacities $C_1 = 112$ and $C_2 = 126$. The total mean demands on each leg (over the two classes and seven periods) are varied in $\{93.3, 112, 126, 140\}$. We assume that the total mean demand for class 1 customers (over the seven periods) is 20% of the total mean demand on the leg (over the seven periods), and the remaining 80% corresponds to that for class 2 customers. We also assume that the fare of class 1 is 3 times larger than the fare of class 2. Recall that class fares are the same on each leg. All these data are based, again, on the analysis of our industry partner's data. It is an important future research direction to perform an extensive design of experiments and run more simulation experiments.

Next we briefly summarize the simulation results for the two fare-class model and compare them with the results obtained for the single fare-class model. Figure 5.1 shows how the expected gain behaves under the limited, delayed, and perfect swapping strategies. The difference in expected gain between the perfect and limited swapping strategies and between the perfect and delayed swapping strategies are depicted in Figure 5.2. Finally, Figure 5.3 shows the average number of swaps under the limited and the delayed swapping strategies. Based on these figures, we make the following observations (please refer to Chapter 4 for comparison of these results with those of the single fare-class model).

1. The expected gain patterns under the perfect swapping strategy versus demand imbalance for the two fare-class model are similar to those of the single fare-class model. Observe also that the expected gain of all the swapping strategies converge to the same value as demand imbalance increases (see Figure 5.2 (a)).

2. The average number of swaps in the limited case is very similar to those obtained for the single fare-class model. This number is close to 0 for large negative demand imbalance and close to 1 for large positive demand imbalance. The average number of swaps in the delayed strategy is less than 2 (out of a possible 6) for the case where the system demand standard deviation is 15 (see Figure 5.3). These numbers are comparable to the results for the single fare-class model.

As a last note, we repeat that the results from our simulation model are based on one data set of demand patterns and fares. In order to fully capture the effectiveness of these swappign strategies for a multiple fare-class model, it is necessary to perform a careful design of experiments.

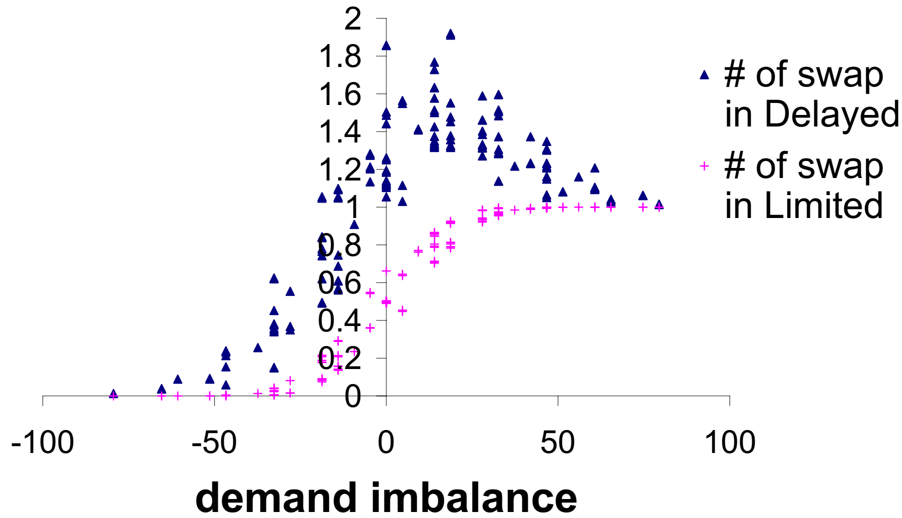

Figure 5.3: Number of swaps in the limited and delayed swapping strategies.

Chapter 6

Conclusions and Future Research Directions

An airline's supply management (i.e., fleet assignment and crew scheduling) decisions greatly impact its revenues. Certain crew regulations and operating characteristics of the airline companies dictate that these decisions be made well in advance of departures, at a time when demand is highly uncertain. However, demand forecasts improve markedly over time as more information on demand patterns is obtained. Thus, it becomes essential for an airline to update its supply management decisions dynamically over time. This provides the airline with an improved ability to match its supply and demand. Although such a dynamic, demand-driven supply management process is constrained by the initial decisions and is difficult to implement, effective supply management strategies can still be devised by taking advantage of the inherent flexibilities in the system. In this research, we collaborate with the United Airlines Research and Development Division.

In this thesis, we analyze a Demand Driven Swapping (DDS) approach that aims at improving the airline's revenue by reducing the supply-demand mismatches through dynamically swapping aircraft. Due to the proximity to departures, the DDS problem is restricted by two main connstraints: 1) the initial crew schedule needs to be kept intact (due to certain union contracts); and 2) airport services and operations need to be preserved to the greatest extent possible. As a result, only a limited number of simple swaps can be performed between aircraft types of the same family (i.e., crew-compatible aircraft types). However, the swaps can be potentially performed on a daily basis given the initial fleet assignments. Clearly, the swapping criteria, frequency, and timing will highly impact the revenue benefits of the DDS approach. When the swapping decisions are made several weeks prior to departures (i.e., 4-6 weeks before departures), they will not cause much disturbance to operations, but will be performed under highly uncertain demand information. On the other hand, swapping decisions that are delayed to a later time (i.e., 1-3 weeks before departures) will decrease the possibility of bad swaps, but will result in larger costs due to higher disruptions to airport services and operations. Thus, our research objective is to understand the critical parameters that affect the benefits of a DDS strategy so as to analyze the effectiveness of different several demand-driven aircraft swapping strategies as a way to hedge against the demand uncertainty in the system. For this purpose, we consider different swapping strategies, characterized in terms of their frequency, and study simple analytical models to gain insights into the critical parameters that affect the expected revenue benefits of each strategy.

Our analytical results suggest that strategies that make the swapping decision early in time (in order to minimize disturbances to the operations) perform very well on routes, where the demand uncertainty is low and the expected demands on the legs are well-balanced. Otherwise, a swapping strategy, which revises the swapping decision over time, should be implemented. We then extend our analysis to more general models through simulation. Our simulation results, based on real data obtained from United Airlines, confirm the analytical findings.

Several extensions to our models deserve further analysis, as summarized in the following section.

6.1 Future Research Directions

In this research, we consider simple analytical models under certain (and restrictive) assumptions for analytical tractability. For example, we consider a single fare-class on each leg and assume that leg demand distributions are independent, having parameters (mean and variance) that are known with certainty at the outset. However, in reality, many fare-classes will exist on each leg, each consisting of customers with different arrival patterns over time. Considering more realistic patterns of customer demand and including customer cancellation and overbooking policies in our models would be important extensions to this research. In addition, considering the cost of rescheduling would be useful from a cost/benefit analysis of swapping. Furthermore, demand patterns will be correlated between legs and over time, and demand parameters will not be known with certainty at the outset. Although in our simulation models some of these assumptions were relaxed to some extent (i.e., our simulation model allows demand forecast updating in the delayed swapping strategy and considers a model having two fare-classes), these aspects have not been considered in the analytical models. Thus, studying analytical models that relax some of these assumptions would be interesting and worthy extensions of the ones studied in this thesis. In addition, our simulation models need to be extended considering a variety of demand patterns.

In practice, the implementation of a DDS approach requires the integration of the airline's revenue management and aircraft swapping decisions. To our knowledge, the revenue management problem under swappable capacities has not been studied in the extensive airline revenue management literature. Thus, considering multiple fare-classes on each leg, a challenging and interesting research direction would be to devise effective revenue management strategies under swappable aircraft capacities.

In this research, we consider a small number of swapping strategies, all of which make decisions based on the revenue expectation of the swaps. An interesting future research direction would be to study some other strategies that make their decisions based on different criteria. This research is one of the first attempts to analyze the effectiveness of different demand-driven aircraft swapping strategies as a way to hedge against the demand uncertainty in the system; and to understand the drivers of these benefits. Thus, we believe that this thesis provides an important basis and guidelines for future research.

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Appendix

Fundamentals

In this section of the Appendix, we derive some basic expressions that will be essential in our subsequent derivations.

We assume that D is a normal random variable with mean μ, standard deviation σ, probability density function (pdf) $f(.)$ and cumulative distribution function (CDF) $F(.)$. We let y, y_1 and y_2 denote some constants.

In what follows, we derive the first and second moments of D; and the first and second derivatives of its pdf and CDF with respect to μ and σ.

First Order Moments

In what follows, we denote the first order partial moments (expectations) of random variable D as

$$E_{-\infty}^{y}(D) = \int_{-\infty}^{y} Df(D)dD, \quad \text{and} \qquad E_{y}^{+\infty}(D) = \int_{y}^{+\infty} Df(D)dD$$

From Winkler et al. (1972), we have:

$$E_{-\infty}^{y}(D) = \mu\ F(y) - \sigma^2 f(y) \tag{6.1}$$

$$E_{y}^{+\infty}(D) = \mu\ [1 - F(y)] + \sigma^2 f(y) \tag{6.2}$$

Also, we observe that,

$$E(D|D \le y) = \frac{1}{F(y)}\ E_{-\infty}^{y}(D) = \mu - \sigma^2\ \frac{f(y)}{F(y)} \tag{6.3}$$

$$E(D|D \ge y) = \frac{1}{[1-F(y)]}\ E_{y}^{+\infty}(D) = \mu + \sigma^2\ \frac{f(y)}{[1-F(y)]} \tag{6.4}$$

$$\begin{aligned} E(D|y_1 < D < y_2) &= \frac{1}{[F(y_2)-F(y_1)]}\ [\ E_{-\infty}^{y_2}(D) - E_{-\infty}^{y_1}(D)\ ] \\ &= \mu + \frac{1}{[F(y_2)-F(y_1)]}\ \sigma^2(f(y_1) - f(y_2)) \end{aligned} \tag{6.5}$$

Second Order Moments

We define the second order partial moments of random variable D as

$$E_{-\infty}^{y}(D^2) = \int_{-\infty}^{y} D^2 f(D)dD, \quad \text{and} \qquad E_{y}^{+\infty}(D^2) = \int_{y}^{+\infty} D^2 f(D)dD$$

From Winkler et al. (1972), we have:

$$E_{-\infty}^{y}(D^2) = (\mu^2 + \sigma^2)\ F(y) - \sigma^2\ (y+\mu)\ f(y) \tag{6.6}$$

$$E_{y}^{+\infty}(D^2) = (\mu^2 + \sigma^2)\ [1-F(y)] + \sigma^2\ (y+\mu)\ f(y) \tag{6.7}$$

Again, we also observe that,

$$\begin{aligned} E(D^2|D \le y) &= \frac{1}{F(y)}\ E_{-\infty}^{y}(D^2) \\ &= \mu^2 + \sigma^2 - \sigma^2\ (y+\mu)\ \frac{f(y)}{F(y)} \end{aligned} \tag{6.8}$$

$$\begin{aligned} E(D^2|D \ge y) &= \frac{1}{1-F(y)}\ E_{y}^{+\infty}(D^2) \\ &= \mu^2 + \sigma^2 + \sigma^2\ (y+\mu)\ \frac{f(y)}{[1-F(y)]} \end{aligned} \tag{6.9}$$

$$\begin{aligned} E(D^2|y_1 < D < y_2) &= \frac{1}{[F(y_2)-F(y_1)]}\ [\ E_{-\infty}^{y_2}(D^2) - E_{-\infty}^{y_1}(D^2)\ ] \qquad (6.10) \\ &= \mu^2 + \sigma^2 + \frac{1}{[F(y_2)-F(y_1)]}\ \sigma^2[(y_1+\mu)f(y_1) - (y_2+\mu)f(y_2)] \end{aligned}$$

First Derivatives of f(.) and F(.) with respect to μ and σ

We derive:

$$\frac{\delta F(d)}{\delta \mu} = -f(d); \quad \frac{\delta F(d)}{\delta \sigma} = (\mu - d)f(d) \tag{6.11}$$

Similarly, we derive:

$$\frac{\delta f(d)}{\delta \mu} = \frac{(d-\mu)}{\sigma^2}f(d); \quad \frac{\delta f(d)}{\delta \sigma} = \frac{(d-\mu)^2}{\sigma^3}f(d) - \frac{1}{\sigma}f(d) \tag{6.12}$$

This fundamentals will be used in our subsequent derivations.

Derivations in Section 3.5.1

In what follows, we consider that D_i are independently normally distributed with mean μ_i, standard deviation σ_i, pdf $f_i(.)$ and CDF $F_i(.)$, for $i \in L$.

Derivation of p_{ij}^{+}

Recall, Equation (3.6). We can write p_{ij}^{+} as follows:

$$\begin{aligned}
p_{ij}^{+} &= Pr(D_j < C_1 < D_i < C_2) + Pr(C_1 < D_j < D_i < C_2) \\
&\quad + Pr(D_j < C_1 < C_2 < D_i) + Pr(C_1 < D_j < C_2 < D_i) \\
&= [F_i(C_2) - F_i(C_1)]F_j(C_1) + \int_{C_1}^{C_2} \int_{C_1}^{d_i} f_j(d_j) f_i(d_i) dd_j dd_i \\
&\quad + F_j(C_1)[1 - F_i(C_2)] + [F_j(C_2) - F_j(C_1)][1 - F_i(C_2)] \\
&= [F_i(C_2) - F_i(C_1)]F_j(C_1) + [1 - F_i(C_2)]F_j(C_2) + \int_{C_1}^{C_2} (F_j(d_i) - F_j(C_1)) f_i(d_i) dd_i \\
&= [1 - F_i(C_2)]F_j(C_2) + [F_i(C_2) - F_i(C_1)]F_j(C_1) \\
&\quad - F_j(C_1)[F_i(C_2) - F_i(C_1)] + \int_{C_1}^{C_2} f_i(d) F_j(d) dd \\
&= F_j(C_2)[1 - F_i(C_2)] + \int_{C_1}^{C_2} f_i(d) F_j(d) dd \qquad (6.13)
\end{aligned}$$

The derivation of the expression on p_{ij}^{-}, given in Equation (3.7), is similar, and therefore, will be omitted.

Derivation of $E[\omega_{ij}^+]$

Following Equation (3.8), $E[\omega_{ij}^+]$ can be expressed as:

$$
\begin{aligned}
E[\omega_{ij}^+] &= E[D_i - C_1 | D_j < C_1 < D_i < C_2] Pr(D_j < C_1 < D_i < C_2) \\
&\quad + E[D_i - D_j | C_1 < D_j < D_i < C_2] Pr(C_1 < D_j < D_i < C_2) \\
&\quad + E[C_2 - C_1 | D_j < C_1 < C_2 < D_i] Pr(D_j < C_1 < C_2 < D_i) \\
&\quad + E[C_2 - D_j | C_1 < D_j < C_2 < D_i] Pr(C_1 < D_j < C_2 < D_i) \\
&= Pr(D_j < C_1 < D_i < C_2)\{E[D_i | C_1 < D_i < C_2] - C_1\} \\
&\quad + \int_{C_1}^{C_2} \int_{d_j}^{C_2} (d_i - d_j) f_j(d_j) f_i(d_i) dd_j dd_i + Pr(D_j < C_1 < C_2 < D_i)[C_2 - C_1] \\
&\quad + Pr(C_1 < D_j < C_2 < D_i)\{C_2 - E[D_j | C_1 < D_j < C_2]\}
\end{aligned}
\tag{6.14}
$$

Using Equation (6.5), $E[\omega_{ij}^+]$ can be rewritten as:

$$
\begin{aligned}
E[\omega_{ij}^+] &= F_j(C_1)[F_i(C_2) - F_i(C_1)]\{\mu_i + \frac{1}{[F_i(C_2) - F_i(C_1)]}\sigma_i^2[f_i(C_1) - f_i(C_2)] - C_1\} \\
&\quad + \int_{C_1}^{C_2} \int_{d_j}^{C_2} (d_i - d_j) f_j(d_j) f_i(d_i) \; dd_i \; dd_j + [C_2 - C_1][1 - F_i(C_2)]F_j(C_1) \\
&\quad + [1 - F_i(C_2)][F_j(C_2) - F_i(C_1)]\{C_2 - [\mu_j + \frac{1}{[F_j(C_2) - F_j(C_1)]}\sigma_j^2[f_j(C_1) - f_j(C_2)]]\} \\
&= C_2 F_j(C_2)[1 - F_i(C_2)] - C_1 F_j(C_1)[1 - F_i(C_1)] \\
&\quad - \mu_j[F_j(C_2) - F_j(C_1)][1 - F_i(C_2)] - \sigma_j^2[f_j(C_1) - f_j(C_2)][1 - F_i(C_2)] \\
&\quad + \mu_i F_j(C_1)[F_i(C_2) - F_i(C_1)] + \sigma_i^2[f_i(C_1) - f_i(C_2)]F_j(C_1) \\
&\quad + \int_{C_1}^{C_2} \int_{d_j}^{C_2} (d_i - d_j) f_j(d_j) f_i(d_i) \; dd_i \; dd_j
\end{aligned}
\tag{6.15}
$$

Now we simplify the term $\int_{C_1}^{C_2}\int_{d_j}^{C_2}(d_i - d_j)f_j(d_j)f_i(d_i)\ dd_i\ dd_j$ using Equation (6.5):

$$\begin{aligned}
&\int_{C_1}^{C_2}\int_{d_j}^{C_2}(d_i - d_j)f_j(d_j)f_i(d_i)\ dd_i\ dd_j \\
&\qquad = \int_{C_1}^{C_2}\int_{d_j}^{C_2} d_i f_j(d_j)f_i(d_i)\ dd_i\ dd_j - \int_{C_1}^{C_2}\int_{d_j}^{C_2} d_j f_j(d_j)f_i(d_i)\ dd_i\ dd_j \\
&\qquad = \int_{C_1}^{C_2}[\mu_i[F_i(C_2) - F_i(d_j)] + \sigma_i^2[f_i(d_j) - f_i(C_2)]]f_j(d_j)\ dd_j \\
&\qquad\quad - \int_{C_1}^{C_2} d_j f_j(d_j)[F_i(C_2) - F_i(d_j)]\ dd_j \\
&\qquad = \mu_i F_i(C_2)[F_j(C_2) - F_j(C_1)] - \mu_i\int_{C_1}^{C_2} F_i(d)f_j(d)\ dd \\
&\qquad\quad + \sigma_i^2\int_{C_1}^{C_2} f_i(d)f_j(d)\ dd - \sigma_i^2 f_i(C_2)[F_j(C_2) - F_j(C_1)] \\
&\qquad\quad - F_i(C_2)[\mu_j[F_j(C_2) - F_j(C_1)] + \sigma_j^2[f_j(C_1) - f_j(C_2)]] \\
&\qquad\quad + \int_{C_1}^{C_2} d\ f_j(d)F_i(d)\ dd
\end{aligned}$$

Finally, using the equations given above, we can obtain the expression for $E[\omega_{ij}^+]$ as follows:

$$\begin{aligned}
E[\omega_{ij}^+] = {} & C_2F_j(C_2)[1 - F_i(C_2)] - C_1F_j(C_1)[1 - F_i(C_1)] \\
& - \mu_j[F_j(C_2) - F_j(C_1)] - \sigma_j^2[f_j(C_1) - f_j(C_2)] \\
& + \mu_i[F_i(C_2)F_j(C_2) - F_i(C_1)F_j(C_1)] + \sigma_i^2[f_i(C_1)F_j(C_1) - f_i(C_2)F_j(C_2)] \\
& + \sigma_i^2\int_{C_1}^{C_2} f_i(d)f_j(d)\ dd - \mu_i\int_{C_1}^{C_2} f_j(d)F_i(d)\ dd \\
& + \int_{C_1}^{C_2} d\ f_j(d)F_i(d)\ dd
\end{aligned}$$

Derivation of $E[\omega_{ij}^{-}]$

Function ω_{ij}^{-} can be written as follows:

$$\omega_{ij}^{-} = \begin{cases} C_1 - D_j, & \text{if } D_i \leq C_1 \leq D_j \leq C_2, \\ C_1 - C_2, & \text{if } D_i \leq C_1 \leq C_2 \leq D_j, \\ D_i - C_2, & \text{if } C_1 \leq D_i \leq C_2 \leq D_j, \\ D_i - D_j, & \text{if } C_1 \leq D_i \leq D_j \leq C_2, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, using Equation (6.5), the expected value of ω_{ij}^{-} can be derived as follows:

$$\begin{aligned} E[\omega_{ij}^{-}] = {} & E[C_1 - D_j | D_i \leq C_1 \leq D_j \leq C_2] Pr(D_i \leq C_1 \leq D_j \leq C_2) \\ & + E[C_1 - C_2 | D_i \leq C_1 \leq C_2 \leq D_j] Pr(D_i \leq C_1 \leq C_2 \leq D_j) \\ & + E[D_i - C_2 | C_1 \leq D_i \leq C_2 \leq D_j] Pr(C_1 \leq D_i \leq C_2 \leq D_j) \\ & + E[D_i - D_j | C_1 \leq D_i \leq D_j \leq C_2] Pr(C_1 \leq D_i \leq D_j \leq C_2) \\ = {} & Pr(D_i \leq C_1 \leq D_j \leq C_2)\{E[C_1 - D_j | C_1 \leq D_j \leq C_2]\} \\ & + (C_1 - C_2) Pr(D_i \leq C_1 \leq C_2 \leq D_j) \\ & + Pr(C_1 \leq D_i \leq C_2 \leq D_j)\{E[D_i - C_2 | C_1 \leq D_i \leq C_2]\} \\ & + Pr(C_1 \leq D_i \leq D_j \leq C_2) E[D_i - D_j | C_1 \leq D_i \leq D_j \leq C_2] \\ = {} & F_i(C_1) \int_{C_1}^{C_2} (C_1 - d) f_j(d)\ dd \\ & + (C_1 - C_2) F_i(C_1)[1 - F_j(C_2)] + \int_{C_1}^{C_2} \int_{C_1}^{d_j} (d_i - d_j) f_j(d_j) f_i(d_i)\ dd_i\ dd_j \\ & + [1 - F_j(C_2)] \int_{C_1}^{C_2} (d - C_2) f_i(d)\ dd \end{aligned}$$

Next, we simplify the expressions for $\int_{C_1}^{C_2} \int_{C_1}^{d_j} (d_i - d_j) f_j(d_j) f_i(d_i)\ dd_i\ dd_j$, $\int_{C_1}^{C_2} (C_1 - d) f_j(d)\ dd$, and $\int_{C_1}^{C_2} (d - C_2)\ f_i(d)\ dd$. We get:

$$
\begin{aligned}
&\int_{C_1}^{C_2}\int_{C_1}^{d_j}(d_i-d_j)\ f_j(d_j)f_i(d_i)\ dd_i\ dd_j \\
&\qquad = \int_{C_1}^{C_2}\int_{C_1}^{d_j} d_i\ f_j(d_j)f_i(d_i)\ dd_i\ dd_j - \int_{C_1}^{C_2}\int_{C_1}^{d_j} d_j\ f_j(d_j)f_i(d_i)\ dd_i\ dd_j \\
&\qquad = \int_{C_1}^{C_2}[\mu_i[F_i(d_j)-F_i(C_1)]+\sigma_i^2[f_i(C_1)-f_i(d_j)]]f_j(d_j)\ dd_j \\
&\qquad\quad - \int_{C_1}^{C_2} d_j f_j(d_j)[F_i(d_j)-F_i(C_1)]\ dd_j \\
&\qquad = \mu_i\int_{C_1}^{C_2} F_i(d)f_j(d)\ dd - \mu_i F_i(C_1)[F_j(C_2)-F_j(C_1)] \\
&\qquad\quad + \sigma_i^2 f_i(C_1)[F_j(C_2)-F_j(C_1)] - \sigma_i^2\int_{C_1}^{C_2} f_i(d)f_j(d)\ dd \\
&\qquad\quad - \int_{C_1}^{C_2} df_j(d)F_i(d)\ dd \\
&\qquad\quad + F_i(C_1)[\mu_j[F_j(C_2)-F_j(C_1)]+\sigma_j^2[f_j(C_1)-f_j(C_2)]]
\end{aligned}
$$

$$
\int_{C_1}^{C_2}(C_1-d)f_j(d)\ dd = C_1[F_j(C_2)-F_j(C_1)] - \mu_j[F_j(C_2)-F_j(C_1)] + \sigma_j^2[f_j(C_1)-f_j(C_2)]
$$

$$
\int_{C_1}^{C_2}(d-C_2)f_i(d)\ dd = \mu_i[F_i(C_2)-F_i(C_1)] + \sigma_i^2[f_i(C_1)-f_i(C_2)] - C_2[F_i(C_2)-F_i(C_1)]
$$

Thus, the expression of $E[\omega_{ij}^-]$ can be written as:

$$
\begin{aligned}
E[\omega_{ij}^-] &= C_1\ F_i(C_1)[1-F_j(C_1)] - C_2\ F_i(C_2)[1-F_j(C_2)] \\
&\quad + \mu_i[F_i(C_2)(1-F_j(C_2)) - F_i(C_1)(1-F_j(C_1)] + \sigma_i^2 f_i(C_1)[1-F_j(C_1)] \\
&\quad - \sigma_i^2 f_i(C_2)[1-F_j(C_2)] + \mu_i\int_{C_1}^{C_2} f_j(d)F_i(d)\ dd \\
&\quad - \sigma_i^2\int_{C_1}^{C_2} f_i(d)f_j(d)\ dd - \int_{C_1}^{C_2} d\ f_j(d)F_i(d)\ dd
\end{aligned}
\tag{6.16}
$$

Derivations of $E[\omega_{ij}]$

Recall that for $i \in L$ and $k = 1, 2$,

$$a_{ki} = min\{C_k, D_i\} = \begin{cases} C_k, & \text{if } C_k < D_i, \\ D_i, & \text{otherwise.} \end{cases}$$

Thus,

$$\begin{aligned} E[a_{ki}] &= C_k \ Pr(C_k < D_i) + E[D_i | D_i \leq C_k] P(D_i \leq C_k) \\ &= C_k[1 - F_i(C_k)] + F_i(C_k)[\mu_i - \frac{1}{F_i(C_k)} \sigma_i^2 f_i(C_k)], \text{by Equation (6.3)} \\ &= C_k[1 - F_i(C_k)] + F_i(C_k)\mu_i - \sigma_i^2 f_i(C_k) \end{aligned} \tag{6.17}$$

Therefore, $E[\omega_{ij}]$ can be written as follows:

$$\begin{aligned} E[\omega_{ij}] &= E[\Delta_i - \Delta_j] \\ &= E[a_{2i} - a_{1i}] - E[a_{2j} - a_{1j}] \\ &= C_2[1 - F_i(C_2)] + F_i(C_2)\mu_i - \sigma_i^2 f_i(C_2) - \{C_1[1 - F_i(C_1)] + F_i(C_1)\mu_i - \sigma_i^2 f_i(C_1)\} \\ &\quad - \{C_2[1 - F_j(C_2)] + F_j(C_2)\mu_j - \sigma_j^2 f_j(C_2)\} + C_1[1 - F_j(C_1)] + F_j(C_1)\mu_j - \sigma_j^2 f_j(C_1) \\ &= F_i(C_2)(\mu_i - C_2) + F_i(C_1)(C_1 - \mu_i) + \sigma_i^2[f_i(C_1) - f_i(C_2)] \\ &\quad - F_j(C_2)(\mu_j - C_2) - F_j(C_1)(C_1 - \mu_j) - \sigma_j^2[f_j(C_1) - f_j(C_2)] \end{aligned} \tag{6.18}$$

Derivations of $Var(\Delta_i)$ and $Var(\omega_{ij})$

Using Equations 6.6 - 6.10, we obtain the following:

$$
\begin{aligned}
Var(\Delta_i) = {} & (\mu_i^2 + \sigma_i^2)\ [F_i(C_2) - F_i(C_1)] + \sigma_i^2\ [(C_1 + \mu_i) f_i(C_1) - (C_2 + \mu_i) f_i(C_2)] \\
& - 2C_1\mu_i\ [F_i(C_2) - F_i(C_1)] - 2C_1\sigma_i^2\ [f_i(C_1) - f_i(C_2)] \\
& + (C_2^2 - 2C_1C_2)[1 - F_i(C_2)] + C_1^2\ [1 - F_i(C_1)] \\
& - \{(C_2 - C_1) + F_i(C_2)\ (\mu_i - C_2) + F_i(C_1)(C_1 - \mu_i) + \sigma_i^2\ [f_i(C_1) - f_i(C_2)]\}^2 \\
Var[\omega_{ij}] = {} & Var(\Delta_i) + Var(\Delta_j) \\
= {} & \{(\mu_i^2 + \sigma_i^2)\ [F_i(C_2) - F_i(C_1)] + \sigma_i^2\ [(C_1 + \mu_i) f_i(C_1) - (C_2 + \mu_i) f_i(C_2)] \\
& - 2C_1\mu_i\ [F_i(C_2) - F_i(C_1)] - 2C_1\sigma_i^2\ [f_i(C_1) - f_i(C_2)] \\
& + (C_2^2 - 2C_1C_2)\ [1 - F_i(C_2)] + C_1^2\ [1 - F_i(C_1)] \\
& - \{(C_2 - C_1) + F_i(C_2)(\mu_i - C_2) + F_i(C_1)(C_1 - \mu_i) + \sigma_i^2\ [f_i(C_1) - f_i(C_2)]\}^2\} \\
& - \{(\mu_j^2 + \sigma_j^2)\ [F_j(C_2) - F_j(C_1)] + \sigma_j^2\ [(C_1 + \mu_j) f_j(C_1) - (C_2 + \mu_j) f_j(C_2)] \\
& - 2C_1\mu_j[F_j(C_2) - F_j(C_1)] - 2C_1\sigma_j^2[f_j(C_1) - f_j(C_2)] \\
& + (C_2^2 - 2C_1C_2)\ [1 - F_j(C_2)] + C_1^2[1 - F_j(C_1)] \\
& - \{(C_2 - C_1) + F_j(C_2)(\mu_i - C_2) + F_j(C_1)(C_1 - \mu_j) + \sigma_j^2\ [f_j(C_1) - f_j(C_2)]\}^2\}
\end{aligned}
$$

The next section shows the details of all derivations in Preliminaries II of Chapter 3.

Derivations in Section 3.5.2

Most results in Section 3.5.2 can be derived by making use of the derivations in Section 6.1. Therefore, here we only detail the expressions that are obatained differently.

First two derivatives of $E[\omega_{ij}^+]$ with respect to C_2

Using Equation (3.8) in Chapter 3, we can rewrite function ω_{ij}^+ as follows:

$$\omega_{ij}^+ = \begin{cases} C_2 - D_j, & \text{if } C_1 \leq D_j \leq C_2 \leq D_i, \\ C_2 - C_1, & \text{if } D_j \leq C_1 \leq C_2 \leq D_i, \\ D_i - C_1, & \text{if } D_j \leq C_1 \leq D_i \leq C_2, \\ D_i - D_j, & \text{if } C_1 \leq D_j \leq D_i \leq C_2, \\ 0, & \text{otherwise.} \end{cases}$$

We will derive $\frac{\delta E[\omega_{ij}^+]}{\delta C_2}$ using the definition of derivative. That is, if we let $\omega_{ij}^+(C_2)$ be the value of function ω_{ij}^+ at parameter C_2, then

$$\frac{\delta E[\omega_{ij}^+]}{\delta C_2} = \lim_{\epsilon \to 0}[\frac{E[\omega_{ij}^+(C_2+\epsilon)] - E[\omega_{ij}^+(C_2)]}{\epsilon}]$$

We let $\Delta = \omega_{ij}^+(C_2+\epsilon) - \omega_{ij}^+(C_2)$. We have:

If $C_1 \leq D_j \leq C_2 \leq D_i$, then

- If $C_1 \leq D_j \leq C_2 + \epsilon \leq D_i \Rightarrow \Delta = \epsilon$
- If $C_1 \leq D_j \leq D_i \leq C_2 + \epsilon \Rightarrow \Delta = D_i - C_2 \leq \epsilon$

If $D_j \leq C_1 \leq C_2 \leq D_i$, then

- If $D_j \leq C_1 \leq C_2 + \epsilon \leq D_i \Rightarrow \Delta = \epsilon$
- If $D_j \leq C_1 \leq D_i \leq C_2 + \epsilon \Rightarrow \Delta = D_i - C_2 \leq \epsilon$

If $D_j \leq C_1 \leq D_i \leq C_2$, then

- $D_j \leq C_1 \leq D_i \leq C_2 + \epsilon \Rightarrow \Delta = 0$

If $C_1 \leq D_j \leq D_i \leq C_2$, then

- $C_1 \leq D_j \leq D_i \leq C_2 + \epsilon \Rightarrow \Delta = 0$

Thus,

$$
\begin{aligned}
\frac{\delta E[\omega_{ij}^{+}]}{\delta C_2} &= \lim_{\epsilon \to 0} [\frac{E[\omega_{ij}^{+}(C_2 + \epsilon)] - E[\omega_{ij}^{+}(C_2)]}{\epsilon}] = \lim_{\epsilon \to 0} [\frac{1}{\epsilon} E(\Delta)] \\
&= \lim_{\epsilon \to 0} \frac{1}{\epsilon} [\epsilon Pr(C_1 \leq D_j \leq C_2 + \epsilon \leq D_i, C_1 \leq D_j \leq C_2 \leq D_i) \\
&\quad + (D_i - C_2) Pr(C_1 \leq D_j \leq D_i \leq C_2 + \epsilon, C_1 \leq D_j \leq C_2 \leq D_i) \\
&\quad + \epsilon Pr(D_j \leq C_1 \leq C_2 + \epsilon \leq D_i, D_j \leq C_1 \leq C_2 \leq D_i) \\
&\quad + (D_i - C_2) Pr(D_j \leq C_1 \leq D_i \leq C_2 + \epsilon, D_j \leq C_1 \leq C_2 \leq D_i)] \\
&= Pr(C_1 \leq D_j \leq C_2 \leq D_i) + Pr(D_j \leq C_1 \leq C_2 \leq D_i) \\
&= Pr(D_i \geq C_2)\ [Pr(C_1 \leq D_j \leq C_2) + Pr(D_j \leq C_1)] \\
&= [1 - F_i(C_2)]\ F_j(C_2)
\end{aligned}
$$

Similarly, we obtain:

$$
\begin{aligned}
\frac{\delta^2 E[\omega_{ij}^{+}]}{\delta C_2^2} &= -f_i(C_2) F_j(C_2) +\ [1 - F_i(C_2)] f_j(C_2) \\
&= -f_i(C_2) F_j(C_2) - F_i(C_2) f_j(C_2) + f_j(C_2)
\end{aligned}
$$

Next, we study the relationship of the probability functions, p_{ij}^{+} and p_{ij}^{-}, with respect to the demand standard deviations.

Consider any leg $i \in L_1$ and $j \in L_2$. We have:

$$\frac{\delta p_{ij}^{+}}{\delta \sigma_i} = F_j(C_2) f_i(C_2) \frac{(C_2 - \mu_i)}{\sigma_i} + \frac{1}{\sigma_i^3} \int_{C_1}^{C_2} F_j(d) f_i(d) \; [\; (d - \mu_i)^2 - \sigma_i^2 \;] \; dd$$

$$\frac{\delta p_{ij}^{+}}{\delta \sigma_j} = \frac{(\mu_j - C_2)}{\sigma_j} f_j(C_2)[1 - F_i(C_2)] + \int_{C_1}^{C_2} f_i(d) f_j(d) \frac{(\mu_j - d)}{\sigma_j} \; dd$$

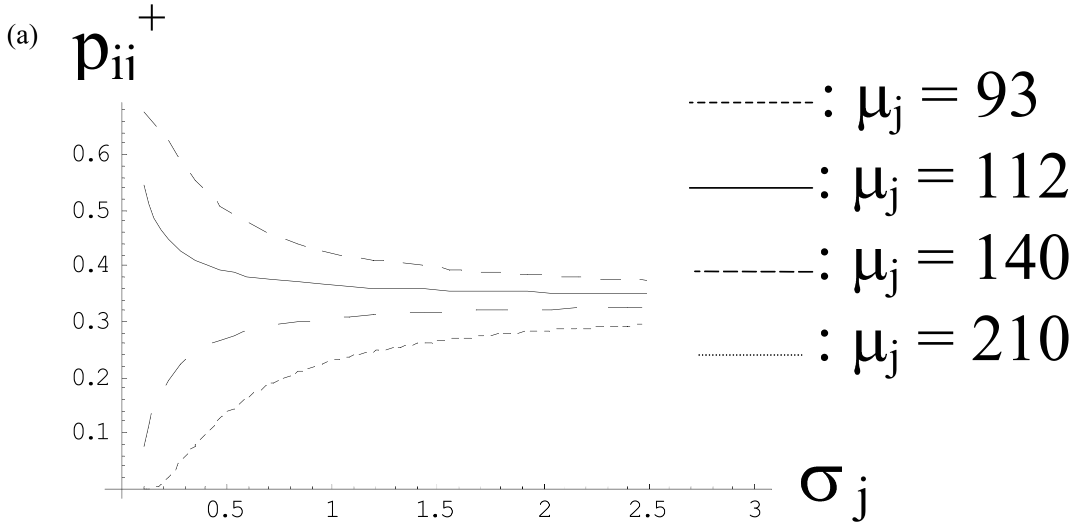
$$\frac{\delta p_{ij}^{-}}{\delta \sigma_i} = \frac{(\mu_i - C_2)}{\sigma_i} f_i(C_2)[1 - F_j(C_2)] + \int_{C_1}^{C_2} f_i(d) f_j(d) \frac{(\mu_i - d)}{\sigma_i} \; dd$$

$$\frac{\delta p_{ij}^{-}}{\delta \sigma_j} = F_i(C_2) f_j(C_2) \frac{(C_2 - \mu_j)}{\sigma_j} + \frac{1}{\sigma_j^3} \int_{C_1}^{C_2} F_i(d) f_j(d) \; [\; (d - \mu_j)^2 - \sigma_j^2 \;] \; dd$$

Through numerical integration, we evaluate the probabilities p_{ij}^{+} and p_{ij}^{-} as σ_i varies, considering $\mu_j = 119, \sigma_j = 15$ and $\mu_i = \{93, 112, 140, 210\}$; see Figure 3.13. Similarly, Figure 6.1 depicts how p_{ij}^{+} and p_{ij}^{-} change as σ_j varies, considering $\mu_i = 119, \sigma_i = 15$ and $\mu_j = \{93, 112, 140, 210\}$. From the graphs, we can see that functions p_{ij}^{+} and p_{ij}^{-} can be either increasing or decreasing in σ_i or σ_j. However, the first derivative converts to zero as σ_i or σ_j becomes large enough; that is, the corresponding probability function converges to a constant. Interestingly, the convergence points of p_{ij}^{+} are similar for different values of μ_i, and we have a similar observation for p_{ij}^{-}.

Figures

In this section, we depict several figures that are used in our analysis.



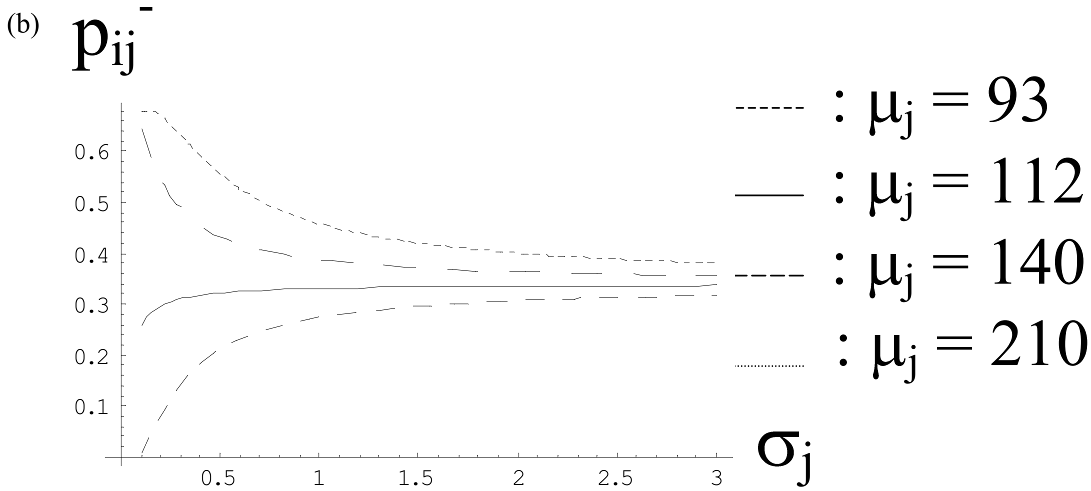


Figure 6.1: a) p_{ij}^{+} versus σ_j; b) p_{ij}^{-} versus σ_j.

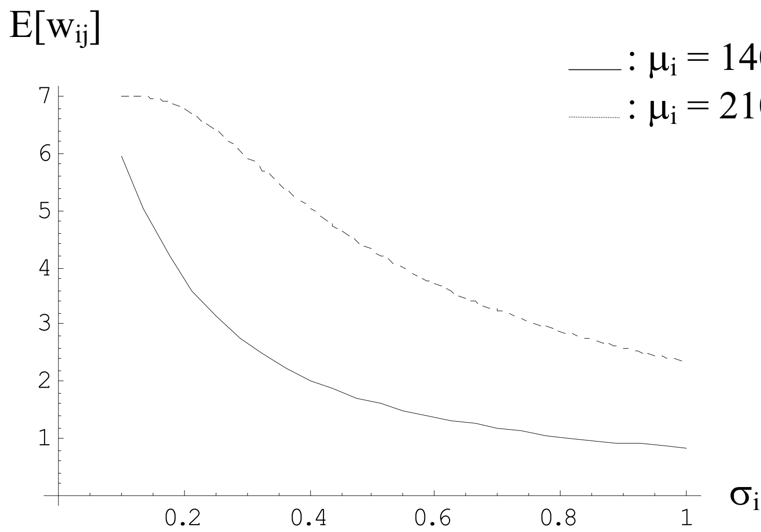


Figure 6.2: $E[G_{ij}^L] = \max\{E[\omega_{ij}], 0\}$ versus σ_i.

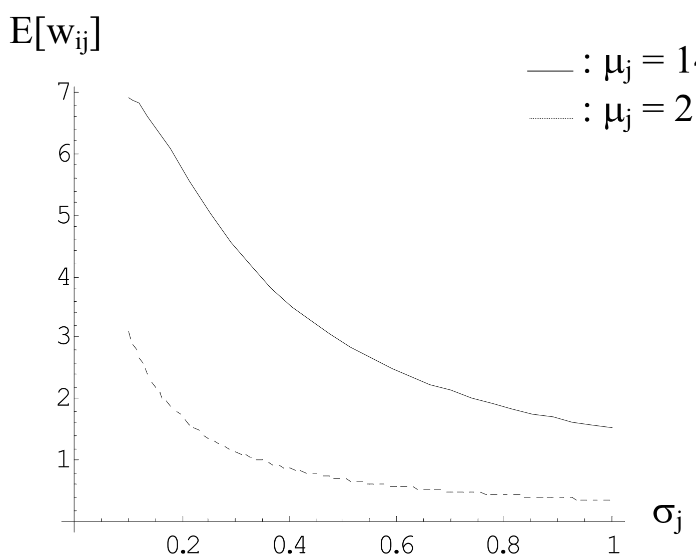


Figure 6.3: $E[G_{ij}^L] = \max\{E[\omega_{ij}], 0\}$ versus σ_j.

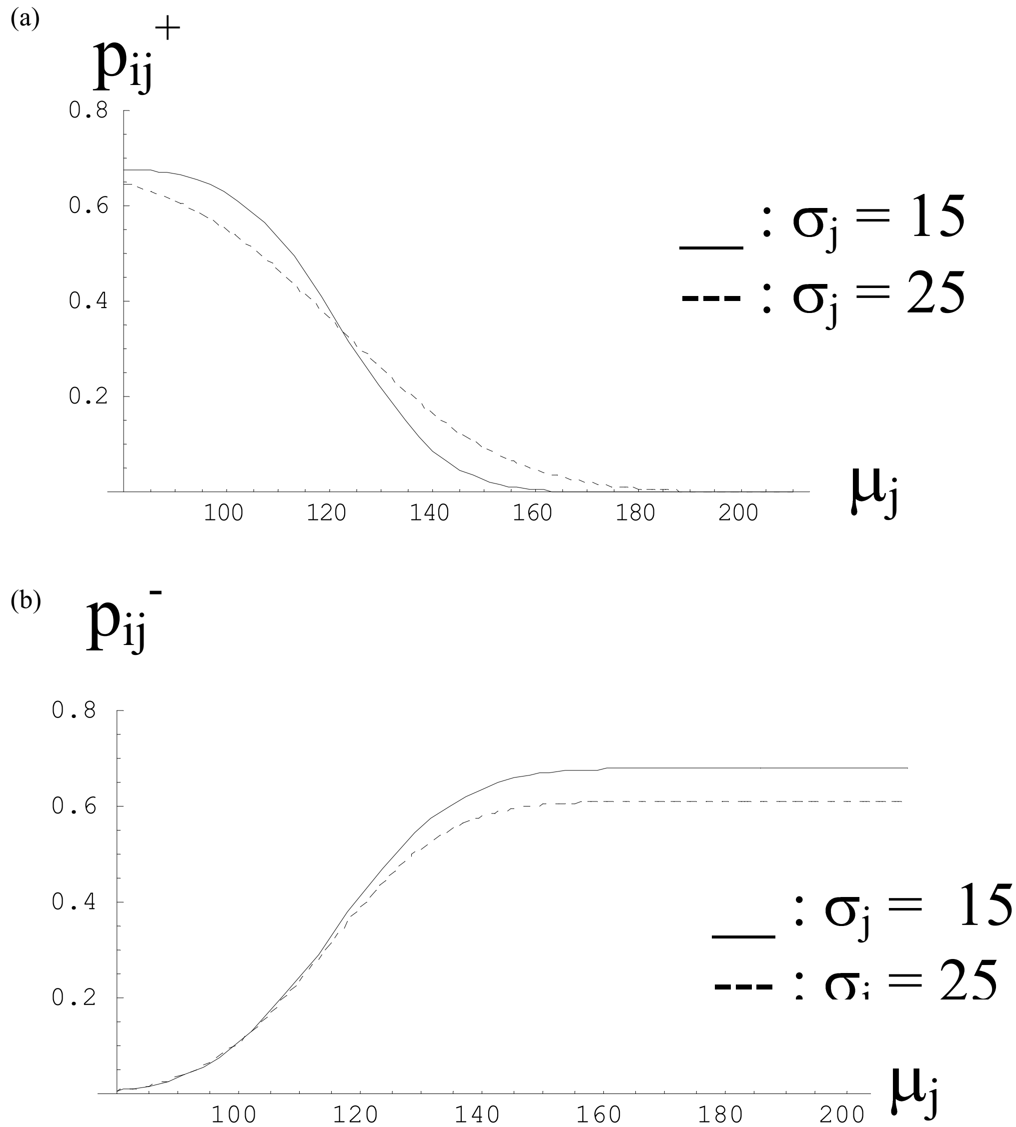


Figure 6.4: a) p_{ij}^{+} versus μ_j; b) p_{ij}^{-} versus μ_j.

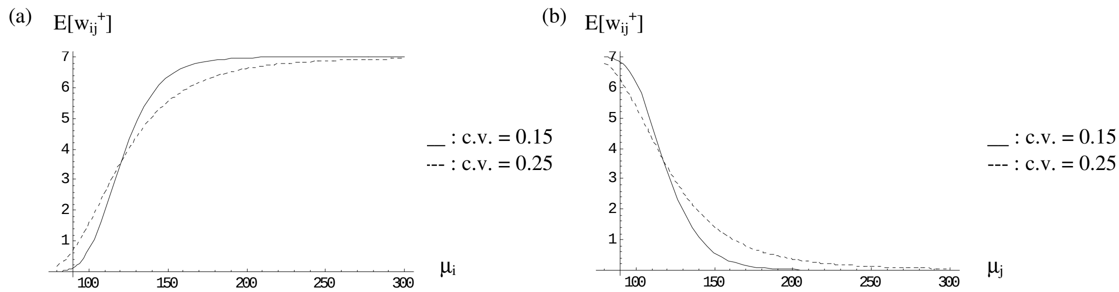


Figure 6.5: a) $E[G_{ij}^P]$ versus μ_i; b) $E[G_{ij}^P]$ versus μ_j.

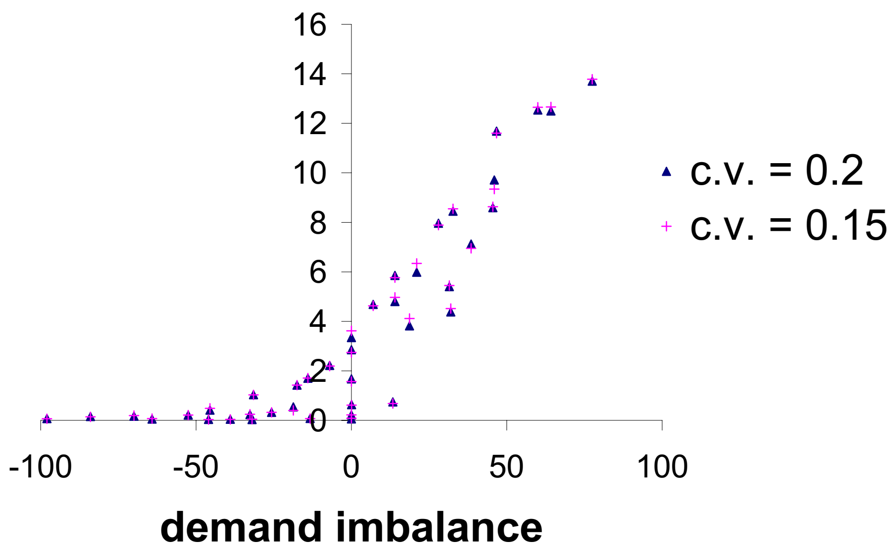


Figure 6.6: $E[G_{ij}^P]$ versus demand imbalance for 2 legs.

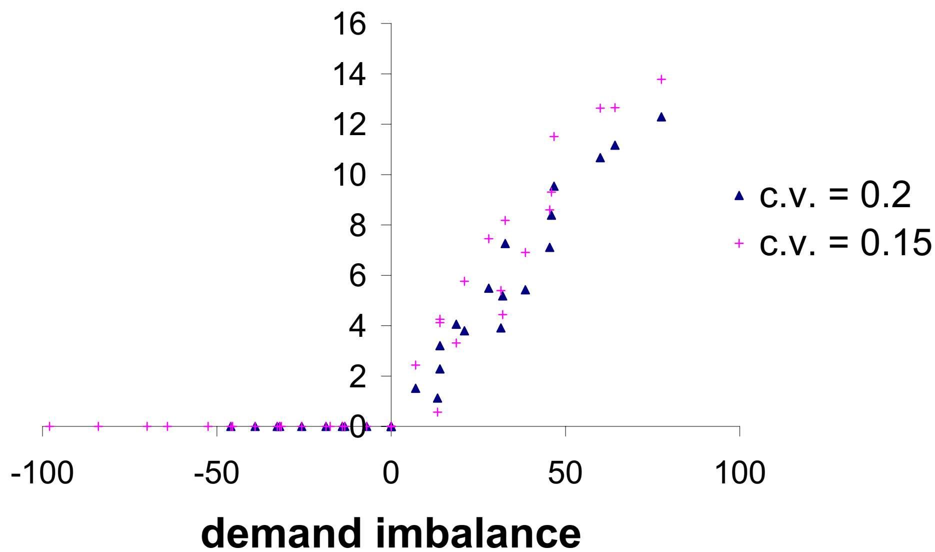


Figure 6.7: $E[G_{ij}^L]$ versus demand imbalance for 2 legs.

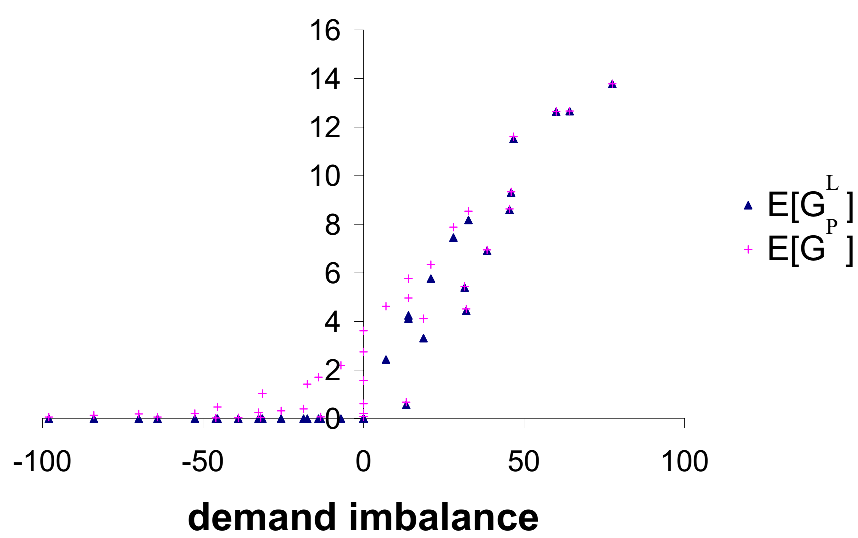


Figure 6.8: $E[G_{ij}^P]$ and $E[G_{ij}^L]$ versus demand imbalance for 2 legs.

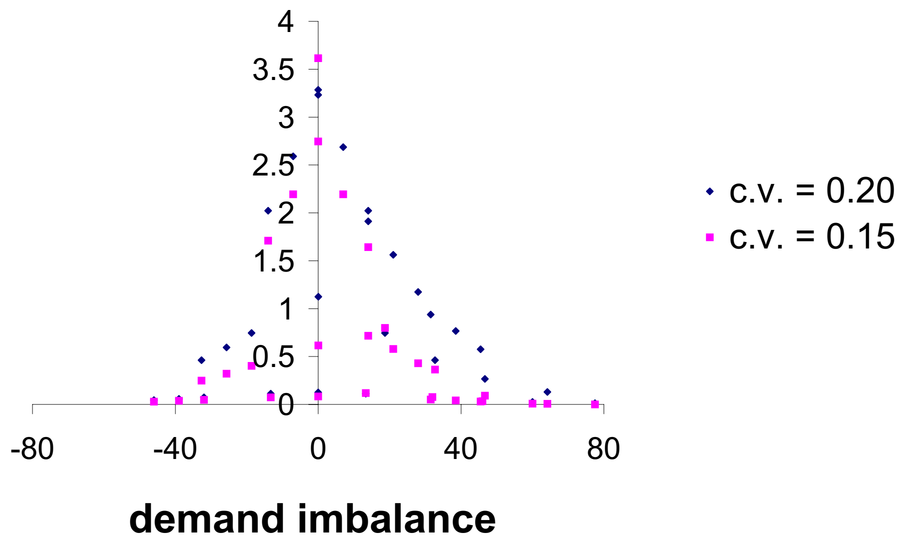


Figure 6.9: $E[G_{ij}^P] - E[G_{ij}^L]$ versus demand imbalance for 2 legs.

Vita

Rawee Suwandechochai was born on March 11, 1976 in Thailand. She got a Development of the Promotion of Science and Talent Project (DPST) Scholarship since 1991. In 1994, she was selected to study abroad in Mathematics. She spent one year at St. Mark's school, MA and then got her BS in Mathematics from University of Rochester, NY. She worked as a Research Assistant for one year in the Physics Department at University of Rochester, NY, and also as a Teaching Assistant in the Mathematics Department when she was a senior. After her graduation in May 1999, she continued her graduate studies in Operations Research at Rutgers University, NJ and worked as a Teaching Assistant for one year. In August 2000, she transferred to Grado Department of Industrial and Systems Engineering at Virginia Tech and pursued Master's degree in Operations Research. She worked three semesters as a Teaching Assistant and one semester as a Research Assistant. In year 2000, she was awarded the Paul J. Woo, Jr. Industrial and Systems Engineering Scholarship. Her Master's thesis focuses on the analysis of decision postponement strategies for aircraft assignment.