MATHEMATICAL MODELING FOR DATA ENVELOPMENT ANALYSIS WITH FUZZY RESTRICTIONS ON WEIGHTS

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(ABSTRACT)

Data envelopment analysis (DEA) is a relative technical efficiency measurement tool, which uses operations research techniques to automatically calculate the weights assigned to the inputs and outputs of the production units being assessed. The actual input/output data values are then multiplied with the calculated weights to determine the efficiency scores. Recent variants of the DEA model impose upper and lower bounds on the weights to eliminate certain drawbacks associated with unrestricted weights. These variants are called weight restriction DEA models. Most weight restriction DEA models suffer from a drawback that the weight bound values are uncertain because they are determined based on either incomplete information or the subjective opinion of the decision-makers. Since the efficiency scores calculated by the DEA model are sensitive to the values of the bounds, the uncertainty of the bounds gets passed onto the efficiency scores. The uncertainty in the efficiency scores becomes unacceptable when we consider the fact that the DEA results are used for making important decisions like allocating funds and taking action against inefficient units.

In order to minimize the effect of the uncertainty in bound values on the decision-making process, we propose to explicitly incorporate the uncertainty in the modeling process using the concepts of fuzzy set theory. Modeling the imprecision involves replacing the bound values by fuzzy numbers because fuzzy numbers can capture the intuitive conception of approximate numbers very well. Amongst the numerous types of weight restriction DEA models developed in the research, two are more commonly used in real-life applications compared to the others. Therefore, in this research, we focus on these

two types of models for modeling the uncertainty in bound values. These are the absolute weight restriction DEA models and the Assurance Region (AR) DEA models.

After developing the fuzzy models, we provide implementation roadmaps for illustrating the development and solution methodology of those models. We apply the fuzzy weight restriction models to the same data sets as those used by the corresponding crisp weight restriction models in the literature and compare the results using the two-sample paired t-test for means. We also use the fuzzy AR model developed in the research to measure the performance of a newspaper preprint insertion line.

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LIST OF ACRONYMS

AR	Assurance Region
BCC	Banker, Charnes and Cooper
CCR	Charnes Cooper and Rhodes
CR	Cone Ratio
DEA	Data Envelopment Analysis
DMU	Decision Making Unit
FDH	Full Disposal Hull
HMO	Health Maintenance Office
LP	Linear Program(ming)
NNSS	Non-News Sensitive Section
RCG	Roll, Cook, and Golany
SBDC	Small Business Development Center

Chapter 1

INTRODUCTION

1.1 OBJECTIVE

This research has five objectives. The first objective is to minimize the effect of bound uncertainty on the decision-making in Data Envelopment Analysis (DEA) by explicitly incorporating the uncertainty in the modeling process through fuzzy weight restriction DEA models. The second objective is to develop a solution methodology for those fuzzy models. The third objective is to provide implementation roadmaps for illustrating the proposed fuzzy models. The fourth objective is to apply the proposed models to the same data sets as those used by corresponding crisp weight restriction models from the literature and compare their results. The fifth objective is to use the results of the fuzzy models to modify the specified bounds so that the borderline [1] decision-making units (DMUs) become part of the efficient set.

1.2 MOTIVATION

Since it's conception by Charnes *et al.* (1978), the original DEA model has undergone many modifications and developments. Most of the developments occurred when some of the deficiencies of the original model were exposed during its application to solving real life problems. One such development occurred when the complete flexibility accorded by the original model to the input/output weights was found to be unacceptable when using DEA for certain applications (see Thompson *et al.* 1986). Thompson *et al.* (1986) tried to use DEA to choose one best site from amongst six probable sites for locating a high-energy physics lab. The original DEA model with complete weight flexibility identified five out of the six sites to be efficient. To identify one best site, Thompson *et al.* (1986) had to modify the existing model by imposing suitable Assurance Region (AR) constraints on the input/output weights. This led to the genesis of a whole new series of models called weight restriction DEA models in which constraints imposing bounds on the input/output weights are added to the original model. A number of weight restriction models have been proposed in the literature so far (see Dyson and Thanassoulis 1988, Wong and Beasley (1990), Thompson *et al.* (1990)). These models differ from one another in the purpose and form of the weight restriction constraints. Chapter 2 provides a comprehensive review of all the weight restriction DEA models proposed in the literature so far.

In this research, we focus our attention only on the following two weight restriction DEA models:

1. The absolute weight restriction DEA model and
2. The Assurance Region DEA model.

The reason we choose these models is that they are implemented in most real-life applications of weight restriction DEA models and they are discussed most commonly in the literature on DEA with weight restrictions.

The absolute weight restriction model involves adding additional constraints to the existing DEA model, which impose upper and lower limits on the weights of the inputs and outputs. This model is used when the objective is to minimize the disparity in the weights assigned to the different inputs and outputs. In other words, absolute weight restrictions ensure that the model does not assign excessively high weights to certain factors while completely ignoring other factors. The first step in the procedure for determining the values of absolute bounds is to run the unbounded model. This is followed by a close scrutiny of the results of the model to identify anomalies in the weight values calculated. Finally, appropriate bounds are assigned to deal with the identified anomalies.

Several methods are available to the decision-maker to calculate the weight bound values. However, the choice of the method, which governs the bound values and the subsequent

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[1] These are DMUs whose membership in the efficient set is highly sensitive to slight changes in bound values.

efficiency scores calculated, rests entirely with the decision-maker. This introduces an element of subjectivity in the DEA analysis, which until this point was entirely objective. In fact Farrell (1957) introduced the concept of the empirical production frontier, which forms the basis of DEA, so as to eliminate the use of human judgement for determining factor weights, the main drawback of previous efficiency measurement techniques.

To compound the problem of subjectivity associated with the choice of the method for determining the bounds, there are several opportunities for the subjectivity to make its appearance in every step of the existing procedures used for setting the bounds. We elaborate this statement by using the method proposed by Roll and Golany (1993) as an illustrative example. The authors clarify that their procedure is intended only to provide general guidelines for setting bounds. Obviously, there are several steps in the procedure where the decision-makers have to use their own discretion. Examples of such steps are:

- Step 1, in which the decision-maker has to choose a value n which is the number of extreme weight values that he/she wishes to truncate from the top and the bottom of the unbounded weight matrix before taking the average -- The decision-maker could choose to eliminate any number of values.

- Step 2, which requires the decision-maker to take an average of the weight values remaining after truncation -- Some decision-makers might choose to use the median as a measure of central tendency.

- Step 3, which requires the decision-maker to choose a ratio (d:1) between the upper and lower bounds -- Since there are no guidelines for choosing a particular value of d, different decision-makers could choose different values of d. In fact Roll and Golany (1993), while demonstrating their proposed method, themselves use two different values of d (2 and 3) to produce two different sets of bounds and two different sets of efficiency scores for the same data set.

Based on the above discussion, we can conclude that overall subjectivity[2] is introduced in the values of absolute weight bounds because of two primary reasons:

- absence of a single standardized technique for setting the bounds and
- presence of ambiguity in several steps of the existing techniques.

Let us now turn our attention to the other type of weight restriction model -- the assurance region (AR) DEA model. **The Assurance Region (AR) model** involves setting bounds on the ratios of weights (see Thompson *et al.* (1990) for the definition of Assurance Regions.) There are two ways in which the AR bound values are determined.

1. One method is based on expert opinion. It involves setting AR bounds on the basis of the magnitude of relative importance of the different inputs/outputs as perceived by the experts. Zhu (1996) used the analytic hierarchy process (AHP) to gather expert opinion when setting the AR bounds for the model, which measures the efficiencies of the different plants of the Nanjing Textile corporation. In Chilingerian and Sherman (1997), the bounds were determined by first running the unbounded DEA model and then using the ranges of the multipliers assigned by the efficient DMUs (primary care physicians) which satisfied certain performance conditions stipulated by the HMO (health maintenance organization) director.

2. Another method for setting AR bounds involves utilizing the fact that the input/output weight (or multiplier) values in the dual of the DEA model are the prices/costs of the inputs/outputs. Therefore, economic information about the price/cost ranges of the inputs/outputs can be used to set AR bounds. Setting AR bounds of this type represents a move from pure technical efficiency measurement to overall efficiency measurement.

The drawback with the first procedure is that the bound values are highly subjective because they are based on human judgment. Moreover, the expert may not have sufficient

[2] Subjectivity is the dependence of the bound values on the judgement of a particular decision-maker. When the decision-maker changes, the bound values also change.

information about the underlying process to make precise judgements about the relative importance of the factors.

The drawback of the second procedure is that the bound values are influenced by factors such as price volatility and presence of inflation (see Taylor *et al.* (1997).) Another drawback is that the price information is often inaccurate or incomplete. For example, Thompson *et al.* (1996b), in the absence of price information for natural gas, determine the bounds for natural gas by multiplying the bounds for crude oil by the gas to oil price ratio. In another application, Thompson *et al.* (1996c) in the absence of price information for one of the input (total number of branches) weights, use the same bounds as those for another input (total physical capital) weight. The overall effect of price volatility, inflation and absence of sufficient information about prices is that the AR bounds based on prices are imprecise.

Based on the discussion so far of the different weight restriction DEA models we can draw the following conclusions about the weight bound values used in the models:

1. **The bound values are imprecise[3].**
2. **The bound values are subjective[4].**

Statements 1 and 2 above are supported by the following quote from Schaffnit *et al.* pp. 281 (1997) "In some cases, the information introduced in the models through the multiplier constraints is highly subjective or contains a considerable degree of inaccuracy or uncertainty."

Lewin *et al.* (1982) point out that since DEA requires only a single observation for each output and input per DMU to construct the efficiency frontier, it is more sensitive compared to statistical techniques to errors in the data. Also Epstein and Henderson (1989) point out that since DEA is an estimation technique relying on extremal points, it could be extremely sensitive to variable selection, model specification, and data errors.

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[3] The values are not precisely known.
[4] Values vary from decision-maker to decision-maker

We would like to extend this discussion further by saying that DEA is also sensitive to the weight bound values whose uncertainty (imprecision + subjectivity) gets passed onto the results. This becomes unacceptable when we consider the fact that DEA is a decision-making tool whose results are used for making important decisions like allocating funds or taking stringent action against inefficient DMUs.

1.3 METHODOLOGY

The objective of this research is to explicitly incorporate the uncertainty in the modeling process so that the effect of the uncertainty on the decision-making process is minimized. Two approaches exist in the literature for modeling uncertainty. The more conventional approach is the stochastic approach that involves specifying a probability distribution function (e.g. Normal) for the error process (Sengupta (1992)). However, as pointed out by Sengupta (1992), the stochastic approach has certain drawbacks associated with modeling the uncertainty in DEA problems. These drawbacks are:

1. When using the stochastic approach, one has to assume a specific error distribution e.g. normal or exponential to derive specific results and this assumption may not be realistic because on *a priori* basis there is very little empirical evidence to choose one type of distribution over another.
2. Stochastic DEA models always emphasize point solutions whereas from the point of view of carrying out a sensitivity analysis, one would be more interested in DEA models that provide interval solutions.
3. Small sample sizes in DEA make it difficult to use stochastic models.

The more recent approach for modeling uncertainty has been fuzzy set theory. Sengupta (1992) who was the first to incorporate fuzzy set theory in DEA proposed a fuzzy mathematical programming approach for dealing with imprecise data in DEA problems. According to Sengupta (1992) the advantages of using such an approach are:

1. Fuzzy set theory allows us to apply the "principle of incompatibility," which has the ability to arrive at decisions based on qualitative data and linguistic information.

2. Fuzzy set theory lends itself easily to be incorporated in LP models. Since DEA involves solving a series of LP models, relatively fewer changes have to be made, under conditions of uncertainty, to the original DEA formulation to incorporate the methods of fuzzy mathematical programming.

Due to these advantages, it is proposed to use fuzzy set theory in the current research for **modeling the uncertainty** in weight bound values. Fuzzy set theory is introduced in the analysis by replacing the crisp weight bounds by fuzzy numbers. The justification is that the imprecise weight bounds need to be represented as approximate numbers (i.e. "numbers close to the specified values") and fuzzy numbers capture the intuitive concept of approximate numbers very well (Yuan and Klir (1995)).

A fuzzy number is a set of values (instead of a single value) close to the value that is being approximated. All values encompassed by the fuzzy number do not belong to it to the same degree. The degree of belongingness of each value is dependent upon the degree of closeness of that value to the value being approximated. Since a fuzzy number represents a range of values (instead of a single value), fuzzy numbers representing the bounds specified by different decision-makers are likely to be a compromise between the different bounds. Therefore, it is hypothesized that using fuzzy numbers for bounds will have an added advantage of **minimizing the sensitivity** of the results to the **subjectivity** in the bound values.

Using fuzzy numbers instead of crisp numbers for the bounds has an added advantage of **increasing the flexibility** in the bound setting process because it allows the decision-maker to specify a range of values instead of one value. The lack of flexibility in the crisp weight restriction models can often put the decision-maker in a tight spot especially when sufficient information does not exist for him/her to make a crisp judgement.

From this point onwards, the models obtained by replacing the crisp weight bounds by fuzzy numbers will be referred to as fuzzy weight restriction DEA models or simply fuzzy models. In this research, we develop fuzzy models to capture the bound uncertainty in the two most commonly used weight restriction DEA models:

1) The DEA model with absolute weight restrictions (see Dyson and Thanasoulis (1988), Roll *et al.* (1991), and Roll and Golany (1993).)

2) The Assurance Region (AR) DEA model (see Thompson *et al.* (1986), Thompson *et al.* (1990).)

We also provide implementation roadmaps for illustrating the development and solution methodology of the fuzzy models. The roadmaps are developed in response to Almond's (1995) criticism that a number of fuzzy approaches lack implementation roadmaps.

Finally, we apply the fuzzy weight restriction models to the same data sets as those used by the corresponding crisp weight restriction models in the literature. This is so that we can compare the results of the two models. We also apply the fuzzy AR model to a real life manufacturing system because sufficient information is available to set the crisp and fuzzy bounds. For comparing the results of the fuzzy models with those of the corresponding crisp models, we use the two-sample paired t-test for means (Bain and Englehardt (1992)).

1.4 Research Results

In this section, we provide a high-level overview of the research results. The results of the two sample paired t tests in each case show that the efficiency scores calculated by the fuzzy model are significantly different from the efficiency scores calculated by the corresponding crisp model. This implies that the operational decisions based on the results of the fuzzy models will be different from those taken based on the results of the crisp models. The fuzzy models ensure that the decisions are taken after the uncertainty has been accounted for. The efficiency scores calculated by the fuzzy model represent a compromise between maximization of the efficiency scores and the satisfaction of the decision-maker with the bounds.

In some cases, it is found that DMUs move from the inefficient set to the efficient set when the bounds are changed from crisp to fuzzy. In some of those cases, even a relaxation of bounds to a 90% satisfaction level of the original values (i.e. just a 10% relaxation of the bounds) is enough to move some DMUs (referred to as borderline DMUs) from the inefficient set to the efficient set. Since the bound values are based on incomplete information, the decision-maker is not expected to resist changing the existing values to values that are at the 90% satisfaction level. Thus, the fuzzy model gives the decision-maker a second chance to revise the bounds and make them favorable to the borderline DMUs.

1.5 ORGANIZATION OF THE DOCUMENT

This chapter provided an overview of the research undertaken in this thesis. The remainder of the document is organized as follows. Chapter 2 entitled "Literature Review" summarizes research in the area of efficiency measurement, data envelopment analysis, data envelopment analysis with weight restrictions, fuzzy set theory, fuzzy linear programming and fuzzy data envelopment analysis. Chapter 3 entitled "Methodology" describes the fuzzy weight restriction DEA models that are developed in this research. Chapter 3 also describes how those fuzzy models are converted into crisp equivalent models. Finally, it illustrates the development and solution methodology of the fuzzy models using implementation roadmaps. Chapter 4, entitled "Application, Results and Discussion", presents and analyzes the results obtained from solving the fuzzy weight restriction models for the data sets used by corresponding crisp weight restriction models in the literature. Chapter 5 entitled "Conclusion" concludes the discussion by highlighting the salient contributions of the research and making recommendations for future research.

Chapter 2

LITERATURE REVIEW

This chapter reviews the literature in all fields that are germane to this research. It touches on all topics in the evolution of Data Envelopment Analysis (DEA) right from the traditional efficiency measurement techniques through the seminal work of M.J. Farrell (1957) to the first DEA model developed by Charnes *et al.* (1978). It then delves into the literature on weight restriction DEA models. A part of the section on weight restriction models is dedicated to describing the applications of those models to real-life examples. The last two sections of the chapter contain a discussion on fuzzy set theory, fuzzy numbers, fuzzy linear programming and applications of fuzzy set theory in DEA.

The chapter has been divided into the following six sections:

2.1 Traditional definitions of efficiency.

2.2 Technical efficiency using the production function - Review of Michael Farrell's (1957) seminal work in the field of efficiency measurement.

2.3 Introduction to Data Envelopment analysis (DEA).

2.4 Review of literature on weight restrictions and value judgements in DEA.

2.5 Introduction to concepts of fuzzy set theory, fuzzy numbers and fuzzy linear programming.

2.6 Review of literature on fuzzy set theory and fuzzy decision-making used in DEA.

2.1 TRADITIONAL DEFINITIONS OF EFFICIENCY

Since DEA is a technical efficiency measurement technique, we start this chapter with a review of the traditional techniques used for efficiency measurement. The objective of this and the subsequent section of this chapter is to trace the evolution of the DEA approach.

2.1.1 Average Productivity of Labor

For a long time, efficiency was assessed by measuring the average productivity of labor (Farrell (1957)). Though this was a very popular measure, it had a drawback. The drawback was that it ignored all inputs except labor and was found to be unsatisfactory when the process or organization being evaluated had multiple inputs and outputs.

2.1.2 Indices of Efficiency

Because of the unsatisfactory nature of the labor productivity measure, attempts were made to develop measures of efficiency, which combined all the factors by aggregating a firm's inputs. One set of measures developed as a result of those efforts is called indices of efficiency. Here, the input vectors are first stripped of their dimensions. The dimensionless quantities are weighted and then added up. Thus, indices of efficiency involve a comparison of weighted-average of inputs with the output. The weighted-average is equivalent to a valuation of the inputs at prices proportional to the weights in the index. Thus, an attempt to compare efficiency by this measure can be regarded as making a cost comparison. The choice of a set of prices introduces an arbitrary element into the measure and the difficulty lies in choosing a suitable set of weights. Even if all the firms use the same set of prices, the measure still boils down to a mere cost comparison (Farrell (1957)).

2.2 TECHNICAL EFFICIENCY USING THE PRODUCTION FUNCTION

To eliminate the above mentioned drawbacks associated with traditional efficiency measures, Farrell (1957) introduced a new measure of (technical) efficiency, which employs the concept of the *efficient production function*. This method of measuring technical efficiency of a firm consists in comparing it with a hypothetical perfectly efficient firm represented by the production function. The efficient production function is some postulated standard of perfect efficiency and is defined as the output that a perfectly efficient firm could obtain from any given combination of inputs.

The first step in calculating the technical efficiency by this method is determining the efficient production function. There are two ways in which the production function can be determined. It could either be a theoretical function or an empirical one. The problem with using a theoretical function is that it is very difficult to define a realistic theoretical function for a complex process. The empirical efficient production function, on the other hand, is estimated from observations of inputs and outputs of a number of firms. Therefore, it is far easier to compare performances with the best actually achieved (the empirical production function) than to compare with some unattainable ideal (the theoretical function).

To understand the concept of an efficient production function, we take the example of a set of firms employing two factors of production (inputs) to produce a single product (output) under conditions of constant returns to scale. Constant returns to scale means that increase in the inputs by a certain proportion results in a proportional increase in the output. An isoquant diagram is the one in which all firms producing the same output lie in the same plane. Each firm in an isoquant diagram is represented by a point so that a set of firms yields a scatter of points. An efficient production function is a curve, which joins all the firms in an isoquant diagram utilizing the inputs most efficiently.

While drawing the isoquant from the scatter plot, two more assumptions, in addition to constant returns to scale are made:

1. The isoquant is convex to the origin. This means that if two points are attainable in practice then so is their convex combination.
2. The slope of the isoquant is nowhere positive which ensures that an increase in both inputs does not result in a decrease in the output.

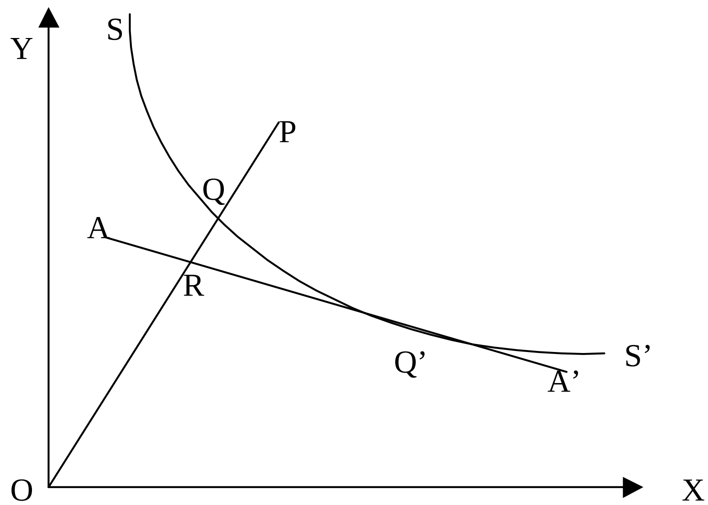

Figure 2.1 Representation of the Production Function (Isoquant) SS’

In Figure 2.1, isoquant SS’ represents a production function. Point P represents an inefficient firm, which uses the two inputs per unit of output in a certain proportion. Point Q represents an efficient firm which produces the same output as P, uses the two inputs in the same proportion as P but uses only a fraction OQ/OP as much of each input. Point Q could also be thought of as producing OP/OQ times as much output from the same inputs. Therefore, the ratio OQ/OP is defined as the **technical efficiency** of firm P. This measure of efficiency ignores the information about the prices of the factors. To incorporate the price information, use is made of the other type of efficiency measure called price (or allocative) efficiency. Price efficiency is a measure of the extent to which a firm uses the various factors of production in the best proportions, in view of their prices.

In Figure 2.1, if AA’ has a slope equal to the ratio of the prices of the two input factors, then Q’ and not Q is an optimal method of production. Although both Q and Q’ represent 100 percent technical efficiency, the costs of production at Q’ will only be a fraction OR/OQ of those at Q. Therefore, the ratio OR/OQ is called the **price efficiency** of both firms P and Q. The product of technical efficiency and price efficiency is called overall efficiency. In Figure 2.1, the ratio OR/OP represents the overall efficiency of firm P.
We see that an important feature of Farrell's (1957) method outlined above is the distinction between price and technical efficiency. While the price efficiency measures a

firm's success in choosing an optimal set of inputs which minimize the cost of production, the technical efficiency measures its success in producing maximum output from a given set of inputs.

2.3 DATA ENVELOPMENT ANALYSIS (DEA)

DEA is an extension of Farrell's (1957) idea of linking the computation of technical efficiency with production frontiers. The first DEA model was developed by Charnes Cooper and Rhodes (1978) (CCR). The CCR model is a fractional programming model, which measures the relative technical efficiency of a firm by calculating the ratio of weighted sum of its outputs to the weighted sum of its inputs. The fractional program is run for each firm to determine the set of input-output weights, which maximizes the efficiency of that firm subject to the condition that no firm can have a relative efficiency score greater than unity for that set of weights. Thus, the DEA model calculates a unique set of factor weights for each firm. The set of weights has the following characteristics:

- It maximizes the efficiency of the firm for which it is calculated and
- It is feasible for all firms.

Since DEA does not incorporate price information in the efficiency measure, it is appropriate for not for profit organizations where price information is not available. These not for profit organizations are referred to as Decision-Making Units (DMUs) by Charnes Cooper and Rhodes (1978).

Since the efficiency of each DMU is calculated in relation to all other DMUs and using actual observed input-output values, the efficiency calculated in DEA is called relative efficiency. Charnes, Cooper and Seiford (pp.6, 1994) define DEA as "DEA produces a piecewise empirical extremal production surface which in economic terms represents the revealed best-practice production frontier – the maximum output empirically obtainable from any DMU in the observed population, given its level of inputs."

In addition to calculating the efficiency scores, DEA also determines the level and amount of inefficiency for each of the inputs and outputs. The amount of inefficiency is determined by comparison with a convex combination of two or more DMUs, which lie on the efficient frontier, utilize the same level of inputs, and produce the same or higher level of outputs.

Several models have been proposed in the DEA field. All DEA models utilize the concept mentioned above. Differences amongst the various models occur only in the shape of the frontier and in the method used for projecting the inefficient DMUs onto the frontier. The very first model proposed in the DEA literature is called the Charnes Cooper and Rhodes (1978) model also known as the CCR model. This model is still the most commonly referenced one in the literature and will be used in the proposed research for demonstrating the fuzzy weight restrictions method.

2.3.1 The CCR Model

Model Definition

This model is an extension of the ratio technique used in traditional efficiency measurement approaches. The measure of efficiency of any DMU is obtained as the maximum of a ratio of weighted output to weighted input subject to the condition that similar ratios for every DMU be less than or equal to unity.

In a more precise form:

$$\max \quad h_0 = \frac{\sum_{r=1}^{s} u_r y_{r0}}{\sum_{i=1}^{m} v_i x_{i0}}$$

Subject to:

$$\frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \leq 1; \qquad j = 1, \ldots, n \qquad (2.1)$$

$$u_r, v_i \geq 0; \qquad r = 1, \ldots, s; \qquad i = 1, \ldots, m$$

Where n represents the number of DMUs, s, the number or outputs and m, the number of inputs. y_{rj}, x_{ij} (all positive) are the known outputs and inputs of the j^{th} DMU and u_r, v_i >=0 are the variable weights to be determined by the solution of this problem. The input-output values are obtained by collecting information on the resourcs used and outputs produced from past observations.

The efficiency of one of the DMUs from the set $j = 1, .., n$ is to be evaluated relative to the others. It is therefore represented in the objective function (for optimization) as well as in the constraints. In the objective function it is distinguished by assigning the subscript 0 to its inputs and outputs,.

Reduction to Linear Programming Forms

Model (2.1) is a fractional programming problem. In its current form, it is computationally intractable when the number of DMUs (*n*) is large and the number of inputs (*m)* and outputs (*s)* is small. Therefore, Charnes *et al*. (1978) convert it into a linear programming form which is as follows:

$$\min g_0 = \sum_{i=1}^{m} \eta_i x_{i0}$$

Subject to:

$$-\sum_{r=1}^{s} \mu_r y_{rj} + \sum_{i=1}^{m} \eta_i x_{ij} \geq 0 \qquad (2.2)$$

$$\sum_{r=1}^{s} \mu_r y_{r0} = 1$$

$$\eta_i, \mu_r \geq 0$$

Where

$$\eta_i = tv_i; \qquad i = 1, ..., m,$$

$$\mu_r = tu_r; \qquad r = 1, ..., s,$$

$$t^{-1} = \sum u_r y_{r0},$$

The dual of (2.2) (as obtained by Charnes *et al.* (1978)) is:

$$\max z_0$$

subject to:

$$\begin{aligned}
-\sum_{j=1}^{n} y_{rj}\lambda_j + y_{r0}z_0 \le 0; \qquad & r = 1,\ldots\ldots,s, \\
\sum_{j=1}^{n} x_{ij}\lambda_j \le x_{i0}; \qquad & i = 1,\ldots,m. \\
\lambda_j \ge 0; \qquad & j = 1,\ldots.,n.
\end{aligned} \tag{2.3}$$

The purpose of the dual is to determine the amount of inefficiency of the inefficient DMUs by projecting them onto the efficient frontier. From this point onwards, to keep the naming convention in line with that used in the DEA literature, the dual will be referred to as the primal and the primal will be referred to as the dual.

The drawback with the CCR model is that it compares DMU's only based on overall efficiency assuming constant returns to scale. It ignores the fact that different DMU's could be operating at different scales. To overcome this drawback, Banker, Charnes and Cooper (1984) developed the BCC model, which considers variable returns to scale and compares DMUs purely on the basis of technical efficiency. The discussion of the BCC model is beyond the scope of this document. Interested readers are referred to Banker *et al*. (1984) for details of the BCC model.

2.3.2 Classification Scheme for the DMUs

At this point, we would like to take a minor digression and discuss a scheme for classifying DMUs based on where they are projected onto the efficient frontier. According to Charnes *et al*. (1986), all DMUs can be classified into two broad sets – Efficient (RE) and Inefficient (N). The set RE further partitions into sets E, E' and F. Set E is the set of extreme efficient DMUs. These are the DMUs that form the vertices of the efficiency frontier. The DMUs belonging to set E' are non-extreme efficient because they can be represented as a convex combination of extreme-efficient DMUs. The DMUs in set F have some slack and hence lie on the extended frontier.

Set N partitions into sets NF, NE', and NE. The DMUs in NF are inefficient DMUs that project onto the extended frontier. The inefficient DMUs in NE' project onto the set E' and the DMUs in NE project onto the vertices of the efficiency frontier i.e. the set E. For a better understanding of this classification scheme, we have illustrated it using an example consisting of eight DMUs. Each of these DMUs consumes two inputs and produces one output. Both inputs have been divided by the output to obtain inputs per unit output and have been plotted in Figure 2.2.

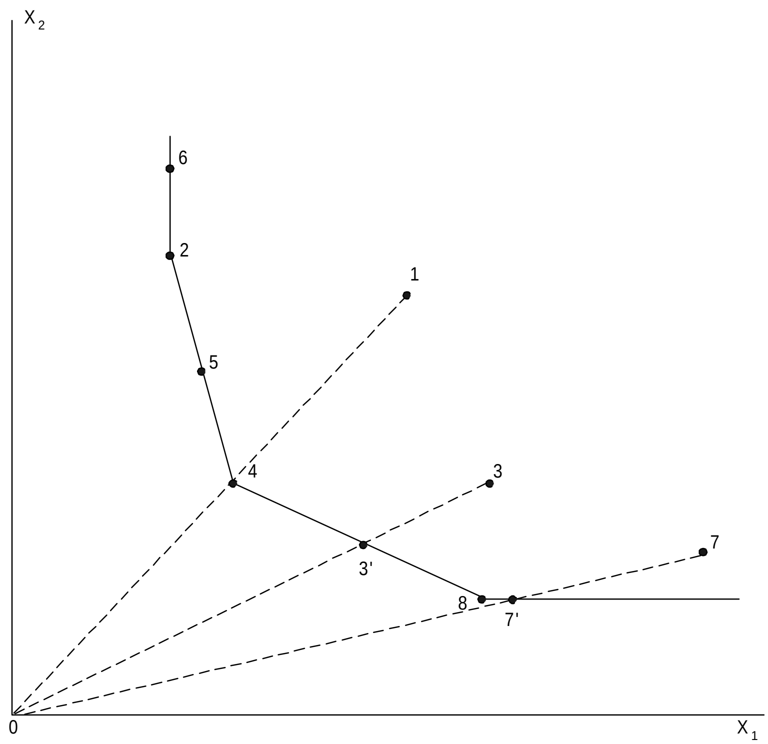

Figure 2.2 Scatter Plot Illustrating Efficiency Classification of DMUs

Based on the classification scheme discussed, we make the following observations:

- DMUs 2, 4 and 8 belong to set E.
- DMU 6 is in set F.
- DMU 5 is in set E' because it can be expressed as a convex combination of DMUs 2 and 4.
- DMU 1 lies in set NE, DMU 3 in NE' and DMU 7 in the set NF.

2.4 WEIGHT RESTRICTIONS AND VALUE JUDGEMENTS IN DEA

Most of the methodological extensions of DEA have been application driven i.e. they have been a result of the application of the method to real life problems (Allen*et al.* 1997). One such development is the use of weight restrictions and value judgements. The intention of incorporating value judgements is to incorporate prior views or information regarding the assessment of efficiency of DMUs. In this section we seek to review the evolution of the methodology of weight restrictions.

2.4.1 Motivation for Incorporating Value Judgements in DEA

A conventional DEA model involves calculating the relative efficiency of a DMU by assigning such weights to its inputs and outputs so that the ratio of its weighted output to weighted input is maximized. Apart from the condition that the weights should be non-zero (type I), the only other condition that restricts the weights is that the efficiency of none of the DMUs should exceed unity (type II). Thus, DEA in its purest form allows almost total flexibility in the selection of weights, especially if fewer[5] DMUs are included in the analysis. This allows each DMU to achieve the maximum feasible efficiency rating with its existing levels of inputs and outputs. An argument in favor of total weight flexibility is that if a DMU is identified as inefficient in spite of using a favorable set of weights, it is a strong statement about the inefficiency of that DMU. Another argument in favor of total flexibility is that the efficiency of different DMUs is evaluated using different sets of weights allowing DMUs to express their different circumstances and different objectives.

Total weight flexibility, however, also has numerous drawbacks. The salient drawbacks are:

- The efficiency measure in DEA is derived relative to the performance of other DMUs and not to some ideal production frontier. As a result, a DMU that is superior to all other units in only a single output - input ratio will receive an efficiency score of one

[5] With fewer DMUs, there are fewer constraints of type II and hence more freedom to the weights.

by placing very high weights on that particular output-input ratio. Thus, factors of secondary importance may dominate a DMU's efficiency assessment and some factors may be ignored. This may be unacceptable given the fact all factors are meticulously selected. In addition, the relative efficiency of a DMU may not really reflect its performance with respect to the inputs and outputs taken as a whole. There might also be an unfounded emphasis on efficient use of relatively unimportant inputs or on a higher production of relatively unimportant outputs, thus concealing inefficiencies in the most important activities undertaken by the DMU (Pedraja *et al.* (1997)).

- Weight flexibility allows different DMUs to assign vastly different weights to the same factor. The argument in favor of this is that different DMUs have different circumstances and therefore one factor may be more important to one DMU compared to another DMU. Thus, some degree of weight flexibility may be desirable to allow DMUs to reflect their particular circumstances. However, complete flexibility becomes unacceptable as most of the DMUs employ similar technologies, pay similar prices for inputs, produce the same kind of outputs and have the same overall objectives (Pedraja *et al.* (1997)).
- Unbounded weight restriction models do not allow us to incorporate into the analysis any *a priori* information that might be available regarding the importance of inputs and outputs.

The other extreme of unbounded weight restriction models is the complete lack of flexibility, which converts the problem to that of ratio analysis and obviates the need for DEA. An in between solution involves setting upper and lower bounds within which factor weights are allowed to vary. The imposition of restrictions on the weights implies the formulation of value judgements about the relative importance of the different outputs and about the relative opportunity costs of the inputs that produce these outputs. Weight restrictions reduce the region of search for the weights thus possibly reducing the efficiency of the DMUs. As the restrictions become increasingly severe, the measure of efficiency derived moves from one of relative *technical* efficiency to one of relative *overall* efficiency. At the extreme, with no flexibility in weights, DEA becomes classical ratio analysis, in which a unit's efficiency is measured as the ratio of weighted outputs to weighted inputs with weights being equal to the prices (Pedraja *et al.* (1997)).

In addition to eliminating the drawbacks of unbounded DEA models, weight restrictions also serve some additional purposes. Listed below are some of the purposes for which weight restriction models could be used or have already been used in real-life applications:

- *To ensure incorporation of all inputs and outputs in the assessment of performance*

By putting upper and lower bounds on the weights, weight restriction models ensure that all factors are considered in the analysis.

- *To incorporate prior views on the values of individual inputs and outputs*

By assigning specific values to weight bounds, the decision-maker can express his/her opinion about the relative importance of the factors. In this way weight restriction models, overcome the drawback of unbounded models of not allowing *a priori* information to be incorporated in the analysis. For example, in Chilingerian and Sherman (1997), weight restrictions were used to enclose the factor weights in a cone, which represented a particular physician practice pattern. This cone was constructed using criteria specified by the health maintenance office (HMO) director. This ensured that only those primary care physicians (DMUs) whose practice styles lay inside the preferred cone i.e. were in line with the preference of the HMO director were identified as efficient. In another example, Dyson and Thanassoulis (1988) imposed restrictions on the weights to incorporate the audit commission's management's perspectives on the relative importance of the inputs and outputs used in the assessment.

- *To relate values of certain inputs with values of certain outputs*

Thanassoulis *et al.* (1995) assessing the efficiency of perinatal care units in the U.K., required the weight on "babies at risk" (input) to be the same as the weight on "number of survivals" (output). The unbounded model allowed them to vary the importance of the ratio of the number of survivals to number of babies at risk relative to other output-input ratios but it did not allow them to vary the relative importance of the individual components of the ratio. To address this problem, Thanassoulis *et al.* (1995) developed a weight restriction model in which one of the constraints equated the weight on babies at risk with the weight on number of survivals.

- *To incorporate prior views on efficient and inefficient DMUs*

Often management has prior perceptions as to which DMUs it considers to be "good" and which ones it considers to be "poor" performers. Weight restriction models allow management to incorporate these prior perceptions into the analysis. For example, while assessing the performance of banks, Charnes *et al.* (1990) found that the original CCR model (1978) recognized some notoriously inefficient banks as efficient. Therefore, Charnes *et al.* (1990) developed the cone-ratio weight restriction model, which assessed the performance of all banks based on input/output values of three preselected banks, which were recognized by management as very good performers. In Chilingerian and Sherman (1997), the weight bounds for the AR/cone-ratio model were determined based on weight values assigned to the factors by those efficient DMUs (PCPs) whose practice styles met the criteria specified by the HMO director.

- *To move from technical efficiency measurement to overall efficiency measurement*

Traditional DEA models measure only technical efficiency i.e. they ignore the information about input/output prices. Farrell (1957) defined overall efficiency as the product of technical efficiency and allocative efficiency. Allocative efficiency attempts to measure how well a DMU selects the combination of inputs so that the total cost is minimized. Obviously for measuring allocative efficiency one requires information about the prices of inputs and outputs. Since DEA is used for non-profit organizations, obtaining price information is difficult in most situations. However in almost all cases, it is certainly possible to determine ranges of prices if not exact values and this fact is leveraged by weight restriction models called Assurance Region (AR) models (Thompson *et al.* (1990)). The AR model recognizes the fact that the input/output multipliers in the dual DEA model are the prices of the inputs/outputs. Therefore, the price ranges obtained from the market can be used to set bounds on the multipliers. Quite often, however, market information may not be readily available (Zhu (1996)). In such circumstances, AR bounds are determined using expert opinion. Zhu (1996) uses the analytic hierarchy process (AHP) to gather expert opinion for setting AR bounds.

- *To switch from the points of view of the individual DMUs to that of central management.*

DEA models with complete weight flexibility weigh the same factor differently while evaluating different DMUs (Roll *et al.* (1991)). This difference in weights may not be acceptable to central management, as they would be interested in knowing how the DMUs perform using similar sets of weights. Appropriate weight restrictions ensure that all the DMUs are evaluated with similar (if not common) sets of weights. This represents a switch from the points of view of individual DMUs to that of central management and offers a compromise between complete weight flexibility on one hand and fixed weights on another.

- *To enable discrimination among efficient units.*

Sometimes DEA may be used to choose one best DMU from amongst the available alternatives. For example, Thompson *et al.* (1986) tried to use DEA to determine the best location for a nuclear physics facility in Texas and discovered that five out of six alternative facilities were found relatively efficient by the free weights model. To narrow the choice down to a single site, Thompson *et al*. (1986) determined assurance region constraints based on expert opinion. The AR model identified only one DMU as efficient.

2.4.2 Approaches for Imposing Weight Bounds

In this section, we will use the same classification scheme as that used by Allen *et al*. (1997) for classifying the different approaches for imposing weights restrictions.
The approaches for imposing restrictions on weights can be classified into the following three broad categories:

1. Direct Restrictions on weights (Absolute Weight Restrictions, Assurance Region I, Assurance Region II)

2. Adjusting the observed input-output levels to capture value judgements (Cone-Ratio and Ordinal Relations).
3. Restricting weight flexibility by restricting the weighted inputs and outputs.

Let us now look at each of the approaches in detail. All the approaches use the CCR (1978) model as the base model.

2.4.2.1 Direct Restrictions on Weights

Direct restrictions on the weights are applied by adding additional constraints involving the weights to the existing DEA model. There are three ways in which direct restrictions have been applied in the literature.

2.4.2.1.1 Absolute Limits on Weights

This type of model uses constraints, which impose upper and lower limits on the input-output weights (see Roll *et al*. (1991) and Roll and Golany (1993)). These constraints are primarily employed to prevent the inputs or outputs from being over or under emphasized. In usual notation, a CCR model with absolute limits on weights can be represented as:

$$\text{Max} \quad \frac{\sum_{r=1}^{s} u_r y_{r0}}{\sum_{i=1}^{m} v_i x_{i0}}$$

$$s.t.$$

$$\frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \le 1 \qquad j = 1,\dots,N$$

$$\underline{v}_i \le v_i \le \overline{v}_i \qquad (2.4)$$

$$\underline{u}_r \le u_r \le \overline{u}_r$$

$$-v_i \le -\varepsilon \qquad i = 1,\dots,m$$

$$-u_r \le -\varepsilon \qquad r = 1,\dots.,s$$

where $\underline{v}_i$ and $\overline{v}_i$ are the user-specified lower and upper bounds respectively on input weights and $\underline{u}_r$ and $\overline{u}_r$ are the lower and upper bounds respectively on the output weights. The key difficulty in using this approach is the determination of the values of the bounds. Roll and Golany (1993) recommend three methods for specifying absolute weight bounds:

1. General restriction of weight variation

This method is used when no information (about the relative importance of weights) is available. It should be noted that incorporating information about the relative importance of the different factors is not the only purpose of using weight bounds. An equally important purpose of introducing weight bounds could be to limit the span of variation of the weights and ensure that the weights do not take extreme values. The method recommended below taken from Roll and Golany (1993) serves this exact purpose.

a) Run an unbounded CCR model, compile a "weight matrix" and find the average weights u'_r and v'_i given to each factor, across all DMUs.

b) Determine the amount of allowable variation in weights for each factor. For example, let the ratio of the highest value to the lowest one is d: 1.

c) Extend the basic CCR model by adding a set of bounding constraints of the type[6]:

$$\frac{2u'_r}{1+d} \leq u_r \leq \frac{2du'_r}{1+d}$$

Apply similar constraints on the input weights.

d) Run the "bounded" model.

As a possible variation to this technique, Roll and Golany (1993), propose cutting off certain percentage of extreme values from both sides of each vector of weights before finding the average in step a).

---

[6] The intuition behind using these formulas for the upper and lower bounds is that they cause the lower bound to take a value smaller than 1 and the upper bound to take a value greater than 1 with the ratio between the bounds being d:1.

2. Judgmental Restriction of weight variation

This method is used when some *a priori* information about the relative importance of the factor weights exists. Just like the previous method, this method also uses the "weight matrix" obtained from the unbounded runs as a starting point. The analyst's opinion about relative importance is brought forward by adopting central values that are different from the arithmetic average (i.e. are either above or below the average) and by choosing different ratios (d_r) between upper and lower bounds for different factors. Graphically these weight bounds are represented in Figure 2.3. Factors, which, in the analyst's view, are more important, will have higher weight spans compared to the less important ones.

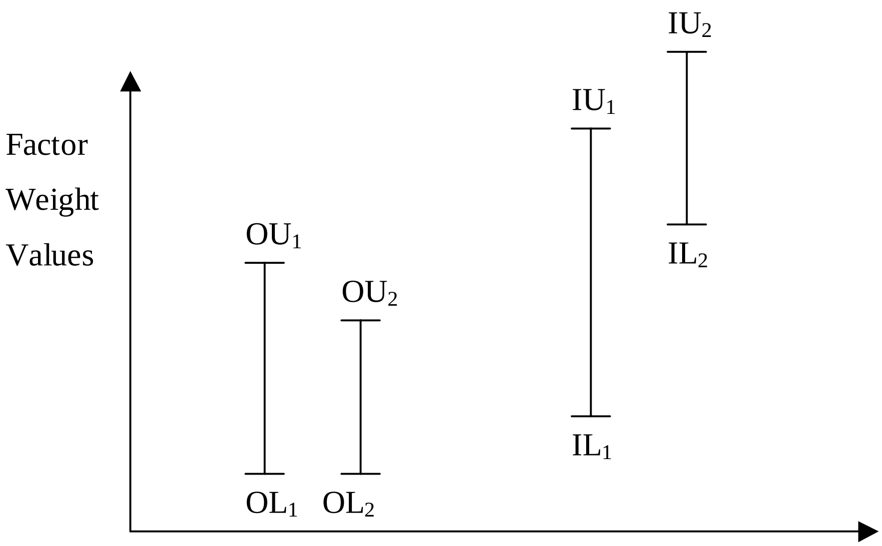

Figure 2.3 Judgmental Restrictions on Weights produce Weight Bounds with Different Spans for Different Factors

3. Advance setting of bounds

Sometimes bounds are determined *a priori* without running the unbounded model. This implies a strong initial position on the relative importance of factors and the allowed spread of weights.

2.4.2.1.1.1 Modification to Absolute weight bound model proposed by Podinovski and Athanassopoulos (1998)

Podinovski and Athanassopoulos (1998) argue that adding weight restrictions to the CCR model underestimates the relative efficiency of the DMUs. Before we proceed to explain this point let us define the notation that will be used throughout this sub-section. The efficiency of any DMU j will be defined as:

$$E(j,u,v) = u^T Y_j / v^T X_j$$

where X_j and Y_j are input and output vectors respectively and u^T and v^T are the vectors representing the output and input weights respectively. Using this notation, the CCR model can be represented as:

$$\begin{aligned} &\max \quad E(j_0,u,v) \\ &\text{subject to} \\ &E(j,u,v) \le 1, \qquad j = 1,\dots,n; \\ &u,v \ge \varepsilon \end{aligned} \qquad (2.5)$$

Now consider the maximin DEA model

$$\max_{\substack{u\in\varepsilon \\ v\in\varepsilon}} \min_{j=1,\dots,n} (E(j_0,u,v)/E(j,u,v)) \qquad (2.6)$$

Although the objective function of (2.5) maximizes absolute efficiency and the objective function of (2.6) maximizes the relative efficiency, both forms are equivalent because both get converted to the same LP form. Thus, in the absence of weight restrictions, the CCR model and the maximin model both maximize the relative efficiency of DMUs. However, when weight restrictions are added to the CCR model, it maximizes only the absolute efficiency of the assessed DMU and may not maximize its relative efficiency, which is the only important measure. This is because in the presence of weight restrictions, models (2.5) and (2.6) are no longer equivalent and get converted to different LP forms. Since the objective function of (2.5) maximizes the absolute efficiency, it continues to do so in the presence of weight restrictions. This may lead to an underestimation of the relative efficiency of the DMUs being assessed when weight restrictions are added to the CCR model. Therefore, Podinovski and Athanassopoulos (1998) recommend adding weight restrictions to the maximin model instead of the fractional CCR model because the objective function of the maximin model is set explicitly to maximize the relative efficiency of the assessed DMU. To explain the difference between the two models let us first consider the following fractional CCR model with absolute weight restriction constraints:

$$
\begin{aligned}
&\max \quad E(j_0,u,v)\\
&\text{subject to}\\
&E(j,u,v) \le 1, \qquad j=1,\ldots,n;\\
&\underline{u}_r \le u_r \le \bar{u}_r \qquad r=1,\ldots,s\\
&\underline{v}_i \le v_i \le \bar{v}_i \qquad i=1,\ldots,m
\end{aligned}
\tag{2.7}
$$

When (2.7) is converted into an LP form, we get the following model:

$$
\begin{aligned}
&\max \quad \mu^{\mathrm{T}} Y_0\\
&\text{subject to}\\
&\eta^T X_0 = 1\\
&\mu^T Y - \eta^T X \le 0;\\
&p\underline{u}_r \le \mu_r \le p\bar{u}_r \qquad r=1,\ldots,s\\
&p\underline{v}_i \le \eta_i \le p\bar{v}_i \qquad i=1,\ldots,m\\
&p \ge 0, \mu, \eta - \text{free}
\end{aligned}
\tag{2.8}
$$

where

$$\mu_r = pu_r$$
$$\eta_i = pv_i$$

$\mu(mx1)$ and $\eta\ (sx1)$ are the new output and input weights; $X\ (mxn)$ and $Y\ (sxn)$ are input and output vectors respectively; and X_0 and Y_0 are vectors representing input and output levels for the assessed DMU_0.

If instead of starting with the CCR model, we start with the maximin model and then place weight restrictions, then the resulting LP will be as follows:

$$
\begin{aligned}
&\max \quad \mu^{\mathrm{T}} Y_0\\
&\text{subject to}\\
&\eta^T X_0 = 1\\
&\mu^T Y - \eta^T X \le 0;\\
&p\underline{u}_r \le \mu_r \le p\bar{u}_r \qquad r=1,\ldots,s\\
&q\underline{v}_i \le \eta_i \le q\bar{v}_i \qquad i=1,\ldots,m\\
&p,q \ge 0, \mu, \eta - \text{free}
\end{aligned}
\tag{2.9}
$$

where

$\mu_r = pu_r$
$\eta_i = qv_i$

Comparing model (2.8) with model (2.9) we notice that (2.9) has two independent scaling variables *p* and *q*, while (2.8) has one. Model (2.8) may not find the most favorable set of weights for the assessed DMU within the feasible set and may end up underestimating the relative efficiency of the named DMUs.

2.4.2.1.2 Assurance Regions of Type I (AR I)

These relations are introduced in the analysis to accomplish either of the following two purposes:

- Incorporate the relative ordering of inputs/outputs
- Incorporate information on prices or values of inputs/outputs.

In these types of constraints, upper and lower bounds are imposed on the ratios of factor weights. Bounds are determined using market price information (see Thompson *et al*. (1990), Thompson *et al*. (1996a), Thompson *et al*. (1996b), Taylor *et al*. (1997)). Thus, an AR model represents a move from measurement of technical efficiency to measurement of overall efficiency. If price information is not available then expert opinion on the relative importance of the inputs/outputs is used to determine the bounds (see Zhu (1996)). The AR model can be mathematically represented as follows:

$$\text{Max} \sum_{r=1}^{s} u_r y_{rj0}$$

$$s.t. \quad \sum_{i=1}^{m} v_i x_{ij0} = 1$$

$$\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0 \qquad j = 1,..., N$$

$$A_i \leq \frac{v_i}{v_k} \leq B_i \qquad i < k\ ,\ i, k = 1,..., m \tag{2.10}$$

$$a_r \leq \frac{u_r}{u_t} \leq b_r \qquad r < t, \quad r, t = 1,...., s$$

$$-v_i \leq -\varepsilon \qquad i = 1,..., m$$

$$-u_r \leq -\varepsilon \qquad r = 1,...., s$$

where A_i and B_i are the lower and upper bounds on the ratios of input weights and a_r and b_r are the lower and upper bounds on the ratios of output weights.

Usually, one of the inputs (say x_1) is selected as an input numeraire and one of the outputs (say y_1) is selected as output numeraire. Then an AR may be specified as a set of (m + s - 2) homogeneous linear inequalities for separable cones (see section 2.4.2.2.1 for a better understanding of cones):

$$a_r \leq \frac{u_r}{u_1} \leq b_r \qquad r = 2,..., s \tag{2.11}$$

$$A_i \leq \frac{v_i}{v_1} \leq B_i \qquad i = 2,..., m$$

Rearranging the terms in (2.10) we get the following most commonly used form of AR constraints:

$$\begin{aligned} a_r u_1 \leq u_r \leq b_r u_1 \qquad r = 2,..., s \\ A_i v_1 \leq v_i \leq B_i v_1 \qquad i = 2,..., m \end{aligned} \tag{2.12}$$

2.4.2.1.3 Assurance Regions of Type II (AR II)

In this type of AR model, the input and output weights are linked together i.e. bounds are set on the ratios of output weights to input weights. These types of AR constraints are also called linked AR constraints because the input – output cones are linked as opposed to be being separable as in case of AR I constraints. AR II models can be used for two purposes. Either to incorporate information about the relative importance of an output with respect to an input (see Thanassoulis *et al*. (1995)) or to determine the profitability of DMUs (see Thompson *et al*. (1996b)). When the model measures profitability of the DMUs, the bounds are set using market price information.

An ARII DEA model can be mathematically represented as:

$$
\begin{aligned}
&\text{Max } \sum_{r=1}^{s} u_r y_{rj0} \\
&s.t. \quad \sum_{i=1}^{m} v_i x_{ij0} = 1 \\
&\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0 \qquad j = 1,\ldots,N \\
&\gamma v_i \geq u_r \\
&-v_i \leq -\varepsilon \qquad i = 1,\ldots,m \\
&-u_r \leq -\varepsilon \qquad r = 1,\ldots.,s
\end{aligned}
\tag{2.13}
$$

where γ is the upper bound on the ratio of the output weight u_r to the input weight v_i.

2.4.2.2 Adjusting the Observed Input-Output Levels to Capture Value Judgements – The Artificial Data Sets Methods

In the previous section, we discussed models in which weight restrictions were imposed by adding additional constraints to the original basic DEA model. In this section we will discuss models in which the weight restriction is imposed by modifying (multiplying by a vector) the existing input-output data. There are two such approaches where transformed input-output data are used to simulate weight restrictions.

2.4.2.2.1 The "Cone Ratio" Model

The cone-ratio model is another method of bringing expert relative valuational knowledge into the analysis. It involves generating a cone (smaller than the non-negative orthants) spanned by the optimal virtual multipliers of efficient DMUs which satisfy certain conditions specified by the decision-maker. The assurance region constraints (discussed previously) are special cases of intersections of half-spaces restricting the virtual multipliers to closed convex cones. The meaning of half-spaces and closed convex cones will become clearer from the discussion that follows in this section as well as in Section 3.2.1 of Chapter 3.

The cone ratio model was first introduced by Charnes *et al*. (1990). We use the following example to illustrate the concept of convex cones graphically. Consider a company having six (6) factories in six different locations in the country. Let each factory utilize two primary inputs - machine hours (X_1) and labor hours (X_2) and produce a single product as output. Let the machine hour rate be less than the labor hour rate. Naturally, the management of the company would prefer factories to use more machine hours and fewer labor hours. Let us also assume that it is possible to substitute labor hours by machine hours. The change in labor hours (ΔX_2) per unit change in machine hours (ΔX_1) is defined as the marginal rate of technical substitution. Let us assume that each of the six factories produce the same quantity of output of comparable quality. Figure 2.4 shows the scatter plot of the data with the production possibility set identifying efficient and inefficient factories.

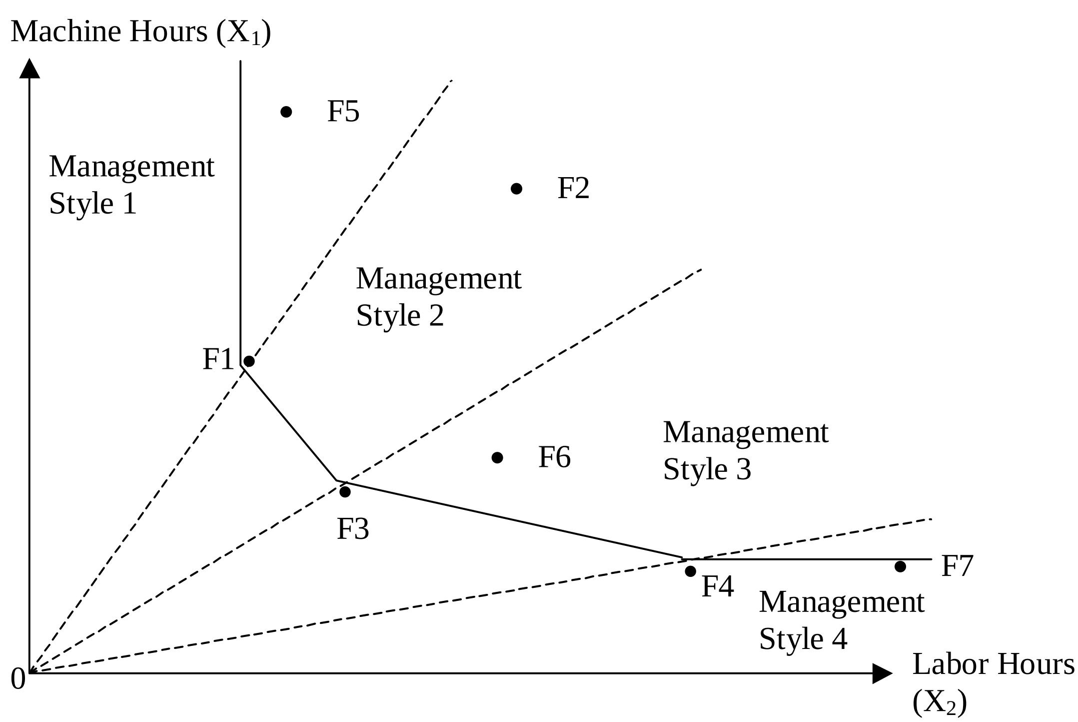

Figure 2.4 Geometric Representation of Convex Cones

Each factory has a "management style" which represents the proportion in which it uses the two inputs. In figure 2.4, convex cones (represented by dotted lines) are used to linearly partition the management styles based on a set of linear constraints such as the ranges of substitution ratios. For example, the ray joining the origin (0) and the point F1 represents all points that use the two inputs in the same ratio as F1. Similarly, the ray joining the origin and the point F3 represents all points that use the inputs in the same proportion as F3. Therefore, a factory lying inside the "Management Style 2" cone will have a ratio of machine hours to labor hours that lies between the corresponding ratios for factories F1 and F3. In general, styles 1 and 2 include factories which use more machine hours compared to labor hours and style 4 contains factories which use relatively more labor hours compared to machine hours. Obviously, since the machine hour rate is lower than the labor hour rate, styles 1 and 2 are more desirable to the company management compared to style 4. Thus, we see that although all factories on the efficiency frontier are technically efficient, not all of them have management styles that would satisfy the company management (example, factory F7). This points out the weakness of using unbounded DEA models when decision-makers have certain preferences or when information about prices exists. Cone-ratio constraints eliminate this drawback of standard models by allowing cones of virtual multipliers to be defined so that decision-makers can incorporate qualitative or price information into the analysis.

Now let's turn to the mathematical representation of cone-ratio constraints. Suppose v_1 and v_2 are input multipliers and suppose that market information sets the range of their ratio as $c_1 \le v_1/v_2 \le c_2$, with $c_2 \ge c_1 > 0$. Then we have:

$$-v_1 + c_1 v_2 \le 0 \quad \text{and} \quad v_1 - c_2 v_2 \le 0 \tag{2.14}$$

The polyhedral convex cone V for the input multipliers would then be defined as:

$$v \in V = \{v : Cv \ge 0, v \ge 0\}. \tag{2.15}$$

Where

$$C = \begin{bmatrix} c_2 & -1 \\ -c_1 & 1 \end{bmatrix}, \text{ and } v = \begin{bmatrix} v_2 \\ v_1 \end{bmatrix}$$

Similarly if F is the matrix equivalent to C for the output weights, then the output closed convex cone will be defined as:

$$U = \{u : Fu \geq 0, u \geq 0\} \qquad (2.16)$$

When the input-output weights are enclosed in cones, the resulting cone-ratio DEA model is as follows:

$$\begin{aligned} &Max \quad u^T Y_0 \\ &s.t. \quad v^T X_0 = 1 \\ &\quad -v^T X + u^T Y \leq 0 \\ &\quad v \in V, \quad u \in U \end{aligned} \qquad (2.17)$$

where X (mxn) and Y (sxn) are input and output vectors respectively, μ ($sx1$) and v ($mx1$) are output and input weight vectors respectively, X_0 and Y_0 are vectors representing input and output levels for the assessed DMU j_0. The closed convex cones $V \subseteq E^m$ and $U \subseteq E^s$ that have already been defined in (2.15) and (2.16) contain the weight restriction information. E^m and E^s are the non-negative orthants used in the unbounded DEA model. Thus if $V = E^m$ and $U = E^s$, then the model becomes equivalent to the standard DEA model (Charnes *et al*. (1990)). The matrix representation of cones shown in (2.15) and (2.16) is called the intersection form and is used in assurance region models. An alternative representation defined in Charnes *et al*. (1990) is called sum form. For small matrices, the intersection form can be easily converted to the sum form and vice-versa.

For example, if C and F are 2 x 2 matrices in intersection form, then the equivalent matrices A and B in sum form can be obtained by carrying out the following transformations:

$A^T = (C^T C)^{-1} C$ and $B^T = (F^T F)^{-1} F$.

Charnes *et al*. (1990) introduced the sum form because by multiplying the input-output data by the matrices in sum form (A and B), the cone-ratio model (2.17) is converted into a form similar to the standard DEA model. The advantage of converting the cone-ratio model to the standard form is that it can be solved using standard DEA packages. Such a cone-ratio model with modified data sets is referred to as the cone-ratio model with artificial data sets and will be represented as follows:

$$\begin{aligned} Max \quad & g^T(BY_0) \\ s.t. \quad & w^T(AX_0) = 1 \\ & -w^T(AX) + g^T(BY) \leq 0 \\ & w \geq 0, g \geq 0 \end{aligned} \tag{2.18}$$

For more details on this discussion, see Charnes *et al*. (1990)

2.4.2.2.2 The Ordinal Relations Approach

Golany (1988) proposed imposing ordinal relations of the form $v_1 \geq v_2 \geq v_3 \geq \varepsilon$ among the DEA weights. Golany (1988) also proposed equivalent transformations on the data which would allow us to incorporate ordinal relations without adding additional constraints. For example the equivalent of the constraint $v_1 \geq v_2 \geq v_3 \geq \varepsilon$ is replacing x_{2j} by $x_{2j} + x_{1j}$ and x_{3j} by $x_{3j} + x_{2j} + x_{1j}$ $\forall j$, where x_{ij} is the level of i^{th} input for the j^{th} DMU.

2.4.2.3 Restricting Weight Flexibility by Restricting the Weighted Inputs and Outputs

Two approaches proposed in the literature impose limitations on weights by restricting the weighted inputs and outputs. One is the "contingent weight restrictions" approach proposed by Pedraja *et al*. (1997) and the other is the approach proposed by Wong and Beasley (1990) which imposes limitations on the "relative importance of factors to a DMU."

2.4.2.3.1 Contingent Restrictions on Weights

Pedraja *et al*. (1997) argue that weight restrictions ought to be imposed taking into account the DMU's level of inputs and outputs to ensure that only those inputs or outputs which contribute "significantly" to the total costs or benefits of a DMU are included in the analysis.

For an input space, Pedraja *et al*. (1997) propose the following form for the constraints:

$$c_i V_1 X_{1j} \leq V_i X_{ij} \leq d_i V_1 X_{1j}$$

where c_i and d_i are to be chosen by the analyst. Similar constraints can be applied in the output space. Constraints of this sort require the proportion of total benefits ascribed to one input (output) to not exceed those ascribed to another input (output) by more than a certain multiple.

Pedreja *et al*. (1997) term this approach "contingent" weight restriction approach to emphasize the fact that the pattern of weights selected depends on the levels of inputs and outputs chosen by the DMU. Because of the dependence, the DMU puts more weight on inputs whose levels are low (i.e. ones which it consumes efficiently) and less weight on inputs whose levels are high (i.e. ones that increase its inefficiency). Thus, the efficiency calculated by the contingent model tends to be more than that calculated by models, which put limits on prices of inputs and outputs.

2.4.2.3.2 Restrictions on Relative Importance of Factors

This method was developed by Wong and Beasley (1990) and involves putting restrictions on the "importance" attached to a certain output (or input) measure by a DMU. The importance attached by a DMU to a particular output is the proportion of the total output devoted to that output. Thus, the importance attached by DMU j to output measure r can be given by $\frac{u_r y_{rj}}{\sum_{r=1}^{s} u_r y_{rj}}$ where u_r is the weight on the r^{th} (r=1,.., s) output and y_{rj} is the level of output r for DMU j.

Wong and Beasley (1990) assume that the decision-maker can set limits [a_r, b_r] on the importance of output measure r in DMU j. Using these values of the bounds, the following constraint can be added to the original DEA model.

$$a_r \leq \frac{u_r y_{rj}}{\sum_{r=1}^{s} u_r y_{rj}} \leq b_r$$

Specification of $[a_r, b_r]$ is a value judgement and is arrived at by seeking a consensus amongst those familiar with the situation being modeled, on the relative importance of each output measure in the total output.

2.4.3 Applications of Weight Restriction Models

This section describes in detail the various applications of weight restriction models that have been published in the DEA literature. For each application, we have tried to provide such details as how the bound values were determined, how the weight restrictions affected the results, etc.

2.4.3.1 A DEA Model for Measuring the Relative Efficiency of Highway Maintenance Patrols – Cook *et al*. (1990)

This is the only real-life application of the absolute weight restrictions DEA model. The rest of the applications that follow in this section are applications of the AR-DEA model. In this application, a pilot DEA study was carried out to measure the efficiency of 14 highway maintenance patrols. Two inputs and two outputs were included in the analysis. The first output called the assignment size factor was a composite measure of all factors that were indicators of the "size of the system" such as surface, shoulder, right of way and median, and winter operations. The other output was the Average Traffic Serviced. The two inputs included in the analysis were Total Expenditure and Average Pavement condition Rating. In the first run of the model, the weights were allowed to vary freely. After investigating the weight matrix of the unbounded run, the authors found that different DMUs were assigning vastly different weights to the same factor. To control this variation, the authors determined an absolute set of bounds based on the unbounded weight matrix (see Roll *et al*. (1993) for different methods for determining absolute bounds based on the unbounded weight matrix). The model was solved again with the weight controlled by bounds.

The consequences of imposing weight bounds were:

- All efficiency ratings fell below the previous (unbounded) levels.

- Two of the DMUs, which were on the frontier in the unbounded model, fell under the frontier.
- There were fewer different peer groups compared to the unbounded model.

2.4.3.2 DEA/AR Efficiency and Profitability of Mexican Banks - A Total Income Model - Taylor *et al*. (1997)

DEA and linked-cone assurance region models were used in this paper to investigate the efficiency and profitability potential of Mexican banks as they engaged in activities that incurred interest and non-interest expenses and produced income. The study had only one output called total income which was the sum of a bank's interest income, which included interest from loans and non-interest income, which in turn included dividends, fees, and others.

The two inputs in the study were:

- Total deposits, which included the banks' interest paying deposit liabilities.
- Total non-interest expense, which included personnel and administrative costs, commissions paid, banking support fund contributions and other non-interest operating costs.

The bounds were set using price/cost data. The price information was obtained from the range of nominal interest rates for the loan and deposit portfolios of all the banks, so they were consistent with the market interest rates for the data years. Since the second input and the single output were expressed as total nominal pesos, their upper and lower bounds were both equal to one.

2.4.3.3 Comparative Site Evaluations for Locating a High-Energy Physics Lab in Texas - Thompson *et al*. (1986)

In this paper a comparative evaluation of six competing sites was carried out using Data Envelopment analysis to determine the ideal site for locating a high energy physics lab. The inputs that were incorporated in the study were project cost, user time delay and environmental impact.

1. The project cost included capital cost of the tunnel, land costs, real estate improvement costs, operating costs for 20 years, and cost of main ring and injector.
2. The user time delay index measured the percentage increase in the time required for the user to complete a given research plan when the lab is located at a site other than the ideal site - where an ideal site is the one which is close to the airport and has a large center of technical support. Index values greater than 1.00 represented a measure of the site's inefficiency.
3. The environmental index measured the effect of the facility on the environment and the effect of the environment on the viability of the facility. Factors representing both the effect of the facility on the environment and that of the environment on the SSC were determined and each site was ranked for each factor. The environmental index was constructed for each site by computing the weighted average of the assigned ranks.

The objective of the project was to select a site, which maximized the net benefit i.e. the difference between the benefits and costs. For that purpose all costs and benefits had to be estimated in dollars and this caused problems because user time delay and environmental impacts are not generally expressed in dollars. This is where DEA came to the rescue because the DEA method, while consistent with the criterion of maximizing the net benefit, did not require that all costs and benefits be denoted in dollars. However, the drawback with applying the basic DEA model, which measures only the technical efficiency, to the existing data was that it identified more than one site (in fact 5 out of 6) as efficient.

To identify the preferred site from amongst the technically efficient sites, Thompson *et al*. (1986) resorted to economics i.e. they modified the DEA model so that it would identify the most economically efficient site. To determine the economically efficient site(s), Thompson *et al*. (1986) carried out an analysis[7] of the space consisting of virtual weights (or prices) of inputs. Before carrying out the weight space analysis, the virtual weights for the user-cost and environmental indices were normalized on facility costs,

[7] For more details on weight space analysis refer to section 3.2.1 in Chapter 3

which was the only input measured in dollars. With the normalization on facility costs, the weight space became two-dimensional and could be divided into regions of site preference i.e. each site had its own region of dominance in the weight space. To determine the best site an assurance region of weights was defined. The site whose region of dominance in the weight space contains the assurance region would be the best site. The "assurance region" was delineated by determining upper and lower bounds for the weights (prices). In the model, v_1 was the price on facility cost; v_2 was the price on user costs and v_3 was the price on environmental costs. Because v_2 and v_3 were uncertain, variations in the values of these prices had to be allowed.

To determine the "assurance interval" for v_2, an expected value of 5 was used for v_2. Using a confidence level of 99 percent and the value of standard error of the mean for the ideal site plus the underlying literal loss of efficiency assumption, the assurance interval for v_2 was determined as (3.6, 6.5). The "assurance interval" for v_3 was found by using the upper bound for ameliorating the environmental impact which was provided by the Texas A&M environmental study for the project. The Texas A&M environment study concluded that the negative environmental impacts at each site can be ameliorated at a cost which will not exceed the costs of tunnel construction. The cost for ameliorating the environmental impact would be given by $(x_3)^*(v_3)$ where x_3 is the environmental index. The maximum difference between the worst and the best values of the cost of ameliorating the environmental impact will be $(\Delta x_3)(v_3)$ where Δx_3 is the maximum difference between the worst and best values of the environmental index. Using the conclusion arrived at by Texas A&M environmental study we can say that:
$(\Delta x_3)(v_3) \leq$ maximum difference in tunneling costs between the worst site and the best site.

The value of Δx_3 was known to be 1.6 and the value of "maximum difference in tunneling costs between the worst site and the best site" was known to be $5.4 million (say 0.5 billion). Thus, $v_3 \leq .313$. Assuming an error of magnitude of 3 in the environmental index (x_3), the relevant range for v_3 was found to be 0.104 to 0.939. Thus, the assurance region constraints for the SSC site location problem were:

$v_1 = 1;$
$3.6 \le v_2 \le 6.5;$ and
$0.104 \le v_3 \le 0.939$

When this assurance region was applied to the site location problem, it was found that the region of dominance of one of the sites, in the weight space, completely enclosed the assurance region. Therefore, this site was the only one with efficiency score of 1 and was also the preferred site.

2.4.3.4 DEA/AR Efficiency and Profitability of 14 Major Oil companies in U.S. Exploration and Production - Thompson *et al*. (1996b)

In this paper, the efficiency and profit potential of 14 integrated oil companies were measured using data envelopment analysis. The definitions of the outputs and inputs for the producers were as follows:

Outputs

$y_{1,}\, y_3$ - Additions made to crude oil (Mbbls) and natural gas (MMCF) reserves, respectively by exploration.

$y_{2,}\, y_4$ - Crude oil (Mbbls) and natural gas (MMCF) production for sale from its respective proved reserves.

Inputs

x_1 - Total costs incurred (M$); and

$x_{2,}\, x_3$ - proved crude oil (Mbbls) and natural gas (MMCF) reserves at previous year-end.

Separable input and output AR bounds were placed on the modeled prices (multipliers) to proceed from technical toward overall efficiency. The "price/cost" data used for setting the AR bounds in this paper were organized from several sources. The following points explain how the price/cost ranges for the inputs and outputs were determined:

1. For $y_{1,}$ the lower and upper endpoints of the range values were estimated by use of Arthur Anderson's annual minimum and maximum discounted after-tax future net cash flows per barrel of crude oil for the majors.

2. For y_2, the lower and upper-endpoints of the range of values were the smallest and largest respective monthly crude oil prices (1^{st} domestic) reported by the U.S. Department of Energy (DOE) in its *Monthly Energy Review*.
3. For y_4, the lower and upper-endpoints of the value range were the smallest and largest respective monthly natural gas prices (wellhead dry) reported by DOE in its *Monthly Energy Review*.
4. For y_3, the ratio of the average annual wellhead natural gas price (dry) to the average annual first domestic crude oil price was multiplied times the lower and upper-endpoints of the value range for y_1 in each year to derive the respective range for y_3.
5. For x_1, the lower and upper-endpoints of the value range were equal to one in each year, presuming that each dollar of costs paid for inputs, e.g. labor, was worth a dollar;
6. For x_2, product of the respective largest values for y_1 and the monthly Corporate Bond AAA interest rate, as reported in the Economic Report of the President, was used to specify the upper-bound for the second input. Similarly, the product of the smallest respective values was used to specify the lower-bound for x_2.
7. For x_3, the upper and lower-endpoints of the value range were derived from the upper and lower-endpoints for x_2 by the use of gas to oil price ratio as for y_4 above.

The bounds for x_2 and x_3 were specified to reflect the rental cost of carrying a unit of the respective reserves from year t-1 into year t.

2.4.3.5 DEA/AR Efficiency of U.S. Independent Oil/Gas Producers over Time - Thompson *et al*. (1992)

In this study, a DEA/AR efficiency analysis of 7 years (1980 - 1986) was made for 45 oil / gas firms called independents. The outputs were y_1 (total crude oil production in barrels (bbls)) and y_2 (total natural gas production in thousand cubic feet (MCF)). The inputs were x_1 (total production costs in dollars), x_2 (total proven crude oil reserves in bbls), x_3 (total proven natural gas reserves in MCF) and x_4 (total net wells drilled). Total oil/gas wells drilled was the sum of all wells drilled (wet and dry), including fractions drilled as joint ventures i.e. it was the total net wells drilled.

The AR constraints on output multipliers were of the following form:

$A_2 \le u_2 / u_1 \le B_2$

where u_2 / u_1 was the oil/gas muliplier ratio and the nonnegative numbers A_2, B_2 were based on the historical price/cost data and expert opinion.

The AR constraints on the input multipliers were as follows:

$$\begin{aligned} &\alpha_2 \le v_2 / v_1 \le \beta_2, \\ &\alpha_3 \le v_3 / v_1 \le \beta_3 \qquad (2.19) \\ &\alpha_4 \le v_4 / v_1 \le \beta_4 \end{aligned}$$

where the nonnegative numbers $\alpha_i, \beta_i, i = 2, 3, 4$ were based on historical price/cost data and expert opinion.

Determination of the AR bounds for output virtual multipliers:

In estimating A_2 and B_2, the monthly minimum and maximum "spot" natural gas prices were divided by the monthly average wellhead West Texas Intermediate crude oil prices for all months from Nov. 1983 - Sept. 1988. For each of the years 1984-1986, the annual minimum and maximum price ratios (A_2, B_2) were estimated by averaging the corresponding monthly ratios.

For the years 1980 - 1983, comparable monthly gas/oil price data were not available, except for Nov. and Dec. 1983. Hence use was made of regression analysis to estimate the values of the maximum and minimum gas/oil price ratios. The monthly minimum and maximum gas/oil prices ratios for the period Nov. 1983 to Sept. 1988 were regressed against the monthly Index of Industrial Production. This Index of Industrial production was then used to calculate the monthly minimum and maximum gas/oil price ratios for the period 1980 - Oct. 1983. For the years 1980 - 1983, A_2 and B_2 were estimated by averaging the respective monthly estimates.

Determination of the AR bounds for input virtual multipliers:

For each year 1980 - 1986, the minimum reserve value and also the maximum reserve value was found from the data of Arthur Anderson for all the firms analyzed; and these minimum and maximum values were used to estimate α_2, β_2, respectively. Pair wise, the bounds A_2, B_2 were multiplied times α_2, β_2, respectively to estimate α_3, β_3. Similarly, for each year, the onshore and offshore drilling cost data represented the lower- and upper bounds for the range of observed costs per well. The range endpoints were used to estimate/specify α_4, β_4, respectively, year by year. The variable production cost x_1 was measured in dollars and the value of its multiplier v_1 was specified to be 1 because a dollar of labor cost is worth a dollar.

2.4.3.6 DEA/AR Analysis of the 1988 - 1989 Performance of the Nanjing Textiles Corporation - Zhu (1996)

This article employed the data envelopment analysis/assurance region (DEA/AR) methods to evaluate the efficiency of the 35 textile factories of the Nanjing Textiles Corporation (NTC), Nanjing China. By specifying input and output cones, a cone-ratio assurance region (CR-AR) was set up. While most existing approaches involving Assurance Regions use "price/cost" data to determine values of the bounds, that approach was not used in this paper because the concepts of price and cost could not be used in Chinese economic planning. The problem being that the prices of many important industrial raw materials and products, and necessities, etc., were controlled by the government and were fixed at certain levels for relatively long periods (e.g. five years or more).

Therefore, in this article, AR's were developed based on expert opinions on the relative importance between various inputs/outputs. NTC uses the Analytic Hierarchy Process (AHP) to gather and present expert opinion for systematically evaluating the overall industrial performance. The results from the AHP were used in this paper to set bounds on the weights. Two CR-ARs (CR-AR1 and CR-AR2) were developed to reflect two different economies (central planning and market) as China was transitioning from central planning economies to a mixture of central planning and market economies. CR-AR1 reflected the evaluation under the assumption of central planning economies which laid more emphasis on net industrial output value while CR-AR2 reflected the evaluation under the assumption of market economies which laid more emphasis on profit/taxes and revenue.

The two AR's were useful in studying the effect of change in the economic conditions on the performance of the factories and also in identifying the factories, which were more flexible than others in adopting the change. Thus, the CR-ARs not only refined the DEA efficiency results but also could be used to analyze the textiles industrial behavior in the face of evolving market economies in Chinese economic reforms.

2.4.3.7 Best Practice Analysis of Bank Branches: An Application of DEA in a Large Canadian Bank - Schaffnit *et al*. (1997)

This paper presents a best practice analysis of the Ontario based branches of a large Canadian bank. The analysis was focused on the performance of branch personnel. To sharpen the efficiency estimates, constraints were imposed on the output multipliers. To find cost efficient branches, i.e. to measure allocative efficiency, a model with similar constraints on the input multipliers was used. The inputs considered in the analysis were the number of personnel of each type (there were total five types) working in the branches and the outputs were the number of transactions and number of maintenance activities of each type. The values of the average standard times for all output activities were used for setting the AR bounds on the output multipliers. Management estimated that the large majority of transaction and maintenance activities fell within a ± 25% range of the standard times. Using this information, the upper and lower time bounds were determined as follows: for each output y_r. r = 1, …, s with standard time $\bar{t}_r$, upper and lower time bounds were $t_r^{\pm} = (1 \pm p)\bar{t}_r$, with p = 25%. From this, the following sets of s*(s-1) / 2 constraints in the output multipliers were obtained:

$$\frac{t_j^-}{t_r^+} \le \frac{\mu_j}{\mu_r} \le \frac{t_j^+}{t_r^-} \qquad r = 1, \ldots, s-1;\ j = r+1, \ldots, s \tag{2.20}$$

To study the cost-minimizing behavior of the branches, values or "prices" were introduced for each of the inputs. The salary range for each type of staff was used to set the bounds. The range of the salaries was estimated by management to be ± 20% of the average salaries. The input multiplier constraints were similar to the output multiplier constraints shown in (2.20). When these input multiplier constraints were added to the model with output multiplier constraints, the resulting model measured the overall efficiency. The ratio of the score given by this model to that given by the model with only output weight constraints measured the allocative efficiency.

2.4.3.8 Exploring Output Quality Targets in the Provision of Perinatal Care in England using Data Envelopment Analysis - Thanassoulis *et al*. (1995)

This paper explores the use of Data Envelopment Analysis to assess units providing perinatal care (District health Authorities, DHAs) in England. The input set for the DHAs consisted of five controllable inputs and one uncontrollable input. The controllable inputs were (for more details see Thanassoulis *et al*. (1995)):

- Whole Time Equivalent (WTE) obstetricians;
- WTE pediatricians;
- General Practitioner's (GP) fees;
- WTE midwives;
- WTE nurses;

The uncontrollable input was Number of babies at risk. This input was included because it was important for monitoring the survival rate of babies at risk.

The output set incorporated both activity levels and quality measures. The output set was classified into three categories:

1. Outputs related to activity levels (these were exogenously fixed):
    - Total number of birth episodes performed in the DHA;
    - Number of Deliveries to mothers resident in the DHA;
    - Number of special care consultant episodes;
    - Number of intensive care consultant episodes;
    - Number of abortions.

2. Outputs related to service quality:
   - Number of very satisfied mothers;
   - Number of satisfied mothers.
3. Output related to Quality of medical outcome:
   - Number of babies at risk surviving.

Conventional DEA model allows complete flexibility to the weights assigned to the inputs and outputs. However, this freedom of choice of weights was found unacceptable in this case because of the presence of output quality measures in the model. Therefore to incorporate information on relative importance of outputs and inputs, additional constraints were added to the original DEA model. Four alternative preference structures over weights on the input-output variables were applied. The structures in tabular form have been reproduced here:

Constraint Set	Constraints
I	No weight constraints imposed
II	$v_{Risk} = \mu_{\text{Survivals}}$ $\mu_{\text{Survivals}} \geq \mu_{\text{Deliveries to Resident mothers}}$ $\mu_{\text{Survivals}} \geq \mu_{\text{Abortions}}$ $\mu_{\text{Survivals}} \geq \mu_{\text{Delivery episodes in DHA}}$
III	$v_{Risk} = \mu_{\text{Survivals}}$ $\mu_{\text{Very satisfied}} \geq 1.5\mu_{\text{Satisfied}}$ $\mu_{\text{Satisfied}} \geq 1.5\mu_{\text{Deliveries to Resident mothers}}$ $\mu_{\text{Survivals}} \geq 1.25\mu_{\text{Very Satisfied}}$
IV	$v_{Risk} = \mu_{\text{Survivals}}$ $\mu_{\text{Very satisfied}} \geq 5\mu_{\text{Satisfied}}$ $\mu_{\text{Satisfied}} \geq 5\mu_{\text{Deliveries to Resident mothers}}$ $\mu_{\text{Survivals}} \geq 100\,\mu_{\text{Very Satisfied}}$ $\mu_{\text{Satisfied}} \geq 5\mu_{\text{Delivery episodes in DHA}}$

The first set allowed complete freedom to the input-output weights. The first constraint in the second set required that babies at risk (input) have the same weight as number of survivals (output). The logic behind equating the two weights was that those two factors jointly defined the survival rate of babies at risk, which was an important quality measure of medical outcome. The other constraints in set II were of the ordinal type and simply ensured that the weight on the number of survivals was at least equal to that on the other outputs. In sets III and IV, the preference information was further restricted to reflect the strength of the preferences. In both cases, the measures of quality (of both service and medical outcomes) were given more importance than the measures of outcome levels and amongst quality measures, the outcome quality was given a stronger emphasis than service quality. The difference between sets III and IV was that set IV represented a situation where strong information on preferences was available. The information on preferences and their strengths was gathered from expert opinion.

2.4.3.9 DEA and Primary Care Physician Report Cards: Deriving Preferred Practice Cones from Managed Care Service Concepts and Operating Strategies – Chilingerian, J.A. and H.D. Sherman (1997)

Chilingerian and Sherman (1997) used the assurance region model to spot inefficiencies in the practice patterns of primary care physicians (PCPs). The primary inputs were the number of primary care visits, number of medical/surgery visits, number of referrals to sub-specialists and the number of ambulatory surgery visits. Tradeoffs existed amongst these inputs. For example, for many patients, the office visits and ambulatory surgeries were substitutes for expensive hospitalizations. The preferred practice pattern for the physicians according to the director of the health maintenance organization (HMO) had two dimensions – financial and clinical. The financial dimension required the PCPs to operate within their budgets and the clinical dimension required them to use fewer than average hospital days and referrals, and provide neither too few, nor too many office visits. The average values of all the inputs were determined by analyzing one year of utilization data. The AR bounds (bounds on marginal rates of substitutions of the inputs) were determined by running the unbounded DEA model for the data. The ratios of input multipliers for PCPs who were on the efficient frontier in the unbounded run and satisfied certain conditions specified by the HMO director were used to set the bounds. Satisfying the HMO director's conditions implied operating within the budget, below the mean medical surgical days and mean referral rates, and at or above mean primary care visits.

Although a standard CCR model produces a proportional reduction of inputs, the purpose of the cone ratio model was to force more than a proportional reduction in hospital days to align the physician practice styles with the preferences of the HMO director. Large deviations from the unbounded to the bounded models helped the medical director to identify trouble spots in the primary care physicians who were practicing outside the preferred practice cone.

2.4.3.10 Computing DEA/AR Efficiency and Profit Ratio Measures with an Illustrative Bank Application – Thompson *et al*. (1996c)

Thompson *et al*. (1996c) solved the AR-DEA model for 48 banks for the years 1980 – 1990. The inputs and outputs included in the analysis were are follows:

Outputs

Y_1 – Total loans including commercial/industrial, installment, and real estate loans.

Y_2 – Total non-interest income.

Inputs

X_1 – Total labor in terms of number of employees

X_2 – Total physical capital in terms of book value of bank premises, furniture, and equipment.

X_3 – Total purchased funds including federal funds purchased, large (> $100k) certificates of deposits (CDs), foreign deposits, and other liabilities for borrowed money.

X_4 – Total number of branches, including main office.

X_5 – Total deposits including demand deposits, time and savings deposits, and small CDs.

The information that was used to determine the AR bounds is summarized in the following table:

Factor	Factor Multiplier	Values used in CR – AR
y_1	u_1	Interest rate on loans expressed as total interest income divided by total loans.
y_2	u_2	Both upper and lower bound equal 1, because a dollar of non-interest income is worth a dollar
x_1	v_1	Price of labor expressed as total salary plus employee benefits divided by total number of employees
x_2	v_2	Price of capital in user cost terms (cost of office space replacement).
x_3	v_3	Interest rate on purchased funds
x_4	v_4	Same as for x_2
x_5	v_5	Interest rate paid on deposits

Table 2.1 Information used to determine AR bounds in Thompson *et al*. (1996c)

Due to the application of the ARs, the number of efficient DMUs reduced significantly. In fact, the AR eliminated 90% of the extreme-efficient DMUs.

2.5 INTRODUCTION TO CONCEPTS OF FUZZY SET THEORY, FUZZY NUMBERS AND FUZZY LINEAR PROGRAMMING

This section provides an overview of the fuzzy mathematical programming approach, which has been used in this research. The following concepts are important for understanding the fuzzy mathematical programming approach.

2.5.1 Fuzzy Sets

The concept of fuzzy sets was first introduced by Zadeh (1965) to deal with the issue of uncertainty in systems modeling. Zadeh defined fuzzy sets as sets with boundaries that are not precise. "The membership in a fuzzy set is not a matter of affirmation or denial, but rather a matter of degree." The concept of fuzzy set theory challenged conventional two-valued logic as follows:

When A is a fuzzy set and x is a relevant object, the proposition "x is a member of A" is not necessarily either true or false, as required by two-valued logic, but it may be true only to some degree - the degree to which x is actually a member of A.

The degrees of membership in fuzzy sets are most commonly expressed by numbers in the closed unit interval [0,1]. Thus fuzzy sets express gradual transitions from membership (membership value of 1) to non-membership (membership value of 0) and vice versa.

Suppose X is a space of positive real values associated with a variable and x is a generic element of X. Mathematically, a fuzzy set A in X is defined as the set of ordered pairs:
$A = \{(x, \mu_A(x)) \mid x \in X\}$,
where $\mu_A: X \rightarrow M$ is the membership function and M is the membership space. M is usually assumed to vary in the interval [0,1].

2.5.2 Membership Functions

A membership function is a function which assigns to each element x of X a number, $\mu_A(x)$, in the closed unit interval [0,1] that characterizes the degree of membership of x in A. The closer the value of $\mu_A(x)$ is to one, the greater the membership of x in A. Thus, a fuzzy set A can be defined precisely by associating with each element$x$, a number between 0 and 1, which represents its grade of membership in A. The membership function of a fuzzy set A can also be represented as $A(x)$.

2.5.3 α-cut and Strong α-cut (Klir and Yuan (1995))

Given a fuzzy set A defined on X and any number $\alpha \in [0,1]$, the α -cut of the fuzzy set A is the crisp set ${}^{\alpha}A$ that contains all the elements of the universal set X whose membership grades in A are greater than or equal to the specified value of α.

Mathematically: ${}^{\alpha}A = \{x \mid A(x) \geq \alpha\}$

On the other hand, the strong α-cut of a fuzzy set A is the crisp set ${}^{\alpha+}A$ that contains all the elements of the universal set X whose membership grades in A are greater than the specified value of α.

Mathematically: ${}^{\alpha+}A = \{x \mid A(x) > \alpha\}$

2.5.3.1 Special Cases of α-cuts:

1-cut: The 1-cut of a fuzzy set A is the crisp set which contains all elements of X whose membership grades in A are equal to 1. The 1-cut is often called the core of A. Mathematically: ${}^{1}A = \{x \mid A(x) = 1\}$.

Support: The support of a fuzzy set A within a universal set X is the crisp set that contains all the elements of X that have nonzero membership grades in A. Clearly the support of A is exactly the same as the strong α-cut of A for α=0.

2.5.4 Intersection of Fuzzy Sets

The intersection of fuzzy set A and fuzzy set B is the largest fuzzy set contained in both A and B. Such a set is denoted A ∩ B. The membership function of A ∩ B, for all $x \in X$, can be given by:

$$A \cap B(x) = \min(A(x), B(x)) = A(x); \qquad if\ A(x) \le B(x)$$
$$= \min(A(x), B(x)) = B(x); \qquad if\ A(x) \ge B(x)$$

2.5.5 Fuzzy Numbers

Fuzzy sets that are defined on the set **R** of real numbers are called fuzzy numbers (Klir and Yuan (1995)). Membership functions of these sets have a quantitative meaning and are represented as:

A: $\mathbf{R} \to [0,1]$

The membership functions of fuzzy numbers tend to capture the intuitive conception of approximate numbers i.e. "numbers close to a given real number." Therefore, they are useful for characterizing states of fuzzy variables.

To qualify as a fuzzy number, a fuzzy set A on **R** must possess at least the following three properties:

1. A must be a normal fuzzy set as defined in section 2.5.1;
2. ${}^{\alpha}A$ must be a closed interval for every $\alpha \in (0,1]$;

3. The support of A, ^{0+}A, must be bounded.

The most commonly used shapes for fuzzy numbers are the triangular and trapezoidal. The triangular functions express the proposition "close to a real number r." The trapezoidal membership function represents a fuzzy interval. Graphically the triangular and trapezoidal membership functions are represented as follows:

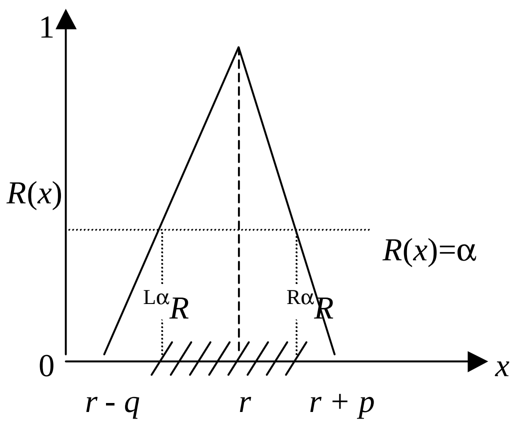

Figure 2.5 Triangular Fuzzy Number *R* "close to crisp number *r*"

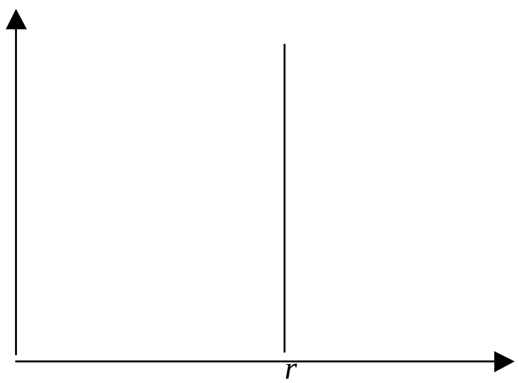

Figure 2.6 Crisp Number *r*

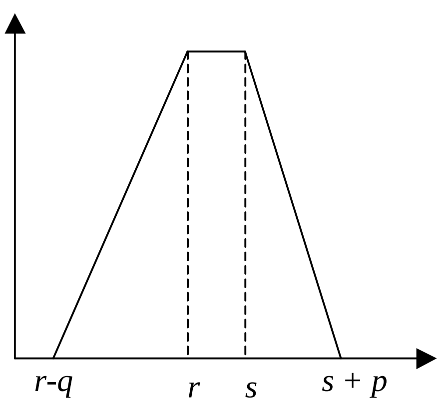

Figure 2.7 Fuzzy Interval - *r* - *s*

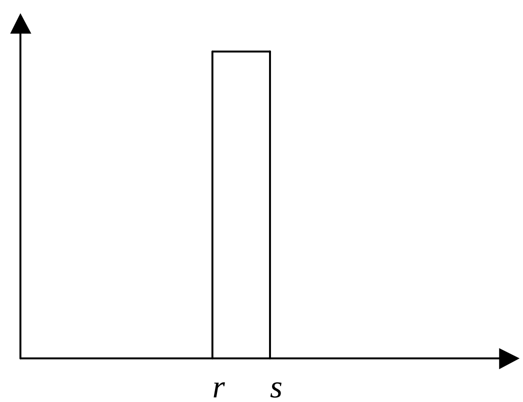

Figure 2.8 Crisp Interval *r* - *s*

2.5.5.1 α-Cuts of Fuzzy Numbers

The α-cut of a fuzzy number is a closed interval and is defined completely by specifying its ends. The ends of an α-cut are points of intersection of the line "membership degree = α" and the rightmost and leftmost lines in the graphical representation of the membership function of the fuzzy number. Refer to figure 2.5 where the α-cut of the fuzzy number *R* approximating the real number r is the crisp set (marked by sloping lines) containing x values between ${}^{L\alpha}R$ and ${}^{R\alpha}R$. ${}^{L\alpha}R$ is the left end of the α-cut given by the intersection of the line $R(x) = \alpha$ and the line $x = r - q + qR(x)$, representing the change in the membership function between $r - q$ and r. Similarly the right-end of the α-cut, ${}^{R\alpha}R$, is the intersection of the line $R(x) = \alpha$ and the line $x = r + p - pR(x)$. Therefore:

$${}^{\alpha}R = [{}^{L\alpha}R\ ,\ {}^{R\alpha}R]$$

$$= [r - q + \alpha q,\ r + p - \alpha p]$$

The 1-cut of R will be:

$${}^{1}R = [r, r]$$

The support R will be:

$${}^{0+}R = [r - q,\ r + p]$$

Note that the 1-cut of the fuzzy number contains only its most desirable element (r), while the support contains all elements belonging to the fuzzy number. Obviously, the ends of the support are the least desirable elements of the fuzzy number. This concept will be utilized later in the discussion in Chapter 3.

2.5.5.2 Arithmetic Operations on Fuzzy Numbers

Fuzzy arithmetic is based on two properties of fuzzy numbers (Klir and Yuan (1995)):
Each fuzzy number can fully and uniquely be represented by its α- cuts and;
All α-cuts ($\alpha \in [0,1]$) of each fuzzy number are closed intervals of real numbers.

Arithmetic operations on fuzzy numbers are therefore defined in terms of arithmetic operations on their α-cuts i.e. arithmetic operations on closed intervals.
Let fuzzy numbers A and B be represented in terms of their α-cuts as:

$${}^{\alpha}A = [a, b]$$

$${}^{\alpha}B = [c, d]$$

In general, if * represents an arithmetic operation between two fuzzy numbers, then we define a fuzzy set $A*B$ on **R** by defining its α-cut ${}^{\alpha}(A*B)$ as

$^{\alpha}(A*B) = {}^{\alpha}A * {}^{\alpha}B$

for any $\alpha \in (0,1]$.

Since $^{\alpha}$(A*B) is a closed interval for each $\alpha \in (0,1]$ and A, B are fuzzy numbers, A*B is also a fuzzy number.

In terms of α-cuts, the four arithmetic operations on the fuzzy numbers A & B would then be defined as follows:

Addition:

$^{\alpha}(A + B) = {}^{\alpha}A + {}^{\alpha}B$

$= [a, b] + [c, d]$

$= [a + c, b + d]$

Subtraction:

$^{\alpha}(A - B) = {}^{\alpha}A - {}^{\alpha}B$

$= [a, b] - [c, d]$

$= [a - c, b - d]$

Multiplication:

$^{\alpha}(A*B) = {}^{\alpha}A * {}^{\alpha}B$

$= [a, b]*[c, d]$

$= [\min(ad, ac, bd, bc), \max(ad, ac, bd, bc)]$

Division:

$^{\alpha}(A/B) = {}^{\alpha}A / {}^{\alpha}B$

$= [a, b]/[c, d]$

$= [\min(a/d, a/c, b/d, b/c), \max(a/d, a/c, b/d, b/c)]$

$c, d \neq 0$

2.5.5.3 Lattice of Fuzzy Numbers

Klir and Yuan (1995) define the MIN and MAX operations on fuzzy numbers A & B as follows:

$$\mathrm{MIN}\ (A, B)(z) = \sup_{z=\min(x,y)} \min\ [A(x),B(y)]$$

$$\mathrm{MAX}\ (A, B)(z) = \sup_{z=\max(x,y)} \min\ [A(x),B(y)]$$

for all z ε **R**

2.5.5.4 Partial Ordering of Fuzzy Numbers

Klir and Yuan (1995) define the partial ordering between those two numbers as

$A \leq B$ iff ${}^{\alpha}A \leq {}^{\alpha}B$ for all $\alpha \in (0,1]$.

$A \geq B$ iff ${}^{\alpha}A \geq {}^{\alpha}B$ for all $\alpha \in (0,1]$,

where the partial ordering of closed intervals is defined in the following way:

$[a_1, a_2] \leq [b_1, b_2]$ iff $a_1 \leq b_1$ and $a_2 \leq b_2$,

$[a_1, a_2] \geq [b_1, b_2]$ iff $a_1 \geq b_1$ and $a_2 \geq b_2$,

2.5.5.5 Fuzzy Relations between Real Numbers and Fuzzy Numbers

While comparing a real number with a fuzzy number, we cannot say, like in the crisp case, that one is strictly greater than the other. A real number can be greater (or smaller) than a fuzzy number to only a certain degree. In this research, we define this degree in the following way:

The fuzzy relation $ax \,\tilde{\leq}\, B$ (where B is a fuzzy number and ax is a real number) will be defined in the following way:

ax satisfies $ax \,\tilde{\leq}\, B$ to a degree equal to λ iff $\lambda = \sup\{\alpha : ax \leq {}^{R\alpha}B\}$.

Similarly, the fuzzy relation $ax \,\tilde{\geq}\, C$ (C is a fuzzy number) will be defined in the following way:

ax satisfies $ax \,\tilde{\geq}\, C$ to a degree equal to λ iff $\lambda = \sup\{\alpha : ax \geq {}^{L\alpha}C\}$.

The following statements follow from the above definitions:

1. $ax \,\tilde{\leq}\, B$ is satisfied to a degree greater than or equal to λ if $ax \leq {}^{R\lambda}B$.
2. $ax \,\tilde{\geq}\, C$ is satisfied to a degree greater than or equal to λ if $ax \geq {}^{L\lambda}C$.

2.5.6 Fuzzy Decision-making

According to Zimmerman (1996) a decision is characterized by:

- A set of decision alternatives (the decision space). The decision space can be described by enumeration or be defined by a number of constraints.
- A set of states of nature (the state space);
- A relation assigning to each pair of a decision and state a result;
- A utility function or objective function that orders the decision space via the one-to-one relationship of results to decision alternatives.

Bellman and Zadeh (1970) suggest a model for decision making in a fuzzy environment. They consider a situation of decision making in which the objective function as well as the constraint(s) are fuzzy. The fuzzy objective function and the fuzzy constraints are both characterized by their membership functions. Since we want to satisfy the objective function as well as the constraints, a decision in a fuzzy environment is defined as the selection of activities that simultaneously satisfy the objective function "*and*" the constraints. In other words, decision making in a fuzzy environment seeks a compromise between satisfying the objective function and satisfying the constraints.

Assuming that the constraints are non-interactive (independent), the logical *"and"* corresponds to intersection. Thus a fuzzy decision can be viewed as the intersection of fuzzy constraints and fuzzy objective function. We see that the relationship between objective functions and constraints is fully symmetric because both can be represented using membership functions. The relationship would have been unsymmetrical if one of them was not expressed as a membership function.

A formal definition of a decision in a fuzzy environment stated by Bellman and Zadeh (1970) is as follows:
Assume that we are given a fuzzy goal[8] $\tilde{G}$ and a fuzzy constraint $\tilde{C}$ in a space of alternatives X. Then $\tilde{G}$ and $\tilde{C}$ combine to form a decision $\tilde{D}$, which is a fuzzy set resulting from intersection of $\tilde{G}$ and $\tilde{C}$. In symbols, $\tilde{D} = \tilde{G} \cap \tilde{C}$ and correspondingly, $D(x) = \min \{G(x), C(x)\}$.

[8] A Goal is a broader notion than an Objective Function

More generally, suppose that we have n goals $\tilde{G}_1,...,\tilde{G}_n$ and m constraints $\tilde{C}_1,...\tilde{C}_m$. Then the resultant decision is the intersection of the given goals $\tilde{G}_1,...,\tilde{G}_n$ and the given constraints $\tilde{C}_1,...\tilde{C}_m$. That is,

$$\tilde{D} = \tilde{G}_1 \cap \tilde{G}_2 \cap ... \cap \tilde{G}_n \cap \tilde{C}_1 \cap \tilde{C}_2 \cap ... \cap \tilde{C}_m$$

and correspondingly.

$$D(x) = \min(G_1(x), G_2(x),..., G_n(x), C_1(x), C_2(x),..., C_m(x))$$

$$= \min \{G_i(x), C_j(x)\}$$

$$= \min \{A_i(x)\}$$

Where $A_i(x)$ is a generalized representation for the membership functions of goals and constraints.

According to Zimmerman (1996), the above definition implies essentially three assumptions:

1. The "and" connecting the goals and the constraints in the model corresponds to the "logical and".
2. The logical "and" corresponds to the set-theoretic intersection.
3. The intersection of fuzzy sets is defined by the "min"-operator.

Bellman and Zadeh (1970) indicated that the min-interpretation of the intersection might have to be modified depending upon the context. Therefore, they stated the following broad definition of the concept of decision: "Decision = Confluence of Goals and Constraints."

2.5.7 Fuzzy Linear Programming

Linear programming models are special kinds of decision models where the decision space is defined by linear constraints and the "goal" is defined by a linear objective function.

A typical linear programming model (Bazaraa *et al*. (1990)) is expressed as follows: Find x which:

$$\begin{aligned} &\text{maximizes } f(x) = c^T x \\ &\text{subject to } Ax \le b \\ &\qquad\qquad x \ge 0 \\ &\text{with } c, x \in R^n, b \in R^m, A \in R^{mxn} \end{aligned} \qquad (2.21)$$

where $f(x) = c^T x$ defines the objective function, $Ax \le b$ the constraints, and $x \ge 0$, the decision variables. $c = (c_1, c_2, \ldots, c_n)$ is known as the revenue coefficient vector, $x = (x_1, x_2, \ldots, x_n)$ as the vector of decision variables, $b = (b_1, b_2, \ldots, b_m)$ as the right-hand-side (resource) vector, and A = $[a_{ij}]$ as the n x m constraint matrix. The a_{ij} elements of A are called technological coefficients.

The above classical model makes the following assumptions:

- All the coefficients A, b, and c are crisp numbers,
- $\le$ is meant in a crisp sense,
- "Maximize" is a strict imperative.

If the classical linear program in (2.21) is used to model decisions in a fuzzy environment, Zimmerman (1996) suggests quite a number of possible modifications to it. Firstly, the decision-maker might not want to actually maximize or minimize the objective function. He/she might just be interested in "improving the present cost situation." Therefore, he/she might end up specifying some aspiration levels for the objective function that may not be definable crisply.

Secondly, the constraints might be vague in one of the following ways:

- The constraints may represent aspiration levels or sensory requirements that cannot adequately be approximated by a crisp constraint. The $\le$ sign may not be meant in the strictly mathematical sense and smaller violations might well be acceptable.
- The coefficients of the vectors b or c or of the matrix A can have fuzzy character either because they are fuzzy in nature or because the perception of them is fuzzy.

Finally, the decision-maker might attach different degrees of importance to violations of different constraints. As a result, the role of the constraints in fuzzy linear programming can be different from that in classical linear programming, where the violation of any single constraint by any amount renders the solution infeasible.

2.5.7.1 Types of Fuzzy Linear Programming Models

In contrast to classical linear programming, "fuzzy linear programming" is not a uniquely defined type of model and many variations are possible, depending on the assumptions or features of the real situation being modeled. In this thesis, we use two types of fuzzy LP models:

1. Zimmerman's (1996) basic fuzzy LP models which can be either symmetric or unsymmetrical and
2. Fuzzy models with fuzzy coefficients of the matrix A.

Depending upon whether the objective function is crisp or fuzzy, Zimmerman (1996) classifies his basic fuzzy LPs into the following two types:

- Symmetric Fuzzy LP where both the objective function and the constraints are fuzzy.
- Unsymmetrical Fuzzy LP where the constraints are fuzzy but the objective function is crisp.

2.5.7.1.1 Symmetric Fuzzy LP (Zimmerman (1996))

In this model, it is assumed that the decision maker can establish an aspiration level, z, for the value of the objective function and that each of the constraints is modeled as a fuzzy set. The fuzzy LP then becomes:

Find *x* such that

$$
\begin{aligned}
&c^T x \mathrel{\tilde{\geq}} \mathrm{z} \\
&Ax \mathrel{\tilde{\leq}} b \qquad\qquad (2.21)\\
&x \geq 0
\end{aligned}
$$

Here the relation $\tilde{\leq}$ [9] denotes the fuzzified version of $\leq$ and has the linguistic interpretation "the real number on the LHS is essentially smaller than or equal to the real number on the RHS." The relation $\tilde{\geq}$ [10] denotes the fuzzified version of $\geq$ and has the linguistic interpretation "the real number on the LHS is essentially greater than or equal to the real number on the RHS". Model (2.22) is fully symmetric with respect to objective function and constraints. This can be made more obvious by substituting

$\begin{pmatrix} -c \\ A \end{pmatrix} = B$ and $\begin{pmatrix} -z \\ b \end{pmatrix} = d$. After making these substitutions, model (2.22) becomes:

Find x such that

$$\begin{aligned} &Bx \tilde{\leq} d \\ &x \geq 0 \end{aligned} \qquad (2.23)$$

Each of the (m+1) rows of model (2.23) shall now be represented by the fuzzy set $\mu_i(x)$. $\mu_i(x)$ can be interpreted as the degree to which x fulfills (satisfies) the fuzzy inequality $B_i x \tilde{\leq} d_i$ (where B_i denotes the i^{th} row of B).

Zimmerman assumes $\mu_i(x)$ to take a value 0 if the constraints (or the objective function) are strongly violated and a value 1 if they are very well satisfied i.e. satisfied in the crisp sense. The values between 0 and 1 represent the "in between" satisfaction.

$$\mu_i(x) = \begin{cases} 1 & \text{if } B_i x \leq d_i \\ \in [0,1] & \text{if } d_i < B_i x \leq d_i + p_i \\ 0 & \text{if } B_i x > d_i + p_i \end{cases} \qquad (2.24)$$

where p_i are subjectively chosen constants of admissible violations of the constraints and the objective function. Zimmerman (1996) assumes that the membership function of the fuzzy set corresponding to constraint (or objective function) i increases linearly over the "tolerance interval" $[d_i, d_i + p_i]$ and is given by:

[9&6] These fuzzy relations are different from those defined in Section 2.5.5.5. The relations in 2.5.5.5 compare real numbers with fuzzy numbers whereas the relations in (2.21) compare two real numbers. We use the same notations for both types of relations but their definitions are clearly different.

$$\mu_i(x) = \begin{cases} 1 & \text{if } B_i x \le d_i \\ 1 - \dfrac{B_i x - d_i}{p_i} & \text{if } d_i < B_i x \le d_i + p_i \\ 0 & \text{if } B_i x > d_i + p_i \end{cases} \tag{2.25}$$

The membership function of the fuzzy set "decision" of model (2.23) is equal to the intersection of the fuzzy sets μ_i and is given by.

$$D(x) = \bigcap_{i=1}^{m+1} \mu_i = \min\{\mu_i(x)\} \tag{2.26}$$

Since the decision-maker is interested not in a fuzzy set but in a crisp "optimal" solution, Zimmerman (1996) suggests finding the "maximizing solution" to equation (2.26). The maximizing solution to (2.26) would be the solution to the following problem:

$$\text{Max } D(x) = \text{Max min } \{\mu_i(x)\} \tag{2.27}$$

Replacing $D(x)$ by a new variable λ, we arrive at the following aggregate model:

$$\begin{aligned} &\text{maximize } \lambda \\ &\text{such that} \quad \lambda p_i + B_i x \le d_i + p_i \\ &\qquad\qquad\quad \lambda \le 1 \\ &\qquad\qquad\quad x \ge 0 \end{aligned} \tag{2.28}$$

The aggregate model (2.28) is a problem of finding a point (say x_0), which satisfies all the constraints and the goal (objective function) with the maximum degree. The point x_0 is the maximizing solution of model (2.22).

2.5.7.1.2 Unsymmetrical Fuzzy LP (Fuzzy LP with Crisp Objective Function)

If the objective function HAS to be either maximized or minimized, it is considered crisp. A model, in which the constraints are fuzzy and the objective function is crisp, is no longer symmetric because the constraints and the objective function play different roles (Zimmerman (1996)). The former define the decision space and the latter induces an order of decision alternatives just like in classical LP models. Therefore, the approach used for arriving at the solution in the symmetric case is not applicable here. To arrive at a solution in unsymmetrical models, we need to somehow aggregate the crisp objective function with fuzzy constraints. For that purpose, Zimmerman (1996) proposes determining an extremum of the crisp function over a fuzzy domain. To determine the extremum of the objective function, we use the notion of "maximizing set" introduced by Zadeh (1972). After the maximizing set for the objective function is determined, the model becomes symmetric and can be solved like the symmetric case by determining a "maximizing solution." Let us now digress a little and understand the concepts of extremum of fuzzy functions and maximizing sets.

Traditionally, the extremum (maximum or minimum) of a crisp function f over a domain D is attained at the same point x_0 at which the function achieves an optimal value when it is the objective function of a decision model. The point x_0 in the latter case is called the "optimal decision." Thus, in classical theory, there is an almost unique relationship between the extremum of a function and the notion of optimal decision of the decision model. However, in case of fuzzy models, this unique relationship does not exist (Zimmerman (1996)). According to Bellman and Zadeh (1970, p.150), "In decision models, the optimal decision is often considered to be the crisp set, D_m, that contains those elements of the fuzzy set decision attaining the maximum degree of membership." When considering functions in general (not as part of a decision model), the concept of a "maximizing set" is equivalent to the notion of an optimal decision defined above.

Zadeh (1972) provides the following definition for the maximizing set:

Let f be a real-valued function in X. Also, let f be bounded from below by inf (f) and from above by sup (f). The fuzzy set $\tilde{M} = \{(x, M(x)\}, x \in X$ where

$$M(x) = \frac{f(x) - \inf\,(f)}{\sup\,(f) - \inf\,(f)} \qquad (2.29)$$

is called the *maximizing set*.

where

- sup stands for supremum (upper bound or maximum);
- inf stands for infrenum (lower bound or minimum)

Let us now return to our unsymmetrical fuzzy LP. Using the concept of maximizing set, Werners (1984) provides the following definition for the membership function of the goal (objective function) given a fuzzy solution space:

Let f: X $\rightarrow \mathbf{R}^1$ be the objective function, $\tilde{R}$ = fuzzy feasible region, S($\tilde{R}$) = support of $\tilde{R}$, and R_1=1-cut of $\tilde{R}$. Then the membership function of the goal (objective function) given solution space $\tilde{R}$ is defined as

$$G(x) = \begin{cases} 0 & \text{if } f(x) \le \sup\limits_{R_1} f \\ \dfrac{f(x) - \sup\limits_{R_1} f}{\sup\limits_{S(\tilde{R})} f - \sup\limits_{R_1} f} & \text{if } \sup\limits_{R_1} f < f(x) < \sup\limits_{S(\tilde{R})} f \\ 1 & \text{if } \sup\limits_{S(\tilde{R})} f \le f(x) \end{cases} \tag{2.30}$$

where

- $\sup\limits_{R_1} f$ represents the supremum of f over R_1 (1-cut of the fuzzy region);
- $\sup\limits_{S(\tilde{R})} f$ represents the supremum of f over S($\tilde{R}$) (the support of the fuzzy region).

S ($\tilde{R}$) includes all possible values in a fuzzy number (or set) while R_1 includes only those values which belong to the set to a degree of 1. Therefore, S ($\tilde{R}$) encompasses the largest possible area of the fuzzy constraint space whereas R_1 encompasses the smallest area possible. This implies that S ($\tilde{R}$) represents constraints when they are most relaxed and R_1 represents the constraints when they are most restrictive. Obviously, if the objective function is of the maximization type, then its value over S ($\tilde{R}$) is the highest value possible (upper bound) and its value over R_1 is the lowest value possible (lower bound). This explanation makes it clear why Werners' (1984) definition of maximizing set in (2.30) is equivalent to Zadeh's (1972) definition of it in (2.29).

Zimmerman (1996), leverages Werners' (1984) definition of maximizing set for determining the solution to the unsymmetrical fuzzy LP. To illustrate Zimmerman's (1996) approach, let us consider an unsymmetrical fuzzy LP model having a crisp objective function, some crisp constraints and some fuzzy constraints.

$$\begin{array}{ll} \text{maximize} & f(x) = c^T x \\ \text{such that} & \left.\begin{array}{l} Ax \mathrel{\tilde{\leq}} b \\ Dx \leq b' \\ x \geq 0 \end{array}\right\} \tilde{R} \end{array} \tag{2.31}$$

The fuzzy sets corresponding to the fuzzy constraints will be again:

$$\mu_i(x) = \begin{cases} 1 & \text{if } A_i x \leq b_i \\ 1 - \dfrac{B_i x - b_i}{p_i} & \text{if } b_i < A_i x \leq b_i + p_i \\ 0 & \text{if } A_i x > b_i + p_i \end{cases} \tag{2.32}$$

$\mu_i(x)$ is the degree to which x satisfies the i^{th} (i=1,…, m) constraint. The intersection of these fuzzy sets $\bigcap_{i=1}^{m} \mu_i$, is a *fuzzy feasible set*.

The membership function of the objective function can be determined by solving the following two crisp LPs:

$$\begin{array}{ll} \text{maximize} & f(x) = c^T x \\ \text{such that} & Ax \leq b \\ & Dx \leq b' \\ & x \geq 0 \end{array} \tag{2.33}$$

yielding $\sup_{R_1} f = (c^T x)_{opt} = f_1$; and

$$\begin{array}{ll} \text{maximize} & f(x) = c^T x \\ \text{such that} & Ax \leq b + p \\ & Dx \leq b' \\ & x \geq 0 \end{array} \tag{2.34}$$

yielding $\sup_{S(\tilde{R})} f = (c^T x)_{opt} = f_0$

The membership function of the objective function using Werners' (1984) definition (2.30) is:

$$G(x)=\begin{cases}1 & \text{if} \quad f_0 \le c^T x \\ \dfrac{c^T x - f_1}{f_0 - f_1} & \text{if} \quad f_1 < c^T x < f_0 \\ 0 & \text{if} \quad c^T x \le f_1\end{cases} \tag{2.35}$$

Then the maximizing solution to the problem involves finding x such that

$$[(\bigcap_{i=1}^{m} \mu_i \cap G](x)$$

reaches the maximum value; that is, a problem of finding a point which satisfies the constraints and goal with the maximum degree. Now (2.31) becomes the following classical optimization problem:

$$\begin{array}{lll} \text{maximize} & \lambda & \\ \text{such that} & \lambda(f_0 - f_1) - c^T x \le -f_1 & \\ & \lambda p \quad + Ax \le b + p & \\ & Dx \le b' & \\ & \lambda \quad \le 1 & \\ & \lambda, x \ge 0 & \end{array} \tag{2.36}$$

2.6 FUZZY DEA

Sengupta (1992) was the first to explore the use of fuzzy set theory in DEA. He used the concepts of fuzzy linear programming to fuzzify the objective function and the constraints of a CCR model under conditions of imprecise data. Following Sengupta's (1992) work, there has been considerable research in the fuzzy DEA field. Triantis and Girod (1998) modified the radial DEA model and the FDH model to incorporate imprecision in measurement of data (i.e. values of inputs and outputs). Triantis (1999) fuzzified the non-radial DEA models to incorporate imprecise data. Sheth (1999) developed a fuzzy GoDEA model, which uses goal programming to solve the DEA problem in a fuzzy environment.

During the review of fuzzy Linear Programming (LP), it was mentioned that LP's can be fuzzified in two ways. One approach is to represent the fuzzy objective function and each of the fuzzy constraints by fuzzy sets. The other approach involves replacing the coefficients A, b and c of the LP by fuzzy sets. To deal with the issue of imprecise data in DEA, Sengupta (1992) uses the former approach while Triantis and Girod (1998) use the latter approach. We will now review the fuzzy DEA model proposed by Sengupta (1992) and the fuzzy radial model with imprecise production plans proposed by Triantis and Girod (1998). The review of the fuzzy FDH model (Triantis and Girod (1998)), the fuzzy non-radial model (Triantis 1999) and the fuzzy GoDEA model (Sheth 1999) is beyond the scope of this document.

2.6.1 Sengupta's (1992) Fuzzy DEA model

Sengupta (1992) proposes two approaches for solving DEA models, which have input-output data subject to inadequate knowledge. One is the stochastic approach i.e. the one that uses a probabilistic efficiency frontier. The other is the fuzzy systems approach. If one assumes that the imprecise data is generated by a stochastic generating mechanism, it seems logical to use stochastic DEA models with a probabilistic efficiency frontier. However there are some drawbacks associated with using probabilistic efficiency frontier:

1. One has to assume a specific error distribution e.g. normal, exponential, etc. to compute specific results and this assumption may not be realistic because on *a priori* basis there is very little empirical evidence to choose one type of distribution. In addition, the normal distribution cannot be used due to non-negativity restrictions on the input-output space.
2. The lack of robustness of the stochastic efficiency frontier and the probabilistic feasibility of the inequality constraints of the DEA model, cause problems.
3. Stochastic DEA models always emphasize point solutions whereas from the point of view of carrying out a data sensitivity analysis, one would be more interested in DEA models, which give interval solutions.
4. Because the sample sizes in DEA are small, it becomes difficult to use the stochastic models.

Sengupta (1992) proposes using the fuzzy mathematical programming approach in DEA problems with imprecise data because of the above disadvantages associated with stochastic methods and the following advantages of the fuzzy systems approach:

1. Fuzzy set theory allows us to apply the "principle of incompatibility," which has the ability to arrive at decisions based on qualitative data.
2. Fuzzy set theory lends itself to be incorporated in LP models. Since DEA involves solving a series of LP models, it is more robust to apply the methods of fuzzy mathematical programming and determine an optimal solution under conditions of inadequate knowledge.
3. By using suitable membership functions, the stochastic transformations of the DEA model may be given a fuzzy programming interpretation, which may be more robust in suitable cases.

Sengupta (1992) proposes two types of membership functions for the fuzzy mathematical programming model - Linear Membership function and Non-linear Membership function. In this review we will only look at the model with linear membership function. For the model with nonlinear membership functions, the reader is referred to Sengupta (1992). In the linear case, the DEA model is written as:

$$\begin{aligned} &\underset{\sim}{\min}\; X'_0 \beta \\ &\text{s.t.} \\ &X'_j \beta \stackrel{\sim}{\geq} y_j, \qquad j \in I_n \\ &\beta \geq 0 \end{aligned} \tag{2.37}$$

where X_j' is the vector of inputs for DMU j and y_j is the output of DMU j. The notation ~ indicates fuzziness in both the objective function and the n constraints. By making the constraints fuzzy, we accept tolerances in their realization. Sengupta (1992) assumes that it is possible to specify an aspiration level (g_0) for the efficiency score. He also assumes that it is possible to specify maximal levels of tolerance violations for the constraints (d_j, $j \in I_n$) and the efficiency score (d_0). Using the information on aspiration level, equation (2.37) can be rewritten as:

$$
\begin{aligned}
& X'_0\,\beta \,\tilde{\leq}\, g_0 \\
& \text{s.t.} \\
& X'_j\,\beta \,\tilde{\geq}\, y_j, \qquad j \in I_n \\
& \beta \geq 0
\end{aligned}
\tag{2.38}
$$

Using the information on maximal levels of tolerance violations, the membership functions of the fuzzy sets corresponding to all the constraints can be written as:

$$
\mu_j(\beta) = 1 - \frac{y_j - X'_j\,\beta}{d_j}, \qquad j \in I_n \tag{2.39}
$$

Similarly the membership function for the fuzzy objective function will be given by:

$$
\mu(\beta) = 1 - \frac{X'_0\,\beta - g_0}{d_0} \tag{2.40}
$$

The decision problem (2.37) is then to find a solution vector β, which maximizes the membership function of the decision, which is given by:

$$
\lambda = \bigcap_{j=1}^{n} \mu_j(\beta) \cap \mu_0(\beta) = \bigcap_{j=0}^{n} \mu_j(\beta) = \min_{j=0}^{n}[\mu_j(\beta)], \qquad \beta \in \mathrm{R}^{\mathrm{m}}
$$

The solution can be reformulated as an LP model:

$$
\begin{aligned}
& \max \lambda \\
& \text{s.t.} \quad \lambda d_0 + X'_0\,\beta \leq g_0 + d_0 \\
& \qquad \lambda d_j \leq d_j + X'_j\,\beta - y_j, \qquad j \in I_n, \\
& \qquad 0 \leq \lambda \leq 1, \qquad \beta \geq 0
\end{aligned}
\tag{2.41}
$$

Sengupta (1992) proposes to parametrically analyze the sensitivity of the optimal β* = β* ($d_0, d_1, \ldots, d_n$) to tolerance variations. It is clear that there will always exist an optimal solution β* (d) for some tolerance vector d.

2.6.2 Triantis and Girod (1998) Radial DEA Models with Fuzzy Production Plans

Triantis and Girod (1998) proposed an approach that uses traditional data envelopment analysis framework and then merges concepts developed in fuzzy parametric programming by Carlsson and Korhonen (1986). Traditional technical efficiency studies assume that production plans (input output data) are known precisely. This may not always be the case. The approach proposed in this paper considers production plans that are not crisp but fuzzy. Since the input-output data values appear as coefficients in the constraints and the objective function (i.e. matrix A and vector c) of the DEA model, the fuzzy approach proposed in this paper is the one of replacing the coefficients by fuzzy sets.

The approach presupposes that the decision-maker can define the risk free and impossible bounds for each fuzzy input and output. Risk-free bounds are the conservative values that are most realistically attainable in production, whereas impossible bounds are associated with those values, which represent production scenarios that are the least realistic. The risk-free and impossible bounds are used for determining the membership functions for the input and output data. All membership functions are assumed to vary linearly between the bounds. In addition, all membership functions have a value equal to zero at the impossible bounds and a value equal to one at the risk-free bounds.

If superscripts 0 and 1 represent impossible and risk-free bounds for input data, then the membership function associated with the i^{h} fuzzy input ($x_{i,h}$) for the k^{th} DMU is given by:

$$\mu_X(x_{i,h}) = \frac{x^0{}_{i,h} - x_{i,h}}{x^0{}_{i,h} - x^1{}_{i,h}} \qquad i = \{1,2,...,I\} \quad h = \{1,2,...,N\} \tag{2.42}$$

Further if superscripts 0 and 1 represent risk-free and impossible bounds for output data, then the membership function associated with the j^{th} output ($y_{j,h}$) for the h^{th} DMU is given by:

$$\mu_Y(y_{j,h}) = \frac{y_{j,h} - y^1_{j,h}}{y^0_{j,h} - y^1_{j,h}} \qquad j = \{1,2,...,J\} \qquad h = \{1,2,...,N\} \tag{2.43}$$

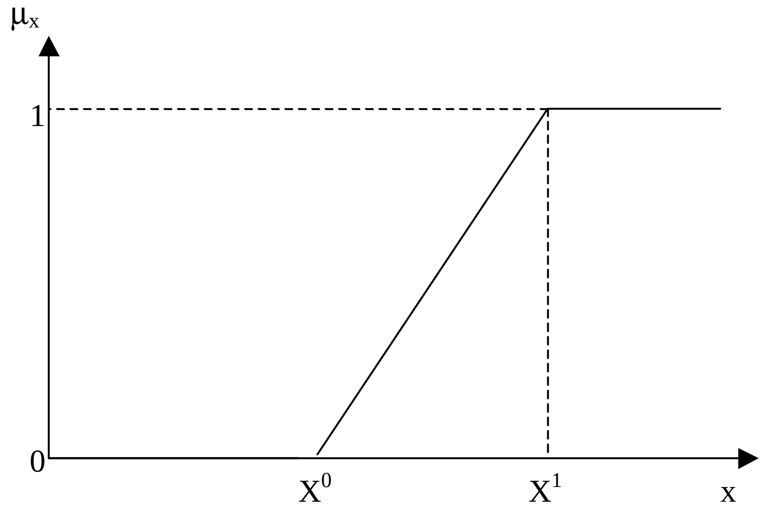

Figure 2.9 Input Membership Function used by Triantis and Girod (1998)

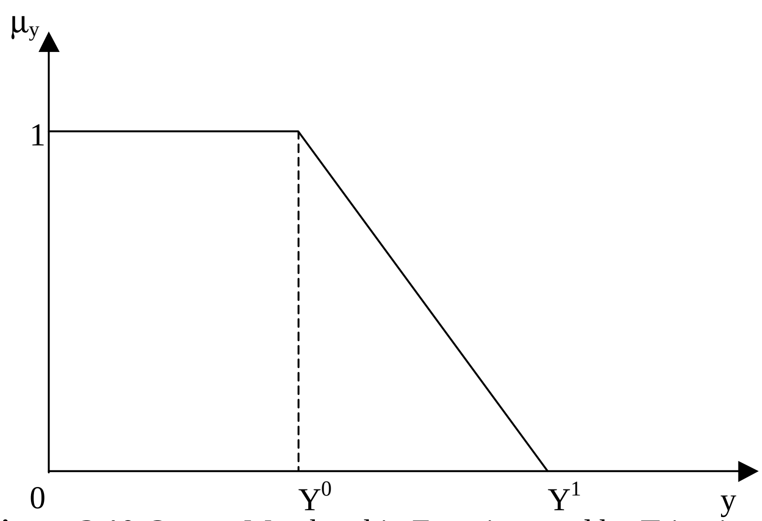

Figure 2.10 Output Membership Function used by Triantis and Girod (1998)

Both $x_{i,h}$ and $y_{j,h}$ can be expressed in terms of the risk-free and impossible bounds and the membership functions as follows:

$$
\begin{aligned}
x_{i,h} &= x^0_{i,h} - (x^0_{i,h} - x^1_{i,h})\mu_X \\
y_{j,h} &= (y^0_{j,h} - y^1_{j,h})\mu_Y + y^1_{j,h}
\end{aligned}
\qquad (2.44)
$$

Using these definitions, the original Charnes, Cooper and Rhodes (1978) model can be modified as follows: Find u and v such that,

$$\text{Max } \frac{\sum_{j=1}^{J} v_j((y_{j,p}^0 - y_{j,p}^1)\mu_Y + y_{j,p}^1)}{\sum_{i=1}^{I} u_i(x_{i,p}^0 - (x_{i,p}^0 - x_{i,p}^1)\mu_X)}$$

subject to :

$$\frac{\sum_{j=1}^{J} v_j((y_{j,h}^0 - y_{j,h}^1)\mu_Y + y_{j,h}^1)}{\sum_{i=1}^{I} u_i(x_{i,h}^0 - (x_{i,h}^0 - x_{i,h}^1)\mu_X)} \le 1 \qquad h = \{1,\ldots,N\} \qquad (2.45)$$

$$u_i \ge 0 \qquad i = \{1,\ldots,I\}$$
$$v_j \ge 0 \qquad j = \{1,\ldots,J\}$$

The fractional model (2.45) can be easily converted into a linear form (see Charnes and Cooper (1962)) as follows: Find vectors η and ω such that

$$\text{Max } \sum_{j=1}^{J} \omega_j((y_{j,p}^0 - y_{j,p}^1)\mu_Y + y_{j,p}^1)$$

subject to :

$$\sum_{i=1}^{I} \eta_i(x_{i,p}^0 - (x_{i,p}^0 - x_{i,p}^1)\mu_X) = 1 \text{ for each } h = \{1,\ldots,N\}$$

$$\sum_{j=1}^{J} \omega_j((y_{j,h}^0 - y_{j,h}^1)\mu_Y + y_{j,h}^1) - \sum_{i=1}^{I} \eta_i(x_{i,h}^0 - (x_{i,h}^0 - x_{i,h}^1)\mu_X) \le 0 \qquad (2.46)$$

for each $h = \{1,\ldots,N\}$

$$u_i \ge 0 \qquad i = \{1,\ldots,I\}$$
$$v_j \ge 0 \qquad j = \{1,\ldots,J\}$$

where $\omega_j = v_j(\sum_{i=1}^{I} u_i(x_{i,h}^0 - (x_{i,h}^0 - x_{i,h}^1)\mu_X))^{-1}$ and $\eta_i = u_i(\sum_{i=1}^{I} u_i(x_{i,h}^0 - (x_{i,h}^0 - x_{i,h}^1)\mu_X))^{-1}$

From Carlsson and Korhonen (1986), the decision for the above model is reached when $\mu_X = \mu_Y = \mu = \min(\mu_X, \mu_Y)$. The above equation can therefore be rewritten as:

$$
\begin{aligned}
&\text{Max} \sum_{j=1}^{J} \omega_j ((y_{j,p}^0 - y_{j,p}^1)\mu + y_{j,p}^1) \\
&\text{subject to :} \\
&\sum_{i=1}^{I} \eta_i (x_{i,p}^0 - (x_{i,p}^0 - x_{i,p}^1)\mu) = 1 \text{ for each } h = \{1,\ldots,N\} \\
&\sum_{j=1}^{J} \omega_j ((y_{j,h}^0 - y_{j,h}^1)\mu + y_{j,h}^1) - \sum_{i=1}^{I} \eta_i (x_{i,h}^0 - (x_{i,h}^0 - x_{i,h}^1)\mu) \le 0 \\
&\text{for each } h = \{1,\ldots,N\} \\
&u_i \ge 0 \qquad i = \{1,\ldots,I\} \\
&v_j \ge 0 \qquad j = \{1,\ldots,J\}
\end{aligned}
\tag{2.47}
$$

Finally, the membership function μ (which is the parameter here) is varied at pre-specified intervals to observe the variations of the efficiency profile. $\mu = 0$ would yield overly optimistic values of technical efficiency and $\mu = 1$ would yield ultra conservative values.

Chapter 3

METHODOLOGY

In Chapter 2, we looked at the various weight restriction DEA models. Two of those models are more commonly used compared to other models. These models are:

1. The DEA model with absolute weight restrictions (see Dyson and Thanasoulis (1988), Roll *et al.* (1991), and Roll and Golany (1993).)
2. The Assurance Region (AR) DEA model (see Thompson *et al.* (1986), Thompson *et al.* (1990).)

In this chapter, we develop fuzzy models for modeling the uncertainty in bound values for these two types of models. It should be noted that the approach is a general one and with slight modifications can be easily applied to all types of weight restriction DEA models discussed in Chapter 2.

This chapter is divided into two sections. In section 3.1, we develop and solve the fuzzy model for the absolute weight restriction DEA problem and in section 3.2, we develop and solve the fuzzy model for the AR-DEA problem.

3.1 FUZZY MODEL FOR THE ABSOLUTE WEIGHT RESTRICTION DEA PROBLEM

The purpose of the absolute weight restriction model is to put upper and lower bounds on factor weights so that none of the factors are ignored or assigned excessively high weights. In order to determine appropriate values for the bounds, one has to first run the unbounded model, identify the anomalies in the results and then calculate the bounds. Once determined, the bounds are added as upper and lower boundconstraints to the original DEA (CCR) model to obtain the absolute weight restriction model.

This section is divided into three main sub-sections. The first sub-section is dedicated to developing and solving the fuzzy model for modeling the uncertainty in the absolute weight restriction DEA model. The second sub-section contains a discussion on the geometric representation of the fuzzy absolute weight bounds and their effect on the efficiency frontier. The third sub-section provides a roadmap for illustrating the development and implementation methodology of the fuzzy model. The roadmap is developed in response to Almond's (1995) criticism that a number of fuzzy approaches lack implementation roadmaps.

3.1.1 Development and Solution Methodology for the Fuzzy Absolute Weight Restriction DEA Model

Mathematically, the absolute weight restriction model as developed by Roll, Cook and Golany (1991) is represented as follows:

$$
\begin{aligned}
&\max \ \frac{u^T Y_0}{v^T X_0} \\
&\text{such that} \\
&\frac{u^T Y}{v^T X} \le 1 \\
&LB_r \le u_r \le UB_r \qquad \forall r \\
&LB_i \le v_i \le UB_i \qquad \forall i \\
&u, v \ge 0
\end{aligned}
\tag{3.1}
$$

where

$Y =$ set of output values

$X =$ set of input values

$Y_0 =$ output values for DMU_0 (DMU being evaluated)

$\mathrm{X}_0 =$ input values for DMU_0

$u =$ output weights that maximize the efficiency of DMU_0

$v =$ input weights that maximize the efficiency of DMU_0

$UB_r =$ upper bound on weight of output r.

$LB_r =$ lower bound on weight of output r.

$UB_i =$ upper bound on weight of input i.

$LB_i =$ lower bound on weight of input i

There are N DMUs i.e. $j = 1 \ldots$ N; s outputs i.e. $r = 1 \ldots$ s; and m inputs i.e. $i = 1 \ldots$ m.

To model the uncertainty of the bound values, we replace the crisp bounds UB_r, LB_r, UB_i, LB_i by the fuzzy numbers $UB^f{}_r\ , LB^f{}_r, UB^f{}_i\ , LB^f{}_i$ respectively. The superscript *f* signifies a fuzzy number. The resulting fractional fuzzy model is as follows:

$$
\begin{aligned}
&\max \ \frac{u^T Y_0}{\mathrm{v}^T X_0} \\
&\text{such that} \\
&\frac{u^T Y}{\mathrm{v}^T X} \le 1 && \forall \mathrm{j} \\
&u_r \le UB_r^f && \forall r \\
&u_r \ge LB_r^f && \forall r \\
&v_i \le UB_i^f && \forall i \\
&v_i \ge LB_i^f && \forall i \\
&u, v \ge 0
\end{aligned}
\tag{3.2}
$$

3.1.1.1 Definitions of the Fuzzy Numbers

The fuzzy numbers in (3.2) express the concept "close to the original crisp bounds." To express such a concept, Yuan and Klir (1995) propose using triangular membership functions. To completely describe triangular membership functions we need to specify the following:

- The most desirable value, which gets a membership grade of 1;
- Two least desirable values - one on either side of the most desirable value which are assigned membership grades of 0; and
- The form of the membership function as it varies between the most desirable and the least desirable values.

For our model, the most desirable bound values are those specified by the decision-maker. The least desirable values are determined by using one of two methods proposed by us later in this section. The membership function is assumed to be linear because linear membership functions are sufficient in most practical applications and are easy to use (Kaufmann and Gupta (1988)). Based on this description, the membership functions of the fuzzy weight bounds can be graphically depicted as:

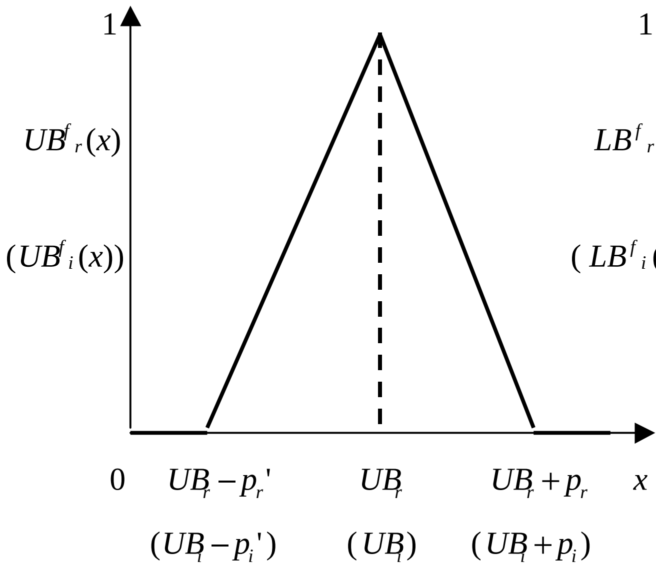

Figure 3.1 Proposed Membership Function of *UB*

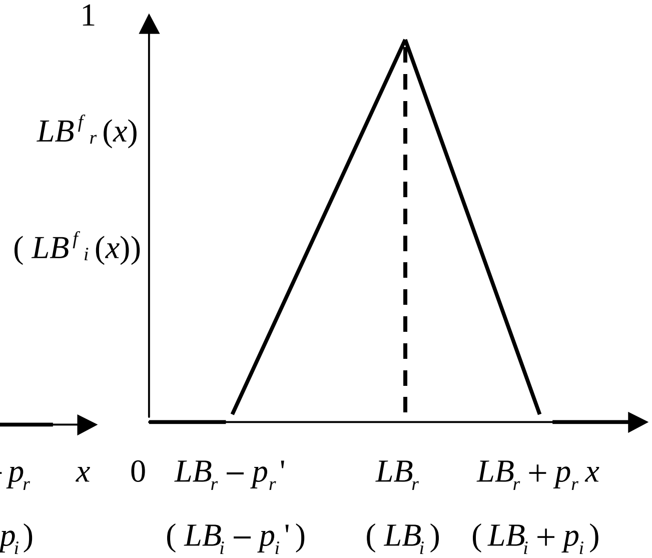

Figure 3.2 Proposed Membership Function of *LB*

p_r (or p_i) and p'_r (or p'_i) are the differences between the most desirable and the least desirable bound values.

Referring to equation (3.2), we note that the fuzzy numbers in Figures 3.1 and 3.2 are right-hand sides of constraints whose left-hand sides are crisp. Therefore, the bounds have the effect of relaxing the weight restriction constraints when they take the values [11] $UB + p$ and $LB - p'$ and tightening the constraints when they take the values $UB - p'$ and $LB + p$. The value of the objective function of a linear program (or a fractional program) is optimized (maximized in this case) when the constraints are most relaxed. Therefore, the membership function (which does not exist yet) of the objective function of (3.2) will favor the bound values which relax the constraints. On the other hand, the membership functions of the fuzzy constraints will favor the bounds specified by the decision-maker. Neither of them will favor the tight bounds. The maximizing solution (solution which maximizes the desirability of both the objective function and the constraints – see section 2.5.7.1.1) of the fuzzy model will be a compromising solution between the relaxed bounds and the specified (most desirable) bounds. This will make the

[11] We have dropped the subscripts *r* and *i* which distinguish between the output and input values because the same discussion applies to both.

tight bounds redundant. Accordingly, we drop the tight bounds from further analysis by modifying the membership functions of the fuzzy numbers as follows:

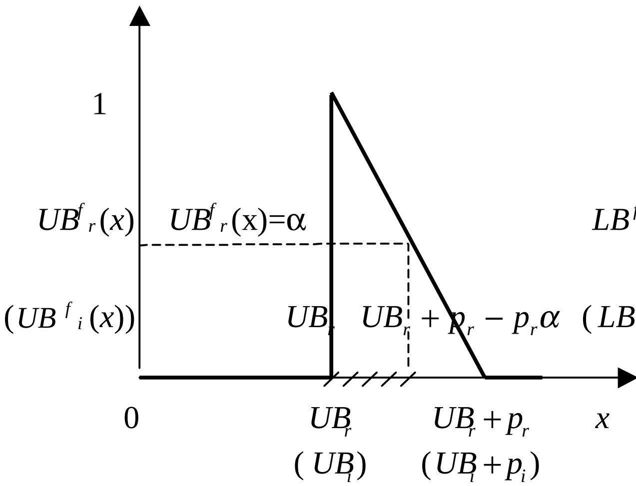

Figure 3.3 Membership Function of *UB*

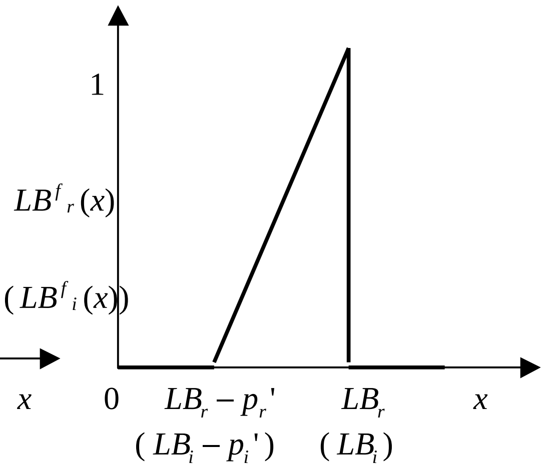

Figure 3.4 Membership Function of *LB*

Thus, the least desirable bounds in our model will always be more relaxed compared to the specified bounds. We propose two methods for determining least desirable bounds with such a characteristic.

Method I: Use the same procedure as that used for determining the most desirable bounds but make a different choice in each step of the procedure. Let us use the procedure proposed by Roll *et al.* (1991) (described in section 2.4.2.1.1) to illustrate this point. In step 2 of the procedure, the authors recommend selecting a ratio d: 1 between the upper and lower bounds and then plugging the value of d in a formula in Step 3 for determining the bounds. Let’s say that in a particular situation, the decision-makers choose d = 2 to determine the (most desirable) bounds. We recommend using d =3 to determine the least desirable bounds. Our justification for labeling the bounds determined using d = 3 as least desirable is that d=3 is not the first choice of the decision-maker.

Method II: Use the highest and lowest values of optimal multipliers obtained for efficient DMUs in the unbounded weight matrix as the least desirable upper and lower bounds respectively. In other words, the least desirable bounds will be bounds, which are just permissible enough so as not to affect the efficiency scores of any of the efficient DMUs. Our justification for calling these bounds as least desirable is that they do not affect the efficiency scores of the pseudo[12] efficient DMUs.

The membership functions can be mathematically represented as:

$$UB^{f}{}_{r}(x) = \begin{cases} 0 & if \;\; x \geq UB_r + p_r ; x < UB_r \\ \dfrac{UB_r + p_r\text{-}x}{p_r} & if \; UB_r < x < UB_r + p_r \\ 1 & if \;\; x = UB_r \end{cases} \tag{3.3}$$

$$UB^{f}{}_{i}(x) = \begin{cases} 0 & if \;\; x \geq UB_i + p ; x < UB_i \\ \dfrac{UB_i + p\text{-}x}{p} & if \; UB_i < x < UB_i + p \\ 1 & if \;\; x = UB_i \end{cases} \tag{3.4}$$

where $x \in \mathbf{R}$ (Figure 3.3) are the values of the bounds

$$LB^{f}{}_{r}(x) = \begin{cases} 0 & if \, x \;\leq\; LB_r - p_r' ; x > LB_r \\ \dfrac{x - LB_r + p_r'}{p_r'} & if \, LB_r - p_r' < x < LB_r \\ 1 & if \, x = LB_r \end{cases} \tag{3.5}$$

$$LB^{f}{}_{i}(x) = \begin{cases} 0 & if \, x \;\leq\; LB_i - p_i' ; x > LB_i \\ \dfrac{x - LB_i + p_i'}{p_i'} & if \, LB_i - p_i' < x < LB_i \\ 1 & if \, x = LB_i \end{cases} \tag{3.6}$$

Where $x \in \mathbf{R}$ (Figure 3.4) are the values of the weight bounds.

The α-cut of $UB^{f}{}_{r}$ is given by (refer to section 2.5.5.1 for the definition of α-cut of a fuzzy number)

$$\begin{aligned} {}^{\alpha}UB^{f}{}_{r} &= [{}^{\mathrm{L}\,\alpha}UB^{f}{}_{r}, {}^{\mathrm{R}\,\alpha}UB^{f}{}_{r}] \\ &= [UB_r, UB_r + p_r - p_r\alpha] \end{aligned} \tag{3.7}$$

Graphically, ${}^{\alpha}UB^{f}{}_{r}$ is represented using hatched lines in Figure (3.3).

[12] These are DMUs which appear efficient because of their good performance on a single output-input ratio.

Similarly, the α-cut representations of the other fuzzy numbers will be:

$$^{\alpha}LB^{f}{}_{r} = [LB_r - p_r' + p_r'\alpha, LB_r] \tag{3.8}$$

$$^{\alpha}UB^{f}{}_{i} = [UB_i, UB_i + p_i - p_i\alpha] \tag{3.9}$$

$$^{\alpha}LB^{f}{}_{i} = [LB_i - p_i' + p_i'\alpha, LB_i] \tag{3.10}$$

3.1.1.2 Degrees of Satisfaction of the Constraints

Replacing the RHS of a constraint by a fuzzy number is equivalent to replacing the constraint by a fuzzy set (Klir and Yuan (1995)). The membership function of this fuzzy set is called the degree of satisfaction of the fuzzy constraint[13]. The degree of satisfaction of a fuzzy constraint can be obtained from the membership function of the fuzzy number (on its RHS) by replacing the argument (*x*) in that membership function by the LHS of the constraint (Klir and Yuan (1995)). For example, the degree of satisfaction of the constraint $u_r \le UB^{f}{}_{r}$ can be obtained from the membership function of $UB^{f}{}_{r}$ defined in (3.3) by replacing the x by u_r:

$$D_r^{UB}(u_r) = \begin{cases} 0 & \text{if } u_r \ge UB_r + p_r \\ \dfrac{UB_r + p_r - u_r}{p_r} & \text{if } UB_r < u_r < UB_r + p_r \\ 1 & \text{if } u_r \le UB_r \end{cases} \tag{3.11}$$

The right-end of the α-cut of this fuzzy set will be:

$^{R\alpha}D_r^{UB} = UB_r + p_r - p_r\alpha$

The right-end of the 1-cut of this fuzzy set will be:

$^{R1}D_r^{UB} = UB_r$

The right-end of the support will be:

$^{R0+}D_r^{UB} = UB_r + p_r$

Refer to section 2.5.3.1 for definitions of 1-cut and support.

---

[13] A constraint whose RHS is a fuzzy number.

Note that the above definition of the satisfaction of the constraint $u_r \le UB^f{}_r$ is equivalent to the definition of the fuzzy relation "$\tilde{\le}$" comparing a real number with a fuzzy number introduced in section 2.5.5.5.

On similar lines, the degree of satisfaction of the constraint $u_r \ge LB^f{}_r$ will be:

$$D_r^{LB}(u_r) = \begin{cases} 0 & if\, u_r \le LB_r - p_r' \\ \dfrac{u_r - LB_r + p_r'}{p_r'} & if\, LB_r - p_r' < u_r < LB_r \\ 1 & if\, u_r \ge LB_r \end{cases} \qquad (3.12)$$

Left-end of the α-cut of the fuzzy set for $u_r \tilde{\ge} LB^f{}_r$ will be:

$^{L\alpha}D_r^{LB} = LB_r - p_r' + p_r'\alpha$

Left-end of the 1-cut will be: $^{L1}D_r^{LB} = LB_r$

Left-end of the support will be: $^{L0+}D_r^{LB} = LB_r - p'$

Note that the definition of the satisfaction of the constraint $u_r \ge LB^f{}_r$, coincides with the definition of the fuzzy relation "$\tilde{\ge}$" introduced in section 2.5.5.5.

The degrees of satisfaction of the input weight constraints are defined as:

$$D_i^{UB}(v_r) = \begin{cases} 0 & if\ v_i \ge UB_i + p_i \\ \dfrac{UB_i + p_i - v_i}{p_i} & if\, UB_i < v_i < UB_i + p_i \\ 1 & if\ v_i \le UB_i \end{cases} \qquad (3.13)$$

$$D_i^{LB}(v_r) = \begin{cases} 0 & if\, v_i \le LB_i - p_i' \\ \dfrac{v_i - LB_i + p_i'}{p_i'} & if\, LB_i - p_i' < v_i < LB_i \\ 1 & if\, v_i \ge LB_i \end{cases} \qquad (3.14)$$

The definitions of the 1-cuts and the supports for the input weight constraints will be similar to those for the output weight constraints.

The membership functions defined in (3.11) to (3.14) are of fuzzy sets corresponding to fuzzy constraints on $\mathbb{S}^n$. The intersection of those fuzzy sets given by $\bigcap_{k=1}^{2s+2m} D_k$ is a *fuzzy feasible set* $(\tilde{R})$ (Yuan and Klir (1995)).

The 1-cut of $\tilde{R}$, denoted as R_1, consists of 1-cuts of the fuzzy sets corresponding to all the fuzzy constraints contained in $\tilde{R}$. Based on the discussion in section 2.5.5.1, the 1-cuts of the fuzzy sets will contain only the most desirable (specified) bounds (UB_r).

The support of $\tilde{R}$, denoted as $S(\tilde{R})$ consists of supports of fuzzy sets corresponding to all the fuzzy constraints contained in $\tilde{R}$. The supports of the fuzzy sets will contain all possible bound values enclosed between the least and most desirable bound values (UB_r+ p_r).

3.1.1.3 Linear Fuzzy Formulation

Model (3.2) is a fractional programming model and is difficult to solve in its current form. To make it easier to solve, we convert it into a linear programming model by

- equating the denominator of the objective function to 1 and adding it as a constraint (see Charnes *et al.* (1978)),
- rearranging the constraints with ratio terms to eliminate the fractions and
- multiplying the objective function and all the constraints by the transformation factor $T_0 = (v^T X_0)^{-1}$ where $v^T X_0$ is the denominator of the objective function (see Charnes *et al.* (1962)).

The resulting linear fuzzy formulation is as follows:

$$
\begin{aligned}
& f = \max \quad \mu^{T} Y_0 \\
& \text{such that} \\
& \eta^{T} X_0 = 1 \\
& \mu^{T} Y - \eta^{T} X \le 0 && \forall j \\
& \mu_r \le T_0 UB^{f}{}_{r} && \forall r \\
& \mu_r \ge T_0 LB^{f}{}_{r} && \forall r \\
& \eta_i \le T_0 UB^{f}{}_{i} && \forall i \\
& \eta_i \ge T_0 LB^{f}{}_{i} && \forall i \\
& \mu, \eta \ge 0 && \forall r,i
\end{aligned}
\tag{3.15}
$$

where $\mu_r = T_0 u_r$ and $\eta_i = T_0 v_i$

Multiplication of both sides of the fuzzy constraints by the positive real number T_0 does not affect the definitions of the fuzzy relations.

3.1.1.4 Conversion to Crisp Linear Formulation

Fuzzy models in which a fuzzy aspiration level is specified for the objective function are called symmetrical models because one can determine a fuzzy set for both the objective function and the constraints (Zimmerman (1996)). Symmetrical models are easy to solve because the membership function of the fuzzy set "decision" is simply the intersection of the fuzzy sets of the objective function and the constraints. The "maximizing solution" (see section 2.5.7.1.1) of the decision equation gives the crisp optimal solution to the model (Zimmerman (1996)).

In model (3.15), there is no basis for determining an aspiration level for the objective function and therefore it is not possible to represent it as a fuzzy set. Such models in which the constraints are represented as fuzzy sets but the objective functions remain crisp are called unsymmetrical fuzzy models. Unsymmetrical fuzzy models can be solved using the same technique as the symmetric models if the crisp objective function can be represented as a "maximizing set" (concept proposed by Zadeh (1972)). The maximizing set is constructed by determining the extremum (upper and lower bounds) of the crisp function over the fuzzy domain. The reader is referred to section 2.5.7.1.2 for more details on the concept of extremum of functions. For model (3.15), we use Werners' (1984) definition to determine the maximizing set of the objective function. Once the maximizing set is determined, we proceed in the same way as we would for symmetric models and determine the crisp "maximizing solution" of the decision equation.

Werners' (1984) definition of "maximizing set" is:

Let f: $X \rightarrow \mathbf{R}^1$ be the objective function, $\tilde{R}$ = fuzzy feasible region, S ($\tilde{R}$) = support of $\tilde{R}$, and R_1=1-cut of $\tilde{R}$. The membership function of the goal (objective function) given solution space $\tilde{R}$ is then defined as

$$G(x) = \begin{cases} 0 & \text{if } f(x) \le \sup\limits_{R_1} f \\ \dfrac{f(x) - \sup\limits_{R_1} f}{\sup\limits_{S(\tilde{R})} f - \sup\limits_{R_1} f} & \text{if } \sup\limits_{R_1} f < f(x) < \sup\limits_{S(\tilde{R})} f \\ 1 & \text{if } \sup\limits_{S(\tilde{R})} f \le f(x) \end{cases} \tag{3.16}$$

where sup stands for supremum.

Let $f_0 = \sup\limits_{S(\tilde{R})} f$ and $f_1 = \sup\limits_{R_1} f$

To determine the maximizing set for any objective function using (3.16), we need to determine two values of the objective function by solving two LPs. The two LPs have the same set of fuzzy constraints but satisfied to different degrees.

3.1.1.4.1 Determination of f_1

f_1 is the supremum of f (the objective function) over R_1. Using the definitions of 1-cuts of the fuzzy sets derived in section 3.1.1.2, R_1 will be given by:

$$
\begin{aligned}
&\eta^T X_0 = 1 \\
&\mu^T Y - \eta^T X \le 0 && \forall j \\
&\mu_r \le T_0 UB_r && \forall r \\
&\mu_r \ge T_0 LB_r && \forall r \\
&\eta_i \le T_0 UB_i && \forall i \\
&\eta_i \ge T_0 LB_i && \forall i \\
&\mu, \eta \ge 0 && \forall r, i
\end{aligned}
\tag{3.17}
$$

The supremum of f over this region is:

$$\sup_{\tilde{R}_1} f = (\mu^T Y_0)_{opt} = f_1$$

3.1.1.4.2 Determination of f_0

f_0 is the supremum of f over $S(\tilde{R})$, which is the support of the fuzzy region. Using the definitions of the supports of the fuzzy sets derived in section 3.1.1.2, $S(\tilde{R})$ will be given by:

$$
\begin{aligned}
&\eta^T X_0 = 1 \\
&\mu^T Y - \eta^T X \le 0 && \forall j \\
&\mu_r \le T_0 (UB_r + p_r) && \forall r \\
&\mu_r \ge T_0 (LB_r - p_r') && \forall r \\
&\eta_i \le T_0 (UB_i + p_i) && \forall i \\
&\eta_i \ge T_0 (LB_i - p_i') && \forall i \\
&\mu, \eta \ge 0 && \forall r, i
\end{aligned}
\tag{3.18}
$$

The supremum of f over this region is:

$$\sup_{S(\tilde{R})} f = (\mu^T Y_0)_{opt} = f_0$$

3.1.1.4.3 Membership Function of the Objective Function

Using (3.16), the fuzzy set of optimal values or the membership function of the objective function will be given by:

$$G(w)=\begin{cases}1 & \text{if } f_0 \le \mu^T Y_0 \\ \dfrac{\mu^T Y_0 - f_1}{f_0 - f_1} & \text{if } f_1 < \mu^T Y_0 < f_0 \\ 0 & \text{if } \mu^T Y_0 \le f_1\end{cases} \tag{3.19}$$

where w= the set of all factor weights = (μ, η)

f_0 is the value of the objective function when the weight bound constraints are the most relaxed i.e. when they have p added to their upper bounds and p' subtracted from their lower bounds (see (3.18)). f_1 is the value of the objective function when the weight bound constraints are the tightest i.e. they have the specified bounds on their RHS (see (3.17)). Since the objective function is of the maximization type, f_0 will be its upper bound and f_1 will be its lower bound. In addition, the upper bound (f_0) will be most desirable (G (w) = 1) and the lower bound (f_1) will be least desirable (G (w) = 0). When the bounds are in-between the most relaxed and the tightest values, the objective function takes a value ($\mu^T Y_0$) in-between f_0 and f_1 and the degree of satisfaction with that value is determined using (3.19). The following figure graphically depicts the variation of the membership function G (w) between 0 and 1 as the objective function varies between f_0 and f_1.

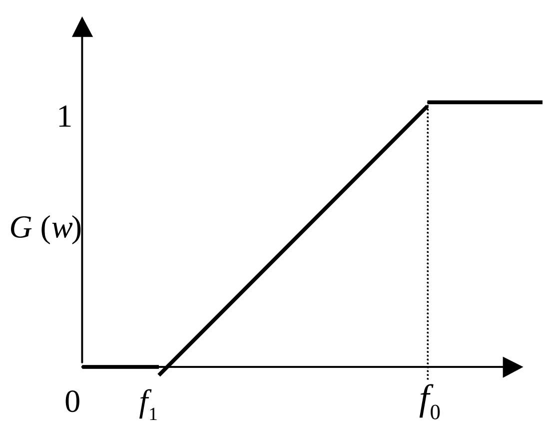

Figure 3.5 Membership function of the goal

3.1.1.4.4 Equivalent Crisp Formulation

Now that we have achieved "symmetry" between the constraints and the objective function, the crisp formulation equivalent to (3.15) is simply a problem of finding the set of weights, $w \in \mathbb{S}^{2s+2m}$ that give the "maximizing solution" i.e. a solution in which the constraints and the objective are satisfied to the maximum degree. Mathematically it is a problem of finding a set of weights so that

$$\lambda = [\bigcap_{k=1}^{2s+2m} D_k \cap G](w) \tag{3.20}$$

reaches its maximum value. Using the min operator to represent intersection, we can say that the objective of the crisp equivalent model is to

$$\text{Maximize } \lambda = \min\{D_k(w), G(w)\};\ k = 1 \text{ to } 2s + 2m \tag{3.21}$$

Or

$$\begin{aligned}\text{Maximize } \lambda = \min\{&\frac{\mu^T Y_0 - f_1}{f_0 - f_1}, \frac{T_0(UB_r + p_r)\text{-}\mu_r}{T_0 p_r}, \frac{\mu_r - T_0(LB_r - p_r')}{T_0 p_r'}, \\ &\frac{T_0(UB_i + p_i)\text{-}\eta_i}{T_0 p_i}, \frac{\eta_i - T_0(LB_i - p_i')}{T_0 p_i'}\}\end{aligned} \tag{3.22}$$

Thus, the crisp model equivalent of (3.15) is as follows:

$$\begin{aligned}
&\text{Max } \lambda \\
&\text{such that} \\
&\eta^T X_0 = 1 \\
&\mu^T Y - \eta^T X \le 0 && \forall j \\
&\lambda \le \frac{\mu^T Y_0 - f_1}{f_0 - f_1} \\
&\lambda \le \frac{T_0(UB_r + p_r)\text{-}\mu_r}{T_0 p_r} && \forall r \\
&\lambda \le \frac{T_0(UB_i + p_i)\text{-}\eta_i}{T_0 p_i} && \forall i \\
&\lambda \le \frac{\mu_r - T_0(LB_r - p_r')}{T_0 p_r'} && \forall r \\
&\lambda \le \frac{\eta_i - T_0(LB_i - p_i')}{T_0 p_i'} && \forall i \\
&\lambda \le 1 \\
&\mu, \eta, \lambda \ge 0 && \forall r, i
\end{aligned} \tag{3.23}$$

Rearranging the terms in (3.23), we get:

Max λ

such that

$$
\begin{array}{lll}
\lambda(f_0 - f_1) - \mu^T Y_0 \le -f_1 & & \\
\eta^T X_0 = 1 & & \\
\mu^T Y - \eta^T X \le 0 & \forall j & \\
\lambda T_0 p_r + \mu_r \le T_0(UB_r + p_r) & \forall r & (3.24) \\
\lambda T_0 p_i + \eta_i \le T_0(UB_i + p_i) & \forall i & \\
-\lambda T_0 p_r' + \mu_r \ge T_0(LB_r - p_r') & \forall r & \\
-\lambda T_0 p_i' + \eta_i \ge T_0(LB_i - p_i') & \forall i & \\
\lambda \le 1 & & \\
\mu, \eta, \lambda \ge 0 & \forall r, i &
\end{array}
$$

(3.24) is a quadratic programming model since the weight bound constraints are non-linear. Since λ is one of the variables in those constraints and we have information about the bounds on λ, we can use the parametric algorithm from Sakawa (1984) to solve the model. Solving (3.24) using the parametric algorithm simply means checking the feasibility of the model for different values of λ (determined by the algorithm) and choosing the solution corresponding to the maximum feasible value of λ. The parametric algorithm provides an efficient method for jumping from one λ value to another and reaching the maximum feasible value in the fastest way. The parametric algorithm is described below:

1. Set λ=0 and check the feasibility of the problem.
2. If the problem is feasible, go to 3. Otherwise STOP.
3. Set λ=1. Check the feasibility.
4. If the problem is feasible, that is the solution – STOP. Otherwise, go to 5.
5. Set λ_{max}=1, λ_{min} = 0.
6. If $\lambda_{max} - \lambda_{min} < \varepsilon$, STOP, otherwise go to 7.
7. Set $\lambda=(\lambda_{max} + \lambda_{min})/2$
8. Check the feasibility for λ.
9. If the problem is infeasible, set $\lambda_{max}=\lambda$ and go to 6. Otherwise set $\lambda_{min}=\lambda$ and go to 6.

The solution of the parametric algorithm gives the optimal weight values which when plugged into the expression $\mu^T Y_0$ give the efficiency scores of the DMUs. In the next section we look at how replacing the crisp bounds by fuzzy bounds affects the efficiency frontier.

3.1.2 Geometric Representation of Fuzzy Bounds and their Effect on the Efficiency Frontier

Using an example with two inputs and one output, Roll *et al.* (1991) geometrically illustrate how crisp absolute weight bounds affect the efficiency frontier. In this section, we extend that discussion to illustrate how fuzzy absolute weight bounds affect the efficiency frontier.

We start by repeating the discussion from Roll *et al.* (1991). Roll *et al.* (1991) modify the existing DEA problem by dividing both the inputs by the single output. The CCR model for a two input-one output problem with the inputs divided by the outputs will be as follows:

$$\begin{array}{lll} \text{Min} & \sum_i v_i X_{i0} & \\ \text{ST} & \sum_i v_i X_{ij} \geq 1 & \forall j \\ & lb_i \leq v_i \leq ub_i & i = 1,2 \end{array} \tag{3.25}$$

To see the effect of the lower bound constraints on the efficiency frontier, Roll *et al.* (1991) convert them to the form $v_1(1/lb_1) \geq 1$ and $v_2(1/lb_2) \geq 1$. These are the same as $v_1(1/lb_1) + 0v_2 \geq 1$ and $0v_1 + v_2(1/lb_2) \geq 1$. Now the lower bound constraints have become exactly like the main set of constraints $v_1 X_{1j} + v_2 X_{2j} \geq 1$. There is one main constraint for each DMU j. Therefore we can say that the two constraints introduced by the lower bounds are equivalent to adding two more DMUs (j+1 and j+2) to the model where $X_{1(j+1)} = 1/lb_1$ and $X_{2(j+1)}=0$ and $X_{1(j+2)} = 0$ and $X_{2(j+2)}=1/lb_2$. Like the other DMUs, these two DMUs can be represented as points on the efficiency frontier.

Now let us turn to the upper bound constraints. The upper bound constraints can be converted to the form $v_1(1/ub_1) + 0v_2 \leq 1$ and $0v_1 + v_2(1/ub_2) \leq 1$. These are not equivalent to the main set of constraints. Thus, the upper bound constraints are not equivalent to adding additional DMUs to the analysis and therefore cannot be directly used to modify the frontier. However, Roll *et al.* (1991) show that using the main set of constraints ($\sum_i v_i X_{ij} \geq 1$), for a two input problem, the upper bound on the weight of one input can be converted into an equivalent lower bound on the weight of the other input and vice-versa. The formula for converting the upper bound of v_1 into an equivalent lower bound on v_2 is as follows:

$$\text{Lower bound}\{v_2\} = \text{Max}_{\text{j}}\{(1 - ub_1 X_{1j}) / X_{2j}\}$$

The reason we choose the maximum of the RHS as a lower bound is that the maximum value is always the most binding as a lower bound.

Once the upper bound on a particular input weight is converted to an equivalent lower bound, the value of the specified lower bound for that input weight is compared with the value determined using the upper bound on the other input weight. The maximum of the two values is then used an effective lower bound on that input weight.

The data set used by Roll *et al.* (1991) for the single-output two-input example is shown in Table 3.1.

j	1	2	3	4	5	6
X_{1j}	2	3	4	2	1	5
X_2j	3	2	1	2	4	1

Table 3.1 Data set[14] used by Roll *et al.* (1991) for illustrating the Absolute Weight Restriction DEA Model Geometrically

Roll *et al.* (1991) use the following absolute weight bound constraints for their example:

[14] All the inputs have been divided by the single output to give input values per unit of output.

$0.15 \le v_1 \le 0.3$
$0.1 \le v_2 \le 0.6$

Before proceeding, we need to convert the upper bounds into equivalent lower bounds. The upper bound of 0.3 on v_1 is equivalent to a lower bound of 0.2 on v_2. Since a lower bound of 0.2 on v_2 is more binding than the original lower bound of 0.1, we use 0.2 as the new lower bound on v_2. The upper bound of 0.6 on v_2 however does not impose a more binding lower bound than the existing value of 0.15 on v_1. Therefore, we retain 0.15 as the lower bound on v_1. The effective weight bound constraints are:

$v_1 \ge 0.15, \qquad v_2 \ge 0.2$

The above constraints add two points (or DMUs) - (1/0.15,0) and (0,1/0.2) to the frontier. In Figure 3.6, we join these points to the rest of the frontier using dotted lines. Therefore the frontier represented by dotted lines is the frontier for the bounded model.

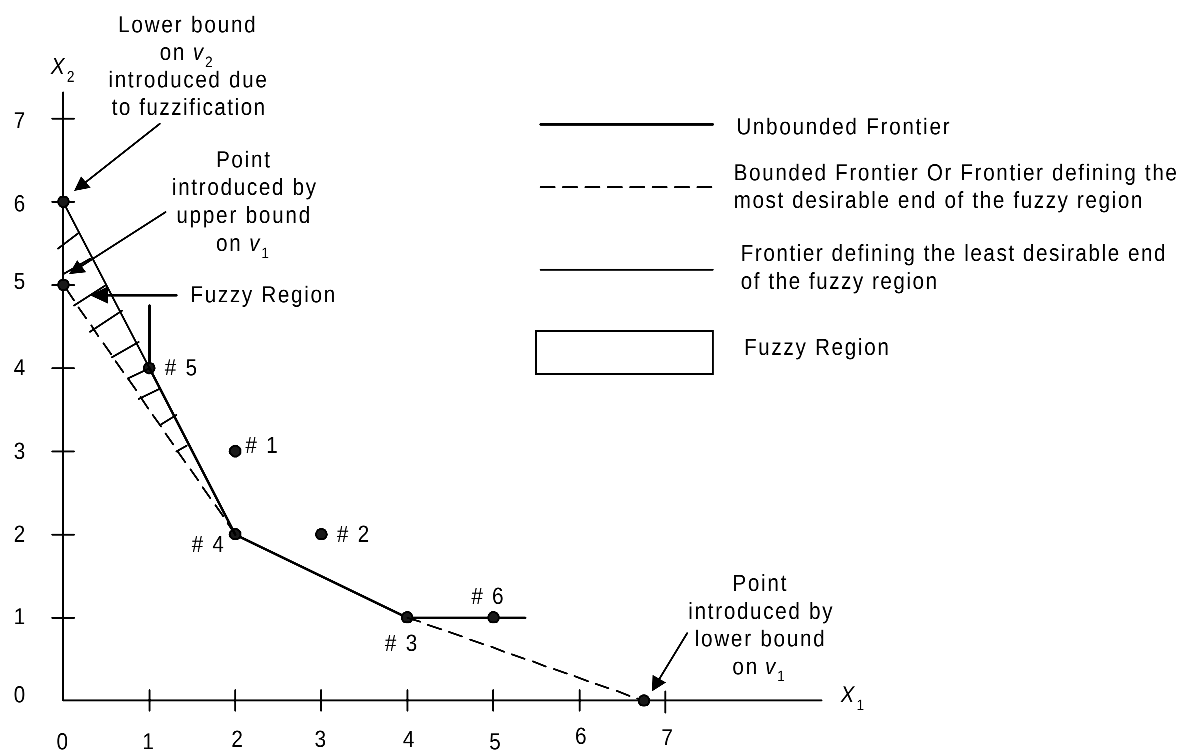

Figure 3.6 Geometric Representation of Crisp and Fuzzy Absolute Weight Bounds

Since Roll *et al.* (1991) do not specify how the weight bound values were determined, we assume that they were arbitrarily determined and are therefore imprecise. To model this imprecision, we propose to replace the specified bounds by fuzzy numbers. The fuzzy numbers will be bounded by the specified values (treated as most desirable) on one end and least desirable bounds on the other end. Restating the conclusion of our discussion in section 3.1.1.1, the least desirable bounds will be less binding compared to the specified bounds.

For this example, we use *Method II* proposed by us in section 3.1.1.1 to determine the least desirable bound values. In other words, the least desirable bounds will be bounds, which are just permissible enough so as not to affect the efficiency scores of any of the efficient DMUs.

On running the unbounded model, we observe that:

- DMU 3 is efficient for all values of v_1 between 0 and 0.16667,
- DMU 4 is efficient for all values of v_1 between 0.16667 and 0.5, and
- DMU 5 is efficient for all values of v_1 between 0.333 and 1.

Thus, a lower bound of 0.1667 and an upper bound of 0.333 for v_1 will not alter the efficiency scores of any of the efficient DMUs.

Similarly,

- DMU 3 is efficient for any value of v_2 between 0.333 and 1.0,
- DMU 4 is efficient for any value of v_2 between 0 and 0.333, and
- DMU 5 is efficient for any value of v_2 between 0 and 0.1667.

Thus lower and upper bounds of 0.1667 and 0.333 respectively for v_2 will not alter the efficiency scores of any efficient DMUs.

The upper bound of 0.333 on v_1 is equivalent to a lower bound of 0.1667 on v_2. In addition, the upper bound of 0.333 on v_2 is equivalent to a lower bound of 0.1667 on v_1. Thus, the effective LEAST desirable bounds on v_1 and v_2 will be:

$v_1 \geq 0.1667$ and $v_2 \geq 0.1667$

Comparing these with the effective MOST desirable bounds, $v_1 \geq 0.15$ and $v_2 \geq 0.2$, we note that the least desirable lower bound of 0.1667 on v_1 is tighter than the most desirable lower bound of 0.15. If we were to choose 0.1667 as the least desirable bound for v_1, we would be violating the definitions of the fuzzy numbers from section 3.1.1.1, which require that the least desirable bound be more relaxed than the most desirable bound. In case of v_2, the least desirable lower bound of 0.1667 is more relaxed than the existing lower bound of 0.2 and in line with our definitions of the fuzzy numbers. Accordingly, we choose to keep the lower bound on v_1 crisp and only replace the lower bound on v_2 by a fuzzy number.

In Figure 3.6, the least desirable lower bound of 0.1667 on v_2 introduces an additional point (0,1/0.1667). We join this point to the rest of the frontier by a solid line thinner than that used for the unbounded frontier. The (shaded) region enclosed between this line and the dotted line representing the frontier corresponding to the specified bounds is called the fuzzy region. The optimal bound value calculated by the fuzzy model will lie between these two extreme values and seek a compromise between maximization of the efficiency score (of DMU 5 in this case) and maximization of proximity to the specified bounds.

The following subsection uses a roadmap to illustrate the implementation methodology of the fuzzy model.

3.1.3 Roadmap for Developing and Solving the Fuzzy Absolute Weight Restriction DEA Model

We demonstrate the roadmap for developing and solving the fuzzy model for the absolute weight restriction DEA model with the use of an example. This example is the same as that used by Roll and Golany (1993) to demonstrate the absolute weight restriction DEA model. The implementation of the fuzzy model has the following steps:

Step 1: ***Collect the raw data***

As mentioned earlier, the raw data consisting of input and output values are taken directly from Roll and Golany (1993). There are 15 DMUs each using 4 inputs to produce three outputs. The data are presented in Table 3.2.

DMU	O_1	O_2	O_3	I_1	I_2	I_3	I_4
1	15500	460	0.85	521	3130	1859	80
2	13700	340	0.63	747	5075	3491	44
3	18000	1080	0.37	935	1483	2984	93
4	8900	490	0.56	205	4583	1736	65
5	10800	960	0.14	177	2990	1823	87
6	17300	890	0.47	584	5467	1775	98
7	21000	2930	0.91	634	7734	1700	58
8	9500	240	0.78	456	6552	503	73
9	9100	370	0.74	471	1855	2528	42
10	6600	800	0.52	325	4579	818	51
11	11800	610	0.87	364	5713	1178	80
12	26200	3600	0.41	585	4217	2012	84
13	11400	470	0.55	343	4061	2957	91
14	7200	1350	0.39	597	3242	665	73
15	38000	2470	0.68	1126	7658	1541	57

Table 3.2 Input / Output Data for the Roadmap Example illustrating the Fuzzy Absolute Weight Restriction DEA Model

Step 2: *Run the unbounded model and determine the most and least desirable bounds*

The data presented in Table 3.2 are plugged into a CCR model without weight restrictions. The optimal input/output weights and efficiency scores for all DMUs calculated by the CCR model are presented in Table 3.3. Looking at the table we realize that on numerous occasions, some inputs and/or outputs are assigned zero weights. By assigning zero weights to some of the inputs and outputs, the conventional DEA model completely ignores these factors and in this way disregards (not intentionally) the decision-maker's opinion that all factors are important for the efficiency evaluation of the given DMUs. Not only that, the model also assigns high values to the weights of some other inputs and outputs. To eliminate the extreme weight values and to minimize the variation between the weights assigned to different inputs and outputs, the decision-maker sets bounds on the weight values. In this example we use the same procedure as that used by Roll and Golany (1993) to set bounds. Although this procedure was already described in section 2.4.2.1.1, we would like to reiterate it here. The steps for setting the bounds are enumerated below:

- *Eliminate the extreme values.* As proposed by Roll and Golany (1993), we eliminate the topmost and bottommost extreme values from all columns. In Table 3.3, the values marked with a * are the ones that are eliminated.
- *Take the average of the remaining values.* The averages $\bar{u}_r$ and $\bar{v}_i$ of the remaining 13 values of all weights are taken. The averages are also presented in Table 3.3 in the row titled "Average after Truncation."
- *Choose the desirable ratio between the largest and the smallest weight values.* This will be the same as the ratio between the upper and lower bounds and will be used to determine the bound values based on the averages. Roll and Golany (1993) use two different ratios, 2:1 and 3:1 to determine two different sets of bounds and produce two different sets of efficiency scores. For our roadmap, we choose the ratio 2:1 to calculate the most desirable bounds (or the specified bounds) and use the ratio 3:1 to calculate the least desirable bounds required by the fuzzy model.
- *Determine the values of the bounds.* Using a value of d=2 and using the formulas $LB_r = \frac{2\bar{u}_r}{1+d}$, $UB_r = \frac{2d\bar{u}_r}{1+d}$, we calculate the most desirable bounds *UB* and *LB*.

For determining the least desirable bounds required by the fuzzy model, we use the two methods already described in section 3.1.1.1. The methods are restated below:

Method I: In this method, we use the same procedure as that used to determine the most desirable bounds except that we make different choices than those made while determining the most desirable bounds in every step of the procedure. In this case, the only change we make is we choose 3:1 (instead of 2:1) as the ratio between the least desirable upper and lower bounds. For example, since $\bar{u}_1$ = 3.38E-05, UB_1 = (2* 3.38E-05)/(1+3) = 5.069E-05 and LB_1 = (2*3*3.38E-05)/(1+3) = 1.69E-05.

Method II: As explained in 3.1.1.1, we use the highest and lowest values of optimal weights assigned to the factors by efficient DMUs in the unbounded runs as the least desirable upper and lower bounds respectively. Since u_1 varies between 0 and .0000926 for efficient DMUs, $UB_1 + p_1$ will be .0000926 and $LB_1 - p_1'$ will be 0.

DMU	u_1	U_2	u_3	v_1	v_2	v_3	v_4	Efficiency
1	3.669E-05	0*	0.50738	1.522E-03	6.622E-05	0*	0*	1
2	1.411E-05	0	1.11636	0*	0*	0	2.273E-02*	0.89667
3	5.556E-05	0	0*	6.693E-04	2.523E-04*	0	0	1
4	8.644E-05	0	0.41195	3.566E-03	0	0	4.139E-03	1
5	9.259E-05*	0	0	5.650E-03*	0	0	0	1
6	3.629E-05	0	0.2411	1.090E-03	0	2.047E-04	0	0.74111
7	3.428E-05	0	0.30787	1.272E-03	0	0	3.333E-03	1
8	0*	0	1.28205	2.912E-04	3.061E-05	1.658E-04	7.990E-03	1
9	1.237E-05	0	1.19925	1.740E-03	9.732E-05	0	0	1
10	0	2.987E-04	1.2458	1.673E-03	0	5.579E-04*	0	0.8868
11	5.586E-05	0	0.39181	2.285E-03	0	1.430E-04	0	1
12	3.817E-05	0	0	1.248E-03	6.397E-05	0	0	1
13	4.617E-05	0	0.65064	1.940E-03	8.238E-05	0	0	0.88417
14	0	3.020E-04*	1.51888*	2.064E-04	1.902E-04	3.911E-04	0	1
15	2.341E-05	0	0.16251	7.388E-04	0	0	2.949E-03	1
Average	3.546E-05	4.005E-05	0.602373	1.593E-03	5.220E-05	9.750E-05	2.743E-03	
Average after Truncation	3.380E-05	2.298E-05	0.5782	1.403E-03	4.082E-05	6.959E-05	1.416E-03	
UB	4.495E-05	3.056E-05	0.769018	1.866E-03	5.430E-05	9.255E-05	1.884E-03	=1.33 x Avg
LB	2.264E-05	1.540E-05	0.3874	9.401E-04	2.735E-05	4.662E-05	9.489E-04	=0.67 x Avg
$^1UB + pr$	5.069E-05	3.447E-05	0.867314	2.105E-03	6.124E-05	1.044E-04	2.124E-03	=1.5 x Avg.
$^1LB - p'$	1.690E-05	1.149E-05	0.289105	7.016E-04	2.041E-05	3.479E-05	7.081E-04	=0.5 x Avg.
$^2UB + pr$	9.26E-05	3.02E-04	1.51888	5.65E-03	2.52E-04	3.91E-04	7.99E-03	
$^2LB - p'$	0	0	0	2.06E-04	0	0	0	

Table 3.3 Results of the Unbounded Runs and Bound Values calculated using those Results for Roadmap Example illustrating the Fuzzy Absolute Weight Restriction DEA Model

Note: $^1UB + p$ and $^1LB - p'$ are the least desirable bounds determined using Method I and $^2UB + p$ and $^2LB - p'$ are the least desirable bounds determined using Method II.

Step 3: ***Solve the fuzzy model***

DEA models with weight restrictions of the type (3.17) are solved using values of *UB* and *LB* from Table 3.3 to obtain f_1 values. Similarly DEA models with weight restrictions of the type (3.18) are solved (twice) with (two different sets of) values of $UB + p$ and $LB - p'$ from Table 3.3 to obtain (two sets of) f_0 values. These values are plugged into the equivalent crisp model (3.24), which is solved using the parametric algorithm with an ε value of 0.1. Model (3.24) is solved twice, once with the set of least desirable bounds determined using Method I and once with those determined using Method II. Table 3.4 compares the results of both models with the results of the crisp weight bound model. Alongside the efficiency scores calculated by the fuzzy models, Table 3.4 also displays λ values obtained in the final iteration of the algorithm. The λ values represent the degree to which the bounds specified by the decision-maker were satisfied in the final solution.

	Efficiency				
DMU	Crisp bounds	[*UB*, *LB*] & [^1UB+p, ^1LB-p']	λ	[*UB*, *LB*] & [^2UB+p, ^2LB-p']	λ
1	1	1	1	1	1
2	0.60241	0.6181	0.4	0.72273	0.4
3	0.57492	0.6015	0.4	0.86451	0.5
4	1	1	1	1	1
5	0.94847	0.97581	0.6	1	0.9
6	0.70723	0.70952	0.5	0.73066	0.5
7	1	1	1	1	1
8	0.82149	0.84767	0.4	0.98496	0.6
9	0.92411	0.95947	0.4	1	0.8
10	0.75715	0.77026	0.5	0.83474	0.5
11	1	1	1	1	1
12	1	1	1	1	1
13	0.81186	0.82269	0.5	0.85659	0.6
14	0.52383	0.54678	0.4	0.78929	0.5
15	0.96241	0.99039	0.6	1	0.9
Average	**0.842259**	**0.856146**		**0.918899**	

Table 3.4 Comparison of Results of Crisp and Fuzzy Absolute Weight Bound Models applied to the Roadmap Example

To show that the difference between the efficiency scores obtained using the crisp model and the fuzzy models is statistically significant, we use the "paired – sample t test" (Bain and Engelhardt (1992)) with $H_0 : E_2 – E_1 = 0$ versus $H_a : E_2 – E_1>0$, where E_2 is the average efficiency calculated by the fuzzy model and E_1 is the average efficiency calculated by the crisp model. When we apply the paired-sample t test to the values in columns 2 and 3 of Table 3.4, we obtain a p-value of 0.000487. This means that we can reject the null hypothesis with an α (probability of type I error) value as low as 0.0005. This allows us to accept the alternative hypothesis that there is a significant difference in the efficiency scores obtained using the two models. Applying the same test to the values in columns 2 and 5, we get a p-value of 0.0033. This means that we can reject the null hypothesis with an α value of 0.005 allowing us to accept the alternative hypothesis that there is a significant difference between the efficiency scores calculated by the crisp and fuzzy models.

Visually comparing the results of columns 2 and 5, we see that three of the DMUs (5,9 and 15) which were not part of the efficient set in the original crisp model, entered the efficient set when the weight bounds were made fuzzy. In fact, two of them (5 & 15) entered the efficient set while satisfying the specified bounds to a degree as high as 90%. Thus, if the decision-maker were willing to change the original bounds to values calculated by the 90% satisfaction level, the state of some of the DMUs would change. We do not anticipate any resistance from the decision-maker to making these slight changes to the bounds since the bounds were determined by subjective methods in the first place. We change the original set of bounds in Table 3.5 to the new set in Table 3.6 using the criterion of 90% satisfaction of the original bounds.

Factor	u_1	u_2	u_3	v_1	v_2	v_3	v_4
Upper bound	4.495E-05	3.056E-05	0.769018	1.866E-03	5.430E-05	9.255E-05	1.884E-03
Lower bound	2.264E-05	1.540E-05	0.3874	9.401E-04	2.735E-05	4.662E-05	9.489E-04

Table 3.5 Original Set of Bounds for Roadmap Example illustrating the Fuzzy Absolute Weight Restriction DEA Model

Factor	u_1	u_2	u_3	v_1	v_2	v_3	v_4
Upper bound	4.97E-05	5.77E-05	0.8440042	2.24E-03	7.41E-05	1.22E-04	2.49E-03
Lower bound	2.21E-05	1.50E-05	0.3775705	9.16E-04	2.67E-05	4.54E-05	9.25E-04

Table 3.6 Modified Set of Bounds (at the 90% Satisfaction Level of Original Bounds)

Using the new set of bounds (from Table 3.6), we solve the crisp weight bound DEA model. Table 3.7 compares the results of this model with the results of the model with the original bounds from Table 3.5.

DMU	Efficiency Scores	
	With Original set of bounds	With Modified set of bounds
1	1	1
2	0.60241	0.6227
3	0.57492	0.61306
4	1	1
5	0.94847	1
6	0.70723	0.71048
7	1	1
8	0.82149	0.8555
9	0.92411	0.97298
10	0.75715	0.78203
11	1	1
12	1	1
13	0.81186	0.82952
14	0.52383	0.57358
15	0.96241	1
Avg.	**0.842259**	**0.86399**

Table 3.7 Comparison of Efficiency Scores obtained using Original and Modified Sets of Bounds for Absolute Weight Bound Roadmap Example

DMUs 5 & 15 are perfect examples of DMUs that were penalized by the imprecision in the bounds. It took only a little tweaking of the bounds in order to restore them to the efficient set.

3.2 FUZZY MODEL FOR THE ASSURANCE REGION (AR) DEA PROBLEM

The AR weight restrictions are bounds on the ratios of the weights. They may be introduced in the analysis

- to align the calculated efficiency scores with the preference of the decision-maker by restricting the multipliers into cones (Chilingerian and Sherman (1997)) or
- to incorporate information about the prices and costs of the inputs and outputs (Thompson *et al.* (1986), (1990), (1992), (1996a), (1996b), and (1996c)) or
- to incorporate expert opinion about relative importance of the factors (Zhu (1996))

This section is divided into three main sections. In the first section, the concepts of AR and fuzzy AR are explained geometrically using the concept of weight space analysis proposed by Seiford and Thrall (1990). In the second section, we distinguish between two different types of AR I constraints and call them AR I constraints in Form 1 and AR I constraints in Form 2. The second section is accordingly divided into two major subsections with each sub-section dedicated to developing a fuzzy model for each form of AR I constraint. The third section contains implementation roadmaps for both types of fuzzy AR models.

3.2.1 Geometric Representation of AR and Fuzzy AR

To geometrically illustrate the concept fuzzy AR, we use the notion of multiplier space *W*, first introduced by Seiford and Thrall (1990) and then used by Thompson *et al.* (1990) to demonstrate their crisp AR approach. The data set for our example is the same as that used by Thompson *et al.* (1990) in their example.

	DMU					
	1	2	3	4	5	6
Output, y	1	1	1	1	1	1
Input 1, x_1	4	2	1	5	4	3
Input 2, x_2	1	2	4	1	4	1.5

Table 3.8 Data set from Thompson *et al.* (1990) used for the Geometric Illustration of the AR Approach

Figure 3.7 shows the input-output graph and the efficiency frontier for the data set in Table 3.8. From the graph we see that DMUs 1,2,3 and 6 are DEA efficient because they lie on the frontier. DMU 5 is DEA inefficient and DMU 4 is DEA-slack-inefficient. Amongst the DEA efficient DMUs, 1, 2 and 3 are extreme efficient while 6 is non-extreme efficient because it can be expressed as a linear combination of DMUs 1 and 2. For more details on this classification scheme, refer to Seiford and Thrall (1990).

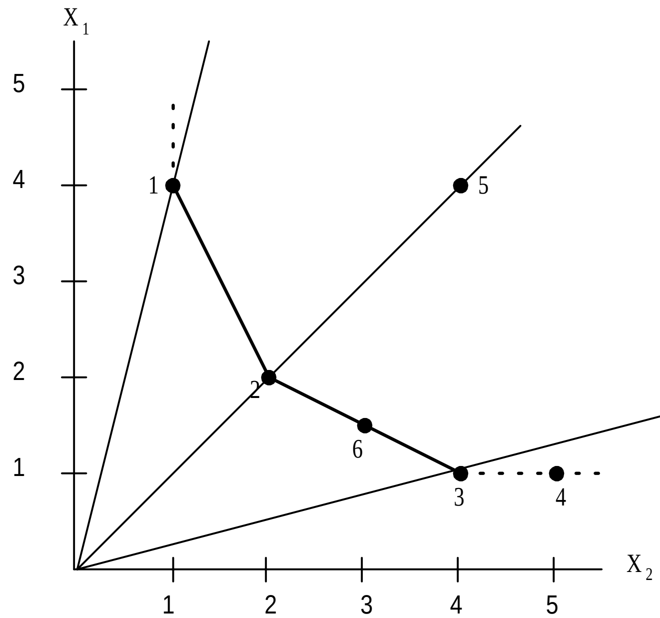

Figure 3.7 Efficiency Frontier for the Example from Thompson *et al.* (1990) for illustrating the AR Approach

Seiford and Thrall (1990) define the concept of optimal multiplier space W_j for each DMU_j as:

$W_j = \{(u, v)$ for which $h_j(u, v)$ is maximal$\}$

where $h_j(u, v)$ is the objective function.

The entire space W of multipliers is the union of multiplier spaces W_j of individual DMUs.

For our example, the multiplier spaces for the various DMUs obtained by solving the ratio form of DEA model are as follows:

DMU_1:	$W_1 = (1, v_1, (1 - 4v_1))$,	$v_1 \leq 1/6$
DMU_2:	$W_2 = (1, v_1, 1/2(1 - 2v_1))$,	$1/6 \leq v_1 \leq 1/3$
DMU_3:	$W_3 = (1, v_1, 1/4(1 - v_1))$,	$1/3 \leq v_1 \leq 1$
DMU_4:	$W_4 = (1, 0, 1)$	
DMU_6:	$W_6 = (1, 1/6, 1/3)$	

In the above definitions of multiplier spaces, we have substituted v_1 for v_2 using the constraints of the DEA model. The ranges for v_1 are the values of v_1 between which the DMU under consideration remains efficient.

From the definitions of the multiplier spaces, we can make the following observations:

- Since v_1 and v_2 are not fixed in the definitions of W_1, W_2, and W_3, we can conclude that the dimension of W_1, W_2, and W_3 is 3. This is the maximal dimension[15] possible (3= 2+1).
- Values of v_1 and v_2 are fixed in the definitions of W_4 and W_6. Therefore, dim W_6 = dim W_4 = 2 < 3, thus proving that DMUs 4 and 6 are not extreme efficient.
- $W_6 = W_3 \cap W_2$. This proves that we can express the multiplier space of any efficient DMU as the intersection of multiplier spaces of DMUs which are extreme efficient and contain it.

Figure 3.8 illustrates the partition of the multiplier space W into sets W_1, W_2, W_3, and W_6. Since dim W_1 = dimW_2 = dimW_3 = 3, they appear as cones in the figure and since dim W_6 = dim W_4 = 2, they appear as lines.

When the DEA model is solved as an LP (instead of ratio form) by adding the normalization constraint $vX_0 = 1$, the multiplier sets obtained previously, reduce in dimension by one and are represented using the same symbols as those from the ratio form with a superscript m added to the symbols. Thus W_1^m, W_2^m, W_3^m appear as lines and W_6^m and W_4^m appear as points in the multiplier space.

W_1^m is the line joining the points (1/3,1/6) and (1,0),
W_2^m is the line joining (1/6,1/3) and (1/3,1/6),
W_3^m is the line joining the points (0,1) and (1/6,1/3)
W_4^m is the point (0,1) and

[15] Maximal dimension is the sum of total number of inputs and total number of outputs

W_6^m is the point (1/6,1/3).

The multiplier sets of DMUs 1,2 and 3 are straight lines instead of points because of the existence of alternative solutions. Any combination of v_1 and v_2 which satisfies the equation of the line joining (0,1) and (1/6,1/3) will be optimal for DMU_3. Similar remarks hold for $DMU_1$and DMU_2. Suppose market price information puts the following restrictions on the weights:

$v_2 / v_1 \leq 1.5$

$v_2 / v_1 \geq 0.75$

This gives the AR = $\{(v_1, v_2)$: $-1.5v_1 + v_2 \leq 0$, $0.75v_1 - v_2 \leq 0$, $v_1>0$, $v_2>0\}$, which may be adjoined to the LP. In Figure 3.8, the AR is the shaded region enclosed by the dotted lines. We notice that the AR only partially encloses the region W_2 and completely excludes the regions W_1 and W_3. Naturally, only DMU_2 will be AR efficient. The results obtained by running this AR-DEA model are shown in Table 3.9.

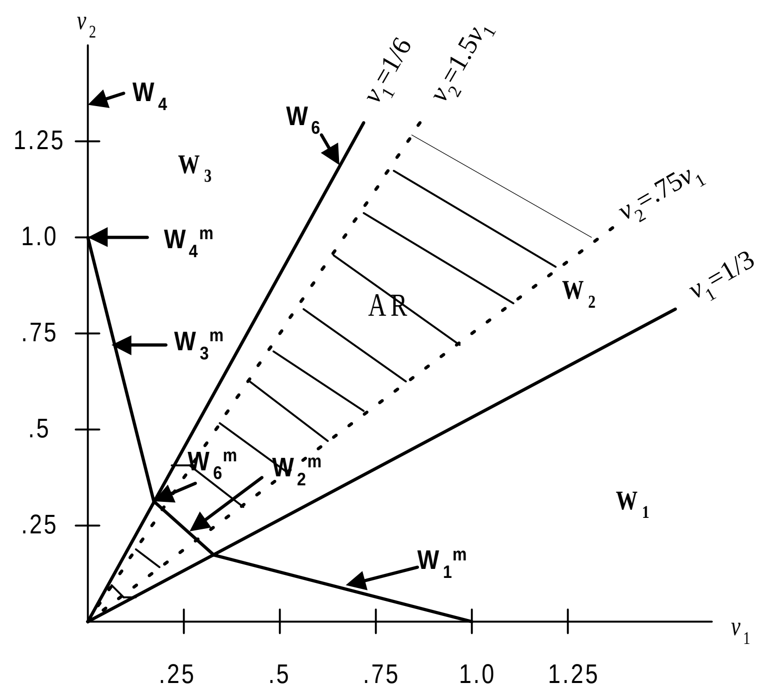

Figure 3.8 Geometric Representation of Multiplier Space and AR

DMU	AR efficiency score
1	0.90909
2	1
3	0.875
4	0.76923
5	0.5
6	0.95238

Table 3.9 Results of the Illustrative AR Model from Thompson *et al.* (1990)

As expected, DMUs 1, 3 and 6 are no longer efficient given the AR. At this point it is appropriate to introduce the definitions of Assurance Regions (AR) and AR efficiency proposed by Thompson *et al.* (1990) pp. 100.

"Assurance Region (AR) Definition - For DEA problems with a finite number of DMUs and a well-defined data domain, an AR is a subset of W such that vectors v excluded from AR are not reasonable input and output virtual multipliers."

"AR Efficiency Definition - A DMU_j *in E (set of extreme efficient DMUs) is said to be AR-efficient, relative to an AR, if the intersection of* W_j $(j=1,2,...,n)$ *and AR is not empty; and it is said to be not AR-efficient otherwise."*

Our results for the example AR-DEA model are consistent with the above definition of AR efficiency. Since the AR does not intersect with W_1, W_6 and W_3, DMUs 1,6 and 3 become AR-inefficient.

The results provoke us to question the accuracy of the market information which was used to determine the ARs because if the value of the lower bound of the ratio v_2/v_1 had been 0.5 instead of 0.75, DMU_1 would have been AR-efficient. For that matter if the upper bound of the same ratio had been 2 instead of 1.5, DMUs 3 and 6 would have been AR-efficient. Thus, given the fact that the efficiency scores of the DMUs are so sensitive to the values of the bounds, it seems inappropriate to use crisp numbers for those bounds when the information for determining them is not accurate. In this research, we propose to replace the bound values by fuzzy numbers so that we can model the imprecision in the bound values. The fuzzy numbers will allow the model to explore a broader region for locating the bounds while still treating the specified bounds as most desirable. The region spanned by the fuzzy numbers will have the specified values at one end and the least desirable values at the other end. The least desirable upper and lower bounds on the ratios will be the highest and lowest values respectively of the ratios of the optimal factor weights for the efficient DMUs in the unbounded runs. The objective of the fuzzy model is to seek a compromise between the solution that maximizes the satisfaction of the decision-maker with the bounds and the solution that maximizes the efficiency scores of all the DMUs.

Coming back to our illustrative example, the fuzzy region is the shaded region in Figure 3.9. At one end of the fuzzy region we have the specified bounds determined from market information and at the other end we have bounds determined from ratios of weights assigned to the factors by the efficient DMUs in the unbounded runs.

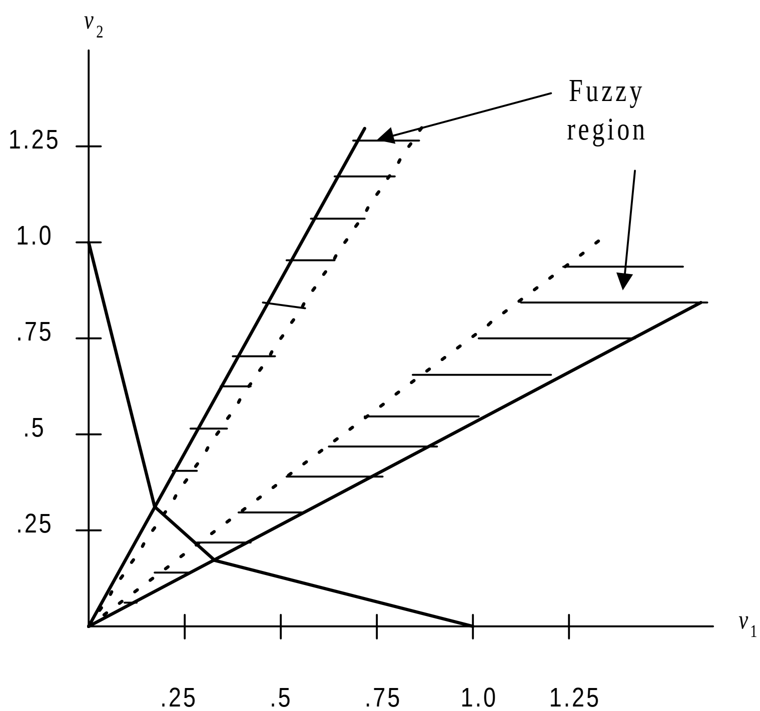

Figure 3.9 Geometric Representation of the Fuzzy AR Region

In the next subsection, we develop and solve the fuzzy AR DEA model.

3.2.2 Development and Solution Methodology of the Fuzzy AR Model

The AR constraints impose bounds on the ratios of the weights. While the absolute weight bounds, discussed in section 3.1, are added to the fractional DEA model, the Assurance Region (AR) constraints are added to the linear form of the DEA model. Therefore, the weights in the AR model are represented using the symbols μ and η instead of the symbols *u* and *v*. The bounds are determined using value data like market price information or expert opinion. This value data typically exists in the following form:

		"Price/cost" data	
Factor	Multiplier	Lower bound	Upper bound
y_1	μ_1	lb_1	ub_1
.	.	.	.
.	.	.	.
y_s	μ_s	lb_s	ub_s
x_1	η_1	LB_1	UB_1
.	.	.	.
.	.	.	.
x_m	η_m	LB_m	UB_m

Table 3.10 Price/Cost Information or Expert Opinion used for Setting AR bounds

At this point, we would like to distinguish between two forms of AR constraints that can be obtained using the information in Table 3.10:

Form 1 is the one in which the upper and lower bounds on the ratios of weights are not themselves in ratio form. Expressing the AR in this form becomes inevitable when information is available in the form of relative importance of inputs and outputs. For example, if the experts specify that input one is 1.5 times more important than input two, then the AR constraint expressing this opinion appears as $v_1 \geq 1.5 v_2$. More generally, AR of this form is expressed as:

$$a_{rt} \leq \mu_r / \mu_t \leq b_{rt}, \qquad r < t, \qquad r,t = 1,\ldots,s. \tag{3.26}$$

$$A_{ij} \leq \eta_i / \eta_j \leq B_{ij}, \qquad i < j, \qquad i,j = 1,\ldots,m.$$

If using "price/cost" information or expert opinion from Table 3.10 for setting AR bounds of Form 1, then $a_{rt} = lb_r / lb_t$, $b_{rt} = ub_r / ub_t$, $A_{ij} = LB_i / LB_j$ and $B_{ij} = UB_i / UB_j$.

Quite often (see Thompson *et al.* (1986) and (1990)), one of the inputs, say x_1, may be selected as an input numeraire and one of the outputs, say y_1, may be selected as an output numeraire. The AR in (3.26) can then be specified as a set of (m + s - 2) homogeneous linear inequalities for separable cones:

$$\begin{aligned} &a_r \mu_1 \le \mu_r \le b_r \mu_1, \qquad r = 2,3,\ldots,s \quad \text{(output cone)} \\ &A_i \eta_1 \le \eta_i \le B_i \eta_1, \qquad i = 2,3,\ldots,m \quad \text{(input cone)} \end{aligned} \tag{3.27}$$

Form 2 is the one in which the bounds on the ratios of multipliers are also expressed as ratios. It is easy to obtain the constraints in this form when information is available in the form shown in Table 3.10. Constraints of this type are represented as:

$$\frac{lb_r}{lb_t} \le \frac{\mu_r}{\mu_t} \le \frac{ub_r}{ub_t}, \qquad r < t, \qquad r,t = 1,\ldots,s. \tag{3.28}$$

$$\frac{LB_i}{LB_j} \le \frac{\eta_i}{\eta_j} \le \frac{UB_i}{UB_j}, \qquad i < j, \qquad i,j = 1,\ldots,m.$$

Expressing in the form of separable input/output cones we get:

$$lb_r \mu_t \le lb_t \mu_r, \qquad ub_t \mu_r \le ub_r \mu_t, \qquad r < t, \qquad r,t = 1,\ldots,s. \text{ (output cone) (3.29)}$$

$$LB_i \eta_j \le LB_j \eta_i, \quad UB_i \eta_j \le \eta_i UB_j, \quad i < j, \qquad i,j = 1,\ldots,m. \text{ (input cone)}$$

We make this distinction between the two forms of AR constraints because the fuzzy approach proposed in this paper differs markedly for solving models with these two types of constraints. The fuzzy relations in the fuzzy model with AR constraints in Form 1 compare real numbers on the LHS with fuzzy numbers on the RHS. On the other hand, the fuzzy relations in the fuzzy model with AR constraints in Form 2 compare two fuzzy numbers. Since the constraints in Form 2 can be easily converted to Form1, the decision-maker has a choice of two methods for solving the fuzzy AR model when the original constraints are in Form 2 and one method when they are in form 1.

Since the two forms of AR constraints are different mathematically, the fuzzy models for solving them are also markedly different and therefroe dealt with in separate subsections – section 3.2.2.1 and section 3.2.2.2.

3.2.2.1 Fuzzy Model for AR Constraints in Form 1

The constraints in Form 1, when added to a CCR model give the following CCR-AR model:

$$\begin{aligned}
&\text{Max } \mu^T Y_0 \\
&\text{Subject to} \\
&\eta^T X_0 = 1 \\
&\mu^T Y - \eta^T X \leq 0 \\
&a_r \leq \frac{\mu_r}{\mu_1} \leq b_r \quad r = 2,\dots,s \\
&A_i \leq \frac{\eta_i}{\eta_1} \leq B_i \quad i = 2,\dots,m
\end{aligned} \tag{3.30}$$

where input 1 (X_1) and output 1 (Y_1) are input and output numeraire respectively.

As stated earlier, the values of the bounds are imprecise. To model the imprecision associated with the bound values, we propose to replace the crisp bound values with fuzzy numbers that express the concept of approximate numbers close to the specified bounds.

3.2.2.1.1 Definitions of Fuzzy Numbers

Let $a^f{}_r, b^f{}_r$, $A^f{}_i$ and $B^f{}_i$ be the fuzzy numbers corresponding to the crisp bounds a_r, b_r, A_i and B_i respectively. Then the fuzzy model corresponding to (3.30) will be:

$$\begin{aligned}
&\text{Max } \mu^T Y_{j0} \\
&\text{Subject to} \\
&\eta^T X_{j0} = 1 \\
&\mu^T Y - \eta^T X \leq 0 \\
&a_r^f \leq \frac{\mu_r}{\mu_1} \leq b_r^f \quad r = 2,\dots,s \\
&A_i^f \leq \frac{\eta_i}{\eta_1} \leq B_i^f \quad i = 2,\dots,m
\end{aligned} \tag{3.31}$$

The fuzzy constraints in the above model are similar in form to the ones in the absolute weight restriction model (3.2) i.e. they have fuzzy numbers on their RHS and crisp numbers on their LHS. Hence, we can use the same argument presented in section 3.1.1.1

to say that the membership functions of the fuzzy numbers in (3.31) will vary between the specified (most desirable) bounds and the more relaxed least desirable bounds.

Graphically, the membership functions can be represented as follows:

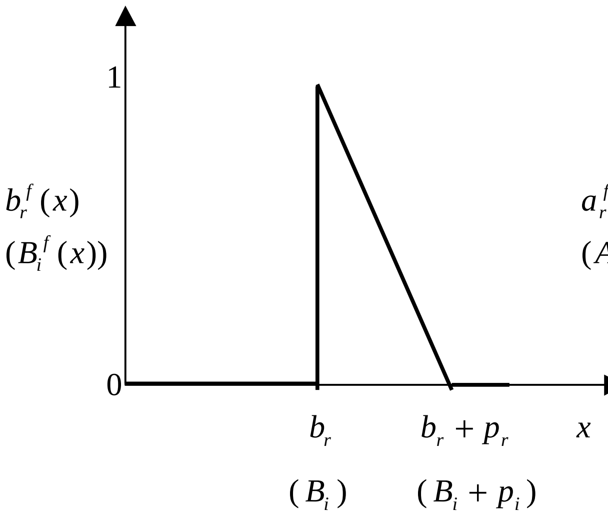

Figure 3.10 Membership Functions of Fuzzy Numbers $b_r^f(x), B_i^f(x)$

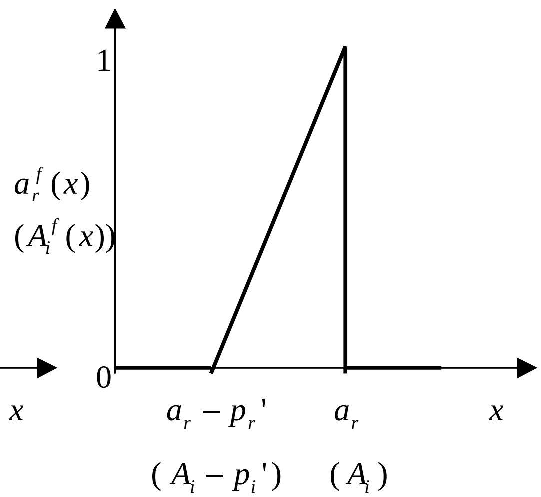

Figure 3.11 Membership Functions of Fuzzy Numbers $a_r^f(x), A_i^f(x)$

p and *p'* are the differences between the most and the least desirable bound values.

Mathematically, the fuzzy bounds can be represented as:

$$b_r^f(x) = \begin{cases} 0 & if\ x \geq b_r + p_r; x < b_r \\ \dfrac{b_r + p_r\text{-}x}{p_r} & if\, b_r < x < b_r + p_r \\ 1 & if\ x = b_r \end{cases} \tag{3.32}$$

$$a_r^f(x) = \begin{cases} 0 & if\, x \leq a_r - p_r'; x > a_r \\ \dfrac{x - a_r + p_r'}{p_r'} & if\, a_r - p_r' < x < a_r \\ 1 & if\, x = a_r \end{cases} \tag{3.33}$$

where $x \in \mathbf{R}$ (Figure 3.10) are the values of the ratios of the bounds

$$B_i^f(x) = \begin{cases} 0 & if \;\; x \geq B_i + p_i; x < B_i \\ \dfrac{B_i + p_i \text{-} x}{p_i} & if \; B_i < x < B_i + p_i \\ 1 & if \;\; x = B_i \end{cases} \tag{3.34}$$

$$A_i^f(x) = \begin{cases} 0 & if \; x \;\leq\; A_i - p_i'; x > A_i \\ \dfrac{x - A_i + p_i'}{p_i'} & if \; A_i - p_i' < x < A_i \\ 1 & if \; x = A_i \end{cases} \tag{3.35}$$

Where $x \in$ **R** (Figure 3.11) are the values of the ratios of the weight bounds.

The α-cut representations of the fuzzy numbers are as follows:

$$^{\alpha}b_r^f = [b_r, b_r + p_r - p_r\alpha] \tag{3.36}$$

$$^{\alpha}a_r^f = [a_r - p_r' + p_r'\alpha, a_r] \tag{3.37}$$

$$^{\alpha}B_i^f = [B_i, B_i + p_i - p_i\alpha] \tag{3.38}$$

$$^{\alpha}A_i^f = [A_i - p_i' + p_i'\alpha, A_i] \tag{3.39}$$

3.2.2.1.2 Degrees of Satisfaction of the Constraints

Fuzzy constraints that contain fuzzy numbers on their RHS can be represented as fuzzy sets. The membership functions of these fuzzy sets, called the “degrees of satisfaction of the constraints”, can be derived from the membership functions of the fuzzy numbers by replacing the argument (x) by the LHS of the constraint (Klir and Yuan (1995)).

The degree of satisfaction of the fuzzy constraint $\dfrac{\mu_r}{\mu_1} \leq b_r^f$ obtained by replacing x by $\dfrac{\mu_r}{\mu_1}$ in (3.32) is given by:

$$\lambda_r^{\,b} = \begin{cases} 0 & if \; \dfrac{\mu_r}{\mu_1} \geq b_r + p_r \\ \dfrac{\mu_1 b_r + \mu_1 p_r \text{-} \mu_r}{\mu_1 p_r} & if \; b_r < \dfrac{\mu_r}{\mu_1} < b_r + p_r \\ 1 & if \; \dfrac{\mu_r}{\mu_1} \leq b_r \end{cases} \tag{3.40}$$

The definition of satisfaction of the constraint $\frac{\mu_r}{\mu_1} \le b_r^f$ coincides with the definition of the fuzzy relation "$\tilde{\le}$"from section 2.5.5.5 comparing a real number on the LHS with a fuzzy number on the RHS.

The degree of satisfaction of the fuzzy constraint $\frac{\mu_r}{\mu_1} \ge a_r^f$ is defined as:

$$\lambda_r^{\ a} = \begin{cases} 0 & if \ \frac{\mu_r}{\mu_1} \le a_r - p'_r \\ \frac{\mu_r - \mu_1 a_r + \mu_1 p'_r}{\mu_1 p'_r} & if \ a_r - p'_r < \frac{\mu_r}{\mu_1} < a_r \\ 1 & if \ \frac{\mu_r}{\mu_1} \ge a_r \end{cases} \tag{3.41}$$

The definition of satisfaction of the constraint $\frac{\mu_r}{\mu_1} \ge a_r^f$ coincides with the definition of the fuzzy relation "$\tilde{\ge}$" from section 2.5.5.5.

The degrees of satisfaction of the input weight constraints are defined as:

$$\lambda_i^{\ B} = \begin{cases} 0 & if \ \frac{\eta_i}{\eta_1} \ge B_i + p_i \\ \frac{\eta_1 B_i + \eta_1 p_i - \eta_i}{\eta_1 p_i} & if \ B_i < \frac{\eta_i}{\eta_1} < B_i + p_i \\ 1 & if \ \frac{\eta_i}{\eta_1} \le B_i \end{cases} \tag{3.42}$$

$$\lambda_i^{\ A} = \begin{cases} 0 & if \ \frac{\eta_i}{\eta_1} \le A_i - p'_i \\ \frac{\eta_i - \eta_1 A_i + \eta_1 p'_i}{\eta_1 p'_i} & if \ A_i - p'_i < \frac{\mu_i}{\eta_1} < A_i \\ 1 & if \ \frac{\eta_i}{\eta_1} \ge A_i \end{cases} \tag{3.43}$$

The membership functions defined in (3.40) to (3.43) are of the fuzzy sets representing the fuzzy constraints. The intersection of these fuzzy sets, $\bigcap_{k=1}^{2s+2m} \lambda_k$, gives the *fuzzy feasible set* $(\tilde{R})$ (Yuan and Klir (1995)).

3.2.2.1.3 Conversion to Crisp Linear Formulation

Model (3.30) is an unsymmetrical fuzzy model. To convert it to an equivalent crisp form we construct the maximizing set of its objective function using Werners' (1984) definition. To determine the maximizing set for the objective function we first need to determine its upper (f_0) and lower (f_1) bounds.

3.2.2.1.3.1 Determination of f_1

f_1 is the supremum of f (the objective function) over R_1, the 1-cut of $\tilde{R}$. As in the case of the absolute weight restriction model, R_1 consists of fuzzy constraints satisfied to a degree 1 i.e., constraints having the most desirable bound values (e.g., a_r) on their RHS. Therefore f_1 is found by solving the following LP:

$$
\begin{aligned}
& f_1 = \text{Max } \mu^T Y_{j0} \\
& \text{Subject to} \\
& \eta^T X_{j0} = 1 \\
& \mu^T Y - \eta^T X \le 0 \\
& \mu_1 a_r \le \mu_r \le \mu_1 b_r \quad r = 2,\ldots,s \\
& \eta_1 A_i \le \eta_i \le \eta_1 B_i \quad i = 2,\ldots,m \\
& \mu_r, \eta_i \ge 0 \qquad \forall r, i
\end{aligned}
\tag{3.44}
$$

3.2.2.1.3.2 Determination of f_0

f_0 is the supremum of the objective function over $S(\tilde{R})$,which is the support of $\tilde{R}$. $S(\tilde{R})$ consists of constraints that have the least desirable bound values (e.g., $a_r - p'_r$) on their RHS. Therefore, f_0 is obtained by solving the following LP:

$$
\begin{aligned}
& f_0 = \text{Max } \mu^T Y_{j0} \\
& \text{Subject to} \\
& \eta^T X_{j0} = 1 \\
& \mu^T Y - \eta^T X \leq 0 \\
& \mu_1 (a_r - p'_r) \leq \mu_r \leq \mu_1 (b_r + p_r) \qquad r = 2,\ldots,s \\
& \eta_1 (A_i - p'_i) \leq \eta_i \leq \eta_1 (B_i + p_i) \qquad i = 2,\ldots,m \\
& \mu_r, \eta_i \geq 0 \qquad \forall r, i
\end{aligned}
\tag{3.45}
$$

3.2.2.1.3.3 The Equivalent Crisp Model

As in the case of the absolute weight restriction model, the crisp model equivalent to (3.30) is a problem of finding the set of weights which give a "maximizing solution" i.e. a solution which satisfies the constraints and the goal with the maximum degree.

The equivalent crisp model is as follows:

$$
\begin{aligned}
& \text{Maximize } \lambda \\
& \text{s.t.} \\
& \eta^T X_0 = 1 \\
& \mu^T Y - \eta^T X \leq 0 \\
& \lambda \leq \frac{\mu^T Y_0 - f_1}{f_0 - f_1} \\
& \lambda \leq \frac{\mu_1 b_r + \mu_1 p_r - \mu_r}{\mu_1 p_r} \\
& \lambda \leq \frac{\mu_r - \mu_1 a_r + \mu_1 p'_r}{\mu_1 p'_r} \\
& \lambda \leq \frac{\eta_1 B_i + \eta_1 p_i - \eta_i}{\eta_1 p_i} \\
& \lambda \leq \frac{\eta_i - \eta_1 A_i + \eta_1 p'_i}{\eta_1 p'_i} \\
& 0 \leq \lambda \leq 1 \\
& \mu_r, \eta_i \geq 0
\end{aligned}
\tag{3.46}
$$

$$
\begin{aligned}
&\text{Max } \lambda \\
&\text{such that} \\
&\lambda(f_0 - f_1) - \mu^T Y_0 \le -f_1 \\
&\eta^T X_0 = 1 \\
&\mu^T Y - \eta^T X \le 0 && \forall j \\
&\lambda\mu_1 p_r + \mu_r \le \mu_1(b_r + p_r) && \forall r \qquad (3.47) \\
&\lambda\eta_1 p_i + \eta_i \le \eta_1(B_i + p_i) && \forall i \\
&- \lambda\mu_1 p'_r + \mu_r \ge \mu_1(a_r - p'_r) && \forall r \\
&- \lambda\eta_1 p'_i + \eta_i \ge \eta_1(A_i - p'_i) && \forall i \\
&0 \le \lambda \le 1 \\
&\mu_r, \eta_i \ge 0 && \forall r, i
\end{aligned}
$$

(3.47) is a quadratic programming model since the weight bound constraints are non-linear. Since λ is one of the variables in the weight bound constraints and we know that it is bounded between 0 and 1, we can solve (3.47) using the parametric algorithm described in section 3.1.1.5.4. The solution of the parametric algorithm gives the optimal weight values which when plugged into the expression $\mu^T Y_0$ give the efficiency scores of the DMUs.

3.2.2.2 The Fuzzy Model with AR Constraints in Form 2

Adding the AR-DEA constraints in (3.28) to the DEA model gives the following AR-DEA model:

$$
\begin{aligned}
&\text{Max } \mu^T Y_{j0} \\
&\text{Subject to} \\
&\eta^T X_{j0} = 1 \\
&\mu^T Y - \eta^T X \le 0 && \qquad (3.48) \\
&\frac{lb_r}{lb_t} \le \frac{\mu_r}{\mu_t} \le \frac{ub_r}{ub_t} && r < t, \quad r, t = 1, \ldots, s \\
&\frac{LB_i}{LB_j} \le \frac{\eta_i}{\eta_j} \le \frac{UB_i}{UB_j} && i < j, \quad i, j = 1, \ldots, m
\end{aligned}
$$

Rearranging the terms in (3.43) to eliminate the fractions, we get the following model:

$$
\begin{aligned}
&\text{Max } \mu^T Y_{j0} \\
&\text{Subject to} \\
&\eta^T X_{j0} = 1 \\
&\mu^T Y - \eta^T X \le 0 \\
&lb_r \mu_t \le lb_t \mu_r && r < t, \quad r, t = 1, \ldots, s \\
&ub_t \mu_r \le ub_r \mu_t && r < t, \quad r, t = 1, \ldots, s \\
&\eta_j LB_i \le \eta_i LB_j && i < j, \; i, j = 1, \ldots, m \\
&UB_j \eta_i \le UB_i \eta_j && i < j, \quad i, j = 1, \ldots, m
\end{aligned}
\tag{3.49}
$$

The values of the price ranges are determined using economic information, which may be either inexact or volatile (Taylor *et al.* (1997)). In fact, the reason we use price "ranges" instead of "exact values" is that enough information is not available to determine exact price values. In this research, we assume that even the price ranges cannot be accurately determined using the available information. To represent the imprecision in the values of the price ranges (referred to as weight bounds from this point onwards), we replace them with fuzzy numbers, which express the concept of approximate numbers. The fuzzy numbers are represented by adding a superscript *f* (signifying fuzziness) to the existing symbols of the crisp bounds which they replace. The resulting model called the fuzzy AR (Form 2) model is as follows:

$$
\begin{aligned}
&\text{Max } \mu^T Y_{j0} \\
&\text{Subject to} \\
&\eta^T X_{j0} = 1 \\
&\mu^T Y - \eta^T X \le 0 \\
&lb_r^f \mu_t \le lb_t^f \mu_r && r < t, \; r, t = 1, \ldots, s \\
&ub_t^f \mu_r \le ub_r^f \mu_t && r < t, \; r, t = 1, \ldots, s \\
&\mathrm{LB}_i^f \eta_j \le LB_j^f \eta_i && i < j, \; i, j = 1, \ldots, m \\
&UB_j^f \eta_i \le UB_i^f \eta_j && i < j, \; i, j = 1, \ldots, m
\end{aligned}
\tag{3.50}
$$

3.2.2.2.1 Definitions of the Fuzzy Numbers and their α-cuts

The fuzzy weight bound constraints in (3.50) are different from those in the previous two types of fuzzy weight restriction models (3.2) and (3.31). The difference is that in (3.2) and (3.31), the fuzzy inequalities compare crisp LHS's with fuzzy RHS's whereas in (3.50), they compare two fuzzy numbers. Therefore, the argument made in section 3.1.1.1 to drop the tighter side of the triangular membership functions of the fuzzy numbers does not hold in this case. The membership functions of the fuzzy bounds in (3.50) are defined by specifying least desirable bounds that are both tighter and more relaxed than the specified bounds. In other words, the least desirable values lie on either side of the most desirable value in the graphical representations of the membership functions of the fuzzy bounds.

For example, the fuzzy number corresponding to the bound lb_r will be: $lb_r^f = \{lb_r, p'_r, q'_r\}$ where p'_r and q'_r are the differences between lb_r and the least desirable values on either side of it.

Graphically, the membership function of lb_r^f is depicted as:

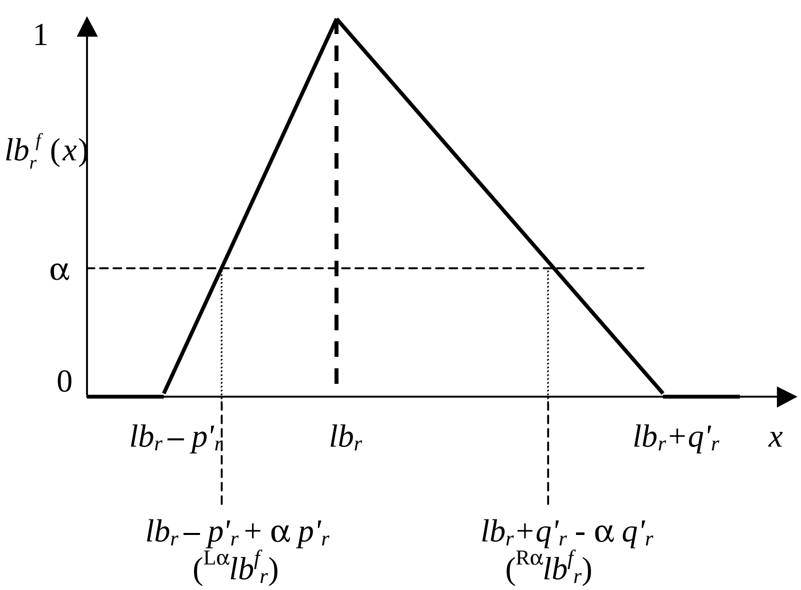

Figure 3.12 Graphical Representation of Fuzzy Number coresponding to lb_r

The mathematical representation of the membership function $lb_r^f(x)$ is:

$$lb_r^f(x) = \begin{cases} \dfrac{x - lb_r + p'_r}{p'_r} & lb_r - p'_r \le x \le lb_r \\ \dfrac{lb_r + q'_r - x}{q'_r} & lb_r < x \le lb_r + q'_r \\ 0 & x < lb_r - p'_r,\ x > lb_r + q'_r \end{cases} \tag{3.51}$$

Similarly, we can define the fuzzy numbers corresponding to the other bound values as:

$$ub_r^f(x) = \begin{cases} \dfrac{x - ub_r + p_r}{p_r} & ub_r - p_r \le x \le ub_r \\ \dfrac{ub_r + q_r - x}{q_r} & ub_r < x \le ub_r + q_r \\ 0 & x < ub_r - p_r,\ x > ub_r + q_r \end{cases} \tag{3.52}$$

$$LB_i^f(x) = \begin{cases} \dfrac{x - LB_i + p'_i}{p'_i} & LB_i - p'_i \le x \le LB_i \\ \dfrac{LB_i + q'_i - x}{q'_i} & LB_i < x \le LB_i + q'_i \\ 0 & x < LB_i - p'_i,\ x > LB_i + q'_i \end{cases} \tag{3.53}$$

$$UB_i^f(x) = \begin{cases} \dfrac{x - UB_i + p_i}{p_i} & UB_i - p_i \le x \le UB_i \\ \dfrac{UB_i + q_i - x}{q_i} & UB_i < x \le UB_i + q_i \\ 0 & x < UB_i - p_i,\ x > UB_i + q_i \end{cases} \tag{3.54}$$

The α-cut of lb_r^f is given by:

$$^{\alpha}lb_r^f = [lb_r - p'_r + \alpha p'_r, lb_r + q'_r - \alpha q'_r] \tag{3.55}$$

Similarly, the α-cuts of the other fuzzy numbers are:

$$^{\alpha}ub_r^f = [ub_r - p_r + \alpha p_r, ub_r + q_r - \alpha q_r] \tag{3.56}$$

$$ {}^{\alpha}UB_i^f = [UB_i - p_i + \alpha p_i, UB_i + q_i - \alpha q_i] \tag{3.57} $$

$$ {}^{\alpha}LB_i^f = [LB_i - p'_i + \alpha p'_i, LB_i + q'_i - \alpha q'_i] \tag{3.58} $$

3.2.2.2.2 Crisp Linear Programming Model

Using the definition of partial orders in terms of α-cuts from section 2.5.5.4 and the definitions of the α-cuts in (3.55) - (3.58), we get the following equivalent of model (3.50).

$$
\begin{array}{ll}
\text{Max } \mu^T Y_{j0} & \\
\text{Subject to} & \\
\eta^T X_{j0} = 1 & \\
\mu^T Y - \eta^T X \le 0 & \\
(lb_r - p'_r + \alpha p'_r)\mu_t \le (lb_t - p'_t + \alpha p'_t)\mu_r & r < t,\ r,t = 1,\ldots,s \\
(lb_r + q'_r - \alpha q'_r)\mu_t \le (lb_t + q'_t - \alpha q'_t)\mu_r & r < t,\ r,t = 1,\ldots,s \\
(ub_t - p_t + \alpha p_t)\mu_r \le (ub_r - p_r + \alpha p_r)\mu_t & r < t,\ r,t = 1,\ldots,s \\
(ub_t + q_t - \alpha q_t)\mu_r \le (ub_r + q_r - \alpha q_r)\mu_t & r < t,\ r,t = 1,\ldots,s \\
(LB_i - p'_i + \alpha p'_i)\eta_j \le (LB_j - p'_j + \alpha p'_j)\eta_i & i < j,\ i,j = 1,\ldots,m \\
(LB_i + q'_i - \alpha q'_i)\eta_j \le (LB_j + q'_j - \alpha q'_j)\eta_i & i < j,\ i,j = 1,\ldots,m \\
(UB_j - p_j + \alpha p_j)\eta_i \le (UB_i - p_i + \alpha p_i)\eta_j & i < j,\ i,j = 1,\ldots,m \\
(UB_j + q_j - \alpha q_j)\eta_i \le (UB_i + q_i - \alpha q_i)\eta_j & i > j,\ i,j = 1,\ldots,m \\
\mu_i, \eta_j \ge 0 & \forall r, i
\end{array}
\tag{3.59}
$$

Since all numbers in (3.59) are real numbers, this is a crisp parametric linear programming model. The α in each case represents the degree of proximity to the specified bound. According to the principle of confluence of goals and constraints proposed by Bellman and Zadeh (1972), the maximizing solution of (3.59) will be attained when all goals and constraints are satisfied to the maximum degree. Therefore, we solve (3.59) for the same degree of satisfaction (λ) of all constraints and try to maximize λ. The model equivalent to (3.59) is then given by:

$$
\begin{array}{ll}
\text{Max } \mu^T Y_{j0} & \\
\text{Subject to} & \\
\eta^T X_{j0} = 1 & \\
\mu^T Y - \eta^T X \le 0 & \\
(lb_r - p'_r + \lambda p'_r)\mu_t \le (lb_t - p'_t + \lambda p'_t)\mu_r & r < t, \ r,t = 1,\ldots,s \\
(lb_r + q'_r - \lambda q'_r)\mu_t \le (lb_t + q'_t - \lambda q'_t)\mu_r & r < t, \ r,t = 1,\ldots,s \\
(ub_t - p_t + \lambda p_t)\mu_r \le (ub_r - p_r + \lambda p_r)\mu_t & r < t, \ r,t = 1,\ldots,s \\
(ub_t + q_t - \lambda q_t)\mu_r \le (ub_r + q_r - \lambda q_r)\mu_t & r < t, \ r,t = 1,\ldots,s \\
(LB_i - p'_i + \lambda p'_i)\eta_j \le (LB_j - p'_j + \lambda p'_j)\eta_i & i < j, \ i,j = 1,\ldots,m \\
(LB_i + q'_i - \lambda q'_i)\eta_j \le (LB_j + q'_j - \lambda q'_j)\eta_i & i < j, \ i,j = 1,\ldots,m \\
(UB_j - p_j + \lambda p_j)\eta_i \le (UB_i - p_i + \lambda p_i)\eta_j & i < j, \ i,j = 1,\ldots,m \\
(UB_j + q_j - \lambda q_j)\eta_i \le (UB_i + q_i - \lambda q_i)\eta_j & i > j, \ i,j = 1,\ldots,m \\
0 \le \lambda \le 1 & \\
\mu_i, \eta_j \ge 0 & \forall r, i
\end{array}
\tag{3.60}
$$

The model (3.60) will always be feasible for all values of λ between 0 and 1. Since 1 is the maximum value possible for λ, we can replace λ by 1 in (3.60). However, we refrain from doing so because leaving λ in the model and varying it between 0 and 1 allows us to determine the efficiency scores for different degrees of closeness to the specified bounds.

3.2.3 Roadmaps for Fuzzy AR Models

In this section, we illustrate the implementation methodology of the fuzzy AR models of both forms using roadmaps. Section 3.2.3.1 contains the roadmap for the fuzzy model with AR constraints in Form 1 and section 3.2.3.2 contains the roadmap for the fuzzy model with AR constraints in Form 2.

3.2.3.1 Roadmap for Developing and Solving the Fuzzy Model with AR Constraints in Form 1

The example used in this section from Roll and Golany (1993) is the same as that used in the roadmap for illustrating the fuzzy absolute weight restriction model. The steps involved in developing and solving the fuzzy AR (Form 1) model are as follows:

Step1: ***Normalize the raw data***

When AR constraints are used, close attention needs to be paid to the units in which the respective factors are measured (Roll and Golany (1993)). To resolve this difficulty, Roll and Golany (1993) recommend bringing all factor values to the same order of magnitude through normalization (make the average of all columns 100). The normalized version of the data in Table 3.2 is shown in Table 3.11 below:

DMU	y_1	y_2	y_3	x_1	x_2	x_3	x_4
1	103	40	144	97	69	101	112
2	91	30	107	139	111	190	61
3	120	95	63	174	33	162	130
4	59	43	95	38	101	94	91
5	72	84	24	33	66	99	121
6	115	78	79	109	120	97	137
7	140	258	154	118	170	92	81
8	63	21	132	85	144	27	102
9	61	33	125	88	41	138	59
10	44	70	88	60	101	45	71
11	79	54	147	68	125	64	112
12	175	317	69	109	93	109	117
13	76	41	93	64	89	161	127
14	48	119	66	111	71	36	102
15	253	217	115	209	168	84	79
Avg.	100	100	100	100	100	100	100

Table 3.11 Normalized Input /Output Data for the Roadmap Example Illustrating the Fuzzy AR (Form 1) Model

Step 2: ***Determine the most desirable bounds on the ratios of weights***

Roll and Golany (1993) treat input 1 as the input numeraire and output 1 as the output numeraire. Although it's not explicitly stated how the bounds on the ratios of weights are determined, we assume that expert opinion is used. The AR bounds used by Roll and Golany (1993) are shown in Table 3.12.

Factor	Upper bound (b, B)	Lower bound (a, A)
μ_2/μ_1	1.0	0.2
μ_3/μ_1	0.5	0.1
η_2/η_1	4.0	0.25
η_3/η_1	4.0	0.25
η_4/η_1	0.4	0.1

Table 3.12 AR Bounds Used by Roll and Golany (1993)

Step 3: ***Determine the fuzzy bounds***

Since the bound values are based on human judgement, there is uncertainty associated with their values. To minimize the effect of uncertainty on the results of the model, we replace the crisp bounds by fuzzy bounds. The fuzzy bounds use the values specified in Table 3.12 as the most desirable. The least desirable bounds are determined from the results of the unbounded model. Table 3.13 shows the results of the unbounded model along with the ratio of each input weight to the weight of input 1 (numeraire) and of each output weight to the weight of output 1. The least desirable upper and lower bounds are the highest and lowest values of these ratios for efficient DMUs.

	μ_1	μ_2	μ_3	η_1	η_2	η_3	η_4		μ_2/μ_1	μ_3/μ_1	η_2/η_1	η_3/η_1	η_4/η_1
1	0.0025	0	0.0052	0.0079	0.0033	0	0	1	0	2.0616	0.4177	0	0
2	0.0021	0	0.0067	0	0	0	0.0164	0.9077	0	3.2711	-	-	-
3	0.0069	0	0.0027	0.0037	0.0107	0	0	1	0	0.3824	2.8623	0	0
4	0	0	0.0105	0.0263	0	0	0	1	-	-	0	0	0
5	0.0049	0.0077	0	0.0303	0	0	0	1	1.5885	0	0	0	0
6	0.0054	0	0.0014	0.0058	0	0.0038	0	0.7349	0	0.2634	0	0.648	0
7	0	0.0024	0.0025	0.0083	0	0	0.0002	1	-	-	0	0	0.0277
8	0	0	0.0076	0.0033	0	0.0029	0.0063	1	-	-	0	0.8741	1.8874
9	0	0	0.008	0.0073	0	0	0.0061	1	-	-	0	0	0.8328
10	0	0.0034	0.0074	0.0091	0	0.0101	0	0.884	-	-	0	1.1203	0
11	0	0	0.0068	0.0095	0.0028	0	0	1	-	-	0.294	0	0
12	0	0.0032	0	0.0092	0	0	0	1	-	-	0	0	0
13	0.0069	0	0.0039	0.0104	0.0038	0	0	0.8847	0	0.5561	0.3611	0	0
14		0.0034	0.009	0.0011	0.0087	0.0071	0	1	-	-	7.5648	6.1741	0
15	0.0035	0	0.001	0.004	0	0	0.0021	1	0	0.2775	0	0	0.529

Table 3.13 Results of the Unbounded DEA model for the AR (Form 1) Roadmap Example

Note: The "-" in the table represents a value with a 0 denominator. These values are ignored in the analysis.

Considering only efficient DMUs, the lowest value of the ratio μ_2/μ_1 is 0 and the highest value is 1.5885. Therefore, the least desirable lower and upper bounds on the ratio μ_2/μ_1 for the purpose of the fuzzy model will be 0 and 1.5885 receptively. Similarly, the least desirable lower and upper bounds on the ratio μ_3/μ_1 will be 0 and 2.062[16] respectively. The rest of the values calculated similarly are shown in Table 3.14.

Factor Ratio	Least Desirable Upper bound ($b+p$, $B+p$)	Least Desirable Lower bound ($a-p'$, $A-p'$)
μ_2/μ_1	1.5885	0
μ_3/μ_1	2.062	0
η_2/η_1	7.56	0
η_3/η_1	6.17	0
η_4/η_1	1.89	0

Table 3.14 Least Desirable Weight Bound Values used in the Roadmap Example for Illustrating the Fuzzy AR (Form 1) Model

Step 4: *Solve the crisp equivalent model*

f_1 is obtained by solving the AR-DEA model with bound values from Table 3.12. f_0 is obtained by solving the AR-DEA model with bound values from Table 3.14. Using the f_0 and f_1 values, the crisp model (3.47) is solved using the parametric algorithm described earlier. ε is chosen to be 0.1. The results of the fuzzy model are compared with the results of the crisp model in Table 3.15.

[16] Although the ratio 3.27 is higher than 2.062, it is not considered because it is obtained from the weights

DMU	Efficiency Scores		
	Crisp bounds	Fuzzy bounds	λ
1	0.85155	1	0.8
2	0.47673	0.67941	0.6
3	0.90362	1	0.8
4	0.75333	0.96234	0.7
5	0.73712	0.9116	0.5
6	0.63773	0.69607	0.5
7	0.99046	1	0.9
8	0.93607	1	0.9
9	0.8533	1	0.8
10	0.70533	0.81129	0.7
11	0.88081	1	0.8
12	1	1	1
13	0.58903	0.77286	0.6
14	0.77929	0.87624	0.5
15	1	1	1
Average	**0.806291**	**0.913987**	

Table 3.15 Comparison of Results of Crisp and Fuzzy AR (Form 1) Weight Bound Models for the Roadmap Example

Applying a paired two sample t test (H_0: $E_2 - E_1 = 0$ versus H_a: E_2-E_1>0) to the two sets of efficiency scores in Table 3.15, we get a p-value of 0.00002. Thus, with an α value as low as 0.00005, we can reject the null hypothesis. This implies that we accept the alternative hypothesis that there is a significant difference between the efficiency scores calculated by the two models.

We see that several DMUs move from the inefficient set to the efficient set when the bounds are made fuzzy. Two of those DMUs (7 & 8) become efficient at a 90% degree of satisfaction with the specified bounds. Therefore, as in section 3.1.3, we modify the existing bounds so that they are at the 90% satisfaction level in order to restore DMUs 7 and 8 to the efficient set. The original set of bounds is shown in Table 3.12. The modified set of bounds is shown in Table 3.16.

---

assigned to the factors by an inefficient DMU.

Factor	Upper bound (b, B)	Lower bound (a, A)
μ_2/μ_1	1.05885	0.18
μ_3/μ_1	0.6562	0.09
η_2/η_1	4.356	0.225
η_3/η_1	4.217	0.225
η_4/η_1	0.549	0.09

Table 3.16 Modified Set of Bounds at a 90% Level of Satisfaction of Original Bounds for the AR (Form 1) Roadmap Example

Note that there is only a small difference between bound values in the two tables. Table 3.17 compares the results of the AR model with specified bounds with those of the AR model with the modified bounds.

	Efficiency Scores	
DMU	Original crisp bounds	Modified set of bounds
1	0.85155	0.94179
2	0.47673	0.54552
3	0.90362	0.96566
4	0.75333	0.85255
5	0.73712	0.76305
6	0.63773	0.66127
7	0.99046	1
8	0.93607	0.963
9	0.8533	0.98115
10	0.70533	0.76341
11	0.88081	0.96944
12	1	1
13	0.58903	0.65236
14	0.77929	0.81145
15	1	1
Avg.	**0.842259**	**0.86051**

Table 3.17 Comparison of results of AR (Form 1) Models with Original and Modified Bounds applied to the Roadmap Example

We see that just a slight modification of the bounds allowed DMU 7 to move from the inefficient set to the efficient set. DMU 8 however did not move to the efficient set as expected.

3.2.3.2 Roadmap for Developing and Solving the Fuzzy Model with AR Constraints in Form 2

For demonstrating this roadmap, we use the AR-DEA model developed by Taylor *et al.* (1997) for calculating the efficiencies of 13 Mexican banks.

Step 1: ***The Bank Model and the Data***

The model used in this study had total income (y_1) as the single output. Total deposits (x_1) and total non-interest expense (x_2) were the two inputs used to generate the output. The data for thirteen Mexican commercial banks are shown in Table 3.18. These data were obtained from the annual financial reports from these banks. Although Taylor *et al.* (1997) solve the AR DEA problem for 3 years of data from 1989 to 1991 we develop and solve the fuzzy model only for the year 1989 since the roadmap is for illustration purposes only.

Bank #	Bank Name	Deposits (X_1)	Non-int. expense (X_2)	Int. income plus non-int. income (Y_1)
1	Banamex	31451.9	1540.8	9648.5
2	Bancomer	24267.8	1491.1	9396.9
3	Serfin	16609	1072.6	6884.8
4	Internac	4109.14	561.6	1924
5	Cremi	1657.57	1842.5	2427.9
6	Banceser	2124	85.2	617.3
7	MercNort	1540	198.3	737.7
8	BCH	1750	152.3	907.9
9	Confia	1728.88	151.2	65.5
10	Bancern	1313.48	129.7	705.4
11	Promex	1410.97	173.5	674.3
12	Banoro	586.29	127.3	505.1
13	Banorie	302.14	45.3	211

Table 3.18 Input / Output Values for 13 Mexican banks in 1989 (Billions of Nominal Pesos) – from Taylor *et al.* (1997)

Step 2: ***Determine "price/cost" ranges or AR bounds***

The AR bounds are obtained from the range of nominal interest rates charged on loans and paid on deposits in the portfolios of the thirteen banks. The nominal interest rate ranges were determined using the following information:

- The weighted average cost-of-funds for mortgage finance institutions,

- The average deposit rate on three-month term deposits and
- The dealer money-market interest rate.

Table 3.19 shows the upper and lower bounds for the input and output multipliers that were used in the AR analysis by Taylor *et al.* (1997). Since the second input (non-interest expense) and the single output (total income) are expressed in price units (total nominal pesos), their multipliers (weights) will not be prices. Therefore, we cannot use price information to put bounds on the multipliers. The bounds are accordingly set to one.

	1989		1990		1991	
	LB (lb)	*UB (ub)*	*LB (lb)*	*UB (ub)*	*LB (lb)*	*UB (ub)*
Y_1	1	1	1	1	1	1
X_1	0.20874	0.51356	0.2207	0.60871	0.14973	0.42784
X_2	1	1	1	1	1	1

Table 3.19 Upper and Lower "price/cost" Data Bounds for Multipliers in the Roadmap Example illustrating the Fuzzy AR (Form 2) Model

The inflation rate in Mexico was relatively high during the study period. Due to this, the interest rates were very volatile. The volatility makes the ranges of prices shown in Table 3.19 uncertain. To model the uncertainty in the bounds, we replace them by fuzzy numbers.

Step 3: *Determine the fuzzy bounds*

In order to define the fuzzy numbers used in the model, we need to define the most desirable bound values and the least desirable values. Since the model is being solved for the 1989 data, the price/cost ranges prevalent in that year (from Table 3.19) will be used as the most desirable values for the fuzzy bounds. For determining the least desirable bounds, we need price/cost values that are different from values prevalent in 1989. Since the price/cost information for the years 1990-91 is readily available, we use it to determine the least desirable bound values. From Table 3.19, we observe that the 1991 price/cost values are lower than those for 1989 and therefore can be used as the least desirable lower ends ($lb - p$ and $ub - p$) of the fuzzy numbers. We also observe that the 1990 values are higher than those for 1989 and can be used as the least desirable upper ends (*lb* + *q* and *ub* + *q*) of the fuzzy numbers. Table 3.20 shows the bound values that are plugged into model (3.61).

	lb	*ub*	*lb + q*	*ub + q*	*lb - p*	*ub - p*
X_1	0.20874	0.51356	0.2207	0.60871	0.14973	0.42784
X_2	1	1	1	1	1	1

Table 3.20 Least Desirable Weight Bound Values for the Roadmap Example Illustrating the Fuzzy AR (Form 2) Model

Model (3.61) is solved for six different values of λ (0, 0.2, 0.4, 0.6, 0.8, 1) where λ represents the degree of closeness to the specified bound value. Table 3.21 compares the results of the model using different values of λ with the results obtained by Taylor *et al.* (1997) for the crisp AR model (λ=1).

Bank #	Bank Name	Efficiency Scores						
		Unbounded CCR	Crisp AR	Fuzzy AR				
			$\lambda = 1$	$\lambda = 0.8$	$\lambda = 0.6$	$\lambda = 0.4$	$\lambda = 0.2$	$\lambda = 0$
1	Banamex	0.914	0.588	0.58625	0.58415	0.58209	0.58007	0.57808
2	Bancomer	0.971	0.708	0.70617	0.70394	0.70174	0.69959	0.69747
3	Serfin	1	0.75	0.74737	0.74508	0.74283	0.74062	0.73844
4	Internac	0.709	0.67	0.66921	0.66835	0.66751	0.66667	0.66585
5	Cremi	1	0.764	0.75446	0.74426	0.73385	0.72321	0.71233
6	Banceser	1	0.577	0.57503	0.57278	0.57058	0.56843	0.56631
7	MercNort	0.751	0.702	0.70057	0.69956	0.69857	0.69758	0.69661
8	BCH	1	0.867	0.86495	0.86286	0.8608	0.85878	0.85678
9	Confia	0.74	0.642	0.64085	0.6393	0.63779	0.63629	0.63482
10	Bancern	0.972	0.863	0.86152	0.85969	0.8579	0.85613	0.85439
11	Promex	0.769	0.712	0.71106	0.70995	0.70885	0.70777	0.70671
12	Banoro	1	1	1	1	1	1	1
13	Banorie	1	0.962	0.96147	0.96047	0.95949	0.95853	0.95757
	Average	**0.9021667**	**0.736917**	**0.734787**	**0.732493**	**0.730209**	**0.727928**	**0.725649**

Table 3.21 Comparison of Efficiency Scores of Unbounded CCR model, Crisp AR model and the Fuzzy AR (Form 2) model for the Roadmap Example

We use the two sample paired t test to compare the results of the crisp AR model with those of the fuzzy AR model. The results of the t tests are shown in Table 3.22.

	λ=0.8	λ=0.6	λ=0.4	λ=0.2	λ=0
P – value	0.005014	0.004491	0.004605	0.004852	0.005162
Result of hypothesis	Reject null at α=0.01	Reject null at α=0.005	Reject null at α=0.005	Reject null at α=0.005	Reject null at α=0.01
Conclusion	Difference in efficiency scores significant	Difference in efficiency scores significant	Difference in efficiency scores significant	Difference in efficiency scores significant	Difference in efficiency scores significant

Table 3.22 Results of the Two Sample Paired t Test comparing the Efficiency Scores of the Crisp and Fuzzy AR (Form 2) Models for the Roadmap Example

Although the difference between the efficiency scores using the fuzzy AR and crisp AR models is statistically significant, the difference is not very obvious when one eyeballs the results. The difference is apparent only beyond the second decimal place. Because of the small difference between the efficiency scores, the results do not provide the decision-maker with as much insight as did the results of the previous two fuzzy models. This is in direct contrast to the assumptions made by us (not stated) that the fuzzy model corresponding to AR in Form 2 is more elegant and easier to solve compared to the one corresponding to Form 1 AR. We call the fuzzy model with AR constraints in Form 2 as more elegant because it uses triangular membership functions, which represent the true notion of "close to the specified bounds" and allow us to model the imprecision on both sides of the specified bounds. The reason we think this type of model is easier to solve compared to the fuzzy model with Form 1 AR constraints is that it does not require solving three LP's before getting to the solution and it does not require using the parametric algorithm.

The triangular membership function, which was thought to be this model’s greatest strength, turned out to be it’s greatest weakness. That’s because, while one end of the triangular membership function which tightens the bounds has the effect of decreasing the objective function (efficiency scores), the other end which relaxes the bounds has the effect of increasing the objective function. The overall effect is that the objective function changes very little. This explains why the efficiency scores calculated by the fuzzy (Form 2) AR model are not very diferent from those calculated by the corresponding crisp model.

Chapter 4

APPLICATION, RESULTS, AND DISCUSSION

In this chapter, we illustrate the three fuzzy models developed in Chapter 3 by applying them to data sets from the DEA literature on crisp weight restriction models. Table 4.1, at a glance, provides information about the source of the data set used for illustrating each fuzzy weight restriction model.

Fuzzy Weight Restriction Model	**Source of Data Set**
Absolute Weight Restriction Model (see Section 3.1)	*A DEA Model for Measuring the Relative Efficiency of Highway Maintenance Patrols* – Cook *et al.* (1990)
AR-DEA model with AR Bounds in Form 1 (usually determined based on expert opinion) (see Section 3.2.1)	*Measuring Technical Efficiency in a Fuzzy Environment* [17]– Girod (1996)
AR-DEA model with AR Bounds in Form 2 obtained Using "price/cost" Data (see Section 3.2.2)	*DEA/Assurance region SBDC Efficiency and Unique Projections* – Thompson *et al.* (1996a)

Table 4.1 Sources of Data Sets for Illustrating the Fuzzy Weight Restriction Models

This chapter is divided into three sections, with each section dedicated to one type of fuzzy weight restriction model. At the beginning of each section, some background information about the data set is provided. This is followed by a discussion on how the weight bounds for that data set were determined. Then, fuzzy weight bounds are determined and applied to the same data set. Finally, the section is concluded by comparing the results of the fuzzy weight restriction model with those of the corresponding crisp model.

[17] In this reference, no weight restriction model is solved. However, information about the relative importance of the inputs was available. This information was used to set crisp and fuzzy bounds.

4.1 FUZZY ABSOLUTE WEIGHT RESTRICTION MODEL APPLIED TO MEASURING RELATIVE EFFICIENCY OF HIGHWAY MAINTENANCE PATROLS

4.1.1 Background

Cook *et al.* (1990) used DEA to measure the relative efficiency of 14 highway maintenance patrols. Two inputs and two outputs were included in the analysis. The first output called the assignment size factor was a composite measure of all factors that were indicators of the "size of the system" such as surface, shoulder, right of way and median, and winter operations. The other output was the Average Traffic Serviced. The two inputs included in the analysis were Total Expenditure and Average Pavement condition Rating. In the first run of the model, the weights were allowed to vary freely. The results of the unbounded runs are shown in Table 4.2.

DMU	Efficiency	u_1x10^5	u_2 x10^5	v_1 x10^5	v_2 x10^5
1	1	1436	10	913	4690
2	0.999	1621	10	1030	5292
3	0.803	1760	10	1688	1312
4	1	1623	10	1557	1210
5	0.86	1535	10	976	5013
6	0.931	2087	10	2246	10
7	0.885	1585	2501	1804	10
8	1	2032	10	2187	10
9	0.913	1635	10	1039	5339
10	0.724	1778	10	1130	5806
11	0.874	708	17883	1697	10
12	1	389	9815	930	10
13	1	808	3291	90	12208
14	0.619	742	13041	1114	2805

Table 4.2 Unbounded Weight Matrix for Highway Maintenance Patrol Data from Cook *et al.* (1990)

After investigating the weight matrix of the unbounded run, the authors found that different DMUs were assigning vastly different weights to the same factor. For example u_2 = 0.00001 for DMU#1 but u_2 = 0.017883 for DMU#11 i.e. patrol #11 was allowed to attach much greater importance to the second output compared to patrol #1. Based on this observation, the authors reached the agreement that although some degree of flexibility is desirable to enable the DMUs to express their own circumstances, the flexibility should fall within some reasonable limits. These limits were imposed in the absolute weight restrictions model.

4.1.2 Crisp Absolute Weight Restrictions Model

Using the weight matrix in Table 4.2, Cook *et al.* (1990) determined the absolute weight bounds, which are shown in Table 4.3.

	u_1	u_2	v_1	v_2
Upper Bounds	2100	10000	2500	6000
Lower Bounds	800	500	900	300

Table 4.3 Absolute Weight Bounds Used by Cook *et al.* (1990)

The model was solved again with the weights controlled by bounds. The results of the bounded model are shown in Table 4.4.

DMU	Efficiency
1	1
2	0.995
3	0.8
4	1
5	0.854
6	0.929
7	0.884
8	1
9	0.91
10	0.722
11	0.803
12	0.913
13	0.876
14	0.614

Table 4.4 Results of the Crisp Absolute Weight Restrictions Model from Cook *et al.* (1990)

The consequences of imposing weight bounds were:

- All efficiency ratings fell below the previous (unbounded) levels.

- DMUs 12 and 13, which were on the frontier in the unbounded model, fell below the frontier.

4.1.3 Fuzzy Absolute Weight Restrictions Model

To model the uncertainty associated with the bound values shown in Table 4.3, we replace them with fuzzy numbers. Fuzzy numbers are sets in which different members have different degrees of membership. As stated in section 3.1.1.1, the most desirable values (values with highest degree of membership) in the fuzzy numbers will be the user specified crisp bounds from Table 4.3. As stated in section 3.1.1.1, the least desirable values will be determined from the unbounded weight matrix using Method II.

The least desirable upper and lower bounds will be the highest and lowest values respectively assigned to the weights by the efficient DMUs. In other words, the least desirable bound values are such that they allow all efficient DMUs in the unbounded run to remain efficient in the bounded run. They are referred to as least desirable bounds because by allowing unreasonable multipliers to dominate the analysis, they defeat the purpose of imposing weight bounds. Looking at the unbounded weight matrix in Table 4.2, we see that u_1 varies from 389 to 2032 for efficient DMUs. Therefore, we assign 389 as the least desirable lower bound and 2032 as the least desirable upper bound for u_1. The least desirable bounds for other factors are calculated on similar lines and displayed in Table 4.5.

Multiplier	u_1	u_2	v_1	v_2
Least Desirable Upper Bound	2032	9815	2187	12208
Least Desirable Lower Bound	389	10	90	10

Table 4.5 Least Desirable Upper and Lower Bounds for the Absolute Weight Bound Model from Cook et al. (1990)

Comparing Table 4.5 with Table 4.3, we see that in some cases (e.g. upper bounds on u_1, u_2, and v_1), the most desirable bounds are more permissive than the least desirable bounds. In section 3.1.1.1, we showed that for the fuzzy model to seek a compromise between maximization of the efficiency scores and maximization of the satisfaction of the decision-maker with the bounds, the least desirable bounds should be more relaxed compared to the most desirable bounds. Hence, we do not use the values in Table 4.5 as least desirable upper bounds on u_1, u_2 and v_1. In fact, since we think that the existing bounds themselves are so lenient, we allow the upper bounds on u_1, u_2 and v_1 to remain crisp. Thus, the only bounds, which we replace by fuzzy numbers, are the lower bounds of all weights and the upper bound of v_2.

The lower bound on the objective function, f_1, is obtained by using the following weight bound constraints:

$$\begin{aligned} &800 \le u_1 \le 2100 \\ &500 \le u_2 \le 10000 \\ &900 \le v_1 \le 2500 \\ &300 \le v_2 \le 6000 \end{aligned} \qquad (4.1)$$

Similarly, the upper bound of the objective function, f_0, is obtained when the following weight bound constraints are used:

$$\begin{aligned} &389 \le u_1 \le 2100 \\ &10 \le u_2 \le 10000 \\ &90 \le v_1 \le 2500 \\ &10 \le v_2 \le 12208 \end{aligned} \qquad (4.2)$$

Finally, the objective function of the crisp equivalent of the fuzzy model is obtained by using the following set of constraints:

$$\begin{aligned} &389 + 411\lambda \le u_1 \le 2100 \\ &10 + 490\lambda \le u_2 \le 10000 \\ &90 + 810\lambda \le v_1 \le 2500 \\ &10 + 290\lambda \le v_2 \le 12208 - 6208\lambda \\ &0 \le \lambda \le 1 \end{aligned} \qquad (4.3)$$

The coefficients of λ in equation (4.3) are the differences between the bound values in (4.2) and (4.1). Table 4.6 compares the results of the fuzzy model with those of the crisp model from Table 4.3.

DMU	Efficiency		
	Crisp model	Fuzzy model	λ
1	1	1	1
2	0.995	0.997	0.5
3	0.8	0.802	0.4
4	1	1	1
5	0.854	0.857	0.5
6	0.929	0.930	0.4
7	0.884	0.884	0.4
8	1	1	1
9	0.91	0.912	0.5
10	0.722	0.723	0.5
11	0.803	0.840	0.4
12	0.913	1.000	0.7
13	0.876	0.952	0.5
14	0.614	0.617	0.6
Avg.	**0.879**	**0.894**	

Table 4.6 Comparison of Efficiency Scores of Crisp and Fuzzy Absolute Weight Restriction models applied to the Highway Maintenance Patrol Data from Cook *et al.* (1990)

Note that for each DMU, the efficiency score obtained using the fuzzy model is greater than that obtained using the crisp model. Especially notable is DMU 12, which moved from the inefficient set to the efficient set when the bounds were changed from crisp to fuzzy. DMU 12 is the best example of a borderline DMU that was penalized by the imprecision in bound values and was rescued by the fuzzy model. The average efficiency using crisp bounds is 0.879 while that using fuzzy bounds is 0.894.

To check if the difference in efficiency scores calculated by the two models is significantly greater than 0, we use the paired two Sample t-Test for means (Bain and Engelhardt (1992)). The details of the test are H_0: E_2-E_1=0; H_a: E_2-E_1>0 where E_2 is the average efficiency score generated by the fuzzy model and E_1 is the average efficiency score generated by the crisp bounded model. The result of the test is a p-value of 0.037

allowing us to reject the null hypothesis using an α value of 0.05. Consequently, we accept the alternative hypothesis that the efficiency scores obtained using the fuzzy model are significantly greater than those obtained using the crisp model. The detailed results of the test are shown in Table 4.7.

	Fuzzy Model	Crisp Model
Mean	0.894	0.879
Variance	0.014	0.013
Observations	14	14
Pearson Correlation	0.968	
Hypothesized Mean Difference	0	
Df	13	
t Stat	1.938	
P(T<=t) one-tail	0.037	
t Critical one-tail (assuming α = 0.05)	1.771	

Table 4.7 Results of t-Test: Paired Two Sample for Means comparing the Efficiency Scores of the Crisp and Fuzzy Absolute Weight Restriction Models applied to the Highway Maintenance Patrol Data from Cook *et al.* (1990)

Clearly, t stat > t critical allowing us to reject the null hypothesis.

4.2 FUZZY AR (Form 1) DEA MODEL APPLIED TO EVALUATING PRODUCTIVE EFFICIENCY OF A NEWSPAPER PREPRINT INSERTION PROCESS

4.2.1 Background

Girod (1996) uses a fuzzy set-based methodology to accommodate the measurement inaccuracies associated with production plans generated by a newspaper preprint insertion manufacturing process. Because the values of the inputs and outputs are imprecise, he replaces them with fuzzy numbers. The membership functions of these fuzzy numbers vary between the risk free and impossible bounds. The risk free bound for a particular input or output is the most pessimistic value of that input or output. The impossible bound is the most optimistic value. Naturally, the risk free bound corresponds to a membership grade of 1 and the impossible bound corresponds to a membership grade of 0. In addition to the risk-free and impossible values for all inputs and outputs, Girod (1996) also calculates intermediate values, which he obtains by varying the membership grade between 0 and 1 in increments of 0.2.

In the current research, we are looking at fuzzy weight bounds and not fuzzy input/output data. Therefore, we assume that the input/output data is crisp. To be able to use the data from Girod (1996) for this research, we need to pick one set of values for each factor from the available range. The natural choice is the central value, which happens to be the value corresponding to a membership grade of 0.6.

The production process analyzed in this paper is the "Newspaper preprint insertion process". Newspaper preprint insertion involves merging incoming newspaper sections and commercial preprints into bundles ready for delivery to newspaper distributors. All newspapers are divided into two major sections - the news section and the nonnews sensitive section. The commercial preprints are inserted in the nonnews sensitive section referred to as NNSS from this point onwards. The preprint insertion typically occurs only once a week. Production data was gathered for 48 weeks. The preprint insertion line analyzed by Girod (1996) works as follows.

Two line operators manually position NNSS's on a mechanical loader that conveys them to a feeding hopper that in turn feeds them into the preprint insertion machine's steel pockets. Once in the steel pocket, the NNSS fold is mechanically opened. In the meantime, other line-operators position commercial pamphlets into the preprint insertion hoppers which drop the inserts in the NNSS fold, producing the intermediate packages. Girod (1996) considered three inputs and one output for the study. The first input is direct labor and is defined as the number of hourly workers dedicated to the preprint insertion production line multiplied by the total production time and the worker hourly rate. This input is fuzzy because the varying package mix causes the manpower requirements to fluctuate from week to week.

The second input used is rework. Rework represents the number of packages that would have to be reprocessed if the preprint-inserting machine produced only accurate packages. It is a proxy variable for the amount of labor that would have to be committed to retrieving nonconforming packages from the production stream and reworking them. The objective of including this input is to minimize the number of defective packages. There are two types of preprint defects – "preprint misses" (no preprint inserted) and "preprint multiples" (more than one preprint inserted). Therefore, rework is defined as the number of "preprint misses" plus the number of "preprint multiples". Both types of defects are recorded by sensors. Unfortunately, the sensor for only one of the types is accurate. The sensor for the other type is inaccurate and therefore this input is treated as fuzzy.

The third input, called raw material, is included to ensure that the waste of NNSS’s due to defects is minimized. The NNSS waste, which is the difference between the number of NNSS’s at the start and the number of defect-free packages at the end, is captured by the raw material variable. There are three types of NNSS defects: miss, multiple; or unopened. These defects are detected and recorded by a sensor at the end of the line. Since the sensor is inaccurate, this input is also treated as fuzzy.

The output variable for the line is defined as the quantity of packages produced by the preprint insertion line per production day minus the amount of incomplete packages. Incomplete packages are caused by preprint shortages.

4.2.2 Crisp AR (Form 1) DEA Model

In Girod (1996), no bounds were imposed on the multipliers. For the purpose of this research, the decision-makers were asked to provide pertinent information for setting the AR bounds. The decision-makers were of the opinion that conserving the labor input was:

a) 1.5 as important as conserving the rework input and
b) twice as important as conserving the raw material input.

Based on this information, the following crisp AR bounds were applied to the Washingtonpost production data.

$$\begin{aligned} 1.5v_2 - v_1 &\leq 0 \\ 2v_3 - v_1 &\leq 0 \end{aligned} \tag{4.4}$$

The data had to be normalized before applying the above bounds. The pre and post normalization data can be found in Appendix A. Table 4.8 compares the results of the bounded model with the results of the unbounded model calculated by Girod (1996).

DMU	Efficiency Scores	
	Unbounded	Bounded
1	1.000	0.744
2	0.716	0.665
3	0.892	0.871
4	0.605	0.582
5	0.716	0.591
6	0.956	0.836
7	0.815	0.729
8	0.845	0.539
9	0.939	0.809
10	0.642	0.560
11	0.716	0.715
12	0.759	0.540
13	0.673	0.645
14	0.677	0.523
15	0.772	0.522
16	0.705	0.439
17	0.631	0.489
18	0.713	0.537
19	1.000	1.000
20	0.582	0.536
21	0.510	0.508
22	0.438	0.431
23	0.805	0.765
24	0.893	0.693
25	0.928	0.750
26	0.872	0.613
27	1.000	0.879
28	0.620	0.566
29	0.983	0.777
30	0.932	0.825
31	0.955	0.789
32	0.862	0.639
33	0.740	0.623
34	0.908	0.748
35	0.964	0.691
36	0.927	0.776
37	0.936	0.835
38	0.758	0.624
39	0.848	0.746
40	0.807	0.731
41	0.720	0.644
42	0.816	0.782
43	0.726	0.585
44	0.614	0.437
45	0.687	0.537
46	0.690	0.394
47	0.962	0.801
48	1.000	1.000
Average	**0.797**	**0.668**

Table 4.8 Comparison of Results of Unbounded and AR DEA Models for the Washingtonpost Data from Girod (1996)

4.2.3 Fuzzy AR (Form 1) DEA Model

The decision-makers admit that conditions a) and b) are based on subjective opinion as opposed to precise information. This implies that the bound values in (4.4) could be imprecise. To model the imprecision, we propose to replace the crisp bounds by fuzzy bounds. According to the decision-makers, condition a) can be violated to the extent of 0.5 and condition b) can be violated to the extent of 1. Based on this information, the following fuzzy AR constraints are created:

$$\begin{aligned}
&(1+0.5\lambda)v_2 - v_1 \le 0 \\
&(1+\lambda)v_3 - v_1 \le 0 \\
&0 \le \lambda \le 1
\end{aligned} \tag{4.5}$$

In Table 4.9, the results of the fuzzy AR model, which applies these constraints to the Washingtonpost data, are compared with the results of the crisp AR model.

DMU	Efficiency Scores	
	Crisp AR	Fuzzy AR
1	0.744	0.779
2	0.665	0.676
3	0.871	0.886
4	0.582	0.594
5	0.591	0.612
6	0.836	0.849
7	0.729	0.751
8	0.539	0.570
9	0.809	0.823
10	0.560	0.576
11	0.715	0.716
12	0.540	0.574
13	0.645	0.655
14	0.523	0.556
15	0.522	0.571
16	0.439	0.485
17	0.489	0.520
18	0.537	0.578
19	1.000	1.000
20	0.536	0.558
21	0.508	0.510
22	0.431	0.436
23	0.765	0.771
24	0.693	0.736
25	0.750	0.778
26	0.613	0.643
27	0.879	0.923
28	0.566	0.597
29	0.777	0.809
30	0.825	0.865
31	0.789	0.833
32	0.639	0.682
33	0.623	0.657
34	0.748	0.785
35	0.691	0.739
36	0.776	0.819
37	0.835	0.874
38	0.624	0.659
39	0.746	0.773
40	0.731	0.752
41	0.644	0.674
42	0.782	0.806
43	0.585	0.616
44	0.437	0.463
45	0.537	0.563
46	0.394	0.430
47	0.801	0.845
48	1.000	1.000
Average	**0.668**	**0.695**

Table 4.9 Comparison of Efficiency Scores of Crisp and Fuzzy AR Models applied to the WashingtonPost Data from Girod (1996)

We observe that for each DMU, the efficiency score increases when we change the bounds from crisp to fuzzy. To check if the increase in efficiency scores is significant, we apply the paired two-sample t test for means (Bain and Engelhardt (1992)). The results of the test are shown in Table 4.10 below.

	Fuzzy AR	*Crisp AR*
Mean	0.695137	0.667951
Variance	0.020684	0.021151
Observations	48	48
Pearson Correlation	0.995366	
Hypothesized Mean Difference	0	
Df	47	
t Stat	13.43801	
P(T<=t) one-tail	5.15E-18	
t Critical one-tail	1.677927	

Table 4.10 Results of Paired Two sample t-test for Means comparing the Efficiency Scores of the Crisp and Fuzzy AR Models applied to the WashingtonPost Data

Based on the above table, we can reject the null hypothesis and accept the alternative hypothesis that the efficiency scores obtained using the fuzzy AR model are significantly greater than the efficiency scores obtained using the crisp AR model.

4.3 FUZZY AR (Form 2) DEA MODEL APPLIED TO MEASURING THE EFFICIENCY OF SMALL BUSINESS DEVELOPMENT CENTERS (SDBCs)

4.3.1 Background

Thompson *et al.* (1996a) used the DEA and AR methods to measure the relative efficiency of 13 Small Business Development Centers (SDBCs) of the University of Houston (UH) for the years 1991 and 1992. An SBDC conducts research; consults with business owners at no cost; provides training to business people in management, finance, and operations of small businesses; and provides comprehensive information services and access to experts in numerous fields. It was decided to measure the efficiency of the

SDBCs to ensure efficient allocation of resources, which had been far exceeded by the demand for the services of the SDBCs. However, no standardized objective for a fair evaluation of all SBDC programs could be arrived at because SDBCs in different locations had different objectives and different priorities. Because of this variety and flexibility in the SBDC services, DEA was an ideal tool for evaluating their performance. The objective of applying the DEA/AR model to the UH SBDC data, according to the authors, was to evaluate the relationship of an efficient use of distributed resources to a quantifiable result.

The study period was three years (1990-92). Two models were considered in the study -- Model I and Model II. The difference between these models was that Model II had an additional input for which data was available only for the last year of the study period. Thus, Model I was applied to all three years of study (1990-92) and Model II was applied to only the third year (1992). In this research, we will focus only on Model II, applied to the data for 1992 since Model II has more number of inputs and the year 1992 has more number of DMUs. The outputs and inputs used in Model II, are as follows:

Outputs:

- Total number of clients – y_1
- Total number of training hours – y_2
- Total number of counseling hours – y_3

Inputs:

- Total amount of federal funds allocated – x_1
- Population density – x_2

The actual data values for the year 1992 can be found in the Appendix A. The results of the unbounded Model II are shown in Table 4.11:

DMU	CCR Efficiency - 1992
1	1
2	0.57
3	0.69
4	0.66
5	1
6	1
7	0.76
8	1
9	0.54
10	1
11	0.68
12	1
13	0.66
Avg.	0.81

Table 4.11 Efficiency Scores for the Unbounded Model II applied to the SBDC Data from Thompson et al. (1996a)

4.3.2 The Crisp AR (Form 2) Model

The AR bounds were determined using the price/cost data ranges displayed in Table 4.12.

Factor	Multiplier	"Price/Cost" data range
y_1	u_1	\$120 to \$8,030
y_2	u_2	\$87 to \$4,075
y_3	u_3	\$114 to \$750
x_1	v_1	7.7%/yr to 15%/yr
x_2	v_2	10,000 to 15,300

Table 4.12 "Price/Cost" Data Ranges from Thompson *et al.* (1996a) for the SBDC Data

The AR constraints added to the CCR model based on the price/cost ranges stated in Table 4.12 are as follows:

$$
\begin{aligned}
&87u_1 - 120u_2 \geq 0 \\
&8030u_2 - 4075u_1 \geq 0 \\
&114u_1 - 120u_3 \geq 0 \\
&8030u_3 - 750u_1 \geq 0 \\
&114u_2 - 87u_3 \geq 0 \\
&4075u_3 - 750u_2 \geq 0 \\
&10000v_1 - 7.7v_2 \geq 0 \\
&15v_2 - 15300v_1 \geq 0
\end{aligned}
\tag{4.6}
$$

In matrix (intersection) form, these constraints are represented as:

$$
\begin{bmatrix} 87 & -120 & 0 \\ -4075 & 8030 & 0 \\ 114 & 0 & -120 \\ -750 & 0 & 8030 \\ 0 & 114 & -87 \\ 0 & -750 & 4075 \end{bmatrix} \mathrm{x} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \geq 0 \tag{4.7}
$$

$$
\begin{bmatrix} 10000 & -7.7 \\ -15300 & 15 \end{bmatrix} \mathrm{x} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \geq 0
$$

It was found that when the AR model with the above set of constraints was solved, the efficiency scores were very much different (much smaller) compared to those obtained by Thompson *et al.* (1996a). Therefore, it was decided to drop the AR constraints on the input weights. The results of the AR model with the AR constraints applied to just the output weights are shown in Table 4.13.

DMU	AR Efficiency
1	1
2	0.383
3	0.367
4	0.263
5	1
6	1
7	0.370
8	0.835
9	0.448
10	0.993
11	0.646
12	0.333
13	0.628
Avg.	**0.636**

Table 4.13 Results of the AR DEA Model for 1992 SBDC Data from Thompson *et al.* (1996a)

4.3.3 The Fuzzy AR (Form 2) Model

In the proposed research, we assume that the price/cost ranges used to set AR bounds are imprecise and to incorporate the imprecision in the modeling process, we replace the crisp AR bounds by fuzzy AR bounds. Ideally, additional information about the price/cost ranges should have been used to determine the fuzzy AR bounds. However, Thompson *et al.* (1996a), do not provide any information as to how the price/cost ranges were determined. Therefore, to determine the fuzzy bounds, we assume the imprecision amount to be a certain percentage of the specified bound value. The following table gives the percentage imprecision values assumed[18] for each output price and the values of price variations (p and q) calculated based on those percentages.

Output	Multiplier	Percentage variation on either side	Lower bound on price varies between	p and q	Upper bound on price varies between	p and q
y_1	u_1	5%	114 and 126	6	7628.5 and 8431.5	401.5
y_2	u_2	10%	78.3 and 95.7	8.7	3667.5 and 4482.5	407.5
y_3	u_3	15%	96.9 and 131.1	17.1	637.5 and 862.5	112.5

Table 4.14 Variation in Price/Cost Ranges assumed for determining Fuzzy Bounds on the SBDC Data from Thompson *et al.* (1996a)

[18] The percentages are arbitrarily assigned.

Figure 4.1 graphically depicts the fuzzy number corresponding to u_1. Since we assume an imprecision of 5% on either side of the price/cost range for y_1, the lower bound on the price, instead of being fixed at 120, will vary between 114 and 126 with the desirability increasing between the values 114 and 120 (most desirable value) and then diminishing between the values 120 and 126.

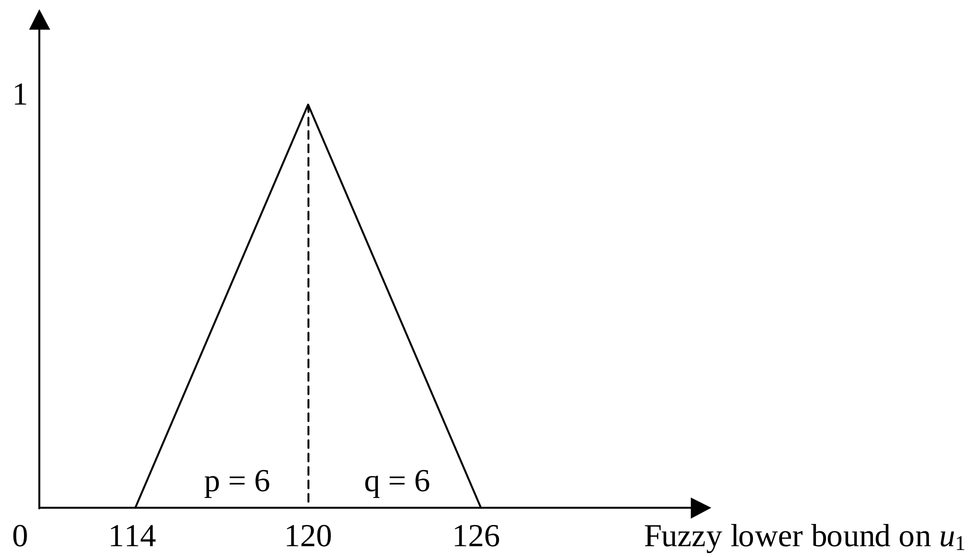

Figure 4.1 Fuzzy Lower bound on u_1 (price of output 1 for the SBDC Data from Thompson *et al.* (1996a))

The values calculated in Table 4.14 are used as the least desirable values in the definitions of the fuzzy numbers. The most desirable values are the ones specified by the decision-makers in Table 4.12.

The fuzzy AR constraints added to the CCR model are as follows:

$$
\begin{aligned}
&(78.3+8.7\lambda)u_1-(114+6\lambda)u_2 \geq 0\\
&(95.7-8.7\lambda)u_1-(126-6\lambda)u_2 \geq 0\\
&(7628.5+401.5\lambda)u_2-(3667.5+407.5\lambda)u_1 \geq 0\\
&(8431.5-401.5\lambda)u_2-(4482.5-407.5\lambda)u_1 \geq 0\\
&(96.9+17.1\lambda)u_1-(114+6\lambda)u_3 \geq 0\\
&(131.1-17.1\lambda)u_1-(126-6\lambda)u_3 \geq 0\\
&(7628.5+401.5\lambda)u_3-(637.5+112.5\lambda)u_1 \geq 0\\
&(8431.5-401.5\lambda)u_3-(862.5-112.5\lambda)u_1 \geq 0\\
&(96.9+17.1\lambda)u_2-(78.3+8.7\lambda)u_3 \geq 0\\
&(131.1-17.1\lambda)u_2-(95.7-8.7\lambda)u_3 \geq 0\\
&(3667.5+407.5\lambda)u_3-(637.5+112.5\lambda)u_2 \geq 0\\
&(4482.5-407.5\lambda)u_3-(862.5-112.5\lambda)u_2 \geq 0
\end{aligned}
\tag{4.8}
$$

Table 4.15 compares the results of the fuzzy AR model (for different values of λ) with those of the crisp AR model from Table 4.13. Since λ represents the degree of satisfaction of the decision-maker with the bounds, the higher the value of λ, the higher is the degree of satisfaction.

DMU	Efficiency				
	Fuzzy AR				Crisp AR
	λ=0.2	λ=0.4	λ=0.6	λ=0.8	
1	1	1	1	1	1
2	0.378	0.379	0.380	0.381	0.383
3	0.364	0.364	0.365	0.366	0.367
4	0.260	0.261	0.261	0.262	0.263
5	1	1	1	1	1
6	1	1	1	1	1
7	0.365	0.366	0.368	0.369	0.370
8	0.820	0.824	0.827	0.831	0.835
9	0.444	0.445	0.446	0.447	0.448
10	0.985	0.987	0.989	0.991	0.993
11	0.646	0.646	0.646	0.646	0.646
12	0.330	0.331	0.332	0.332	0.333
13	0.625	0.625	0.626	0.627	0.628
Average	**0.632**	**0.633**	**0.634**	**0.635**	**0.636**

Table 4.15 Comparison of Efficiency Scores of Crisp and Fuzzy bounds applied to the 1992 SBDC Data from Thompson *et al.* (1996a)

Notice that the efficiency scores obtained using the crisp AR model are higher than those obtained using the fuzzy AR. This is contrary to the results of the previous two fuzzy models (absolute weight restriction and AR Form 1). The explanation lies in the fact that the forms of the membership functions of the fuzzy bounds in the previous models are different from those used in this model. The membership functions of the fuzzy bounds (see Figure 4.1) in this model allow variation on both sides of the specified bound while the membership functions in the previous models allow variation in only that direction which has the effect of relaxing the constraints. Therefore, the efficiency scores calculated by the fuzzy model considered in this section could be either higher or lower compared to the corresponding crisp model while those calculated by the fuzzy models discussed earlier will always be higher compared to the corresponding crisp models.

To check if the efficiency scores obtained using the fuzzy models are significantly different from those obtained using the crisp model, we use the paired two-sample t-test for means. The results of the tests are shown in Tables 4.16 to Table 4.19.

	Crisp AR	*Fuzzy AR -*
Mean	0.6359	0.632001
Variance	0.08652	0.086597
Observations	13	13
Pearson Correlation	0.999906	
Hypothesized Mean	0	
Df	12	
t Stat	3.476479	
P(T<=t) one-tail	0.002288	
t Critical one-tail	1.782287	

Table 4.16 Results of Paired Two Sample t-Test comparing the Efficiency scores of the Crisp AR and Fuzzy AR with λ=0.2 for the SBDC Data from Thompson *et al.* (1996a)

	Crisp AR	*Fuzzy AR-*
Mean	0.6359	0.6329285
Variance	0.08652	0.086576
Observations	13	13
Pearson Correlation	0.999945	
Hypothesized Mean	0	
Df	12	
t Stat	3.4708382	
P(T<=t) one-tail	0.0023119	
t Critical one-tail	1.7822867	

Table 4.17 Results of Paired Two Sample t-Test comparing the Efficiency scores for the Crisp AR and Fuzzy AR with λ=0.4 for the SBDC Data from Thompson *et al.* (1996a)

	Crisp AR	*Fuzzy AR-*
Mean	0.6359	0.6338854
Variance	0.08652	0.0865568
Observations	13	13
Pearson Correlation	0.9999747	
Hypothesized Mean	0	
Df	12	
t Stat	3.4688929	
P(T<=t) one-tail	0.0023202	
t Critical one-tail	1.7822867	

Table 4.18 Results of Paired Two Sample t-Test comparing the Efficiency scores of Crisp AR and Fuzzy AR with λ=0.6 for the SBDC Data from Thompson *et al.* (1996a)

	Crisp AR	*Fuzzy AR -*
Mean	0.6359	0.634875
Variance	0.08652	0.086537
Observations	13	13
Pearson Correlation	0.999993	
Hypothesized Mean	0	
Df	12	
t Stat	3.467247	
P(T<=t) one-tail	0.002327	
t Critical one-tail	1.782287	

Table 4.19 Results of Paired Two Sample t-Test comparing the Efficiency scores of the Crisp AR and Fuzzy AR with λ=0.8 for the SBDC Data from Thompson *et al.* (1996a)

In each of the above tests, we reject the null hypothesis thus accepting the alternative hypothesis that there is a significant difference between the mean efficiency scores of the crisp and fuzzy models.

4.4 CONCLUSION

For each type of fuzzy model, it is found that the efficiency scores calculated by the fuzzy model are significantly different from the efficiency scores calculated by the corresponding crisp model. This implies that the operational decisions based on the results of the fuzzy models will be different from those taken based on the results of the crisp models. The fuzzy models ensure that the decisions are taken after the uncertainty has been accounted for. The efficiency scores calculated by the fuzzy models represent a compromise between maximization of the efficiency scores and the satisfaction of the decision-maker with the bounds.

Chapter 5

CONCLUSION

This chapter concludes the presentation of this research with the following three sections. The first section summarizes the research effort of this thesis. The second section describes the major contribution of this research and includes some concluding comments. The third section outlines some recommendations for future research.

5.1 SUMMARY

This research has five objectives. The first objective is to minimize the effect of bound uncertainty on the decision-making in Data Envelopment Analysis (DEA) by explicitly incorporating the uncertainty in the modeling process through fuzzy weight restriction DEA models. The second objective is to develop a solution methodology for those fuzzy models. The third objective is to provide implementation roadmaps for illustrating the proposed fuzzy models. The fourth objective is to apply the proposed models to the same data sets as those used by corresponding crisp weight restriction models from the literature and compare their results. The fifth objective is to use the results of the fuzzy models to modify the specified bounds in order to move the borderline[19] decision-making units (DMUs) from the inefficienct set to the efficient set.

Although numerous types of weight restriction models have been developed in the DEA literature, the following two are more commonly used compared to the rest:

1. The absolute weight restriction DEA model and
2. The Assurance Region DEA model.

Therefore, in this research, we focus our attention on only these two models.

Both these models suffer from the following two shortcomings:

1. **Their weight bound values are imprecise** and

2. **Their weight bound values are subjective.** The subjectivity leads to different decision-makers specifying different values for the bounds thus producing different efficiency scores even though the data set is the same.

The results of the model are sensitive to the values of the bounds, and therefore, the uncertainty (imprecision + subjectivity) in bound values gets passed onto the results of the model. This becomes unacceptable when we consider the fact that DEA is a decision-making tool whose results are used for making important decisions like allocating funds or taking stringent action against inefficient DMUs.

The objective of this research is to explicitly incorporate the uncertainty in the modeling process so that the effect of the uncertainty on the decision-making process is minimized. Two approaches have been commonly used in the past to model uncertainty. The more conventional approach is the stochastic approach that involves specifying a probability distribution function (e.g. Normal) for the error process (Sengupta 1992). However, when it comes to using stochastic processes for modeling the uncertainty in DEA problems, there are certain drawbacks, as pointed out by Sengupta (1992). The more recent approach for dealing with uncertainty has been the use of fuzzy set theory. Sengupta (1992) was the first to incorporate fuzzy set theory in DEA by proposing a fuzzy mathematical programming approach for dealing with imprecise data in DEA problems.

In the current research also, we propose to use fuzzy set theory for **modeling the uncertainty** in weight bound values. Fuzzy set theory is introduced by replacing the crisp weight bounds by fuzzy numbers. The justification is that the imprecise weight bounds need to be represented as approximate numbers (i.e. "numbers close to the specified values") and fuzzy numbers capture the intuitive concept of approximate numbers very well (Yuan and Klir 1995).

---

[19] These are DMUs whose membership in the efficient set is highly sensitive to slight changes in bound values.

A fuzzy number is a set of values (instead of a single value) close to the value that is being approximated. Because a fuzzy number represents a range of values (instead of a single value), it is likely to be a compromise between the bounds set by different decision-makers. It is hypothesized that using fuzzy numbers for bounds will have an added advantage of **minimizing the sensitivity** of the results to the **subjectivity** in the bound values.

Using fuzzy numbers instead of crisp numbers for the bounds has an added advantage of **increasing the flexibility** in the bound setting process because it allows the decision-maker to specify a range of values instead of one value. The lack of flexibility in the crisp weight restriction problems can often put the decision-maker in a tight spot especially when enough information does not exist for him/her to make a crisp judgement.

In this research, we develop fuzzy models to model the bound uncertainty in the two most commonly used weight restriction models, discussed earlier:

1. The DEA model with absolute weight restrictions (See Dyson and Thanasoulis (1988), Roll *et al.* (1991), and Roll and Golany (1993)).
2. The Assurance Region (AR) DEA model (see Thompson *et al.* (1986), Thompson *et al.* (1990)).

To illustrate the development and solution methodology of the fuzzy models, we make use of implementation roadmaps. We further illustrate the fuzzy weight restriction models by applying them to data sets from the weight restriction DEA literature. The reason we choose the same data sets as those used by the crisp models in the literature is to enable us to compare the results of the crisp models with those of our fuzzy model. We also apply the fuzzy AR model to a real life manufacturing system in which sufficient information is available to define both crisp and fuzzy bounds.

We compare the results of the fuzzy models with those of the corresponding crisp models using the two sample paired t test for means (Bain and Engelhardt (1992)). In each case, it is found that the difference between the efficiency scores generated by the fuzzy model are significantly different from the efficiency scores calculated by the corresponding crisp model. This implies that the operational decisions based on the results of the fuzzy models will be different from those taken based on the results of the crisp models. The fuzzy models ensure that the decisions are taken after the uncertainty has been accounted for. The efficiency scores calculated by the fuzzy model represent a compromise between maximization of the satisfaction of the decision-maker with the bounds and maximization of the efficiency scores.

In some cases, it is also found that DMUs move from the inefficient set to the efficient set when the bounds are changed from crisp to fuzzy. In some of those cases, even a relaxation of bounds to a 90% satisfaction level of the original values (i.e. just a 10% relaxation of bounds) is enough to move some DMUs (referred to as borderline) from the inefficient set to the efficient set. It is assumed that since the bound values are based on incomplete information, the decision-maker would not resist changing them slightly if the change is going to allow some DMUs to move from the inefficient set to the efficient set. Thus, the fuzzy model gives the decision-maker a second chance to revise the bounds.

The objective of the proposed fuzzy model of trying to move the borderline DMUs from the inefficient set to the efficient set is contrary to the objective of the research that has been carried out so far in the field of weight restriction DEA. The objective of all past weight restriction models has been to discriminate between the DMUs by reducing the efficiency scores by making the bounds tighter and tighter. However, we are of the opinion that since the results of DEA are used for making important decisions like allocating funds or taking stringent action against inefficient DMUs, the model should be fair to all DMUs. The conventional DEA model, in its purest form, is fair to all DMUs since it does not impose any synthetic constraints on the weights and allows each DMU to choose a set of weights which optimizes its performance. Imposing the bounds takes away the fairness and brings in the bias of the decision-maker. On the other hand, getting rid of the bounds, takes away the only opportunity the decision-maker has to express his/her opinion about the relative importance of the factors. To resolve this dilemma, we propose the fuzzy weight restriction model, which seeks a compromise between a bounded and unbounded DEA model. The proposed fuzzy model satisfies the decision-maker by treating the bounds specified by him/her as most desirable and also allows all DMUs to express their different circumstances to the best extent possible by stretching the bounds.

5.2 RESEARCH CONTRIBUTION AND CONCLUDING COMMENTS

Prior to this research, fuzzy set theory in DEA was restricted to modeling either the imprecision in the realization of constraints and objective function (Sengupta (1992)) or the imprecision in the input/output data (Triantis and Griod (1998)). This is the first time fuzzy set theory was used to model the imprecision in the weight (multiplier) bound values. In fact, this is the first time imprecision in bound values was ever considered or modeled in DEA. The proposed approach ensures that decisions are taken after the uncertainty has been accounted for.

As mentioned in the previous section, the fuzzy model helps in identifying borderline DMUs i.e. DMUs, which could move from the inefficient set to the efficient set if the bounds were only slightly relaxed. This gives the decision-maker a second chance to change the bounds and do justice to the borderline DMUs. It should be noted that this approach is very much different from a sensitivity analysis. In a sensitivity analysis, the decision-maker studies the effect of moving the bounds on the efficiency scores. Sensitivity analysis produces a range of efficiency scores for the given range of bound values. The fuzzy approach produces only one value of efficiency score, which is the maximum possible efficiency value at which the satisfaction of the decision-maker with the bounds is maximized.

The research elucidates the fuzzy approach by providing a geometric representation of the fuzzy bounds. The geometric representation helps bring out the intuition behind the fuzzy approach.

5.3 RECOMMENDATIONS FOR FUTURE RESEARCH

The current research can be extended and further investigated with respect to one or more of its components, namely, fuzzy set theory, DEA, and weight restriction DEA.

With respect to fuzzy set theory, the suitability of the form of the membership function is an issue of interest. Moreover, the impact of the form of the membership function on the efficiency results also warrants attention. The physical interpretation of the membership functions requires investigation. The linear membership function may not be satisfactory in all applications. As further research, one could experiment with other forms of membership functions like hyperbolic, logistic, S-shaped, etc.

In this research, we used the “intersection” or the “min” operator for aggregating the degrees of satisfaction of the constraints and the membership function of the objective function and arriving at the membership function of the fuzzy set “decision.” One of the objections against the min operator (see Zimmermann and Zysno (1980)) is the fact that neither the logical "and" nor the min operator is compensatory i.e. increases in the degree of membership in the "intersected" fuzzy sets do not influence the membership in the resulting fuzzy set (aggregated fuzzy set or intersection). To cure this weakness, the (limitational) min operator as a model for the logical "and" can be combined with the fully compensatory “max” operator as a model for the inclusive "or." Developing a model in which the min and max operators are combined has a potential for further research.

There is also potential for combining the proposed approach with other fuzzy DEA approaches like the one proposed by Sengupta (1992) or the one proposed by Triantis and Girod (1998). If all constraints (including weight bound) and the objective function are capable of being violated, we can combine our approach with Sengupta’s (1992) approach. On the other hand, if in addition to the weight bound values, even the input/output data are imprecise; we can combine our approach with the Triantis and Girod (1998) approach.

With respect to DEA, the approach could be applied to other DEA models apart from the CCR model (e.g. BCC, FDH, etc.).

With respect to weight restriction DEA, the approach could be applied to other types of weight restriction models like the Wong and Beasley (1990) model; the "contingent weight restrictions" model proposed by Pedraja *et al.* (1997); or the "ordinal relations" model proposed by Golany *et al.* (1990).

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Appendix A

DATA SETS

Patrol	Assignment Size Factor	Average Traffic Serviced	Total Expenditure	Average Pavement Condition Rating
1	751	67	696	39
2	611	70	616	26
3	538	70	456	17
4	584	75	616	31
5	665	70	560	16
6	445	75	446	16
7	554	76	517	26
8	457	72	492	18
9	582	74	558	23
10	556	64	407	18
11	590	78	402	33
12	1074	75	350	88
13	1072	74	581	64
14	696	80	413	24

Table A.1 Highway Maintenance Patrol Data from Cook *et al.* (1990)

Prod. Period	Labor	Rework	RM	Package
1	81	891	3105	68290
2	137	8245	7535	135912
3	71	11232	5709	112072
4	125	9634	7451	113281
5	121	6889	5548	100288
6	94	2997	5239	109138
7	147	11237	7725	159789
8	115	2837	3324	70314
9	86	2466	4473	93620
10	134	7989	6689	108329
11	104	5517	6458	114816
12	154	9938	5707	110757
13	100	6727	5830	98913
14	184	24813	7821	136878
15	187	25019	6581	131440
16	213	19589	6740	122799
17	170	13224	7197	116327
18	157	13559	6386	117373
19	41	5414	4154	82693
20	180	18193	10214	150834
21	176	15038	13391	153800
22	173	20424	14589	137414
23	113	4991	6302	126152
24	173	22727	7410	171139
25	127	6976	5635	132051
26	124	7515	4506	100233
27	129	14990	6646	171852
28	142	19470	7972	125699
29	147	9944	6375	159379
30	159	20673	8321	200038
31	158	16950	7377	182109
32	125	15161	5005	111538
33	140	14512	6783	129451
34	165	14393	7692	179118
35	187	32664	7132	177811
36	144	22463	6864	164597
37	155	18944	8248	198428
38	179	22150	8342	163419
39	174	14359	8930	192897
40	177	13359	9499	194364
41	200	28483	10694	197582
42	187	21969	12399	243260
43	232	20471	10485	195572
44	292	17521	10794	168828
45	233	17935	9988	175602
46	295	18632	8072	144030
47	87	10502	4145	103106
48	95	4742	5807	145253

Table A.2 WashintonPost Preprint Insertion Line Data from Girod (1996)

Prod. Period	Labor	Rework	RM	Package
1	53.21	6.34	42.18	48.21
2	90.30	58.69	102.38	95.96
3	46.93	79.95	77.56	79.13
4	82.24	68.58	101.24	79.98
5	79.83	49.04	75.38	70.81
6	61.98	21.33	71.18	77.05
7	96.81	79.98	104.95	112.82
8	75.52	20.20	45.16	49.64
9	56.87	17.55	60.78	66.10
10	88.35	56.86	90.88	76.48
11	68.33	39.27	87.75	81.06
12	101.65	70.74	77.54	78.20
13	66.11	47.88	79.22	69.84
14	121.01	176.62	106.26	96.64
15	122.99	178.08	89.41	92.80
16	140.32	139.43	91.57	86.70
17	112.16	94.12	97.79	82.13
18	103.02	96.51	86.77	82.87
19	26.85	38.54	56.44	58.38
20	118.35	129.49	138.77	106.49
21	115.71	107.04	181.93	108.59
22	113.78	145.37	198.22	97.02
23	74.66	35.52	85.62	89.07
24	114.15	161.76	100.68	120.83
25	83.85	49.66	76.57	93.23
26	81.63	53.49	61.22	70.77
27	84.99	106.69	90.30	121.33
28	93.63	138.58	108.32	88.75
29	97.00	70.78	86.61	112.53
30	104.95	147.15	113.05	141.23
31	103.66	120.64	100.22	128.57
32	82.17	107.91	67.99	78.75
33	92.26	103.30	92.16	91.40
34	108.51	102.45	104.51	126.46
35	122.99	232.49	96.90	125.54
36	94.58	159.89	93.26	116.21
37	102.30	134.84	112.06	140.10
38	117.71	157.66	113.33	115.38
39	114.33	102.20	121.33	136.19
40	116.73	95.09	129.06	137.23
41	131.65	202.73	145.29	139.50
42	122.99	156.37	168.46	171.75
43	152.79	145.71	142.46	138.08
44	192.33	124.71	146.66	119.20
45	153.64	127.65	135.70	123.98
46	194.09	132.62	109.67	101.69
47	57.59	74.75	56.31	72.80
48	62.31	33.76	78.90	102.55

Table A.3 WashingtonPost Data Normalized

SBDC	# of Clients	# of Training Hrs.	# of Counseling Hrs.	Federal Funds	Population Density
1	162970	4872	1454	128	10087
2	95960	3610	421	47	1173
3	115314	2987	434	69	1611
4	115573	1431	288	59	844
5	70450	303	496	43	3582
6	69637	75	159	18	985
7	54275	295	186	26	500
8	108553	1235	960	52	1984
9	93207	273	257	10	1014
10	48800	43	106	17	359
11	39209	112	110	13	886
12	46000	1475	148	54	597
13	36482	52	71	8	334

Table A.4 Small Business Development Center (SBDC) Data from Thompson *et al.* (1996a)

Appendix B

SAS CODES

B.1 SAS CODE FOR THE ROADMAP EXAMPLE OF SECTION 3.1.3 ILLUSTRATING THE FUZZY ABSOLUTE WEIGHT RESTRICTION DEA MODEL

The roadmap example in section 3.1.3[20] used to illustrate the fuzzy absolute weight restriction DEA model, requires solving the following three LPs in the given sequence.

The first LP, which is used to calculate f_1 values for all DMUs, uses the most desirable weight bound values in the weight bound constraints (B):

$$Max\ \mu^T Y_0 = f_1$$

$$s.t.$$

$$\eta^T X_0 = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots N$$

$$\mu^T Y - \eta^T X \leq 0 \qquad \forall j \ldots\ldots\ldots\ldots M$$

$$\begin{aligned}
&0.00002264T_0 \leq \mu_1 \leq 0.00004495T_0 \\
&0.0000154T_0 \leq \mu_2 \leq 0.00003056T_0 \\
&0.3874T_0 \leq \mu_3 \leq 0.769018T_0 \ldots\ldots\ldots\ldots B \\
&0.0009401T_0 \leq \eta_1 \leq 0.001866T_0 \\
&0.00002735T_0 \leq \eta_2 \leq 0.0000543T_0 \\
&0.00004662T_0 \leq \eta_3 \leq 0.00009255T_0 \\
&0.0009489T_0 \leq \eta_4 \leq 0.001884T_0 \\
&\mu, \eta \geq 0
\end{aligned}$$

The following LP, which is used for obtaining f_0 values, uses the least desirable weight bound values in the weight bound constraints (B):

[20] Although this roadmap example is solved with two different sets of least desirable bounds, here we describe the SAS code only for the model with least desirable bounds determined using Method II. The SAS code for the model with the least desirable bounds determined using Method I is the same with the exception of the values of the bounds.

$$Max\ \mu^T Y_0 = f_0$$

$$s.t.$$

$$\eta^T X_0 = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots N$$

$$\mu^T Y - \eta^T X \le 0 \qquad \forall j \ldots\ldots\ldots\ldots M$$

$$0 \le \mu_1 \le 0.0000926 T_0$$

$$0 \le \mu_2 \le 0.000302 T_0$$

$$0 \le \mu_3 \le 1.51888 T_0 \ldots\ldots\ldots\ldots B$$

$$0.000206 T_0 \le \eta_1 \le 0.00565 T_0$$

$$0 \le \eta_2 \le 0.000252 T_0$$

$$0 \le \eta_3 \le 0.000252 T_0$$

$$0 \le \eta_4 \le 0.00799 T_0$$

$$\mu, \eta \ge 0$$

Finally, the following crisp equivalent of the fuzzy model is solved:

$$Max\ \lambda$$

$$s.t.$$

$$\lambda(f_0 - f_1) - \mu^T Y_0 \le -f_1 \ldots\ldots\ldots\ldots\ldots\ldots F$$

$$\eta^T X_0 = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots N$$

$$\mu^T Y - \eta^T X \le 0 \qquad \forall j \ldots\ldots\ldots\ldots M$$

$$(0.00002264 - 0.00002264\,\lambda) T_0 \le \mu_1 \le (0.00004495 + 0.00004765\,\lambda) T_0$$

$$(0.0000154 - 0.0000154\,\lambda) T_0 \le \mu_2 \le (0.00003056 + 0.0002714\,\lambda) T_0$$

$$(0.3874 - 0.3874\,\lambda) T_0 \le \mu_3 \le (0.769018 + 0.7499\,\lambda) T_0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots B$$

$$(0.0009401 - 0.0007341\,\lambda) T_0 \le \eta_1 \le (0.001866 + 0.003784\,\lambda) T_0$$

$$(0.00002735 - 0.00002735\,\lambda) T_0 \le \eta_2 \le (0.0000543 + 0.0001977\,\lambda) T_0$$

$$(0.00004662 - 0.00004662\,\lambda) T_0 \le \eta_3 \le (0.00009255 + 0.0002985\,\lambda) T_0$$

$$(0.0009489 - 0.0009489\,\lambda) T_0 \le \eta_4 \le (0.001884 + 0.006106\,\lambda) T_0$$

$$0 \le \lambda \le 1$$

$$\mu, \eta \ge 0$$

Note that this model is a quadratic-programming model because its weight bound constraints (B) contain expressions in which two variables are multiplied to each other. Since one of these variables is λ and we know the range of variation of λ, we can use the parametric algorithm described in section 3.1.1.5.4 to solve it. Steps 1,3 and 8 of the algorithm require us to solve the model with λ fixed at a certain value. This is tantamount to solving an LP with a fixed objective function since λ is the objective function of the crisp equivalent LP. We cannot solve an LP with a fixed objective function. Therefore, we need to modify the model so that the objective function contains an expression involving decision variables as opposed to a fixed value. The most logical choice for the objective function is the efficiency term. The modified model, which can now be solved using the parametric algorithm, is as follows:

$$Max\ \mu^T Y_0$$

$$s.t.$$

$$\lambda(f_0 - f_1) - \mu^T Y_0 \leq -f_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots F$$

$$\eta^T X_0 = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots N$$

$$\mu^T Y - \eta^T X \leq 0 \qquad \forall j \ldots\ldots\ldots\ldots M$$

$$\begin{aligned}
&(0.00002264 - 0.00002264\,\lambda)T_0 \leq \mu_1 \leq (0.00004495 + 0.00004765\,\lambda)T_0 \\
&(0.0000154 - 0.0000154\,\lambda)T_0 \leq \mu_2 \leq (0.00003056 + 0.0002714\,\lambda)T_0 \\
&(0.3874 - 0.3874\lambda)T_0 \leq \mu_3 \leq (0.769018 + 0.7499\lambda)T_0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots B \\
&(0.0009401 - 0.0007341\,\lambda)T_0 \leq \eta_1 \leq (0.001866 + 0.003784\,\lambda)T_0 \\
&(0.00002735 - 0.00002735\,\lambda)T_0 \leq \eta_2 \leq (0.0000543 + 0.0001977\,\lambda)T_0 \\
&(0.00004662 - 0.00004662\,\lambda)T_0 \leq \eta_3 \leq (0.00009255 + 0.0002985\,\lambda)T_0 \\
&(0.0009489 - 0.0009489\,\lambda)T_0 \leq \eta_4 \leq (0.001884 + 0.006106\,\lambda)T_0 \\
&0 \leq \lambda \leq 1 \\
&\mu, \eta \geq 0
\end{aligned}$$

Instead of using the parametric algorithm directly, we solve the model for different values of λ between 0 and 1 at intervals of 0.1 and choose the solution corresponding to the maximum feasible value of λ. The justification is that this technique is easier to implement using SAS compared to the parametric algorithm and gives the same solution as the parametric algorithm using $\varepsilon = 0.1$.

The description of the constraint types used in the three LP's is as follows:

F – Constraint introduced by the membership function of the objective function

N – Normalization Constraint

M – Main set of Constraints

B – Weight Bound Constraints

All three LPs are solved in the same SAS code. The code with embedded descriptions of the data sets is as follows:

```
data bound1;
input UM1 UM2 UM3 UN1 UN2 UN3 UN4 LM1 LM2 LM3 LN1 LN2 LN3 LN4 pUM1
pUM2 pUM3 pUN1 pUN2 pUN3 pUN4 pLM1 pLM2 pLM3 pLN1 pLN2 pLN3 pLN4;
cards;
0.0000926 0.000302 1.51888 0.00565 0.000252 0.000391 0.00799 0 0 0 0.000206 0 0 0
0.00004765 0.00027144 0.749862 0.003784 0.0001977 0.00029845 0.006106
0.00002264 0.0000154 0.3874 0.0007341 0.00002735 0.00004662 0.0009489
;
```

***This data set contains the least desirable weight bounds determined using Method II* [21]. *The values with prefix 'p' are the differences between the least desirable upper (most desirable lower) bounds and the most desirable upper (least desirable lower) bounds. These values will be the coefficients of lambda in the weight bound constraints (B) of the final LP.*

```
data bound2;
input UM1 UM2 UM3 UN1 UN2 UN3 UN4 LM1 LM2 LM3 LN1 LN2 LN3 LN4;
cards;
0.00004495 0.00003056 0.769018 0.001866 0.0000543 0.00009255 0.001884
0.00002264 0.0000154 0.3874 0.0009401 0.00002735 0.00004662 0.0009489;
```

[21] The data set corresponding to least desirable bounds determined using Method I is:

```
data bound1;
input UM1 UM2 UM3 UN1 UN2 UN3 UN4 LM1 LM2 LM3 LN1 LN2 LN3 LN4 pUM1 pUM2 pUM3
pUN1 pUN2 pUN3 pUN4 pLM1 pLM2 pLM3 pLN1 pLN2 pLN3 pLN4;
cards;
0.00005069 0.00003447 0.867314 0.002105 0.00006124 0.0001044 0.002124 0.0000169 0.00001149
0.289105 0.0007016 0.00002041 0.00003479 0.0007081 0.00000574 0.00000391 0.098296 0.000239
0.00000694 0.00001185 0.00024 0.00000574 0.00000391 0.098296 0.000239 0.00000694 0.00001185
0.00024
;
```

This is the only difference between the SAS codes for solving models with bounds determined using the two methods.

```
**This data set contains the most desirable bounds.

data lambda;
input l1 l2 l3 l4 l5 l6 l7;
cards;
0.4 0.5 0.6 0.7 0.8 0.9 1
;
**This data set contains the different lambda values for which the final LP will be solved.

data dea;
input y1 y2 y3 x1 x2 x3 x4;
cards;
15500  460   0.85  521   3130  1859  80
13700  340   0.63  747   5075  3491  44
18000  1080  0.37  935   1483  2984  93
8900   490   0.56  205   4583  1736  65
10800  960   0.14  177   2990  1823  87
17300  890   0.47  584   5467  1775  98
21000  2930  0.91  634   7734  1700  58
9500   240   0.78  456   6552  503   73
9100   370   0.74  471   1855  2528  42
6600   800   0.52  325   4579  818   51
11800  610   0.87  364   5713  1178  80
26200  3600  0.41  585   4217  2012  84
11400  470   0.55  343   4061  2957  91
7200   1350  0.39  597   3242  665   73
38000  2470  0.68  1126  7658  1541  57
;
**This data set contains the input-output data.

data constr;
set dea;
array x{4} x1-x4;
array y{3} y1-y3;
length _type_ $ 8 _row_ $ 16 _col_ $ 8;
keep _type_ _row_ _col_ _coef_;

_type_ = 'LE';
_row_ = 'DMU'||put(_n_,2.);
_col_='_rhs_';
_coef_=0;
output;
do i=1 to 4;
_col_ = 'v'||put(i,1.);
_coef_= -x{i};
output;
```

```
end;

do i=1 to 3;
_col_='u'||put(i,1.);
_coef_=y{i};
output;
end;
run;
```

***This data set constructs the main set of constraints.*

```
data final;
input _value_;
cards;
;
run;
```

***This data set will be eventually used for displaying the results of the model.*

```
%macro runbound;
%do c=1 %to 2;
```

***This macro constructs the weight bound constraints for the first two LPs.*

```
data bounds&c;
set bound&c;
length _type_ $ 8 _row_ $ 16 _col_ $ 8;
keep _type_ _row_ _col_ _coef_;
array UM{3} UM1-UM3;
array LM{3} LM1-LM3;
array UN{4} UN1-UN4;
array LN{4} LN1-LN4;

do i = 1 to 3;
_row_='uubound'||put(i,1.);
_type_='LE';
_col_='_rhs_';
_coef_=0;
output;
_col_='u'||put(i,1.);
_coef_=1;
output;
_col_='T0';
_coef_=-UM{i};
output;
end;

do j=1 to 4;
_row_='vubound'||put(j,1.);
```

```
_type_='LE';
_col_='_rhs_';
_coef_=0;
output;
_col_='v'||put(j,1.);
_coef_=1;
output;
_col_='T0';
_coef_=-UN{j};
output;
end;

do i = 1 to 3;
_row_='ulbound'||put(i,1.);
_type_='GE';
_col_='_rhs_';
_coef_=0;
output;
_col_='u'||put(i,1.);
_coef_=1;
output;
_col_='T0';
_coef_=-LM{i};
output;
end;

do j=1 to 4;
_row_='vlbound'||put(j,1.);
_type_='GE';
_col_='_rhs_';
_coef_=0;
output;
_col_='v'||put(j,1.);
_coef_=1;
output;
_col_='T0';
_coef_=-LN{j};
output;
end;
run;
```

***This data set constructs the input & output weight bound constraints (B) for the first two LPs.*

```
%end;
%mend runbound;
%runbound;
```

```
%macro runlp;
%do a=1 %to 15;
```
***This macro constructs and runs all three LPs for all 15 DMUs.*

```
data obj&a;
set dea;
array x{4} x1-x4;
array y{3} y1-y3;
length _type_ $ 8 _row_ $ 16 _col_ $ 8;
keep _type_ _row_ _col_ _coef_;

if _n_ = &a then do;
do i = 1 to 3;
_type_ = 'MAX';
_row_ = 'obj';
_col_='u'||put(i,1.);
_coef_= y{i};
output;
end;

_row_='DMU0';
_type_='EQ';
_col_='_rhs_';
_coef_= 1;
output;
do i=1 to 4;
_col_='v'||put(i,1.);
_coef_= x{i};
output;
end;
end;
run;
```
***This data set constructs the objective function and the normalization constraint (N) for the first two LPs.*

```
data main&a.1;
set obj&a constr bounds1;
run;
```
***This data set aggregates the objective function and the constraints and constructs the LP with the least desirable bounds.*

```
proc lp data=main&a.1 printlevel = -2 sparsedata primalout = prim&a.1;
run;
```
***This statement solves main&a.1 and stores the results in prim&a.1.*

```
data main&a.2;
set obj&a constr bounds2;
run;
```
***This data set constructs the LP with the most desirable bounds.*

```
proc lp data=main&a.2 printlevel = -2 sparsedata primalout = prim&a.2;
run;
```
***This statement runs main&a.2 and stores the results in prim&a.2.*

```
data difobj&a;
keep f0 f1 diff;

set prim&a.1;
if _VAR_='obj' then f0 = _VALUE_;

set prim&a.2;
if _VAR_='obj' then f1 = _VALUE_;

diff= f0 - f1;
output;
run;
```
***This data set extracts the f_1 and f_0 values and calculates their difference.*

```
data newobj&a;
keep newf1 newdiff;
set difobj&a;
if f1>0 then do;
newf1 = f1;
newdiff = diff;
output;
end;
run;
```
***This data set gets rid of all the null values from difobj&a.*

```
data objective&a;
length _type_ $ 8 _row_ $ 16 _col_ $ 8;
keep _type_ _row_ _col_ _coef_;
set dea;
array y{3} y1-y3;
if _n_ = &a then do;
_type_ = 'MAX';
_row_ = 'obj';
do i = 1 to 3;
_col_='u'||put(i,1.);
_coef_= y{i};
output;
```

```
end;
end;
run;
```
***This data set constructs the objective function of the third & final LP.*

```
%macro runlam;
%do b=1 %to 7;
```
***This macro causes the final LP to run for seven different values of* λ*.*

```
data firstcons&a&b;
length _type_ $ 8 _row_ $ 16 _col_ $ 8;
keep _type_ _row_ _col_ _coef_ lam f1 diff;
set lambda;
array l{7} l1-l7;
lam = l{&b};

set newobj&a;
f1 = newf1;
diff = newdiff;

_type_ = 'GE';
_row_ = 'fuzzy';
_col_='_rhs_';
_coef_= f1 + diff*lam;
output;
run;
```
***This data set constructs part of the constraint F for the final LP.*

```
data secondcons&a&b;
length _type_ $ 8 _row_ $ 16 _col_ $ 8;
keep _type_ _row_ _col_ _coef_;
set dea;
array x{4} x1-x4;
array y{3} y1-y3;

if _n_ = &a then do;
_type_ = 'GE';
_row_ = 'fuzzy';
do i = 1 to 3;
_col_='u'||put(i,1.);
_coef_= y{i};
output;
end;

_row_='DMU0';
_type_='EQ';
```

```
_col_='_rhs_';
_coef_= 1;
output;

do i=1 to 4;
_col_='v'||put(i,1.);
_coef_= x{i};
output;
end;
end;
run;
```

***This data set constructs the remaining part of constraint F and the normalization constraint for the final LP.*

```
data aggbound&a&b;
length _type_ $ 8 _row_ $ 16 _col_ $ 8;
keep _type_ _row_ _col_ _coef_ lam;

set lambda;
array l{7} l1-l7;
lam = l{&b};

set bound1;
array UM{3} UM1-UM3;
array LM{3} LM1-LM3;
array UN{4} UN1-UN4;
array LN{4} LN1-LN4;
array pUM{3} pUM1-pUM3;
array pLM{3} pLM1-pLM3;
array pUN{4} pUN1-pUN4;
array pLN{4} pLN1-pLN4;

do i = 1 to 3;
_row_='uubound'||put(i,1.);
_type_='LE';
_col_='_rhs_';
_coef_=0;
output;
_col_='u'||put(i,1.);
_coef_=1;
output;
_col_='T0';
_coef_=-UM{i} + pUM{i}*lam;
output;
end;
```

```
do j=1 to 4;
_row_='vubound'||put(j,1.);
_type_='LE';
_col_='_rhs_';
_coef_=0;
output;
_col_='v'||put(j,1.);
_coef_=1;
output;
_col_='T0';
_coef_=-UN{j}+pUN{j}*lam;
output;
end;

do i = 1 to 3;
_row_='ulbound'||put(i,1.);
_type_='GE';
_col_='_rhs_';
_coef_=0;
output;
_col_='u'||put(i,1.);
_coef_=1;
output;
_col_='T0';
_coef_=-LM{i}-pLM{i}*lam;
output;
end;

do j=1 to 4;
_row_='vlbound'||put(j,1.);
_type_='GE';
_col_='_rhs_';
_coef_=0;
output;
_col_='v'||put(j,1.);
_coef_=1;
output;
_col_='T0';
_coef_=-LN{j}- pLN{j}*lam;
output;
end;
run;
```
***This data set constructs the weight bound constraints (B) of the final LP.*

```
data aggregate&a&b;
```

```
set objective&a firstcons&a&b (keep = _type_ _row_ _col_ _coef_ lam)
secondcons&a&b constr aggbound&a&b;
**This data set aggregates all constraints and the objective function to construct the final
LP.

proc lp data = aggregate&a&b printlevel=-2 sparsedata primalout = final&a&b (keep =
_VAR_ _Value_);
run;
**This statement runs the final LP and stores the results in final&a&b.

data temp (keep=_VALUE_);
set final&a&b;
if _VAR_='obj';
run;
**This data set temporarily stores the objective function value of the final LP.

proc append base=final data=temp;
run;
%end;
**This statement appends the objective function values from 'temp' to 'final'.

%mend runlam;
%runlam;
%end;

%mend runlp;
%runlp;

proc print data=final;
run;
**This statement prints the results of the model.
```

B.2 SAS CODE FOR ROADMAP EXAMPLE OF SECTION 3.2.3.1 ILLUSTRATING THE FUZZY MODEL WITH AR CONSTRAINTS IN FORM 1

The roadmap example in section 3.2.2.1 used to illustrate the fuzzy AR (Form 1) DEA model, requires solving the following three LPs in the given sequence.

The first LP, which is used to calculate f_1 values, uses the most desirable weight bound values in the weight bound constraints (B):

$$Max\ \mu^T Y_0 = f_1$$

$s.t.$

$$\eta^T X_0 = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots N$$

$$\mu^{\mathrm{T}} Y \text{ - } \eta^T X \le 0 \qquad \forall j \ldots\ldots\ldots\ldots M$$

$$-\mu_1 + \mu_2 \le 0$$
$$0.2\mu_1 - \mu_2 \le 0$$
$$-0.5\mu_1 + \mu_3 \le 0$$
$$0.1\mu_1 - \mu_3 \le 0$$
$$-4\eta_1 + \eta_2 \le 0$$
$$0.25\eta_1 - \eta_2 \le 0 \ldots\ldots\ldots\ldots\ldots\ldots B$$
$$-4\eta_1 + \eta_3 \le 0$$
$$0.25\eta_1 - \eta_3 \le 0$$
$$-0.4\eta_1 + \eta_4 \le 0$$
$$0.1\eta_1 - \eta_4 \le 0$$
$$\mu, \eta \ge 0$$

The following LP, which calculates f_0 values, uses the least desirable weight bound values in the weight bound constraints (*B*):

$$Max\ \mu^T Y_0 = f_0$$

$s.t.$

$$\eta^T X_0 = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots N$$

$$\mu^{\mathrm{T}} Y \text{ - } \eta^T X \le 0 \qquad \forall j \ldots\ldots\ldots\ldots M$$

$$-1.5885\mu_1 + \mu_2 \le 0$$
$$-2.062\mu_1 + \mu_3 \le 0$$
$$-7.5\eta_1 + \eta_2 \le 0 \ldots\ldots\ldots\ldots\ldots\ldots B$$
$$-6.17\eta_1 + \eta_3 \le 0$$
$$-1.89\eta_1 + \eta_4 \le 0$$
$$\mu, \eta \ge 0$$

Finally, the following crisp equivalent of the fuzzy model is solved.

Max λ

such that

$\lambda(f_0 - f_1) - \mu^T Y_0 \leq -f_1$....................$F$

$\eta^T X_0 = 1$..N

$\mu^T Y - \eta^T X \leq 0$...............................M

$-(1+0.5885\lambda)\mu_1 + \mu_2 \leq 0$

$(0.2-0.2\lambda)\mu_1 - \mu_2 \leq 0$

$-(0.5+1.562\lambda)\mu_1 + \mu_3 \leq 0$

$(0.1-0.1\lambda)\mu_1 - \mu_3 \leq 0$

$-(4+3.56\lambda)\eta_1 + \eta_2 \leq 0$

$(0.25-0.25\lambda)\eta_1 - \eta_2 \leq 0$............................B

$-(4+2.17\lambda)\eta_1 + \eta_3 \leq 0$

$(0.25-0.25\lambda)\eta_1 - \eta_3 \leq 0$

$-(0.4+1.49\lambda)\eta_1 + \eta_4 \leq 0$

$(0.1-0.1\lambda)\eta_1 - \eta_4 \leq 0$

$0 \leq \lambda \leq 1$

$\mu, \eta \geq 0$

Note that this model is a quadratic-programming model because its weight bound constraints (B) contain expressions in which two variables are multiplied to each other. Since one of these variables is λ and we know the range of variation of λ, we can use the parametric algorithm described in section 3.1.1.5.4 to solve it. Steps 1,3 and 8 of the algorithm require us to solve the model with λ fixed at a certain value. This is tantamount to solving an LP with a fixed objective function since λ is the objective function of the crisp equivalent LP. We cannot solve an LP with a fixed objective function. Therefore, we need to modify the model so that the objective function contains an expression involving decision variables as opposed to a fixed value. The most logical choice for the objective function is the efficiency term. The modified model, which can now be solved using the parametric algorithm, is as follows:

Max $\mu^T Y_0$

such that

$\lambda(f_0 - f_1) - \mu^T Y_0 \le -f_1$....................$F$

$\eta^T X_0 = 1$.......................................N

$\mu^T Y - \eta^T X \le 0$..............................M

$-(1+0.5885\lambda)\mu_1 + \mu_2 \le 0$

$(0.2 - 0.2\lambda)\mu_1 - \mu_2 \le 0$

$-(0.5 + 1.562\lambda)\mu_1 + \mu_3 \le 0$

$(0.1 - 0.1\lambda)\mu_1 - \mu_3 \le 0$

$-(4 + 3.56\lambda)\eta_1 + \eta_2 \le 0$

$(0.25 - 0.25\lambda)\eta_1 - \eta_2 \le 0$............................B

$-(4 + 2.17\lambda)\eta_1 + \eta_3 \le 0$

$(0.25 - 0.25\lambda)\eta_1 - \eta_3 \le 0$

$-(0.4 + 1.49\lambda)\eta_1 + \eta_4 \le 0$

$(0.1 - 0.1\lambda)\eta_1 - \eta_4 \le 0$

$0 \le \lambda \le 1$

$\mu, \eta \ge 0$

Instead of using the parametric algorithm directly, we solve the model for different values of λ between 0 and 1 at intervals of 0.1 and choose the solution corresponding to the maximum feasible value of λ. The justification is that this technique is easier to implement using SAS compared to the parametric algorithm and gives the same solution as the parametric algorithm using $\varepsilon = 0.1$.

The description of the constraint types used in the three LP's is as follows:

F – Constraint introduced by the membership function of the objective function

N – Normalization Constraint

M – Main set of Constraints

B – Weight Bound Constraints

All three LPs are solved in the same SAS code. The code with embedded descriptions of the data sets is as follows:

```
data ubound;
input UM1 UM2 UM3;
cards;
-1 1 0
.2 -1 0
-.5 0 1
.1 0 -1
;
```
***This data set contains the coefficients of the most desirable output weight AR constraints.*

```
data vbound;
input VM1 VM2 VM3 VM4;
cards;
-4 1 0 0
.25 -1 0 0
-4 0 1 0
.25 0 -1 0
-.4 0 0 1
.1 0 0 -1
;
```
***This data set contains the coefficients of the most desirable input weight AR constraints.*

```
data fuzzyubound;
input UM1 UM2 UM3 p1 p2 p3;
cards;
-1.5885 1 0 .5885 0 0
0 -1 0 .2 0 0
-2.062 0 1 1.562 0 0
0 0 -1 .1 0 0
;
```
***This data set contains the coefficients of the least desirable output weight AR constraints.*

```
data fuzzyvbound;
input VM1 VM2 VM3 VM4 p1 p2 p3 p4;
cards;
-7.56 1 0 0 3.56 0 0 0
```

```
0 -1 0 0 .25 0 0 0
-6.17 0 1 0 2.17 0 0 0
0 0 -1 0 .25 0 0 0
-1.89 0 0 1 1.49 0 0 0
0 0 0 -1 .1 0 0 0
;
**This data set contains the coefficients of the least desirable input weight AR
constraints.

data dea;
input y1 y2 y3 x1 x2 x3 x4;
cards;
103   40   144   97   69   101   112
91    30   107   139  111  190   61
120   95   63    174  33   162   130
59    43   95    38   101  94    91
72    84   24    33   66   99    121
115   78   79    109  120  97    137
140   258  154   118  170  92    81
63    21   132   85   144  27    102
61    33   125   88   41   138   59
44    70   88    60   101  45    71
79    54   147   68   125  64    112
175   317  69    109  93   109   117
76    41   93    64   89   161   127
48    119  66    111  71   36    102
253   217  115   209  168  84    79
;
**This is the normalized input-output data set

data lambda;
input l1 l2 l3 l4 l5 l6 l7;
cards;
0.4 0.5 0.6 0.7 0.8 0.9 1
;
**This is the data set of all λ values for which the crisp equivalent model is solved.

data final;
input _VALUE_;
cards;
;
**This creates an empty data set which will be used later to display the results.

data constr;
length _type_ $ 8 _row_ $ 16 _col_ $ 8;
keep _type_ _row_ _col_ _coef_;
```

```
set dea;
array x{4} x1-x4;
array y{3} y1-y3;
_type_ = 'LE';
_row_ = 'DMU'||put(_n_,2.);
_col_='_rhs_';
_coef_=0;
output;
do i=1 to 4;
_col_ = 'v'||put(i,1.);
_coef_= -x{i};
output;
end;

do j = 1 to 3;
_col_='u'||put(j,1.);
_coef_=y{j};
output;
end;
run;
```

***This data set constructs the main set of constraints (M)*

```
data ubounds;
length _type_ $ 8 _row_ $ 16 _col_ $ 8;
keep _type_ _row_ _col_ _coef_;
set ubound;
array UM{3} UM1-UM3;
_type_='LE';
_row_ = 'aru'||put(_n_,1.);
_col_='_rhs_';
_coef_=0;
output;
do i = 1 to 3;
_col_='u'||put(i,1.);
_coef_=UM{i};
output;
end;
run;
```

***This data set constructs the most desirable output weight AR constraints (B) using the coefficients from 'ubound'.*

```
data vbounds;
length _type_ $ 8 _row_ $ 16 _col_ $ 8;
keep _type_ _row_ _col_ _coef_;
set vbound;
```

```
array VM{4} VM1-VM4;
_type_='LE';
_row_ = 'arv'||put(_n_,1.);
_col_='_rhs_';
_coef_=0;
output;
do i = 1 to 4;
_col_='v'||put(i,1.);
_coef_=VM{i};
output;
end;
run;
```
***This data set constructs the most desirable input weight AR constraints (B) using the coefficients from 'vbound'.*

```
data fuzzyubounds;
length _type_ $ 8 _row_ $ 16 _col_ $ 8;
keep _type_ _row_ _col_ _coef_;
set fuzzyubound;
array UM{3} UM1-UM3;
_type_='LE';
_row_ = 'aru'||put(_n_,1.);
_col_='_rhs_';
_coef_=0;
output;
do i = 1 to 3;
_col_='u'||put(i,1.);
_coef_=UM{i};
output;
end;
run;
```
***This data set constructs the least desirable output weight AR constraints (B) using the coefficients from 'fuzzyubound'.*

```
data fuzzyvbounds;
length _type_ $ 8 _row_ $ 16 _col_ $ 8;
keep _type_ _row_ _col_ _coef_;
set fuzzyvbound;
array VM{4} VM1-VM4;
_type_='LE';
_row_ = 'arv'||put(_n_,1.);
_col_='_rhs_';
_coef_=0;
output;
do i = 1 to 4;
_col_='v'||put(i,1.);
```

```
_coef_=VM{i};
output;
end;
run;
```
***This data set constructs the least desirable input weight AR constraints (B) using the coefficients from 'fuzzyubound'.*

```
%macro runlp;
%do a=1 %to 15;
```
***This macro runs the same LPs for all 15 DMUs.*

```
data obj&a;
length _type_ $ 8 _row_ $ 16 _col_ $ 8;
keep _type_ _row_ _col_ _coef_;

set dea;
array x{4} x1-x4;
array y{3} y1-y3;

if _n_ = &a then do;
_type_ = 'MAX';
_row_ = 'obj';
do i = 1 to 3;
_col_='u'||put(i,1.);
_coef_= y{i};
output;
end;

_row_='DMU0';
_type_='EQ';
_col_='_rhs_';
_coef_= 1;
output;
do i=1 to 4;
_col_='v'||put(i,1.);
_coef_= x{i};
output;
end;
end;
run;
```
***This data set constructs the objective function and the normalization constraint (N) for the LPs used for obtaining f_1 and f_0 values.*

```
data main&a.1;
set obj&a constr fuzzyubounds fuzzyvbounds;
run;
```

***This data set aggregates all the constraints and the objective function to construct LPs that are used for calculating f_0 values.*

```
proc lp data=main&a.1 printlevel = -2 sparsedata primalout = prim&a.1;
run;
```

***This statement solves the LP main&a.1 and saves the results in prim&a.1*

```
data main&a.2;
set obj&a constr ubounds vbounds;
run;
```

***This data set aggregates all the constraints and the objective function to construct LPs that are used for calculating f_1 values.*

```
proc lp data=main&a.2 printlevel = -2 sparsedata primalout = prim&a.2;
run;
```

***This statement solves main&a.2 and saves the results in prim&a.2*

```
data difobj&a;
keep f0 f1 diff;

set prim&a.1;
if _VAR_='obj' then f0 = _VALUE_;

set prim&a.2;
if _VAR_='obj' then f1 = _VALUE_;

diff= f0 - f1;
output;
run;
```

***This data set extracts the f_0 and f_1 values from prim&a.1 and prim&a.2 and calculates their difference.*

```
data newobj&a;
keep newf1 newdiff;
set difobj&a;
if f1>0 then do;
newf1 = f1;
newdiff = diff;
output;
end;
run;
```

***This data set is 'difobj&a' without the null values.*

```
data objective&a;
length _type_ $ 8 _row_ $ 16 _col_ $ 8;
keep _type_ _row_ _col_ _coef_;
```

```
set dea;
array y{3} y1-y3;
if _n_ = &a then do;
_type_ = 'MAX';
_row_ = 'obj';
do i = 1 to 3;
_col_='u'||put(i,1.);
_coef_= y{i};
output;
end;
end;
run;
```
***This data set constructs the objective function of the final LP.*

```
%macro runlam;
%do b=1 %to 7;
```
***This macro runs the final LP for seven different values of λ.*

```
data firstcons&a&b;
length _type_ $ 8 _row_ $ 16 _col_ $ 8;
keep _type_ _row_ _col_ _coef_ lam f1 diff;
set lambda;
array l{7} l1-l7;
lam = l{&b};

set newobj&a;
f1 = newf1;
diff = newdiff;

_type_ = 'GE';
_row_ = 'fuzzy';
_col_='_rhs_';
_coef_= f1 + diff*lam;
output;
run;
```
***This data set constructs part of the constraint (F).*

```
data secondcons&a&b;
length _type_ $ 8 _row_ $ 16 _col_ $ 8;
keep _type_ _row_ _col_ _coef_;
set dea;
array x{4} x1-x4;
array y{3} y1-y3;
if _n_ = &a then do;
_type_ = 'GE';
```

```
_row_ = 'fuzzy';
do i=1 to 3;
_col_='u'||put(i,1.);
_coef_= y{i};
output;
end;

_row_='DMU0';
_type_='EQ';
_col_='_rhs_';
_coef_= 1;
output;

do i=1 to 4;
_col_='v'||put(i,1.);
_coef_= x{i};
output;
end;
end;
run;
```

***This data set constructs the normalization constraint and the remaining part of the constraint F.*

```
data aggubound&a&b;
length _type_ $ 8 _row_ $ 16 _col_ $ 8;
keep _type_ _row_ _col_ _coef_;

set lambda;
array l{7} l1-l7;
lam=l{&b};

set fuzzyubound;
array UM{3} UM1-UM3;
array p{3} p1-p3;

_row_='fuzzyaru'||put(_n_,1.);
_type_='LE';
_col_='_rhs_';
_coef_=0;
output;
do i = 1 to 3;
_col_='u'||put(i,1.);
_coef_=UM{i}+p{i}*lam;
output;
end;
run;
```

***This data set constructs the constraints (B) pertaining to the output weights for the final LP.*

```
data aggvbound&a&b;
length _type_ $ 8 _row_ $ 16 _col_ $ 8;
keep _type_ _row_ _col_ _coef_;

set lambda;
l{7}=l1-l7;
lam=l{&b};

set fuzzyvbound;
array VM{4} VM1-VM4;
array p{4} p1-p4;
_row_='fuzzyarv'||put(_n_,1.);
_type_='LE';
_col_='_rhs_';
_coef_=0;
output;
do i = 1 to 4;
_col_='v'||put(i,1.);
_coef_=VM{i}+p{i}*lam;
output;
end;
run;
```

***This data set constructs the constraints B pertaining to input weights for the final LP.*

```
data aggregate&a&b;
set objective&a firstcons&a&b (keep = _type_ _row_ _col_ _coef_ lam)
secondcons&a&b constr aggubound&a&b aggvbound&a&b;
run;
```

***This data set aggregates the objective function and the constraints to construct the final LP.*

```
proc lp data = aggregate&a&b printlevel=-2 sparsedata primalout = final&a&b (keep =
_VAR_ _Value_);
run;
```

***This statement solves the final LP for different DMUs and different values of lambda and saves the results in the data set 'final&a&b'. 'a' indicates the DMU and 'b' indicates the value of lambda.*

```
data temp (keep=_VALUE_);
set final&a&b;
if _VAR_='obj';
run;
```

***This data set temporarily stores the values of the objective function.*

```
proc append base=final data=temp;
run;
%end;
%mend runlam;
%runlam;
```
***This statement appends the objective function values to the data set 'final'.*

```
%end;
%mend runlp;
%runlp;

proc print data=final;
run;
```
***This statement prints the results of the model.*

B.3 SAS CODE FOR THE ROADMAP EXAMPLE OF SECTION 3.2.3.2 ILLUSTRATING THE FUZZY MODEL WITH AR CONSTRAINTS IN FORM 2

The fuzzy AR (Form 2) model for the roadmap example in section 3.2.3.2 is as follows:

$$Max\ \mu^T Y_0$$

$$s.t.$$

$$\eta^T X_0 = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots N$$

$$\mu^T Y - \eta^T X \le 0 \qquad \forall j \ldots\ldots\ldots\ldots M$$

$$-\eta_1 + (0.15 + 0.059\lambda)\eta_2 \le 0$$
$$\eta_1 - (0.428 - 0.086\lambda)\eta_2 \le 0$$
$$-\eta_1 + 0.209\eta_2 \le 0 \ldots\ldots\ldots\ldots\ldots\ldots B$$
$$\eta_1 - 0.514\eta_2 \le 0$$
$$-\eta_1 + (0.22 - 0.011\lambda)\eta_2 \le 0$$
$$\eta_1 - (0.609 + 0.095\lambda)\eta_2 \le 0$$

$$0 \le \lambda \le 1$$
$$\mu, \eta \ge 0$$

The SAS code used for solving this model is as follows:

```
data bound;
input VM1 VM2 p1 p2 l1 l2 l3 l4 l5 l6;
cards;
-1 .15 0 .059 0 .2 .4 .6 .8 .9
1 -.428 0 -.086 0 .2 .4 .6 .8 .9
-1 .209 0 0 0 .2 .4 .6 .8 .9
1 -.514 0 0 0 .2 .4 .6 .8 .9
-1 .22 0 -.011 0 .2 .4 .6 .8 .9
1 -.609 0 .095 0 .2 .4 .6 .8 .9
;
```

***This data set represents the coefficients of the variables in the weight bound constraints (B). The p's which are the differences between the least desirable and most desirable bounds, are the coefficients of λ.*

```
data dea;
input x1 x2 y1;
cards;
31451.9 1540.8 9648.5
24267.8 1491.1 9396.9
16609 1072.6 6884.8
4109.14 561.6 1924
1657.57 1842.5 2427.9
2124 85.2 617.3
1540 198.3 737.7
1750 152.3 907.9
1728.88 151.2 665.5
1313.48 129.7 705.4
1410.97 173.5 674.3
586.29 127.3 505.1
302.14 45.3 211
;
```

***This is the input-output data set.*

```
data constr;
set dea;
array x{2} x1-x2;
length _type_ $ 8 _row_ $ 16 _col_ $ 8;
keep _type_ _row_ _col_ _coef_;

_type_ = 'LE';
_row_ = 'DMU'||put(_n_,2.);
_col_='_rhs_';
_coef_=0;
output;
```

```
do i=1 to 2;
_col_ = 'v'||put(i,1.);
_coef_= -x{i};
output;
end;

_col_='u1';
_coef_=y1;
output;
run;
```
***This data set constructs the main set of constraints (M).*

```
%macro alpha;
%do b=1 %to 6;

data vbound;
set bound;
array VM{2} VM1-VM2;
array p{2} p1-p2;
array l{6} l1-l6;
length _type_ $ 8 _row_ $ 16 _col_ $ 8;
keep _type_ _row_ _col_ _coef_;

_type_ = 'LE';
_row_='arv'||put(_n_,1.);
_col_='_rhs_';
_coef_=0;
output;

do i = 1 to 2;
_col_= 'v'||put(i,1.);
_coef_=VM{i}+l{&b}*p{i};
output;
end;
run;
```
***This data set constructs the weight bound constraints (B).*

```
%macro runlp;
%do a=1 %to 13;
```
***This macro runs the LP for each of the thirteen DMUs.*

```
data obj&a&b;
set dea;
array x{2} x1-x2;
length _type_ $ 8 _row_ $ 16 _col_ $ 8;
keep _type_ _row_ _col_ _coef_;
```

```
if _n_ = &a then do;
_type_ = 'MAX';
_row_ = 'obj';
_col_='u1';
_coef_= y1;
output;

_row_='DMU0';
_type_='EQ';
_col_='_rhs_';
_coef_= 1;
output;

do i=1 to 2;
_col_='v'||put(i,1.);
_coef_= x{i};
output;
end;
end;
run;
```
***This data set constructs the objective function and the normalization constraint (N).*

```
data main&a&b;
set obj&a&b constr vbound&b;
run;
```
***This data set constructs the LP by aggregating all the constraints and the objective function.*

```
proc lp sparsedata data = main&a&b primalout = final&a&b;
run;
```
***This statement solves the LP and saves the results in the data set 'final&a&b'.*

```
proc print data=final&a&b;
run;
```
***This statement prints the results of the model.*

```
%end;
%mend runlp;
%runlp;

%end;
%mend alpha;
%alpha;
```

B.4 SAS CODE FOR SOLVING THE FUZZY ABSOLUTE WEIGHT RESTRICTION DEA MODEL APPLIED TO THE HIGHWAY MAINTENANCE PATROL DATA FROM COOK *ET AL.* (1990)

The fuzzy model corresponding to the absolute weight restriction DEA model applied to the highway maintenance patrol data is solved in three steps.

In the first step, the following model is solved for each DMU to obtain f_1 values:

$$Max\, \mu^T Y_0 = f_1$$

$$s.t.$$

$$\eta^T X_0 = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots N$$

$$\mu^T Y - \eta^T X \le 0 \qquad \forall j \ldots\ldots\ldots\ldots M$$

$$800T_0 \le \mu_1 \le 2100T_0$$

$$500T_0 \le \mu_2 \le 10000T_0 \ldots\ldots\ldots\ldots B$$

$$900T_0 \le \eta_1 \le 2500T_0$$

$$300T_0 \le \eta_2 \le 6000T_0$$

$$\mu, \eta \ge 0$$

In the next step, the following model is solved for each DMU to obtain f_0 values:

$$Max\, \mu^T Y_0 = f_0$$

$$s.t.$$

$$\eta^T X_0 = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots N$$

$$\mu^T Y - \eta^T X \le 0 \qquad \forall j \ldots\ldots\ldots\ldots M$$

$$389T_0 \le \mu_1 \le 2100T_0$$

$$10T_0 \le \mu_2 \le 10000T_0 \ldots\ldots\ldots\ldots B$$

$$90T_0 \le \eta_1 \le 2500T_0$$

$$10T_0 \le \eta_2 \le 12208T_0$$

$$\mu, \eta \ge 0$$

Finally, the following crisp equivalent of the fuzzy model is solved:

Max λ

such that

$$\lambda(f_0 - f_1) - \mu^T Y_0 \le -f_1 \ldots\ldots\ldots\ldots\ldots\ldots F$$

$$\eta^T X_0 = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots N$$

$$\mu^T Y - \eta^T X \le 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots M$$

$$(389 + 411\lambda)T_0 \le \mu_1 \le 2100T_0$$
$$(10 + 490\lambda)T_0 \le \mu_2 \le 10000T_0 \ldots\ldots\ldots\ldots\ldots\ldots B$$
$$(90 + 810\lambda)T_0 \le \eta_1 \le 2500T_0$$
$$(10 + 290\lambda)T_0 \le \eta_2 \le (12208 - 6208\lambda)T_0$$

$$\mu, \eta \ge 0$$
$$0 \le \lambda \le 1$$

Note that this model is a quadratic-programming model because its weight bound constraints (B) contain expressions in which two variables are multiplied to each other. Since one of these variables is λ and we know the range of variation of λ, we can use the parametric algorithm described in section 3.1.1.5.4 to solve it. Steps 1,3 and 8 of the algorithm require us to solve the model with λ fixed at a certain value. This is tantamount to solving an LP with a fixed objective function since λ is the objective function of the crisp equivalent LP. We cannot solve an LP with a fixed objective function. Therefore, we need to modify the model so that the objective function contains an expression involving decision variables as opposed to a fixed value. The most logical choice for the objective function is the efficiency term. The modified model, which can now be solved using the parametric algorithm, is as follows:

Max $\mu^T Y_0$

such that

$\lambda(f_0 - f_1) - \mu^T Y_0 \leq -f_1$....................$F$

$\eta^T X_0 = 1$.......................................N

$\mu^T Y - \eta^T X \leq 0$..............................M

$(389 + 411\lambda)T_0 \leq \mu_1 \leq 2100T_0$

$(10 + 490\lambda)T_0 \leq \mu_2 \leq 10000T_0$....................$B$

$(90 + 810\lambda)T_0 \leq \eta_1 \leq 2500T_0$

$(10 + 290\lambda)T_0 \leq \eta_2 \leq (12208 - 6208\lambda)T_0$

$\mu, \eta \geq 0$

$0 \leq \lambda \leq 1$

Instead of using the parametric algorithm directly, we solve the model for different values of λ between 0 and 1 at intervals of 0.1 and choose the solution corresponding to the maximum feasible value of λ. The justification is that this technique is easier to implement using SAS compared to the parametric algorithm and gives the same solution as the parametric algorithm using $\varepsilon = 0.1$.

The description of the constraints used in the three above models is as follows:

F – Constraint introduced by the membership function of the objective function

N – Normalization Constraint

M – Main set of Constraints

B – Weight Bound Constraints

The three LPs are solved within the same SAS code. The SAS code is described below.

```
data bound1;
input UM1 UM2 UN1 UN2 LM1 LM2 LN1 LN2 pUM1 pUM2 pUN1 pUN2 pLM1
pLM2 pLN1 pLN2;
```

```
cards;
2100 10000 2500 12208 389 10 90 10 0 0 0 6208 411 490 810 290
;
```

***This data set represents the least desirable bound values. 'U' represents upper bound and 'L' represents lower bound. Also M represents output weight and N represents input weight. For example 'UM1' represents upper bound on output weight 1. The values with the prefix 'p' represent the differences between the least (most) desirable bounds and the most (least) desirable bounds. For example, 'pUM1' represents the difference between the least desirable and most desirable values of 'UM1' and 'pLM1' represents the difference between the most desirable and least values of 'LM1'.*

```
data bound2;
input UM1 UM2 UN1 UN2 LM1 LM2 LN1 LN2;
cards;
2100 10000 2500 6000 800 500 900 300
;
```

***This data set represents the most desirable bound values. It uses the same notation as the data set for least desirable bound values.*

```
data lambda;
input l1 l2 l3 l4 l5 l6 l7;
cards;
0.4 0.5 0.6 0.7 0.8 0.9 1
;
```

***This data set represents the different values of λ for which the crisp equivalent of the fuzzy model will be solved.*

```
data dea;
input x1 x2 y1 y2;
cards;
751 67 696 39
611 70 616 26
538 70 456 17
584 75 616 31
665 70 560 16
445 75 446 16
554 76 517 26
457 72 492 18
582 74 558 23
556 64 407 18
590 78 402 33
1074 75 350 88
1072 74 581 64
696 80 413 24
;
```

***This data set represents the input – output data values.*

```
data constr;
set dea;
array x{2} x1-x2;
array y{2} y1-y2;
length _type_ $ 8 _row_ $ 16 _col_ $ 8;
keep _type_ _row_ _col_ _coef_;

_type_ = 'LE';
_row_ = 'DMU'||put(_n_,2.);
_col_='_rhs_';
_coef_=0;
output;

do i=1 to 2;
_col_ = 'v'||put(i,1.);
_coef_= -x{i};
output;
end;

do i=1 to 2;
_col_='u'||put(i,1.);
_coef_=y{i};
output;
end;
run;
```

*** This data set is used for constructing the main set of constraints (M). Since the main set of constraints are the same for all three LP's, this data set will be used for all three LP's.*

```
data final;
input _value_;
cards;
;
run;
```

***This statement is used to initiate a data set called 'final' which will be used later.*

```
%macro runbound;
%do c=1 %to 2;
```

***This macro is used to construct the weight bound constraints (B). The macro runs twice and creates data sets 'bounds1' and 'bounds2'. 'bounds1' constructs the bound constraints for calculating f_1 values while 'bounds2' constructs the bound constraints for calculating f_0 values.*

```
data bounds&c;
```

```
set bound&c;
length _type_ $ 8 _row_ $ 16 _col_ $ 8;
keep _type_ _row_ _col_ _coef_;
array UM{2} UM1-UM2;
array LM{2} LM1-LM2;
array UN{2} UN1-UN2;
array LN{2} LN1-LN2;

do i = 1 to 2;
_row_='uubound'||put(i,1.);
_type_='LE';
_col_='_rhs_';
_coef_=0;
output;
_col_='u'||put(i,1.);
_coef_=1;
output;
_col_='T0';
_coef_=-UM{i};
output;
end;

do j=1 to 2;
_row_='vubound'||put(j,1.);
_type_='LE';
_col_='_rhs_';
_coef_=0;
output;
_col_='v'||put(j,1.);
_coef_=1;
output;
_col_='T0';
_coef_=-UN{j};
output;
end;

do i = 1 to 2;
_row_='ulbound'||put(i,1.);
_type_='GE';
_col_='_rhs_';
_coef_=0;
output;
_col_='u'||put(i,1.);
_coef_=1;
output;
_col_='T0';
```

```
_coef_=-LM{i};
output;
end;

do j=1 to 2;
_row_='vlbound'||put(j,1.);
_type_='GE';
_col_='_rhs_';
_coef_=0;
output;
_col_='v'||put(j,1.);
_coef_=1;
output;
_col_='T0';
_coef_=-LN{j};
output;
end;
run;

%end;
%mend runbound;
%runbound;


%macro runlp;
%do a=1 %to 14;
**This macro is used for solving the LP's corresponding to f1 and f0 for all DMUs.

data obj&a;
set dea;
array x{2} x1-x2;
array y{2} y1-y2;
length _type_ $ 8 _row_ $ 16 _col_ $ 8;
keep _type_ _row_ _col_ _coef_;

if _n_ = &a then do;
do i = 1 to 2;
_type_ = 'MAX';
_row_ = 'obj';
_col_='u'||put(i,1.);
_coef_= y{i};
output;
end;

_row_='DMU0';
_type_='EQ';
```

```
_col_='_rhs_';
_coef_= 1;
output;

do i=1 to 2;
_col_='v'||put(i,1.);
_coef_= x{i};
output;
end;
end;
run;
```

***This data set is used for constructing the objective function and the normalization constraint (N) for LP's used for calculating f_1 and f_0.*

```
data main&a.1;
set obj&a constr bounds1;
run;
```

***This data set combines all the constraints (N, M, and B) and the objective function for obtaining f_1 values.*

```
proc lp data=main&a.1 printlevel = -2 sparsedata primalout = prim&a.1;
run;
```

***This statement solves the LP 'main&a.1' and saves the results in 'prim&a.1'.*

```
data main&a.2;
set obj&a constr bounds2;
run;
```

***This data set combines all the constraints (N, M, and B) and the objective function for obtaining f_0 values.*

```
proc lp data=main&a.2 printlevel = -2 sparsedata primalout = prim&a.2;
run;
```

***This statement solves the LP 'main&a.2' and saves the results in 'prim&a.2'.*

```
data difobj&a;
keep f0 f1 diff;

set prim&a.1;
if _VAR_='obj' then f0 = _VALUE_;

set prim&a.2;
if _VAR_='obj' then f1 = _VALUE_;

diff= f0 - f1;
output;
run;
```

***This data set extracts f0 and f1 values from prim&a.1 and prim&a.2 respectively. It also calculates the difference between f_0 and f_1.*

```
data newobj&a;
keep newf1 newdiff;
set difobj&a;
if f1>0 then do;
newf1 = f1;
newdiff = diff;
output;
end;
run;
```

***'newobj&a' is a cleaner version of 'difobj&a' without the null values.*

```
data objective&a;
length _type_ $ 8 _row_ $ 16 _col_ $ 8;
keep _type_ _row_ _col_ _coef_;
set dea;
array y{2} y1-y2;
if _n_ = &a then do;
_type_ = 'MAX';
_row_ = 'obj';
do i = 1 to 2;
_col_='u'||put(i,1.);
_coef_= y{i};
output;
end;
end;
run;
```

***This data set constructs the objective function of the third LP - the crisp equivalent of the fuzzy model.*

```
%macro runlam;
%do b=1 %to 7;
```

***This macro is used for running the crisp equivalent LP 7 times for seven different values of λ.*

```
data firstcons&a&b;
length _type_ $ 8 _row_ $ 16 _col_ $ 8;
keep _type_ _row_ _col_ _coef_ lam f1 diff;
set lambda;
array l{7} l1-l7;
lam = l{&b};

set newobj&a;
f1 = newf1;
```

```
diff = newdiff;

_type_ = 'GE';
_row_ = 'fuzzy';
_col_='_rhs_';
_coef_= f1 + diff*lam;
output;
run;
```
***This data set is used for constructing part of the constraint F.*

```
data secondcons&a&b;
length _type_ $ 8 _row_ $ 16 _col_ $ 8;
keep _type_ _row_ _col_ _coef_;
set dea;
array x{2} x1-x2;
array y{2} y1-y2;

if _n_ = &a then do;
_type_ = 'GE';
_row_ = 'fuzzy';
do i = 1 to 2;
_col_='u'||put(i,1.);
_coef_= y{i};
output;
end;

_row_='DMU0';
_type_='EQ';
_col_='_rhs_';
_coef_= 1;
output;
do i=1 to 2;
_col_='v'||put(i,1.);
_coef_= x{i};
output;
end;
end;
run;
```
***This data set is used for constructing the normalization constraint (N) and the remaining part of the constraint F.*

```
data aggbound&a&b;
length _type_ $ 8 _row_ $ 16 _col_ $ 8;
keep _type_ _row_ _col_ _coef_ lam;
set lambda;
array l{7} l1-l7;
```

```
lam = l{&b};

set bound1;
array UM{2} UM1-UM2;
array LM{2} LM1-LM2;
array UN{2} UN1-UN2;
array LN{2} LN1-LN2;
array pUM{2} pUM1-pUM2;
array pLM{2} pLM1-pLM2;
array pUN{2} pUN1-pUN2;
array pLN{2} pLN1-pLN2;
do i = 1 to 2;
_row_='uubound'||put(i,1.);
_type_='LE';
_col_='_rhs_';
_coef_=0;
output;
_col_='u'||put(i,1.);
_coef_=1;
output;
_col_='T0';
_coef_=-UM{i} + pUM{i}*lam;
output;
end;

do j=1 to 2;
_row_='vubound'||put(j,1.);
_type_='LE';
_col_='_rhs_';
_coef_=0;
output;
_col_='v'||put(j,1.);
_coef_=1;
output;
_col_='T0';
_coef_=-UN{j}+pUN{j}*lam;
output;
end;

do i = 1 to 2;
_row_='ulbound'||put(i,1.);
_type_='GE';
_col_='_rhs_';
_coef_=0;
output;
_col_='u'||put(i,1.);
```

```
_coef_=1;
output;
_col_='T0';
_coef_=-LM{i}-pLM{i}*lam;
output;
end;

do j=1 to 2;
_row_='vlbound'||put(j,1.);
_type_='GE';
_col_='_rhs_';
_coef_=0;
output;
_col_='v'||put(j,1.);
_coef_=1;
output;
_col_='T0';
_coef_=-LN{j}-pLN{j}*lam;
output;
end;
run;
```

***This data set constructs the weight bound constraints (B) for the crisp equivalent of the fuzzy LP.*

```
data aggregate&a&b;
set objective&a firstcons&a&b (keep = _type_ _row_ _col_ _coef_ lam)
secondcons&a&b constr aggbound&a&b;
```

***This data set joins all the constraints and the objective function and creates the crisp equivalent LP.*

```
proc lp data = aggregate&a&b printlevel=-2 sparsedata primalout = final&a&b (keep =
_VAR_ _Value_);
run;
```

***This statement runs the crisp equivalent LP and saves the results in final&a&b.*

```
data temp (keep=_VALUE_);
set final&a&b;
if _VAR_='obj';
run;
```

***This data set temporarily stores the objective function value of final&a&b before it is appended to the data set 'final' which was created earlier.*

```
proc append base=final data=temp;
run;
```

***This statement appends the objective function value from 'temp' to 'final'.*

```
%end;
%mend runlam;
%runlam;

%end;
%mend runlp;
%runlp;

proc print data=final;
run;
```
***This statement prints the results of the model.*

B.5 SAS CODE FOR THE FUZZY AR (FORM 1) DEA MODEL APPLIED TO THE WASHINGTONPOST DATA

The fuzzy model corresponding to the AR (Form 1) DEA model applied to the Washingtonpost production data requires solving the following three LP's.

The first LP is used to obtain f_1 values for all DMUs:

$$Max\ \mu^T Y_0 = f_1$$

$$s.t.$$

$$\eta^T X_0 = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots N$$

$$\mu^T Y - \eta^T X \leq 0 \qquad \forall j \ldots\ldots\ldots\ldots M$$

$$1.5\eta_2 - \eta_1 \leq 0$$

$$2\mu_3 - \eta_1 \leq 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots B$$

$$\mu, \eta \geq 0$$

Then the following LP is used to obtain f_0 values:

$$Max\,\mu^T Y_0 = f_0$$

$s.t.$

$$\eta^T X_0 = 1 \text{..................................} N$$

$$\mu^T Y - \eta^T X \le 0 \qquad \forall j \text{..............} M$$

$$\eta_2 - \eta_1 \le 0$$

$$\eta_3 - \eta_1 \le 0 \text{..............................} B$$

$$\mu, \eta \ge 0$$

Finally, the following crisp equivalent of the fuzzy model is solved using the parametric algorithm:

Max λ

such that

$$\lambda(f_0 - f_1) - \mu^T Y_0 \le -f_1 \text{....................} F$$

$$\eta^T X_0 = 1 \text{.......................................} N$$

$$\mu^T Y - \eta^T X \le 0 \text{...............................} M$$

$$(1+0.5\lambda)\eta_2 - \eta_1 \le 0$$

$$(1+\lambda)\eta_3 - \eta_1 \le 0 \text{..............................} B$$

$$0 \le \lambda \le 1$$

$$\mu, \eta \ge 0$$

Note that this model is a quadratic-programming model because its weight bound constraints (B) contain expressions in which two variables are multiplied to each other. Since one of these variables is λ and we know the range of variation of λ, we can use the parametric algorithm described in section 3.1.1.5.4 to solve it. Steps 1,3 and 8 of the algorithm require us to solve the model with λ fixed at a certain value. This is tantamount to solving an LP with a fixed objective function since λ is the objective function of the crisp equivalent LP. We cannot solve an LP with a fixed objective function. Therefore, we need to modify the model so that the objective function contains an expression involving decision variables as opposed to a fixed value. The most logical choice for the objective function is the efficiency term. The modified model, which can now be solved using the parametric algorithm, is as follows:

Max $\mu^T Y_0$

such that

$\lambda(f_0 - f_1) - \mu^T Y_0 \leq -f_1$....................$F$

$\eta^T X_0 = 1$...N

$\mu^T Y - \eta^T X \leq 0$...............................M

$(1 + 0.5\lambda)\eta_2 - \eta_1 \leq 0$

$(1 + \lambda)\eta_3 - \eta_1 \leq 0$..............................B

$0 \leq \lambda \leq 1$

$\mu, \eta \geq 0$

Instead of using the parametric algorithm directly, we solve the model for different values of λ between 0 and 1 in increments of 0.1 and choose the solution corresponding to the maximum feasible value of λ. The justification is that this technique is easier to implement using SAS compared to the parametric algorithm and gives the same solution as the parametric algorithm at ε = 0.1.

The description of the constraint types used in the three above LP's is as follows:

F – Constraint introduced by the membership function of the objective function

N – Normalization Constraint

M – Main set of Constraints

B – Weight Bound Constraints

All three LPs are solved in the same SAS code. The code with embedded descriptions of the data sets is as follows:

```
data dea;
input x1 x2 x3 y1;
cards;
53.21   6.34   42.18   48.21
90.30   58.69   102.38   95.96
46.93   79.95   77.56   79.13
82.24   68.58   101.24   79.98
79.83   49.04   75.38   70.81
61.98   21.33   71.18   77.05
96.81   79.98   104.95   112.82
75.52   20.20   45.16   49.64
56.87   17.55   60.78   66.10
88.35   56.86   90.88   76.48
68.33   39.27   87.75   81.06
101.65   70.74   77.54   78.20
66.11   47.88   79.22   69.84
121.01   176.62   106.26   96.64
122.99   178.08   89.41   92.80
140.32   139.43   91.57   86.70
112.16   94.12   97.79   82.13
103.02   96.51   86.77   82.87
26.85   38.54   56.44   58.38
118.35   129.49   138.77   106.49
115.71   107.04   181.93   108.59
113.78   145.37   198.22   97.02
74.66   35.52   85.62   89.07
114.15   161.76   100.68   120.83
83.85   49.66   76.57   93.23
81.63   53.49   61.22   70.77
84.99   106.69   90.30   121.33
93.63   138.58   108.32   88.75
97.00   70.78   86.61   112.53
104.95   147.15   113.05   141.23
103.66   120.64   100.22   128.57
82.17   107.91   67.99   78.75
92.26   103.30   92.16   91.40
108.51   102.45   104.51   126.46
122.99   232.49   96.90   125.54
94.58   159.89   93.26   116.21
102.30   134.84   112.06   140.10
117.71   157.66   113.33   115.38
114.33   102.20   121.33   136.19
116.73   95.09   129.06   137.23
131.65   202.73   145.29   139.50
122.99   156.37   168.46   171.75
152.79   145.71   142.46   138.08
```

```
192.33    124.71    146.66    119.20
153.64    127.65    135.70    123.98
194.09    132.62    109.67    101.69
57.59   74.75   56.31   72.80
62.31   33.76   78.90   102.55
;
```
***This is the input – output data set.*

```
data lambda;
input l1 l2 l3 l4 l5 l6 l7;
cards;
0.4 0.5 0.6 0.7 0.8 0.9 1
;
```
***This data set contains the different λ values for which the fuzzy model is solved.*

```
data final;
input _VALUE_;
cards;
;
```
***This data set will be used later to append to and display the results of the model.*

```
data constr;
length _type_ $ 8 _row_ $ 16 _col_ $ 8;
keep _type_ _row_ _col_ _coef_;

set dea;
array x{3} x1-x3;
_type_ = 'LE';
_row_ = 'DMU'||put(_n_,2.);
_col_='_rhs_';
_coef_=0;
output;

do i=1 to 3;
_col_ = 'v'||put(i,1.);
_coef_= -x{i};
output;
end;

_col_='u1';
_coef_=y1;
output;
run;
```
***This data set constructs the main set of constraints (M).*

```
data bounds1;
```

```
length _type_ $ 8 _row_ $ 16 _col_ $ 8;
keep _type_ _row_ _col_ _coef_;

_type_='LE';
_row_ = 'ar1';
_col_='_rhs_';
_coef_=0;
output;

_col_='v1';
_coef_=-1;
output;

_col_='v2';
_coef_=1.5;
output;

_type_='LE';
_row_ = 'ar2';
_col_='_rhs_';
_coef_=0;
output;

_col_='v1';
_coef_=-1;
output;

_col_='v3';
_coef_=2;
output;
run;
```
***This data set constructs the weight bound constraints (B) for the LP, which is used for calculating the* f_1 *values.*

```
data bounds2;
length _type_ $ 8 _row_ $ 16 _col_ $ 8;
keep _type_ _row_ _col_ _coef_;

_type_='LE';
_row_ = 'ar1';
_col_='_rhs_';
_coef_=0;
output;

_col_='v1';
_coef_=-1;
```

```
output;

_col_='v2';
_coef_=1;
output;

_type_='LE';
_row_ = 'ar2';
_col_='_rhs_';
_coef_=0;
output;

_col_='v1';
_coef_=-1;
output;

_col_='v3';
_coef_=1;
output;
run;
```

***This data set constructs the weight bound constraints (B) for the LP used for calculating the f_0 values.*

```
%macro runlp;
%do a=1 %to 48;
```

***This macro is used for running the three LP's for all 48 DMUs.*

```
data obj&a;
length _type_ $ 8 _row_ $ 16 _col_ $ 8;
keep _type_ _row_ _col_ _coef_;

set dea;
array x{3} x1-x3;

if _n_ = &a then do;
_type_ = 'MAX';
_row_ = 'obj';
_col_='u1';
_coef_= y1;
output;

_row_='DMU0';
_type_='EQ';
_col_='_rhs_';
_coef_= 1;
output;
```

```
do i=1 to 3;
_col_='v'||put(i,1.);
_coef_= x{i};
output;
end;
end;
run;
```

***This data set constructs the objective function and the normalization constraints (N) for all three LP's.*

```
data main&a.1;
set obj&a constr bounds1;
run;
```

***This data set joins the objective function and the constraints to create the LP for calculating f_1 values.*

```
proc lp data=main&a.1 printlevel = -2 sparsedata primalout = prim&a.1;
run;
```

***This statement runs the LP and extracts the results into the data set 'prim&a.1'.*

```
data main&a.2;
set obj&a constr bounds2;
run;
```

***This data set joins the objective function and the constraints to create the LP for calculating f_0 values.*

```
proc lp data=main&a.2 printlevel = -2 sparsedata primalout = prim&a.2;
run;
```

***This statement runs the LP and stores the results in 'prim&a.2'.*

```
data difobj&a;
keep f0 f1 diff;

set prim&a.1;
if _VAR_='obj' then f0 = _VALUE_;

set prim&a.2;
if _VAR_='obj' then f1 = _VALUE_;

diff= f0 - f1;
output;
run;
```

***This data set extracts the f_1 and f_0 values and calculates their difference.*

```
data newobj&a;
keep newf1 newdiff;
```

```
set difobj&a;
if f1>0 then do;
newf1 = f1;
newdiff = diff;
output;
end;
run;
```
***This data set clears all the null values from the data set 'difobj&a'.*

```
data objective&a;
length _type_ $ 8 _row_ $ 16 _col_ $ 8;
keep _type_ _row_ _col_ _coef_;

set dea;
if _n_ = &a then do;
_type_ = 'MAX';
_row_ = 'obj';
_col_='u1';
_coef_= y1;
output;
end;
run;
```
***This data set constructs the objective function of the crisp equivalent of the fuzzy model.*

```
%macro runlam;
%do b=1 %to 7;
```
***This macro is used for running the crisp equivalent of the fuzzy model seven times with seven different values of λ.*

```
data firstcons&a&b;

length _type_ $ 8 _row_ $ 16 _col_ $ 8;
keep _type_ _row_ _col_ _coef_ lam f1 diff;
set lambda;
array l{7} l1-l7;
lam = l{&b};

set newobj&a;
f1 = newf1;
diff = newdiff;

_type_ = 'GE';
_row_ = 'fuzzy';
_col_='_rhs_';
_coef_= f1 + diff*lam;
```

```
output;
run;
```
***This data set constructs part of the constraint F.*

```
data secondcons&a&b;
length _type_ $ 8 _row_ $ 16 _col_ $ 8;
keep _type_ _row_ _col_ _coef_;
set dea;
array x{3} x1-x3;

if _n_ = &a then do;
_type_ = 'GE';
_row_ = 'fuzzy';
_col_='u1';
_coef_= y1;
output;

_row_='DMU0';
_type_='EQ';
_col_='_rhs_';
_coef_= 1;
output;

do i=1 to 3;
_col_='v'||put(i,1.);
_coef_= x{i};
output;
end;
end;
run;
```
***This data set constructs the normalization constraint (N) and the other part of the constraint F.*

```
data aggbound&a&b;
length _type_ $ 8 _row_ $ 16 _col_ $ 8;
keep _type_ _row_ _col_ _coef_ lam;

set lambda;
array l{7} l1-l7;
lam = l{&b};

_row_='fuzzyar1';
_type_='LE';
_col_='_rhs_';
_coef_=0;
output;
```

```
_col_='v1';
_coef_=-1;
output;

_col_='v2';
_coef_=1+0.5*l&b;
output;

_type_='LE';
_row_ = 'fuzzyar2';
_col_='_rhs_';
_coef_=0;
output;

_col_='v1';
_coef_=-1;
output;

_col_='v3';
_coef_=1+l&b;
output;
run;
```

***This data set constructs the weight bound constraints (B) of the crisp equivalent of the fuzzy model.*

```
data aggregate&a&b;
set objective&a firstcons&a&b (keep = _type_ _row_ _col_ _coef_ lam)
secondcons&a&b constr aggbound&a&b;
run;
```

***This data set combines all the constraints and the objective function to create the crisp equivalent of the fuzzy model.*

```
proc lp data = aggregate&a&b printlevel=-2 sparsedata primalout = final&a&b (keep =
_VAR_ _Value_);
run;
```

***This statement solves the crisp equivalent of the fuzzy model and saves the results in 'final&a&b'.*

```
data temp (keep=_VALUE_);
set final&a&b;
if _VAR_='obj';
run;
```

***This data set temporarily stores the objective function value.*

```
proc append base=final data=temp;
run;
```

***This statement appends the objective function value stored in 'temp' into the data set 'final'.*

```
%end;
%mend runlam;
%runlam;

%end;
%mend runlp;
%runlp;

proc print data=final;
run;
```

***This statement displays the efficiency scores calculated by the crisp equivalent of the fuzzy model.*

B.6 SAS CODE FOR THE FUZZY AR (FORM 2) DEA MODEL APPLIED TO THE SDBC DATA FROM THOMPSON *ET AL.* (1996a)

The fuzzy AR DEA model applied to the SDBC data is shown below:

Max $\mu^T Y_0$

such that

$\eta^T X_0 = 1$.......................................N

$\mu^T Y - \eta^T X \leq 0$..............................M

$(78.3 + 8.7\lambda)\mu_1 - (114 + 6\lambda)\mu_2 \geq 0$............................01
$(95.7 - 8.7\lambda)\mu_1 - (126 - 6\lambda)\mu_2 \geq 0$............................02
$(7628.5 + 401.5\lambda)\mu_2 - (3667.5 + 407.5\lambda)\mu_1 \geq 0$........03
$(8431.5 - 401.5\lambda)\mu_2 - (4482.5 - 407.5\lambda)\mu_1 \geq 0$........04
$(96.9 + 17.1\lambda)\mu_1 - (114 + 6\lambda)\mu_3 \geq 0$..........................05
$(131.1 - 17.1\lambda)\mu_1 - (126 - 6\lambda)\mu_3 \geq 0$..........................06
$(7628.5 + 401.5\lambda)\mu_3 - (637.5 + 112.5\lambda)\mu_1 \geq 0$...........07......$B$
$(8431.5 - 401.5\lambda)\mu_3 - (862.5 - 112.5\lambda)\mu_1 \geq 0$...........08
$(96.9 + 17.1\lambda)\mu_2 - (78.3 + 8.7\lambda)\mu_3 \geq 0$.......................09
$(131.1 - 17.1\lambda)\mu_2 - (95.7 - 8.7\lambda)\mu_3 \geq 0$.....................10
$(3667.5 + 407.5\lambda)\mu_3 - (637.5 + 112.5\lambda)\mu_2 \geq 0$..........11
$(4482.5 - 407.5\lambda)\mu_3 - (862.5 - 112.5\lambda)\mu_2 \geq 0$..........12

$\mu, \eta \geq 0$

$0 \leq \lambda \leq 1$

The SAS code for solving this LP is as follows:

```
data bound1;
input U101 U102 U103 U104 U105 U106 U107 U108 U109 U110 U111 U112 pu101
pu102 pu103 pu104 pu105 pu106 pu107 pu108 pu109 pu110 pu111 pu112;
cards;
78.3 95.7 -3667.5 -4482.5 96.9 131.1 -637.5 -862.5 0 0 0 0 8.7 -8.7 -407.5 407.5 17.1 -
17.1 -112.5 112.5 0 0 0 0
;
```

***This data set represents the coefficients of output weight u_1 in the weight bound constraints (B). The values with the prefix 'p' are the coefficients of λ while the values without the 'p' are the values added to the λ term. The last two digits of all symbols represent the number of the constraint. For example, 'U101' is the first term in the coefficient of u_1 in constraint 01 and 'pu101' multiplied by λ is the second term. Thus, 'U101 + pu101*λ' is the coefficient of u_1 in constraint 01.*

```
data bound2;
input U201 U202 U203 U204 U205 U206 U207 U208 U209 U210 U211 U212 pu201
pu202 pu203 pu204 pu205 pu206 pu207 pu208 pu209 pu210 pu211 pu212;
cards;
-114 -126 7628.5 8431.5 0 0 0 0 96.9 131.1 -637.5 -862.5 -6 6 401.5 -401.5 0 0 0 0 17.1 -
17.1 -112.5 112.5
;
```
***This set represents the coefficients of u_2 in B.*

```
data bound3;
input U301 U302 U303 U304 U305 U306 U307 U308 U309 U310 U311 U312 pu301
pu302 pu303 pu304 pu305 pu306 pu307 pu308 pu309 pu310 pu311 pu312;
cards;
0 0 0 0 -114 -126 7628.5 8431.5 -78.3 -95.7 3667.5 4482.5 0 0 0 0 -6 6 401.5 -401.5 -8.7
8.7 407.5 -407.5
;
```
***This set represents the coefficients of u_3 in B.*

```
data lambda;
input l1 l2 l3 l4 l5 l6;
cards;
0.2 0.4 0.5 0.6 0.8 1
;
```
***This data set represents the different values of λ for which the model will be solved.*

```
data dea;
input x1 x2 y1 y2 y3;
cards;
162970 4872 1454 128 10087
95960 3610 421 47 1173
115314 2987 434 69 1611
115573 1431 288 59 844
70450 303 496 43 3582
69637 75 159 18 985
54275 295 186 26 500
108553 1235 960 52 1984
93207 273 257 10 1014
48800 43 106 17 359
39209 112 110 13 886
46000 1475 148 54 597
36482 52 71 8 334
;
```
***This data set represents the input – output data values.*

```
data constr;
set dea;
array y{3} y1-y3;
array x{2} x1-x2;
length _type_ $ 8 _row_ $ 16 _col_ $ 8;
keep _type_ _row_ _col_ _coef_;

_type_ = 'LE';
_row_ = 'DMU'||put(_n_,2.);
_col_='_rhs_';
_coef_=0;
output;
do j=1 to 2;
_col_ = 'v'||put(j,1.);
_coef_= -x{j};
output;
end;

do i = 1 to 3;
_col_='u'||put(i,1.);
_coef_=y{i};
output;
end;
run;
```
***This data set constructs the main set of constraints (M).*

```
data final;
input _VALUE_;
cards;
;
run;
```
***Null data set 'final' is created here so that it can be used to append values.*

```
%macro runbounds;
%do b=1 %to 6;
```
***This macro is used for creating weight bound constraints (B) with different values of λ.*

```
data bounds&b;
length _type_ $ 8 _row_ $ 16 _col_ $ 8;
keep _type_ _row_ _col_ _coef_;

set lambda;
k=l&b;

set bound1;
array U1{12} U101-U112;
```

```
array pu1{12} pu101-pu112;

do i = 01 to 12;
_row_='c'||put(i,2.);
_type_='GE';
_col_='_rhs_';
_coef_=0;
output;
_col_='u1';
_coef_=U1{i}+pu1{i}*k;
output;
end;

set bound2;
array U2{12} U201-U212;
array pu2{12} pu201-pu212;

do i = 01 to 12;
_row_='c'||put(i,2.);
_type_='GE';
_col_='u2';
_coef_=U2{i}+pu2{i}*k;
output;
end;

set bound3;
array U3{12} U301-U312;
array pu3{12} pu301-pu312;

do i = 01 to 12;
_row_='c'||put(i,2.);
_type_='GE';
_col_='u3';
_coef_=U3{i}+pu3{i}*k;
output;
end;

run;
**This data set constructs the weight bound constraints B.

%macro runlp;
%do a=1 %to 13;
**This macro is used for solving the LP for each of the 13 DMUs.

data obj&a;
set dea;
```

```
array y{3} y1-y3;
array x{2} x1-x2;
length _type_ $ 8 _row_ $ 16 _col_ $ 8;
keep _type_ _row_ _col_ _coef_;

if _n_ = &a then do;
_type_ = 'MAX';
_row_ = 'obj';
do i=1 to 3;
_col_='u'||put(i,1.);
_coef_= y{i};
output;
end;

_row_='DMU0';
_type_='EQ';
_col_='_rhs_';
_coef_= 1;
output;
do i = 1 to 2;
_col_='v'||put(i,1.);
_coef_= x{i};
output;
end;
end;
run;
```

***This data set constructs the objective function and the normalization constraint (N)*

```
data main&a;
set obj&a bounds&b constr;
run;
```

***This data set combines the objective function and the constraints N, B, and M.*

```
proc lp sparsedata printlevel=-2 data = main&a primalout = final&a;
run;
```

***This statement solve 'main&a' and stores the results in 'final&a'*

```
data temp (keep=_VALUE_);
set final&a;
if _VAR_='obj';
run;
```

***The data set 'temp' temporarily stores the objective function value of 'final&a'.*

```
proc append base = final data = temp;
run;
```

***This statement appends the contents of the data set 'temp' to the data set 'final' which was created earlier.*

```
%end;
%mend runlp;
%runlp;

proc print data=final;
run;

%end;
%mend runbounds;
%runbounds;
```

VITA

Amit K. Kabnurkar was born on September 30, 1975 in Amaravati, India. He did his schooling in Bombay, India. He did his Bachelor's in Production Engineering from VJTI, which is affiliated to the University of Bombay. During his undergraduate program, he interned in Larsen and Toubro Ltd., Bombay as a "Project Planner" in their heavy equipment manufacturing plant. During his tenure in Larsen and Toubro, he planned and optimally assigned resources to multiple heat exchanger manufacturing projects. He had an excellent academic record during his undergraduate studies, which culminated in him featuring in the top five of 300 graduating students appearing for the final year examination of the University of Bombay.

After completion of his Bachelor's, he worked for a year as a research engineer in the field of Rapid Prototyping in the Indian Institute of Technology (IIT), Bombay. As a research engineer, his mandate was to advance the field of Rapid Tooling through research and experimentation and also increase the awareness about Rapid Prototyping in the Indian industry. This research experience in IIT inspired him to pursue a Master's degree in Industrial and Systems Engineering at Virginia Tech, which is a top ten ranked program in the United States. During his MS, he maintained the same strong academic track record that he had demonstrated during his undergraduate studies, and completed the course with a GPA of 3.95. After graduation, he is working as a Project Manager in Capital One Financial Corporation. His work involves coordination of IT related projects initiated to resolve software conflicts on desktops.

His research interests are process improvement; and performance and productivity measurement.