Non-unique Product Groups on Two Generators

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(ABSTRACT)

The main purpose of this paper is to better understand groups that do not have the unique product property. In particular, the goal is to better understand Promislow's example, G, of such a group. In doing so, we will develop methods for generating examples of other sets that do not have the unique product property. With these methods we can show that there exists other distinct 14 element, square, non-unique product sets in G that are not inversions or translations. Also, this paper answers the question as to whether every non-unique product set can have only 14 elements in the negative by producing a 17 element square n.u.p. set. The secondary purpose of this paper is to demonstrate that in the group ring $K[G]$, there are no units of support size 3.

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Chapter 1

Introduction

Two of the major problems in the theory of group rings of infinite groups are the famous zero divisor conjecture of Kaplanski and the, closely related, nontrivial units conjecture. The zero divisor conjecture states that if a group G is a torsion-free group and K is a field, then the group ring $K[G]$ has no zero divisors. Similarly, the nontrivial units conjecture states that if G is a torsion-free group and K is a field, then the group ring $K[G]$ has only trivial units, i.e. elements of the form kg where $0 \neq k \in K$ and $g \in G$. For so called unique product groups, both conjectures hold. Specifically, in the case of a right orderable group, under some total order $\leq$, then for any product of nonempty finite sets there is a uniquely determined element g of maximal or minimal magnitude. Naturally, the question was raised as to whether or not every torsion-free group was a u.p. group since this would give an easy proof for both conjectures [4]. This question was answered in the negative by Rips and Segev. In their paper, it was shown how the existence of a counterexample could be constructed and small cancellation theory was used to show that such a group was, in fact, torsion free. Although the method described was concrete, no explicit group has been constructed via this method. Later, Promislow demonstrated, by direct methods, the existence of a more elementary counterexample. In his paper, he produced an explicit subset A for the group

$$\langle x, y \mid x^{-1}y^2x = y^{-2}, y^{-1}x^2y = x^{-2} \rangle$$

so that the square product, (A, A), did not satisfy the u.p. property.

We will refer to groups that do not satisfy the u.p. property as non-unique product groups. For non-unique product groups, the zero divisor conjecture and the nontrivial units conjecture are both still open problems. One of the major reasons for this is that currently the two examples given above are the only known specimens of such groups. Still, very little is known about these two examples. This leaves many questions to be answered about both of these examples and the structure of generalized groups of this nature. The purpose of this paper is a modest attempt to better understand some of the properties of non-unique product groups and to develop techniques that generate new non-unique product sets. In particular, the crux of this paper is to better understand Promislow's example.

Chapter 2

Technical Introduction

The following chapter is given as a reasonable desire for self-containment. Such an attempt is by no means exhaustive. However, a full attempt at representation of such material would be quite taxing and detract from its overall purpose. References to proofs and additional ideas represent the author's personal taste, but many of which can be found in standard texts of the pertaining material.

2.1 U.P. Groups

Although we are primarily interested in non-unique product groups, in some cases it will be helpful for us to look at u.p. groups; for instance, a subgroup may be a u.p. group. In order to deduce that such a group is a u.p. group, it is helpful to have some criterion; for our purposes we will use right orderability.

Definition 1. *A torsion-free group G is said to have the unique product property if for any nonempty finite sets X and Y in G there exists an element g uniquely expressed in the form xy where $x \in X$ and $y \in Y$. Any group that satisfies this property is called a unique product group (or a u.p. group). Conversely, we refer to groups that do not have this property as non-unique product groups (or n.u.p. groups).*

Definition 2. *A group G is said to be right orderable (or a RO-group) if there exists a total ordering $\leq$ on G such that for all $x, y, z \in G$ if $x \leq y$, then $xz \leq yz$.*

Remark
By this definition it follows that every RO-group must also be torsion-free

Theorem 3. *Every right orderable group has the unique product property.*

Proof
Let G be a right orderable group and $A, B \subset G$ be nonempty finite subsets. Since G is right orderable, there exists a largest element $\hat{a} \in A$. Choose $\hat{b} \in B$ so that $\hat{a}\hat{b}$ is the largest element of $\hat{a}B$. Then for $a \in A$ and $b \in B$, $a \leq \hat{a} \Rightarrow ab \leq \hat{a}b \leq \hat{a}\hat{b}$. Moreover, if $ab = \hat{a}\hat{b}$, then this implies $a = \hat{a}$, by definition of $\hat{a}$. So, $b = \hat{b}$. Therefore, it follows then that $\hat{a}\hat{b}$ is uniquely represented. Since A and B were arbitrary, then G has the unique product property. $\square$
Remark
At the time of writing, it is still an open problem as to if there exists a non-right orderable u.p. group.

2.2 Rudiments of Combinatorial Group Theory

This section consists mostly of definitions and statements of theorems that will be used freely in this work. The proofs of the stated theorems can be found in most standard texts on Combinatorial Group Theory.

Definition 4. *Let $\langle S \mid R \rangle$ be a presentation for a group G. We define a Tietze transformation on G to be any of the following transformations.*
1. *Insertion of a relation that is a consequence of other relations in R: $\langle S \mid R \rangle \rightarrow \langle S \mid R \cup \{r\} \rangle$ where r is a consequence of elements of R.*
2. *Deletion of a redundant relation: $\langle S \mid R \cup \{r\} \rangle \rightarrow \langle S \mid R \rangle$ where r is a consequence of elements of R.*
3. *Addition of a new generator: $\langle S \mid R \rangle \rightarrow \langle S \cup \{x\} \mid R \cup \{xy^{-1}\} \rangle$ where y is an expression of x in S.*
4.*Removal of a redundant generator: $\langle S \cup T \mid R \cup R_T \rangle \rightarrow \langle S \mid R' \rangle$ where R_T are relations on the elements of T and R' denotes replacement by equivalent words on S.*

Theorem 5 (Tietze's Theorem)**.** *Two presentations define isomorphic groups if and only if one can be transformed into another by a sequence of Tietze transformations.*

Definition 6. *Let $R \subset F$ for a free group F. Then R is said to be symmetrized if every element of R is cyclically reduced and $r \in R$ implies every cyclically reduced conjugate of r or r^{-1} is also in R. Here cyclically reduced means that a word $w(x_1, \ldots, x_n)$ in F does not simultaneously begin with x_i^{ϵ} and end with $x_i^{-\epsilon}$. Intuitively, we can think of this as saying w is not reduced when written on a circle.*

Definition 7. *A group $G = \langle X \mid R \rangle$ satisfies the metric condition $C'(\lambda)$ if every r contained in the symmetrized set of relations, R', has the property that if r is equivalent, in G, to a word, bc (denoted $r \equiv bc$) where b is a piece of a relator, then $\mid b \mid < \lambda \mid r \mid$. Here b is a piece of a relator if there exists relators x and y in R' where $x \equiv bw$ and $y \equiv bv$ for distinct words w, $v \in G$.*

Theorem 8 (The Torsion Theorem)**.** *Let F be the free group and let R be a symmetrized subset of F satisfying $C'(1/8)$. Let $\nu : F \to F/N$ be the natural map. If w is a word in $G = F/N$ of finite order, then:*
(1) $w = v(w')$ where w' is a word of finite order in F, or
(2) some $r \in R$ has the form $r = rv^n$ where w is conjugate to a power of v in G.

Definition 9. *We define the Cayley graph for a group G with a generating set X to be a connected 1-complex constructed in the following way:*
(1) For each $g_i \in G$ we assign a vertex v_i.
(2) Next each element x_i of X is assigned a color c_i.
(3) Then if $g_k = g_j x_i$ in G we construct a directed edge of color c_i from v_k to v_j.

Similarly, we define the notion of a map.

Definition 10. *We define a map Λ over a group $G = \langle X \mid R \rangle$ to be a 2-complex along with a labeling function L, such that every cell in Λ is either a 0-cell or an R-cell. Here a cell p of the diagram is an R-cell if the label $L(p)$ of its boundary is visually equal to a relator in G. Note that here 0-cell does not mean the 0-cell in the topological sense, but rather a cell that is visually equal to 0 in $F(X)$.*

Chapter 3

Rips and Segev Example

The first example of such a non-unique product group was given by Rips and Segev. However, it is worth noting that by means of a construction, they posited the existence of such a group, in the sense that a choice of parameters will give such a group with the desired property. For reasons that are partially graph theoretic in nature, at the time of writing no explicit example of such a group has been computed. It is also worth noting that such a construction is completely generalizable. With these in mind we will look at their construction. Since we are mainly interested in the construction of such a group and not necessarily the more technical specifics such as proving that such groups are torsion free, we direct the curious reader to [6]. Specifically the reason we omit these more technical points is the group that we will be considering for our constructions is a well known torsion-free group.

Let $F = F(a, b)$ be the free group on two generators. Choose base points $x_1, \ldots, x_n$ (with some moderate restrictions noted below). Next, construct paths $K_1, \ldots, K_n$ in F so that $K_i = \{x_i, x_i a, x_i a^2, \ldots, x_i a^m\}$ and $m_i = m + 1$ is a distinct integer for each $i \in \{1, \ldots n\}$. We want to construct a group H where $K = \bigcup_{i=1}^{n} K_i$ and $J = \{1, a, b, ab\}$ are subsets of H that do not satisfy the unique product property. As indicated by the figure below, in each (K_i, J) every edge is matched twice in F with the exception to the circled edges represented by $M_i = \{x_i, x_i b, x_i a^{m_i}, x_i a^{m_i} b\}$.

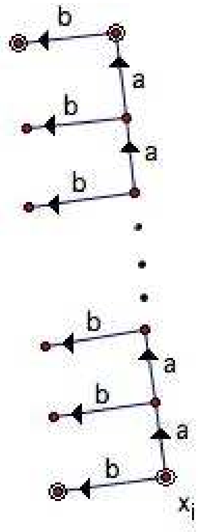

Figure 3.1: Illustration of a Single Set, $(K_i,\ J)$

To construct H it is pretty clear that we want to define relations on F so that we can match the elements of each M_i to other elements in $(K,\ J)$. The difficulty lies in the fact we must ensure that the resulting group H is torsion free.

The first parameter we consider is n, the number of paths K_i in F. Define Γ to be the directed graph with n vertices, $v_1, \ldots v_n$ and $4n$ edges, $a_1, \ldots, a_n$, $b_1, \ldots, b_n$, $c_1, \ldots, c_n$, $d_1, \ldots, d_n$. Let α, β, γ, $\delta \in S_n$, the permutation group on n letters. For each permutation, say α, draw the edge a_i from v_i to $v_{\alpha(i)}$. We place the condition that n must be chosen large enough so that with a proper choice of α, β, γ, and δ, the graph Γ has $4n$ distinct edges and no nontrivial closed path of length less than 16, we denote this condition as P_1. As noted above, P_1 is the graph theoretic problem in constructing an explicit example of such a group. The existence of such a graph is not difficult to show, but the large size of such a construction adds to its intractability. Regardless, we use the graph $\Gamma(\alpha, \beta, \gamma, \delta)$ to match the $4n$ elements of $M = \bigcup_{i=1}^{n} M_i$ to other elements in $(K,\ J)$. The second condition P_2 to be satisfied requires that we choose $4n$ integers, $0 < D_i < C_i < B_i < A_i < m_i$ so that A_i, B_i, C_i, D_i, $A_i - B_i$, $A_i - C_i$, $A_i - D_i$, $B_i - C_i$, $B_i - D_i$, $C_i - D_i$ are distinct among themselves for all i.

After satisfying the conditions P_1, and P_2, define L to be a labeling function on Γ, so that:

$$L(a_i) = b^{-1} a^{A_{\alpha(i)}},$$

$$L(b_i) = b a^{-B_{\beta(i)}},$$

$$L(c_i) = a^{m_i} b^{-1} a^{C_{\gamma(i)}},$$
$$L(d_i) = a^{m_i} b a^{D_{\delta(i)}}.$$

Using this labeling function, our initial choice of base points is given by $x_i = x_1 L(\omega_i)$ where $x_1 \in F$ is an arbitrary choice and each ω_i denotes a path in Γ from v_1 to v_i. This restriction prevents pathological choices that can be made in our selection of initial base points; for instance, $x_1 = a, \ldots, x_n = a^n$.

Now define $H = \langle a, b \mid \{L(\rho) \mid \rho$ is a closed path in $\Gamma\rangle$. In particular, it can be shown that the following relations hold in H:

$$x_i = x_{\alpha(i)} a^{A_{\alpha(i)}} b,$$

$$x_i b = x_{\beta(i)} a^{B_{\beta(i)}},$$
$$x_i a^{m_i} = x_{\gamma(i)} a^{C_{\gamma(i)}},$$
$$x_i a^{m_i} b = c_{\delta(i)} a^{D_{\delta(i)}}.$$

Clearly, this shows that every element of each M_i can be matched up to another element inside of $(K,\ J)$. As mentioned before, the remaining part of the argument is to show that H is torsion free. The argument given by Rips and Segev uses the labeling function to deduce that the group, H, satisfies the small cancellation condition $C'(1/8)$ and by the Torsion Theorem must be torsion free. Despite the graph theoretic difficulty, the minimal conditions on the construction makes it is easy to see that this method could be used to generate infinitely many of such groups.

Chapter 4

Promislow's Example

The second example of a non-unique product group,

$$G = \langle x, y \mid x^{-1}y^2x = y^{-2}, y^{-1}x^2y = x^{-2} \rangle$$

was discovered by David Promislow. One of the key reasons for looking at this group was that it was known to be torsion free and non-right orderable. Earlier, Fox conjectured that this group might be a counterexample; but as the Fibonacci group $F(2,6)$ [1]. We will show G itself is not a RO-group. For that, we will use the following theorems are found in [4].

Theorem 11. *A group G is an RO-group if and only if for all nonidentity elements x_1, x_2, …, $x_n \in G$ there exist suitable signs $\epsilon_i = \pm 1$ such that $1 \notin S^G(x_1^{\epsilon_1}, \ldots, x_n^{\epsilon_n})$ where $S^G(x_1^{\epsilon_1}, \ldots, x_n^{\epsilon_n})$ denotes the subsemigroup generated by x_1, x_2, …, x_n.*

Theorem 12. *Promislow's group, G, is not an RO-group.*

Proof
Let $\epsilon = \pm 1$ and $\delta = \pm 1$ be arbitrary. Then for $x, y \in G$ we can write

$$(x^\epsilon y^\delta)^2 (y^\delta x^\epsilon)^2 = (x^\epsilon y^\delta x^\epsilon y^\delta)(y^\delta x^\epsilon y^\delta x^\epsilon) = (x^\epsilon y^\delta)(x^\epsilon y^{2\delta})(x^{-\epsilon} x^\epsilon)(x^\epsilon y^\delta x^\epsilon) =$$

$$(x^\epsilon y^\delta)(x^\epsilon y^{2\delta} x^{-\epsilon})(x^{2\epsilon} y^\delta x^\epsilon) = x^\epsilon (y^{-\delta} x^{2\epsilon} y^\delta) x^\epsilon = x^\epsilon x^{-2\epsilon} x^\epsilon = 1.$$

Thus, $1 \in S^G(x^\epsilon, y^\delta)$ for any $\epsilon, \delta = \pm 1$. It follows then by the previous theorem that G is not an RO-group.$\square$

4.1 Group as described by Promislow

We adopt the same convention used in [5] to describe G, Promislow's example of a torsion-free, non-unique product group, as a subgroup of $D \times D \times D$ where D is the infinite dihedral

group and $G = E_0 \cup E_1 \cup E_2 \cup E_3$, where E_0, E_1, E_2, and E_3 satisfy the following patterns:
E_0: (even, even, even),
E_1: (even*, odd, odd*),
E_2: (odd, even*, even*),
E_3: (odd*, odd*, odd),
where we have the relations; $nm = n + m$, $nm^* = (n + m)^*$, $n^*\ m = (n - m)^*$, and $n^*m^* = (n - m)$ for some m, $n \in \mathbb{Z}$. Thus, using the canonical generators $x = (1,\ 0^*,\ 0^*)$ and $y = (0^*,\ 1,\ 1^*)$, we see that the presentation above also defines G. It is worth noting that "*" is not an operation, but rather represents the element of the dihedral group of order 2.

Under this representation, Promislow exhibited that the following 14 element subset

$$A = \{(0,0,2), (0,0,-2), (2^*,1,1^*), \\ (2^*,-1,-1^*), (0^*,1,1^*), (0^*,1,-1^*), (0^*,-1,1^*), (0^*,-1,-1^*), (1^*,2^*,0^*), \\ (-1^*,2^*,0^*), (1^*,0^*,-2^*), (-1^*,0^*,2^*), (1^*,0^*,0^*), (-1^*,0^*,0^*)\ \}$$

has the property that $(A,\ A)$ has no uniquely represented elements. Beyond stating that such a set must contain elements from at least 3 sets above (we actually show this below), very little information was given as to the construction of such a set. The remaining two sections of this chapter will be devoted to understanding the structure of such a set. However, before doing so we will illustrate that elements from at least three of the sets E_0, E_1, E_2, or E_3 are needed to produce any n.u.p. set in G.

Theorem 13. *The set E_0 is an abelian normal subgroup of G and hence a RO-group.*

Proof
By definition $E_0 \subseteq G$ and $1 \in E_0$. Let $a, b \in E_0$ with $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$. So, $ab = (a_1, a_2, a_3)(b_1, b_2, b_3) = (a_1 + b_1,\ a_2 + b_2,\ a_3 + b_3)$. Since $a_1,\ a_2,\ a_3,\ b_1,\ b_2,\ b_3 \in 2\mathbb{Z}$ this implies $a_1 + b_1,\ a_2 + b_2,\ a_3 + b_3 \in 2\mathbb{Z}$ and so $ab \in E_0$. Also, if $a = (a_1, a_2, a_3)$ contained in G this implies $a_1,\ a_2,\ a_3 \in 2\mathbb{Z}$ which says that $-a_1,\ -a_2,\ -a_3 \in 2\mathbb{Z}$ and so $-a = (-a_1, -a_2, -a_3) \in E_0$. Thus, $E_0 \leq G$. Moreover, let $c,\ d,\ e,\ f \in G$ with $c \in E_0$, $d \in E_1$, $e \in E_2$, and $f \in E_3$. Let $a \in E_0$ be arbitrary. Then $cac^{-1} \in E_0$ is obvious. Also, $dad^{-1} = (-a_1,\ a_2,\ -a_3) \in E_0$, $eae^{-1} = (a_1,\ -a_2,\ -a_3) \in E_0$, and $faf^{-1} = (-a_1,\ -a_2,\ a_3) \in E_0$. Since these elements are arbitrary and $G = E_0 \cup E_1 \cup E_2 \cup E_3$ this implies for all $g \in G$, $gag^{-1} \in E_0$ for all $a \in E_0$. Therefore, $gE_0g^{-1} \subseteq E_0$. Hence $E_0 \triangleleft G$. Also, E_0 has cosets E_0, E_1, E_2, and E_3 and so $|G : E_0| = 4$. From the above definition of E_0 we see that $E_0 \cong \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ and hence abelian. Moreover, since every torsion-free abelian group is also right orderable [3], we know that E_0 is also a RO-group.$\square$

Theorem 14. *The set $E_0 \cup xE_0$ is a right orderable normal subgroup of G for all $x \in G$.*

Proof
Let $H = E_0 \cup xE_0$. If $x \in E_0$, then there is nothing to prove by the previous theorem. Thus,

we may assume that $x \notin E_0$. By the previous theorem we know that $1 \in H$. For $a, b \in H$, there are only 3 possible cases to consider. If both a, $b \in E_0$, then obviously $ab \in E_0 \subseteq H$, we just proved this. If one element is in E_0 and the other is in xE_0, we will only consider one case since showing both are similar. Suppose $a \in E_0$ and $b \in xE_0$. Then $b \in xE_0 = E_0x$. So, $bx^{-1} \in E_0$. So, by Theorem 13, this says that $abx^{-1} \in E_0$. Hence, $ab \in xE_0$. Finally, if both $a, b \in xE_0$, then $a = xa_1$ and $b = xb_1$ where $a_1, b_1 \in E_0$. Since $a_1 \in E_0$, then $x^{-1}a_2x = a_1$ for some $a_2 \in E_0$. So, $a = xa_1 = a_2x$. Thus, $ab = a_2xxb_1 = a_2x^2b_1$, but for all $y \in G$, $y^2 \in E_0$. Therefore, $ab \in E_0$. Also, let $a \in xE_0 = E_0x$. Then by the above description, it is obvious that a^{-1}, must also lie in xE_0. Moreover, since $|G : H|$=2 then this automatically implies that $H \lhd G$. Also, $2 = |H : E_0$ |implies the factor group $H/E_0 \cong C_2$. Thus, $E_0 \lhd H$, and H/E_0 cyclic. So, by [2], there exists a subnormal series $1 = H_1 \lhd H_2 \lhd \ldots \lhd H_n = H$, where $H_{i+1}/H_i \cong \mathbb{Z}$ for all i $\in \{1, 2, \ldots, n\}$. That is, H is said to be poly-$\mathbb{Z}$ and hence right orderable.□
Therefore, it follows then that.

Corollary 15. *Any finite subset of G that does not possess the u.p. property must have elements contained in at least three of the sets E_0, E_1, E_2, or E_3.*

Although this group is well studied, remarkably, very little is known about the structure of this set. In order to develop some intuition about the construction of such a set, we will look beyond the construction in terms of cosets. The primary goal of the remaining part of the chapter is to show that other square n.u.p. sets can be generated inside the group that are not the result of translations or the inverse map. To do so, we will look at the group (and also the set) from two other view points.

4.2 Group as a Fibonacci Group

One of the problems with understanding Promislow's example in such a way that new examples can be formed is that very little is known about this set. It was found by a computer search and little or no information exists about either its construction or structure. The purpose of this section is to answer the question, is there a structure to the set given by Promislow? Idealistically, we would like a generalizable pattern on the set, or, at the very least, some definable structure. To get some intuition about the construction of Promislow's set we will look at it inside the Fibonacci group,

$$F(2,6) = \langle x_0, x_1, x_2, x_3, x_4, x_5 \mid x_0x_1 = x_2, x_1x_2 = x_3,$$

$$x_2x_3 = x_4, x_3x_4 = x_5, x_4x_5 = x_0, x_5x_0 = x_1\rangle.$$

So, the first thing to do is to show that G and $F(2,6)$ are isomorphic.

Theorem 16. *The groups G and $F(2,6)$ are isomorphic.*

Proof
We define $F(2,6) =$
$\langle x_0,\, x_1,\, x_2,\, x_3,\, x_4,\, x_5 \mid x_0x_1 = x_2,\, x_1x_2 = x_3,\, x_2x_3 = x_4,\, x_3x_4 = x_5,\, x_4x_5 = x_0,\, x_5x_0 = x_1\rangle$.
By Tietze transformations:

$$
\begin{gathered}
\langle x_0,\, x_1,\, x_2,\, x_3,\, x_4,\, x_5 \mid x_0x_1 = x_2,\, x_1x_2 = x_3,\, x_2x_3 = x_4,\, x_3x_4 = x_5,\, x_4x_5 = x_0,\, x_5x_0 = x_1\rangle = \\
\langle x_0,\, x_1,\, x_2,\, x_3,\, x_4,\, x_5 \mid x_0x_1 = x_2,\, x_1x_2 = x_3,\, x_2x_3 = x_4,\, x_3x_4 = x_5,\, x_4x_5 = x_0,\, x_5x_0 = x_1, \\
x_4x_5x_1 = x_2,\, x_5x_4x_5 = x_1\rangle = \\
\langle x_0,\, x_1,\, x_2,\, x_3,\, x_4,\, x_5 \mid x_1x_2 = x_3,\, x_2x_3 = x_4,\, x_3x_4 = x_5,\, x_4x_5 = x_0,\, x_4x_5x_1 = x_2, \\
x_5x_4x_5 = x_1\rangle = \\
\langle x_1,\, x_2,\, x_3,\, x_4,\, x_5 \mid x_1x_2 = x_3,\, x_2x_3 = x_4,\, x_3x_4 = x_5,\, x_4x_5x_1 = x_2,\, x_5x_4x_5 = x_1\rangle = \\
\langle x_1,\, x_2,\, x_3,\, x_4,\, x_5 \mid x_1x_2 = x_3,\, x_2x_3 = x_4,\, x_3x_4 = x_5,\, x_4x_5x_1 = x_2,\, x_5x_4x_5 = x_1, \\
x_4x_5x_3x_2^{-1} = x_2,\, x_5x_4x_5 = x_3x_2^{-1}\rangle = \\
\langle x_1,\, x_2,\, x_3,\, x_4,\, x_5 \mid x_1x_2 = x_3,\, x_2x_3 = x_4,\, x_3x_4 = x_5,\, x_4x_5x_3x_2^{-1} = x_2, x_5x_4x_5 = x_3x_2^{-1}\rangle = \\
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\langle x_3,\, x_4,\, x_5 \mid x_3x_4 = x_5,\, x_4x_5x_3x_3x_4^{-1} = x_4x_3^{-1},\, x_5x_4x_5 = x_3x_3x_4^{-1}\rangle = \\
\langle x_3,\, x_4,\, x_5 \mid x_3x_4 = x_5,\, x_4x_3x_4x_3x_3x_4^{-1} = x_4x_3^{-1},\, x_3x_4x_4x_3x_4 = x_3x_3x_4^{-1}\rangle = \\
\langle x_3,\, x_4 \mid x_4x_3x_4x_3x_3x_4^{-1} = x_4x_3^{-1},\, x_3x_4x_4x_3x_4 = x_3x_3x_4^{-1}\rangle = \\
\langle x_3,\, x_4 \mid x_3^2x_4x_3^2x_4^{-1} = 1,\, x_3^{-1}x_4^2x_3x_4^2 = 1\rangle = G.
\end{gathered}
$$

So, by Tietze's theorem, it follows then that $G \cong F(2,\, 6)$.□
There are two advantages to viewing Promislow's set A inside of $F(2,\, 6)$. The first is that in the Cayley graph $\Gamma(F(2,\, 6))$, A is connected. The second advantage is the set in $F(2,\, 6)$ is even parameterizable, allowing us to construct new sets based on the structure of the group.

We start by setting $x_3 = (1, 0^*, 0^*)$ and $x_4 = (0^*, 1, 1^*)$. Using these two elements, we make the following identifications:

$$
\begin{gathered}
x_1 = (2^*, -1, 1^*), \\
x_2 = (1^*, 1^*, 1), \\
x_3 = (1, 0^*, 0^*), \\
x_4 = (0^*, 1, 1^*), \\
x_5 = (1^*, -1^*, -1), \\
x_0 = (-1, 0^*, 2^*)
\end{gathered}
$$

and one checks that these satisfy the Fibonacci relations. So, with a little creativity (there are many ways to write such a set in $F(2,\, 6)$) we can rewrite $A \cup \{1\}$ as:

$$
\begin{gathered}
\{1,\, x_0,\, x_0x_3,\, x_0x_2,\, x_4,\, x_4x_2,\, x_4x_2x_2,\, x_4x_2x_2x_2,\, x_4x_2x_2x_2x_5, \\
x_4x_2x_2x_2x_5x_2,\, x_3,x_3x_5,\, x_3x_5x_2,\, x_3x_0,\, x_3x_0x_3,\, x_3x_2\}.
\end{gathered}
$$

The easiest way to show the set is connected is by demonstrating it directly as a flat map. We can write the set out as indicated in the figure below. The circled vertices with denote the identity vertex and additional edges are added to show the relationship between the elements in the map. Later we will see that this structure is at least partially generalizable. Such a map shows that we can define a structure on Promislow's set.

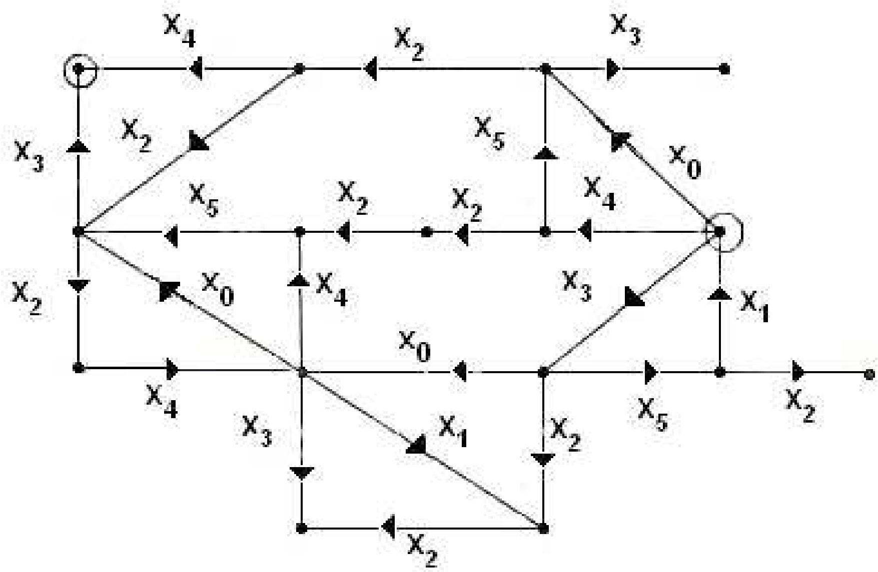

Figure 4.1: Map Showing the Relationship Between the Elements in A

Although it a bit difficult to imagine, there is an advantage to viewing the set as a connected graph. Namely, we can view the multiplication geometrically in the following way. First we construct a spanning tree from the above diagram like the one shown below. At each vertex we glue an identical model of the spanning tree. Next, we use the relations to glue the respective vertices. After the gluings the product graph of $(A \cup \{1\}, A \cup \{1\})$ will have the property that each vertex is contained in at least one closed loop. This gives us in at least some small sense an idea of what happens geometrically in such sets.

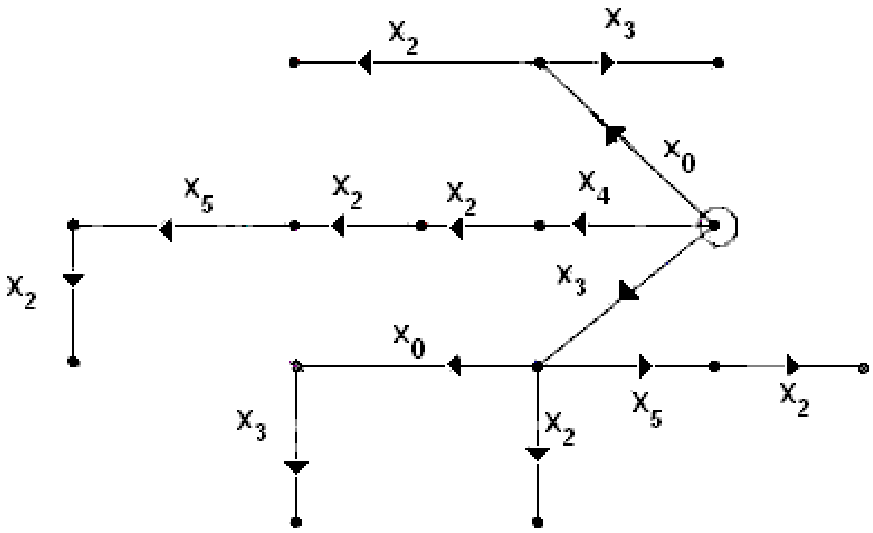

Figure 4.2: A Spanning Tree for Promislow's Set

4.3 Group as Described by Combinatorial Methods

One thing that we would like to do is to mimic the construction used by Rips and Segev. The main reason we want to do this is to be able to generate new examples of non-unique product groups and sets in known n.u.p. groups in a practical way. We begin with some preliminary ideas of how to construct a 17 element square set in Promislow's example that contains elements from all four of the sets E_0, E_1, E_2, and E_3, mentioned earlier.

As in the Rips-Segev example we will define our set by paths in the Cayley graph $\Gamma(G)$ (note that Γ in Chapter 3 does not refer to the Cayley graph). However, we need to be careful in choosing our base points, unlike the Rips-Segev example, we do not have the luxury of defining a series of relations to make our matchings work on the external vertices, as this would change the definition of our group. The up-shot to this is that we already know the group, as defined, is torsion free.

We begin by choosing pieces $x_1, \ldots, x_n$ from the symmetrized set of relators on

$$G = \langle b, c \mid b^{-1}c^2b = c^{-2}, c^{-1}b^2c = b^{-2} \rangle$$

and we construct sets S_i that are paths $S_i = \{x_i, x_ic, x_icc, \ldots, x_ic^{m_i}\}$ in $\Gamma(G)$. To show one of the criteria for choosing such a set $X = \{x_1, \ldots, x_n\}$ we make the following observation, for some i, j, k, l, r, s, t, u we need the following to hold

$$x_i c^r x_j c^s = x_k c^t x_l c^u \Rightarrow$$

$$x_k^{-1} x_i c^r x_j = c^t x_l c^{u-s} \Rightarrow$$

$$x_k^{-1} x_i = c^t x_l c^{u-s} x_j^{-1} c^{-r}$$

if we choose $k = i$, then $\Rightarrow$

$$1 = x_l c^{u-s} x_j^{-1} c^{t-r} \Rightarrow$$

$$x_j = c^n x_l c^m,$$

where $n = t - r$ and $m = s - u$. Thus, we can compute an additional piece x_j given a choice of x_i. We will refer to x_j and x_i as complement pairs. Using the complement pairs we can also get a reasonable idea of the length of the paths by using $r, s, t,$ and u as parameters for m and n with the restriction that r, s, t, $u \geq 0$. It is worth noting that our complement pairs need not be uniquely determined, in the sense that one is free to choose any complement x_j satisfying the above observation given a choice of x_i.

Before we construct such a set there are a few things to keep in mind. First, each complement pair need not be distinct in itself; i.e. self-pairs can exist. Second, by right orderability, any set of choices must be contained in at least two cosets. Third, by construction, most of the matching will occur in the sets of the form $S_i(S_j \cup S_k)$ where S_j and S_k are the sets formed by the complement pairs, but by right orderability, not every element of can match within the set itself. Care should be taken in the initial choice of x_i and length of the paths to ensure proper matching takes place.

With these remarks in mind, we will construct a new example of a non-unique product set in G. We start by choosing the pieces b^{-1}, cbc, and c^{-2}. Next we choose the complement pairs, cb^{-1}, bc, and c^{-2} respectively. We then construct the sets $S_1 = \{b^{-1}, b^{-1}c\}$, $S_2 = \{cb^{-1}, cb^{-1}c, cb^{-1}cc, cb^{-1}ccc\}$, $S_3 = \{c^{-2}, c^{-1}, 1, c, c^2\}$,$S_4 = \{cbc, cbcc, cbccc\}$ and $S_5 = \{bc, bcc, bccc\}$. Our claim is that (S, S) has no unique elements, where $S = \bigcup_{i=1}^{5} S_i$.

We demonstrate this by the multiplication table in the following figure. For convenience, we label $B = b^{-1}$ and $C = c^{-1}$. The corresponding colored sets indicate complementary pairs and the white spaces indicate elements that do not match within each pair. As noted earlier, our choice of base point elements and path lengths allow us to match the words in white spaces up with other elements in the set. This construction also answers the question are the only n.u.p. sets in Promislow's example 14 element sets? Later we will show that this set can be used to generate infinitely many distinct square n.u.p. 17 element sets inside of the group that are not translations or inverse mappings.

	B	Bc	cB	cBc	cBcc	cBccc	CC	C	1	c	cc	bc	bcc	bccc	cbc	cbcc	cbccc
B	BB	BBc	BcB	BcBc	BCB	BCBc	BCC	BC	B	Bc	Bcc	c	cc	ccc	Bcbc	BCb	BCbc
Bc	BcB	BcBc	BBCC	BBC	BB	BBc	BC	B	Bc	Bcc	Bccc	Bcbc	BCb	BCbc	C	1	c
cB	bbc	bbcc	cBcB	cBcBc	cBCB	cBCBc	cccB	cBC	cB	cBc	CB	cc	ccc	cccc	cBcbc	cBCb	cBCbc
cBc	cBcB	cBcBc	bbC	bb	bbc	bbcc	cBC	cB	cBc	CB	CBc	cBcbc	cBCb	cBCbc	1	c	cc
cBcc	bbC	bb	CBcB	CBCBC	cBcB	cbcbc	cB	cBc	cBcc	CBc	CBcc	1	c	cc	cBcbC	cBcb	cBcbc
cBccc	CBcB	CBCBC	bbCCC	bbCC	bbC	bb	cBc	CB	cBccc	CBcc	CBccc	cBcbC	cBcb	cBcbc	CC	C	1
CC	Bcc	Bccc	CB	CBc	CBcc	CBccc	CCCC	CCC	CC	C	1	bccc	bcccc	bccccc	Cbc	Cbcc	Cbccc
C	CB	CBc	B	Bc	Bcc	Bccc	CCC	CC	C	1	c	Cbc	Cbcc	Cbccc	bc	bcc	bccc
1	B	Bc	cB	cBc	cBcc	cBccc	CC	C	1	c	cc	bc	bcc	bccc	cbc	cbcc	cbccc
c	cB	cBc	BCC	BC	B	Bc	C	1	c	cc	ccc	cbc	Cb	Cbc	bC	b	bc
cc	BCC	BC	cccB	cBC	cB	cBc	1	c	cc	ccc	cccc	bC	b	bc	cbC	cb	cbc
bc	bcB	bcBc	CC	C	1	c	bC	b	bc	bcc	bccc	bcbc	BCB	BCBc	bbC	bb	bbc
bcc	CC	C	bcccB	bcBC	bcB	bcBc	b	bc	bcc	bccc	bcccc	bbC	bb	bbc	bcbC	bcb	bcbc
bccc	bcccB	bcBC	CCCC	CCC	CC	C	bc	bcc	bccc	bcccc	bccccc	bcbC	bcb	bcbc	bbCCC	bbCC	bbC
cbc	cbcB	cbcBc	C	1	c	cc	cbC	cb	cbc	Cb	Cbc	cBcBc	cBCB	cBCBc	BB	BBc	BBcc
cbcc	C	1	CbcB	CbCBC	cbcB	cbcBc	cb	cbc	cbcc	Cbc	Cbcc	BB	BBc	BBcc	cbcbC	cbcb	cbcbc
cbccc	CbcB	CbCBC	CCC	CC	C	1	cbc	cbcc	cbccc	Cbcc	Cbccc	cbcbC	cbcb	cbcbc	BBCC	BBC	BB

Figure 4.3: Multiplication Table of the Constructed 17 Element Set

Chapter 5

Generalizations of Promislow's Example

5.1 Inductively Defined Groups That Contain Promislow's Group

Using the construction from section 4.3, we can extend the results to show that there exist other examples of 17 element non-unique product sets in both Promislow's group and other similarly defined groups. For this we use that fact that matchings in our set S only rely on the defining relations. We can use similarly defined relations of the form $R(k,l) = \{a^k b^{2l} a^{-k} b^{2l} = 1, b^l a^{2k} b^{-l} a^{2k} = 1\}$ where $k, l \in \mathbb{N}$ to show other non-unique product sets exist. Define $S(k,l)$ to be the image of the map $\theta : S \to F(a,b)$ sending $a \mapsto a^k$ and $b \mapsto b^l$.

The first question to ask is if we momentarily ignore the criterion of torsion free does $S(k,l)$ define a n.u.p. set in the group $G(k,l) = \langle a, b \mid R(k,l) \rangle$? Surprisingly, this is easy to see; i.e., if we replace a with a^k and b with b^l in Figure 4.3, then under the relations $R(k,l)$ the structure of the table does not change. Moreover, we can demonstrate that if k and l are odd, then $S(k,l)$ is another 17 element square subset without the unique product property in G that is not a translation or inversion. To show this we need only verify that $R(k,l)$ are consequences of of the relations, $ab^2a^{-1}b^2 = ba^2b^{-1}a^2 = 1$ in G. In G we have,

$$ab^2a^{-1}b^2 = 1 \Rightarrow$$

$$ab^2a^{-1} = b^{-2} \Rightarrow$$

$$ab^{2l}a^{-1} = b^{-2l} \Rightarrow$$

$$ab^{2l}a^{-1} = a^{1-k}a^{k-1}b^{-2l} \Rightarrow$$

$$ab^{2l}a^{-1} = a^{1-k}b^{-2l}a^{k-1} \Rightarrow$$

$$a^{k-1}ab^{2l}a^{-1}a^{1-k} = b^{-2l} \Rightarrow$$
$$a^kb^{2l}a^{-k} = b^{-2l} \Rightarrow$$
$$a^kb^{2l}a^{-k}b^{2l} = 1.$$

Similarly, $b^la^{2k}b^{-l}a^{2k} = 1$ in G. Therefore, $S(k,l)$ is also a n.u.p. set in G. Since square elements commute inside of G when k and l are even, right orderability implies that, $S(k,l)$ does not embed in G as a n.u.p. set. Unfortunately, these groups do not generate interesting new examples of non-unique product groups since the maps sending $a \mapsto a^k$ and $b \mapsto b^l$ also embed a copy of Promislow's group in each $G(l,k)$. One could explore several other similarly defined groups. However, we will content ourselves showing infinitely many distinct square n.u.p. sets exist inside of G.

5.2 Inductively defined Subsets of the Fibonacci Group

We end this chapter by noting that we can make a smaller generalization of Promislow's example. The map in Figure 4.1 shows a structure exists for Promislow's example. What's interesting is that the cyclically defined relators in $F(2,6)$ imply that such relationships also exists in a map if we replace x_0, x_1, x_2, x_3, x_4, and x_5 by x_i, x_i+1, x_i+2, x_i+3, x_i+4, and x_i+5 respectively, where $i \in \{0,\ 1,\ 2,\ 3, 4,\ 5\}$ and the subscripts are taken modulo 6. Furthermore, since the structure of the map determines the matchings, it is possible to show that these are also n.u.p. sets. Whence, we can view this structure as a generalized map for matchings in the Promislow-like examples.

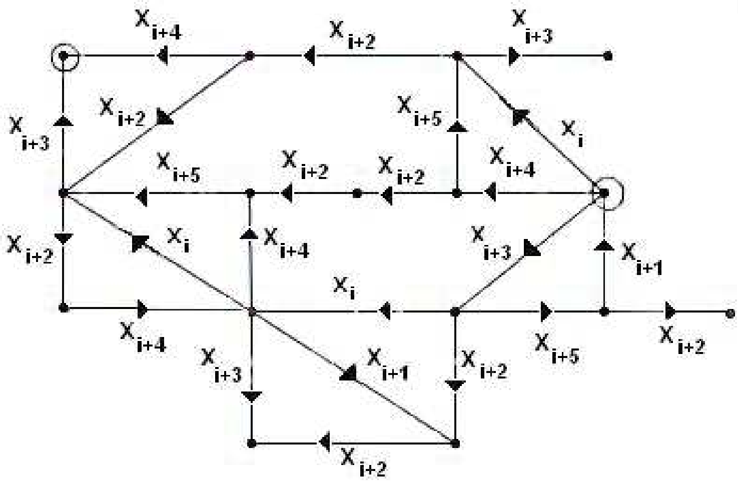

Figure 5.1: Generalized Promislow-Like Map

Chapter 6

Preliminary Results in Showing Promislow's Group Satisfies the Nontrivial Units Conjecture

Although this chapter differs in both technique and flavor from the earlier chapters, it seems only fitting to include these preliminary results showing the group ring, $K[G]$, satisfies the nontrivial units conjecture. In particular, we show that there are no nontrivial units up to support size three.

Definition 17. *Let $K[G]$ be a group ring over a field K and G a multiplicative group, then $K[G]$ is an associative K-algebra with basis $g \in G$. That is, $K[G]$ consists of all finite formal sums*

$$\alpha = \sum_{x \in G} a_x x$$

that satisfy the following operations for addition and multiplication.
If

$$\alpha = \sum_{x \in G} a_x x, \beta = \sum_{x \in G} b_x x \in K[G],$$

then

$$\alpha + \beta = \sum_{x \in G} a_x x + \sum_{x \in G} b_x x = \sum_{x \in G} (a_x + b_x) x,$$

$$\alpha\beta = (\sum_{x \in G} a_x x)(\sum_{x \in G} b_y y) = \sum_{x,y \in G} a_x b_y xy.$$

Furthermore, we can define scalar multiplication in the following way, if $a \in K$ and $\alpha \in K[G]$, then

$$a\alpha = a \sum_{x \in G} a_x x = \sum_{x \in G} (a a_x) x.$$

Definition 18. *If*

$$\alpha = \sum_{x \in G} a_x x \in K[G],$$

then we define the support of α*, denoted supp* α*, to be* $\{x \in G \mid a_x \neq 0\}$*. Also, if* H *is a subgroup of* G*, then we define a right transversal (similarly a left transversal) to be the set of right coset representatives for* H *in* G*.*

Definition 19. *For a group ring* $K[G], \alpha \in K[G]$ *is said to be nontrivial provided* $|supp\ \alpha| > 1$*.*

Definition 20. *Let* G *be a group with subgroup* H *and let* X *be a right transversal for* H *in* G*. We define the natural projection map* $\pi_H : K[G] \to K[H]$ *to be*

$$\pi_H(\sum_{x \in G} a_x x) = \sum_{x \in H} a_x x.$$

6.1 Consequences of the Group Structure

In looking at the consequences of the group structure, it is convenient to represent the group as Promislow did in [5].

Theorem 21. *Let* $\alpha \in K[G]$ *such that supp* α *is contained exclusively in one of the sets* E_0*,* E_1*,* E_2*, or* E_3 *and* α *is a nontrivial element, then* α *can not have an inverse contained in exactly one of the sets* E_0*,* E_1*,* E_2*, or* E_3*.*

Proof
The result is obvious for supp $\alpha \subset E_0$. Therefore, we need only consider E_1, E_2, and E_3. Let $\alpha \in K[G]$ such that $|\text{supp } \alpha| > 1$ and supp $\alpha \subseteq E_l$ where $l \in \{1, 2, 3\}$. Suppose that α has an inverse β contained in exactly one of the sets E_0, E_1, E_2, or E_3. Suppose first that β is contained in the same set, say, E_m. However, E_m is a coset of E_0 . So, we can write α as $\gamma 1x$ where supp $\gamma \subseteq E_0$ and $x \in E_m$. Similarly, we can write β as $1y\delta$ where supp $\delta \subseteq E_0$ and $y \in E_m$. Thus,

$$1 = \alpha\beta = (\gamma 1x)(1y\delta) = \gamma(1x1y)\delta = \gamma(1xy)\delta.$$

However, $x, y \in E_m$ implies that xy $\in E_0$. It follows then that E_0 is an RO-group with a nontrivial unit which is a contradiction. Thus, $\beta \in E_n$, where $n \neq m$, but this is obviously false since $\alpha\beta \in E_m E_n$ and $E_m E_n \neq E_0$ since $G/E_0 \cong V_4$ implies every factor group element has order 2. Thus, $E_m E_n = E_0$ says $m = n$ which is also a contradiction. The result follows.
□

Theorem 22. *If supp* $\alpha \subseteq E_m$ *where* $m \in \{0, 1, 2, 3\}$*, then* α *is not invertible in* $K[G]$*.*

Proof
Let $\alpha \in K[G]$ for some field K with supp α contained exclusively in one of the cosets E_0, E_1, E_2, or E_3 and α nontrivial. Suppose that α were invertible. Then from Theorem 21, we know that $\alpha^{-1} = \beta$ must have its support contained in at least 2 of the above cosets. So, we can write β as $\beta = \beta_0 + \beta_1 + \beta_2 + \beta_3$ where supp $\beta_m \subseteq E_m$ for m = 0,1,2,3. Note that some of these supports may be empty. Thus,

$$\alpha\beta = \alpha(\beta_0 + \beta_1 + \beta_2 + \beta_3) = \alpha\beta_0 + \alpha\beta_1 + \alpha\beta_2 + \alpha\beta_3.$$

Now consider any β_l with $|$supp β_l $| \geq 1$. Then we know that $\alpha\beta_l \neq 1x$ where x $\in G$. This places the restriction that $|$supp $\alpha\beta_l$ $| \geq 2$. However, each $\alpha\beta_l$ lies in a distinct coset and so each set supp $\alpha\beta_l$ must be pair wise disjoint for each l. Thus, since at most 2 of the supports may be empty this says

$$1 = |\text{supp } \alpha(\beta_0 + \beta_1 + \beta_2 + \beta_3)| \geq 4 > 1.$$

So, $1 > 1$. Contradiction.$\square$

Theorem 23. *If $\alpha \in K[E_0 \cup xE_0]$ and α is nontrivial, then α is not invertible in $K[G]$ for all $x \in G$.*

Proof
Once again if $x \in E_0$, then there is nothing to prove. Thus, we may assume that $x \notin E_0$. Let $\alpha \in K[E_0 \cup xE_0]$ be nontrivial. Then we may write α as $\alpha = \alpha_0 + \alpha_x$, where supp $\alpha_0 \subseteq E_0$ and supp $\alpha_x \subseteq xE_0$. Suppose that α is invertible with inverse $\beta = \beta_0 + \beta_x + \beta_y + \beta_z$ where supp $\beta_0 \subseteq E_0$, supp $\beta_x \subseteq xE_0$, supp $\beta_y \subseteq yE_0$, and supp $\beta_z \subseteq zE_0$. That is, we may write the inverse as individual sums from the 4 distinct cosets E_0, xE_0, yE_0, zE_0. Thus, $1_{K[G]} = \alpha\beta = (\alpha_0+\alpha_x)(\beta_0+\beta_x+\beta_y+\beta_z) = (\alpha_0\beta_0+\alpha_x\beta_x)+(\alpha_0\beta_x+\alpha_x\beta_0)+(\alpha_0\beta_y+\alpha_x\beta_z)+(\alpha_0\beta_z+\alpha_x\beta_y)$, by writing the product as the sum of the elements from each individual coset. Now if $\alpha\beta = 1_{K[G]}$, then this implies supp $((\alpha_0\beta_y+\alpha_x\beta_z)+(\alpha_0\beta_z+\alpha_x\beta_y)) = 0$, otherwise $\alpha\beta = 1_{K[G]}$ is impossible. However, this says that $\alpha(\beta_y + \beta_z) = 0$. It follows then that $(\beta_y + \beta_z) = 0$, since, $K[G]$ has no zero divisors. However, if $(\beta_y + \beta_z) = 0$ this implies $\beta \in K[E_0 \cup xE_0]$. Thus, $K[E_0 \cup xE_0]$ has a nontrivial unit, contradicting the fact that $E_0 \cup xE_0$ is a RO-group.$\square$
This places the restriction that if α is contained in exactly any two cosets, then α is not invertible. Otherwise, we could generate a nontrivial unit β with supp $\beta \subseteq E_0 \cup xE_0$ for some $x \in G$. Since G is itself not a RO-group, we have exhausted every possible case using right orderability. Therefore, we will need to develop a new plan to examine the general element in $K[G]$ of support size 3.

6.2 Embedding $K[G]$ in a Matrix Ring and the Determinant Criterion for a Unit

As an introduction, we can make an observation that if K is a field of 2 elements, then $K[G]$ has no invertible elements of support size 3.

Theorem 24. *Let K be the field of two elements. Then $K[G]$ has no invertible elements of support size 3.*

Proof
Let K be the field with 2 elements. Suppose that there is an invertible element in $K[G]$ of support size 3. Write $\alpha + x$ as that element, with $\alpha \in K[H]$, with supp $\alpha \subset H$, a subgroup of index 2, and $x \notin H$. Since $\alpha + x$ is invertible we write its inverse as $\beta + \gamma x$ with supp $\beta \in H$ possibly zero and supp $\gamma x \notin H$. Then, $(\alpha + x)(\beta + \gamma x) = 1$. Hence $\alpha\beta + \alpha\gamma x + x\gamma x + x\beta = 1$. So, $\alpha\beta + x\gamma x = 1$ and $x\beta + \alpha\gamma x = 0$. Which says that $\alpha x^{-1}\alpha\gamma x + x\gamma x = 1$. Thus, $(\alpha x^{-1}\alpha x^{-1} + 1)(x\gamma x) = 1$ is the product of elements in $K[H]$. The first inclusion is obvious, but then again, so is the second since H has index 2. It follows then that $x\gamma x$ must be a trivial unit, by previous results. Hence, |supp γ|=1. Also, $x\beta + \alpha\gamma x = 0$ implies that |supp β | = 2. It follows then that an inverse must have support size exactly 3.

We now prove that if $\breve{\alpha}$ and $\breve{\beta}$ both have support size 3, then $\breve{\alpha}\breve{\beta} \neq 1$. Without loss of generality, we may write $\breve{\alpha} = p + x$ and $\breve{\beta} = q + y$ where $p, q \in H$ and $x, y \notin H$. Suppose to the contrary that $\breve{\alpha}\breve{\beta} = 1$. Then we must have $pq = 1 + xy$. If $xy = 1$, then $pq = 0$ and we are finished so we may assume that $xy \neq 1$. Write $p = a + b$ and $q = c + d$. Then without loss of generality, we may assume that $ac = 1$. So, by multiplying on the left by c and the right by a, we may assume that $a = c = 1$, in other words we may write $p = 1 + b$ and $q = 1 + d$. If $b \neq d$, then either the order of the support of pq is 4, or $bd = 1$ in which case, 1 is not in the support of pq. Since neither of these cases are possible, we must have $b = d$. We now have $(1 + b + x)(1 + b + y) = 1$. Therefore, $(1 + b)y = x(1 + b)$, and $b^2 = xy$. So, $y + by = x + xb$. If $x = y$, then $1 = (1 + b + x)^2$. So, expanding the product, we have $0 = b^2 + bx + xb + x^2$, whence $b(x + b) = x(x + b)$. But this says that $x = b$, contrary to our choice of x. Therefore, $x \neq y$, so $by = x$ and $y = xb$. Then $xy^{-1} = b = x^{-1}y$. We can deduce that $xy = byxb$, whence $b^2 = byxb$. However, this says that $yx = 1$, contrary to our choice of x and y.□
This handles the case with a field of two elements. As we will see, it turns out that the characteristic of the field does not make a difference. So, as this section suggests, the first step is to show that we can embed $K[G]$ in a matrix ring. To show this we need a Lemma. This may be found in [4].

Lemma 25. *Let G be a group with a subgroup H and let Y be a left transversal for H in G.*

Then every $\alpha \in K[G]$ can be written uniquely as a finite sum in the form

$$\alpha = \sum_{y \in Y} y\alpha_y$$

with $\alpha_y \in K[H]$. In fact, $\alpha_y = \pi_H(y^{-1}\alpha)$. Thus, $K[G]$ is a right $K[H]$-module with Y as a free basis.

Proof
Let $\alpha \in K[G]$. Because supp α is finite, it is contained in only finitely many left cosets, say of H, say, $y_1H, y_2H, \ldots, y_nH$ with $y_i \in Y$. We can write

$$\alpha = \alpha_1 + \alpha_2 + \ldots + \alpha_n,$$

where each α_i is the partial sum of those $\alpha_x x$ with $y \in y_iH$. Now observe that $x \in y_iH$ implies $y_i^{-1}x \in H$ so that the expression $\alpha = \alpha_1 + \alpha_2 + \ldots + \alpha_n$ becomes

$$\alpha = \sum_1^n y_i(y_i^{-1}\alpha_i)$$

we see that $y_i^{-1}\alpha_i \in K[H]$. Thus, α can be written as a sum of the form $\sum y\alpha_y$ with $\alpha_x \in K[H]$. To show uniqueness, let $\alpha = \sum \alpha_y y$, and let $y_0 \in Y$. Then $y_0^{-1}\alpha = \sum y_0^{-1}y\alpha_y$, and certainly $y_0^{-1}y \in H$ if and only if $y = y_0$. Thus, clearly,

$$\pi_H(y_0^{-1}\alpha) = y_0^{-1}y_0\alpha_{y_0} = \alpha_{y_0}$$

and the result follows.□
The following theorem may also be found in [4]

Theorem 26. *Let H be a normal subgroup of G of finite index n, then $K[G] \subset M_n(K[H])$. More specifically if $\alpha = \sum_{g \in X} \bar{x}a_x$, then we may write the (h,g)-entry of the matrix α is $f(hg^{-1}, g)\breve{g}^{-1}(a_{hg^{-1}})$.*

Proof
Let H be a normal subgroup of a group G having finite index n. Let X be a left transversal for H in G and as such it follows from Lemma 25 that $K[G]$ is a right $K[H]$-module with basis X. Furthermore, $K[G]$ is also a left $K[H]$-module. Because right and left multiplications commute as operators of $K[G]$, this makes $K[G]$ into a set of $K[H]$-linear transformations on an n-dimensional free $K[H]$-module $K[G]$. Thus, $K[G] \subset M_n(K[H])$. Also by Lemma 25, we have the formula

$$\alpha\bar{g} = \sum_h \pi_H(\bar{h}\bar{h}^{-1}\alpha\bar{g}) = \sum_h \bar{h}\pi_H(\bar{h}^{-1}\alpha\bar{g}),$$

where $\pi_H(\bar{h}^{-1}\alpha\bar{g})$ is the (h,g)-entry of our matrix. So, if $\alpha = \sum_{x\in G}\bar{x}a_x$, then the (h,g) entry becomes $f(hg^{-1},g)\check{g}^{-1}(a_{hg^{-1}})$ as desired.□
So, as an embedding we know that there is some injective map

$$\theta : K[G] \to M_{|X|}(K[H]).$$

Thus, we will use $\theta(\alpha)$to denote the matrix representation for α. For any $g \in G$ we write $M(g)$ monomial matrix given by $M(g)(v_h) = f(hg, g^{-1})v_{hg}$. As an embedding we know that obviously, Im$\theta \neq M_{|X|}(K[H])$, particularly this is true since no matrix of Imθ has determinant zero. So, the next question to ask is what matrices make up Imθ?

The next theorem, proof and corollary were provided by Dan Farkas, using the same notation in the proof above. In relation to the structure of G here we will assume that our group G is a crystallographic group and that our subgroup H is the translation group.

Theorem 27. *Let $C \in M_{|X|}(K[H])$, the ring of matrices $|X| \times |X|$ with entries in $K[H]$, then $C \in Im\theta$ if and only if $\check{w}(C) = M(w)CM(w)^{-1}$ for all $w \in X$.*

Proof
Let $C \in M_{|X|}(K[H])$. We begin by computing the (h,g)-entry of $M(w)CM(w)^{-1}$ in terms of $c_{h,g}$, the (h,g)-entry of C. First, from the above notation for the monomial matrices, $v_u = f(uw, w^{-1})^{-1}M(w)^{-1}v_{uw}$. So, $v_{tw^{-1}} = f(t,w)^{-1}M(w)^{-1}(v_t)$.
Thus,$M(w)^{-1}(v_t) = f(t,w^{-1})(v_{tw^{-1}})$. Now,

$$M(w)CM(w)^{-1}(v_g) = M(w)C(f(g,w^{-1})v_{gw^{-1}})$$

$$= f(g,w^{-1})M(w)(\sum_h c_{h,gw^{-1}}v_h) = f(g,w^{-1})\sum_h c_{h,gw^{-1}}M(w)(v_h)$$

$$= f(g,w^{-1})\sum_h c_{h,gw^{-1}}f(hw,w^{-1})^{-1}v_{hw}.$$

Thus,the (h,g)-entry of $M(w)CM(w)^{-1}$ is $f(g,w^{-1})f(h,w^{-1})^{-1}c_{hw^{-1},gw^{-1}}$. It follows then that the equational condition is equivalent to

$$\check{w}(c_{h,g}) = f(g,w^{-1})f(h,w^{-1})^{-1}c_{hw^{-1},gw^{-1}}.$$

Consider $C = \theta(m)$ where $m = \sum_x \bar{x}a_x \in K[G]$. Its (hw^{-1}, gw^{-1})-entry is $f(hg^{-1}, gw^{-1})\check{wg}^{-1}(a_{hg^{-1}})$.
Thus,the right hand side of

$$\check{w}(c_{h,g}) = f(g,w^{-1})f(h,w^{-1})^{-1}c_{hw^{-1},gw^{-1}}$$

becomes

$$f(g,w^{-1})f(h,w^{-1})^{-1}f(hg^{-1},gw^{-1})\check{w}(g\check{(}-1)(a_{hg^{-1}})$$
$$= f(g,w^{-1})f(h,w^{-1})^{-1}f(hg^{-1},gw^{-1})\check{w}(f(hg^{-1},g)^{-1}c_{h,g})$$

$$= f(h, w^{-1})^{-1}((\check{w^{-1}})^{-1})f(hg^{-1}, g)^{-1}\dot{f}(g, w^{-1})f(hg^{-1}, gw^{-1})\check{w}(c_{h,g}).$$

If we apply the cocycle identities with $x = hg^{-1}$, $y = g$, $z = w^{-1}$ we see that this becomes $\check{w}(c_{h,g})$, as desired. Conversely, we must show that if C satisfies

$$\check{w}(c_{h,g}) = f(g, w^{-1})f(h, w^{-1})^{-1}c_{hw^{-1},gw^{-1}},$$

then it is the image under θ of some member of the group algebra. We examine
$\check{g}(f(x, g)^{-1}f(g, g^{-1})f(xg, g^{-1})^{-1}c_{x,1} = f(x, 1)^{-1}f(g, g^{-1})^{-1}f(g, g^{-1})c_{x,1}$
$= f(x, 1)^{-1}c_{x,1}$.
In other words, $\check{g}(f(x, g)^{-1}c_{xg,g}$ is independent of $g \in G$. It is now a straight forward calculation to see that $\theta(\sum \bar{x}f(x, 1)^{-1}c_{x,1}) = C$.□
As a corollary to this we have.

Corollary 28. *The element $\alpha \in K[G]$ is a unit if and only if det($\theta(\alpha)$) $\in K^\times \cdot H$.*

Proof Clearly, if α is a unit, then $\theta(\alpha)$ is a unit, and so $\det(\theta(\alpha))$ is a unit. Conversely, suppose that $\theta(\alpha)$ is a unit. Let D be its inverse matrix in $M_{|X|}(K[H])$. For all $x \in X$, we have $\check{x}(\theta(\alpha)) = M(x)\theta(\alpha)M(x)^{-1}$. It follows that $\check{x}(D) = M(x)DM(x)^{-1}$. Applying the theorem a second time we see that $D = \theta(\beta)$ for some $\beta \in K[G]$. Hence $\alpha\beta = \beta\alpha = 1$.□
Let X denote transversal for E_0 in G and let K be a field. By these results we see that we can embed $K[G]$ in $M_{|X|}(K[E_0])$. Now we want to make use of the determinant criteria for units to look at the general element support size 3. In this case we see that the determinant maps invertible elements into $K^\times$.

6.3 The General Element of $K[G]$ of support size 3

We begin by choosing our transversal. We choose $X = \{(0, 0, 0), (0^*, 1, 1^*), (1, 0^*, 0^*), (1^*, 1^*, 1)\}$ as our coset representatives. The next step is to defining a "multiplication table" for our cocycle condition. Set
1=(0,0,0), a=(0*,1 ,1*), b=(1, 0*, 0*), c=(1*, 1*, 1),
then using the relations
$\bar{u}\bar{v} = \bar{u}\bar{v}f(u, v)$, and $f(1, x) = f(x, 1) = 1$.
We generate the following table.

f	1	a	b	c
1	(0,0,0)	(0,0,0)	(0,0,0)	(0,0,0)
a	(0,0,0)	(0,2,0)	(2,0,0)	(-2,-2,0)
b	(0,0,0)	(0,2,-2)	(2,0,0)	(-2,-2,2)
c	(0,0,0)	(0,0,-2)	(0,0,0)	(0,0,2)

It is tedious, but straight forward to show that the above relations satisfy the cocycle condition

$$f(xy,z)\check{z}^{-1}(f(x,y)) = f(x,yz)f(y,z).$$

We now make an observation about the conjugation action of the elements of G on an arbitrary element of E_0. let $(a_1,a_2,a_3) \in E_0$ be arbitrary. Then:
obviously $\check{1}(a_1,a_2,a_3) = (a_1,a_2,a_3)$,
$\check{a}^{-1}(a_1,a_2,a_3) =$ (0*,-1 ,1*) (a_1,a_2,a_3)(0*,1 ,1*)$= (-a_1,a_2,-a_3)$,
$\check{b}^{-1}(a_1,a_2,a_3) =$ (-1, 0*, 0*) (a_1,a_2,a_3)(1, 0*, 0*)$= (a_1,-a_2,-a_3)$,
$\check{c}^{-1}(a_1,a_2,a_3) =$ (1*, 1*, -1) (a_1,a_2,a_3)(1*, 1*, 1)$= (-a_1,-a_2,a_3)$.
In addition, we can express every element in E_0 as a Laurent polynomial in three variables r,s,t; i.e. $(a_1,a_2,a_3) \rightsquigarrow r^{a_1}s^{a_2}t^{a_3}$. With these ideas in mind, we can represent each of the matrices in $K[G]$ that we want to consider.
For example, using the above mentioned generators for G in our definition, it can easily be shown that the matrices generated do in fact satisfy the relations for

$$G = \langle x,y \mid x^{-1}y^2x = y^{-2}, y^{-1}x^2y = x^{-2}\rangle.$$

Having obtained a matrix representation for $K[G]$, we want to show that there are no invertible elements of support size 3. Many of the cases have already been taken care of by the previous theorems. Hence, we need only consider those elements in $K[G]$ having support in exactly 3 distinct cosets of E_0. Writing them with respect to our chosen transversal, they look like:

$$k_u(u_1,u_2,u_3)\cdot\bar{1} + k_a(a_1,a_2,a_3)\cdot\bar{a} + k_b(b_1,b_2,b_3)\cdot\bar{b},$$
$$k_u(u_1,u_2,u_3)\cdot\bar{1} + k_a(a_1,a_2,a_3)\cdot\bar{a} + k_c(c_1,c_2,c_3)\cdot\bar{c},$$
$$k_u(u_1,u_2,u_3)\cdot\bar{1} + k_b(b_1,b_2,b_3)\cdot\bar{b} + k_c(c_1,c_2,c_3)\cdot\bar{c},$$
$$k_a(a_1,a_2,a_3)\cdot\bar{a} + k_b(b_1,b_2,b_3)\cdot\bar{b} + k_c(c_1,c_2,c_3)\cdot\bar{c},$$

with k_u, k_a, k_b, $k_c \in K$ and (u_1,u_2,u_3), (u_1,u_2,u_3), (b_1,b_2,b_3), $(c_1,c_2,c_3) \in E_0$.

We now examine the cases where K is any field using the arbitrary elements just described. For the first case,
$m_1 = k_u(u_1,u_2,u_3)\cdot\bar{1} + k_a(a_1,a_2,a_3)\cdot\bar{a} + k_b(b_1,b_2,b_3)\cdot\bar{b}$, we generate the matrix

$$\theta(m_1) = \begin{pmatrix} k_ur^{u_1}s^{u_2}t^{u_3} & k_ar^{-a_1}s^{2+a_2}t^{-a_3} & k_br^{2+b_1}s^{-b_2}t^{-b_3} & 0 \\ k_ar^{a_1}s^{a_2}t^{a_3} & k_ur^{-u_1}s^{u_2}t^{-u_3} & 0 & k_br^{-2-b_1}s^{-2-b_2}t^{2+b_3} \\ k_br^{b_1}s^{b_2}t^{b_3} & 0 & k_ur^{u_1}s^{-u_2}t^{-u_3} & k_ar^{-2-a_1}s^{-2-a_2}t^{a_3} \\ 0 & k_br^{-b_1}s^{2+b_2}t^{-2-b_3} & k_ar^{2+a_1}s^{-a_2}t^{-a_3} & k_ur^{-u_1}s^{-u_2}t^{u_3} \end{pmatrix}$$

with determinant

$$\det(\theta(m_1)) = k_a^4 + k_b^4 + k_u^4 - k_b^2k_u^2r^{2+2b_1-2u_1} - k_b^2k_u^2r^{-2-2b_1+2u_1} - k_a^2k_u^2s^{2+2a_2-2u_2}$$
$$-k_a^2k_u^2s^{-2-2a_2+2u_2} - k_a^2k_b^2t^{-2+2a_3-2b_3} - k_a^2k_b^2t^{2-2a_3+2b_3}.$$

Now, for $\det(\theta(m_1)) \in K \cdot E_0$, we need the exponents for r, s, and t to be zero. So, suppose that they are all zero. Then this says that we have (by equating the individual exponents for each r, s, and t respectively):

$$2 + 2b_1 - 2u_1 = -2 - 2b_1 + 2u_1,$$

$$2 + 2a_2 - 2u_2 = -2 - 2a_2 + 2u_2,$$

$$-2 + 2a_3 - 2b_3 = 2 - 2a_3 + 2b_3.$$

Which implies that $u_1 = b_1 + 1, u_2 = a_2 + 1$, and $a_3 = 1 + b_3$. This is impossible, otherwise it implies that $2\mathbb{Z}$ has odd elements. It follows then that there are no units of the form $m_1 = k_u(u_1, u_2, u_3) \cdot \bar{1} + k_a(a_1, a_2, a_3) \cdot \bar{a} + k_b(b_1, b_2, b_3) \cdot \bar{b}$.

Similarly, we generate matrix representations for

$$m_2 = k_u(u_1, u_2, u_3) \cdot \bar{1} + k_a(a_1, a_2, a_3) \cdot \bar{a} + k_c(c_1, c_2, c_3) \cdot \bar{c},$$

$$m_3 = k_u(u_1, u_2, u_3) \cdot \bar{1} + k_b(b_1, b_2, b_3) \cdot \bar{b} + k_c(c_1, c_2, c_3) \cdot \bar{c},$$

$$m_4 = k_a(a_1, a_2, a_3) \cdot \bar{a} + k_b(b_1, b_2, b_3) \cdot \bar{b} + k_c(c_1, c_2, c_3) \cdot \bar{c},$$

respectively as

$$\theta(m_2) = \begin{pmatrix} k_u r^{u_1} s^{u_2} t^{u_3} & k_a r^{-a_1} s^{2+a_2} t^{-a_3} & 0 & k_c r^{-c_1} s{-c_2} t^{2+c_3} \\ k_a r^{a_1} s^{a_2} t^{a_3} & k_u r^{-u_1} s^{u_2} t^{-u_3} & k_c r^{c_1} s^{-c_2} t^{-c_3} & 0 \\ 0 & k_c r^{-c_1} s^{c_2} t^{-2-c_3} & k_u r^{u_1} s^{-u_2} t^{-u_3} & k_a r^{-2-a_1} s^{-2-a_2} t^{a_3} \\ k_c r^{c_1} s^{c_2} t^{c_3} & 0 & k_a r^{2+a_1} s^{-a_2} t^{-a_3} & k_u r^{-u_1} s^{-u_2} t^{u_3} \end{pmatrix},$$

$$\theta(m_3) = \begin{pmatrix} k_u r^{u_1} s^{u_2} t^{u_3} & 0 & k_b r^{2+b_1} s^{-b_2} t^{-b_3} & k_c r^{-c_1} s{-c_2} t^{2+c_3} \\ 0 & k_u r^{-u_1} s^{u_2} t^{-u_3} & k_c r^{c_1} s^{-c_2} t^{-c_3} & k_b r^{-2-b_1} s^{-2-b_2} t^{2+b_3} \\ k_b r^{b_1} s^{b_2} t^{b_3} & k_c r^{-c_1} s^{c_2} t^{-2-c_3} & k_u r^{u_1} s^{-u_2} t^{-u_3} & 0 \\ k_c r^{c_1} s^{c_2} t^{c_3} & k_b r^{-b_1} s^{2+b_2} t^{-2-b_3} & 0 & k_u r^{-u_1} s^{-u_2} t^{u_3} \end{pmatrix},$$

and

$$\theta(m_4) = \begin{pmatrix} 0 & k_a r^{-a_1} s^{2+a_2} t^{-a_3} & k_b r^{2+b_1} s^{-b_2} t^{-b_3} & k_c r^{-c_1} s{-c_2} t^{2+c_3} \\ k_a r^{a_1} s^{a_2} t^{a_3} & 0 & k_c r^{c_1} s^{-c_2} t^{-c_3} & k_b r^{-2-b_1} s^{-2-b_2} t^{2+b_3} \\ k_b r^{b_1} s^{b_2} t^{b_3} & k_c r^{-c_1} s^{c_2} t^{-2-c_3} & 0 & k_a r^{-2-a_1} s^{-2-a_2} t^{a_3} \\ k_c r^{c_1} s^{c_2} t^{c_3} & k_b r^{-b_1} s^{2+b_2} t^{-2-b_3} & k_a r^{2+a_1} s^{-a_2} t^{-a_3} & 0 \end{pmatrix}.$$

Then, we have respective determinants,

$$\det(\theta(m_2)) = k_a^4 + k_c^4 + k_u^4 - k_a^2 k_c^2 r^{2+2a_1-2c_1} - k_a^2 k c^2 r^{-2-2a_1+2c_1} - k_a^2 k_u^2 s^{2+2a_2-2u_2}$$
$$-k_a^2 k_u^2 s^{-2-2a_2+2u_2} - k_c^2 k_u^2 t^{2+2c_3-2u_3} - k_c^2 k_u^2 t^{-2-2c_3+2u_3},$$

$$\det(\theta(m_3)) = k_b^4 + k_c^4 + k_u^4 - k_b^2 k_u^2 r^{2+2b_1-2u_2} - k_b^2 k_u^2 r^{-2-2b_1+2u_2} - k_b^2 k_c^2 s^{2+2b_2-2c_2}$$
$$-k_b^2 k_c^2 s^{-2-2b_2+2c_2} - k_c^2 k_u^2 t^{2+2c_3-2u_4} - k_c^2 k_u^2 t^{-2-2c_3+2u_4},$$

and

$$\det(\theta(m_4)) = k_a^4 + k_b^4 + k_c^4 - k_a^2k_c^2r^{2+2a_1-2c_1} - k_a^2k_c^2r^{-2-2a_1+2c_1} - k_b^2k_c^2s^{2+2b_2-2c_2}$$
$$-k_b^2kc^2s^{-2-2b_2+2c_2} - k_a^2k_b^2t^{-2+2a_3-2b_3} - k_a^2k_b^2t^{2-2a_3+2b_3}.$$

In relation to the above argument, we see that similar conditions apply to each of the determinants. Hence, it is impossible for $\theta(m_2))$, $\theta(m_3))$, or $\theta(m_4))$ to be in $K \cdot E_0$, in this case, actually just $K\cdot\{1\}$. In other words there are no invertible elements of support size 3 in $K[G]$.

Chapter 7

Conclusions

In Summary, we have answered some questions as to the basic nature of the more tractable example of a non-unique product group. It was shown that there is a structure to Promislow's 14 element set, shown that there exists a 17 element n.u.p. set and a method for creating other n.u.p. sets within the group. From this we were able to demonstrate the existence of infinitely many 17 element square sets inside of the group. Using the methods constructed it could be possible to modify these results to any torsion-free group that includes relations of the form $xy^l = y^{-l}x$, $x^k y = yx^{-k}$ for words x and y and nonzero integers k and l to show that it does not satisfy the unique product property. It is possible, since generally it is only the end points that have a problem in the matchings, we can extend the results in section 5.1 to demonstrate even larger examples of such sets. Also, it could be possible to extend the maps constructed in section 5.2 for Promislow's example to other Fibonacci groups with some modifications. Finally, in regards to the trivial units conjecture we showed that there are no units up to support size 3.

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