Complex Analysis on Planar Cell Complexes

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(ABSTRACT)

This paper is an examination of the theory of discrete complex analysis that arises from the framework of a planar cell complex. Construction of this theory is largely integration-based. A combination of two cell complexes, the double and its associated diamond complex, allows for the development of a discrete Cauchy Integral Formula.

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Introduction

Analytic functions and harmonic functions are the fundamental objects of study in complex analysis. The Cauchy Integral Formula is arguably the most important result. This theorem and the classes of analytic and harmonic functions are proof that complex analysis relies heavily on what are known as geometric differential operators. Analytic functions are the functions that lie in the kernel of the Cauchy-Riemann operator $\bar{\partial}$; harmonic functions are those in the kernel of the Laplace operator Δ; the Cauchy Integral Formula is derived from Green's Theorem, and hence from exterior differentiation acting on differential forms.

These geometric operators also play a role in physics, for example in models of electromagnetism. Consequently, there is interest in the discretization of these operators in order to do numerical computation [2, 4, 10]. This paper is an exploration of the discrete theory of complex analysis that arises as a result of the discretization of the Cauchy-Riemann operator. Its foundation is the work of Christian Mercat [8, 9], who further developed ideas of J. Lelong-Ferrand [7]. The original results in this thesis are identified as original in this introduction. All results stated without attribution can be found in [8, 9].

Mercat, motivated by interest in the Ising Model, develops discrete, or combinatorial, versions of the geometric differential operators important in complex analysis. He follows a well-established convention in defining discrete differential k-forms as functionals on a vector space generated by the k-cells of a cell complex. The analogue of the exterior derivative is dual to the cell complex's boundary map. This is natural in that standard differential k-forms evaluate on oriented k-dimensional submanifolds by integration. Furthermore, the cochain complex dual to the cell complex represents the cell complex's cohomology, just as the complex of differential forms and exterior derivatives represents the de Rham cohomology of a manifold.

Mercat's innovation is in using the cell complex and its dual cell complex to capture the analogue of a complex structure. Previously, most discrete or combinatorial analogues of differential operators have involved operators with a differential topological definition. Mercat's work suggests the possibility of including more geometric content in discrete, or combinatorial, constructions.

In this paper, we begin in Chapter 1 with the underlying foundation of a planar cell decomposition along with its dual (together they are called the *double*) and a discrete metric that assigns each edge (x, x') a length $\ell(x, x')$. We explore the differential k-forms for $k = 0, 1, 2$. Differential 0-forms are defined on the vertices, 1-forms on the edges, and 2-forms on the faces of the double.

Once the framework of our theory is established, we give the definition of a discrete analytic function based on the discretization of the Cauchy-Riemann equation. We also investigate properties of discrete analytic functions and discover that polynomials in z are not necessarily discrete analytic.

We explore operations on and between k-forms. We define the coboundary on the double to satisfy Green's Theorem and also establish the definitions of multiplication between forms, with the exception of 1-form multiplication. We prove that the coboundary satisfies a product rule with respect to multiplication of functions, a property to be expected of an analogue of exterior differentiation.

We provide a derivation of a discrete Hodge star operator, a linear operator which aids us in deriving a basis for the space of discrete differential 1-forms on the double. Consequently, we see that the direct sum of the eigenspaces of the Hodge star is equivalent to the space of 1-forms on the double. This direct sum is known as the Hodge Decomposition. Our featured result in this section is the development of analogues of dz and $d\bar{z}$. Local versions of these analogues form a basis for differential 1-forms and are therefore helpful in representing an arbitrary 1-form. The fundamental results of the theory are discussed in [8], but the discussion of the Hodge star operator and the emphasis on and discussion of the analogues of dz and $d\bar{z}$ are our own.

Via the Hodge decomposition, we give the definition of a holomorphic 1-form. We show that contrary to the continuous case, f discrete analytic does not guarantee $f dz$ is holomorphic. We state and prove some original results about the product of functions and 1-forms, and we explore properties of a 1-form that are necessary and sufficient for it to be holomorphic. We give one such property, an original result, in which we use a vertical half-shift of a square complex. We find that a 1-form on a square double is holomorphic if and only if a vertical half-shift has analytic crossings. This result demonstrates an alternative discrete connection between holomorphic 1-forms and analyticity.

We also address the difficulties in defining a discrete meromorphic function. This discussion and the observation that the discrete theory is based on the analogue of integration explain Mercat's decision to focus on discrete meromorphic 1-forms rather than discrete meromorphic functions. The definition of discrete meromorphic 1-forms allows us to recover a notion analogous to the residue of a pole via integration over the boundary of a 2-cell in the double. The major result in this section is the existence of a meromorphic 1-form with a single pole at a vertex x. This meromorphic 1-form is the discrete analogue of $\dfrac{dz}{z - z_0}$ on the double, where $z_0 = x$. The power of this result is later utilized in our development of a discrete Cauchy Integral Formula.

We conclude Chapter 1 with an original attempt at the derivation of a wedge product of 1-forms on the double complex. This wedge product requires 1-forms to be evaluated on edges that are not part of the double. Hence, it illustrates the restrictive nature of the double cell complex. While the double complex has great advantages, namely the rendering of a discrete Cauchy-Riemann equation and a Hodge star operator, its weaknesses beg for the development of an associated cell complex.

In Chapter 2, we provide an exposition of the diamond complex, a cell complex that is derived from a double complex, and its associated discrete theory. Each diamond face justifies its name by having only four vertices. This offers a major advantage over the double, where

2-cells may have an arbitrary number of vertices.

Our highlighted original discovery in this chapter is the mixed wedge product of 1-forms. Previously, no wedge product was defined on the double. We are able to define a wedge product of a diamond 1-form and a double 1-form that yields a double 2-form. With respect to the mixed wedge product, the coboundary is a derivation. This development allows for the recovery of an original Cauchy Integral Formula on the double.

In general, we see that the diamond provides a stronger foundation from which to develop discrete theory. Unlike the double, it allows a wedge product of its 1-forms. We also can define a wedge product of double 1-forms that yields a 2-form on the diamond. This result is made useful under the averaging map.

The averaging map is defined to be a mapping from k-forms on the diamond to k-forms on the double. This mapping is vital in utilizing the strengths of the double and the diamond simultaneously. It allows us to use both the power of the wedge product on the diamond and of the Cauchy-Riemann equation and Hodge decomposition on the double. More importantly, we are able to implement the analogue of $\dfrac{dz}{z - z_0}$ in a Cauchy Integral Formula on the diamond.

The third and final chapter of this paper is an exploration of the Cauchy Integral Formula. The first section provides a review for the reader of the continuous CIF. We give its proof to establish the process we will use as our guideline in the proof of a discrete CIF, namely Green's Theorem. In Section 3.2, we develop an original result: The Cauchy Integral Formula on the double. As in the continuous case, we multiply the discrete analogue of $\frac{dz}{z-z_0}$ by a function f and apply Green's Theorem off of the face containing $z_0 = x$. This renders a formula that formally resembles the continuous CIF. However, in a discussion that follows, we unmask the major drawback of our formula. It does not simplify to a line-integral representation of $f(x)$ when f is discrete analytic. The class of functions for which our formula simplifies in that way is restricted and probably of limited interest.

This seeming failure provides a motivation for the Cauchy Integral Formula on the diamond that is discussed in [8]. We construct a 1-form on the diamond whose average is a meromorphic 1-form on the double with two poles at vertices that form an interior diamond edge. The construction reveals that we are forced to work with two vertices on the diamond, instead of the usual Cauchy kernel involving a pole at a single point. Consequently the formula expressed on the diamond provides an average of two function values, instead of the value at a single point. However, unlike the Cauchy Integral Formula on the double, the diamond formula recovers the simplification of the Cauchy Integral Formula when f is discrete analytic.

And so, we see that a cell decomposition, along with its dual, is not strong enough to recover important results from the continuous setting. Although the double does provide a good foundation from which to start, the diamond gives our discrete analytic function theory the final boost that it needs to successfully "discretize" results in complex analysis. However, we do note that neither cell complex is sufficient in the absence of the other.

Throughout this paper we discover that our discrete theory has a linear structure, but not an algebra structure. For example, products of discrete analytic functions with each other or with discrete analytic 1-forms may fail to be analytic. In several cases, however, we see that difficulties arising from the absence of linearity can be overcome by using the guideline of integration. One such example is the failure to develop a theory of discrete meromorphic functions. Using integration and motivated by the continuous definition of the residue of a pole, we are able to define discrete meromorphic 1-forms. In fact, we see that our strongest results are integration-based, often times in ways that go far beyond the constructions on which the theory is based.

Chapter 1

Discrete Geometric Analysis

In this chapter, we begin by laying the foundation of the discrete version of complex analysis: a cell decomposition along with its dual, together called the double complex. We define complex-valued differential k-forms as follows: functions are defined on the set of vertices, 1-forms on the set of edges, and 2-forms on the set of faces of the double.

The first major construction we describe is the discrete Cauchy-Riemann equation. It is an analogue of the continuous Cauchy-Riemann equation, $i\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y}$, defined on a crossing of a dual and regular edge. As in the continuous case, we define a discrete analytic function as a function for which the Cauchy-Riemann equation holds at each such crossing. However, under this definition we show that polynomials in z are not necessarily analytic, an initially troubling aspect of the discrete theory.

We define operations on forms, with the exception of a wedge product of 1-forms, and illustrate the motivation behind the definition of multiplication of 1-forms by functions. We define the coboundary operator on k-forms via Green's Theorem and prove that, as an exterior derivative should, it satisfies a product rule with respect to function multiplication. We also provide a motivation for the discrete Hodge star operator and give the resulting Hodge decomposition, which plays a significant role in the discrete theory. The Hodge decomposition leads to a basis for discrete 1-forms, of which we provide an original construction. Furthermore, it provides a foundation on which to define discrete holomorphic and meromorphic 1-forms.

We note that an analytic function multiplied with the differential dz is not necessarily such that fdz is discrete holomorphic. We provide original work in exploring 1-form properties that are equivalent to holomorphic. We define a vertical half-shift on the square double complex and prove the result that a 1-form is holomorphic if and only if a vertical half-shift of this double has analytic crossings.

We provide an original exposition of the difficulties in defining a meromorphic function and give Mercat's definition of a meromorphic 1-form, noting integration as a motivating factor. We state the existence of the discrete analogue of the continuous meromorphic 1-form $\dfrac{dz}{z - z_0}$.

Finally, we conclude this chapter with an original investigation of the difficulty in defining, for pairs of 1-forms on the double, a wedge product with respect to which the exterior derivative satisfies a product rule on the product of functions and 1-forms. An obstacle that we encounter in this examination provides an important motivation for venturing beyond the double to develop more discrete theory.

1.1 The Double Defined

The framework upon which we build our discrete theory is a cell decomposition. Thus, we begin by defining a cell decomposition along with its associated discrete metric, the length of an edge.

Definition 1.1.1. Let Σ be an oriented surface without boundary. A **cell decomposition Γ of Σ** is a partition of Σ into disjoint connected sets, called cells, of three types:

(1) Γ_0, a discrete set of points, called vertices.

(2) Γ_1, a set of nonintersecting oriented paths running between vertices and called edges.

(3) Γ_2, a set of topological discs bounded by a finite number of edges and vertices and called faces.

Note: Let $x, y \in \Gamma_0$. In this paper we denote the oriented edge from x to y as $(x, y) \in \Gamma_1$. We use only paths for which a length can be defined. The length of an edge $(x, y) \in \Gamma_1$ is denoted $\ell(x, y)$. Throughout the entirety of this paper we focus only on cell decompositions of the complex plane, although much of the theory discussed can be generalized to Riemann surfaces. We assume that our edges are straight segments, in which case we may define $\ell(x, y) = |y - x|$. In any case, $\ell(e) = \ell(-e)$, where $-e$ denotes the edge denoted by e with the opposite orientation.

Also, a face with vertices $x_1, \ldots, x_n \in \Gamma_1$ may be denoted $(x_1, \ldots, x_n) \in \Gamma_2$ with boundary $(x_1, x_2) \cup \cdots \cup (x_{n-1}, x_n) \cup (x_n, x_1)$. Under this notation, each (x_i, x_{i+1}) is assumed to be an edge in Γ_1.

For each cell decomposition Γ, we may construct a **dual cell decomposition** Γ^*. Begin by defining a dual vertex $F^* \in \Gamma_0^*$ inside each face $F \in \Gamma_2$ to be the image of the origin of the Euclidean plane by some parameterization of the face. Now, each edge $e \in \Gamma_1$ separates two adjacent faces $F_1, F_2 \in \Gamma_2$. Its dual edge $e^* \in \Gamma_1^*$ is defined to be a chosen simple path between the vertices F_1^* and F_2^* in Γ_0^*, lying in the faces F_1 and F_2, that cuts $e \in \Gamma_1$ once and transversely and cuts no other edge. We orient e^* so that the orientation of e followed by the orientation of e^* agrees with the orientation of Σ, which in our case is the standard orientation of the plane. If $v \in \Gamma_0$ has adjacent vertices $v_1, v_2, ..., v_n \in \Gamma_0$, a face $v* \in \Gamma_2^*$ is defined by the 1-cells in its boundary $\partial v^* = \cup_{k=1}^{n}(v, v_k)^*$. We can now state a formal definition.

Definition 1.1.2. A **dual cell decomposition** Γ^* of Γ is a cell decomposition with vertices Γ_0^*, edges Γ_1^*, and faces Γ_2^* unique to and determined by Γ as described above.

Definition 1.1.3. The **double** Λ is the union of the cell decomposition Γ and its dual Γ^*.

Note that Γ is also the dual of Γ^*, $\Gamma = \Gamma^{**}$. In particular, $e^{**} = -e$.

Examples of the double Λ are shown in Figure 1.1.

In this paper we work only on the plane, and we assume that Γ and Γ^* are such that all edges in Λ are straight segments.

A bounded cell complex in the plane has to have a boundary, near which the duality between a cell complex and its dual must break down. A bounded double complex is constructed by starting with a double complex on the entire plane. From Γ_2, choose finitely many 2-cells

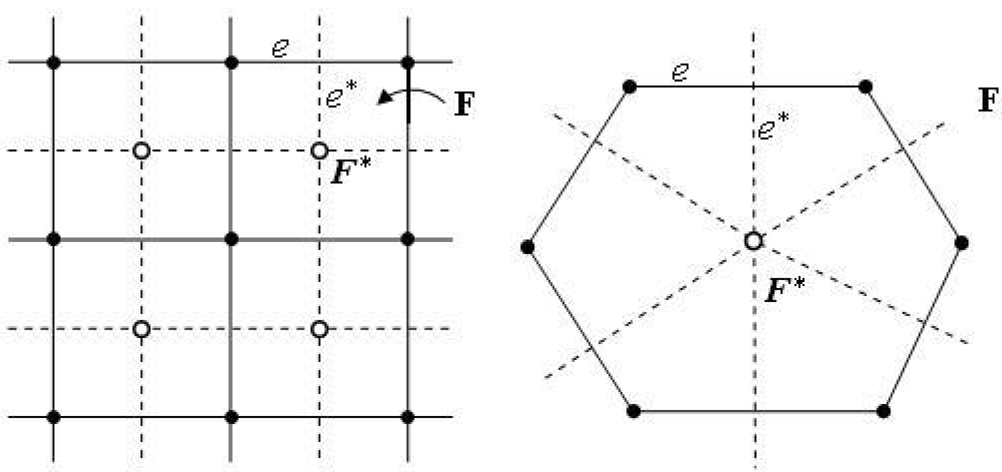

Figure 1.1: The Double Λ

and all the cells from $\Gamma_0 \cup \Gamma_1$ that are in their closures. (Alternatively, one could make all of these choices from Γ^*.) For each chosen cell other than the boundary vertices, choose its dual cell. Also, choose the dual cells in the closure of these chosen dual cells. The collection of cells chosen above forms what we call a **bounded double complex**.

Throughout this paper, we frequently refer to the standard orientation of a regular and dual edge crossing. We give a definition below with the aid of Figure 1.2.

Definition 1.1.4. We say that the crossing of a regular edge $(x, x') \in \Gamma_1$ with its dual $(y, y') \in \Gamma_1^*$ is of **standard orientation** if (x, x') is followed by (y, y') in the counterclockwise direction as shown in Figure 1.2.

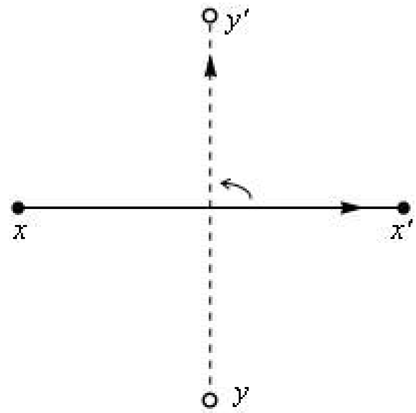

Figure 1.2: The standard orientation of a regular and dual edge crossing in the double complex

1.2 The Cauchy-Riemann Equation and Analytic Functions

Now that we have developed the structure of our cell decomposition, a natural question is how to define functions on the double. In particular, how do we define an analogue of an analytic function?

Let Λ_0 denote the set of vertices in Λ, $\Lambda_0 = \Gamma_0 \cup \Gamma_0^*$. A function f is defined on the vertices of the double. We denote the set of such functions by $C^0(\Lambda)$. An element of $C^0(\Lambda)$ assigns a complex number to each vertex in Λ_0. Function multiplication is pointwise as expected. Hence, if $f, g \in C^0(\Lambda)$ and $x \in \Lambda_0$ then $(f \cdot g)(x) = f(x) \cdot g(x)$. In taking the boundary of an edge, we give vertices orientations, denoted by signs. Functions $f \in C^0(\Lambda)$ are defined on oriented vertices by the requirement that $f(-v) = -f(v)$.

Next, to establish the notion of a discrete analytic function, we recall a theorem from the continuous case.

Theorem 1.2.1. *Let $f(z)$ be a continuously differentiable function on a domain D. Then $f(z)$ is analytic if and only if f(z) satisfies the complex Cauchy-Riemann equation* $\frac{\partial f}{\partial \bar{z}} = 0$.

Recall that if $z = x + iy$, the Cauchy-Riemann equation above is equivalent to $i\dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial y}$. It is this form that allows us to define an analogous discrete Cauchy-Riemann equation.

Definition 1.2.2. Let $(x, x') \in \Gamma_1$ and let $(y, y') \in \Gamma_1^*$ be the edge dual to (x, x') with standard orientation. Then the **discrete Cauchy-Riemann equation** is

$$i\frac{f(x') - f(x)}{\ell(x, x')} = \frac{f(y') - f(y)}{\ell(y, y')}.$$

Throughout this paper, most examples are given on a double complex of squares, with each vertex at the center of its dual 2-cell. On such a complex, Λ, we may assume that $\ell(e) = 1\ \forall e \in \Lambda_1$ for simplification purposes. Thus, in those cases, the discrete C-R equation reduces to $i(f(x') - (x)) = f(y') - f(y)$.

Extending the results of Theorem 1.2.1, to the discrete case we may now define a discrete analytic function.

Definition 1.2.3. A function $f : \Lambda_0 \to \mathbb{C}$ is **discrete analytic** if, for every pair of dual edges $(x, x') \in \Gamma_1$ and $(y, y') = (x, x')^* \in \Gamma_1^*$, it satisfies the discrete Cauchy-Riemann equation.

Despite the fact that we have founded the notion of discrete analyticity on the analogous discrete C-R equation, there is a very key property of continuous analytic functions that is seemingly absent in the discrete case. This property is described in the following theorem.

Theorem 1.2.4. *Suppose that $f(z)$ is analytic for $|z - z_0| < \rho$. Then $f(z)$ is represented by the power series*

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k, \quad |z - z_0| < \rho,$$

where the power series has radius of convergence $R \geq \rho$.

The results of this theorem are far-reaching in complex analysis. Thus, a similar result in the discrete case is desirable.

Proposition 1.2.5. *The function $f(z) = z$ is discrete analytic on the double Λ, whenever Λ is such that every regular edge, dual edge crossing is orthogonal.*

Proof. **Case.** Let Λ be a square cell decomposition with edges of length one parallel to the coordinate axes. Let $(x, x') \in \Lambda_1$ and $(x, x')^* = (y, y') \in \Lambda_1^*$ with standard orientation (Figure 1.2). Then $i(f(x') - f(x)) = i(x' - x) = y' - y = f(y') - f(y)$.

General Case. Let Λ be any cell decomposition. Let $(x, x') \in \Lambda_1$ and $(x, x')^* = (y, y') \in \Lambda_1^*$ with standard orientation. Then $f(x') - f(x) = x' - x$ and $f(y') - f(y) = y' - y$. Further, $\dfrac{|x' - x|}{\ell(x, x')} = 1 = \dfrac{|y' - y|}{\ell(y, y')}$ since $\ell(a, b) = |b - a|$ by definition. Now, because (x, x') and (y, y') are orthogonal, we have $\dfrac{i(x' - x)}{\ell(x, x')} = \dfrac{y' - y}{\ell(y, y')}$ and the C-R equation is satisfied.

□

Thus, for many double complexes the discrete function $f(z) = z$ is analogous to its continuous analytic counterpart. However, unlike in the continuous case, the function $f(z) = z^n,\ n > 2$ is not necessarily discrete analytic. A simple example illustrating the failure of the case when $n = 3$ is given below.

Example 1.2.6. Define $f(z) = z^3$. Consider the cell decomposition pictured in Figure 1.3. Then $(1, 1+i)$ is dual to $(\frac{1}{2} + \frac{i}{2}, \frac{3}{2} + \frac{i}{2})$ with

$f(1) = 1, f(1+i) = 2i - 2, f(\frac{1}{2} + \frac{i}{2}) = \frac{i}{4} - \frac{1}{4}$, and $f(\frac{3}{2} + \frac{i}{2}) = \frac{9}{4} + \frac{13i}{4}$.

Hence, $\quad i[f(\frac{3}{2} + \frac{i}{2}) - f(\frac{1}{2} + \frac{i}{2})] = \frac{5}{2}i - 3 \quad$ and $f(1+i) - f(1) = 2i - 3$.

$\therefore$ The discrete C-R equation is not satisfied.

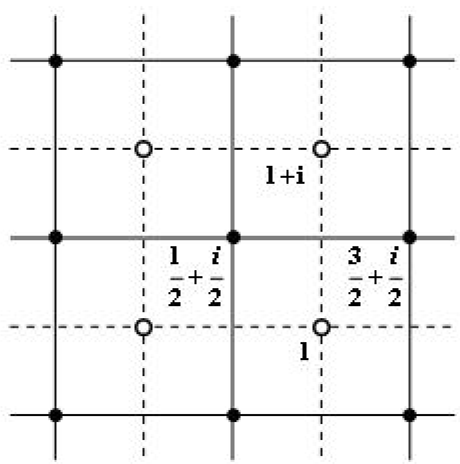

Figure 1.3: The cell complex from Example 1.2.6

When constructing an analytic function on a cell complex on the complex plane we typically have infinitely many degrees of freedom. For example, on a square complex, we may arbitrarily choose the values of a function on a row of regular vertices and an adjacent row of dual vertices. Then we can solve the system of equations that results from the Cauchy-Riemann equation row by row. Each vertex value is in a single equation in which it is the only variable. Thus, we see that the space of discrete analytic function is typically quite large.

1.3 Differential k-forms

Because much of the theory developed around this discrete cell decomposition is based on integration, it is necessary to define discrete differential 1-forms and 2-forms.

Definition 1.3.1. A **discrete 1-form** $\alpha \in C^1(\Lambda)$ is a complex-valued function defined on the set of edges Λ_1. Such an α is required to satisfy $\alpha(-e) = -\alpha(e)$.

Definition 1.3.2. A **discrete 2-form** $\omega \in C^2(\Lambda)$ is a complex-valued function defined on the set of faces Λ_2. We do not use different orientations on faces F. If we did, we would require $\omega(-F) = -\omega(F)$.

Note: from the above definitions we see that functions can naturally be called discrete 0-forms.

Throughout this paper we will use the following notation:

$$\alpha(e) = \int_e \alpha \qquad \alpha \in C^1(\Lambda), e \in \Lambda_1 \qquad \text{and} \qquad \omega(F) = \iint_F \omega \qquad \omega \in C^2(\Lambda), F \in \Lambda_2.$$

As in the continuous case, we can define a coboundary function which maps from $C^k(\Lambda)$ to $C^{k+1}(\Lambda)$ as follows.

Definition 1.3.3. The **coboundary** $d : C^k(\Lambda) \to C^{k+1}(\Lambda)$ is defined by

$$\int_{(x,x')} df := f(\partial(x,x')) = f(x') - f(x) \quad \textit{and} \quad \iint_F d\alpha := \oint_{\partial F} \alpha.$$

Here ∂F must be oriented so that the outer normal, followed by the direction of ∂F, agrees with the orientation of F, as in the usual Green's Theorem.

The definition of the coboundary is modeled after The Fundamental Theorem of Calculus and Green's Theorem from the continuous case.

Theorem 1.3.4. (The Fundamental Theorem of Calculus) *If $F(t)$ is an antiderivative for the continuous function $f(t)$, then*

$$\int_a^b f(t)dt = F(b) - F(a).$$

Theorem 1.3.5. (Green's Theorem) *Let $D \subseteq \mathbb{C}$ be a bounded domain with C^1 boundary. Let α be a 1-form on $C^1(\bar{D})$. Then*

$$\iint_D d\alpha = \oint_{\partial D} \alpha.$$

where ∂D has induced orientation from D.

1.4 Operations on the Double Λ

Now that we have established the notion of discrete functions, 1-forms, and 2-forms, we want to define a discrete multiplication between forms. Let $e = (x, x') \in \Lambda_1, F \in \Lambda_2, f \in C^0(\Lambda), \alpha \in C^1(\Lambda)$, and $\omega \in C^2(\Lambda)$. Then

$$\int_e f \cdot \alpha := \frac{f(x) + f(x')}{2} \int_e \alpha \qquad \textit{and} \qquad \iint_F f \cdot \omega := f(F^*) \iint_F \omega.$$

Note: The failure to define a wedge product of 1-forms at this point is intentional. We will see later that it is not easily defined on the double, and we will introduce an alternative cell complex that does admit a wedge product that is essential to the further development of the discrete theory.

Our definition of the product of a function and a 1-form is based on our desire for the coboundary to be a derivation with respect to function multiplication. In other words, for any edge $e = (x, x') \in \Lambda_1$ we want $d(f \cdot g)(e) = f \cdot dg(e) + g \cdot df(e)$. This goal forces the definition of function and 1-form multiplication given above. Before offering a proof of this claim, we first state two conditions that we require of the product $f \cdot \alpha$ where $f \in C^0(\Lambda)$ and $\alpha \in C^1(\Lambda)$.

1. For an edge $e = (x, x') \in \Lambda_1$, $(f \cdot \alpha)(e)$ depends only on $f(x), f(x')$, and $\alpha(e)$.

2. $f \cdot \alpha$ is bilinear in f and α, i.e. $\exists u, v, k \in \mathbb{C}$ such that $(f \cdot \alpha)(e) = [uf(x) + vf(x')](k\alpha(e))$ $\forall f \in C^0(\Lambda)$ and $\forall \alpha \in C^1(\Lambda)$. Substituting $s = uk$ and $t = vk$, this condition simplifies to $(f \cdot \alpha)(e) = [sf(x) + tf(x')]\alpha(e)$.

Both of these requirements are natural. The second is simply a result of our desire to have a vector space of functions with operations that behave similarly to their continuous counterparts. We are now ready to state our proposition.

Proposition 1.4.1. *Assume the conditions above. Let* $f, g \in C^0(\Lambda)$ *and* $e = (x, x') \in \Lambda_0$. *If* $d(f \cdot g) = f \cdot dg + g \cdot df$ *then* $s = t = \frac{1}{2}$.

Proof. For simplicity, let $f(x) = a, f(x') = b, g(x) = c$, and $g(x') = d$. Then $d(f \cdot g) = f \cdot dg + g \cdot df$ becomes $bd - ac = [sa + tb](d - c) + [sc + td](b - a) \Rightarrow bd - ac = (s - t)ad - 2sac + 2tbd + (s - t)bc$. Consider the case $a = d = 1$ and $b = c = 0$. Then the above reduces to $0 = s - t \implies s = t$. Consider, also, the case where $b = d = 1$ and $a = c = 0$. Then $bd - ac = (s - t)ad - 2sac + 2tbd + (s - t)bc$ reduces to $1 = 2t \implies t = \frac{1}{2}$. Since both cases must be satisfied simultaneously, the only possible solution is $s = t = \frac{1}{2}$.

□

The following proposition shows that $(f \cdot \alpha)(e) = [sf(x) + tf(x')]\alpha(e)$ where $s = t = \frac{1}{2}$ is indeed a solution that satisfies the product rule for functions.

Proposition 1.4.2. *d is a derivation with respect to function multiplication.*

Proof. Suppose $f, g \in C^0(\Lambda)$. Let $e \in \Lambda_1$ s.t. $e = (x, x')$. Then

$$\begin{aligned}
\int_e d(f \cdot g) &= (f \cdot g)(\partial(x, x')) \\
&= (f \cdot g)(x') - (f \cdot g)(x) \\
&= f(x')g(x') - f(x)g(x) \\
&= \frac{1}{2}[2f(x')g(x') - f(x)g(x') + f(x)g(x') - f(x')g(x) + f(x')g(x) - 2f(x)g(x) \\
&= \frac{f(x) + f(x')}{2}[g(x') - g(x)] + \frac{g(x) + g(x')}{2}[f(x') - f(x)] \\
&= \int_e f \cdot dg + \int_e g \cdot df
\end{aligned}$$

□

Remark 1.4.3. At this time we present no justification for the definition of $f \cdot w$ when $f \in C^0(\Lambda)$ and $\omega \in C^2(\Lambda)$. As defined the product has the property that $(fg)\omega = f(g\omega)$. The 1-form case suggests replacing $f(F^*)$ by the average of f's values on vertices in the boundary of F. The definition given at the beginning of this subsection has some good properties described in Chapter 2. The alternative definition does not resolve the difficulties that arise in Subsection 1.8.

1.5 The Hodge Star, the Hodge Decomposition, and a Basis for $C^1(\Lambda)$

Now that we have defined discrete differential forms and established the exterior derivative d on the double Λ, we consider the building blocks of continuous 1-forms: dz and $d\bar{z}$. We would like to develop local analogues of dz and $d\bar{z}$ on the double, a construction that will work on general Riemann surfaces as well as on the complex plane. To help us better determine an appropriate analogy, we recall an operator on continuous 1-forms, the Hodge star.

1.5.1 The Hodge Star and the Hodge Decomposition

In the continuous setting, the Hodge star is defined to be $*fdx = fdy$ and $*fdy = -fdx$. Therefore,

$$*df = *\left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\right) = \frac{\partial f}{\partial x}dy - \frac{\partial f}{\partial y}dx.$$

If we now assume that f is analytic, the Cauchy-Riemann equations yield

$$*df = -i\frac{\partial f}{\partial y}dy - i\frac{\partial f}{\partial x}dx = -i\left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\right) = -idf.$$

Thus, the continuous Cauchy-Riemann equation may be written $*df = -idf$.

We want our discrete definition of the Hodge star to satisfy the equations $*dx = dy$ and $*dy = -dx$. Here we define the discrete 1-forms $dx(e) = Re(x') - Re(y')$ and $dy(e) = Im(x') - Im(x)$ for an edge $e = (x, x') \in \Lambda_1$. To motivate the general definition of the discrete Hodge star operator, we explore the consequences of requiring $*dx = dy$ and $*dy = -dx$ on a cell complex in which every edge is either horizontal or vertical. Let Λ be such a cell complex. We'll use the notation $e \in \Gamma_1$ for a horizontal edge oriented to the right and the notation $\tilde{e} \in \Gamma_1$ for a vertical edge oriented upward. Then e^* is vertical, oriented upward, and $\tilde{e}^*$ is horizontal, oriented to the left. We'll explore this specific case to gain insight into a generalized Hodge star definition. To simplify notation, we define $\rho(e) = \dfrac{\ell(e^*)}{\ell(e)}$ for any edge $e \in \Lambda_1$. If we assume that the discrete Hodge star satisfies $*dx = dy$ and $*dy = -dx$, then we have the following.

$$*dx(\tilde{e}) = dy(\tilde{e}) = \ell(\tilde{e}) = \frac{\ell(\tilde{e})}{\ell(\tilde{e}^*)}\ell(\tilde{e}^*) = -\rho(\tilde{e}^*)dx(\tilde{e}^*)$$

$$*dx(e) = dy(e) = 0 = dx(e^*) = -\rho(e^*)dx(e^*)$$

$$*dy(e) = -dx(e) = -\ell(e) = -\rho(e^*)dy(e^*)$$

$$*dy(\tilde{e}) = -dx(\tilde{e}) = 0 = dy(\tilde{e}) = -\rho(\tilde{e}^*)dy(\tilde{e}^*).$$

From these equations we see that if our edge e is either horizontal or verical, we arrive at the same result.

$$*dx(e) = -\rho(e^*)dx(e^*)$$

$$*dy(e) = -\rho(e^*)dy(e^*).$$

Since $*$ is a linear operator, we should also have the following.

$$*dz(e) = *(dx + idy)(e) = (*dx + i * dy)(e) = -\rho(e^*)(dx(e^*) + idy(e^*)) = -\rho(e^*)dz(e^*)$$

Dropping the restriction to dx, dy, and dz on a cell complex in which every edge is parallel to a coordinate axis, we can define a discrete Hodge star operator on an arbitrary 1-form $\alpha \in C^1(\Lambda)$.

Definition 1.5.1. The **discrete Hodge star** $* : C^1(\Lambda) \to C^1(\Lambda)$ is a linear map defined by

$$\int_e *\alpha = -\rho(e^*) \int_{e^*} \alpha, \qquad e \in \Lambda_1,$$

where the crossing at e and e^* has standard orientation, as shown earlier in Figure 1.2.

Recall that in the continuous case, if f is analytic then the Cauchy-Riemann equation $*df = -idf$ is satisfied. The same is true in the discrete setting.

Proposition 1.5.2. *Suppose $f \in C^0(\Lambda)$ is discrete analytic. Then $*df = -idf$.*

Proof. Let $e \in \Lambda_1$ and e^* its dual. Recall, that by the Cauchy-Riemann equation we have

$$\frac{idf(e)}{\ell(e)} = \frac{df(e^*)}{\ell(e^*)}.$$

Then

$$*df(e) = -\rho(e^*)df(e^*) = -\ell(e)\frac{df(e^*)}{\ell(e^*)} = -idf(e).$$

□

Next, since the Hodge star is linear, we can investigate the eigenspaces of 1-forms it induces. In the continuous case, an eigenvalue λ of a linear map L is such that $L\mathbf{v} = \lambda\mathbf{v}$, where $\mathbf{v}$ is an eigenvector of λ. If we apply L twice to $\mathbf{v}$, we have $L(L\mathbf{v}) = \lambda^2\mathbf{v}$. Therefore, since $*$ is linear, applying $*$ twice to a 1-form will help us discover the possible eigenvalues of the discrete Hodge star.

Let $\alpha \in C^1(\Lambda)$ and $e \in \Lambda_1$. Then

$$* * \alpha(e) = *\big(- \rho(e^*)\alpha(e^*)\big) = -\rho(e^*)(*\alpha(e^*)) = -\rho(e^*)(-\rho(e)\alpha(-e)) = -\alpha(e)$$

Any eigenvalue λ of the Hodge star must satisfy $\lambda^2 = -1$. Consequently, the only possible eigenvalues are $\pm i$.

Definition 1.5.3. Let $\alpha \in C^1(\Lambda)$. We say α is of **type (1,0)** ($\alpha \in C^{(1,0)}(\Lambda)$) if $*\alpha(e) = -i\alpha(e) \;\; \forall e \in \Lambda_1$ and α is of **type (0,1)** ($\alpha \in C^{(0,1)}(\Lambda)$) if $*\alpha(e) = i\alpha(e) \;\; \forall e \in \Lambda_1$. This is called the **Hodge decomposition**.

In other words, α is of type $(1,0)$ if α belongs to the $-i$ eigenspace, and α is of type $(0,1)$ if α belongs to the $+i$ eigenspace.

Proposition 1.5.4. *Let Λ be a cell complex on $\mathbb{C}$. Then dz is of type (1,0) and $d\bar{z}$ is of type (0,1).*

Proof. First, we'll show that dz is of type (1,0). Consider an edge $e = (x, x') \in \Gamma_1$ and its dual $e^* = (y, y') \in \Gamma_1^*$ as pictured in Figure 1.4. Then $dz(e) = x' - x$ and $dz(e^*) = y' - y = i(x' - x)\rho(e)$ since the function z is discrete analytic. Therefore $-idz(e) = -\rho(e^*)dz(e^*) = *dz(e)$ and dz is of type (1,0). Now consider $d\bar{z}$ on the same crossing. Then $d\bar{z}(e) = \overline{x' - x}$ and $d\bar{z}(e^*) = \overline{y' - y} = \overline{i(x' - x)\rho(e)} = -i\rho(e)(\overline{x' - x})$. Hence, $id\bar{z}(e) = -\rho(e^*)d\bar{z}(e^*) = *d\bar{z}(e)$ and $d\bar{z}$ is of type (0,1).

□

1.5.2 A Basis for $C^1(\Lambda)$

The continuous 1-forms dz and $d\bar{z}$ are also of type (1,0) and (0,1), respectively, under the continuous Hodge star. Because z is analytic, the Cauchy Riemann equation $*dz = -idz$ holds. We may show $d\bar{z}$ is type (0,1) directly.

$$*d\bar{z} = *(dx - idy) = *dx - i * dy = dy + idx = i(dx - idy) = id\bar{z}.$$

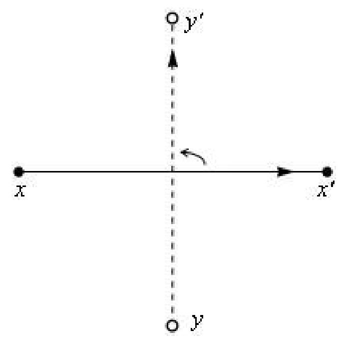

Figure 1.4: A regular and dual edge crossing.

In the continuous case, dz and $d\bar{z}$ generate the collection of all differential 1-forms on the complex plane. In the discrete setting we can show that local analogues of dz and $d\bar{z}$ form a basis for the vector space of all discrete differential 1-forms. To do this, we need to show that the $\pm i$ eigenspaces span $C^1(\Lambda)$.

We begin with the specific case where our cell complex is given on the complex plane. With the continuous case as our guide, we are looking to develop a notation for the discrete 1-forms dz and $d\bar{z}$ that will point toward a general definition for our $\pm i$ eigenspace representatives when working of off the plane. We observe that in the discrete case, continuity is not of concern, and we can therefore introduce a localized definition specific to each regular and dual edge crossing. Let $e = (x, x')$ be a regular edge in Γ_1, $e^* = (y, y')$ its dual, and let their crossing be of standard orientation. We define the discrete 1-form dz localized at this crossing for an arbitrary edge $e' \in \Lambda_1$ by

$$dz_e(e') = \begin{cases} dz(e) & \text{for } e' = e \\ dz(e^*) & \text{for } e' = e^* \\ 0 & \text{otherwise.} \end{cases}$$

As previously shown, dz is of type (1,0) and therefore satisfies $*dz(e) = -idz(e)$ at each crossing. By the definition of the Hodge star, this yields $-\rho(e^*)dz(e^*) = -idz(e)$. Hence, $dz(e^*) = i\rho(e)dz(e)$ and we may rewrite our definition of dz_e as follows

$$dz_e(e') = \begin{cases} dz(e) & \text{for } e' = e \\ i\rho(e)dz(e) & \text{for } e' = e^* \\ 0 & \text{otherwise.} \end{cases}$$

A similar process shows that we may define the discrete 1-form $d\bar{z}$ localized at the crossing associated with e for an arbitrary edge $e' \in \Lambda_1$ by

$$d\bar{z}_e(e') = \begin{cases} d\bar{z}(e) & \text{for } d' = e \\ -i\rho(e)d\bar{z}(e) & \text{for } e' = e^* \\ 0 & \text{otherwise.} \end{cases}$$

And so, above we see that dz_e returns a complex constant $c = x' - x$, and the value returned on e^* is found using the Hodge star relationship of a (1,0) 1-form. This observation shows us that the only essential ingredients in defining a $\pm i$ eigenspace representative localized at a crossing is the relationship of the values of the 1-form (complex constants) on e and e^* given by the definition of the Hodge star. So we can choose two collections $\mathbb{K}$ and $\mathbb{J}$ of constants k_e and j_e indexed by edges $e \in \Gamma_1$ and define our eigenspace representatives for the general case. We will use the notation $\zeta_{\mathbb{K},e} \in C^{(1,0)}(\Lambda)$ and $\bar{\zeta}_{\mathbb{J},e} \in C^{(0,1)}(\Lambda)$ to denote the respective representatives of the $\mp i$ eigenspaces. For simplification purposes, we'll suppress $\mathbb{K}$ and $\mathbb{J}$ from the notation and define ζ_e and $\bar{\zeta}_e$ localized at the crossing of e and e^* for an arbitrary edge $e' \in \Lambda_1$.

$$\zeta_e(e') = \begin{cases} k_e & \text{for } e' = e \\ ik_e\rho(e) & \text{for } e' = e^* \\ 0 & \text{otherwise} \end{cases} \qquad \bar{\zeta}_e(e') = \begin{cases} j_e & \text{for } e' = e \\ -ij_e\rho(e) & \text{for } e' = e^* \\ 0 & \text{otherwise.} \end{cases}$$

Thus, dz_e and $d\bar{z}_e$ are ζ_e and $\bar{\zeta}_e$, respectively, where

$$\mathbb{K} = \{k_e : k_e = x' - x \text{ for } e = (x, x') \in \Gamma_1\}.$$

and

$$\mathbb{J} = \{j_e : j_e = \overline{x' - x} \text{ for } e = (x,' x) \in \Gamma_1\}.$$

Proposition 1.5.5. *ζ_e is of type (1,0) and $\bar{\zeta}_e$ is of type (0,1).*

Proof.

$$
\begin{aligned}
*\zeta_e(e') &= -\rho(e'^*)\zeta_e(e'^*) \\
&= \begin{cases} -\rho(e^*)\zeta_e(e^*) & \text{for } e' = e \\ -\rho(-e)\zeta_e(-e) & \text{for } e' = e^* \\ 0 & \text{otherwise} \end{cases}
= \begin{cases} -\rho(e^*)(ik_e\rho(e)) & \text{for } e' = e \\ -\rho(e)(-k_e) & \text{for } e' = e^* \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} -ik_e & \text{for } e' = e \\ -i\rho(e)(ik_e) & \text{for } e' = e^* \\ 0 & \text{otherwise} \end{cases} \\
&= -i\zeta_e(e') \qquad \text{and } \zeta_e \text{ is of type (1,0).}
\end{aligned}
$$

$$
\begin{aligned}
*\bar{\zeta}_e(e') &= -\rho(e'^*)\bar{\zeta}_e(e'^*) \\
&= \begin{cases} -\rho(e^*)\bar{\zeta}_e(e^*) & \text{for } e' = e \\ -\rho(-e)\bar{\zeta}_e(-e) & \text{for } e' = e^* \\ 0 & \text{otherwise} \end{cases}
= \begin{cases} -\rho(e^*)(-ij_e\rho(e)) & \text{for } e' = e \\ -\rho(e)(-j_e) & \text{for } e' = e^* \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} ij_e & \text{for } e' = e \\ i\rho(e)(-ij_e) & \text{for } e' = e^* \\ 0 & \text{otherwise} \end{cases} \\
&= i\bar{\zeta}_e(e') \qquad \text{and } \bar{\zeta}_e \text{ is of type (0,1).}
\end{aligned}
$$

□

And so, we have found our $\pm i$ eigenspace representatives, respectively $\bar{\zeta}_e$ and ζ_e.

Proposition 1.5.6. *For any chosen collections of nonzero constants $\mathbb{K}$ and $\mathbb{J}$, indexed by the elements e of Γ_1, the associated collection $\{\zeta_e : e \in \Gamma_1\} \cup \{\bar{\zeta}_e : e \in \Gamma_1\}$ is a basis for $C^1(\Lambda)$.*

Proof. By construction, the ζ_e's and $\bar{\zeta}_e$'s are independent. Therefore, it suffices to show that they span $C^1(\Lambda)$. Let $\alpha \in C^1(\Lambda)$. Because each coefficient matrix $\left(\begin{smallmatrix} k_e & j_e \\ k_e & -j_e \end{smallmatrix}\right)$ is nonsingular, we may choose collections $\mathbb{A}$ and $\mathbb{B}$ of complex constants a_e and b_e, respectively, indexed by the edges $e \in \Gamma_1$ such that, for every e, the following system of equations is satisfied.

$$\begin{cases} a_e k_e + b_e j_e = \alpha(e) \\ a_e k_e - b_e j_e = \dfrac{\alpha(e^*)}{i\rho(e)}. \end{cases}$$

The solutions are $a_e = \dfrac{i\alpha(e) + \rho(e^*)\alpha(e^*)}{2ik_e}$ and $b_e = \dfrac{i\alpha(e) - \rho(e^*)\alpha(e^*)}{2ij_e}$ for $e \in \Gamma_1$.

Claim. $\alpha = \sum_{e\in\Gamma_1} \left[a_e\zeta_e + b_e\bar{\zeta}_e\right]$.

Case 1. $e' \in \Gamma_1$.

$$\begin{aligned} \sum_{e\in\Gamma_1} \left[a_e\zeta_e(e') + b_e\bar{\zeta}_e(e')\right] &= a_{e'}\zeta_{e'}(e') + b_{e'}\bar{\zeta}_{e'}(e') \\ &= a_{e'}k_{e'} + b_{e'}j_{e'} \\ &= \left(\frac{i\alpha(e') + \rho(e'^*)\alpha(e'^*)}{2ik_{e'}}\right)k_{e'} + \left(\frac{i\alpha(e') - \rho(e'^*)\alpha(e'^*)}{2ij_{e'}}\right)j_{e'} \\ &= \frac{i\alpha(e') + \rho(e'^*)\alpha(e'^*)}{2i} + \frac{i\alpha(e') - \rho(e'^*)\alpha(e'^*)}{2i} \\ &= \alpha(e'). \end{aligned}$$

Case 2. $e' \in \Gamma_1^*$

$$\begin{aligned}
\sum_{e\in\Gamma_1}\left[a_e\zeta_e(e') + b_e\bar{\zeta}_e(e')\right] &= a_{e'^*}\zeta_{e'^*}(e') + b_{e'^*}\bar{\zeta}_{e'^*}(e') \\
&= a_{e'^*}(i\rho(e'^*)k_{e'}) + b_{e'^*}(-i\rho(e'^*)j_{e'}) \\
&= \left(\frac{i\alpha(e'^*) + \rho(e')\alpha(e')}{2ik_{e'}}\right)(i\rho(e'^*)k_{e'}) + \left(\frac{i\alpha(e'^*) - \rho(e')\alpha(e')}{2ij_{e'}}\right)(-i\rho(e'^*)j_{e'}) \\
&= \frac{i\rho(e'^*)\alpha(e'^*) + \alpha(e')}{2} - \frac{i\rho(e'^*)\alpha(e'^*) - \alpha(e')}{2} \\
&= \alpha(e').
\end{aligned}$$

Hence, every $\alpha \in C^1(\Lambda)$ can be written as a linear combination of ζ_e's and $\bar{\zeta}_e$'s. $\{\zeta_e\} \cup \{\bar{\zeta}_e\}$ spans $C^1(\Lambda)$ and is therefore a basis for $C^1(\Lambda)$.

□

The assertion that $\{\zeta_e\} \cup \{\bar{\zeta}_e\}$ is a basis of $C^1(\Lambda)$ leads to the conclusion that we may write the space of discrete differential 1-forms on Λ as a direct sum of $*$'s eigenspaces.

$$C^1(\Lambda) = C^{(1,0)}(\Lambda) \oplus C^{(0,1)}(\Lambda).$$

The associated projections with this direct sum are as follows:

$$\begin{aligned}
\pi_{(1,0)} &= \frac{1}{2}(Id + i*) : C^1(\Lambda) \to C^{(1,0)}(\Lambda), \\
\pi_{(0,1)} &= \frac{1}{2}(Id - i*) : C^1(\Lambda) \to C^{(0,1)}(\Lambda).
\end{aligned}$$

Thus, for $\alpha \in C^1(\Lambda)$ with $\alpha = \sum_{e\in\Gamma_1} a_e\zeta_e + b_e\bar{\zeta}_e$, under these projections we have

$$\pi_{(1,0)} \circ \alpha = \frac{1}{2}\Bigg(\sum_{e\in\Gamma_1} a_e\zeta_e + b_e\bar{\zeta}_e + i\big(\sum_{e\in\Gamma_1} a_e(-i\zeta_e) + b_e(i\bar{\zeta}_e)\big)\Bigg) = \sum_{e\in\Gamma_1} a_e\zeta_e.$$

And similarly,

$$\pi_{(0,1)} \circ \alpha = \sum_{e \in \Gamma_1} b_e \bar{\zeta}_e.$$

We may conclude that

$$\alpha \in C^{(1,0)}(\Lambda) \iff \alpha = \textstyle\sum_{e \in \Gamma_1} a_e \zeta_e \text{ and } \alpha \in C^{(0,1)} \iff \alpha = \textstyle\sum_{e \in \Gamma_1} b_e \bar{\zeta}_e.$$

Recall that in the continuous case, $df = \frac{\partial f}{\partial z} dz + \frac{\partial}{\partial \bar{z}} d\bar{z}$ where f is a differentiable complex-valued function. The (1,0) projection of df is simply $\frac{\partial f}{\partial z} dz$, and the (0,1) projection is $\frac{\partial}{\partial \bar{z}} d\bar{z}$. This stems from our earlier result that dz and $d\bar{z}$ are of type (1,0) and (0,1) respectively. We may analogously define the (1,0) and (0,1) projections of the coboundary d in the discrete case.

Definition 1.5.7. Let $d : C^0(\Lambda) \to C^1(\Lambda)$ be the coboundary on the double Λ. Then we define the **(1,0) and (0,1) projections of d** as follows.

$$d' := \pi_{(1,0)} \circ d, \qquad d'' := \pi_{(0,1)} \circ d$$

Theorem 1.5.8. *Let $d : C^0(\Lambda) \to C^1(\Lambda)$ be the coboundary. Then $d = d' + d''$.*

Proof. This result is a direct consequence of $C^1(\Lambda) = C^{(1,0)}(\Lambda) \oplus C^{(0,1)}(\Lambda)$ and the definition of d' and d''.

□

The development of ζ and $\bar{\zeta}$ allow us to establish another correspondence between the continuous and discrete theory. As noted earlier, in the continuous case we have $\frac{\partial f}{\partial \bar{z}} = 0 \iff \frac{\partial f}{\partial \bar{z}} d\bar{z} = 0 \iff i\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y}$. The same is true in the discrete case.

Theorem 1.5.9. *$\pi_{(0,1)} \circ df(e) = 0$ for every edge $e \in \Lambda_1 \iff f$ is discrete analytic.*

Proof. Let $e = (x, x') \in \Lambda_1$ be arbitrary with dual edge $e^* = (y, y')$. Then $\pi_{(0,1)} \circ df(e) = 0 \iff *df(e) = -idf(e) \iff -\rho(e^*)df(e^*) = -idf(e) \iff -\frac{\ell(x,x')}{\ell(y,y')}(f(y') - f(y)) = -i(f(x') - f(x)) \iff \frac{i(f(x) - f(x'))}{\ell(x,x')} = \frac{f(y) - f(y')}{\ell(y,y')} \iff f$ is discrete analytic.

□

Let $\zeta_e = dz_e$. We have already established that a generic 1-form of type (1,0) has the form $\sum_{e \in \Gamma_1} a_e \zeta_e$. What happens if we multiply such a 1-form by a function? Will it still be of type (1,0)? We know that the answer to this question in the continuous case is yes independent of our choice of function. However, because function and 1-form multiplication does not behave as nicely in the discrete case, we must introduce some restrictions on our function in order to guarantee that its product with a type (1,0) 1-form will remain of that same type.

Proposition 1.5.10. *Suppose $\alpha = f \sum_{e \in \Gamma_1} a_e \zeta_e$ where $f \in C^0(\Lambda)$. Then $\alpha \in C^{(1,0)}(\Lambda) \iff f$ has equal averages at each crossing of a regular edge and dual edge.*

Proof. Let $e' = (x, x')$ and $e'^* = (y, y')$. Then

$$\begin{aligned}
\alpha \in C^{(1,0)}(\Lambda) &\iff *\alpha(e') = -i\alpha(e') \\
&\iff -\rho(e'^*)\frac{f(y)+f(y')}{2}\sum_{e\in\Gamma_1} a_e\zeta_e(e'^*) = -i\frac{f(x)+f(x')}{2}\sum_{e\in\Gamma_1} a_e\zeta_e(e') \\
&\iff \frac{f(x)+f(x')}{2} = \frac{f(y)+f(y')}{2},
\end{aligned}$$

since

$$-i\sum_{e\in\Gamma_1} a_e\zeta_e(e') = -\rho(e'^*)\sum_{e\in\Gamma_1} a_e\zeta_e(e'^*) \text{ by } \sum_{e\in\Gamma_1} a_e\zeta_e \text{ of type (1,0).}$$

□

Proposition 1.5.11. *Suppose* $\alpha = f\sum_{e\in\Gamma_1} a_e\zeta_e$ *where* $f \in C^0(\Lambda)$*. Then* $\alpha \in C^{(0,1)}(\Lambda) \iff f$ *has averages summing to zero at each regular edge, dual edge crossing.*

Proof. Let $e' = (x, x')$ and $e'^* = (y, y')$. Then

$$\begin{aligned}
\alpha \in C^{(0,1)}(\Lambda) &\iff *\alpha(e') = i\alpha(e') \\
&\iff -\rho(e'^*)\frac{f(y)+f(y')}{2}\sum_{e\in\Gamma_1} a_e\zeta_e(e'^*) = i\frac{f(x)+f(x')}{2}\sum_{e\in\Gamma_1} a_e\zeta_e(e') \\
&\iff \frac{f(x)+f(x')}{2} = -\frac{f(y)+f(y')}{2},
\end{aligned}$$

since

$$-i\sum_{e\in\Gamma_1} a_e\zeta_e(e') = -\rho(e'^*)\sum_{e\in\Gamma_1} a_e\zeta_e(e'^*) \text{ by } \sum_{e\in\Gamma_1} a_e\zeta_e \text{ of type (1,0).}$$

□

Proposition 1.5.12. *Suppose* $\alpha = f \sum_{e \in \Gamma_1} b_e \bar{\zeta}_e$ *where* $f \in C^0(\Lambda)$*. Then* $\alpha \in C^{(0,1)}(\Lambda) \iff f$ *has equal averages at each regular edge, dual edge crossing.*

Proposition 1.5.13. *Suppose* $\alpha = f \sum_{e \in \Gamma_1} b_e \bar{\zeta}_e$ *where* $f \in C^0(\Lambda)$*. Then* $\alpha \in C^{(1,0)}(\Lambda) \iff f$ *has averages summing to zero at each regular edge, dual edge crossing.*

Proofs similar to those given for function multiplication with a 1-form of type (1,0) show these last two claims are also true.

1.6 Holomorphic 1-Forms

We have already developed the notion of a discrete analytic function. Now, because the driving force of the discrete theory is integration, we wish to establish the notion of a holomorphic discrete 1-form. Recall that in the differentiable case we have, for any function f, fdz is of type (1,0). Further, fdz is holomorphic $\iff$ f is analytic $\iff$ $\dfrac{\partial f}{\partial \bar{z}} = 0 \iff d(fdz) = 0$, i.e. fdz is closed. It is this continuous property that we use as the motivation for the definition of a holomorphic discrete 1-form.

Definition 1.6.1. A 1-form $\alpha \in C^1(\Lambda)$ is **holomorphic** if it is closed and of type (1,0). Thus, if $\Omega^1(\Lambda)$ is the space of all holomorphic 1-forms, $\alpha \in \Omega^1(\Lambda) \iff d\alpha = 0$ and $*\alpha(e) = -i\alpha(e)\ \forall e \in \Lambda_1$.

Since we have used the equivalence of $f dz$ holomorphic and f analytic as the foundation of our definition of a holomorphic 1-form, it is interesting to identify the functions f for which the analogous relationship holds in the discrete setting. To simplify our investigation, assume that Λ is a cell complex of squares on the complex plane with edges of length one parallel to the coordinate axes. Consider the type (1,0) 1-form $\alpha = f \sum_{e \in \Gamma_1} \zeta_e$ for some $f \in C^0(\Lambda)$ with equal averages at each crossing, so as to guarantee that α is indeed type (1,0). In defining ζ_e choose constants so that ζ_e is equal to the discrete 1-form dz_e. So $\alpha = f \sum_{e \in \Gamma_1} dz_e = f dz$. Then α is holomorphic $\iff d(f dz) = 0$, by definition of a holomorphic 1-form. Since we have yet to define a working wedge product of 1-forms, applying the product rule to $d(f dz)$ yields no further information. Instead, let us consider the calculation directly on a given face $F \in \Lambda_2$ with bounding edges $(x_1, x_2), (x_2, x_3), (x_3, x_4)$, and (x_4, x_1). For simplicity, we'll refer to the regular edges as e_1, e_2, e_3, and e_4, respectively. Their dual edges are $(x_1', x^*), (x_2', x^*), (x_3', x^*)$, and (x_4', x^*), respectively (see Figure 1.5).

Proposition 1.6.2. *Under the assumptions given above, suppose $\alpha = f dz$ for $f \in C^0(\Lambda)$ with equal averages at each crossing in Λ. Let $F \in \Lambda_2$ be an arbitrary face shown in Figure 1.5. Then α is holomorphic $\iff i\big[f(x_1') - f(x_3')\big] = f(x_2') - f(x_4')$.*

Proof. We need to show that α is closed on each face $F \in \Lambda_2$, i.e. $d(f dz)(F) = 0$ for all

$F \in \Lambda_2$. We have $d(fdz)(F) = 0$

$$\begin{aligned}
&\iff (fdz)(\partial F) = 0 \\
&\iff (fdz)(e_1) + (fdz)(e_2) + (fdz)(e_3) + (fdz)(e_4) = 0 \\
&\iff \frac{f(x_1) + f(x_2)}{2}(i) + \frac{f(x_2) + f(x_3)}{2}(-1) + \frac{f(x_3) + f(x_4)}{2}(-i) + \frac{f(x_4) + f(x_1)}{2}(1) = 0.
\end{aligned}$$

Now, since we have assumed that f has equal averages at each crossing, we may substitute the dual averages into our equation. Hence, we have

$$\begin{aligned}
d(fdz)(F) = 0 \quad &\iff \quad i\frac{f(x^*) + f(x_1')}{2} - \frac{f(x^*) + f(x_2')}{2} - i\frac{f(x^*) + f(x_3')}{2} + \frac{f(x^*) + f(x_4')}{2} = 0 \\
&\iff \quad \frac{1}{2}\big[if(x_1') - f(x_2') - if(x_3') + f(x_4')\big] = 0 \\
&\iff \quad i\big[f(x_1') - f(x_3')\big] = f(x_2') - f(x_4').
\end{aligned}$$

□

This last equality has a strange resemblance of the discrete Cauchy-Riemann equation. If we were to ignore the vertex x^* in the center of our face F, then the function values involved in the similar equation look as if they are derived from a crossing. However, these edges are all dual and thus it is not the defining crossing necessary to satisfy the Cauchy-Riemann equation. Nonetheless, we have discovered an interesting result that suggests to a relationship of holomorphic 1-forms and analytic functions on a broader cell decomposition. Without the development of a wedge product of discrete 1-forms, it seems that this is the best direct approach we can offer for now.

Despite our apparent failure to obtain a discrete relationship between holomorphic 1-forms

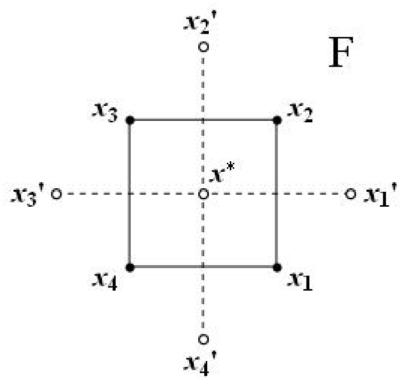

Figure 1.5: A face $F \in \Lambda_2$

and analyticity completely analogous to that of the continuous case, the above proposition offers a suggestion for a different approach in developing such a relationship. Because we have found a resemblance of the Cauchy-Riemann equation, the result of Proposition 1.6.2 points to the idea that we can possibly recover analyticity by shifting our cell decomposition around in some way.

Definition 1.6.3. For a square cell complex Λ with horizontal and vertical edges, a **vertical half-shift** is a new complex rendered by sliding each horizontal edge $e \in \Gamma_1$ up so that its new position horizontally bisects the face it originally bounded below (see Figure 1.6).

Let e be an edge in such a complex Λ. After a vertical half-shift of Λ, a vertex will be located at the midpoint of the original location of e (see Figure 1.6). In fact, every vertex in the half-shift is a midpoint of an edge in the original complex. Let α be an arbitrary 1-form in $C^{(1,0)}(\Lambda)$. Then we may write $\alpha = \sum_{e \in \Gamma_1} a_e dz_e$, as shown earlier, where a_e is a constant belonging to the collection $\mathbb{A}$. We define an associated function on the vertices of the vertical half-shift of Λ as follows. Let x be a vertex in the shifted Λ. Then x is the midpoint of an edge $e \in \Lambda_1$. Let $f(x) = a_e$.

Definition 1.6.4. Let Λ be a square cell complex with horizontal and vertical edges of length one, and let $\alpha \in C^{(1,0)}(\Lambda)$. Then we say that the function f on the set of vertices of the shifted complex as defined above is the **function associated with α under a vertical half-shift**.

Proposition 1.6.5. *Let Λ be a square cell decomposition with edges of length one parallel to the coordinate axes. Let $\alpha \in C^{(1,0)}(\Lambda)$. Then α is holomorphic $\iff$ its associated function under a vertical half-shift is discrete analytic on the shifted complex.*

Proof. Let Λ' denote the complex rendered from a vertical half-shift of Λ. Let $\alpha \in C^{(1,0)}(\Lambda)$. Then $\alpha = \sum_{e \in \Gamma_1} a_e dz_e$ where a_e is a constant belonging to the collection $\mathbb{A}$. Consider an arbitrary face $F \in \Lambda_2$ with bounding edges $e_1, e_2, e_3, e_4 \in \Lambda_1$. The case $F \in \Gamma_2$ is pictured in Figure 1.6. Thinking of a similar picture with the roles of Γ and Γ^* reversed, we can use the same notation for the case $F \in \Gamma_2^*$. For simplicity, we'll denote a_{e_i} as a_i. Then, $\alpha(e_1) = ia_1$, $\alpha(e_2) = -a_2$, $\alpha(e_3) = -ia_3$, and $\alpha(e_4) = a_4$. For each i, $1 \le i \le 4$, let x_i be the vertex in Λ_0' located at the midpoint of each respective edge e_i in the unshifted Λ. Then α's associated function f under this vertical half-shift is such that $f(x_i) = a_i$, by definition. Thus, α is closed on F $\iff d\alpha(F) = \alpha(\partial F) = 0 \iff ia_1 - a_2 - ia_3 + a_4 = 0 \iff i(a_1 - a_3) = a_2 - a_4 \iff i\big(f(x_1) - f(x_3)\big) = f(x_2) - f(x_4)$. When $F \in \Gamma_2$ so that (x_3, x_1) is the regular edge and (x_4, x_2) is the dual edge, this is the Cauchy-Riemann equation at the crossing of these edges. When $F \in \Gamma_2^*$, we can multiply the final equation

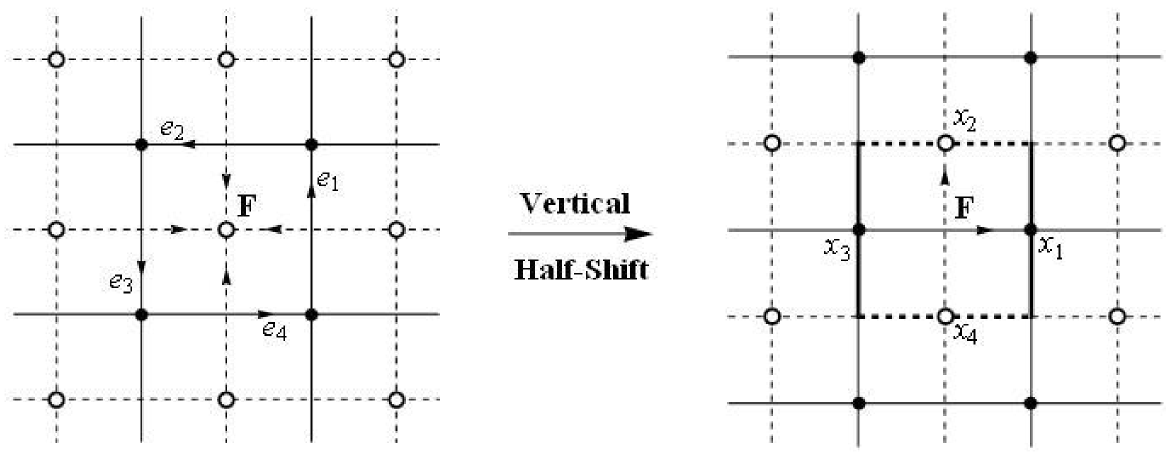

Figure 1.6: A vertical half-shift of Λ

by i to get $f(x_3) - f(x_1) = i\big(f(x_2) - f(x_4)\big)$. Because the oriented regular edge (x_2, x_4) has oriented dual edge (x_3, x_1), this is the Cauchy-Riemann equation at this crossing. Because all crossings of the vertical half-shift lie within a face $F \in \Lambda_2$, this argument shows that α is holomorphic $\iff$ its associated function f under a vertical half-shift is discrete analytic on the shifted complex.

□

Thus, we see that Proposition 1.6.2 and Proposition 1.6.5 are related in that both recover a relationship of holomorphic 1-forms and analytic functions via a change in cell complex. The former deals with the product fdz and merely suggests changing the complex, while the latter looks at a general (1,0)-form and specifies the relevant change of complex.

1.7 Meromorphic 1-Forms

Much of the function theory in complex analysis relies on analyticity. However, many functions fail to have this property. Consequently, analysts have categorized singularities of functions and developed an alternative way to classify functions. In the continuous case, the concept of a meromorphic function allows for a plethora of theoretical results to be extended to a larger class of functions. The significance of this result is far-reaching in complex analysis. Therefore, creating a definition of a discrete meromorphic function is a worthy goal. Recall the traditional definition of a meromorphic function.

Definition 1.7.1. A function $f(z)$ is **meromorphic** on a domain D if $f(z)$ is analytic on D except possibly at isolated singularities, each of which is a pole. Or alternatively, $f(z)$ is meromorphic on D with a pole of order N at $z_0 \iff$ on some neighborhood of z_0, $g(z) = (z - z_0)^N f(z)$ is analytic but $(z - z_0)^{N-1} f(z)$ is not.

Consider the following analogous definition for a discrete meromorphic function.

Definition 1.7.2. A function $f \in C^0(\Lambda)$ is **discrete meromorphic** with a pole of order N at a vertex $z_0 \iff g(z) = (z - z_0)^N f(z)$ is discrete analytic and $(z - z_0)^{N-1} f(z)$ is not.

In the continuous case, this definition makes sense because $\dfrac{1}{(z - z_0)^N}$ is the prototypical meromorphic function with a pole at $z = z_0$. However, in the discrete setting, $f(z) = \dfrac{1}{(z - z_0)^N}$ does not behave as expected. Consider the following example.

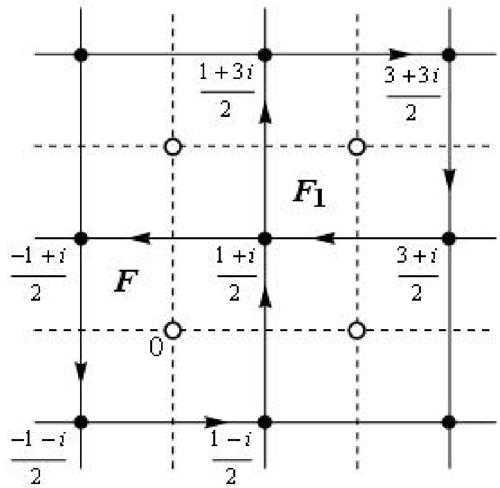

Figure 1.7: The square complex Λ with pole at 0.

Example 1.7.3. Let Λ be the square cell complex on $\mathbb{C}$ as shown in Figure 1.7. Let $f(z) = \frac{1}{z}$. In the continuous plane, we know that f is meromorphic with a pole at $z = 0$ with residue 1. Also, outside a neighborhood of 0, f is analytic. Since f qualifies as a discrete meromorphic function, we would expect that $\alpha = f(z)dz$ would be meromorphic on the face $F \in \Lambda_2$ with 0 at its center, and that f would satisfy the Cauchy-Riemann equation at every regular, dual edge crossing that does not involve an edge contained in the boundary of 0^*. However, $Res_0(\alpha) = \frac{1}{2\pi i}\oint_{\partial F} \alpha = \frac{2}{\pi} \neq 1$. Furthermore, if we consider the regular, dual edge crossing involving the dual edge $(1, 1+i)$ it is easily shown that $\frac{1}{z}$ fails to satisfy the Cauchy-Riemann equation at this crossing. Thus, we have seen that f and α behave far differently than expected.

It is clear from the above example, that the definition given for a discrete meromorphic function is not a good one. Since $\frac{1}{(z-z_0)^N}$ does not translate its prototypical meromorphic properties to the discrete setting, its significance is lost on the double.

We can now see the difficulty in defining a meromorphic function. However, our discrete function theory would benefit greatly from the ability to recover meromorphic theory. So, since our guideline for the development of the discrete setting is based on integration, we turn to the possibility of obtaining meromorphic 1-forms, instead of meromorphic functions.

In the continuous case, a meromorphic 1-form is of the form fdz, where f is a meromorphic function. Suppose f has a pole of order N at z_0. By definition, there exists N such that $(z-z_0)^N f(z)$ is analytic. Moreover, $(z-z_0)^N f(z)dz$ is closed. Using this knowledge, we attempt to introduce an analogous discrete definition here.

Definition 1.7.4. A differential 1-form $\alpha \in C^{(1,0)}(\Lambda)$ is **discrete meromorphic** with a pole at a vertex $x \in \Lambda_0$ if there exists an analytic $g(x)$ such that $g(x)\alpha$ is closed on the face $x^* \in \Lambda_2$.

The motivation behind the definition of meromorphic in the continuous setting is the necessity to"cancel" the pole through multiplication. However, we have no discrete analogue of higher order zeros, and the product of discrete functions and 1-forms does not behave well. Recall that our definition of this multiplication was chosen to force the coboundary d to be a derivation with respect to function multiplication. Because the definition involves an averaging of the function against the 1-form evaluated on an edge, it will not satisfy $(fg)\alpha = f(g\alpha)$. There appears to be no satisfactory way to define meromorphic 1-forms by "canceling" poles.

It seems that we need to rely on integration to define meromorphic 1-forms. Given below is a final and sufficient attempt at a definition.

Definition 1.7.5. A discrete differential 1-form $\alpha \in C^{(1,0)}(\Lambda)$ is **meromorphic** with a pole at a vertex $x \in \Lambda_0$ if it is not closed on the face $x^* \in \Lambda_2$. The pole at x has **discrete residue** defined by

$$Res_x(\alpha) := \frac{1}{2\pi i} \oint_{\partial x^*} \alpha$$

The definition of the discrete residue of a 1-form is the equivalent version of the continuous definition of the residue of a function.

Definition 1.7.6. The **residue** of $f(z)$ at z_0 is $Res_{z_0}(f) = \frac{1}{2\pi i} \oint_{|z-z_0|=r} f(z)dz$

This definition allows us to recover the definition of the residue of a pole. It also has a direct connection with holomorphicity. However, this meromorphic categorization is far more general than in the continuous case. It states that if a 1-form of type (1,0) is not closed on a face, then it has a pole in its center. In other words, if a type $(1,0)$ 1-form is not holomorphic, it is meromorphic. We know that this is of course not true in the continuous case. Consider $f(z) = e^{1/z}$, for example. fdz is type $(1,0)$, it is not holomorphic, and f is certainly not meromorphic as it has an essential singularity at 0. Another such example is $z\bar{z}dz$. It is of type (1,0), not holomorphic, not meromorphic, and it even has no singularities. We would like for our discrete interpretation to parallel the continuous case as closely as a possible. Unfortunately, from the above exposition, it seems that this is the best we can do for now.

Despite the drawback of its seeming overgeneralization, the definition of a meromorphic 1-form does follow the guideline of relying on integration. In fact, this definition ultimately plays a very important role in the development of the discrete Cauchy Integral Formula, which we will investigate later in this paper. For now, we would like to develop a discrete analogue of the prototypical continuous meromorphic 1-form, $\dfrac{dz}{z - z_0}$. To make sense of its existence, we first state a definition and a result from [9]. Note that vertices are called neighbors if they are joined by an edge.

Definition 1.7.7. A function $f \in C^0(\Lambda)$ is **discrete harmonic** at a vertex x with neighboring vertices $x_1, \ldots, x_n$ if

$$(\Delta f)(x) = \sum_{k=1}^{n} \rho(x, x_k)(f(x) - f(x_k)) = 0.$$

Δ is the **discrete Laplacian**.

Recall that $\rho(e) = \frac{\ell(e^*)}{\ell(e)}$. Note that as in the continuous case, it can be easily checked that $\Delta f = - * d * df$. Furthermore, this definition also has the property that $(\Delta f)(x) = 0 \Rightarrow f(x) = \dfrac{1}{n} \displaystyle\sum_{k=1}^{n} f(x_k)$ on a complex in which every edge has the same length.. Therefore, for such complexes, the value of a harmonic function at a vertex x is the average of its values at neighboring vertices. This is analogous to the continuous case, where we know that a harmonic function value at the center of a disk is the average value around the boundary.

Definition 1.7.8. Let Λ be a bounded double complex and let x be a vertex on the interior of Λ. We say that Λ is **boundary accessible** with respect to x if for every interior vertex

$v \in \Lambda_0 \backslash \{x\}$ there exists a path of edges that runs from v to a boundary vertex which misses x and contains no boundary vertex other than its endpoint.

Proposition 1.7.9. (The Dirichlet Problem.) *Let Λ be a finite, bounded cell complex that is boundary accessible with respect to $x \in \Gamma_0$. Let $f \in C^0(\Gamma)$ such that $f(x) = 1$ and $f(v) = 0$ for all vertices v on the boundary of Γ, $\partial\Gamma$. Then there exists a function $a \in C^0(\Gamma)$ satisfying these boundary conditions such that a is harmonic on $\Gamma_0 \backslash (\{x\} \cup \{v \in \partial\Gamma\})$.*

Note: A similar result is true for Γ^*.

Proof. The set of $\mathbb{R}$-valued functions on Γ is a finite-dimensional vector space with coordinates $u_i = f(x_i)$ representing the values of the functions at vertices $x_i \in \Gamma_0$. Fixing the values at certain x_ℓ's, namely at the vertices of the boundary and at x, amounts to imposing conditions $u_\ell = c_\ell$ for a subset $L = \{\ell : x_\ell \in \{x\} \cup \partial\Gamma\}$ of $I = \{i : x_i \in \Gamma_0\}$. With the conditions imposed, we have an affine subspace of the vector space. This affine subspace can be represented in coordinates by u_k's for $k \in I \backslash L$. Let q be the differentiable function

$$q(\vec{u}) = \sum_{(x_i, x_j) \in \Lambda_1} \frac{1}{2} \rho(x_i, x_j)(u_i - u_j)^2$$

Note here that the orientation of each edge (x_i, x_j) is irrelevant.

On both sides of this equation, if i, respectively j, is equal to some $\ell \in L$ we replace u_i, respectively u_j, by the real constant c_ℓ, in order to restrict our attention to the affine subspace of functions satisfying the boundary conditions. Hence, q is a function of the variables u_i for $i \in I \backslash L$. In taking the gradient, we use these variables. Note that the k^{th} coordinate of ∇q

is

$$\sum_{x_j} \rho(x_k, x_j)(u_k - u_j)$$

where the x_j's are vertices in Λ_0 adjacent to x_k.

Thus, we see that $\nabla q(\vec{v}) = \vec{0} \iff$ the function f represented by its values $\vec{v}$ is harmonic. Hence, to show that a harmonic function exists, it suffices to show that the nonnegative-valued q has a minimum, because $\nabla q = \vec{0}$ at the minimum.

First, note that for the function whose every value not determined by the Dirichlet conditions is zero, q assumes the value $\displaystyle\sum_{x_j} \frac{1}{2}\rho(x, x_j)$, where x_j is adjacent to x. Call this value C. We will exibit a compact subset K of the affine subspace such that $q \geq C+1$ on the complement of the interior of K. This will show that q's minimum is interior to K, hence providing the desired critical point for q.

By Λ boundary accessible with respect to x, for each vertex $x_i \in \Gamma_0 \backslash \big(\{x\} \cup \partial\Gamma\big)$, we may choose a path of edges that misses x, and that runs from x_i to a boundary point, containing no boundary point other than its endpoint. Let $\mathcal{L}$ be the largest number of edges appearing in any of these paths. Let p be the minimum value that ρ attains on an edge in Γ_1. Choose $M \geq \left(\dfrac{2\mathcal{L}^2(C+1)}{p}\right)^{1/2}$. Define K by $|u_i| \leq M \; \forall i \in I \backslash L$. Then K is a compact subset of the affine subspace. Suppose $\vec{u}$ is not in the interior of K. Then there exists u_i such that $|u_i| \geq M$. Therefore, there is an edge along the path between x_i and the boundary point used in calculating $\mathcal{L}$, namely (u_j, u_k), that must be such that $|u_j - u_k| \geq \dfrac{M}{\mathcal{L}}$. Note that j or k may be equal to i, and there may be more than one such edge. Then we have the following.

$$\begin{aligned} q(\vec{u}) &\geq \frac{1}{2}\rho(x_j, x_k)\left(\frac{M}{\mathcal{L}}\right)^2 \\ &\geq \frac{1}{2}\rho(x_j, x_k)\left(\frac{2\mathcal{L}^2(C+1)}{\mathcal{L}^2 p}\right) \\ &\geq C+1. \end{aligned}$$

Thus, q achieves its minimum on the interior of K at some $\vec{v}$ with coordinates v_i. The solution to the Dirichlet problem is the discrete function a such that $a(x_i) = v_i$ for all $x_i \in \Gamma_0$.

□

We are now ready to prove the existence of a discrete analogue of the continuous meromorphic 1-form $\dfrac{dz}{z - z_0}$ on the double complex.

Proposition 1.7.10. *Let Λ be a bounded double complex that is boundary accessible with respect to $x \in \Lambda_0$. Then there exists a meromorphic 1-form μ_x with a single pole at x with residue $+1$.*

Proof. WLOG, suppose $x \in \Gamma_0$. Let B denote the boundary of a finite subcollection, or subcomplex, of faces that contains x^*. Consider the Dirichlet problem where $f(v) = 0$ for $v \in B$ and $f(x) = 1$. Then there exists a harmonic function a that solves the discrete Dirichlet problem with this boundary condition. Let $\mu_x = da$ on Γ_1 and $\mu_x = -i * da$ on Γ_1^*.

Then μ_x is type (1,0). Furthermore, for all $v \in \Gamma_0 \backslash \{x\}$ with neighboring vertices $v_1, \ldots, v_n$,

$$\begin{aligned}\iint_{v^*} d\mu_x &= -i \oint_{\partial v^*} *da \\ &= -i \sum_{k=1}^{n} \rho(v, v_k) \int_{(v,v_k)} da \\ &= -i \sum_{k=1}^{n} \rho(v, v_k)[a(v_k) - a(v)] \\ &= 0 \quad \text{since } a \text{ is harmonic.}\end{aligned}$$

Hence, μ_x is closed on Γ^*. Also, $\mu_x = da$ is closed on Γ. This follows directly from the fact that for a given face $F \in \Gamma_2$, $d \circ da(F) = da(\partial F) == a(\partial(\partial F)) = 0$ by by properties of the boundary. To show that $d\mu_x$ is nonzero on x^* , it suffices to show that $da(v_i, x) > 0$ for each (v_i, x), where v_i is a neighboring vertex of x. This is sufficient because μ_x is type (1,0) and hence

$$\oint_{\partial x^*} \mu_x = \oint_{\partial x^*} -i * da = \sum_{i=1}^{n} da(v_i, x).$$

Note that $da(v_i, x) > 0 \iff a(x) - a(v_i) > 0 \iff a(x) > a(v_i)$ for a neighboring vertex v_i. Therefore, we want to show that a achieves its maximum at the vertex x. Suppose that a attains its maximum at a vertex v where a is harmonic. Note that $v(a) \geq 1$ since a is the solution to the Dirichlet problem where $f(x) = 1$. By assumption, every vertex in Γ_0 and the chosen subcomplex is connected to a boundary vertex by a path of edges in the subcomplex that does not pass through x. By a harmonic at v, all of the neighboring vertices of v must also have the same value. We may propagate this argument along the path through the vertices where a is harmonic that leads to a boundary vertex. But, $a \equiv 0$ on the boundary and we have assumed $a(v) \geq 1$. Contradiction. Hence, a does not achieve its maximum at any point at which a is harmonic and therefore at no neighboring vertex of x. So, $a(x) > a(v_i)$ for all neighboring v_i and $d\mu_x$ is nonzero on x^*. Therefore, an appropriate scaling of μ_x is a meromorphic 1-form with a single pole of residue +1 at x.

□

As stated before, u_x is the analogue of $\dfrac{dz}{z - z_0}$ as it has a single pole of residue +1 located at $z_0 = x$.

Exhibiting an example of such a μ_x can be messy. Its existence depends on the Dirichlet problem. Finding such an a depends on solving a system of linear equations representing the harmonic condition. Therefore, we have the following proposition, which will be useful in an investigation we pursue later on in this paper. This proposition and the following example show that it is easier to construct a discrete analogue of a meromorphic 1-form with a pole at x if one relaxes the assumption that the form be closed away from x^*, and one requires only that, away from x^*, it be closed on Γ (if the pole is in Γ_0^*) or closed on Γ^* (if the pole is in Γ_0).

Proposition 1.7.11. *Let Λ be a square double. Suppose $x \in \Gamma_0^*$ (or $x \in \Gamma_0$). There exists a 1-form λ_x of type (1,0) that is closed on $\Gamma_2 \backslash \{x^*\}$ (or $\Gamma_2^* \backslash \{x^*\}$, respectively), with $\oint_{x^*} \lambda_x \neq 0$. In other words, λ_x is meromorphic with respect to Γ_2 (or Γ_2^*, respectively) with a single pole at the vertex x of residue +1.*

Proof. It suffices to exhibit an example.

□

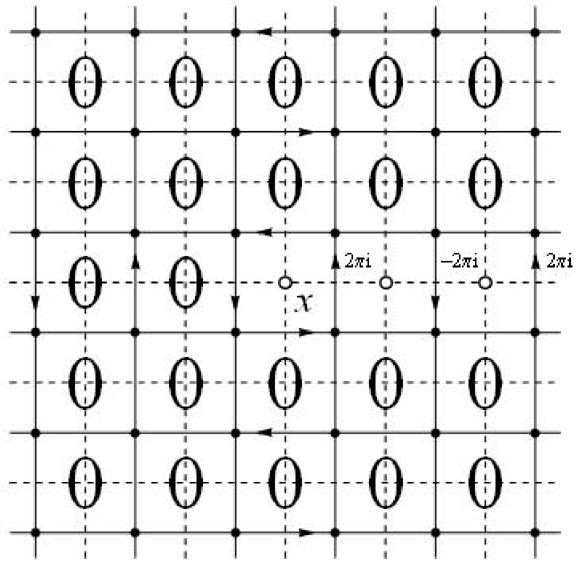

Figure 1.8: The values of the meromorphic 1-form λ_x in Example 1.7.12.

Example 1.7.12. Let Λ be a complex of squares and $x \in \Lambda_0^*$. Let λ_x have values on each edge in Γ_1 as shown in Figure 1.8. λ_x is zero on all edges bounding faces labeled with a zero. Assign values to the dual edges e^* in Γ_1^* by $\lambda_x(e^*) = i\rho(e)\lambda_x(e)$. Then, $*\lambda_x(e) = -i\lambda_x(e)$ and λ_x is of type (1,0). One can easily check that λ_x satisfies the necessary conditions of Proposition 1.7.11.

1.8 Defining a Wedge Product

Previously, we briefly noted the difficulty in defining a wedge product of 1-forms on the double Λ. We are now ready to explore this claim. Let $\alpha \in C^1(\Lambda)$ and $f \in C^0(\Lambda)$. We want the coboundary d to be a derivation with respect to function and 1-form multiplication.

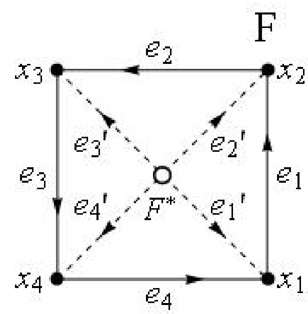

Figure 1.9: The face $F \in \Lambda_2$ on which the wedge product is evaluated.

Therefore, we must define a wedge product such that the following equation holds.

$$d(f \cdot \alpha) = f \cdot d\alpha + df \wedge \alpha$$

Since we have already defined how to evaluate $d(f \cdot \alpha)$ and $f \cdot d\alpha$ on a face $F \in \Lambda_2$, this forces our definition of the wedge product $df \wedge \alpha$. Therefore, if we assume Λ is a cell decomposition of squares for simplicity, we have, in the notation of Figure 1.9,

$$\begin{aligned}
df \wedge \alpha(F) &= d(f \cdot \alpha)(F) - f \cdot d\alpha(F) \\
&= (f \cdot \alpha)(\partial F) - f(F^*)\alpha(\partial F) \\
&= \frac{f(x_1) + f(x_2)}{2}\alpha(e_1) + \cdots + \frac{f(x_4) + f(x_1)}{2}\alpha(e_4) - f(F^*)\big[\alpha(e_1) + \cdots + \alpha(e_4)\big] \\
&= \frac{f(x_1) + f(x_2) - 2f(F^*)}{2}\alpha(e_1) + \cdots + \frac{f(x_4) + f(x_1) - 2f(F^*)}{2}\alpha(e_4) \\
&= \frac{[f(x_1) - f(F^*)] + [f(x_2) - f(F^*)]}{2}\alpha(e_1) + \cdots + \frac{[f(x_4) - f(F^*)] + [f(x_1) - f(F^*)]}{2}\alpha e_4 \\
&= \frac{1}{2}[df(e_1') + df(e_2')]\alpha(e_1) + \cdots + \frac{1}{2}[df(e_4') + df(e_1')]\alpha(e_4).
\end{aligned}$$

Note that this wedge product elicits a 2-form on our square cell complex. So we see that we can define the wedge product of the form $df \wedge \alpha$, but, because $e_i' \notin \Lambda_1$, there is no obvious way to define $\beta \wedge \alpha$ for $\beta \in C^1(\Lambda)$ not of the form df. Therefore, it seems that we cannot generalize the wedge product $\beta \wedge \alpha$ beyond the specific case when $\beta = df$. This hindrance is because our the product rule requires the 1-form df to be evaluated on edges that are not in Λ_1 (Figure 1.9). Instead, these edges belong to a new cell complex, the diamond $\Diamond$, that can be derived from any double cell decomposition Λ.

Chapter 2

The Diamond $\Diamond$

The diamond complex is a new cell complex that is constructed from the double complex. Each double complex has an associated diamond complex. While the double complex permits the definition of discrete analytic functions and of a Hodge star operator, it does not admit a complete theory of exterior multiplication, i.e. of wedge products. In contrast, the diamond complex allows for the definition of three wedge products: one of 1-forms on the diamond, one of 1-forms on the double and one of a 1-form on the diamond and a 1-form on the double. The first two wedge products define a 2-form in the diamond. The last wedge product, namely the mixed wedge product, is an original construction and yields a 2-form on the double. The mixed wedge product is the multiplication of 1-forms that we were in search of in the last section of Chapter 1. It will be of great importance in the development of an original Cauchy Integral Formula on the double complex in Chapter 3.

We will also see in the third chapter that a strong form of a discrete Cauchy Integral Formula arises from the combination of a double complex and its associated diamond complex. The double complex contributes the Cauchy-Riemann equation, hence analytic functions, and the Hodge decomposition. The diamond complex provides a wedge product of 1-forms that satisfies the following property: the wedge product of 1-forms belonging to the same category in the Hodge decomposition is zero. No one complex provides all of the ingredients for the strongest form of the discrete Cauchy Integral Formula.

In this chapter, we develop the discrete theory on the diamond. We define diamond k-forms and their associated operations. We also introduce the averaging map, a way to move from k-forms on the diamond to k-forms on the double. This map plays a vital role in utilizing the strengths of each complex in the discrete CIF.

2.1 The Structure of $\Diamond$

Each double cell decomposition has an associated diamond complex. We give its construction in the definition below.

Definition 2.1.1. The **diamond** $\Diamond$ is a cell complex constructed from the double Λ and defined as follows. Let $(x, x') \in \Lambda_1$ and $(y, y') \in \Gamma_1^*$ its dual. Then the edges (x, y), (y, x'), (x', y'), and (y', x) are edges in $\Diamond_1$ that bound a four-sided polygon, a diamond face in $\Diamond_2$. Hence, the vertices of the diamond are $\Diamond_0 = \Lambda_0$, the edges $\Diamond_1$ are those derived from each crossing, and the faces bounded by these edges comprise $\Diamond_2$ (Figure 2.1).

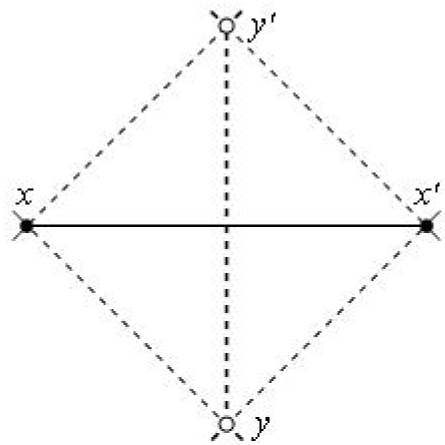

Figure 2.1: The diamond $\Diamond$

Note: The orientation of the edges in $\Diamond_1$ constructed from each crossing is flexible. The orientation given in the definition is merely one possibility.

We may define functions, 1-forms, and 2-forms on the diamond $\Diamond$ in the same way that we have already defined them on the double: $f \in C^0(\Diamond)$ maps the set of vertices $\Diamond_0$ to $\mathbb{C}$, $\alpha \in C^1(\Diamond)$ maps the set of edges $\Diamond_1$ to $\mathbb{C}$, and $\omega \in C^2(\Diamond)$ maps the set of faces $\Diamond_2$ to $\mathbb{C}$. Note that since $\Diamond_0 = \Lambda_0$, a discrete function may be evaluated interchangeably on the diamond $\Diamond$ and the double Λ.

2.2 The Mixed Wedge Product

Now that we have defined the diamond $\Diamond$ and a 1-form $\alpha \in C^1(\Diamond)$, we may return to our derivation of the wedge product. If Λ is a complex of squares, $f \in C^0(\Lambda)$, and $\alpha \in C^1(\Lambda)$, recall that the wedge product

$$df \wedge \alpha = \frac{1}{2}[df(e_1') + df(e_2')]\alpha(e_1) + \cdots + \frac{1}{2}[df(e_4') + df(e_1')]\alpha(e_4)$$

is such that d is a derivation with respect to function and 1-form multiplication. Since $e_1', \ldots e_4' \in \Diamond_1$, we have $df \in C^1(\Diamond)$. Consequently, we can define a mixed wedge product of 1-forms, $\alpha \wedge \beta$, where $\alpha \in C^1(\Diamond)$ and $\beta \in C^1(\Lambda)$ as follows.

$$\alpha \wedge \beta = \frac{\alpha(e_1') + \alpha(e_2')}{2}\beta(e_1) + \cdots + \frac{\alpha(e_4') + \alpha(e_2')}{2}\beta(e_4).$$

Thus, if $e \in \Lambda_1$ is an edge that bounds a face $F \in \Lambda_2$, this wedge product is the sum of multiplications over each edge e of $\beta(e)$ and the average of α on two diamond edges inside the face F. These edges together with e form a half-diamond (Figure 1.9). Although we have simplified to the case where Λ is a square complex, this wedge product is independent of the cell structure of the double because every face elicits a diamond structure.

Definition 2.2.1. Let $\alpha \in C^1(\Diamond)$ and $\beta \in C^1(\Lambda)$. The **mixed wedge product** of α and β on a face $F \in \Lambda_2$ with bounding edges $e_1, \ldots, e_n \in \Lambda_1$ and containing diamond edges $e_1', \ldots, e_n' \in \Diamond_1$ is

$$(\alpha \wedge \beta)(F) = \frac{\alpha(e_1') + \alpha(e_2')}{2}\beta(e_1) + \frac{\alpha(e_2') + \alpha(e_3')}{2}\beta(e_2) + \cdots + \frac{\alpha(e_n') + \alpha(e_1')}{2}\beta(e_n).$$

and $\alpha \wedge \beta \in C^2(\Lambda)$.

By construction, we have the following result.

Proposition 2.2.2. *d is a derivation with respect to the mixed wedge product $\alpha \wedge \beta$, where $\alpha \in C^1(\Diamond)$ and $\beta \in C^1(\Lambda)$, i.e. for $f \in C^0(\Diamond) = C^0(\Lambda)$ and $\beta \in C^1(\Lambda)$, $d(f \cdot \alpha) = f \cdot d\alpha + df \wedge \alpha$.*

The mixed wedge product plays an important role in the discrete Cauchy Integral Formula, which we will discuss further later in this paper.

2.3 Differential k-Forms and Operations on $\Diamond$

The diamond $\Diamond$ is a very useful cell complex. Since there are no restrictions on the number of edges that bound any given face in Λ_2, working with the double can sometimes be quite complicated. The diamond gives us a way to simplify our investigation of any cell structure into a complex of four-sided polygons. In order to work further with the diamond, we must define a few operations. Let $f, g \in C^0(\Diamond)$, $\alpha, \beta \in C^1(\Diamond)$ and $\omega \in C^2(\Diamond)$. Then we have the following operations.

$$(f \cdot g)(x) := f(x) \cdot g(x) \qquad \text{for } x \in \Diamond_0,$$

$$\int_{(x,x')} f \cdot \alpha := \frac{f(x) + f(x')}{2} \int_{(x,x')} \alpha \qquad \text{for } (x, x') \in \Diamond_1,$$

$$\iint_{(x_1,x_2,x_3,x_4)} \alpha \wedge \beta := \frac{1}{4}\sum_{k=1}^{4} \int_{(x_{k-1},x_k)} \alpha \int_{(x_k,x_{k+1})} \beta - \int_{(x_{k+1},x_k)} \alpha \int_{(x_k,x_{k-1})} \beta$$

$$\iint_{(x_1,x_2,x_3,x_4)} f \cdot \omega := \tfrac{f(x_1)+f(x_2)+f(x_3)+f(x_4)}{4} \iint_{(x_1,x_2,x_3,x_4)} \omega \qquad \text{for } (x_1,x_2,x_3,x_4) \in \Diamond_2.$$

So, we see that the product of functions and the product of a function and a 1-form is defined the same on the diamond as it is on the double. Also, note that we do not hesitate to define a wedge product on the diamond. This ability is another, and perhaps the most powerful, advantage of the diamond over the double. If we define the coboundary $d_\Diamond$ on the diamond in the same way as the double (see Definition 1.3.3), then we have the following results.

Proposition 2.3.1. *$d_\Diamond$ is a derivation with respect to function multiplication.*

Proof. The proof is analogous to that of Proposition 1.4.2.

□

Proposition 2.3.2. *$d_\Diamond$ is a derivation with respect to the wedge product $\alpha \wedge \beta$ for $\alpha, \beta \in C^1(\Diamond)$.*

Proof. The proof is simply a computation. Let $f \in C^0(\Diamond)$, $\alpha \in C^1(\Diamond)$ and $F = (x_1, x_2, x_3, x_4) \in \Diamond_2$. Assume $e_i = (x_i, x_{i+1}) \in \Diamond_2$. Then we want to show that $d(f \cdot \alpha)(F) = (f \cdot d\alpha + df \wedge \alpha)(F)$. Using the definitions of these operations, we have

$(f \cdot d\alpha + df \wedge \alpha)(F)$

$$
\begin{aligned}
&= \tfrac{f(x_1)+f(x_2)+f(x_3)+f(x_4)}{4}(\alpha(e_1)+\alpha(e_2)+\alpha(e_3)+\alpha(e_4)) \\
&+ \tfrac{1}{4}[(f(x_1)-f(x_4))\alpha(e_1)-(f(x_2)-f(x_1))\alpha(e_4)+(f(x_2)-f(x_1))\alpha(e_2) \\
&+ (f(x_3)-f(x_2))\alpha(e_1)+(f(x_3)-f(x_2))\alpha(e_3)-(f(x_4)-f(x_3))\alpha(e_4) \\
&- (f(x_4)-f(x_1))\alpha(e_3)] \\
&= \tfrac{f(x_1)+f(x_2)}{2}\alpha(e_1)+\tfrac{f(x_2)+f(x_3)}{2}\alpha(e_2)+\tfrac{f(x_3)+f(x_4)}{2}\alpha(e_3)+\tfrac{f(x_4)+f(x_1)}{2}\alpha(e_4) \\
&= d(f\cdot\alpha)(F) \qquad \text{as desired.}
\end{aligned}
$$

□

2.4 The Averaging Map

We have developed concepts thus far that are valid only on the double Λ. For example, the Hodge star (Definition 1.5.1) is a relationship between the original complex and the dual complex. Since the diamond does not contain its dual, on the diamond complex we lose all of the theory that stems from the Hodge star. On the other hand, the diamond gives us a simplified way to investigate each double cell complex. It also has the advantage a working wedge product between two diamond 1-forms. Hence, ideally we want to be able to utilize freely all of the tools we have on both structures, Λ and $\Diamond$. To do so, we construct the following map from $C^k(\Diamond)$ to $C^k(\Lambda)$ for $k = 0, 1, 2$.

Definition 2.4.1. The **averaging map** $A : C^k(\Diamond) \to C^k(\Lambda)$ for $k = 1, 2$ is defined as follows:

$$\int_{(x,x')} A(\alpha_\Diamond) := \frac{1}{2}\left(\int_{(x,y)} + \int_{(y,x')} + \int_{(x,y')} + \int_{(y',x')}\right)\alpha_\Diamond,$$

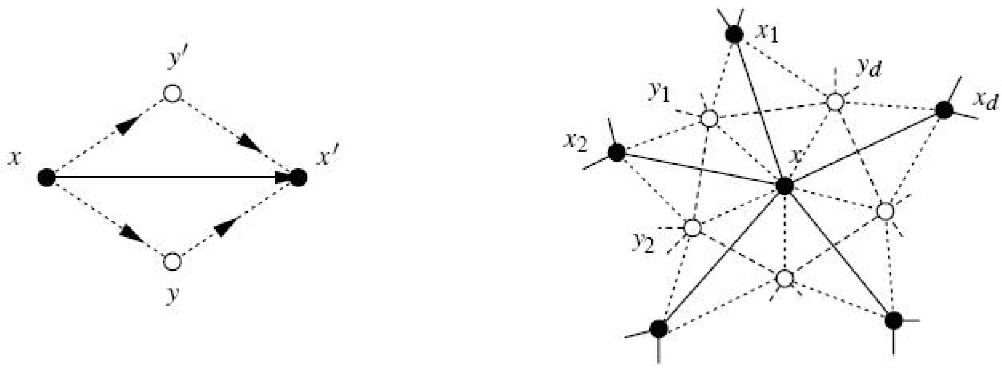

Figure 2.2: Notation in the averaging map.

$$\iint_{x^*} A(\omega_\Diamond) := \frac{1}{2}\sum_{k=1}^{d} \iint_{(x_k, y_k, x, y_{k-1})} \omega_\Diamond.$$

For $k = 0$, $A(f) := f$. See Figure 2.2.

Note: In the definition of $A(\omega_\Diamond)$, $(x, x_k) \in \Lambda_1$ and (y_{k-1}, y_k) is its dual in Λ_1. y_0 is interpreted as y_d.

So the value on an edge of the average map of a 1-form $\alpha_\Diamond$ is simply the average of $\alpha_\Diamond$ evaluated on the two diamond paths that closely approximate the edge in Λ. The average map of a 2-form $\omega_\Diamond = d\alpha_\Diamond$ on a face $F \in \Lambda$ is equivalent to A applied to $\alpha_\Diamond$ on ∂F. Therefore, this definition respects Green's Theorem. From Figure 2.2, we see that $A(\omega_\Diamond)$ evaluated on a double face x^* is a sum of evaluations of $\omega_\Diamond$ on each diamond face that radiates from the vertex x^*.

Lemma 2.4.2. *The averaging map A is such that $d_\Lambda A = A d_\Diamond$.*

Proof. We omit the proof here. However, with the aid of Figure 2.2, one can see that the equality holds. $\square$

Now that we have a way to move from $\Diamond$ to Λ, we define a wedge product of 1-forms, $\alpha, \beta \in C^1(\Lambda)$ such that $\alpha \wedge \beta \in C^2(\Diamond)$ as follows.

$$\iint_{(x,y,x',y')} \alpha \wedge \beta = \frac{1}{2}\Big(\int_{(x,x')} \alpha \int_{(y,y')} \beta + \int_{(y,y')} \alpha \int_{(x',x)} \beta \Big). \tag{2.1}$$

This wedge product offers a way to utilize properties of 1-forms that we have given on the double and not on the diamond. For example, 1-forms of type (1,0) and (0,1), as well as meromorphic 1-forms, only make sense on the double.

Lemma 2.4.3. *Let* $\alpha, \beta \in C^{(1,0)}(\Lambda)$ *and* $(x, y, x', y') \in \Diamond_2$. *Then* $\iint_{(x,y,x',y')} \alpha \wedge \beta = 0.$

Proof. Since α and β are both of type (1,0), $*\alpha = -i\alpha$ and $*\beta = -i\beta$. Thus, by the definition of the Hodge star,

$$\int_{(y,y')} \alpha = i\rho(x,x') \int_{(x,x')} \alpha \quad \text{and} \quad \int_{(y,y')} \beta = i\rho(x,x') \int_{(x,x')} \beta.$$

Hence,

$$\begin{aligned} 2\iint_{(x,y,x',y')} \alpha \wedge \beta &= \int_{(x,x')} \alpha \int_{(y,y')} \beta + \int_{(y,y')} \alpha \int_{(x',x)} \beta \\ &= \Big(\int_{(x,x')} \alpha\Big)\Big(i\rho(x,x') \int_{(x,x')} \beta\Big) + \Big(i\rho(x,x') \int_{(x,x')} \alpha\Big)\Big(-\int_{(x,x')} \beta\Big) \\ &= i\rho(x,x') \int_{(x,x')} \alpha \int_{(x,x')} \beta - i\rho(x,x') \int_{(x,x')} \alpha \int_{(x,x')} \beta \\ &= 0. \end{aligned}$$

□

Lemma 2.4.4. *Let* $\alpha, \beta \in C^{(0,1)}(\Lambda)$ *and* $(x, y, x', y') \in \Diamond_2$. *Then* $\displaystyle\iint_{(x,y,x',y')} \alpha \wedge \beta = 0.$

Proof. The proof is similar to the case when $\alpha, \beta \in C^{(1,0)}(\Lambda)$.

□

These two results are parallel to the continuous case where $dz \wedge dz = 0$ and $d\bar{z} \wedge d\bar{z} = 0$.

We can also see that the above wedge product of 1-forms on the double relates nicely to the wedge product of 1-forms on the diamond.

Lemma 2.4.5. *The wedge product in* $C^2(\Diamond)$ *of 1-forms on the double satisfies*

$$A(\alpha_\Diamond) \wedge A(\beta_\Diamond) = \alpha_\Diamond \wedge \beta_\Diamond.$$

Proof. This can be verified straight from the definition of each wedge product and the averaging map of a 1-form on $\Diamond$. It is left as an exercise for the reader. □

Furthermore, when working on the plane, we may extend the averaging map from a single face to the entire cell decomposition. For every statement about an "infinite sum," we assume that the "sums" are absolutely convergent.

Lemma 2.4.6. *Suppose that* Λ *and* $\Diamond$ *are cell decompositions of the complex plane. Let* A *be the averaging map and let* $\omega_\Diamond \in C^2(\Diamond)$. *Then*

$$\iint_{\Diamond_2} \omega_\Diamond = \iint_{\Gamma_2} A(\omega_\Diamond) = \iint_{\Gamma_2^*} A(\omega_\Diamond) = \frac{1}{2}\iint_{\Lambda_2} A(\omega_\Diamond).$$

Proof. Let $\lozenge_x$ denote the set of diamond 2-cells that radiate out from the vertex $x \in \Lambda_0$.

$$\iint_{\Gamma_2} A(\omega_\lozenge) = \sum_{x\in\Gamma_0^*} \iint_{x^*} A(\omega_\lozenge) = \sum_{x\in\Gamma_0^*} \left(\frac{1}{2}\sum_{F\in\lozenge_x} \iint_F \omega_\lozenge\right) = \sum_{F\in\lozenge_2} \iint_F \omega_\lozenge = \iint_{\lozenge_2} \omega_\lozenge.$$

The middle equality holds because each face in $\lozenge_2$ radiates from two faces in Γ_2. Thus, when we sum over all $x \in \Gamma_0^*$, we are simply counting twice the contributiong of each diamond face in $\lozenge_2$. A similar process is true for Γ_2^*, and so we also have

$$\iint_{\Gamma_2^*} A(\omega_\lozenge) = \iint_{\lozenge_2} \omega_\lozenge.$$

Since

$$\iint_{\Lambda_2} A(\omega_\lozenge) = \iint_{\Gamma_2\cup\Gamma_2^*} A(\omega_\lozenge) = \iint_{\Gamma_2} A(\omega_\lozenge) + \iint_{\Gamma_2^*} A(\omega_\lozenge),$$

we have the following relationship between 2-forms over the diamond and 2-forms over the double.

$$\iint_{\lozenge_2} \omega_\lozenge = \iint_{\Gamma_2} A(\omega_\lozenge) = \iint_{\Gamma_2^*} A(\omega_\lozenge) = \frac{1}{2}\iint_{\Lambda_2} A(\omega_\lozenge).$$

□

A similar result holds for function and 2-form multiplication.

Lemma 2.4.7. *Suppose that* Λ *and* $\lozenge$ *are cell decompositions of the complex plane. Let* $f \in C^0(\lozenge)$ *and* $\omega_\lozenge \in C^2(\Lambda)$. *Then*

$$\iint_{\lozenge_2} f\cdot\omega_\lozenge = \frac{1}{2}\iint_{\Lambda_2} f\cdot A(\omega_\lozenge).$$

Proof. By Lemma 2.4.6, we have the following:

$$\iint_{\Diamond_2} f \cdot \omega_\Diamond = \frac{1}{2} \iint_{\Lambda_2} A(f \cdot \omega_\Diamond).$$

Recall that the averaging map is the identity on functions. Because the product of a function and a 2-form is defined differently on the double and the diamond, we need to check that this operation is well defined, i.e. $\frac{1}{2} \iint_{\Lambda_2} A(f \cdot \omega_\Diamond) = \frac{1}{2} \iint_{\Lambda_2} f \cdot A(\omega_\Diamond)$. We'll investigate each integral separately and show that they evaluate equivalently. We denote the vertices of an arbitrary diamond face F by x_i for $1 \leq i \leq 4$.

$$\begin{aligned} \iint_{\Lambda_2} A(f \cdot \omega_\Diamond) &= \sum_{x^* \in \Lambda_2} \iint_{x^*} A(f \cdot \omega_\Diamond) \\ &= \sum_{x^* \in \Lambda_2} \left(\frac{1}{2} \sum_{F \in \Diamond_x} \iint_F f \cdot \omega_\Diamond \right) \\ &= \frac{1}{2} \sum_{x^* \in \Lambda_2} \sum_{F \in \Diamond_x} \left(\frac{1}{4} \sum_{i=1}^{4} f(x_i) \iint_F \omega_\Diamond \right). \end{aligned}$$

Each face $F \in \Diamond_2$ appears four times under this sum. This is because F radiates from four different vertices, two dual vertices and two regular. Therefore, the integral above reduces to

$$\frac{1}{2} \sum_{F \in \Diamond_2} \sum_{i=1}^{4} f(x_i) \iint_F \omega_\Diamond.$$

Now consider the other integral.

$$\begin{aligned}\iint_{\Lambda_2} f \cdot A(\omega_\Diamond) &= \sum_{x^* \in \Lambda_2} \iint_{x^*} f \cdot A(\omega_\Diamond) \\ &= \sum_{x^* \in \Lambda_2} f(x^*) \iint_{x^*} A(\omega_\Diamond) \\ &= \sum_{x^* \in \Lambda_2} f(x^*) \left(\frac{1}{2} \sum_{F \in \Diamond_x} \iint_F \omega_\Diamond \right) \\ &= \frac{1}{2} \sum_{F \in \Diamond_2} \sum_{i=1}^{4} f(x_i) \iint_F \omega_\Diamond .\end{aligned}$$

This last equality holds since a given face $F \in \Diamond_2$ appears in the calculation four times, as indicated earlier. In particular, this is whenever $x^* = x_i$ for $1 \le i \le 4$. Thus, if we group together all terms involving an integral over a specific $F \in \Diamond_2$, we will have $\sum_{i=1}^{4} f(x_i) \iint_F \omega_\Diamond$. This is carried out over each $F \in \Diamond_2$, hence our operation is well defined.

$$\therefore \iint_{\Diamond_2} f \cdot \omega_\Diamond = \frac{1}{2} \iint_{\Lambda_2} f \cdot A(\omega_\Diamond).$$

□

In the above Lemmas we gave the condition that Λ and $\Diamond$ must be cell decompositions of the complex plane. This restriction was made because the averaging map poses problems when it is evaluated on a cell complex with boundary. Diamond faces on the boundary no longer overlap four faces of the double since not all of those faces are present. Consequently, we see that Lemma 2.4.7 fails to hold in general if we remove the condition that our cell decomposition occurs on the entire complex plane. This complication is illustrated in the example below.

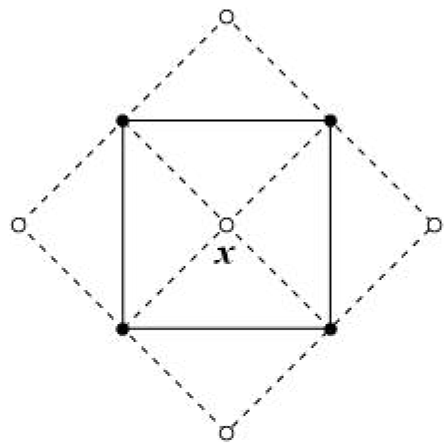

Figure 2.3: Λ and $\Diamond$ of Example 2.4.8.

Example 2.4.8. Let Λ and $\Diamond$ be the cell decompositions pictured in Figure 2.3. Note that each diamond face only overlaps one double face, namely x^*. Suppose $\omega_\Diamond$ is an arbitrary 2-form in $C^2(\Diamond)$ and $f \in C^0(\Diamond)$. Then,

$$\begin{aligned} \iint_{\Lambda_2} A(f \cdot \omega_\Diamond) &= \iint_{x^*} A(f \cdot \omega_\Diamond) \\ &= \frac{1}{2} \sum_{F \in \Diamond_x} \iint_F f \cdot \omega_\Diamond \\ &= \frac{1}{8} \sum_{F \in \Diamond_x} \sum_{i=1}^{4} \iint_F \omega_\Diamond. \end{aligned}$$

However,

$$\begin{aligned} \iint_{\Lambda_2} f \cdot A(\omega_\Diamond) &= \iint_{x^*} f \cdot A(\omega_\Diamond) \\ &= f(x)\Big(\frac{1}{2} \sum_{F \in \Diamond_x} \iint_F \omega_\Diamond\Big). \end{aligned}$$

In general, these two integrals will not be equal.

The above example shows that when working on a cell decomposition with boundary, we no longer have the convenient relationship that each diamond face touches four double faces. Therefore, in general, we lose the result in Lemma 2.4.7 on a bounded collection of diamond 2-cells.

Chapter 3

The Cauchy Integral Formula

We have now developed a working foundation for our discrete function theory. Our understanding of the basic workings of differential forms on both the diamond and the double make it possible to begin our attempt to recover major discrete results analogous to those in the continuous case. This paper will focus on the development of one result in particular. Our guideline of integration offers a natural motivation to recover a discrete version of a very powerful result, the well-known Cauchy Integral Formula (CIF).

In this chapter, we are able to develop a CIF for the double and one for the diamond. The CIF on the double is an original result which is recovered via Green's theorem applied to the product of a function and the meromorphic 1-form in Proposition 1.7.10. The mixed wedge product, also an original finding, is key in obtaining this result. However, as we will see, the CIF on the double merely resembles the formal appearance of the continuous Cauchy

Integral Formula. It fails to represent values of discrete analytic functions by line integrals.

This failure is a key motivating factor behind the need to work on the diamond. We give an original exposition of the construction of the diamond 1-form to which we apply Green's Theorem. With a recipe of results from both the diamond and the double, we are able to develop a CIF on the diamond that behaves analogously to the continuous case when f is discrete analytic. However, we unfortunately must settle for an integral formula that yields an average of two function values, instead of the value at a single point.

3.1 The Continuous Cauchy Integral Formula

We state and prove the continuous result here.

Theorem 3.1.1. (Continuous CIF) *Let $D \subseteq \mathbb{C}$ be a bounded domain with C^1 boundary and $z \in D$. For each function $f \in C^1(\bar{D})$,*

$$f(z) = \frac{1}{2\pi i} \oint_{\partial D} \frac{f(w)}{w-z} dw - \frac{1}{2\pi i} \iint_D \frac{\partial f(w)}{\partial \bar{w}} \cdot \frac{d\bar{w} \wedge dw}{w-z} \ .$$

Proof. Let $z \in D$. Let $\epsilon > 0$ be s.t. $\bar{D}(z,\epsilon) \subseteq D$. Let $D_\epsilon = D \backslash \bar{D}(z,\epsilon)$ (see Figure 3.1).

Apply Green's Theorem (Theorem 1.3.5) to $\dfrac{f(w)}{w-z}$ on D_ϵ.

$$\begin{aligned}
\oint_{\partial D_\epsilon} \frac{f(w)}{w-z} dw &= \iint_{D_\epsilon} \bar{\partial}\Big(\frac{f(w)}{w-z} dw\Big) \\
&= \iint_{D_\epsilon} \Big(\frac{\partial f(w)}{\partial \bar{w}} \cdot \frac{1}{w-z} + \frac{\partial}{\partial \bar{w}}(\frac{1}{w-z}) f(w)\Big) d\bar{w} \wedge dw \\
&= \iint_{D_\epsilon} (\frac{\partial f(w)}{\partial \bar{w}} \cdot \frac{1}{w-z}) d\bar{w} \wedge dw.
\end{aligned}$$

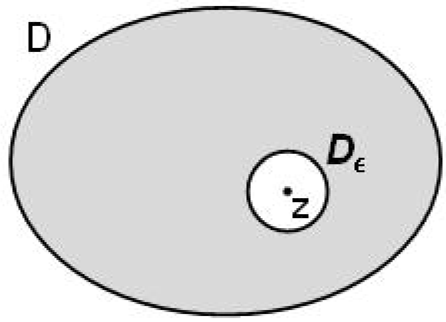

Figure 3.1: The region D_ϵ

By the construction of D_ϵ, as $\epsilon \to 0^+$,

$$\iint_{D_\epsilon} (\frac{\partial f(w)}{\partial \bar{w}} \cdot \frac{1}{w-z}) d\bar{w} \wedge dw \to \iint_{D} (\frac{\partial f(w)}{\partial \bar{w}} \cdot \frac{1}{w-z}) d\bar{w} \wedge dw.$$

Now consider $w \in \partial D(z, \epsilon)$. Parameterizing yields $w = z + \epsilon e^{it}$ and $dw = i\epsilon e^{it} dt$.

$$\therefore \oint_{\partial D(z,\epsilon)} \frac{f(w)}{w-z} dw = \int_0^{2\pi} \frac{f(z+\epsilon e^{it})}{\epsilon e^{it}} \cdot i\epsilon e^{it} dt = i \int_0^{2\pi} f(z+\epsilon e^{it}) dt.$$

$$\text{As } \epsilon \to 0^+, \quad i \int_0^{2\pi} f(z+\epsilon e^{it}) dt \to i \int_0^{2\pi} f(z) = 2\pi i f(z).$$

Thus,

$$\oint_{\partial D} \frac{f(w)}{w-z} dw - \oint_{\partial D(z,\epsilon)} \frac{f(w)}{w-z} dw = \oint_{\partial D_\epsilon} \frac{f(w)}{w-z} dw = \iint_{D_\epsilon} (\frac{\partial f(w)}{\partial \bar{w}} \cdot \frac{1}{w-z}) d\bar{w} \wedge dw,$$

and letting $\epsilon \to 0^+$ yields

$$\oint_{\partial D} \frac{f(w)}{w-z} dw = \iint_D \frac{\partial f(w)}{\partial \bar{w}} \cdot \frac{d\bar{w} \wedge dw}{w-z} + 2\pi i f(z).$$

Rearranging gives the desired result.

□

This theorem has significant consequences. It shows that the value of an analytic function f on a domain is solely determined by the value of the function on the boundary since f is analytic $\iff \dfrac{\partial f(z)}{\partial \bar{z}} = 0$. Furthermore, the well-known Cauchy Integral Formula for Derivatives (Theorem 3.1.2) shows that we may calculate the value of the derivative of a function at a point in a domain via the integral formula, and hence via integration. In addition, it also proves that a complex differentiable function is infinitely differentiable.

Theorem 3.1.2. (CIF for derivatives) *Let $D \subseteq \mathbb{C}$ be a bounded domain with piecewise smooth boundary. If $f(z)$ is an analytic function on D that extends smoothly to the boundary of D, then $f(z)$ has complex derivatives of all orders on D, which are given by*

$$f^{(m)}(z) = \frac{m!}{2\pi i} \oint_{\partial D} \frac{f(w)}{(w-z)^{m+1}} dw, z \in D, m \geq 0.$$

3.2 The Cauchy Integral Formula on the Double

The continuous CIF relies on a multiplication of $f(w)$ against the meromorphic 1-form $\dfrac{dw}{w-z}$ with a single pole at $z \in \mathbb{C}$, the point at which the formula returns a function value. In order to develop a similar result in the discrete case, we recall our discrete analogue of $\dfrac{dw}{w-z}$, the meromorphic 1-form μ_x where $z = x$ (Proposition 1.7.10). Let $f \in C^0(\Lambda)$ and $x \in \Lambda_0$. Then there exists a meromorphic 1-form μ_x with a single pole at x. As in the continuous case, we may apply Green's theorem to $\iint_\Lambda d(f \cdot \mu_x)$. Consequently, we can develop a discrete CIF on the double.

First, we introduce notation. Let D be a subset of Λ. We denote the components of D belonging to the regular, respectively dual, cell decomposition by D_Γ, respectively D_{Γ^*}. So, for example, a dual edge of D is in $D_{\Gamma_1^*}$.

Proposition 3.2.1. (Discrete Double CIF) *Let D be a bounded double complex that is boundary accessible with respect to $x \in D_{\Gamma_0^*}$, a dual vertex on the interior of D. Let μ_x be a meromorphic 1-form with a single pole at x of residue $+1$. For each function $f \in C^0(D)$,*

$$f(x) = \frac{1}{2\pi i} \oint_{\partial D_\Gamma} f \cdot \mu_x - \frac{1}{2\pi i} \iint_{D_\Gamma} d_\Lambda f \wedge \mu_x \ .$$

Note: The wedge product here is the mixed wedge product as defined in Definition 2.2.1.

Proof. Let $x \in D_{\Gamma_0^*}$. Let $x^* \in D_\Gamma$ be the face dual to x. The product rule yields:

$$d_\Lambda(f \cdot \mu_x) = d_\Lambda f \wedge \mu_x + f \cdot d_\Lambda \mu_x \ .$$

Integrating over $D_\Gamma \backslash x^*$,

$$\iint_{D_\Gamma \backslash x^*} d_\Lambda(f \cdot \mu_x) = \iint_{D_\Gamma \backslash x^*} d_\Lambda f \wedge \mu_x + \iint_{D_\Gamma \backslash x^*} f \cdot d_\Lambda \mu_x \ .$$

Applying Green's Theorem,

$$\oint_{\partial D_\Gamma} f \cdot \mu_x - \oint_{\partial x^*} f \cdot \mu_x = \iint_{D_\Gamma \backslash x^*} d_\Lambda f \wedge \mu_x + \sum_{y^* \in D_\Gamma \backslash x^*} f(y) \iint_{y^*} d_\Lambda \mu_x \ .$$

Since μ_x is closed off of x^*, the last term disappears and rearranging yields

$$\oint_{\partial D_\Gamma} f \cdot \mu_x = \iint_{D_\Gamma \backslash x^*} d_\Lambda f \wedge \mu_x + \oint_{\partial x^*} f \cdot \mu_x \ . \qquad (\star)$$

Now, we may apply Green's Theorem to the integral over ∂x^* and use the product rule again to obtain the following,

$$\begin{aligned} \oint_{\partial x^*} f \cdot \mu_x &= \iint_{x^*} d_\Lambda (f \cdot \mu_x) \\ &= \iint_{x^*} d_\Lambda f \wedge \mu_x + \iint_{x^*} f \cdot d_\Lambda \mu_x \\ &= \iint_{x^*} d_\Lambda f \wedge \mu_x + f(x) \iint_{x^*} d_\Lambda \mu_x \\ &= \iint_{x^*} d_\Lambda \mu_x + f(x) \cdot 2\pi i \end{aligned}$$

Substituting this into $(\star)$ we have

$$\oint_{\partial D_\Gamma} f \cdot \mu_x = 2\pi i \cdot f(x) + \iint_{D_\Gamma} d_\Lambda f \wedge \mu_x \ .$$

Rearranging gives the desired result,

$$f(x) = \frac{1}{2\pi i} \oint_{\partial D_\Gamma} f \cdot \mu_x - \frac{1}{2\pi i} \iint_{D_\Gamma} d_\Lambda f \wedge \mu_x \ .$$

□

Note: An analogous formula holds for $x \in D_{\Gamma_0}$, with integration over D_{Γ^*}.

This derivation of the discrete CIF formally resembles that of the continuous case in its appearance. Since it was recovered analogously via Green's theorem, it is logically the most promising attempt at a working CIF on the double. In the continuous case, if f is analytic, the integral over D disappears. If our discrete formula is to be a true analogy, we need to show that our new CIF reduces when f is discrete analytic.

Assume $f \in C^0(\Lambda)$ is analytic. By Theorem 1.5.9, the wedge product $d_\Lambda f \wedge \mu_x = (d'f + d''f) \wedge \mu_x$ reduces to $d'f \wedge \mu_x$, since $d''f = 0$. Therefore, we have now reduced our problem to the question: Is the mixed wedge product such that the wedge product of (1,0) forms is zero? In general, this question doesn't make sense because one of the 1-forms in the wedge product is only defined on the diamond complex. Recall that the classification of a 1-form as type (1,0) exists only the double. However, we know that df is defined on both the diamond and the double. We need to use the defining property of type (1,0) 1-forms, i.e. $*df = -idf$. This gives us a relationship between df on a regular and a dual edge in the double. But, we are unable to utilize this relationship in the context of the mixed wedge product because df is evaluated on diamond edges. Hence, there is no significant information that can be extracted from the (1,0) property. Unfortunately, this means that we cannot establish the claim that the mixed wedge product is such that the wedge product of (1,0) forms is zero. Consequently, we see that f analytic is not sufficient for $df \wedge \mu_x = 0$. Thus, the analogous appearance of the discrete CIF is deceiving. It fails to parallel the continuous CIF in how it reveals analytic function values.

If f discrete analytic is not a sufficient catalyst for the simplification of the CIF on the double, then a natural question is: For which functions f is the CIF interesting? In other words, which functions are such that $\iint_{D_\Gamma} df \wedge \mu_x = 0$?

Exploring this question directly via the definition of the mixed wedge product is a messy calculation that fails to yield significant results. Consequently, we explore the following example.

Example 3.2.2. Let D be a square double of 9 cells with x^* the center face, as pictured in Figure 3.2. Let μ_x be the meromorphic 1-form with values on Γ_1 shown in Figure 3.2. The values of μ_x on Γ_1^* are determined to satisfy $*\mu_x = -i\mu_x$ since μ_x is of type (1,0). Also, μ_x has a single pole of residue $+1$ located at x. We want to determine which functions belong to the class $\left\{f \in C^0(\Lambda) : \iint_{D_\Gamma} df \wedge \mu_x = 0\right\}$. If we consider each face $F \in D$ individually, we can determine a system of equations whose solution is necessary and sufficient for $\iint_F df \wedge \mu_x = 0$. However, this system is too tightly tied to this particular example to yield much insight. Instead we look for a sufficient condition whose statement is the same on every face. The condition is expressed as a system of equations that relates the function values of all vertices associated with F, i.e. the four corners of the cell, x_1, x_2, x_3, x_4, and the dual vertex in its center, x^*. For simplicity, we denote $y_i = f(x_i)$ for $i = 1, 2, 3, 4$ and $f(x^*) = c$. See Figure 3.3 for clarification of this notation.

The values of μ_x on the edges of the 9 cells of D break down into five cases, shown in Figure 3.4. The unlabeled sides of the squares shown are assumed to have value 0. Using the definition of the mixed wedge product, the five cases respectively elicit the following system of equations.

$$\begin{cases} y_1 + y_2 + y_3 + y_4 = 4c \\ y_2 + y_3 - y_4 - y_1 = 0 \\ y_1 + y_2 - y_3 - y_4 = 0 \\ y_2 - y_4 = 0 \\ y_3 - y_1 = 0 \end{cases}$$

The last two equations require our function f to have equal values on the diagonals of a given face. Consequently, a solution of the system is determined by a solution of

$$y_1 + y_2 = 2c.$$

Therefore, we see that our sufficient condition for membership in the class of functions $\left\{f \in C^0(\Lambda) : \iint_{D_\Gamma} df \wedge \mu_x = 0\right\}$ involves only two degrees of freedom. Such an f can be found by choosing a random face F and choosing the value of a corner vertex and the dual vertex. From there, the function values on the remaining vertices of all faces are determined, and the function is constant on the set of dual vertices.

Example 3.2.2 illustrates the weakness of the CIF on the double. The class of functions that makes the CIF interesting is small. While the discrete formula resembles that of the continuous case, all of its power is lost on the double.

However, from the proof of the discrete CIF, we see that μ_x being closed on $D_\Gamma \backslash \{x^*\}$ and $\iint_{x^*} d\mu_x = 2\pi i$ was ultimately what made the result fall out. Therefore, we can loosen the restriction on our 1-form in the CIF and still maintain the resulting formula.

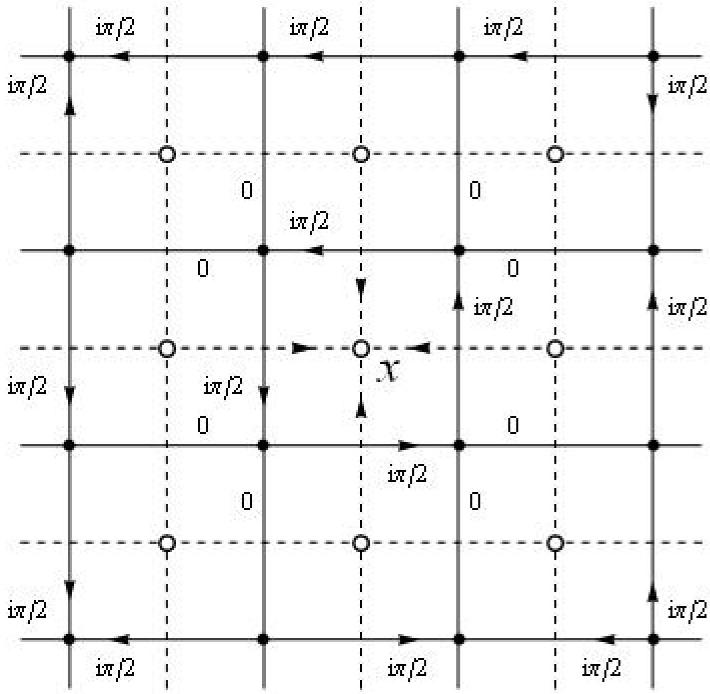

Figure 3.2: The meromorphic 1-form μ_x on the 9-cell D of Example 3.2.2.

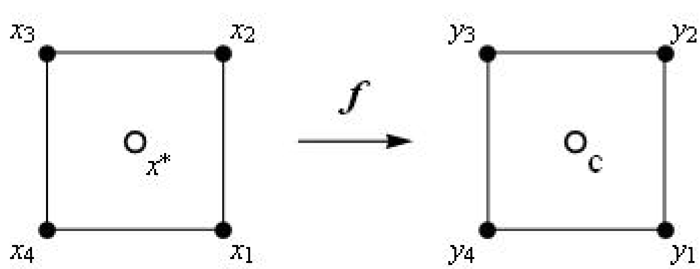

Figure 3.3: Notation for the function values on the vertices associated with a face F in D of Examples 3.2.2 and 3.2.4.

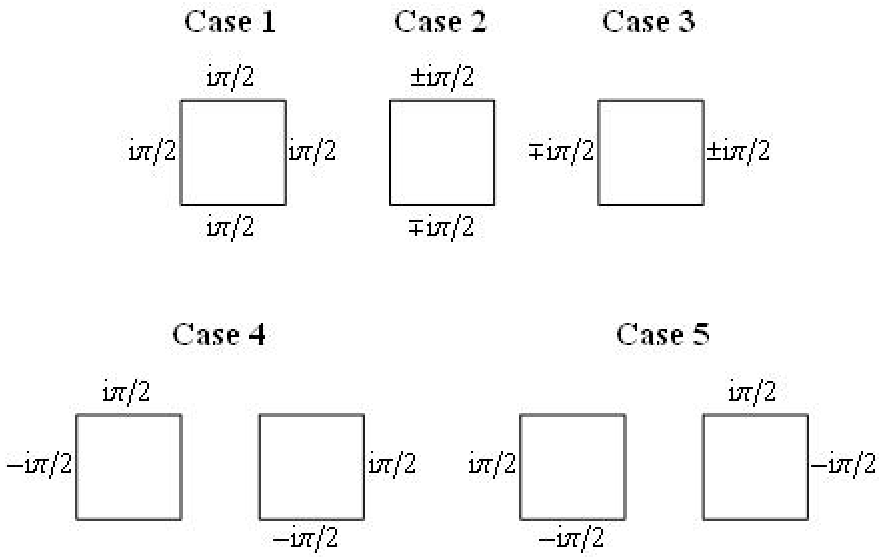

Figure 3.4: The five cases associated with the value of μ_x on the edges of a face F in D

Proposition 3.2.3. *Let D be a compact subset of Λ_2 and $x \in D_{\Gamma_0^*}$ a dual vertex on the interior of D. Let λ_x be a 1-form that is closed on $D_\Gamma \backslash x^*$ with a single pole at x of residue $+1$. For each function $f \in C^0(\Lambda)$,*

$$f(x) = \frac{1}{2\pi i} \oint_{\partial D_\Gamma} f \cdot \lambda_x - \frac{1}{2\pi i} \iint_{D_\Gamma} d_\Lambda f \wedge \lambda_x \ .$$

Proof. The proof of this proposition is identical to that of the CIF on the double.

□

Example 3.2.4. Let D be the subset of Λ_2 and λ_x be the (1,0) 1-form of Example 1.7.12. As in Example 3.2.2 the values of λ_x on each cell break down into cases. This time, however, there are only three cases. One case is trivial since λ_x is zero on all sides of the square, and hence there are no restrictions to our function there. The two nontrivial cases are depicted in Figure 3.5. As in Example 3.2.2, we are looking for sufficient conditions determining functions f belonging to the class $\left\{f \in C^0(\Lambda) : \iint_{D_\Gamma} df \wedge \lambda_x = 0\right\}$. We'll again use the notation illustrated in Figure 3.3. Then the system of equations induced from the restriction $\iint_{D_\Gamma} df \wedge \lambda_x = 0$ from each case is

$$\begin{cases} y_1 + y_2 = 4c \\ y_1 + y_2 - y_3 - y_4 = 0. \end{cases}$$

This reduces to

$$\begin{cases} y_1 + y_2 = 4c \\ y_3 + y_4 = 4c \end{cases}.$$

The functions that satisfy this system of equations is broader than the functions of Example 3.2.2. Begin with an arbitrary face F. We have three degrees of freedom when assigning function values to the vertices associated with F. Those three determine the remaining two vertex values in F. If we next label a face directly below or above F, there is one degree of freedom for each of those. If we continue to label adjacent faces in the vertical direction, there is always one degree of freedom. Next, we consider a face horizontally adjacent to F. The value of the dual vertex at its center is determined. However, we may choose a value for one of the two remaining unlabeled vertices. Such is the case as we continue to label adjacent faces in the horizontal direction. Once we have labeled all faces that are positioned vertically or horizontally in line with F, all other vertices associated with the remaining faces are determined. Thus, the class of functions $\left\{f \in C^0(\Lambda) : \iint_{D_\Gamma} df \wedge \lambda_x = 0\right\}$ identified by our sufficient condition associated with this particular λ_x is much larger than that identified by our sufficient condition associated with μ_x in Example 3.2.2.

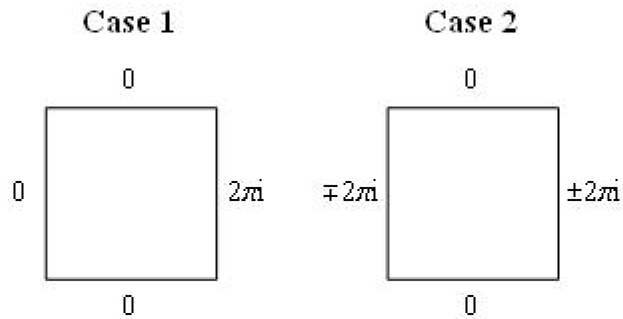

Figure 3.5: The two nontrivial cases associated with the value of λ_x on the edges of a face F in D

Although we have managed to broaden the collection of functions that make our CIF interesting on the double, these functions are again not as interesting as we would wish. For example, their values on dual vertices are constants on rows. Unfortunately, the above examples bear witness to the breakdown of the discrete formula on the double. Because the mixed wedge product is not guaranteed to satisfy the property that $df \wedge \mu_x = 0$ when df is of type (1,0), we lose the ability to simplify the CIF when f is analytic. This difference from the continuous case is the major drawback of the formula on the double.

3.3 The Cauchy Integral Formula on the Diamond

All hope is not yet lost for recovering a discrete CIF. We have seen already that the diamond has been a very powerful tool for recovering analogous discrete theory. Now that we have defined the averaging map A, we have a way to utilize the properties of a meromorphic 1-form on the double and the strength of the diamond wedge product of two 1-forms in $C^1(\Lambda)$.

Recall that our major obstacle in the CIF on the double was the failure of the mixed wedge product of (1,0) forms to equal zero. As we have seen already in Lemma 2.4.3, this is no longer a problem when working on the diamond complex.

A key tool in developing the CIF is the use of Green's theorem on a function f multiplied with a 1-form. The CIF relies on this 1-form having a single pole of residue +1 at the point for which the formula reveals a function value. However, poles and residues only make sense on the double. This is no problem, though, since we can move from the diamond to the double via the averaging map in order to utilize this property. Consequently, we will need to try to prove the existence of a 1-form ν_x such that $A(\nu_x) = \mu_x$, where μ_x is a meromorphic 1-form with a single pole at x of residue +1.

3.3.1 The Construction of $\nu_{x,y}$

In order to construct a ν_x such that $A(\nu_x) = \mu_x$, we need to relate the integration of ν_x over diamond edges to the integration of μ_x over edges of the double. Because each diamond edge contains two vertices, one belonging to Γ_0 and the other belonging to Γ_0^*, it is natural to relate the integration of ν_x to integration of μ_x over Γ_1 and over Γ_1^*. Therefore, it makes sense to relate ν_x evaluated on an edge $(x', y') \in \diamondsuit_1$ to integration of μ_x over a path from x' to x and over a path leading to y'. The latter path must traverse edges in Γ_1^*, therefore it is natural to choose a vertex $y \in \Gamma_0^*$ adjacent x from which to develop our paths over which we'll integrate μ_x to obtain the value of ν_x. Therefore, we will instead use the notation $\nu_{x,y}$

to indicate our choice of the adjacent vertex y.

Let us first exhibit a way to define $\nu_{x,y}$ so that $A(\nu_{x,y}) = \mu_x$ on the edges associated with a single diamond face.

So, WLOG assume that $x \in \Gamma_0$ and $y \in \Gamma_0^*$ such that $(x, y) \in \Diamond_1$. As in the construction of μ_x, we need to restrict ourselves to (x, y) belonging to the interior of $\Diamond$. Let μ_x be the meromorphic 1-form of Proposition 1.7.10 with a single pole at x of residue $+1$. Let (x', y', x'', y'') be a face in $\Diamond_2$. Define $\nu_{x,y}$ on the edges of this diamond as follows.

$$\begin{aligned}
\nu_{x,y}(x', y') &= \mu_x(x'x) + \mu_x(yy') \\
\nu_{x,y}(y', x'') &= \mu_x(y'y) + \mu_x(xx') + \mu_x((x', x'')) \\
\nu_{x,y}(x'', y'') &= \mu_x((x'', x')) + \mu_x(x'x) + \mu_x(yy') + \mu_x((y', y'')) \\
\nu_{x,y}(y'', x') &= \mu_x((y'', y')) + \mu_x(y'y) + \mu_x(xx')
\end{aligned}$$

Notation:

$x'x$ denotes a chosen path from x' to x via the regular edges.

yy' denotes a chosen path from y to y' via the dual edges.

We now need to verify that our definition satisfies $A(\nu_{x,y}) = \mu_x$ on the double edges whose crossing lies inside the diamond (x', y', x'', y'').

$$\begin{aligned}
\int_{(x',x'')} A(\nu_{x,y}) &= \frac{1}{2}\Big(\int_{(x',y')} + \int_{(y',x'')} + \int_{(x',y'')} + \int_{(y'',x'')}\Big)\nu_{x,y} \\
&= \frac{1}{2}\Big(\mu_x(x'x) + \mu_x(yy') + \mu_x(y'y) + \mu_x(xx') + \mu_x((x',x'')) + \mu_x(x'x) \\
&+ \mu_x(yy') + \mu_x((y',y'')) + \mu_x((y'',y')) + \mu_x(y'y) + \mu_x(xx') + \mu_x((x',x''))\Big) \\
&= \frac{1}{2}\Big(\mu_x((x',x'')) + \mu_x((x',x''))\Big) \\
&= \mu_x(x',x'').
\end{aligned}$$

$$\begin{aligned}
\int_{(y',y'')} A(\nu_{x,y}) &= \frac{1}{2}\Big(\int_{(y',x'')} + \int_{(x'',y'')} + \int_{(y',x')} + \int_{(x',y'')}\Big)\nu_{x,y} \\
&= \frac{1}{2}\Big(\mu_x(y'y) + \mu_x(xx') + \mu_x((x',x'')) + \mu_x((x'',x')) + \mu_x(x'x) + \mu_x(yy') \\
&+ \mu_x((y',y'')) + \mu_x(y'y) + \mu_x(xx') + \mu_x(x'x) + \mu_x(yy') + \mu_x((y',y''))\Big) \\
&= \frac{1}{2}\Big(\mu_x((y',y'')) + \mu_x((y',y''))\Big) \\
&= \mu_x(y',y'').
\end{aligned}$$

Note that the calculations show that $\big(\int_{(x',y')} + \int_{(y',x'')}\big)\nu_{x,y} = \mu_x(x',x'')$ and that similar equalities hold for the other half-diamonds.

The construction of $\nu_{x,y}$ on a diamond given above was such that paths to x'' and y'' came into the diamond via x' and y', respectively, first. The calculations above depended on that aspect of the construction. Because each diamond edge is on the boundary of more than one diamond face, it appears that we may have a problem defining $\nu_{x,y}$ so that it has everywhere the property used in those calculations. However, μ_x is closed off of x^*. Hence,

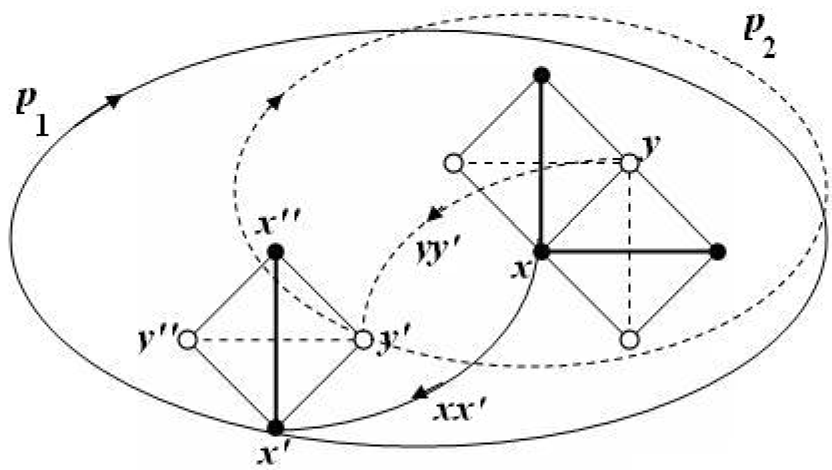

Figure 3.6: The two paths from x to x' and the two paths from y to y'

over many regions, integrals of μ_x are independent of path. This observation provides most of the freedom necessary to construction $\nu_{x,y}$ in the way suggested above so that $\nu_{x,y}$ has the properties needed above. One problem still remains. Integrals of μ_x will not be independent of path for paths that differ by a path with nonzero winding number around x.

It is not possible to avoid the problems associated with winding number. For example, suppose that (x', y', x'', y'') is the first diamond face on which we define $\nu_{x,y}$. As we define $\nu_{x,y}$ on adjacent diamonds, we make choices of paths that allow us to use independence of path. As we move clockwise from one diamond to another, our paths travel clockwise, also. We will eventually come to a diamond containing the edge (x', y') (WLOG) that borders (x', y', x'', y''). According to our construction, when defining $\nu_{x,y}(x', y')$ for this last diamond, we must use paths that differ by a full clockwise circuit from the paths xx' and yy' with which we began.

Let p_1 denote the path from x' to x' that represents the difference of the two regular paths used, and let p_2 denote the path from y' to y' that represents the difference of the two dual paths used (See Figure 3.6). On the one hand we have,

$$\nu_{x,y}(x',y') = \mu_x(x'x) + \mu_x(yy').$$

and on the other,

$$\nu_{x,y}(x',y') = \mu_x(-p_1) + \mu_x(x'x) + \mu_x(yy') + \mu_x(p_2).$$

In order for our construction of $\nu_{x,y}$ to be well defined, we must have μ_x evaluated on the paths where they differ equal to zero. Therefore,

$$\begin{aligned} 0 &= \mu_x(-p_1) + \mu_x(p_2) \\ &= \mu_x(\text{regular path winding counterclockwise around the vertex } y) \\ &+ \mu_x(\text{dual path winding clockwise around the vertex } x) \\ &= 2\pi i[Res_y(\mu_x) - Res_x(\mu_x)] \end{aligned}$$

This last equality holds since μ_x is closed off of x^*, so the integral reduces to the residues. But, $Res_x(\mu_x) = 1$. This says that $Res_y(\mu_x) = 1$, also. However, recall that our choice of μ_x was such that it had a *single* pole at x. Thus, we see that our choice of μ_x is incorrect. However, we also notice that the defining properties of μ_x did not come into play until the last step of our construction. Therefore, there should be an easy fix.

Proposition 3.3.1. *Let Λ be a bounded double complex that is boundary accessible with respect to $x \in \Gamma_0$ and with respect to $y \in \Gamma_0^*$, two adjacent vertices in the diamond complex.*

Then there exists a meromorphic 1-form $\mu_{x,y} \in C^1(\Lambda)$ with only two poles, both of residue +1, located at x and at y.

Proof. By Proposition 1.7.10 there exist meromorphic 1-forms μ_x and μ_y with single poles of residue +1 at x and y, respectively. Let $\mu_{x,y} = \mu_x + \mu_y$. Then $\mu_{x,y}$ is as desired.

□

We now have the necessary meromorphic 1-form with which to construct $\nu_{x,y}$. Before we formally state the existence of $\nu_{x,y}$, we first make an observation. Recall that in our construction, we excluded the two diamonds that share the edge (x, y). Let $R = (abcd)$ be the rectangle comprised of these two faces (see Figure 3.7). Although we exclude R from our construction, we do define $\nu_{x,y}$ on the boundary of R since those edges are contained in diamonds outside of R. This leaves one last edge where we must define $\nu_{x,y}$, namely (x, y). However, we never need any information about $\nu_{x,y}(x, y)$. Therefore, we give the averaging relationship formally off of R.

Proposition 3.3.2. *Let (x, y) be an interior edge of a bounded diamond complex D. Let $\mu_{x,y}$ be the meromorphic 1-form in Proposition 3.3.1. Then there exists a diamond 1-form $\nu_{x,y}$ such that $A(\nu_{x,y}) = \mu_{x,y}$ off of the rectangle R containing the two diamond faces that share the edge (x, y).*

Proof. We'll give a sketch of the key ideas in recovering such a $\nu_{x,y}$. We have already introduced many of the issues earlier in this section. Start by introducing a diamond-edge "slit" (terminology as in complex analysis) running from y to x, along (y, x), and then out to the boundary of the region. Choose a regular vertex $\tilde{x}$ adjacent to x and a dual vertex $\tilde{y}$ adjacent to y such that neither $\tilde{x}$ nor $\tilde{y}$ belongs to the slit. In using $\mu_{x,y}$ to define $\nu_{x,y}$, as indicated earlier in this section, always use paths that travel $(x, \tilde{x})$ to leave x, that travel $(\tilde{x}, x)$ to come to x, and that use $(y, \tilde{y})$ and $(\tilde{y}, y)$ similarly for paths that contain y.

The "outward" orientation of the slit allows us, at each vertex (other than y) on the slit, to label the non-slit edges touching the vertex as right edges (in the region swept out in passing counterclockwise from the slit edge coming into the vertex to the slit edge leaving the vertex) or left edges (otherwise). We say that a path "does not cross" the slit if whenever it intersects the slit, it enters and exits on the same side.

This definition provides a way to give a construction of $\nu_{x,y}$ that avoids the problems caused by residues. In using paths from x' to x and from y to y' to define $\nu_{x,y}(x', y')$, we restrict ourselves to paths for which the associated path made by dropping $(\tilde{x}, x)$ and $(y, \tilde{y})$ and running from $\tilde{y}$ to y', then running along (y', x'), and finally, running from x' to $\tilde{x}$ does not cross the slit. If (x', y') does not lie on the slit, the value of $\nu_{x,y}(x', y')$ is independent of the path in this class used to define it because the difference of any two such paths must have winding number zero around each of the poles x and y. If (x', y') lies on the slit, the difference of two defining paths in our class may have nonzero winding number ± 1 around x, but then it will have a winding number ∓ 1 around y. Hence, by using paths that do not cross the slit, we are able to construct a well-defined $\nu_{x,y}$ satisfying $A(\nu_{x,y}) = \mu_{x,y}$ on $D \backslash R$. $\square$

Note: $\nu_{x,y}$ is the discrete analogue of $\dfrac{dz}{z-z_0}$ on the diamond, where $z_0 = (x,y)$.

Lemma 3.3.3. *Let* $F = (x', y', x'')$ *be a half diamond. Then* $\oint_{\partial F} \nu_{x,y} = \int_{x',x''} \mu_{x,y}$.

Proof.

$$\begin{aligned}
\oint_{\partial F} \nu_{x,y} &= \left(\int_{(x',y')} + \int_{(y',x'')}\right)\nu_{x,y} \\
&= \mu_{x,y}(x'x + yy') + \mu_{x,y}(y'y + xx' + (x',x'')) \\
&= \int_{x',x''} \mu_{x,y}.
\end{aligned}$$

□

Lemma 3.3.4. $\nu_{x,y}$ *is closed off* R.

Proof. Let $F = (x', y', x'', y'') \in \diamondsuit_2$. Then,

$$\begin{aligned}
\iint_F d\nu_{x,y} &= \oint_{\partial F} \nu_{x,y} \\
&= \left(\int_{(x'',y'')} + \int_{(y'',x')} + \int_{(x',y')} + \int (y',x'')\right)\nu_{x,y} \\
&= \int_{x'',x'} \mu_{x,y} + \int_{(x',x'')} \mu_{x,y} \\
&= 0.
\end{aligned}$$

□

We are now ready to derive a Cauchy Integral Formula on the diamond via Green's Theorem.

3.3.2 The Diamond Cauchy Integral Formula

Proposition 3.3.5. (Discrete Diamond CIF) *Let D be a compact connected subset of $\Diamond_2$ defined by the closure of a finite collection of cells in $\Diamond_2$, and let $(x, y) \in D_1$ be an edge that bounds two interior neighboring diamond 2-cells. Suppose that the underlying bounded double complex (whose crossings determine D's 1-cells) satisfies the properties that permit the definition of the meromorphic 1-form $\mu_{x,y}$. Then for each function $f \in C^0(D)$,*

$$\oint_{\partial D} f \cdot \nu_{x,y} = \iint_D d'' f \wedge \mu_{x,y} + 2\pi i \frac{f(x) + f(y)}{2}.$$

Proof. The edge (x, y) bounds two faces in D. Let $R = (abcd)$ be the rectangle comprised of these faces (see Figure 3.7).

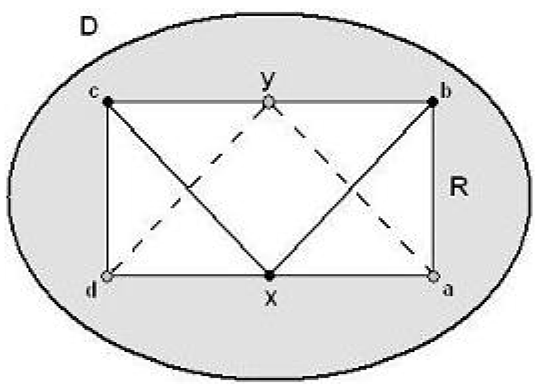

Figure 3.7: The rectangle R in domain D defined by edge $(x, y) \in \Diamond_1$

On $D\backslash R$, the product rule yields

$$d_\Diamond(f \cdot \nu_{x,y}) = d_\Diamond f \wedge \nu_{x,y} + f \cdot d_\Diamond \nu_{x,y}.$$

Note:

$$\begin{aligned} d_{\Diamond} f \wedge \nu_{x,y} &= A(d_{\Diamond} f) \wedge A(\nu_{x,y}) \qquad \text{by Lemma 2.4.5} \\ &= d_{\Lambda} A(f) \wedge \mu_{x,y} \\ &= d_{\Lambda} f \wedge \mu_{x,y} \\ &= (d' f + d'' f) \wedge \mu_{x,y} \\ &= d'' f \wedge \mu_{x,y} \qquad \text{by Lemma 2.4.3 since } \mu_{x,y} \text{ is of type } (1,0). \end{aligned}$$

So,

$$d_{\Diamond}(f \cdot \nu_{x,y}) = d'' f \wedge \mu_{x,y} + f \cdot d_{\Diamond} \nu_{x,y}.$$

Integrating this equation over $D \backslash R$,

$$\iint_{D\backslash R} d_{\Diamond}(f \cdot \nu_{x,y}) = \iint_{D\backslash R} d'' f \wedge \mu_{x,y} + \iint_{D\backslash R} f \cdot d_{\Diamond} \nu_{x,y}.$$

Applying Green's Theorem,

$$\oint_{\partial D} f \cdot \nu_{x,y} - \oint_{\partial R} f \cdot \nu_{x,y} = \iint_{D\backslash R} d'' f \wedge \mu_{x,y} + \sum_{F \in D\backslash R} \frac{f(x_1) + f(x_2) + f(x_3) + f(x_4)}{4} \iint_F d_{\Diamond} \nu_{x,y},$$

where $F = (x_1, x_2, x_3, x_4)$.

Note: $\iint_{D\backslash R} d_{\Diamond} \nu_{x,y} = 0$ since $\nu_{x,y}$ is closed off R by Lemma 3.3.4.

$$\therefore \oint_{\partial D} f \cdot \nu_{x,y} = \iint_{D\backslash R} d'' f \wedge \mu_{x,y} + \oint_{\partial R} f \cdot \nu_{x,y}. \qquad (\star)$$

Now, consider the integral over ∂R.

Claim. $\oint_{\partial R} f \cdot \nu_{x,y} = \iint_R d'' f \wedge \mu + 2\pi i \dfrac{f(x) + f(y)}{2}.$

This claim can be verified by calculating each piece separately and making necessary substitutions. Begin by calculating $\oint_{\partial R} f \cdot \nu_{x,y}$ using the definition of the product of a function and a 1-form on the diamond. In the calculation of $\iint_R d''f \wedge \mu$, first recall that $df \wedge \mu_{x,y} = d''f \wedge \mu_{x,y}$. Therefore, we instead calculate $\iint_R df \wedge \mu$ using the definition of the diamond wedge product of double 1-forms given in Equation 2.1. We may then write out each evaluation of df on an edge via the definition of the coboundary. For each evaluation of $\mu_{x,y}$ on an edge, we substitute an evaluation of $\nu_{x,y}$ over the half diamond which the double edge closes as permitted under Lemma 3.3.3. Our choice of half diamond is such that at least one of its edges is in the boundary of R.

Next, we rewrite $2\pi i \frac{f(x)+f(y)}{2}$ using the definition of the +1 residue of $\mu_{x,y}$ on x^* and y^*. Then,

$$2\pi i \frac{f(x)+f(y)}{2} = \frac{1}{2} f(x) \oint_{\partial x^*} \mu_{x,y} + \frac{1}{2} f(y) \oint_{\partial y^*} \mu_{x,y}.$$

Again, we use Lemma 3.3.3 to substitute for each evaluation of $\mu_{x,y}$ on an edge of the double. As in the calculation of the wedge product, our substitution is such that the half diamond has at least one of its edges in the boundary of R.

Once each piece is written in terms of evaluations of $\nu_{x,y}$ and multiplications of function values, the claim is made clear.

Substituting the result of the claim into our original formula ($\star$) yields the discrete CIF,

$$\oint_{\partial D} f \cdot \nu_{x,y} = \iint_D d''f \wedge \mu_{x,y} + 2\pi i \frac{f(x)+f(y)}{2}.$$

$\square$

So, we have managed to recover a CIF on the diamond in an analogous way to the continuous case via Green's Theorem. Unfortunately, this formula only reveals an average of function values instead of a value at a single vertex. It would appear that our attempt to find a CIF on the double was more successful. But, although the diamond has this one weakness, unlike our findings on the double, it possesses the strength of the continuous formula in a more important way. The diamond formula behaves analogously to the continuous case when f is an analytic function.

Proposition 3.3.6. *Let D be a compact connected subset of $\Diamond_2$ and $(x, y) \in D_1$ an edge that bounds two interior neighbors of D with a non-empty boundary. Suppose $f \in C^0(\Lambda)$ is discrete analytic. Then*

$$\oint_{\partial D_\Diamond} f \cdot \nu_{x,y} = 2\pi i \frac{f(x) + f(y)}{2}.$$

Proof. By Proposition 3.3.5 (the CIF on the diamond),

$$\oint_{\partial D_\Diamond} f \cdot \nu_{x,y} = \iint_{D_\Diamond} d'' f \wedge \mu_{x,y} + 2\pi i \frac{f(x) + f(y)}{2}.$$

Since f is discrete analytic, $d'' f = \pi_{(0,1)} \circ df = 0$ on all edges in D_Λ by Theorem 1.5.9. Therefore, $d'' f \wedge \mu_{x,y} = 0$ on $D_\Diamond$. And so, the CIF on the diamond reduces to

$$\oint_{\partial D_\Diamond} f \cdot \nu_{x,y} = 2\pi i \frac{f(x) + f(y)}{2}.$$

$\square$

This result solidifies the parallel between the CIF on the diamond and the continuous CIF. While the CIF on the double reveals a value at a single point, the diamond behaves analogously where it matters most. The Cauchy Integral Formula on the diamond is a stronger and much more useful result than that of the double.

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