Algebras of Toeplitz Operators

Bartleby Ordonez-Delgado

Master's Thesis submitted to the Faculty of the
Virginia Polytechnic Institute and State University
in partial fulfillment of the requirements for the degree of

Master of Science
in
Mathematics

Peter E. Haskell, Chair
Joseph A. Ball
John F. Rossi

May 3, 2006
Blacksburg, Virginia

Keywords: C^*-algebras, Toeplitz Operators

Algebras of Toeplitz Operators

Bartleby Ordonez-Delgado

(ABSTRACT)

In this work we examine C^*-algebras of Toeplitz operators over the unit ball in $\mathbb{C}^n$ and the unit polydisc in $\mathbb{C}^2$. Toeplitz operators are interesting examples of non-normal operators that generate non-commutative C^*-algebras. Moreover, in the nice cases (depending on the geometry of the domain) of algebras of Toeplitz operators we can recover some analogues of the spectral theorem up to compact operators. In this setting, we can capture the index of a Fredholm operator which is a fundamental numerical invariant in Operator Theory.

Dedication

Dedicated to my parents: Raul and Susana, and to my brothers: Salim, Zenem, Mijail and Zubin.

Acknowledgments

I want to thank Dr. Peter Haskell for advising me during this paper. I have learned a lot of Mathematics from him not only from the courses that he taught me but also from the informal conversations about Mathematics. Also, I would like to thank Dr. Ball and Dr. Rossi for reading this paper and making some corrections.

Contents

Chapter 1

Introduction

The spectral theorem plays a fundamental role in the study of normal, especially self-adjoint, operators. In fact, the spectral theorem can be applied (by simultaneous diagonalizations) to commutative C^*-algebras. The spectrum is a rich source of numerical invariants, such as the index and the trace, when these are defined. The version of the spectral theorem that shows that an operator is unitarily equivalent to a multiplication operator is the foundation for functional analysis.

Some questions arise after this discussion. Are there any "natural" examples of non-normal operators? What kinds of algebras do these operators generate? Clearly a non-normal operator generates a non-commutative C^*-algebra. This leads us to search for families of operators that generate non-commutative C^*-algebras. In function theory we have very natural examples - Toeplitz operators.

In the context of algebras of Toeplitz operators, we can ask under which conditions or over which domains we can recover some analogues or consequences of the spectral theorem. It turns out that for some "nice" cases (algebras of Toeplitz operators over strictly pseudoconvex domains) the commutator ideal coincides with the compact operators; therefore, we have a sort of " spectral theorem up to compact operators". This similarity is enough to recover some results from the spectral theorem modulo compact operators, for example the index of a Fredholm operator, which is one of the primary numerical invariants in operator theory.

The domain where Toeplitz operators are defined plays an important role in the characterization of the algebras of Toeplitz operators. It turns out that if the domain is not "nice" enough (not a strictly pseudoconvex domain), the commutator ideal of the algebra of Toeplitz operators contains more than compact operators. For this case the algebra is not "commutative up to compact operators" and our analogue of the spectral theorem is not a straightforward source of invariants. The end of the last chapter contains some comments pointing towards further developments.

This paper is a survey of the structure theory of algebras of Toeplitz operators associated with continuous, scalar-valued functions. The paper organizes the known results in a way that motivates recent and ongoing research. The main source of this paper is [Upm]. Most of the results in this paper are particular cases of stronger results that can be found in [Upm]. However, many of the proofs have been worked independently from [Upm] by using different techniques.

The first domain we consider is the unit disc because the disc is the simplest domain with "nice" boundary. Then, we consider the natural generalizations of the unit disc, the unit ball and the unit polydisc. In the latter case, the polydisc, we do not have a "nice" boundary, so this will be our example of contrast with the previous cases, where we could apply our analogue of the spectral theorem.

Also, we consider the Hardy and Bergman spaces in L^2 because they are the simplest spaces where the Toeplitz operators are defined and have interesting properties. Through this work we discuss the similarities or differences between the C^*-algebras generated by the Hardy-Toeplitz and Bergman-Toeplitz operators. Eventually we will show that the only difference between them is the commutator ideal.

In the second chapter we study the Bergman space over the unit disc. The Bergman space is defined as the Hilbert space of holomorphic functions that are square-integrable. In this context, we shall show an explicit basis for the Bergman space that will be very useful for characterizing the orthogonal projection from $L^2(\mathbb{D})$ onto the Bergman space, for finding the Bergman Kernel (reproducing kernel), and for finding an integral representation of functions in the Bergman space. In addition, we do the analysis of Bergman-Toeplitz operators over the unit disc, operators which are defined as the compressions to the Bergman space of multiplication operators with continuous symbols. Our main result in this chapter is the characterization of the C^*-algebra generated by the Toeplitz operators. It turns out that the algebra of compact operators on the Bergman space is contained in the C^*-algebra generated by the Toeplitz operators and coincides with the commutator ideal. Furthermore, the quotient C^*-algebra (subalgebra of the Calkin algebra) resulting by modding out the C^*-algebra of Toeplitz operators by the algebra of compact operators is C^*-isomorphic to the algebra of continuous functions over the unit circle. We conclude this chapter by showing a nice characterization of the Fredholm-Toeplitz operators.

In chapter three we study the Hardy space over the unit disc. The Hardy space is defined as the closed subspace of $L^2(S^1)$ generated by the functions z^n for n non-negative integer. Using the Poisson Kernel we can extend every function in the Hardy space to a holomorphic function on the unit disc. As in chapter two, we characterize the C^*-algebra of the Hardy-Toeplitz operators over the unit disc. Here the definition of the Toeplitz operator is similar to the definition in the Bergman case. We obtain the same result as in the Bergman case, i.e., the algebra of compact operators on the Hardy space is equal to the commutator ideal of the C^*-algebra of the Hardy-Toeplitz operators and the quotient C^*-algebra (subalgebra of the Calkin algebra) resulting by modding out the C^*-algebra of Toeplitz operators by the algebra of compact operators is again C^*-isomorphic to the set of continuous functions over the unit circle. In conclusion, the C^*-algebra of the Hardy-Toeplitz operators is isomorphic (up to compact operators) to the C^*-algebra of the Bergman-Toeplitz operators.

In chapter four we start the generalization of the previous cases to higher dimensions. This time we consider the Bergman space over the unit ball in $\mathbb{C}^n$, defined just as in the one-dimensional case. In this case, the nice geometry of the unit ball (strictly pseudoconvex domain) guarantees the existence of peaking functions, which play an important role in the characterization of the C^*-algebra of the Toeplitz operators. At the end of this chapter we get the analogue to the previous cases of the characterization of the C^*-algebra of the Toeplitz operators.

For the Hardy space over the unit ball, in chapter five, we use more tools of function theory of several complex variables e.g. the Szego Kernel formula, which plays the same role as the Poisson Kernel formula in the one-dimensional case. Through this chapter we repeat many arguments used in the Bergman case, and our characterization of the C^*-algebra of the Toeplitz operators is similar in the Bergman and Hardy cases.

In chapter six we consider the Hardy space over the unit polydisc in $\mathbb{C}^2$. The Hardy space is defined as the closed subspace of $L^2(T^2)$ generated by the functions z^α for $z = (z_1, z_2)$, $\alpha = (\alpha_1, \alpha_2)$ with α_1, α_2 non-negative integers. As in the one-dimensional case, the Poisson Kernel formula permits extending functions on the Hardy space to holomorphic functions defined on the unit polydisc.
Unlike the previous cases, the characterization of the C^*-algebra of Toeplitz operators is more complicated due to the failure of the polydisc to be strictly pseudoconvex. Tensor products are introduced to analyze the C^*-algebra of Toeplitz operators.

In chapter seven, we use tools from the one-dimensional case to describe the Bergman space over the unit polydisc in $\mathbb{C}^2$. However, we do not obtain the same characterization of the Bergman-Toeplitz C^*-algebras as in the one-dimensional case.

The aim of the last chapter is to give an application of the previous results. It turns out

that in the unit ball case the Toeplitz operators that are Fredholm operators are the ones whose symbols never vanish on the boundary; therefore, the index is well defined for these operators.
Finally we shall prove an index theorem for Fredholm Toeplitz operators in the case of the unit ball.

Chapter 2

Bergman Space and Toeplitz Operators over the unit disc $\mathbb{D}$

The point of view in this chapter has been influenced by [Upm] and [Zhu]. Some of the assertions in this chapter are particular cases of stronger results that can be found in [Upm]. In this chapter we shall show that Toeplitz operators with non-constant polynomial symbols are natural examples of non-normal operators that arise in operator theory and verify that the algebra generated by these Toeplitz operators is commutative up to compact operators. From this characterization we can tell which operators are Fredholm (invertible up to compact operators) and capture the primary numerical invariant in operator theory, the index of a Fredholm operator. (See Chapter 8 for the discussion of index of Fredholm-Toeplitz Operators).

2.1 Definition and basic properties

Consider $L^2(\mathbb{D})$ the Lebesgue space of square-integrable functions on $\mathbb{D}$ with Lebesgue measure $dA(z) = dxdy = rdrd\theta$, and inner product,

$$\langle f, g\rangle = \int_{\mathbb{D}} \overline{f(z)}g(z)dA(z)$$

Convention: Inner products will be considered conjugate-linear in the first variable.

The following two propositions give the background to show that the Bergman space is a Hilbert space of holomorphic functions and to prove the existence of its reproducing kernel called the Bergman Kernel. Also, it can be proven that these propositions still hold if we replace the unit disc for the unit ball in $\mathbb{C}^n$.

Proposition 2.1.1. *Let $K \subset \mathbb{D}$ be a compact set, then the restriction map,*

$$R : L^2(\mathbb{D}) \cap O(\mathbb{D}) \to C(K),\ R(f) = f|_K$$

is continuous where the metric of $C(K)$ is the metric induced by the infinity norm.

Proof. Say $a \in K$ and $2d = dist(K, \partial\mathbb{D}) > 0$, then we have

$$\bar{B}(a, d) \subset \mathbb{D}$$

Take any $f \in L^2(\mathbb{D}) \cap O(\mathbb{D})$. Since f is holomorphic, we have

$$f(z) = \sum_{j\geq 0} c_j (z-a)^j \text{ converging uniformly on } \bar{B}(a, d)$$

Since for $j \neq k$

$$\int_{B(a,d)} \overline{(z-a)}^j (z-a)^k dA(z) = 0$$

we obtain

$$\int_{\mathbb{D}} |f(z)|^2 dA(z) \geq \int_{B(a,d)} |f(z)|^2 dA(z) \geq \sum_{j\geq 0} \int_{B(a,d)} |c_j (z-a)^j|^2 dA(z)$$

$$\geq |f(a)|^2 Area(B(a, d))$$

Since $Area(B(a, d))$ is constant for every a in K, we have

$$\|f\|_2 \geq \|f|_K\|_\infty \sqrt{Area(B(a, d))}$$

This proves the continuity of R.

□

Proposition 2.1.2. $L^2(\mathbb{D}) \cap O(\mathbb{D})$ *is a closed subspace of* $L^2(\mathbb{D})$

Proof. Take (f_n) a L^2-Cauchy sequence in $L^2(\mathbb{D}) \cap O(\mathbb{D})$. By Proposition 2.1.1, (f_n) is a Cauchy sequence in $C(K)$ for every compact subset of $\mathbb{D}$. Then, there exists $f \in O(\mathbb{D})$ such that $f_n \to f$ normally on $\mathbb{D}$ (i.e. convergence on compact subsets).
On the other hand, we have $\|f_n - h\|_2 \to 0$ for some $h \in L^2(\mathbb{D})$, because $L^2(\mathbb{D})$ is complete. This implies that (f_n) converges in measure to h; therefore, there exists a subsequence of (f_n) that converges a.e. to h. Thus, $f = h$ a.e.; hence, $f \in L^2(\mathbb{D}) \cap O(\mathbb{D})$. This proves that $L^2(\mathbb{D}) \cap O(\mathbb{D})$ is complete; therefore, a Hilbert space. Thus $L^2(\mathbb{D}) \cap O(\mathbb{D})$ is a closed subspace of $L^2(\mathbb{D})$.

□

Definition 2.1.3. $H^2(\mathbb{D}) := \overline{\{f \in C(\bar{\mathbb{D}}) : f \in O(\mathbb{D})\}}^{L^2}$ is called the Bergman Space over the unit disc.

Notice that $H^2(\mathbb{D}) \subset L^2(\mathbb{D}) \bigcap O(\mathbb{D})$ is a Hilbert space of holomorphic functions.

The orthogonal projection $P : L^2(\mathbb{D}) \rightarrow H^2(\mathbb{D})$ is called the Bergman projection. For any fixed $z \in \mathbb{D}$; by Proposition 2.1.1, the evaluation function

$$eval : H^2(\mathbb{D}) \rightarrow \mathbb{C}$$

$$eval(f) = f(z)$$

is continuous. Then, by the Riesz-Frechet Theorem, there exists $K_z \in H^2(\mathbb{D})$ such that

$$f(z) = \langle K_z, f \rangle = \int_{\mathbb{D}} \overline{K_z(w)} f(w) dA(w) \ \forall f \in H^2(\mathbb{D})$$

$K(z, w) := \overline{K_z(w)}$ is called the Bergman Kernel function.

2.2 Characterization of $H^2(\mathbb{D})$

For $0 < r < 1$ denote $\bar{D}_r = \{z \in D : |z| \leq r\}$. Now, for any $f \in L^2(\mathbb{D}) \bigcap O(\mathbb{D})$ with $f(z) = \sum_{j=0}^{\infty} d_j z^j$ converging normally on $\mathbb{D}$, define

$$f_r(z) := f(z)\chi_{\bar{\mathbb{D}}_r}$$

Then, $f_r(z) = \sum_{j=0}^{\infty} d_j z^j$ converges uniformly on $\bar{\mathbb{D}}_r$. It is clear that $f_r \in L^2(\mathbb{D})$

The "restriction" functions f_r are very useful to get information and approximate $f \in H^2(\mathbb{D})$. The proof of the following lemma just requires straightforward calculation; for this reason we will omit it.

Lemma 2.2.1.

$$\int_{\bar{\mathbb{D}}_r} |z|^{2n} dA(z) = \frac{\pi r^{2n+2}}{n+1} \ \textit{and} \ \int_{\bar{\mathbb{D}}_r} z^n \bar{z}^m dA(z) = 0 \ \forall n \neq m$$

Lemma 2.2.2. *$f_r \rightarrow f$ in $L^2(\mathbb{D})$ norm for any $f \in L^2(\mathbb{D}) \bigcap O(\mathbb{D})$*

Proof. Clearly $|f_r(z) - f(z)| \leq 2|f(z)|$, then $|f_r(z) - f(z)|^2 \leq 4|f(z)|^2$. Since $f_r \rightarrow f$ pointwise and $4|f(z)|^2$ is integrable, by the Lebesgue Convergence Theorem we obtain $f_r \rightarrow f$ in $L^2(\mathbb{D})$ norm.

□

Lemma 2.2.3. *For any* $f \in L^2(\mathbb{D}) \bigcap O(\mathbb{D})$ *with* $f(z) = \sum_{j=o}^{\infty} d_j z^j$ *converging normally on* $\mathbb{D}$

$$\langle z^n, f_r \rangle = \frac{\pi}{n+1} d_n r^{2n+2}$$

Proof.

$$\langle z^n, f_r \rangle = \int_{\mathbb{D}} f_r(z) \bar{z}^n dA(z) = \int_{\bar{\mathbb{D}}_r} \sum_{j=0}^{\infty} d_j z^j \bar{z}^n dA(z)$$

By uniform convergence,

$$= \sum_{j=0}^{\infty} \int_{\bar{\mathbb{D}}_r} d_j z^j \bar{z}^n dA(z) = d_n \int_{\bar{\mathbb{D}}_r} z^n \bar{z}^m dA(z) = \frac{\pi}{n+1} d_n r^{2n+2}$$

□

Lemma 2.2.4. *Let* $g \in L^2(\mathbb{D})$ *and* $f \in L^2(\mathbb{D}) \bigcap O(\mathbb{D})$*, then*

$$\langle g, f_r \rangle \to \langle g, f \rangle$$

In particular

$$\langle z^n, f \rangle = \frac{\pi}{n+1} d_n$$

where d_n *is the same as above.*

Proof. By the Cauchy-Schwarz Inequality and Lemma 2.2.2,

$$|\langle g, f_r \rangle - \langle g, f \rangle| = |\langle g, f_r - f \rangle| \leq \|g\|_2 \|f_r - f\|_2 \to 0$$

□

The next proposition shows a basis for $L^2(\mathbb{D}) \bigcap O(\mathbb{D})$. It turns out that the elements of this basis belong to the Bergman space $H^2(\mathbb{D})$; hence, the Bergman space coincides with $L^2(\mathbb{D}) \bigcap O(\mathbb{D})$.

Proposition 2.2.5. *Let* $\phi_n(z) := c_n z^n$ *where* $c_n = \sqrt{\frac{n+1}{\pi}}$*. Then* $\{\phi_n\}$ *is an orthonormal basis of* $L^2(\mathbb{D}) \bigcap O(\mathbb{D})$*.*

Proof. It is easy to check that $\{\phi_n\}$ is an orthonormal set. By Lemma 2.2.4, $\langle \phi_n, f \rangle = \frac{d_n}{c_n}$ for any $f \in L^2(\mathbb{D}) \bigcap O(\mathbb{D})$ with $f(z) = \sum_{j=0}^{\infty} d_j z^j$ converging normally.
It is not hard to verify that $\|f_r\|^2 = \sum_{n=0}^{\infty} \frac{|d_n|^2}{c_n{}^2} r^{2n+2}$. Since $f_r \to f$ in L^2 norm and $f_r(z) \to f(z)$ pointwise, we have $\|f_r\|^2 \to \|f\|^2$; hence, $\|f\|^2 = \sum_{n=0}^{\infty} \frac{|d_n|^2}{c_n{}^2}$. Therefore, we have

$$\sum_{n=0}^{\infty} \langle \phi_n, f \rangle = \|f\|^2$$

Then $f \in span\{\phi_n\}$.
Thus $\{\phi_n\}$ is an orthonormal basis of $L^2(\mathbb{D}) \bigcap O(\mathbb{D})$ and $f(z) = \sum_{n=0}^{\infty} \frac{d_n}{c_n} \phi_n(z) = \sum_{n=0}^{\infty} d_n z^n$ in L^2 norm.

$\square$

Corollary 2.2.6. *Let $f \in L^2(\mathbb{D}) \bigcap O(\mathbb{D})$ with $f(z) = \sum_{j=0}^{\infty} d_j z^j$ converging normally. Then, $f(z) = \sum_{j=0}^{\infty} d_j z^j$ converges in L^2 norm.*

Remark 2.2.1. Since $\phi_n \in H^2(\mathbb{D})$, $H^2(\mathbb{D}) = L^2(\mathbb{D}) \bigcap O(\mathbb{D})$.

Now, we have the tools to calculate the Bergman Kernel and give an explicit representation of the Bergman projection P.

Proposition 2.2.7. *Let $f \in H^2(\mathbb{D})$ with $f(z) = \sum_{j=o}^{\infty} d_j z^j$ converging normally on $\mathbb{D}$. Fix $z \in \mathbb{D}$, then*

$$f(z) = \frac{1}{\pi} \int_{\mathbb{D}} \frac{f(w)}{(1 - \bar{w}z)^2} dA(w)$$

Proof. Using power series

$$\frac{1}{(1 - \bar{w}z)} = \sum_{n=0}^{\infty} \bar{w}^n z^n$$

converges uniformly on $\bar{\mathbb{D}} \times \{z\}$. Notice that

$$\frac{\bar{w}}{(1 - \bar{w}z)^2} = \frac{d}{dz}(\frac{1}{1 - \bar{w}z}) \Rightarrow \frac{1}{(1 - \bar{w}z)^2} = \sum_{n=0}^{\infty} (n+1) \bar{w}^n z^n$$

converges uniformly on $\bar{\mathbb{D}} \times \{z\}$. By uniform convergence of the series, we have

$$\frac{1}{\pi} \int_{\mathbb{D}} f(w) \sum_{n=0}^{\infty} (n+1) \bar{w}^n z^n dA(w) = \frac{1}{\pi} \sum_{n=0}^{\infty} (\int_{\mathbb{D}} f(w)(n+1) \bar{w}^n dA(w)) z^n$$

By Lemma 2.2.4

$$= \sum_{n=0}^{\infty} d_n z^n = f(z)$$

$\square$

Remark 2.2.2. Recall that

$$f(z) = \int_{\mathbb{D}} \overline{K_z(w)} f(w) dA(w) \ \forall f \in H^2(\mathbb{D}) \ , \forall z \in \mathbb{D}$$

Then,

$$K(z, w) = \overline{K_z(w)} = \frac{1}{\pi(1 - \bar{w}z)^2}$$

2.3 Bergman-Toeplitz Operators over the unit disc $\mathbb{D}$

2.3.1 Definitions and basic properties

Definition 2.3.1. For f a continuous function on $\bar{\mathbb{D}}$ the multiplication operator with symbol f $m_f : L^2(\mathbb{D}) \to L^2(\mathbb{D})$ is defined by $m_f(g) := fg$

Remark 2.3.1. $\|m_f(g)\|_2 = \|fg\|_2 \leq \|f\|_\infty \|g\|_2 \Rightarrow \|m_f\|_{op} \leq \|f\|_\infty$

Remark 2.3.2. Since $\langle m_f(h), g\rangle = \langle h, \bar{f}g\rangle$ for every $h, g \in L^2(\mathbb{D})$, ${m_f}^* = m_{\bar{f}}$

Definition 2.3.2. For f a continuous function on $\bar{\mathbb{D}}$ the Bergman-Toeplitz operator with symbol f $T_f : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ is defined by $T_f(g) := P \circ m_f(g)$

Remark 2.3.3. By characterization of the Bergman kernel, we have

$$T_f(g)(z) = P(fg)(z) = \frac{1}{\pi}\int_{\mathbb{D}} \frac{f(w)g(w)}{(1-\bar{w}z)^2} dA(w) \ \forall g \in H^2(\mathbb{D})$$

Proposition 2.3.3. *For every $f \in C(\bar{\mathbb{D}})$ we have*
(i) $\|T_f\|_{op} \leq \|f\|_\infty$
(ii) ${T_f}^* = T_{\bar{f}}$
If $\varphi \in A(\mathbb{D}) := O(\mathbb{D}) \cap C(\bar{\mathbb{D}})$, then
(iii) $T_f T_\varphi = T_{f\varphi}$
(iv) $T_{\bar{\varphi}} T_f = T_{\bar{\varphi} f}$

Proof. (i) Recall that $\|P\|_{op} = 1$. Therefore, $\|T_f\|_{op} = \|P \circ m_f\|_{op} \leq \|P\|_{op}\|m_f\|_{op} = \|m_f\|_{op} \leq \|f\|_\infty$.

(ii) Recall that P is self adjoint. Notice that we can write $T_f = Pm_fP$. Therefore, using Remark 2.3.2 we have ${T_f}^* = P^*{m_f}^*P^* = Pm_{\bar{f}}P = T_{\bar{f}}$

(iii) It is easy to see that $A(\mathbb{D})$ is an algebra. For any $h \in H^2(\mathbb{D})$ there exists (h_n) in $A(\mathbb{D})$ such that $\|h_n - h\|_2 \to 0$. It follows that $\|\varphi h_n - \varphi h\|_2 \to 0$; hence, $\varphi h \in H^2(\mathbb{D})$. Moreover, $T_\varphi(h) = Pm_\varphi(h) = \varphi h$, i.e., $T_\varphi = m_\varphi P$. Then, $T_f T_\varphi = Pm_f m_\varphi P = Pm_{f\varphi}P = T_{f\varphi}$.

(iv) Using (iii) we have $T_{\bar{f}} T_\varphi = T_{\bar{f}\varphi}$. Taking adjoint we obtain ${T_\varphi}^* {T_{\bar{f}}}^* = {T_{\bar{f}\varphi}}^*$
Applying (ii) we have $T_{\bar{\varphi}} T_f = T_{\bar{\varphi} f}$

□

Definition 2.3.4. The Bergman-Toeplitz C*-algebra over $\mathbb{D}$ is defined as the unital C*-algebra $\mathcal{T}(\mathbb{D}) := C^*\langle T_f : f \in C(\bar{\mathbb{D}})\rangle$ generated by all Toeplitz operators with continuous symbols.

Proposition 2.3.5. *$\mathcal{T}(\mathbb{D}) = C^*\langle T_p : p \in P(\mathbb{C})\rangle$, where $P(\mathbb{C})$: polynomials over $\mathbb{C}$*

Proof. Let $f \in C(\bar{\mathbb{D}})$. Applying the Stone-Weierstrass Theorem there exists a sequence of polynomials $p_n(z,\bar{z})$,that can be written as $p_n(z,\bar{z}) = \sum_{j=1}^{m} \overline{p_j(z)} q_j(z)$ where $p_j, q_j \in P(\mathbb{C})$, converging uniformly to f.
Therefore, using Proposition 2.3.3,

$$\|T_f - \sum_{j=1}^{m} T_{p_j}{}^* T_{q_j}\|_{op} = \|T_{f-\sum_{j=1}^{m}\overline{p_j}q_j}\|_{op} \leq \|f - \sum_{j=1}^{m} \overline{p_j} q_j\|_\infty$$

But each $p_n(z,\bar{z})$ can be written as $p_n(z,\bar{z}) = \sum_{j=1}^{m} \overline{p_j(z)} q_j(z)$. Thus

$$T_f \in C^*\langle T_p : p \in P(\mathbb{C})\rangle$$

because p_n converges uniformly to f and $T_{p_j}{}^* T_{q_j} \in C^*\langle T_p : p \in P(\mathbb{C})\rangle$.

□

The following proposition shows that T_p is a non-normal operator for any non-constant polynomial p. This means that we have found our first example of non-commutative C^*-algebra generated (except for constant polynomials) by non-normal operators.
If we form the quotient of $\mathcal{T}(\mathbb{D})$ by its commutator ideal we will obtain a commutative C^*-algebra that is easier to deal with. The key idea in understanding $\mathcal{T}(\mathbb{D})$ is the characterization of its commutator ideal.

Proposition 2.3.6. *T_p is a non-normal operator for any non-constant polynomial p. Moreover, $\mathcal{T}(\mathbb{D})$ is a non-commutative C^*-algebra.*

Proof. It is not difficult to verify that the integral over $\mathbb{D}$ of any non-constant polynomial is zero. Let p any non-constant polynomial. Since $\phi_n p$ is a non-constant polynomial for every n, we have $\langle \bar{p}, \phi_n\rangle = 0$. This means that $\bar{p} \in H^2(\mathbb{D})^\perp$; therefore, $P(\bar{p}) = 0$. On the other hand, $\langle 1, |p|^2\rangle \neq 0$; hence, $P(|p|^2) \neq 0$.
Putting things together we have

$$T_p T_{\bar{p}}(1) = T_p P(\bar{p}) = 0$$

and

$$T_{\bar{p}} T_p(1) = T_{\bar{p}}(p) = P(|p|^2) \neq 0$$

Thus T_p is a non-normal operator.

□

The next proposition is crucial in our characterization of $\mathcal{T}(\mathbb{D})$. At the end of the next section we shall prove as a consequence of the next proposition that the C^*-algebra of compact operators is included in $\mathcal{T}(\mathbb{D})$.

Proposition 2.3.7. *$\mathcal{T}(\mathbb{D})$ acts irreducibly on $H^2(\mathbb{D})$*

Proof. Let $B : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ be an orthogonal projection that commutes with $\mathcal{T}(\mathbb{D})$. In particular, $T_f B = BT_f$ on $H^2(\mathbb{D})$ for every $f \in C(\bar{\mathbb{D}})$.
It is clear that $p(z)B(1) \in H^2(\mathbb{D})$ for any $p \in P(\mathbb{C})$. Then for any $p, q \in P(\mathbb{C})$

$$B(q) = B(T_q(1)) = T_q B(1) = P(qB(1)) = qB(1) \tag{2.3.1}$$

$$\begin{aligned}\langle B(1), \bar{p}q\rangle &= \langle pB(1), q\rangle = \langle T_p(B(1)), q\rangle = \langle (T_p \circ B)(1), q\rangle \\ &= \langle (B \circ T_p)(1), q\rangle = \langle B(p), q\rangle\end{aligned}$$

$$\langle B(p), q\rangle = \langle p, B(q)\rangle = \langle p, qB(1)\rangle = \langle \overline{B(1)}, \bar{p}q\rangle \ \Rightarrow \ \langle B(1) - \overline{B(1)}, \bar{p}q\rangle = 0$$

Since $\{\sum_{j=1}^{n} \overline{p_j(z)} q_j(z) : p_j, q_j \in P(\mathbb{C})\}$ is uniformly dense in $C(\bar{\mathbb{D}})$, $\{\sum_{j=1}^{n} \overline{p_j(z)} q_j(z) : p_j, q_j \in P(\mathbb{C})\}$ is dense in $L^2(\mathbb{D})$.
Then, $B(1) - \overline{B(1)} = 0$. Because B(1) is holomorphic, B(1) is constant function.
Since $B^2 = B$ and $B(1)$ is constant, $B(1) = B^2(1)$ implies $B(1) = 1$ or 0.
Thus; by equation 2.3.1, $B = 0$ or $B = Id$. □

2.4 Toeplitz Operators and Compact Operators

In the first part of this section, we study the compact Toeplitz operators. It turns out that a Toeplitz operator is compact if and only its symbol vanishes on the boundary of $\mathbb{D}$.

Lemma 2.4.1. *Let $\{K_n\}$ be the exhaustion of $\mathbb{D}$ by compact subsets, where $K_n = \overline{B}(0, 1 - \frac{1}{n})$. Then for each $K = K_n$, the family*

$$\mathcal{F} = \{h|_K : h \in H^2(\mathbb{D}), \|h\|_2 \leq 1\} \text{ has compact closure in } C(K)$$

Proof. Say $K = K_n$ and $E = K_{n+1}$ for a fixed n. Denote $2r = dist(K, \partial E)$. Applying Proposition 2.1.1, $\exists c > 0$ such that

$$sup|h(E)| \leq c\|h\|_2 \ \ \forall h \in H^2(\mathbb{D})$$

Now, for any $z_1, z_2 \in K$ with $|z_1 - z_2| \leq r$, we have $[z_1, z_2] \subset K \subset E$
Applying the Mean Value Theorem:

$$|h(z_1) - h(z_2)| \leq sup_{w \in [z_1, z_2]} |h'(w)||z_1 - z_2| \ \ ...(*)$$

Observe that for any $w_0 \in [z_1, z_2]$, $\overline{B}(w_0, r) \subseteq E$. Now, by the Cauchy Integral Inequality

$$|h'(w_0)| \leq \frac{sup\{|h(\overline{B}(w_0, r)|\}}{r} \leq \frac{\sup |h(E)|}{r}$$

Therefore

$$sup_{w \in [z_1, z_2]} |h'(w)| \leq \frac{\sup |h(E)|}{r} \leq \frac{c}{r} \|h\|_2$$

Using (*) and the above result; we get

$$|h(z) - h(w)| \leq \frac{c}{r} \|h\|_2 |z - w| \leq \frac{c}{r} |z - w| \quad \forall h \in \mathcal{F}$$

Then $\mathcal{F}$ is equicontinuous. Moreover, for any $z \in K$ and $h \in \mathcal{F}$, $|h(z)| \leq c$
By the Arzela-Ascoli Theorem, $\mathcal{F}$ has compact closure in $C(K)$.

□

Proposition 2.4.2. *Let $f \in C(\bar{\mathbb{D}})$ with $f|_{\partial \mathbb{D}} = 0$. Then, T_f is a compact operator.*

Proof. Consider $\{K_n\}$ and $\mathcal{F}$ as in the previous lemma. Say $K = K_n$ for some fixed n.
By the previous lemma, the restriction map

$$r : H^2(\mathbb{D}) \to C(K) \ , \ r(h) := h|_K$$

is a compact operator.
Notice that the inclusion

$$i : C(K) \to L^2(\mathbb{D}) \ , \ i(g) := g\chi_K$$

is continuous, because

$$\|g\chi_K\|_2{}^2 = \int_{\mathbb{D}} |g\chi_K|^2 dA \leq \|g\|_\infty{}^2 Area(\mathbb{D})$$

Define $m_{\chi_K} : H^2(\mathbb{D}) \to L^2(\mathbb{D})$ as $m_{\chi_K} = i \circ r$. Since i is continuous and r is compact operator, m_{χ_K} is compact operator.
Now, observe that $m_{f\chi_K} : H^2(\mathbb{D}) \to L^2(\mathbb{D})$ defined as $m_{f\chi_K} := m_f \circ m_{\chi_K}$ is a compact operator, because m_f is a bounded operator.
Note that $f\chi_K$ is not necessarily continuous, but we can denote $T_{f\chi_K} = P \circ m_{f\chi_K}$ which is a compact operator.

Claim: $f\chi_{K_m} \to f$ in $L^\infty(\bar{\mathbb{D}})$ norm as $m \to \infty$
Proof:
By uniform continuity of f, given $\epsilon > 0$, there exists $\delta \in (0, 1)$ such that

$$|z - w| < \delta \Rightarrow |f(z) - f(w)| < \epsilon \ \forall z, w \in \bar{\mathbb{D}}$$

Since $\{K_n\}$ is a nested exhaustion of compact sets of $\mathbb{D}$, $\exists n \in \mathbb{N}$ such that $\bar{B}(0, 1-\delta) \subseteq K_n$.
Therefore, if $z \in \bar{\mathbb{D}} \setminus K_n \Rightarrow \exists z_0 \in \partial\mathbb{D}$ such that $|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| = |f(z)| < \epsilon$.
Hence, $|f(z) - (f\chi_{K_n})(z)| < \epsilon \; \forall z \in \bar{\mathbb{D}}$.
Since $K_n \subseteq K_m$, $\forall m \geq n$, $\|f - (f\chi_{K_m})\|_\infty < \epsilon \; \forall m \geq n$.
Thus $\|f - (f\chi_{K_m})\|_\infty \to 0$

Finally,

$$\|T_f - T_{f\chi_{K_m}}\|_{op} = \|T_{f - f\chi_{K_m}}\|_{op} \leq \|f - (f\chi_{K_m})\|_\infty \to 0$$

Since $T_{f\chi_{K_m}}$ are compact operators, T_f is compact operator.

□

In order to prove the remaining of our characterization of compact Toeplitz operators, we will define and use some properties of the Berezin Transforms. The Berezin transform of a Toeplitz operator recovers information about the symbol; in fact, it is a continuous function that coincides with the symbol in S^1.

Definition 2.4.3. Define for a fixed $a \in \mathbb{D}$, $\varphi_a(z) = \frac{a-z}{1-\bar{a}z} \; \forall z \in \mathbb{D}$.

Clearly φ_a defined above is an automorphism (*Möbius* transformation) of the unit disc.

Remark 2.4.1. Notice that $k_a(z) := \varphi_a'(z) = \frac{1-|a|^2}{(1-z\bar{a})^2}$ and $k_a \in H^2(\mathbb{D})$. Moreover, $Jac_{\varphi_a}(z) = |k_a(z)|^2$.

Remark 2.4.2. Recall that the Bergman Kernel $K_a(z) = \frac{1}{\pi(1-z\bar{a})^2}$ and $K_a(a) = \langle K_a, K_a \rangle = \|K_a\|^2$. Therefore, $k_a(z) = \pi(1-|a|^2)K_a$ has norm $\|k_a\| = \pi(1-|a|^2)\sqrt{K_a(a)} = \sqrt{\pi}$.

Definition 2.4.4. $\tilde{f}(z) := \frac{1}{\pi}\langle k_z, T_f(k_z)\rangle$ is called the Berezin Transform of f.

Remark 2.4.3.

$$\langle k_z, T_f(k_z)\rangle = \langle k_z, P(fk_z)\rangle = \langle P(k_z), fk_z\rangle = \langle k_z, fk_z\rangle = \int_{\mathbb{D}} f(w)|k_z(w)|^2 dA(w)$$

Changing variables,

$$\tilde{f}(z) = \frac{1}{\pi}\int_{\mathbb{D}} (f \circ \varphi_z)(w) dA(w)$$

Proposition 2.4.5. *$\tilde{f} \in C(\bar{\mathbb{D}})$ and $\tilde{f} = f$ on $\partial\mathbb{D}$ for every $f \in C(\bar{\mathbb{D}})$*

Proof. Notice that for any $a \in \partial\mathbb{D}$ we have $\varphi_z(w) = \frac{z-w}{1-\bar{z}w} \to a$ as $z \to a$ $\forall w \in \mathbb{D}$.
By the continuity of f, $(f \circ \varphi_z)(w) \to f(a)$ as $z \to a$ $\forall w \in \mathbb{D}$; and, $|f \circ \varphi_z(w)| \leq \|f\|_\infty$ $\forall w \in \mathbb{D}$ $\forall z \in \mathbb{D}$.
By the Lebesgue Convergence Theorem,

$$\tilde{f}(a) := \lim_{z \to a} \frac{1}{\pi} \int_{\mathbb{D}} (f \circ \varphi_z)(w) dA(w) = \frac{1}{\pi} \int_{\mathbb{D}} f(a) dA(w) = f(a)$$

The continuity of $\tilde{f}$ on $\mathbb{D}$ follows by using the same argument.

□

Lemma 2.4.6. $k_a \to 0$ *weakly as* $|a| \to 1$

Proof. Recall that $k_a(z) = \pi(1-|a|^2)K_a(z)$. Then

$$\langle k_a, z^n \rangle = \pi(1-|a|^2)\langle K_a, z^n \rangle = \pi a^n (1-|a|^2)$$

Then

$$\langle k_a, z^n \rangle \to 0 \text{ as } |a| \to 1$$

Therefore, $\langle k_a, \phi_n \rangle \to 0$ as $|a| \to 1$. This means, $k_z \to 0$ weakly as $|z| \to 1$.

□

Theorem 2.4.7. *Suppose* $f \in C(\bar{\mathbb{D}})$, *then* T_f *is a compact operator if and only if* $f \in C_0(\mathbb{D})$

Proof. $f \in C_0(\mathbb{D})$ implies T_f is compact operator was proven in Proposition 2.4.2.
Suppose T_f is a compact operator. By Remark 2.4.2

$$|\tilde{f}(z)| = \frac{1}{\pi} |\langle k_z, T_f(k_z) \rangle| \leq \frac{1}{\pi} \|k_z\|_2 \|T_f(k_z)\|_2 \leq \sqrt{\pi} \|T_f(k_z)\|_2$$

Since $k_z \to 0$ weakly as $|z| \to 1$ and T_f is a compact operator, $\|T_f(k_z)\|_2 \to 0$ as $z \to 1$.
Thus, $\tilde{f}(z_0) = 0$, $\forall z_0 \in \partial\mathbb{D}$. Since $\tilde{f}$ and f are equal on $\partial\mathbb{D}$, $f \in C_0(\mathbb{D})$. □

Remember that $\{\phi_n\}_{n=0}^{+\infty}$ is an orthonormal basis for $H^2(\mathbb{D})$

$$\text{where} \quad \phi_n(z) = c_n z^n, \quad c_n = \sqrt{\frac{n+1}{\pi}}$$

It is also true that: If $g \in H^2(\mathbb{D})$

$$g(z) = \sum_{n=0}^{+\infty} \langle g, \phi_n \rangle \phi_n(z)$$

Let $P : L^2(\mathbb{D}) \longrightarrow H^2(\mathbb{D})$ the orthogonal projection over $H^2(\mathbb{D})$. Then, we have for any $g \in L^2(\mathbb{D})$,

$$P(g)(z) = \sum_{n=0}^{+\infty} \langle P(g), \phi_n \rangle \phi_n(z)$$

Since P is self-adjoint,

$$\begin{aligned} P(g)(z) &= \textstyle\sum_{n=0}^{+\infty} \langle g, P(\phi_n) \rangle \phi_n(z) \\ &= \textstyle\sum_{n=0}^{+\infty} \langle g, \phi_n \rangle \phi_n(z). \end{aligned}$$

To prove that the semi-commutators $T_{fg} - T_f \circ T_g$ are compact operators, we consider several cases in the following lemmas.

Lemma 2.4.8.

$$P(z^n \overline{z}^m) = \begin{cases} 0, \ n < m \\ \frac{c_{n-m}}{c_n^2} \phi_{n-m}, \ n \geq m. \end{cases}$$

Proof. By the above formula:

$$\begin{aligned} P(z^n \overline{z}^m) &= \textstyle\sum_{k=0}^{+\infty} \langle z^n \overline{z}^m, \phi_k \rangle \phi_k \\ &= \textstyle\sum_{k=0}^{+\infty} \langle z^n, z^m \phi_k \rangle \phi_k \\ &= \textstyle\sum_{k=0}^{+\infty} \langle \frac{1}{c_n} \phi_n, \frac{c_k}{c_{m+k}} \phi_{m+k} \rangle \phi_k \\ &= \textstyle\sum_{k=0}^{+\infty} \frac{c_k}{c_n c_{m+k}} < \phi_n, \phi_{m+k} > \phi_k \\ &= \begin{cases} 0, \ n < m \\ \frac{c_{n-m}}{c_n^2} \phi_{n-m}, \ n \geq m. \end{cases} \end{aligned}$$

□

Lemma 2.4.9. *$T_{z^p \overline{z}^q} - T_{z^p} \circ T_{\overline{z}^q}$ is a compact operator.*

Proof. By the above lemma $T_{z^p \overline{z}^q}(\phi_k) = P(c_k z^{p+k} \overline{z}^q) = \begin{cases} 0, \ p+k < q \\ \frac{c_k c_{p+k-q}}{c_{p+k}^2} \phi_{p+k-q}, \ p+k \geq q. \end{cases}$

$$T_{\overline{z}^q}(\phi_k) = P(c_k z^k \overline{z}^q) = \begin{cases} 0, \ k < q \\ \frac{c_{k-q}}{c_k} \phi_{k-q}(z), \ k \geq q. \end{cases}$$

$$T_{z^p}(T_{\overline{z}^q}(\phi_k)) = \begin{cases} 0, \ k < q \\ \\ P(\frac{c_{k-q}}{c_k} z^p c_{k-q} z^{k-q}), \ k \geq q. \end{cases}$$

$$= \begin{cases} 0, \ k < q \\ \\ \frac{c_{k-q}^2}{c_k c_{p+k-q}} \phi_{p+k-q}(z), \ k \geq q. \end{cases}$$

For $k \geq q$:

$$(T_{z^p\overline{z}^q} - T_{z^p} \circ T_{\overline{z}^q})(\phi_k) = (\frac{c_k c_{p+k-q}}{c_{p+k}^2} - \frac{c_{p-q}^2}{c_k c_{p+k-q}})\phi_{p+k-q}(z)$$

$$a_k := (\frac{c_k c_{p+k-q}}{c_{p+k}^2} - \frac{c_{p-q}^2}{c_k c_{p+k-q}}) = \frac{c_k^2 c_{p+k-q}^2 - c_{k-q}^2 c_{p+k}^2}{c_{p+k}^2 c_k c_{p+k-q}} = \frac{1}{\pi^2} \frac{(k+1)(p+k-q+1)-(k-q+1)(p+k+1)}{c_{p+k}^2 c_k c_{p+k-q}}$$

$$a_k = \frac{pq}{\pi^2 c_{p+k}^2 c_k c_{p+k-q}}$$

$$\Longrightarrow \sum_k^{+\infty} a_k \text{ converges}$$

Let a_k defined as above. Denote $A = T_{z^p\overline{z}^q} - T_{z^p} \circ T_{\overline{z}^q}$, $A(\phi_k) = a_k \phi_{p+k-q}$, $a_k \geq 0$.

Define $B_r(\phi_k) = \begin{cases} a_k \phi_{p+k-q}, \ k \leq r \\ \\ 0, \ k > r \end{cases}$

Let $h \in H^2(D)$ with $\|h\|_2 = 1$ and $h(z) = \sum \alpha_k \phi_k$

$$\Longrightarrow |\ \alpha_k\ | \leq 1$$

$$\|\ (A - B_r)(\sum \alpha_k \phi_k)\ \| = \|\ \sum_{k>r} \alpha_k a_k \phi_{p+k-q}\ \| \leq \sum_{k>r} \|\ \alpha_k a_k \phi_{p+k-q}\ \|$$

$$\leq \sum_{k>r} |\ \alpha_k\ |\ a_k \leq \sum_{k>r} a_k$$

Then

$$\|\ A - B_r\ \|_{op} \leq \sum_{k>r} a_k$$

Since $\sum a_k$ converges,

$$\sum_{k>r} a_k \to 0 \quad as \quad r \to +\infty$$

Then $B_r \xrightarrow{op} A$.
Thus A is a compact operator.

□

Lemma 2.4.10. *$T_{z^p\overline{z}^q} \circ T_{z^r\overline{z}^l} - T_{(z^p z^r)(\overline{z}^q \overline{z}^l)}$ is a compact operator.*

Proof. $T_{z^p\overline{z}^q} = T_{\overline{z}^q z^p} = T_{\overline{z}^q} \circ T_{z^p}$
$T_{z^r\overline{z}^l} = T_{\overline{z}^l z^r} = T_{\overline{z}^l} \circ T_{z^r}$
$T_{(z^p z^r)(\overline{z}^q\overline{z}^l)} = T_{\overline{z}^q} \circ T_{z^p\overline{z}^l z^r} = T_{\overline{z}^q} \circ T_{z^p\overline{z}^l} \circ T_{z^r}$

$T_{z^p\overline{z}^q} \circ T_{z^r\overline{z}^l} - T_{(z^p z^r)(\overline{z}^q\overline{z}^l)} = T_{\overline{z}^q} \circ T_{z^p} \circ T_{\overline{z}^l} \circ T_{z^r} - T_{\overline{z}^q} \circ T_{z^p\overline{z}^l} \circ T_{z^r} = T_{\overline{z}^q} \circ (T_{z^p} \circ T_{\overline{z}^l} - T_{z^p\overline{z}^l}) \circ T_{z^r}$
By previous lemma, $T_{z^p\overline{z}^q} \circ T_{z^r\overline{z}^l} - T_{(z^p z^r)(\overline{z}^q\overline{z}^l)}$ is a compact operator.

□

Lemma 2.4.11. *Let $\varphi(z,\bar{z})$, $\psi(z,\bar{z})$ be polynomials, then*
(a) $T_\varphi \circ T_{z^p\overline{z}^q} - T_{\varphi z^p\overline{z}^q}$ is a compact operator.
(b) $T_{z^p\overline{z}^q} \circ T_\varphi - T_{\varphi z^p\overline{z}^q}$ is a compact operator.
(c) $T_\varphi \circ T_\psi - T_{\varphi\psi}$ is a compact operator.

Proof. **(a)** Say $\varphi(z,\bar{z}) = \sum b_{i,j} z^{\alpha_i}\overline{z}^{\beta_j}$
Then

$$T_\varphi \circ T_{z^p\overline{z}^q} - T_{\varphi z^p\overline{z}^q} = \sum b_{i,j}(T_{z^{\alpha_i}\overline{z}^{\beta_j}} \circ T_{z^p\overline{z}^q} - T_{z^p\overline{z}^q z^{\alpha_i}\overline{z}^{\beta_j}})$$

It follows by using the previous lemma.
(b) and **(c)** follow similarly.

□

Theorem 2.4.12. *Let φ, $\psi \in C(\bar{D})$, then $T_\varphi \circ T_\psi - T_{\varphi\psi}$ is a compact operator.*

Proof. First, let $p(z,\bar{z})$ be a polynomial. Since ψ is continuous, there exists $q_n(z,\bar{z}) \to \psi$ uniformly. Then

$$\|T_p \circ T_{q_n} - T_p \circ T_\psi - (T_{pq_n} - T_{p\psi})\|_{op} = \|T_p \circ T_{q_n-\psi} - T_{p(q_n-\psi)}\|_{op} \leq \|T_p\|_{op}\|T_{q_n\psi}\|_{op} + \|T_{p(q_n-\psi)}\|_{op}$$

$$\leq \|p\|_\infty\|q_n - \psi\|_\infty + \|p\|_\infty\|q_n - \psi\|_\infty \ \to 0 \text{ as } n \to \infty$$

Thus $T_p \circ T_{q_n} - T_{pq_n} \to T_p \circ T_\psi - T_{p\psi}$ in operator norm.
By previous lemma, $T_p \circ T_{q_n} - T_{pq_n}$ are compact operators; therefore, $T_p \circ T_\psi - T_{p\psi}$ is compact operator.
Likewise, we can show that $T_\varphi \circ T_\psi - T_{\varphi\psi}$ is compact operator, by fixing ψ and approximating φ (uniformly) by polynomials.

□

Now, we are ready to prove the main theorem of this chapter, the characterization of $\mathcal{T}(\mathbb{D})$.

Theorem 2.4.13. *The Bergman-Toeplitz C^*-algebra $\mathcal{T}(\mathbb{D})$ has the commutator ideal $\mathcal{K}(H^2(\mathbb{D}))$ (compact operators), and there exists a C^*-isomorphism*

$$\nu : \mathcal{T}(\mathbb{D})/\mathcal{K}(H^2(\mathbb{D})) \to C(S^1)$$

such that for every $f \in C(\bar{\mathbb{D}})$,

$$\nu(T_f + \mathcal{K}(H^2(\mathbb{D}))) = f|_{S^1}$$

Proof. The commutator

$$[T_f, T_g] := T_f T_g - T_g T_f = (T_f T_g - T_{fg}) - (T_g T_f - T_{gf})$$

is a compact operator, by the previous theorem.
Then, the commutator ideal $\mathcal{T}'(\mathbb{D})$ generated by the commutators $[T_f, T_g]$ is contained in $\mathcal{K}(H^2(\mathbb{D}))$
We know that $\mathcal{T}(\mathbb{D})$ is not commutative, then $\mathcal{T}'(\mathbb{D}) \neq 0$
Since $\mathcal{T}(\mathbb{D})$ acts irreducibly on $H^2(\mathbb{D})$ and by Corollary 2 of [Arv, p. 18], we have $\mathcal{K}(H^2(\mathbb{D})) \subseteq \mathcal{T}(\mathbb{D})$. Then $\mathcal{T}'(\mathbb{D})$ is a nonzero ideal of $\mathcal{K}(H^2(\mathbb{D}))$. By Corollary 1 of [Arv, p. 18] $\mathcal{T}'(\mathbb{D}) = \mathcal{K}(H^2(\mathbb{D}))$.
Therefore,

$$\rho : C(\bar{\mathbb{D}}) \to \mathcal{T}(\mathbb{D})/\mathcal{K}(H^2(\mathbb{D})) \ , \ \rho(f) := T_f + \mathcal{K}(H^2(\mathbb{D}))$$

is a (well-defined) C*-homomorphism that is surjective.
Notice that the restriction map

$$r : C(\bar{\mathbb{D}}) \to C(S^1) \ , \ r(f) := f|_{S^1}$$

is a surjective C*-homomorphism. Indeed, for any $f \in C(S^1)$ we can define

$$\tilde{f}(z) := |z| f(\frac{z}{|z|}) \text{ if } z \neq 0 \text{ and } \tilde{f}(0) := 0$$

Clearly, $\tilde{f}(z) \in C(\bar{\mathbb{D}})$ and $\tilde{f}|_{S^1} = f$.
If $f \in Ker(\rho)$ i.e. vanishes on S^1, then T_f is a compact operator (by $Theorem$ 2.4.7). Therefore, there exists $\bar{\rho}$ surjective C*-homomorphism such that $\bar{\rho} \circ r = \rho$.
Now, take $f \in Ker(\bar{\rho})$ i.e. $T_{\tilde{f}}$ compact operator. By $Theorem$ 2.4.7, $\tilde{f}$ vanishes on S^1. This implies $f \equiv 0$. Thus $\bar{\rho}$ is injective; hence, C*-isomorphism.

□

Chapter 3

Hardy Space and Toeplitz Operators over the unit disc $\mathbb{D}$

The viewpoint in the first two sections of this chapter are influenced by [Yng], and the last two sections follow the line of reasoning from [Upm]. As in the Bergman case (Chapter 2) we shall show that Toeplitz operators with non-constant polynomial symbols are non-normal operators and verify that the algebra generated by these Toeplitz operators is commutative up to compact operators. Also, in this case we can tell which operators are Fredholm. We will discuss the index of these operators in Chapter 8. Moreover, we observe that the C^*-algebra of Bergman-Toeplitz operators is C^*-isomorphic (up to compact operators) to the C^*-algebra of Hardy-Toeplitz operators.

3.1 Definition and basic properties

The purpose of this section is to define and show an orthonormal basis of the Hardy space on the unit disc. Once again the "scaled" or "radial" functions are very useful to get information about functions on the Hardy space.

Consider $L^2(S^1) \approx L^2(0, 2\pi)$ the Lebesgue space of square-integrable functions on S^1 with Haar measure $dm(z) = \frac{d\theta}{2\pi}$, and inner product,

$$\langle f, g\rangle = \int_{S^1} \overline{f(z)} g(z) dm(z)$$

Remark 3.1.1.

$$\int_{S^1} f(z) dm(z) := \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) d\theta$$

Definition 3.1.1. $H^2(S^1)$ defined as the $L^2(S^1)$ closure of $\{f|_{S^1} : f \in C(\bar{\mathbb{D}}) \cap O(\mathbb{D})\}$ is called the Hardy Space over $\mathbb{D}$

Theorem 3.1.2. *$(z^n)_{n\in\mathbb{Z}}$ is an orthonormal basis of $L^2(S^1)$*

Proof. It is known that $(e^{in\theta})_{n\in\mathbb{Z}}$ is an orthonormal basis of $L^2(0, 2\pi)$. It follows by the identification $L^2(S^1) \approx L^2(0, 2\pi)$ that $(z^n)_{n\in\mathbb{Z}}$ is an orthonormal basis of $L^2(S^1)$.

□

Lemma 3.1.3. *Let $f \in C(\bar{\mathbb{D}}) \cap O(\mathbb{D})$. Define for any $r \in (0,1)$ $f_r(z) := f(rz)$ $\forall z \in \bar{\mathbb{D}}$. Then $f_r|_{S^1} \to f|_{S^1}$ in $L^2(S^1)$ norm as $r \to 1$.*

Proof. For every $z \in S^1$ we have $f_r(z) \to f(z)$ pointwise and $|f_r(z) - f(z)|^2 \leq 4\|f\|^2_\infty$ (the infinity norm on the right is considered on $\bar{\mathbb{D}}$). It follows by the Lebesgue Convergence Theorem.

□

Theorem 3.1.4. *Let $f \in C(\bar{\mathbb{D}}) \cap O(\mathbb{D})$ with $f(z) = \sum_{j=0}^{\infty} a_j z^j$ converging normally on D. Then $f(z) = \sum_{j=0}^{\infty} a_j z^j$ in $L^2(S^1)$ norm.*

Proof. Notice that $f_r(z) = \sum_{j=0}^{\infty} a_j r^j z^j$ converges uniformly on $\bar{D}$. Then, by the above lemma, $f_r(z) = \sum_{j=0}^{\infty} a_j r^j z^j$ converges in $L^2(S^1)$ norm as $r \to 1$.
Since $f_r \to f$ in $L^2(S^1)$ norm and by the Cauchy-Schwarz Inequality, $\langle z^n, f_r\rangle \to \langle z^n, f\rangle$ as $r \to 1$.
But $\langle z^n, f_r\rangle = a_n r^n$ $\forall n \geq 0$ and 0 otherwise. Therefore, $\langle z^n, f\rangle = a_n$ $\forall n \geq 0$ and 0 otherwise. Thus, $f(z) = \sum_{j=0}^{\infty} a_j z^j$ in L^2 norm, because $(z^n)_{n\in\mathbb{Z}}$ is an orthonormal basis of $L^2(S^1)$.

□

Corollary 3.1.5. *$(z^n)_{n\geq 0}$ is an orthonormal basis of $H^2(S^1)$.*

Proof. By the previous theorem, $\{f|_{S^1} : f \in C(\bar{\mathbb{D}}) \cap O(\mathbb{D})\} \subseteq clos\{(z^n)_{n\geq 0}\}$, where $clos$ means $L^2(S^1)$ closure. Since $clos\{(z^n)_{n\geq 0}\}$ is a closed subspace of $L^2(S^1)$, we have $H^2(S^1) \subseteq clos\{(z^n)_{n\geq 0}\}$.
On the other hand, $z^n \in H^2(S^1)$. Thus $clos\{(z^n)_{n\geq 0}\} \subseteq H^2(S^1)$. □

3.2 Characterization of $H^2(S^1)$:

In this section we are going to look more closely at the Hardy space. We shall prove that we can extend functions in the Hardy space to holomorphic function defined on the unit disc. Moreover, thank to the Poisson kernel we will show an explicit formula for this extension.

Theorem 3.2.1. *Let $f \in H^2(S^1)$ with $f(z) = \sum_{j=0}^{\infty} a_j z^j$ in $L^2(S^1)$ norm. Then $\tilde{f}(z) = \sum_{j=0}^{\infty} a_j z^j$ defines a holomorphic function in $\mathbb{D}$, i.e., converges normally on $\mathbb{D}$*

Proof. Notice that $\sum_{j=0}^{\infty} |a_j|^2 = \|f\|^2 < \infty$. Then $|a_j| \to 0$. This implies that $\exists n_0 \in \mathbb{N}$ such that $|a_j| < 1 \ \forall j > n_0$; therefore, $\limsup \sqrt[j]{|a_j|} \leq 1$.
By the Cauchy-Hadamard formula,

$$R = \frac{1}{\limsup \sqrt[j]{|a_j|}} \geq 1$$

Thus, $\sum_{j=0}^{\infty} a_j z^j$ defines a holomorphic function $\forall |z| < R$; therefore, $\tilde{f}(z)$ is holomorphic in $\mathbb{D}$.

□

Corollary 3.2.2. *Let $f \in H^2(S^1)$, then $\|\tilde{f}_r\|_2$ increases with $r \in (0,1)$ and $\|\tilde{f}_r\|_2 = \|f\|_2$.*

Proof. Let $f \in H^2(S^1)$ with $f(z) = \sum_{j=0}^{\infty} a_j z^j$ in L^2 norm. Then $\tilde{f}_r(z) = \sum_{j=0}^{\infty} a_j r^j z^j$ converges uniformly, so it converges in L^2 norm. This implies,

$$\|\tilde{f}_r\|_2^2 = \sum_{j=0}^{\infty} |a_j r^j|^2 \Rightarrow \lim_{r \to 1} \|\tilde{f}_r\|_2^2 = \sum_{j=0}^{\infty} |a_j|^2 = \|f\|_2^2$$

□

Definition 3.2.3. $\log^+(x) := \log(x)$ if $x \geq 1$ and 0 otherwise.

Theorem 3.2.4 (Fatou's Theorem). *Let $f \in H^2(S^1)$, then $\lim_{r \to 1} \tilde{f}_r(z) = f(z)$ a.e.*

Proof. Observe that

$$\log^+ |\tilde{f}_r(z)| = \frac{1}{2} \log^+ |\tilde{f}_r(z)|^2 \leq \frac{1}{2} |\tilde{f}_r(z)|^2$$

then

$$\begin{aligned} \int_{S^1} \log^+ |\tilde{f}_r(z)| dm(z) &\leq \frac{1}{2} \int_{S^1} |\tilde{f}_r(z)|_2^2 dm(z) \\ &= \frac{1}{2} \|\tilde{f}_r\|_2^2 \leq \frac{1}{2} \|f\|_2^2 \end{aligned}$$

Therefore

$$\sup_{0 \leq r < 1} \int_{S^1} \log^+ |\tilde{f}_r(z)| dm(z) \leq \frac{1}{2} \|f\|_2^2 < \infty$$

By Theorem 3.3.3 of [Rd3, p. 45], $f^*(z) = \lim_{r \to 1} \tilde{f}_r(z)$ exists a.e. Hence f^* is measurable function.
By Fatou's Lemma,

$$\int_{S^1} |f^*(z)|^2 dm(z) = \int_{S^1} \lim_{r \to 1} |\tilde{f}_r(z)|^2 dm(z) \leq \liminf_{r \to 1} \int_{S^1} |\tilde{f}_r(z)|^2 dm(z)$$

$$= \liminf_{r \to 1} \|\tilde{f}_r(z)\|_2^2 \leq \|f\|_2^2$$

Thus $f^* \in L^2(S^1)$. Now, using the Problem 17 of [Roy, p. 127], we have

$$\langle z^n, \tilde{f}_r \rangle \to \langle z^n, f^* \rangle \ \forall n \in \mathbb{Z}$$

Since $(z^n)_{n \in \mathbb{Z}}$ is an orthonormal basis of $L^2(S^1)$ and

$$\langle z^n, \tilde{f}_r \rangle \to \langle z^n, f \rangle \ \forall n \in \mathbb{Z}$$

$f = f^*$ in $L^2(S^1)$. □

Remark 3.2.1. It can be proven that $\tilde{f}_r \to f$ in $L^2(S^1)$ norm (see Theorem 3.4.3. [Rd3, p. 51]).

Theorem 3.2.5 (Poisson's Kernel Formula). *Let* $f \in H^2(S^1)$. *For* $0 \leq r < 1$ *and* $\theta \in \mathbb{R}$,

$$\tilde{f}(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) P_r(\theta - t) dt$$

where

$$P_r(\theta - t) = \frac{1 - r^2}{1 - 2r\cos(\theta) + r^2}$$

Proof. See [Yng, p. 161] . □

Definition 3.2.6. Define

$$P : L^2(S^1) \to H^2(S^1)$$

$$P(g) := \sum_{n=0}^{\infty} \langle z^n, g \rangle z^n \text{ in } L^2 \text{ norm}$$

Remark 3.2.2. P is clearly well-defined and is the orthogonal projection of $L^2(S^1)$ over $H^2(S^1)$

3.3 Hardy-Toeplitz Operators over $\mathbb{D}$

3.3.1 Definitions and basic properties

The most important result of this section is that the C^*-algebra of Hardy-Toeplitz operators acts irreducibly on the Hardy space. The two main ingredients for this proof are the Poisson Kernel formula and Fatou's Theorem.
We start with some definitions and some properties analogous to those appearing in the Bergman case.

Definition 3.3.1. For f a continuous function on S^1 the multiplication operator with symbol f $m_f : L^2(S^1) \to L^2(S^1)$ is defined by $m_f(g) := fg$

Remark 3.3.1. $\|m_f(g)\|_2 = \|fg\|_2 \leq \|f\|_\infty \|g\|_2 \Rightarrow \|m_f\|_{op} \leq \|f\|_\infty$.

Remark 3.3.2. Since $\langle m_f(h), g\rangle = \langle h, \bar{f}g\rangle$ for every $h, g \in L^2(S^1)$, we have ${m_f}^* = m_{\bar{f}}$

Definition 3.3.2. For f a continuous function on S^1 the Hardy-Toeplitz operator with symbol f $T_f : H^2(S^1) \to H^2(S^1)$ is defined by $T_f(g) := P \circ m_f(g)$

Proposition 3.3.3. *For every $f \in C(S^1)$, we have*
(i) $\|T_f\|_{op} \leq \|f\|_\infty$
(ii) ${T_f}^* = T_{\bar{f}}$
If $\varphi \in H^2(S^1) \cap C(S^1)$ *, then*
(iii) $T_f T_\varphi = T_{f\varphi}$
(iv) $T_{\bar{\varphi}} T_f = T_{\bar{\varphi} f}$

Proof. The proof is like the proof in the Bergman case (see Proposition 2.3.3.)

□

Definition 3.3.4. The Hardy-Toeplitz C*-algebra over S^1 is defined as the unital C*-algebra $\mathcal{T}(S^1) := C^*\langle T_f : f \in C(S^1)\rangle$ generated by all Toeplitz operators with continuous symbols.

Proposition 3.3.5. $\mathcal{T}(S^1) = C^*\langle T_p : p \in P(\mathbb{C})\rangle$, *where* $P(\mathbb{C})$ *: polynomials over* $\mathbb{C}$

Proof. The proof is the same as in the Bergman case (see Proposition 2.3.5).

□

Proposition 3.3.6. T_p *is a non-normal operator for any non-constant polynomial* p*. Moreover,* $\mathcal{T}(S^1)$ *is a non-commutative* C^**-algebra.*

Proof. An argument similar to that used in Proposition 2.3.6 works here.

□

Proposition 3.3.7. $\mathcal{T}(S^1)$ *acts irreducibly on* $H^2(S^1)$

Proof. Let $B : H^2(S^1) \to H^2(S^1)$ be an orthogonal projection that commutes with $\mathcal{T}(S^1)$. In particular, $T_f B = B T_f$ on $H^2(S^1)$ for every $f \in C(S^1)$.
Notice that for any $p \in P(\mathbb{C})$ we have $p(z)B(1) \in H^2(S^1)$. Then, for any $p, q \in P(\mathbb{C})$

$$B(q) = B(T_q(1)) = T_q B(1) = P(qB(1)) = qB(1) \tag{3.3.1}$$

$$\begin{aligned}\langle B(1), \bar{p}q\rangle &= \langle pB(1), q\rangle = \langle T_p(B(1)), q\rangle = \langle (T_p \circ B)(1), q\rangle \\ &= \langle (B \circ T_p)(1), q\rangle = \langle B(p), q\rangle\end{aligned}$$

$$\langle B(p), q\rangle = \langle p, B(q)\rangle = \langle p, qB(1)\rangle = \langle \overline{B(1)}, \bar{p}q\rangle \;\Rightarrow\; \langle B(1) - \overline{B(1)}, \bar{p}q\rangle = 0$$

If we choose $p(z) = z^n$ and $q(z) = 1$ and vice versa for $n \geq 0$, we obtain for every $n \in \mathbb{Z}$

$$\langle B(1) - \overline{B(1)}, z^n\rangle = 0$$

Denote $f = B(1)$, then $f = \bar{f}$,i.e., f is real on S^1.
Using Poisson's Kernel formula, we see that

$$\tilde{f}_r(e^{i\theta}) = \tilde{f}(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})P_r(\theta - t)dt$$

Since $f(e^{it})$ and $P_r(\theta - t)$ are reals, $\tilde{f}$ is real on D. Therefore, $\tilde{f}$ is a real constant function, because it is holomorphic.
Now, by Fatou's Theorem, $f \equiv a \in \mathbb{R}$ a.e. Since we are working in $L^2(S^1)$, we can assume $f \equiv a$.
Since $B^2 = B$ and $B(1)$ is constant, $B(1) = B^2(1) = B(B(1)) = B(a.1) = a.B(1)$ implies $B(1) = 1$ or 0.
Thus, by equation 3.3.1, $B = 0$ or $B = Id$. □

3.4 Toeplitz Operators and Compact Operators

The following theorem shows that the Hankel operators with continuous symbols (defined below) are compact. As a consequence of this theorem we will see that the semi-commutator operators are compact,too. Therefore, the commutator ideal is contained in the C^*-algebra of compact operators.

Theorem 3.4.1 (Hartman's Theorem). *For every $f \in C(S^1)$, the Hankel operator*

$$H_f := (I - P) \circ m_f : H^2(S^1) \to H^2(S^1)^{\perp}$$

is compact.

Proof. By the Stone-Weierstrass Theorem, we can approximate f uniformly by polynomials $p_n(z, \bar{z})$. The idea of the proof is to show that the Hankel operators H_{p_n} are compact; therefore, by using uniform convergence of the symbols we will obtain that H_f is a compact operator. We will split the proof into three cases.

Case 1: $f = z^k$, $k \geq 0$
Consider $h \in H^2(S^1)$ with $h(z) = \sum_{n=0}^{\infty} c_n z^n$ converges in L^2 norm.Then

$$m_f(h) = \sum_{n=0}^{\infty} c_n z^{n+k} \Rightarrow (I - P) \circ m_f(h) = 0$$

because $(I-P)\perp P$, so $I-P$ projects over $clos(z^n)_{-\infty}^{-1}$.
Thus $(I-P)\circ m_f = 0.$ is a compact operator.

Case 2: $f = z^{-k}$ **,** $k > 0$
Consider h as above. Then

$$m_f(h) = \sum_{n=0}^{\infty} c_n z^{n-k} \Rightarrow (I-P)\circ m_f(h) = \sum_{n=0}^{k-1} c_n z^{n-k}$$

Thus $(I-P)\circ m_f$ has finite rank, so it is a compact operator.

Case 3: General case
By cases 2 and 3, we have that H_{p_n} is a compact operator. Since $p_n \to f$ uniformly, we have

$$\|(I-P)\circ m_f - (I-P)\circ m_{p_n}\|_{op} = \|(I-P)\circ m_{f-p_n}\|_{op} \leq \|I-P\|_{op}\|m_{f-p_n}\|_{op}$$
$$\leq \|I-P\|_{op}\|f-p_n\|_{\infty} \to 0 \text{ as } k \to 0$$

Thus $(I-P)\circ m_f$ is a compact operator.

□

Corollary 3.4.2. *Let φ, $\psi \in C(S^1)$, then $T_\varphi \circ T_\psi - T_{\varphi\psi}$ is a compact operator.*

Proof.
$$T_\varphi \circ T_\psi - T_{\varphi\psi} = Pm_\varphi Pm_\psi - Pm_{\varphi\psi} = Pm_\varphi(P-I)m_\psi$$
It follows from the fact that Pm_φ is a bounded operator and the previous theorem that $T_\varphi \circ T_\psi - T_{\varphi\psi}$ is a compact operator.

□

Theorem 3.4.3. *Let $f \in C(S^1)$. If T_f is a compact operator then $f \equiv 0$.*

Proof. Clearly $f \in L^2(S^1)$. Then, $f(z) = \sum_{-\infty}^{\infty} a_n z^n$ converges in L^2 norm.
For $k \geq 0$, we have

$$m_f(z^k) = \sum_{-\infty}^{\infty} a_n z^{n+k} \Rightarrow T_f(z^k) = P\circ m_f(z^k) = \sum_{m=0}^{\infty} a_{m-k} z^m$$

Then the Toeplitz matrix $[T_f] = (b_{i,j})$ has constant entries a_{i-j} on diagonals. Therefore, for a fixed $n \in \mathbb{Z}$ we have $b_{i+n,i} = a_n$.
Since T_f is compact operator, $b_{i+n,i} \to 0$ as $i \to \infty$. Thus $a_n = 0$; hence, $f \equiv 0$.

□

Now, we have all the tools to prove our main theorem. The proof basically follows the same pattern as in the Bergman case.

Theorem 3.4.4. *The Hardy-Toeplitz C*-algebra* $\mathcal{T}(S^1)$ *has the commutator ideal* $\mathcal{K}(H^2(S^1))$ *(compact operators), and there exists a C*-isomorphism*

$$\nu : \mathcal{T}(S^1)/\mathcal{K}(H^2(S^1)) \to C(S^1)$$

such that for every $f \in C(S^1)$,

$$\nu(T_f + \mathcal{K}(H^2(S^1))) = f$$

Proof. The commutator

$$[T_f, T_g] := T_f T_g - T_g T_f = (T_f T_g - T_{fg}) - (T_g T_f - T_{gf})$$

is compact operator, by Corollary 3.4.2 .
Then, the commutator ideal $\mathcal{T}'(S^1)$ generated by the commutators $[T_f, T_g]$ is contained in $\mathcal{K}(H^2(S^1)))$
We know that $\mathcal{T}(S^1)$ is not commutative, then $\mathcal{T}'(S^1) \neq 0$
Since $\mathcal{T}(S^1)$ acts irreducibly on $H^2(S^1)$ and by Corollary 2 of [Arv, p. 18] , $\mathcal{K}(H^2(S^1)) \subseteq \mathcal{T}(S^1)$. Then $\mathcal{T}'(S^1)$ is a nonzero ideal of $\mathcal{K}(H^2(S^1)))$. By Corollary 1 of [Arv, p. 18], $\mathcal{T}'(S^1) = \mathcal{K}(H^2(S^1))$.
Therefore,

$$\rho : C(S^1) \to \mathcal{T}(S^1)/\mathcal{K}(H^2(S^1)) \ , \ \rho(f) := T_f + \mathcal{K}(H^2(S^1))$$

is a (well-defined) C*-homomorphism that is surjective.

Now, take $f \in Ker(\rho)$ i.e. T_f is a compact operator. By Theorem 3.4.3, f vanishes on S^1. Thus ρ is injective; hence, C*-isomorphism.

□

Chapter 4

Bergman Space and Toeplitz Operators over the unit ball $B_n \subset \mathbb{C}^n$

In the following chapters I focus my attention on the discussion of Fredholm operators and index.

This chapter follows the line of reasoning from Chapter 2. The unit ball is a natural generalization of the unit disc, and we shall prove that for this case the algebra of Bergman-Toeplitz operators has basically the same structure as in the one-dimensional case. Note that the unit ball has a nice boundary, which allows us to use peaking functions and to get a clear characterization of the commutator ideal. We can ask how far we can go performing our argument used to get parallel results for the unit disc and unit ball. The answer to this question can be found in the context of Chapter 6, the polydisc case, for which the boundary is not nice. The last corollary of this chapter is fundamental for our discussion of index of a Fredholm operator carried in Chapter 8.

Many of the assertions through this chapter are simple generalizations of the one-dimensional case (see Chapter 2); hence, we shall omit some proofs that just require repeating a previous argument.

4.1 Definition and basic properties

Consider $L^2(B_n)$ the Lebesgue space of square integrable functions on B_n with Lebesgue measure $dV(z)$, and inner product,

$$\langle f, g\rangle = \int_{B_n} \overline{f(z)} g(z) dV(z)$$

Lemma 4.1.1. *Let $\alpha \neq \beta$ be non-negative multi-indices, then*

$$\langle z^\alpha, z^\beta\rangle = 0$$

Proof. Assume by simplicity that $\alpha_1 \neq \beta_1$ and denote $w = (z_2, ..., z_n)$, $\alpha_0 = (\alpha_2, ..., \alpha_n)$, $\beta_0 = (\beta_2, ..., \beta_n)$ then

$$\int_{B_n} \bar{z}^\alpha z^\beta dV(z) = \int_{|w|\leq 1} \bar{w}^{\alpha_0} w^{\beta_0} (\int_{|z_1|\leq\sqrt{1-|w|^2}} \bar{z_1}^{\alpha_1} z_1{}^{\beta_1} dV(z_1))dV(w)$$

Clearly

$$\int_{|z_1|\leq\sqrt{1-|w|^2}} \bar{z_1}^{\alpha_1} z_1{}^{\beta_1} dV(z_1) = 0$$

Thus the lemma follows.

□

Proposition 4.1.2. *Let $K \subset B_n$ be a compact set, then the restriction map,*

$$R : L^2(B_n) \cap O(B_n) \to C(K),\ \ R(f) = f|_K$$

is continuous.

Proof. Apply Lemma 4.1.1 and use same argument as in the proof of Proposition 2.1.1.

□

Proposition 4.1.3. *$L^2(B_n) \cap O(B_n)$ is a closed subspace of $L^2(B_n)$*

Definition 4.1.4. $H^2(B_n) := L^2(B_n) \cap O(B_n)$ is called the Bergman Space.

Proposition 4.1.5. *Denote $c_\alpha = \frac{1}{\|z^\alpha\|_2}$ for any non-negative multi-index α and define $\phi_\alpha(z) := c_\alpha z^\alpha$. Then $\{\phi_\alpha\}$ is an orthonormal basis of $H^2(B_n)$.*

Proof. By Lemma 4.1.1 we have that $\{\phi_\alpha\}$ is an orthogonal set. Now, take $f \in H^2(B_n)$ with

$$f(z) = \sum_{|\alpha| \geq 0}^{\infty} d_\alpha z^\alpha$$

converging normally on B_n. Applying a similar argument as in the one-dimensional case we can show that

$$f(z) = \sum_{|\alpha| \geq 0}^{\infty} d_\alpha z^\alpha$$

converges in $L^2(B_n)$ norm.
Thus $\{\phi_\alpha\}$ is an orthonormal basis of $H^2(B_n)$.

$\square$

Definition 4.1.6. The orthogonal projection $P : L^2(B_n) \to H^2(B_n)$ is called the Bergman projection.

Remark 4.1.1. For any fixed $z \in B_n$; by Proposition 4.1.2, we have that the evaluation map $eval : H^2(B_n) \to \mathbb{C}$ $eval(f) = f(z)$ is continuous. Then, by the Riesz-Frechet Theorem, there exists $K_z \in H^2(B_n)$ such that

$$f(z) = \langle K_z, f \rangle = \int_{B_n} \overline{K_z(w)} f(w) dV(w) \ \forall f \in H^2(B_n)$$

Definition 4.1.7. $K(z, w) := \overline{K_z(w)}$ is called the Bergman Kernel function.

Remark 4.1.2. It is possible to find an explicit representation of the Bergman Kernel for the unit ball (see for example [Ran, p. 183]). The Bergman Kernel for the unit ball is given by

$$K(z, w) = \frac{n!}{\pi^n (1 - \langle z, w \rangle)^{n+1}}$$

4.2 Bergman-Toeplitz Operators over the unit ball B_n

4.2.1 Definitions and basic properties

In this section we can still use most of the arguments used in the one-dimensional case and get natural generalizations of the results obtained in Chapter 2. However, in some cases new technicalities arise because of the dimension.

Definition 4.2.1. For f a continuous function on $\bar{B_n}$ the multiplication operator with symbol f , $m_f : L^2(B_n) \to L^2(B_n)$ is defined by $m_f(g) := fg$

Remark 4.2.1. $\|m_f(g)\|_2 = \|fg\|_2 \leq \|f\|_\infty \|g\|_2 \Rightarrow \|m_f\|_{op} \leq \|f\|_\infty$.

Remark 4.2.2. Since $\langle m_f(h), g\rangle = \langle h, \bar{f}g\rangle$ for every $h, g \in L^2(B_n)$, then $m_f{}^* = m_{\bar{f}}$

Definition 4.2.2. For f a continuous function on $\bar{B_n}$ the Bergman-Toeplitz operator with symbol f $T_f : H^2(B_n) \to H^2(B_n)$ is defined by $T_f(g) := P \circ m_f(g)$

Proposition 4.2.3. *For every $f \in C(\bar{B_n})$, we have*
(i) $\|T_f\|_{op} \leq \|f\|_\infty$
(ii) $T_f{}^* = T_{\bar{f}}$
If $\varphi \in A(B_n) := O(B_n) \cap C(\bar{B_n})$, then
(iii) $T_f T_\varphi = T_{f\varphi}$
(iv) $T_{\bar{\varphi}} T_f = T_{\bar{\varphi} f}$

Definition 4.2.4. The Bergman-Toeplitz C*-algebra over B_n is defined as the unital C*-algebra $\mathcal{T}(B_n) := C^*\langle T_f : f \in C(\bar{B_n})\rangle$ generated by all Toeplitz operators with continuous symbols.

Proposition 4.2.5. $\mathcal{T}(B_n) = C^*\langle T_p : p \in P(\mathbb{C}^n)\rangle$*, where* $P(\mathbb{C}^n)$ *: polynomials over* $\mathbb{C}^n$

Proposition 4.2.6. *T_p is a non-normal operator for any non-constant polynomial p. Moreover, $\mathcal{T}(B_n)$ is a non-commutative C^*-algebra.*

Proof. An argument similar to that used in Proposition 2.3.6 works here, but we need to apply Lemma 4.1.1 and use the orthonormal basis $\{\phi_\alpha\}$.

□

Proposition 4.2.7. *$\mathcal{T}(B_n)$ acts irreducibly on $H^2(B_n)$*

Proof. Let $B : H^2(B_n) \to H^2(B_n)$ be an orthogonal projection that commutes with $\mathcal{T}(B_n)$. Then, $T_f B = BT_f$ on $H^2(B_n)$ for every $f \in C(B_n)$.
Notice that $pB(1) \in H^2(B_n)$ for any $p \in A(B_n)$, then for any $p, q \in A(B_n)$

$$B(q) = B(q.1) = B(T_q(1)) = T_q B(1) = P(qB(1)) = qB(1)$$

$$\begin{aligned}\langle B(1), \bar{p}q\rangle &= \langle pB(1), q\rangle = \langle T_p(B(1)), q\rangle = \langle (T_p \circ B)(1), q\rangle \\ &= \langle (B \circ T_p)(1), q\rangle = \langle B(p), q\rangle\end{aligned}$$

$$\langle B(p), q\rangle = \langle p, B(q)\rangle = \langle p, qB(1)\rangle = \langle \overline{B(1)}, \bar{p}q\rangle \;\Rightarrow\; \langle B(1) - \overline{B(1)}, \bar{p}q\rangle = 0$$

Since the algebra generated by $\{\bar{p}q : p, q \in P(\mathbb{C}^n) \subset A(B_n)\}$ is dense in $C(B_n)$ and $C(B_n)$ is dense in $L^2(B_n)$, the algebra generated by $\{\bar{p}q : p, q \in P(\mathbb{C}^n)\}$ is dense in $L^2(\partial B_n)$. Therefore,

$$B(1) - \overline{B(1)} = 0$$

Denote $f = B(1)$, then $f = \bar{f}$,i.e., f is real on ∂B_n.Hence, $f \equiv a$ is constant real function.
Since $B^2 = B$ and $f = B(1)$ is constant, $B(1) = B^2(1) = B(B(1)) = B(a.1) = a.B(1)$ implies $B(1) = 1$ or 0.
Thus $B = 0$ or $B = Id$.

□

4.3 Toeplitz Operators and Compact Operators

In this section we focus on identifying the commutator ideal of $\mathcal{T}(B_n)$. We introduce the concept of Hankel operator which is a sort of "orthogonal complement" of a Toeplitz operator. Properties of Hankel operators imply properties of Toeplitz operators, and vice versa. For our purposes we will use just one result about Hankel operators, but they are studied as much as Toeplitz operators (e.g. see [Pel]).

Lemma 4.3.1. *Let $\{K_m\}$ be the exhaustion of B_n by compact subsets , where $K_m = \overline{B}(0, 1 - \frac{1}{m})$. Then for each $K = K_m$, the family*

$$\mathcal{F} = \{h|_K : h \in H^2(B_n), \|h\|_2 \leq 1\}$$

has compact closure.

Proposition 4.3.2. *Let $f \in C(\bar{B}_n)$ with $f|_{\partial B_n} = 0$. Then, T_f is a compact operator.*

Remark 4.3.1. For each $a \in B_n$ there exist an automorphism (*Möbius* transformation) of B_n satisfying:
(1) $\varphi_a(a) = 0$ and
(2) $\varphi_a \circ \varphi_a = Id_{B_n}$
Moreover for any $z_0 \in \partial B_n$ we have

$$\lim_{a \to z_0} \varphi_a(z) = z_0 \ \forall z \in B_n$$

(see [Rd2]).

Proposition 4.3.3. *Let $f \in C(\bar{B}_n)$. Then the Hankel operator*

$$H_f := (I - P)m_f : H^2(B_n) \to L^2(B_n)$$

is a compact operator.

Proof. For any $z_0 \in \partial B_n$ we have

$$f \circ \varphi_a \to f(z_0) \text{ uniformly as } a \to z_0$$

In particular

$$f \circ \varphi_a \to f(z_0)$$

in $L^2(B_n)$ norm. Then

$$(I - P)(f \circ \varphi_a) \to 0$$

in $L^2(B_n)$ norm, because $I - P$ is continuous.
Therefore, by Theorem 7 of [StZ] H_f is a compact operator. □

Corollary 4.3.4. *Let f, $g \in C(\bar{B}_n)$, then $T_f \circ T_g - T_{fg}$ is a compact operator.*

4.4 Characterization of $\mathcal{T}(B_n)$

Peaking functions (defined below) play a fundamental role in this section. They are very useful for obtaining norm estimates. The existence of peaking functions depends on the domain and is a well-known result in function theory of complex variables.

Definition 4.4.1. Let $w \in B_n$. $h_w \in A(B_n)$ is a peaking function if $|h_w(z)| < 1$ for any $z \in \bar{B}_n$ with $z \neq w$, and $h_w(w) = 1$

Lemma 4.4.2. *There exists a function*

$$\partial B_n \to A(B_n) \subset H^2(B_n)$$

$$w \mapsto h_w$$

where h_w is a peaking function at w.

Proof. It is clear that B_n is a strictly (strongly) pseudoconvex domain with C^∞ boundary. The lemma follows by Theorem 5.2.15 of [Kra, p. 188].

□

Remark 4.4.1. By the continuity of h_w and its peaking property at w, for each open neighborhood $U \subset \bar{B}_n$ of $w \in \partial B_n$ there exists an open neighborhood $V \subset U$ of w relatively compact in U such that

$$\sup_{z \in B_n \setminus U} |h_w(z)| < \inf_{z \in V} |h_w(z)|$$

Proposition 4.4.3. *For every $f \in C(\bar{B}_n)$ and every $w \in \partial B_n$*

$$\frac{\int_{B_n} f(z)|h_w(z)|^{2n} dV(z)}{\int_{B_n} |h_w(z)|^{2n} dV(z)} \to f(w) \ \text{as } n \to \infty$$

Proof. We can assume that f is a real-valued function because we can repeat the same argument for the real and imaginary part of f. Also, correcting by a constant, we can assume that $f(w) = 0$.
Therefore, given $\epsilon > 0$ there exists an open neighborhood $U \subset \bar{B}_n$ of $w \in \partial B_n$ such that $\sup |f|(U) < \epsilon$ .By the above remark, there exists an open neighborhood $V \subset U$ of w relatively compact in U such that

$$\sup_{z \in B_n \setminus U} |h_w(z)| < \inf_{z \in V} |h_w(z)|$$

$$\left| \int_{B_n} f(z)|h_w(z)|^{2n} dV(z) \right| \le \int_U |f(z)||h_w(z)|^{2n} dV(z) + \int_{B_n \setminus U} |f(z)||h_w(z)|^{2n} dV(z)$$

$$< \epsilon \int_{B_n} |h_w(z)|^{2n} dV(z) + Vol(B_n \setminus U)\|f\|_\infty \sup_{z \in B_n \setminus U} |h_w(z)|^{2n}$$

On the other hand

$$\int_{B_n} |h_w(z)|^{2n} dV(z) \geq \int_V |h_w(z)|^{2n} dz \geq Vol(V) \inf_{z \in V} |h_w(z)|^{2n}$$

Then

$$|\frac{\int_{B_n} f(z)|h_w(z)|^{2n} dV(z)}{\int_{B_n} |h_w(z)|^{2n} dV(z)}| < \epsilon + \frac{Vol(B_n \setminus U)}{Vol(V)} \|f\|_\infty \left(\frac{\sup_{z \in B_n \setminus U} |h_w(z)|}{\inf_{z \in V} |h_w(z)|} \right)^{2n}$$

Therefore

$$\frac{\sup_{z \in B_n \setminus U} |h_w(z)|}{\inf_{z \in V} |h_w(z)|} < 1 \Rightarrow \frac{\int_{B_n} f(z)|h_w(z)|^{2n} dV(z)}{\int_{B_n} |h_w(z)|^{2n} dV(z)} \to 0 \text{ as } n \to \infty$$

□

Corollary 4.4.4. *Let $f \in C(\bar{B_n})$ be non-negative real-valued. Then for each $w \in \partial B_n$ there exists a sequence $(h_n) \subset H^2(B_n)$ where*

$$h_n(z) := \frac{{h_w}^n(z)}{\|{h_w}^n\|_2}$$

such that

$$\|fh_n\|_2 \to f(w) \ \textit{as}\ n \to \infty$$

Proposition 4.4.5. *Let p a polynomial in k non-commuting variables. Then*

$$\|p(T_{f_1}, ..., T_{f_k})\|_{op} \geq \|p(f_1|_{\partial B_n}, ..., f_k|_{\partial B_n})\|_\infty$$

where $f_j \in C(\bar{B_n})$ for $1 \leq j \leq k$

Proof. The argument used in Proposition 6.4.6 can be easily adapted to this proof.

□

Now, we have all the machinery needed to get our characterization of the algebra of Toeplitz operators. Again we basically use the argument used in the one-dimensional case; however, the proof of the injectivity of ρ, defined below, requires a different argument, which is supported by the previous proposition.

Theorem 4.4.6. *The Bergman-Toeplitz C*-algebra $\mathcal{T}(B_n)$ has the commutator ideal $\mathcal{K}(H^2(B_n))$ (compact operators), and there exists a C*-isomorphism*

$$\nu : \mathcal{T}(B_n)/\mathcal{K}(H^2(B_n)) \to C(\partial B_n)$$

Proof. The commutator

$$[T_f, T_g] := T_f T_g - T_g T_f = (T_f T_g - T_{fg}) - (T_g T_f - T_{gf})$$

is a compact operator, because each semi-commutator is a compact operator.
Then, the commutator ideal $\mathcal{T}'(B_n)$ generated by the commutators $[T_f, T_g]$ is contained in $\mathcal{K}(H^2(B_n)))$
We know that $\mathcal{T}(B_n)$ is not commutative, then $\mathcal{T}'(B_n) \neq 0$
Since $\mathcal{T}(B_n)$ acts irreducibly on $H^2(B_n)$ and by Corollary 2 of [Arv, p. 18], $\mathcal{K}(H^2(B_n)) \subseteq \mathcal{T}(B_n)$. Then $\mathcal{T}'(B_n)$ is a nonzero ideal of $\mathcal{K}(H^2(B_n)))$. By Corollary 1 of [Arv, p. 18], $\mathcal{T}'(B_n) = \mathcal{K}(H^2(B_n))$.
Therefore,

$$\rho : C(\bar{B_n}) \to \mathcal{T}(B_n)/\mathcal{K}(H^2(B_n)) \ , \ \rho(f) := T_f + \mathcal{K}(H^2(B_n))$$

is a (well-defined) C*-homomorphism that is surjective.
Notice that the restriction map

$$r : C(\bar{B_n}) \to C(\partial B_n) \ , \ r(f) := f|_{\partial B_n}$$

is a surjective C*-homomorphism. Indeed, for any $f \in C(\partial B_n)$ we can define

$$\tilde{f}(z) := |z| f(\frac{z}{|z|}) \text{ if } z \neq 0 \text{ and } \tilde{f}(0) := 0$$

Clearly, $\tilde{f}(z) \in C(\bar{B_n})$ and $\tilde{f}|_{\partial B_n} = f$.
If $f \in Ker(\rho)$ i.e. vanishes on ∂B_n, then T_f is a compact operator (by Proposition 4.3.2).
Therefore, there exists $\bar{\rho}$ surjective C*-homomorphism such that $\bar{\rho} \circ r = \rho$.
Now, take $f \in Ker(\bar{\rho})$ i.e. $T_{\tilde{f}}$ compact operator.
Let $A \in \mathcal{K}(H^2(B_n))$. Then A is in the C^*-ideal generated by the semi-commutators; hence, $A = lim A_n$ where A_n is a finite sum of operators of the form

$$T_{f_1}, ..., T_{f_k}(T_F T_G - T_{FG}) T_{g_1}, ..., T_{g_m}$$

Consider the polynomial with non-commutative variables with terms of the same form than the terms of A_n, i.e., with terms of the form $x_1 x_2 ... x_k (z_1 z_2 - z_3) y_1 ... y_m$.
Using the previous proposition and

$$f_1|_{\partial B_n} ... f_k|_{\partial B_n} (F|_{\partial B_n} G|_{\partial B_n} - F|_{\partial B_n} G|_{\partial B_n}) g_1|_{\partial B_n} ... g_m|_{\partial B_n} = 0$$

we have

$$\|\tilde{f}|_{\partial B_n}\|_\infty \leq \|T_{\tilde{f}} + A_n\|_{op}$$

Therefore $\|\tilde{f}|_{\partial B_n}\|_\infty \leq \|T_{\tilde{f}} + A\|_{op}$
Since $f \in Ker(\bar{\rho})$, $\tilde{f}|_{\partial B_n} \equiv 0$.
Thus $f \equiv 0$

□

Corollary 4.4.7. *T_f is Fredholm if and only if $f(z) \neq 0\ \forall z \in \partial B_n$.*

Proof. The above theorem implies $f|_{\partial B_n}$ is invertible in $C(\partial B_n)$ if and only if T_f is invertible up to compact operators. Thus, the corollary follows. $\square$

Chapter 5

Hardy Space and Toeplitz Operators over the unit ball $\mathbf{B_n}$

There are not many additional difficulties in following the same reasoning used in the Bergman case, except for the assertion that Hankel operators are compact operators. We quote this result because its proof requires sophisticated arguments. On the other hand, we notice that the algebras of Bergman-Toeplitz and Hardy-Toeplitz operators over the unit ball are still C^*-isomorphic up to compact operators. In context of the spectral theorem we can still calculate the numerical invariant that is invariant under compact perturbations, the index of a Fredholm operator (see Chapter 8).

Many of the assertions throughout this chapter are simple generalization of the one-dimensional case (see Chapter 3); hence, we shall omit some proofs that just require repeating a previous argument.

Consider $L^2(\partial B_n)$ the Lebesgue space of square integrable functions on ∂B_n with surface measure $d\sigma(z)$, and inner product,

$$\langle f, g\rangle = \int_{\partial B_n} \overline{f(z)} g(z) d\sigma(z) \tag{5.0.1}$$

Definition 5.0.8. $H^2(\partial B_n)$ defined as the $L^2(\partial B_n)$ closure of $\{f|_{\partial B_n} : f \in A(B_n)\}$ is called the Hardy Space over B_n.

Definition 5.0.9. The orthogonal projection $P : L^2(\partial B_n) \to H^2(\partial B_n)$ is called the Cauchy-Szëgo projection.

5.1 Hardy-Toeplitz Operators over the unit ball B_n

Definition 5.1.1. For f a continuous function on ∂B_n the multiplication operator with symbol f , $m_f : L^2(\partial B_n) \to L^2(\partial B_n)$ is defined by $m_f(g) := fg$

Remark 5.1.1. $\|m_f(g)\|_2 = \|fg\|_2 \leq \|f\|_\infty \|g\|_2$. Thus $\|m_f\|_{op} \leq \|f\|_\infty$.

Remark 5.1.2. Since $\langle m_f(h), g\rangle = \langle h, \bar{f}g\rangle$ for every $h, g \in L^2(B_n)$, then ${m_f}^* = m_{\bar{f}}$

Definition 5.1.2. For f a continuous function on ∂B_n the Hardy-Toeplitz operator with symbol f, $T_f : H^2(\partial B_n) \to H^2(\partial B_n)$ is defined by $T_f(g) := P \circ m_f(g)$

Proposition 5.1.3. *For every $f \in C(\partial B_n)$, we have*
(i) $\|T_f\|_{op} \leq \|f\|_\infty$
(ii) ${T_f}^* = T_{\bar{f}}$
If $\varphi \in H^2(\partial B_n) \cap C(\partial B_n)$ *, then*
(iii) $T_f T_\varphi = T_{f\varphi}$
(iv) $T_{\bar{\varphi}} T_f = T_{\bar{\varphi} f}$

Definition 5.1.4. The Hardy-Toeplitz C*-algebra over ∂B_n is defined as the unital C*-algebra $\mathcal{T}(\partial B_n) := C^*\langle T_f : f \in C(\partial B_n)\rangle$ generated by all Toeplitz operators with continuous symbols.

Proposition 5.1.5. *$\mathcal{T}(\partial B_n)$ is not commutative.*

Proof. Since $f(z) = z_1 \in H^2(\partial B_n)$, we have

$$T_f(1) = P(f) = f$$

It is not difficult to verify that $\bar{f} = \bar{z_1} \in H^2(\partial B_n)^\perp$. We just need to verify that $\langle \bar{f}, p\rangle = 0$ for any polynomial $p \in P(\mathbb{C}^n)$ because $P(\mathbb{C}^n)$ is dense in $A(B_n)$.
Then we have

$$T_{\bar{f}}(1) = P(\bar{f}) = 0$$

Notice that $|f|^2$ is not in $H^2(\partial B_n)^\perp$, because $\langle |f|^2, 1\rangle = \int_{\partial B_n} |f^2(z)| d\sigma(z) \neq 0$. Therefore

$$T_f T_{\bar{f}}(1) = 0 \text{ and } T_{\bar{f}} T_f(1) = T_{\bar{f}}(f) = P(|f|^2) \neq 0$$

Thus, $\mathcal{T}(\partial B_n)$ is not commutative.

□

Proposition 5.1.6. *$\mathcal{T}(\partial B_n) = C^*\langle T_p : p \in P(\mathbb{C}^n)\rangle$, where $P(\mathbb{C}^n)$: polynomials over $\mathbb{C}^n$*

Proposition 5.1.7. *$\mathcal{T}(\partial B_n)$ acts irreducibly on $H^2(\partial B_n)$*

Proof. Let $B : H^2(\partial B_n) \to H^2(\partial B_n)$ be an orthogonal projection that commutes with $\mathcal{T}(\partial B_n)$. Then, $T_f B = BT_f$ on $H^2(\partial B_n)$ for every $f \in C(\partial B_n)$.
Notice that $pB(1) \in H^2(\partial B_n)$ for any $p \in P(\mathbb{C}^n)$, then for any $p, q \in P(\mathbb{C}^n)$

$$B(q) = B(q1) = B(T_q(1)) = T_q B(1) = P(qB(1)) = qB(1)$$

$$\begin{aligned}\langle B(1), \bar{p}q\rangle &= \langle pB(1), q\rangle = \langle T_p(B(1)), q\rangle = \langle (T_p \circ B)(1), q\rangle \\ &= \langle (B \circ T_p)(1), q\rangle = \langle B(p), q\rangle\end{aligned}$$

$$\langle B(p), q\rangle = \langle p, B(q)\rangle = \langle p, qB(1)\rangle = \langle \overline{B(1)}, \bar{p}q\rangle \;\Rightarrow\; \langle B(1) - \overline{B(1)}, \bar{p}q\rangle = 0$$

Since the algebra generated by $\{\bar{p}q : p, q \in P(\mathbb{C}^n)\}$ is dense in $C(\partial B_n)$ and $C(\partial B_n)$ is dense in $L^2(\partial B_n)$, the algebra generated by $\{\bar{p}q : p, q \in P(\mathbb{C}^n)\}$ is dense in $L^2(\partial B_n)$. Therefore,

$$B(1) - \overline{B(1)} = 0$$

Denote $f = B(1)$, then $f = \bar{f}$,i.e., f is real on ∂B_n.
Using Poisson's integral formula we extend f to a holomorphic function on B_n (see [Kra, p. 55])

$$Pf(z) := \int_{\partial B_n} f(w)S(z, w)d\sigma(w) \;\forall z \in B_n$$

defines a holomorphic function in B_n where $S(z, w)$ is the Szego Kernel.
But we know that f is real on ∂B_n, it follows that Pf is real on B_n; hence, it is constant. Thus $f \equiv a$ is a constant real-valued function.
Since $B^2 = B$ and $f = B(1)$ is constant, $B(1) = B^2(1) = B(B(1)) = B(a.1) = a.B(1)$ implies $B(1) = 1$ or 0.
Thus $B = 0$ or $B = Id$. □

5.2 Toeplitz Operators and Compact Operators

Proposition 5.2.1. *Let $f \in C(\bar{B}_n)$. Then*

$$H_f := (I - P)m_f : H^2(\partial B_n) \to L^2(\partial B_n)$$

is a compact operator.

Proof. See Theorem 4.2.17 of [Upm, p. 253].

□

Corollary 5.2.2. *Let f, $g \in C(\bar{B}_n)$, then $T_f \circ T_g - T_{fg}$ is a compact operator.*

5.2.1 Characterization of $\mathcal{T}(\partial B_n)$

Remark 5.2.1. Recall that for any function $g \in A(B_n) = O(B_n) \cap C(\bar{B_n})$, the restriction function $g|_{\partial B_n} \in H^2(\partial B_n)$. For this reason we are going to allow the abuse of notation $A(B_n) \subset H^2(\partial B_n)$ in the following lemma.

Lemma 5.2.3. *There exists a function*

$$\partial B_n \to A(B_n) \subset H^2(\partial B_n)$$

$$w \mapsto h_w$$

where h_w is a peaking function at w.

Proof. It is clear that B_n is a strictly (strongly) pseudoconvex domain with C^∞ boundary. The lemma follows from Theorem 5.2.15 of [Kra, p. 188].

□

Remark 5.2.2. By the continuity of h_w and its peaking property at w, for each open neighborhood $U \subset \partial B_n$ of $w \in \partial B_n$ there exists an open neighborhood $V \subset U$ of w relatively compact in U such that

$$\sup_{z \in \partial B_n \setminus U} |h_w(z)| < \inf_{z \in V} |h_w(z)|$$

Proposition 5.2.4. *For every $f \in C(\partial B_n)$ and every $w \in \partial B_n$*

$$\frac{\int_{\partial B_n} f(z)|h_w(z)|^{2n} d\sigma(z)}{\int_{\partial B_n} |h_w(z)|^{2n} d\sigma(z)} \to f(w) \ \text{as } n \to \infty$$

Proof. The argument used in Proposition 4.4.3 works for this proof.

□

Corollary 5.2.5. *Let $f \in C(\partial B_n)$ be non-negative real valued. Then for each $w \in \partial B_n$ there exists a sequence $(h_n) \subset H^2(\partial B_n)$ where*

$$h_n(z) := \frac{{h_w}^n(z)}{\|{h_w}^n\|_2}$$

such that

$$\|f h_n\|_2 \to f(w) \ \text{as } n \to \infty$$

Proposition 5.2.6. *Let p a polynomial in k non-commuting variables. Then*

$$\|p(T_{f_1}, ..., T_{f_k})\|_{op} \geq \|p(f_1, ..., f_k)\|_\infty$$

where $f_j \in C(\partial B_n)$ for $1 \leq j \leq k$

Proof. The argument used in Proposition 6.4.6 can be easily adapted to this proof. $\square$

Theorem 5.2.7. *The Hardy-Toeplitz C^*-algebra $\mathcal{T}(\partial B_n)$ has the commutator ideal $\mathcal{K}(H^2(\partial B_n))$ (compact operators), and there exists a C^*-isomorphism*

$$\rho : C(\partial B_n) \to \mathcal{T}(\partial B_n)/\mathcal{K}(H^2(\partial B_n)) \ , \ \rho(f) := T_f + \mathcal{K}(H^2(\partial B_n))$$

Proof. The commutator

$$[T_f, T_g] := T_f T_g - T_g T_f = (T_f T_g - T_{fg}) - (T_g T_f - T_{gf})$$

is a compact operator, because each semi-commutator is a compact operator.
Then, the commutator ideal $\mathcal{T}'(B_n)$ generated by the commutators $[T_f, T_g]$ is contained in $\mathcal{K}(H^2(\partial B_n))$
We know that $\mathcal{T}(\partial B_n)$ is not commutative, then $\mathcal{T}'(B_n) \neq 0$
Since $\mathcal{T}(\partial B_n)$ acts irreducibly on $H^2(\partial B_n)$ and by Corollary 2 of [Arv, p. 18], $\mathcal{K}(H^2(\partial B_n)) \subseteq \mathcal{T}(\partial B_n)$. Then $\mathcal{T}'(B_n)$ is a nonzero ideal of $\mathcal{K}(H^2(\partial B_n))$. By Corollary 1 of [Arv, p. 18], $\mathcal{T}'(B_n) = \mathcal{K}(H^2(\partial B_n))$.
Therefore,

$$\rho : C(\partial B_n) \to \mathcal{T}(\partial B_n)/\mathcal{K}(H^2(\partial B_n)) \ , \ \rho(f) := T_f + \mathcal{K}(H^2(\partial B_n))$$

is a (well-defined) C*-homomorphism that is surjective.

Let $f \in Ker(\rho)$ i.e. T_f is a compact operator.
Let $B \in \mathcal{K}(H^2(\partial B_n))$. Then B is in the C^*-ideal generated by the semi-commutators; hence, $A = lim A_n$ where A_n is a finite sum of operators of the form

$$T_{f_1}, ..., T_{f_k}(T_F T_G - T_{FG}) T_{g_1}, ..., T_{g_m}$$

Consider the polynomial with non-commutative variables with terms of the same form than the terms of A_n, i.e., with terms of the form $x_1 x_2 ... x_k (z_1 z_2 - z_3) y_1 ... y_m$.
Using the previous proposition and

$$f_1 ... f_k (FG - FG) g_1 ... g_m = 0$$

we have

$$\|f\|_\infty \leq \|T_f + A_n\|_{op}$$

Therefore $\|f\|_\infty \leq \|T_f + A\|_{op}$
Since $f \in Ker(\rho)$, $f \equiv 0$. $\square$

Corollary 5.2.8. *T_f is Fredholm if and only if $f(z) \neq 0$ $\forall z \in \partial B_n$.*

Proof. The proof is the same as in *Corollary* 4.4.6. $\square$

Chapter 6

Hardy Space and Toeplitz Operators over $\Delta(0,1) \subset \mathbb{C}^2$

The point of view in this chapter has been mainly influenced by [BoS] and [Upm]. The polydisc is the other natural generalization of the unit disc. In contrast to the unit ball, the boundary of a polydisc is not "nice" anymore. This induces some changes in the structure of the algebra of Toeplitz operators. In fact, the algebra of Hardy-Toeplitz operators over the polydisc is our first example where the commutator ideal is not the space of compact operators; consequently, we do not have an analogue of the spectral theorem up to compact operators. We use ideas different from those used in the previous chapters, because of the difficulty of the boundary of the domain. For example, we shall show the existence of peaking functions for the polydisc because the polydisc is not a strictly pseudoconvex domain. Also, tensor products are necessary to characterize the commutator ideal of $\mathcal{T}(T^2)$.
To obtain a description of the Toeplitz-Fredholm operators and the index, we need to use different and more complicated arguments. For this reason we will not discuss the index of a Fredholm operator in this setting, but we give some references about this topic at the end of Chapter 8.

6.1 Definition and basic properties

The purpose of this section is to recover some properties that we obtained in the one-dimensional case, such as Fatou's Lemma and Poisson's Kernel formula.

Notation:
$\Delta(0,1) = \mathbb{D} \times \mathbb{D} \subset \mathbb{C}^2$
$T^2 = S^1 \times S^1$

$z^\alpha = {z_1}^{\alpha_1} {z_2}^{\alpha_2}$ for $z = (z_1, z_2) \in \mathbb{C}^2$ and $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}^2$
For $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}^2$ denote $|\alpha| = |\alpha_1| + |\alpha_2|$ and $\alpha \geq 0 \leftrightarrow \alpha_1, \alpha_2 \geq 0$

Consider $L^2(T^2)$ the Lebesgue space of square integrable functions on T^2 with Haar measure $dm(z) = \frac{d\theta_1 d\theta_2}{(2\pi)^2}$, and inner product,

$$\langle f, g \rangle = \int_{T^2} \overline{f(z)} g(z) dm(z)$$

Remark 6.1.1.

$$\int_{T^2} f(z) dm(z) := \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} f(e^{i\theta_1}, e^{i\theta_2}) d\theta_1 d\theta_2$$

Remark 6.1.2. $(z^\alpha)_{\alpha \in \mathbb{Z}^2} \subset L^2(T^2)$ is an orthonormal set.

Proposition 6.1.1. *$(z^\alpha)_{\alpha \in \mathbb{Z}^2}$ is an orthonormal basis of $L^2(T^2)$.*

Proof. Let $f \in C(T^2)$. By the Stone-Weierstrass Theorem, there exists a sequence of polynomials $p_n(z_1, \bar{z}_1, z_2, \bar{z}_2)$ such that

$$\|p_n - f\|_2 \leq \|p_n - f\|_\infty \to 0 \text{ as } n \to \infty$$

Therefore p_n converges to f in $L^2(T^2)$ norm. On the other hand each p_n is in $span(z^\alpha)_{\alpha \in \mathbb{Z}^2}$; hence , f is in the the $L^2(T^2)$ closure of $span(z^\alpha)_{\alpha \in \mathbb{Z}^2}$.
Thus $L^2(T^2)$ is equal to the $L^2(T^2)$ closure of $(z^\alpha)_{\alpha \in \mathbb{Z}^2}$

$\square$

Definition 6.1.2. $H^2(T^2) := clos\{z^\alpha \in L^2(T^2) : \alpha \geq 0\}$ in the L^2 sense.

Remark 6.1.3. $H^2(T^2)$ is a closed subspace of $L^2(T^2)$. Then $H^2(T^2)$ is a Hilbert space with Hilbert-space basis $(z^\alpha)_{\alpha \geq 0}$

The following theorem says that we can extend any function in the Hardy space to a holomorphic function defined in the unit polydisc. This theorem is the analogue of Theorem 3.2.1.

Theorem 6.1.3. *Let $f \in H^2(T^2)$ with $f(z) = \sum_{|\alpha|=0}^\infty a_\alpha z^\alpha$ in L^2 sense. Then $\tilde{f}(z) := \sum_{|\alpha|=0}^\infty a_\alpha z^\alpha$ defines a holomorphic function in $\Delta(0,1)$, i.e., it converges normally on $\Delta(0,1)$*

Proof. We are going to prove that $\tilde{f}(z_1, z_2) = \sum_{|\alpha|=0}^\infty a_\alpha z^\alpha$ defines a holomorphic function in each variable separately.

Fix $z_1 \in \mathbb{D}$ and define $\tilde{f_{z_1}}(z_2) := \tilde{f}(z_1, z_2)$ as a function of the variable z_2. Then, we can rewrite

$$\tilde{f_{z_1}}(z_2) = \sum_{\alpha_2=0}^{\infty} b_{\alpha_2} {z_2}^{\alpha_2}$$

where $b_{\alpha_2} = \sum_{\alpha_1=0}^{\infty} a_{\alpha_1,\alpha_2} {z_1}^{\alpha_1}$.
Then

$$|b_{\alpha_2}|^2 \leq \sum_{\alpha_1=0}^{\infty} |a_{\alpha_1,\alpha_2}|^2 \Rightarrow \sum_{\alpha_2=0}^{\infty} |b_{\alpha_2}|^2 \leq \sum_{\alpha_2=0}^{\infty} \sum_{\alpha_1=0}^{\infty} |a_{\alpha_1,\alpha_2}|^2 = \sum_{|\alpha|=0}^{\infty} |a_\alpha|^2 < \infty$$

Notice that $\sum_{|\alpha|=0}^{\infty} |a_\alpha|^2 = \|f\|^2 < \infty$. By the Cauchy-Hadamard formula, as in the one-dimensional case, $\tilde{f_{z_1}}$ is a function holomorphic function in the variable $z_2 \in D$.
Likewise, we have that $\tilde{f_{z_2}}$ is a holomorphic function in the variable $z_1 \in \mathbb{D}$. Thus $\tilde{f}(z) = \sum_{|\alpha|=0}^{\infty} a_\alpha z^\alpha$ defines a holomorphic function in $\Delta(0,1)$.

□

Corollary 6.1.4. *Let $f \in H^2(T^2)$ and $\tilde{f}_r(z) := \tilde{f}(rz)$, then $\|\tilde{f}_r\|_2$ increases with r and $\lim_{r\to 1} \|\tilde{f}_r - f\|_2 = 0$.*

Proof. Let $f \in H^2(T^2)$ with $f(z) = \sum_{|\alpha|=0}^{\infty} a_\alpha z^\alpha$ in L^2 norm. Then $\tilde{f}_r(z) = \sum_{|\alpha|=0}^{\infty} a_\alpha r^{|\alpha|} z^\alpha$ converges uniformly, so it converges in L^2 norm. Notice that

$$f - \tilde{f}_r = \sum_{|\alpha|=0}^{\infty} a_\alpha (1 - r^{|\alpha|}) z^\alpha$$

in L^2 sense.
This implies,

$$\|f - \tilde{f}_r\|_2^2 = \sum_{|\alpha|=0}^{\infty} |a_\alpha|^2 (1 - r^{|\alpha|})^2 \Rightarrow \lim_{r\to 1} \|f - \tilde{f}_r\|_2^2 = 0$$

□

Theorem 6.1.5 (Fatou's Theorem). *Let $f \in H^2(T^2)$, then $\lim_{r\to 1} \tilde{f}_r(z) = f(z)$ a.e.*

Proof. Observe that

$$\log^+ |\tilde{f}_r(z)| = \frac{1}{2} \log^+ |\tilde{f}_r(z)|^2 \leq \frac{1}{2} |\tilde{f}_r(z)|^2$$

then

$$\int_{T^2} \log^+ |\tilde{f}_r(z)| dm(z) \leq \frac{1}{2} \int_{T^2} |\tilde{f}_r(z)|_2^2 dm(z)$$
$$= \frac{1}{2} \|\tilde{f}_r\|_2^2 \leq \frac{1}{2} \|f\|_2^2$$

Therefore

$$\sup_{0\leq r<1} \int_{T^2} \log^+ |\tilde{f}_r(z)| dm(z) \leq \frac{1}{2}\|f\|_2^2 < \infty$$

By Theorem 3.3.3 of [Rd3, p. 45] , $f^*(z) := \lim_{r\to 1} \tilde{f}_r(z)$ exists a.e. Hence f^* is measurable function.

By Fatou's Lemma,

$$\begin{aligned}\int_{T^2} |f^*(z)|^2 dm(z) &= \int_{T^2} \lim_{r\to 1} |\tilde{f}_r(z)|^2 dm(z) \leq \liminf_{r\to 1} \int_{T^2} |\tilde{f}_r(z)|^2 dm(z) \\ &= \liminf_{r\to 1} \|\tilde{f}_r(z)\|_2^2 \leq \|f\|_2^2\end{aligned}$$

Thus $f^* \in L^2(T^2)$. Now, since $\tilde{f}_r \to f$ in L^2 norm, it converges in measure; therefore, there exist a subsequence of $\tilde{f}_r$ that converges pointwise to f. Hence $f = f^*$ a.e.

□

Remark 6.1.4. Notice that the Poisson's Kernel Formula (see Theorem 3.2.5) in dimension one can be restated as follows: For $f \in H^2(S^1)$ we have

$$\tilde{f}(z) = \int_{S^1} f(w) P_z(w) dm(w)$$

where $z \in \mathbb{D}$ is fixed and

$$P_z(w) := \sum_{-\infty}^{\infty} \bar{w}^k z^k$$

converges uniformly.

P_z is called Poisson's Kernel.

Now, we are going to prove the analogue to Theorem 3.2.5 .

Theorem 6.1.6 (Poisson's Kernel Formula). *Let $f \in H^2(T^2)$. For $z = (z_1, z_2) \in \Delta(0,1)$ we have*

$$\tilde{f}(z) = \int_{T^2} f(w) P_z(w) dm(w)$$

where

$$P_z(w) := P_{z_1}(w_1) P_{z_2}(w_2)$$

is the product of Poisson's Kernels in dimension one.

Proof. Let $f \in H^2(T^2)$ with $f = \sum_{|\alpha|=0}^{\infty} a_\alpha z^\alpha$ in L^2 norm.

$$P_z(w) = P_{z_1}(w_1) P_{z_2}(w_2) = \sum_{-\infty}^{\infty} \bar{w_1}^{k_1} z_1^{k_1} \sum_{-\infty}^{\infty} \bar{w_2}^{k_2} z_2^{k_2} = \sum_{k\in\mathbb{Z}^2} \bar{w}^k z^k$$

By uniform convergence we have

$$\int_{T^2} f(w)P_z(w)dm(w) = \sum_{k\in\mathbb{Z}^2}(\int_{T^2} f(w)\bar{w}^k dm(w))z^k = \sum_{k\in(\mathbb{Z}^+)^2} a_k z^k$$

$$= \sum_{|\alpha|=0}^{\infty} a_\alpha z^\alpha = \tilde{f}(z)$$

□

Definition 6.1.7. Define

$$P : L^2(T^2) \to H^2(T^2)$$

$$P(g)(z) := \sum_{\alpha\geq 0}^{\infty} \langle z^\alpha, g\rangle z^\alpha \text{ in } L^2 \text{ norm}$$

Remark 6.1.5. P is clearly well-defined and is the orthogonal projection of $L^2(T^2)$ onto $H^2(T^2)$

6.2 Hardy-Toeplitz Operators over $\Delta(0,1) \subset \mathbb{C}^2$

6.2.1 Definitions and basic properties

In this section we shall obtain the same standard results as in the one-dimensional case which are the background for the characterization of the Hardy-Toeplitz C^*-algebra.

Definition 6.2.1. For f a continuous function on T^2 the multiplication operator with symbol f $m_f : L^2(T^2) \to L^2(T^2)$ is defined by $m_f(g) := fg$

Remark 6.2.1. $\|m_f(g)\|_2 = \|fg\|_2 \leq \|f\|_\infty \|g\|_2 \Rightarrow \|m_f\|_{op} \leq \|f\|_\infty$

Remark 6.2.2. Since $< m_f(h), g >=< h, \bar{f}g >$ for every $h, g \in L^2(T^2)$, we have $m_f{}^* = m_{\bar{f}}$

Definition 6.2.2. For f a continuous function on T^2, the Hardy-Toeplitz operator with symbol f $T_f : H^2(T^2) \to H^2(T^2)$ is defined by $T_f(g) := P \circ m_f(g)$

Proposition 6.2.3. *For every $f \in C(T^2)$ we have*
(i) $\|T_f\|_{op} \leq \|f\|_\infty$
(ii) $T_f{}^* = T_{\bar{f}}$
If $\varphi \in H^2(T^2) \cap C(T^2)$ *, then*
(iii) $T_f T_\varphi = T_{f\varphi}$
(iv) $T_{\bar{\varphi}} T_f = T_{\bar{\varphi} f}$

Proof. The proof is similar to the proof in Proposition 2.3.3

$\square$

Definition 6.2.4. The Hardy-Toeplitz C*-algebra over T^2 is defined as the unital C*-algebra $\mathcal{T}(T^2) := C^*\langle T_f : f \in C(T^2)\rangle$ generated by all Toeplitz operators with continuous symbols.

Remark 6.2.3. $\mathcal{T}(T^2)$ is not commutative. Indeed,
Since $z^\alpha \in H^2(T^2)$ $\forall \alpha \geq 0$, we have

$$T_z(z^\alpha) = P(z^{\alpha+1}) = z^{\alpha+1}$$

and

$$T_{\bar{z}}(z^\alpha) = P(\bar{z}z^\alpha) = P(z^{\alpha-1}) = z^{\alpha-1} \text{ if } \alpha > 0 \text{ and } 0 \text{ otherwise}$$

In particular, $T_zT_{\bar{z}}(1) = 0$ and $T_{\bar{z}}T_z(1) = T_{\bar{z}}(z) = 1$
Thus, $\mathcal{T}(T^2)$ is not commutative.

Proposition 6.2.5. *$\mathcal{T}(T^2) = C^*\langle T_p : p \in P(\mathbb{C}^2)\rangle$, where $P(\mathbb{C}^2)$ denotes the algebra of polynomials over $\mathbb{C}^2$*

Proof. Same proof as in Proposition 2.3.5 .

$\square$

Proposition 6.2.6. *$\mathcal{T}(T^2)$ acts irreducibly on $H^2(T^2)$*

Proof. Let $B : H^2(T^2) \to H^2(T^2)$ be an orthogonal projection that commutes with $\mathcal{T}(T^2)$.
Then $T_fB = BT_f$ on $H^2(T^2)$ for every $f \in C(T^2)$.
Notice that $p(z)B(1) \in H^2(T^2)$ for any $p \in P(\mathbb{C}^2)$, then for any $p, q \in P(\mathbb{C}^2)$

$$B(q) = B(T_q(1)) = T_qB(1) = P(qB(1)) = qB(1) \tag{6.2.1}$$

$$\begin{aligned}\langle B(1), \bar{p}q\rangle = \langle pB(1), q\rangle = \langle T_p(B(1)), q\rangle &= \langle (T_p \circ B)(1), q\rangle \\ = \langle (B \circ T_p)(1), q\rangle &= \langle B(p), q\rangle\end{aligned}$$

$$\langle B(p), q\rangle = \langle p, B(q)\rangle = \langle p, qB(1)\rangle = \langle\overline{B(1)}, \bar{p}q\rangle \;\Rightarrow\; \langle B(1) - \overline{B(1)}, \bar{p}q\rangle = 0$$

If we choose $p(z) = z^\alpha$ and $q(z) = 1$ and vice versa for $\alpha \geq 0$, we obtain for every $\alpha \in ({\mathbb{Z}_0}^+)^2 \cup (\mathbb{Z}^-)^2$

$$\langle B(1) - \overline{B(1)}, z^\alpha\rangle = 0$$

Denote $f = B(1)$. Note that $f - \bar{f} \in span(z^\alpha)_{\alpha \in ({\mathbb{Z}_0}^+)^2 \cup (\mathbb{Z}^-)^2}$. Therefore $f - \bar{f} = 0$,i.e., f is real on T^2.
Using Poisson's Kernel formula, for $z \in \Delta(0,1)$

$$\tilde{f}(z) = \int_{T^2} f(w)P_z(w)dm(w) \in \mathbb{R}$$

because $f(w) \in \mathbb{R}$ and $P_z(w) \in \mathbb{R}$. Therefore $\tilde{f}$ is real on $\Delta(0,1)$. Hence, $\tilde{f}$ is a real constant function,because it is holomorphic.
Now, by Fatou's Theorem, $f \equiv a \in \mathbb{R}$ a.e. Since we are working in $L^2(T^2)$, we can assume $f \equiv a$.
Since $B^2 = B$ and $B(1)$ is constant, $B(1) = B^2(1) = B(B(1)) = B(a \times 1) = a \times B(1)$ implies $B(1) = 1$ or 0.
Thus, by equation 6.2.1, $B = 0$ or $B = Id$. □

6.3 Tensor Products

Roughly speaking, tensor products are introduced in this section because they allow us to work in each variable separately, i.e, to split the analysis over the polydisc into the analysis over two independent unit discs.
The most important result of this section is the description of the commutator ideal of $\mathcal{T}(H^2(\Delta(0,1)))$ in terms of tensor products of operators.

Definition 6.3.1. For $f, g \in L^2(S^1)$ define

$$(f \otimes g)(z_1, z_2) := f(z_1)g(z_2) \; z_1, z_2 \in S^1$$

Remark 6.3.1. Clearly $f \otimes g \in L^2(T^2)$ and $\|f \otimes g\|_2 = \|f\|_2 \|g\|_2$

Definition 6.3.2. For $X, Y \subset L^2(S^1)$ define

$$X \odot Y := \{\sum_{j=1}^{n} f_j \otimes g_j : f_j \in X, g_j \in Y\}$$

Denote by $X \otimes Y$ the closure of $X \odot Y$ in $L^2(T^2)$

Proposition 6.3.3. *$L^2(T^2) = \mathcal{P} \otimes \mathcal{P} = L^2(S^1) \otimes L^2(S^1)$ and $H^2(T^2) = \mathcal{P}_+ \otimes \mathcal{P}_+ = H^2(S^1) \otimes H^2(S^1)$, where $\mathcal{P} := \{p : p(z, \bar{z})$is a polynomial in the variables z and $\bar{z}$ with $z \in S^1\}$ and $\mathcal{P}_+ := \{p : p(z)$ is a polynomial in the variable z with $z \in S^1\}$*

Proof. Just observe that the basis elements of $L^2(T^2)$ (resp. $H^2(T^2)$) are in $\mathcal{P} \otimes \mathcal{P}$ (resp. $\mathcal{P}_+ \otimes \mathcal{P}_+$). The other containment is trivial. □

Proposition 6.3.4. *$C(T^2) = \mathcal{P} \otimes \mathcal{P} = C(S^1) \otimes C(S^1)$ where the tensor product spaces on the right are closures with the infinity norm.*

Proof. It follows from the Stone-Weierstrass Theorem.

$\square$

Definition 6.3.5. Let X be $L^2(S^1)$ or $H^2(S^1)$, let Y be $L^2(S^1)$ or $H^2(S^1)$, and $A \in L(X)$, $B \in L(Y)$. For $h = \sum_{j=1}^{n} f_j \otimes g_j \in X \odot Y$ define

$$(A \otimes B)(h) := \sum_{j=1}^{n} A(f_j) \otimes B(g_j)$$

Proposition 6.3.6. *Let X, Y, A and B be as in the above definition. $A \otimes B$ extends to a bounded operator on $X \otimes Y$ and*

$$\|A \otimes B\|_{op} = \|A\|_{op}\|B\|_{op}$$

Proof. Let $h \in X \odot Y$. Straightforward calculation leads to

$$\|(A \odot I)(h)\| \leq \|A\|\|h\| \text{ and } \|(I \odot B)(h)\| \leq \|B\|\|h\|$$

Then

$$\|(A \otimes B)(h)\| = \|(A \odot I)(I \odot B)(h)\| \leq \|A\|\|B\|\|h\|$$

Therefore we can extend $A \otimes B$ to a bounded operator on $X \otimes Y$.
Let $\epsilon > 0$. Choose $f \in X$ and $g \in Y$ both with norm equal to 1 and $\sqrt{1-\epsilon}\|A\| \leq \|A(f)\|$, $\sqrt{1-\epsilon}\|B\| \leq \|B(g)\|$.
Notice that $\|f \otimes g\| = 1$ then

$$\|(A \otimes B)(f \otimes g)\| = \|A(f) \otimes B(g)\| = \|A(f)\|\|B(g)\| \geq (1-\epsilon)\|A\|\|B\|$$

Thus $\|A \otimes B\|_{op} = \|A\|_{op}\|B\|_{op}$

$\square$

Corollary 6.3.7. $P = P \otimes P$ *, where P on the left side of the equality is the orthogonal projection of $L^2(T^2)$ onto $H^2(T^2)$ and P on the right side is the orthogonal projection of $L^2(S^1)$ onto $H^2(S^1)$*

Proof. The equality is true for $\mathcal{P} \odot \mathcal{P}$. Then the equality extends to $L^2(T^2) = \mathcal{P} \otimes \mathcal{P}$

$\square$

Lemma 6.3.8. *Let $f, g, h \in C(S^1)$ then*
(i) $m_f \otimes m_g = m_{f \otimes g}$
(ii) $T_f \otimes T_g = T_{f \otimes g}$
(iii) $T_f \otimes (T_g \circ T_h) = T_{f \otimes g} \circ T_{1 \otimes h}$

Proof. **(i)** The equality is true for the basis elements of $L^2(T^2)$; hence, $m_f \otimes m_g = m_{f \otimes g}$
(ii) $T_f \otimes T_g = (P \circ m_f) \otimes (P \circ m_g) = (P \otimes P) \circ (m_f \otimes m_g) = P \circ m_{f \otimes g} = T_{f \otimes g}$
(iii) $T_f \otimes (T_g \circ T_h) = (T_f \otimes T_g) \circ (T_1 \otimes T_h) = T_{f \otimes g} \circ T_{1 \otimes h}$

$\square$

Proposition 6.3.9. *$\mathcal{T}(T^2) = \overline{\mathcal{T}(S^1) \otimes \mathcal{T}(S^1)}$ where the closure on the right side of the equality is respect to the operator norm.*

Proof. Using (ii) and (iii) in the above lemma we have $\mathcal{T}(S^1) \otimes \mathcal{T}(S^1) \subset \mathcal{T}(T^2)$.
Recall that $C(T^2) = \overline{C(S^1) \otimes C(S^1)}$ in the infinity norm sense. By (ii) in the above lemma we have $T_{\sum_{j=1}^n f_j \otimes g_j} = \sum_{j=1}^n T_{f_j} \otimes T_{g_j} \subset \mathcal{T}(S^1) \otimes \mathcal{T}(S^1)$. Since uniform convergence of symbols implies operator norm convergence of Toeplitz operators, $T_f \in \overline{\mathcal{T}(S^1) \otimes \mathcal{T}(S^1)}$ for any $f \in C(T^2)$.
Thus $\mathcal{T}(T^2) \subset \overline{\mathcal{T}(S^1) \otimes \mathcal{T}(S^1)}$

□

Corollary 6.3.10. *$\mathcal{T}(S^1) \otimes \mathcal{K}(H^2(S^1))$ and $\mathcal{K}(H^2(S^1)) \otimes \mathcal{T}(S^1)$ are two-sided ideals of $\mathcal{T}(T^2)$*

Corollary 6.3.11. *The C^*-algebraic sum $I_2 := \mathcal{T}(S^1) \otimes \mathcal{K}(H^2(S^1)) \bigoplus \mathcal{K}(H^2(S^1)) \otimes \mathcal{T}(S^1)$ is a two-sided ideal of $\mathcal{T}(T^2)$*

Theorem 6.3.12. *I_2 is the semi-commutator ideal $\mathcal{T}''(T^2)$ of $\mathcal{T}(T^2)$*

Proof. Consider any semi-commutator

$$T_{fg} - T_f T_g = P m_f (I - P) m_g = P m_f [(I - P) \otimes I + P \otimes (I - P)] m_g$$

As before, we first assume that $f = f_1 \otimes f_2$ and $g = g_1 \otimes g_2$. Then

$$T_{fg} - T_f T_g = P m_{f_1 \otimes f_2} [(I - P) \otimes I + P \otimes (I - P)] m_{g_1 \otimes g_2}$$

$$= P m_{f_1} (I - P) m_{g_1} \otimes P m_{f_2} m_{g_2} + P m_{f_1} m_{g_1} \otimes P m_{f_2} (I - P) m_{g_2}$$
$$= (T_{f_1 g_1} - T_{f_1} T_{g_1}) \otimes T_{f_2 g_2} + T_{f_1 g_1} \otimes (T_{f_2 g_2} - T_{f_2} T_{g_2}) \in I_2$$

Similar argument works for $f, g \in C(S^1) \odot C(S^1)$. By closure of I_2 we have that $T_{fg} - T_f T_g \in I_2$ for any $f, g \in C(T^2)$. Thus $\mathcal{T}''(T^2) \subset I_2$.
Take any $T_h \otimes (T_f g - T_f T_g)$ "generator" of $\mathcal{T}(S^1) \otimes \mathcal{K}(H^2(S^1))$. Then

$$T_h \otimes (T_f g - T_f T_g) = T_h \otimes T_{fg} - T_h \otimes T_f T_g = T_{h \times fg} - T_{h \otimes f} T_{1 \otimes g}$$

$$= T_{(h \otimes f)(1 \otimes g)} - T_{h \otimes f} T_{1 \otimes g} \in \mathcal{T}''(T^2)$$

Therefore $\mathcal{T}(S^1) \otimes \mathcal{K}(H^2(S^1)) \subset \mathcal{T}''(T^2)$. Similarly $\mathcal{K}(H^2(S^1)) \otimes \mathcal{T}(S^1) \subset \mathcal{T}''(T^2)$

□

Corollary 6.3.13. *I_2 coincides with the commutator ideal $\mathcal{T}'(T^2)$*

Proof. Choose any "generator"

$$T_h \otimes (T_f T_g - T_g T_f) \in I_2$$

Then

$$\begin{aligned} T_h \otimes (T_f T_g - T_g T_f) &= (T_h \otimes T_1)(T_1 \otimes (T_f T_g - T_g T_f)) \\ &= (T_{h\otimes 1})(T_{1\otimes f} T_{1\otimes g} - T_{1\otimes g} T_{1\otimes f}) \in \mathcal{T}'(T^2) \end{aligned}$$

Therefore $I_2 \subset \mathcal{T}'(T^2)$. On the other hand, it is clear that the commutator ideal is included in the semi-commutator ideal. Thus I_2 coincides with the commutator ideal $\mathcal{T}'(T^2)$.

□

Remark 6.3.2. The C^*-algebra generated by the semi-commutators and the commutator ideal $\mathcal{T}'(T^2)$ coincide.

6.4 Characterization of $\mathcal{T}(T^2)$

We start this section by examining some properties of peaking functions over the unit polydisc and giving some applications of those peaking functions.
Some of the following assertions will be proven for the one-dimensional case and can be easily generalized for the two-dimensional case.

Remark 6.4.1. Recall that for any function $g \in A(\mathbb{D}) = O(\mathbb{D}) \cap C(\bar{\mathbb{D}})$ (algebra of continuous functions on $\bar{\mathbb{D}}$ which are holomorphic on $\mathbb{D}$) , the restriction function is $g|_{S^1} \in H^2(S^1)$ (see Theorem 3.1.4). For this reason we are going to allow the abuse of notation $A(\mathbb{D}) \subset H^2(S^1)$ in the following lemma.

Lemma 6.4.1. *There exists a continuous mapping*

$$S^1 \to A(\mathbb{D}) \subset H^2(S^1)$$

$$w \mapsto h_w$$

where h_w is a peaking function at w.

Proof. Define $h_1(z) := \frac{1}{2}(z+1)$ a *Möbius* transform that takes S^1 to the circle centered at $(\frac{1}{2}, 0)$ and radius $\frac{1}{2}$. Clearly h_1 is a peaking function at 1.
For any $w \in S^1$ define

$$h_w(z) := h_1(\frac{z}{w}) = \frac{1}{2}\frac{z+w}{w}$$

It is clear that h_w is a peaking function at w. Let $w, w_0 \in S^1$ then

$$|h_w(z) - h_{w_0}(z)| = \frac{1}{2}|\frac{z+w}{w} - \frac{z+w_0}{w_0}| = \frac{1}{2}|z(w_0 - w)| \leq \frac{1}{2}|w_0 - w|$$

Thus, the lemma follows.

□

Remark 6.4.2. For $s \in S^1$ we have $h_w(zs^{-1}) = h_{ws}(z)$

Remark 6.4.3. By the continuity of h_w and its peaking property at w, for each open neighborhood $U \subset S^1$ of $w \in S^1$ there exists an open neighborhood $V \subset U$ of w relatively compact in U such that

$$\sup_{z \in S^1 \setminus U} |h_w(z)| < \inf_{z \in V} |h_w(z)|$$

Remark 6.4.4. Recall that for any function $g \in A(\Delta(0,1)) = O(\Delta(0,1)) \cap C(\bar{\Delta}(0,1))$, the restriction function $g|_{T^2} \in H^2(T^2)$ (same idea as in Theorem 3.1.4). For this reason we are going to allow the abuse of notation $A(\Delta(0,1)) \subset H^2(T^2)$ in the following lemma.

Lemma 6.4.2. *There exists a continuous mapping*

$$T^2 \to A(\Delta(0,1)) \subset H^2(T^2)$$

$$w \mapsto h_w$$

where h_w is a peaking function at w.

Proof. For $w \in T^2$ define $h_w(z) := h_{w_1}(z_1)h_{w_2}(z_2)$. Clearly $h_w \in A(\Delta(0,1))$ and is a peaking function at w. The continuity of the map

$$T^2 \to A(\Delta(0,1)) \subset H^2(T^2)$$

follows from the lemma in the one-dimensional case and the next inequality

$$|h_w(z) - h_a(z)| = |h_{w_1}(z_1)h_{w_2}(z_2) - h_{a_1}(z_1)h_{a_2}(z_2)|$$

$$\begin{aligned}
&= |h_{w_1}(z_1)h_{w_2}(z_2) - h_{w_1}(z_1)h_{a_2}(z_2) + h_{w_1}(z_1)h_{a_2}(z_2) - h_{a_1}(z_1)h_{a_2}(z_2)| \\
&\le |h_{w_1}(z_1)||h_{w_2}(z_2) - h_{a_2}(z_2)| + |h_{a_2}(z_2)||h_{w_1}(z_1) - h_{a_1}(z_1)| \\
&\le |h_{w_2}(z_2) - h_{a_2}(z_2)| + |h_{w_1}(z_1) - h_{a_1}(z_1)|
\end{aligned}$$

□

Remark 6.4.5. For $s \in T^2$ we have $h_w(zs^{-1}) = h_{ws}(z)$

Remark 6.4.6. By the continuity of h_w and its peaking property at w, for each open neighborhood $U \subset T^2$ of $w \in T^2$ there exists an open neighborhood $V \subset U$ of w relatively compact in U such that

$$\sup_{z \in T^2 \setminus U} |h_w(z)| < \inf_{z \in V} |h_w(z)|$$

Proposition 6.4.3. *For every $f \in C(S^1)$ and every $w \in S^1$*

$$\frac{\int_{S^1} f(z)|h_w(z)|^{2n} dz}{\int_{S^1} |h_w(z)|^{2n} dz} \to f(w) \text{ as } n \to \infty$$

Proof. We can assume that f is a real-valued function because we can repeat the same argument for the real and imaginary part of f. Also, correcting by a constant, we can assume that $f(w) = 0$.
Therefore, given $\epsilon > 0$ there exists an open neighborhood $U \subset T^2$ of $w \in S^1$ such that $\sup|f|(U) < \epsilon$. By the above remark, there exists an open neighborhood $V \subset U$ of w relatively compact in U such that

$$\sup_{z\in S^1\setminus U} |h_w(z)| < \inf_{z\in V} |h_w(z)|$$

$$|\int_{S^1} f(z)|h_w(z)|^{2n}dz| \leq \int_U |f(z)||h_w(z)|^{2n}dz + \int_{S^1\setminus U} |f(z)||h_w(z)|^{2n}dz$$
$$< \epsilon \int_{S^1} |h_w(z)|^{2n}dz + m(S^1\setminus U)\|f\|_\infty \sup_{z\in S^1\setminus U} |h_w(z)|^{2n}$$

On the other hand

$$\int_{S^1} |h_w(z)|^{2n}dz \geq \int_V |h_w(z)|^{2n}dz \geq m(V) \inf_{z\in V} |h_w(z)|^{2n}$$

Then

$$|\frac{\int_{S^1} f(z)|h_w(z)|^{2n}dz}{\int_{S^1} |h_w(z)|^{2n}dz}| < \epsilon + \frac{m(S^1\setminus U)}{m(V)}\|f\|_\infty \left(\frac{\sup_{z\in S^1\setminus U}|h_w(z)|}{\inf_{z\in V}|h_w(z)|}\right)^{2n}$$

Therefore

$$\frac{\sup_{z\in S^1\setminus U}|h_w(z)|}{\inf_{z\in V}|h_w(z)|} < 1 \Rightarrow \frac{\int_{S^1} f(z)|h_w(z)|^{2n}dz}{\int_{S^1} |h_w(z)|^{2n}dz} \to 0 \text{ as } n\to\infty$$

□

Proposition 6.4.4. *For every $f \in C(T^2)$ and every $w \in T^2$*

$$\frac{\int_{T^2} f(z)|h_w(z)|^{2n}dm(z)}{\int_{T^2} |h_w(z)|^{2n}dm(z)} \to f(w) \text{ as } n \to \infty$$

Proof. Repeat the prove of the above proposition.

□

We can say that the purpose of introducing peaking functions is reflected in the following corollary.

Corollary 6.4.5. *Let $f \in C(T^2)$ be non-negative real valued. Then for each $w \in T^2$ there exists a sequence $(h_n) \subset H^2(T^2)$ where*

$$h_n(z) := \frac{{h_w}^n(z)}{\|{h_w}^n\|_2}$$

such that

$$\|fh_n\|_2 \to f(w) \text{ as } n \to \infty$$

Proposition 6.4.6. *Let p a polynomial in k non-commuting variables. Then*

$$\|p(T_{f_1}, ..., T_{f_k})\|_{op} \geq \|p(f_1, ..., f_k)\|_\infty$$

where $f_j \in C(T^2)$ for $1 \leq j \leq k$

Proof. Let $(b_1, ..., b_k) \in Im(f_1, ..., f_k)$. Define

$$f := \sum_{j=1}^{k} |f_j - b_j|^2 \in C(T^2)$$

Clearly f has a zero in T^2. Using the above corollary, given $\epsilon > 0$ there exists $h \in H^2(T^2)$ with norm 1 such that $\|fh\|_2 < \epsilon$. Then

$$\sum_{j=1}^{k} \|T_{f_j}(h) - b_j h\|_2{}^2 = \sum_{j=1}^{k} \|P(f_j h - b_j h)\|_2{}^2 \leq \sum_{j=1}^{k} \|f_j h - b_j h\|_2{}^2 = \|fh\|_2{}^2 < \epsilon^2$$

therefore

$$\|T_{f_j}(h) - b_j h\|_2 < \epsilon \text{ for every } 1 \leq j \leq k$$

Consider operators of the form $T_{f_{j_1}}...T_{f_{j_m}} - b_{j_1}...b_{j_m}$ where $1 \leq j_l \leq k$ for $1 \leq l \leq m$ Notice that

$$T_{f_{j_1}}...T_{f_{j_m}} - b_{j_1}...b_{j_m} = \sum_{l=1}^{m} T_{f_{j_1}}...T_{f_{j_{l-1}}}(T_{f_{j_l}} - b_{j_l})b_{j_{l+1}}...b_{j_m}$$

We can assume that $\|f_j\| \leq 1$ and $|b_j| \leq 1$ for $1 \leq j \leq k$ in order to reduce calculations. Then

$$\|T_{f_{j_1}}...T_{f_{j_m}}(h) - b_{j_1}...b_{j_m} h\|_2 \leq \sum_{l=1}^{m} \|T_{f_{j_1}}...T_{f_{j_{l-1}}}(T_{f_{j_l}}(h) - b_{j_l} h) b_{j_{l+1}}...b_{j_m}\|_2 \leq \sum_{l=1}^{m} \|T_{f_{j_l}}(h) - b_{j_l} h\|$$

$$\leq m\epsilon$$

For any $\eta > 0$ we can choose $\epsilon > 0$ such that

$$\|p(T_{f_1}...T_{f_k})(h) - p(b_1, ..., b_k)h\|_2 \leq \eta$$

Using triangle inequality

$$|p(b_1, ..., b_k)| \leq \|p(T_{f_1}...T_{f_k})(h)\|_2 + \eta \leq \|p(T_{f_1}...T_{f_k})\|_{op} + \eta$$

Since η was arbitrary, we have

$$|p(b_1, ..., b_k)| \leq \|p(T_{f_1}...T_{f_k})\|_{op}$$

Finally, since $(b_1, ..., b_k) \in Im(f_1, ..., f_k)$ was arbitrary, we have

$$\|p(f_1, ..., f_k)\|_\infty \leq \|p(T_{f_1}, ..., T_{f_k})\|_{op}$$

$\square$

The above proposition is crucial for the proof of the injectivity of the C^*-homomorphism ρ defined in the following theorem.

Theorem 6.4.7. *The mapping*

$$\rho : C(T^2) \to \mathcal{T}(T^2)/I_2$$

$$\rho(f) := T_f + I_2$$

is a C^*- isomorphism.

Proof. Recall that I_2 is the commutator ideal of $\mathcal{T}(T^2)$ (see Remark 6.3.2). It is clear that ρ is well-defined and is a surjective C^*- homomorphism. It is suffices to prove that ρ is injective.
Let $B \in I_2$. Then B is in the C^*-ideal generated by the semi-commutators; hence, $B = limB_n$ where B_n is a finite sum of operators of the form

$$T_{f_1}, ..., T_{f_k}(T_F T_G - T_{FG})T_{g_1}, ..., T_{g_m}$$

Consider a polynomial with non-commutative variables with terms of the same form than the terms of B_n, i.e., with terms of the form $x_1x_2...x_k(z_1z_2 - z_3)y_1...y_m$.
Using the previous proposition and

$$f_1...f_k(FG - FG)g_1...g_m = 0$$

we have

$$\|f\|_\infty \le \|T_f + B_n\|_{op}$$

Therefore $\|f\|_\infty \le \|T_f + B\|_{op}$
If $f \in Ker(\rho)$ then $T_f \in I_2$. Thus $f \equiv 0$. □

Chapter 7

Bergman Space and Toeplitz Operators over $\Delta(0,1) \subset \mathbb{C}^2$

This chapter follows the line of reasoning from the previous chapter and generalizes some results in Chapter 2. In the previous chapter we found that the Hardy-Toeplitz algebra over the polydisc is not similar in structure to the Hardy-Toeplitz algebra over the unit ball. In this chapter we shall show a similar contrast in the setting of the Bergman-Toeplitz algebras.

7.1 Definition and basic properties

Notation:
$dV(z)$: Lebesgue measure on $C^2 \approx R^4$.
$L^2(\Delta(0,1))$: Lebesgue space of square-integrable functions.

$$\langle f, g\rangle = \int_{\Delta(0,1)} \overline{f(z)} g(z) dV(z)$$

$A(\Delta(0,1)) := O(\Delta(0,1)) \cap C(\bar{\Delta}(0,1))$

Definition 7.1.1.

$$H^2(\Delta(0,1)) := L^2(\Delta(0,1)) \cap O(\Delta(0,1))$$

is called Bergman Space.

Remark 7.1.1. It is not hard to see that $H^2(\Delta(0,1)) := L^2(\Delta(0,1)) \cap O(\Delta(0,1))$ is a closed subspace of $L^2(\Delta(0,1))$ (similar proof as in Theorem 2.1.2).

Definition 7.1.2. The orthogonal projection $P : L^2(\Delta(0,1)) \to H^2(\Delta(0,1))$ is called the Bergman Projection.

Remark 7.1.2. For any fixed $z \in \Delta(0,1)$, the evaluation map $eval : H^2(\Delta(0,1)) \to \mathbb{C}$ $eval(f) = f(z)$ is continuous (same idea as in Proposition 2.1.1). Then, by the Riesz-Frechet Theorem, there exists $K_z \in H^2(\Delta(0,1))$ such that

$$f(z) = \langle K_z, f\rangle = \int_{\Delta(0,1)} \overline{K_z(w)} f(w) dA(w) \ \forall f \in H^2(\Delta(0,1))$$

$K(z,w) := \overline{K_z(w)}$ is called the Bergman Kernel function.

Remark 7.1.3. Using power series as in the one-dimensional case (see Remark 2.2.2) we have

$$K(z,w) = \overline{K_z(w)} = \frac{1}{\pi^2(1-\bar{w}_1 z_1)^2(1-\bar{w}_2 z_2)^2}$$

Remark 7.1.4. Since the Bergman projection P is self-adjoint, we obtain the following formula

$$P(g)(z) = \int_{\Delta(0,1)} \frac{g(w)}{\pi^2(1-\bar{w}_1 z_1)^2(1-\bar{w}_2 z_2)^2} dA(w)$$

where $g \in L^2(\Delta(0,1))$ and $z \in \Delta(0,1)$

Observations:

1. If $f \in L^2(\Delta(0,1))$, then $\int_{\Delta(0,1)} f(z)dA(z) = \int_{\mathbb{D}} \int_{\mathbb{D}} f(z_1,z_2)dA(z_1)dA(z_2)$

2. Recall that in dimension 1: $\phi_n(z) = c_n z^n$ where $c_n = \sqrt{\frac{n+1}{\pi}}$ is an orthonormal basis of $H^2(\mathbb{D})$

3. Define $\phi_\alpha(z) := \phi_{\alpha_1}(z_1)\phi_{\alpha_2}(z_2)$ where $\alpha = (\alpha_1, \alpha_2) \in {Z_0}^+ \times {Z_0}^+$. Then $(\phi_\alpha)_{\alpha \in {Z_0}^+ \times {Z_0}^+}$ is an orthonormal set of $H^2(\Delta(0,1))$

7.2 Characterization of $H^2(\Delta(0,1))$:

Denote $\bar{\Delta}_r(0,1) = \{z \in \Delta(0,1) : |z_1|, |z_2| \leq r\}$.
For any $f \in H^2(\Delta(0,1))$ with $f(z) = \sum_{\alpha \geq 0}^{\infty} d_\alpha z^\alpha$ converging normally on $\Delta(0,1)$, define $f_r(z) := f(z)\chi_{\bar{\Delta}_r(0,1)}$. Then, $f_r(z) = \sum_{\alpha \geq 0}^{\infty} d_\alpha z^\alpha$ converges uniformly on $\bar{\Delta}_r(0,1)$. Notice that $f_r \in L^2(\Delta(0,1))$

Lemma 7.2.1. *For $\alpha = (\alpha_1, \alpha_2) \geq 0$*

$$\int_{\bar{\Delta}_r(0,1)} |z|^{2\alpha} dA(z) = \frac{\pi r^{2\alpha+2}}{(\alpha_1+1)(\alpha_2+1)} \text{ and } \int_{\bar{\Delta}_r(0,1)} z^\alpha \bar{z}^\beta dA(z) = 0 \ \forall \alpha \neq \beta$$

Proof. The proof just requires straightforward calculation.

□

Lemma 7.2.2. *$f_r \to f$ in $L^2(\Delta(0,1))$ norm for any $f \in H^2(\Delta(0,1))$*

Proof. Notice that $f_r(z) \to f(z)$ pointwise and $|f_r(z) - f(z)| \leq 2|f(z)|$; hence, $|f_r(z) - f(z)|^2 \leq 4|f(z)|^2$ By the Lebesgue Convergence Theorem, $f_r \to f$ in $L^2(\Delta(0,1))$.

□

Lemma 7.2.3. *For any $f \in H^2(\Delta(0,1))$ with $f(z) = \sum_{\alpha \geq 0}^{\infty} d_\alpha z^\alpha$ converging normally on $\Delta(0,1)$,*

$$\langle z^\alpha, f_r \rangle = \frac{\pi}{(\alpha_1+1)(\alpha_2+1)} d_\alpha r^{2\alpha+2}$$

Proof. By uniform convergence,

$$\langle z^\alpha, f_r \rangle = \int_{\Delta(0,1)} f_r(z) \bar{z}^\alpha dA(z) = \int_{\bar{\Delta}_r(0,1)} \sum_{\beta \geq 0}^{\infty} d_\beta z^\beta \bar{z}^\alpha dA(z)$$

$$= \sum_{\beta \geq 0}^{\infty} \int_{\bar{\Delta}_r(0,1)} d_\beta z^\beta \bar{z}^\alpha dA(z) = d_\alpha \int_{\bar{\Delta}_r(0,1)} z^\alpha \bar{z}^\alpha dA(z) = \frac{\pi}{(\alpha_1+1)(\alpha_2+1)} d_\alpha r^{2\alpha+2}$$

□

Lemma 7.2.4. *Let $g \in L^2(\Delta(0,1))$ and $f \in H^2(\Delta(0,1))$, then*

$$\langle g, f_r \rangle \to \langle g, f \rangle$$

In particular

$$\langle z^\alpha, f \rangle = \frac{\pi}{(\alpha_1+1)(\alpha_2+1)} d_\alpha$$

where d_n is the same as in the previous lemma.

Proof. By the Cauchy-Schwarz Inequality and Lemma 7.2.2 ,

$$|\langle g, f_r \rangle - \langle g, f \rangle| = |\langle g, f_r - f \rangle| \leq \|g\|_2 \|f_r - f\|_2 \to 0$$

□

Proposition 7.2.5. *$\{\phi_\alpha\}_{\alpha \geq 0}$, defined as in Obs.3, form an orthonormal basis of $H^2(\Delta(0,1))$.*

Proof. Denote $c_\alpha := \sqrt{\frac{\pi}{(\alpha_1+1)(\alpha_2+1)}}$.
By Obs.3 $\{\phi_\alpha\}$ is an orthonormal set. By Lemma 7.2.4, $<\phi_\alpha, f>= \frac{d_\alpha}{c_\alpha}$ for any $f \in H^2(\Delta(0,1))$ with $f(z) = \sum_{|\alpha|=0}^{\infty} d_\alpha z^\alpha$ converging normally on $\Delta(0,1)$.
It is not hard to verify that $\|f_r\|_2 = \sum_{|\alpha|=0}^{\infty} \frac{|d_\alpha|^2}{c_\alpha{}^2} r^{4n+4}$. Since $f_r \to f$ in L^2 norm and $f_r(z) \to f(z)$ pointwise, we have $\|f_r\|_2 \to \|f\|^2$; hence, $\|f\|^2 = \sum_{|\alpha|=0}^{\infty} \frac{|d_\alpha|^2}{c_\alpha{}^2}$. Therefore, we have

$$\sum_{|\alpha|=0}^{\infty} \langle \phi_\alpha, f \rangle = \|f\|^2$$

Then $f \in span\{\phi_\alpha\}$.
Thus $\{\phi_\alpha\}$ is an orthonormal basis of $H^2(\Delta(0,1))$ and $f(z) = \sum_{|\alpha|=0}^{\infty} \frac{d_\alpha}{c_\alpha} \phi_\alpha(z) = \sum_{|\alpha|=0}^{\infty} d_\alpha z^\alpha$ converges in L^2 norm.

□

Corollary 7.2.6. *Let $f \in H^2(\Delta(0,1))$ with $f(z) = \sum_{\alpha \geq 0}^{\infty} d_\alpha z^\alpha$ converging normally, then $f = \sum_{\alpha \geq 0}^{\infty} d_\alpha z^\alpha$ converges in $L^2(\Delta(0,1))$*

Corollary 7.2.7. *The orthogonal projection P of $L^2(\Delta(0,1))$ onto $H^2(\Delta(0,1))$ is given by*

$$P(g) = \sum_{\alpha \geq 0}^{\infty} \langle \phi_\alpha, g \rangle \phi_\alpha$$

Lemma 7.2.8. *Suppose that*

$$\sum_{\alpha \geq 0}^{\infty} |d_\alpha|^2 \frac{1}{(\alpha_1+1)(\alpha_2+1)} < +\infty$$

Then $f(z) = \sum_{\alpha \geq 0}^{\infty} d_\alpha z^\alpha$ converges normally on $\Delta(0,1)$, i.e., $f \in O(\Delta(0,1))$

Proof. Consider, r_1 and r_2 in $[0,1)$. Then $\exists N > 0$ s.t $\forall \alpha_1, \alpha_2 > N$ we have:

$$r_1^{\alpha_1} r_2^{\alpha_2} < \frac{1}{\sqrt{\alpha_1+1}} \frac{1}{\sqrt{\alpha_2+1}} \quad \Rightarrow |d_\alpha| r_1^{\alpha_1} r_2^{\alpha_2} < |d_\alpha| \frac{1}{\sqrt{\alpha_1+1}} \frac{1}{\sqrt{\alpha_2+1}}$$

$$\text{is bounded for large} |\alpha| \text{ because } |d_\alpha|^2 \frac{1}{(\alpha_1+1)(\alpha_2+1)} \text{ is bounded for large } |\alpha|$$

Therefore, $f(z) = \sum_{\alpha \geq 0}^{\infty} d_\alpha z^\alpha$ converges uniformly on compact subsets of $\Delta(0,1)$ (by Abel's Lemma). Hence f is holomorphic on $\Delta(0,1)$

□

Theorem 7.2.9. $f \in H^2(\Delta(0,1))$ *with*

$$f(z) = \sum_{\alpha \geq 0}^{\infty} d_\alpha z^\alpha \ \textit{normally}$$

if and only if

$$\sum_{\alpha \geq 0}^{\infty} |d_\alpha|^2 \frac{1}{(\alpha_1 + 1)(\alpha_2 + 1)} < +\infty$$

Proof. Suppose that $f \in H^2(\Delta(0,1))$ with

$$f(z) = \sum_{\alpha \geq 0}^{\infty} d_\alpha z^\alpha \text{ normally}$$

Then $f = \sum_{\alpha \geq 0}^{\infty} d_\alpha z^\alpha$ converges in $L^2(\Delta(0,1))$ norm. Rewriting we have

$$f = \sum_{\alpha \geq 0}^{\infty} \frac{d_\alpha \sqrt{\pi}}{\sqrt{(\alpha_1 + 1)(\alpha_2 + 1)}} \phi_\alpha$$

Thus $\|f\|_2 = \pi \sum_{\alpha \geq 0}^{\infty} |d_\alpha|^2 \frac{1}{(\alpha_1+1)(\alpha_2+1)} < +\infty$

Now suppose that $\sum_{\alpha \geq 0}^{\infty} |d_\alpha|^2 \frac{1}{(\alpha_1+1)(\alpha_2+1)} < +\infty$. By the above lemma

$$f(z) = \sum_{\alpha \geq 0}^{\infty} d_\alpha z^\alpha \ \in O(\Delta(0,1))$$

Notice that

$$\int_{\Delta(0,1)} |f_r(z)|^2 dz = \pi \sum_{\alpha \geq 0}^{\infty} |d_\alpha|^2 \frac{1}{(\alpha_1 + 1)(\alpha_2 + 1)} r^{4\alpha + 4}$$

Also observe that $f_r(z) \to f(z)$ pointwise and $|f_r(z)|^2 \leq |f(z)|^2$. By the Monotone Convergence Theorem

$$\lim_{r \to 1} \int_{\Delta(0,1)} |f_r(z)|^2 dz = \int_{\Delta(0,1)} |f(z)|^2 dz$$

But

$$\lim_{r \to 1} \int_{\Delta(0,1)} |f_r(z)|^2 dz = \pi \sum_{\alpha \geq 0}^{\infty} |d_\alpha|^2 \frac{1}{(\alpha_1 + 1)(\alpha_2 + 1)}$$

Thus $f \in L^2(\Delta(0,1))$; hence, $f \in H^2(\Delta(0,1))$

□

7.3 Bergman-Toeplitz Operators over $\Delta(0,1)$

7.3.1 Definitions and basic properties

Definition 7.3.1. For f a continuous function on $\bar{\Delta}(0,1)$ the multiplication operator with symbol f $m_f : L^2(\Delta(0,1)) \to L^2(\Delta(0,1))$ is defined by $m_f(g) := fg$

Remark 7.3.1. $\|m_f(g)\|_2 = \|fg\|_2 \leq \|f\|_\infty \|g\|_2$. This implies $\|m_f\|_{op} \leq \|f\|_\infty$

Remark 7.3.2. Since $\langle m_f(h), g\rangle = \langle h, \bar{f}g\rangle$ for every $h, g \in L^2(D)$, we have $m_f{}^* = m_{\bar{f}}$

Definition 7.3.2. For f a continuous function on $\bar{\Delta}(0,1)$ the Bergman-Toeplitz operator with symbol f, $T_f : H^2(\Delta(0,1)) \to H^2(\Delta(0,1))$ is defined by $T_f(g) := P \circ m_f(g)$

Remark 7.3.3. By characterization of the Bergman projection, we have

$$T_f(g)(z) = P(fg)(z) = \frac{1}{\pi^2}\int_{\Delta(0,1)} \frac{f(w)g(w)}{(1-\bar{w_1}z_1)^2(1-\bar{w_2}z_2)^2} dA(w) \; \forall g \in H^2(\Delta(0,1))$$

Proposition 7.3.3. *For every $f \in C(\bar{\Delta}(0,1))$, we have*
(i) $\|T_f\|_{op} \leq \|f\|_\infty$
(ii) $T_f{}^* = T_{\bar{f}}$
If $\varphi \in A(\Delta(0,1))$ *, then*
(iii) $T_f T_\varphi = T_{f\varphi}$
(iv) $T_{\bar{\varphi}} T_f = T_{\bar{\varphi} f}$

Proof. Similar to the proof in the one-dimensional case (see Proposition 2.3.3). □

Definition 7.3.4. The Bergman-Toeplitz C^*-algebra over $\Delta(0,1)$ is defined as the unital C^*-algebra $\mathcal{T}(\Delta(0,1)) := C^*\langle T_f : f \in C(\bar{\Delta}(0,1))\rangle$ generated by all Toeplitz operators with continuous symbols.

Remark 7.3.4. $\mathcal{T}(\Delta(0,1))$ is not commutative. Indeed,

$$T_z(1) = P(z) = z$$

$$T_{\bar{z}}(1) = P(\bar{z}) = 0$$

$$T_z T_{\bar{z}}(1) = 0 \text{ and } T_{\bar{z}} T_z(1) = T_{\bar{z}}(z) = 1$$

Thus, $\mathcal{T}(\Delta(0,1))$ is not commutative.

Proposition 7.3.5. *$\mathcal{T}(\Delta(0,1)) = C^*\langle T_p : p \in P(\mathbb{C}^2)\rangle$, where $P(\mathbb{C}^2)$: polynomials over $\mathbb{C}^2$*

Proof. Similar to proof in the one-dimensional case.

□

7.4 Characterization of $\mathcal{T}(\Delta(0,1))$

Remark 7.4.1. Recall that $A(\Delta(0,1)) \subset H^2(\Delta(0,1))$

The following lemma was proven in the Hardy case chapter.

Lemma 7.4.1. *There exists a continuous mapping*

$$T^2 \to A(\Delta(0,1)) \subset H^2(\Delta(0,1))$$

$$w \mapsto h_w$$

where h_w is a peaking function at w.

Remark 7.4.2. By the continuity of h_w and its peaking property at w, for each open neighborhood $U \subset \bar{\Delta}(0,1)$ of $w \in T^2$ there exists an open neighborhood $V \subset U$ of w relatively compact in U such that

$$\sup_{z \in \Delta(0,1) \setminus U} |h_w(z)| < \inf_{z \in V} |h_w(z)|$$

Proposition 7.4.2. *For every $f \in C(\bar{\Delta}(0,1))$ and every $w \in T^2$*

$$\frac{\int_{\Delta(0,1)} f(z)|h_w(z)|^{2n} dm(z)}{\int_{\Delta(0,1)} |h_w(z)|^{2n} dm(z)} \to f(w) \text{ as } n \to \infty$$

Proof. The argument used to prove Proposition 6.4.3 can be adapted to make this proof.

□

Corollary 7.4.3. *Let $f \in C(\bar{\Delta}(0,1))$ be non-negative real valued. Then for each $w \in T^2$ there exists a sequence $(h_n) \subset H^2(\Delta(0,1))$ where*

$$h_n(z) := \frac{{h_w}^n(z)}{\|{h_w}^n\|_2}$$

such that

$$\|fh_n\|_2 \to f(w) \text{ as } n \to \infty$$

Proposition 7.4.4. *Let p a polynomial in k non-commuting variables. Then*

$$\|p(T_{f_1}, ..., T_{f_k})\|_{op} \geq \|p(f_1|_{T^2}, ..., f_k|_{T^2})\|_\infty$$

where $f_j \in C(\bar{\Delta}(0,1))$ for $1 \leq j \leq k$

Proof. Let $(b_1, ..., b_k) \in Im(f_1|_{T^2}, ..., f_k|_{T^2})$. Define

$$f := \sum_{j=1}^{k} |f_j - b_j|^2 \in C(\bar{\Delta}(0,1))$$

Clearly f has a zero in T^2. Using the above corollary, given $\epsilon > 0$ there exists $h \in H^2(\Delta(0,1))$ with norm 1 such that $\|fh\|_2 < \epsilon$. Then

$$\sum_{j=1}^{k} \|T_{f_j}(h) - b_j h\|_2{}^2 = \sum_{j=1}^{k} \|P(f_j h - b_j h)\|_2{}^2 \leq \sum_{j=1}^{k} \|f_j h - b_j h\|_2{}^2 = \|fh\|_2{}^2 < \epsilon^2$$

therefore

$$\|T_{f_j}(h) - b_j h\|_2 < \epsilon \text{ for every } 1 \leq j \leq k$$

Consider operators of the form $T_{f_{j_1}}...T_{f_{j_m}} - b_{j_1}...b_{j_m}$ where $1 \leq j_l \leq k$ for $1 \leq l \leq m$ Notice that

$$T_{f_{j_1}}...T_{f_{j_m}} - b_{j_1}...b_{j_m} = \sum_{l=1}^{m} T_{f_{j_1}}...T_{f_{j_{l-1}}}(T_{f_{j_l}} - b_{j_l})b_{j_{l+1}}...b_{j_m}$$

We can assume that $\|f_j\| \leq 1$ and $|b_j| \leq 1$ for $1 \leq j \leq k$ in order to reduce calculations. Then

$$\|T_{f_{j_1}}...T_{f_{j_m}}(h) - b_{j_1}...b_{j_m}h\|_2 \leq \sum_{l=1}^{m} \|T_{f_{j_1}}...T_{f_{j_{l-1}}}(T_{f_{j_l}}(h) - b_{j_l}h)b_{j_{l+1}}...b_{j_m}\|_2 \leq \sum_{l=1}^{m} \|T_{f_{j_l}}(h) - b_{j_l}h\|$$

$$\leq m\epsilon$$

For any $\eta > 0$ we can choose $\epsilon > 0$ such that

$$\|p(T_{f_1}...T_{f_k})(h) - p(b_1, ..., b_k)h\|_2 \leq \eta$$

Using triangle inequality

$$|p(b_1, ..., b_k)| \leq \|p(T_{f_1}...T_{f_k})(h)\|_2 + \eta \leq \|p(T_{f_1}...T_{f_k})\|_{op} + \eta$$

Since η was arbitrary, we have

$$|p(b_1, ..., b_k)| \leq \|p(T_{f_1}...T_{f_k})\|_{op}$$

Finally, since $(b_1, ..., b_k) \in Im(f_1|_{T^2}, ..., f_k|_{T^2})$ was arbitrary, we have

$$\|p(f_1|_{T^2}, ..., f_k|_{T^2})\|_\infty \leq \|p(T_{f_1}, ..., T_{f_k})\|_{op}$$

$\square$

Theorem 7.4.5.

$$C(T^2) \cong \mathcal{T}(\Delta(0,1))/\mathcal{T}''$$

is a C^*- isomorphism, where $\mathcal{T}''$ is the semi-commutator ideal of $\mathcal{T}(\Delta(0,1))$

Proof. Define

$$\rho : C(\bar{\Delta}(0,1)) \to \mathcal{T}(\Delta(0,1))/\mathcal{T}''$$

$$\rho(f) := T_f + \mathcal{T}''$$

It is clear that ρ is well-defined and is a surjective C^*- homomorphism.Take $f \in Ker(\rho)$. Let $B \in \mathcal{T}''$. Then B is in the C^*-ideal generated by the semi-commutators; hence, $B = lim B_n$ where B_n is a finite sum of operators of the form

$$T_{f_1}, ..., T_{f_k}(T_F T_G - T_{FG})T_{g_1}, ..., T_{g_m}$$

Consider a polynomial with non-commutative variables with terms of the same form than the terms of B_n, i.e., with terms of the form $x_1 x_2 ... x_k(z_1 z_2 - z_3)y_1 ... y_m$.
Using the previous proposition and

$$f_1|_{T^2} ... f_k|_{T^2}(F|_{T^2} G|_{T^2} - F|_{T^2} G|_{T^2})g_1|_{T^2} ... g_m|_{T^2} = 0$$

we have

$$\|f|_{T^2}\|_\infty \leq \|T_f + B_n\|_{op}$$

Therefore $\|f|_{T^2}\|_\infty \leq \|T_f + B\|_{op}$
Since $f \in Ker(\rho)$, $f|_{T^2} \equiv 0$.
Thus

$$C(T^2) \cong C(\bar{\Delta}(0,1))/Ker(\rho) \cong \mathcal{T}(\Delta(0,1))/\mathcal{T}''$$

□

Chapter 8

An Index Theorem for Toeplitz Operators

The reading of [StZ] has been the motivation for this chapter. In this chapter we will show an index theorem for Bergman-Toeplitz and Hardy-Toeplitz operators over the unit ball. In dimension one, this theorem gives us a relation between the index of a Freholm-Toeplitz operator and the winding number of its symbol. It turns out that in dimension greater than one the index of a Toeplitz operator is 0. We are going to start by stating some basic properties of Fredholm operators and of the index of a Fredholm operator. The proof of these results can be found in [BsB] and [BoS].

Theorem 8.0.6 (Atkinson's Theorem). *The space of Fredholm operators is closed under composition, the adjoint operation and addition of compact operators.*

Corollary 8.0.7. *Let A, B be Fredholm operators and K be a compact operator, then*
(i) $Ind(A^*) = -Ind(A)$
(ii) $Ind(AB) = Ind(A) + Ind(B)$
(iii) $Ind(A + K) = Ind(A)$

Theorem 8.0.8 (Dieudonne's Theorem). *Let $\mathcal{F}$ be the space of Fredholm operators. The index is constant on the connected components of $\mathcal{F}$.*

Corollary 8.0.9. *Let D be either B_n or ∂B_n. Then, the index of the (Hardy or Bergman)-Toeplitz operators T_{f_t} over B_n is constant under the homotopy*

$$f_t : D \to \mathbb{C}$$

where f_t does not vanish in ∂B_n.

Proof. The hypothesis f_t does not vanish in ∂B_n guarantees that the Toeplitz operators T_{f_t} are Fredholm. Therefore, the homotopy f_t defines a continuous path

$$t \mapsto T_{f_t}$$

in $\mathcal{F}$ because the operator norm of a Toeplitz operator is less than the infinity norm of its symbol.
Thus, the corollary follows by Dieudonne's Theorem.

□

Lemma 8.0.10. *Let $f(z) = z^m \in C(S^1)$. Then $Ind(T_f) = -m$*

Proof. Suppose m is non-negative. For any $h \in H^2(S^1)$ with $h = \sum_{j=0}^{\infty} a_j z^j$ in L^2 norm we have

$$z^m h = \sum_{j=0}^{\infty} a_j z^{m+j}$$

$$T_f(h) = P(z^m h) = \sum_{j=0}^{\infty} a_j z^{m+j}$$

Therefore $dim(Ker(T_f)) = 0$ and $dim(Coker(T_f)) = m$. Thus $Ind(f) = -m$.
If m is negative, $f(z) = \bar{z}^{-m}$. By part (i) in Corollary 8.0.7 we have $Ind(T_f) = -(m) = -m$.
□

Theorem 8.0.11. *Let $f \in C(\partial B_n)$ and $f(z) \neq 0$ $\forall z \in \partial B_n$. Then*

$$Ind(T_f) = -wind(f) \text{ if } n = 1$$

and

$$Ind(T_f) = 0 \text{ if } n > 1$$

Proof. Suppose $n = 1$. Let m be the winding number of f. Then f is homotopic to z^m with images in $\mathbb{C} \setminus \{0\}$. Since any of the functions (or paths) f_t in the homotopy does not attain 0, T_{f_t} is Fredholm for every t.
On the other hand, by Corollary 8.0.9 the index of T_f is invariant under the above homotopy. This means that

$$Ind(T_f) = Ind(T_{z^m})$$

Thus $Ind(T_f) = -m$.
If $n > 1$, $\partial B_n = S^{2n-1}$ is simply connected. Notice that by radial homotopy we have that $f : S^{2n-1} \to \mathbb{C} \setminus \{0\}$ is homotopic to $g(z) := \frac{f(z)}{|f(z)|}$. Then we have

$$g : S^{2n-1} \to S^1$$

and

$$p : \mathbb{R} \to S^1$$

covering space of S^1.
By Proposition 1.33 of [Htc] there exists a lift $\tilde{g} : S^{2n-1} \to \mathbb{R}$ of g, because

$$g_*(\pi_1(S^{2n-1})) = \{0\} \subseteq p_*(\pi_1(\mathbb{R})) = \{0\}$$

Clearly $\tilde{g}$ is homotopic to the constant function 0. Then $g : S^{2n-1} \to S^1$ is homotopic to a constant function $a \in S^1$.
Furthermore, $f : S^{2n-1} \to \mathbb{C} \setminus \{0\}$ is homotopic to the constant function a.
By Corollary 8.0.9 the $Ind(T_f) = Ind(T_a) = 0$.

□

Now, we shall give an index theorem for (Fredholm) Bergman-Toeplitz operators that is a similar to the result obtained in the Hardy case.

Lemma 8.0.12. *Let $f(z) = z^m \in C(\bar{\mathbb{D}})$. Then $Ind(T_f) = -m = -wind(f|_{S^1})$*

Proof. Same idea as in the above lemma.

□

Theorem 8.0.13. *Let $f \in C(\bar{B_n})$ and $f(z) \neq 0\ \forall z \in \partial B_n$. Then*

$$Ind(T_f) = -wind(f|_{S^1}))\ \textit{if}\ n = 1$$

and

$$Ind(T_f) = 0\ \textit{if}\ n > 1$$

Proof. Suppose $n = 1$. Let m be the winding number of $f|_{S^1}$. Then there exist a homotopy f_t with images in $\mathbb{C} \setminus \{0\}$ between $f|_{S^1}$ and z^m (as function defined on S^1). We can extend the homotopy f_t to g_t a homotopy on $\bar{\mathbb{D}}$ such that $g_t|_{S^1} = f_t$. This homotopy is defined as follows

$$\begin{aligned} g_t(z) &:= |z| f_t(\frac{z}{|z|}) \text{ if } z \neq 0 \\ &= 0 \text{ if } z = 0 \end{aligned}$$

Since any of the functions (or paths) $g_t|_{S^1} = f_t$ in the homotopy does not attains 0, T_{g_t} is a Fredholm operator for every t.
Since $g_0(z) = f(z)$ and $g_1(z) = z^m\ \forall z \in S^1$,

$$Ind(T_{g_0}) = Ind(T_f) \text{ and } Ind(T_{g_1}) = Ind(T_{z^m})$$

By Corollary 8.0.9 and the above lemma, we get $Ind(T_f) = -m$.
If $n > 1$ $\partial B_n = S^{2n-1}$ is simply connected. As in the proof of the preceding theorem, there exist a homotopy $f_t : S^{2n-1} \to \mathbb{C} \setminus \{0\}$ between $f|_{S^{2n-1}}$ and $f(p)$ where p is any point in S^{2n-1}.

We can extend the homotopy f_t to g_t a homotopy on $\bar{B}_n$ such that $g_t|_{S^{2n-1}} = f_t$ as above. It follows that

$$Ind(T_f) = Ind(T_{f(p)}) = 0$$

$\square$

Chapter 9

Appendix

In the following we will mention some results (not covered by this work) and references where these assertions are proven.
To calculate the index of a Toeplitz operators over $\Delta(0,1) \subset \mathbb{C}^n$ for $n > 1$ we need to characterize the Fredholm-Toeplitz operators.
For the Hardy case and for $n = 2$, we have

Corollary 9.0.14. *Let $f \in C(T^2)$. Then T_f is Fredholm on $H^2(T^2)$ if and only if*

$$f(z_1, z_2) \neq 0 \ \forall (z_1, z_2) \in T^2$$
$$wind_1(f) = wind_2(f) = 0$$

where $wind_j(f)$ is the winding number of the function obtained by fixing the j variable. Moreover, if T_f is Fredholm then $Ind(T_f) = 0$.

The proof of this Corollary can be found in [BoS, p. 353].

For the Bergman case, assuming compactness of the Hankel operator H_f and Fredholmness of T_f, we have an analogous index theorem to the unit ball case (see [StZ]).

The vanishing of Fredholm indices of Toeplitz operators on higher-dimensional domains is a consequence of our focus on Toeplitz operators made from scalar-valued functions. On these domains Toeplitz operators made from matrix-valued functions can have geometrically significant nonzero Fredholm indices (see [Upm] for details).

For algebras whose commutator ideals are not contained in the ideal of compact operators, it can be useful to generalize the Fredholm index to an index that lives in the K-theory of a C^*-algebra. Some details are in [Upm, Ch. 5].

Bibliography

[Arv] Arveson,William., *An Invitation to C*-algebras*, Springer-Verlag, New York, 1976.

[BsB] Boos, B., Bleecker D.D., *Topology and Analysis: Atiyah-Singer. Index formula Gauge theoretic Physics*, Springer-Verlag, New York, 1985.

[BoS] Böttcher, A., Silbermann B., *Analysis of Toeplitz Operators*, Springer-Verlag, New York, 1990.

[Dou] Douglas, Ronald G., *Banach Algebra Techniques in Operator Theory*, Academic Press, New York, 1972.

[Htc] Hatcher, Allen., *Algebraic Topology*, Cambridge University Press, Cambridge, 2002.

[Hof] Hoffman, K., *Banach Spaces of Analytic Functions*, Prentice-Hall, Englewood Cliffs, 1962.

[Kra] Krantz, Steven G., *Function Theory of Several Complex Variables*, Wiley, New York, 1982.

[Pel] Peller V., *Hankel Operators and Their Applications*, Springer-Verlag, New York, 2003.

[Ran] Range, Michael R., *Holomorphic Functions and Integral Representations in Several Complex Variables*, Springer-Verlag, New York, 1986.

[RRo] Rosenblum M., Rovnyak J., *Hardy Classes and Operator Theory*, Oxford University Press, New York, 1985.

[Roy] Royden, H.L., *Real Analysis*, Prentice-Hall, New York, 1988.

[Rud] Rudin,W., *Function Theory in Polydiscs*, W.A. Benjamin, Inc., New York, 1969.

[Rd2] Rudin,W., *Function Theory in the Unit Ball of* $\mathbb{C}^n$, Springer, Berlin, 1980.

[Rd3] Rudin, W., *Functional Analysis*, McGraw-Hill, New York, 1973.

[StZ] Stroethoff Karel; Zheng Dechao. *Toeplitz and Hankel Operators on Bergman Spaces*, Transactions of the American Mathematical Society, Vol.329, No.2. (Feb., 1992), pp.773-794.

[Upm] Upmeier, Harald., *Toeplitz Operators and Index Theory in Several Complex Variables*, Operator Theory; Vol.81, *Birkhäuser*, Basel, 1996.

[Yng] Young,N., *An Introduction to Hilbert Space*, Cambridge University Press, New York, 1988.

[Zhu] Zhu, K., *Operator Theory in Function Spaces*, Monographs and Textbooks in Pure and Applied Mathematics, Marcel Dekker, New York, 1990.

Vita

Bartleby Ordonez-Delgado is currently a second year graduate student of the Mathematics Department at Virginia Tech. He graduated from San Marcos University (Lima-Peru) in 2002 with a bachelors degree in Mathematics. In 2003, he gained a fellowship from Instituto de Matematicas Puras y Aplicadas in Lima-Peru. He is currently doing research in Functional Analysis and Complex Analysis.