Bott Periodicity

David Manuel Murrugarra Tomairo

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Peter Haskell, Chair
William Floyd
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(ABSTRACT)

Bott periodicity plays a fundamental role in the definition and understanding of K-theory, the generalized cohomology theory defined by vector bundles. This paper examines the proof, given by Atiyah and Bott[3], of the periodicity theorem for the complex case.

We also describe the long exact sequence for K-cohomology in the category of connected finite CW-complexes.

Dedication

To my mother: Eugenia Victoria Tomairo Geronimo.

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Contents

Introduction

The Bott periodicity theorem was discovered by Raoul Bott in 1950 by using elements of Morse theory[4]. After that, many others came up with different proofs of Bott periodicity (see Husemoller[6], p.150). Bott periodicity plays a fundamental role in the definition and understanding of K-theory, the generalized cohomology theory defined by vector bundles. In 1964, Atiyah and Bott gave, as they say in[3], an "elementary proof" of the periodicity theorem. This thesis explains the techniques used by Atiyah and Bott in their proof.

The Bott periodicity theorem can be formulated in many ways. One of the simplest ways to state the Bott Periodicity Theorem is the following: there is an explicit isomorphism between $K(X) \otimes K(S^2)$ and $K(X \times S^2)$ for a compact Hausdorff space X. We examine this version of the Bott periodicity theorem, and also give a version the periodicity theorem for the reduced K-theory: $\widetilde{K}(X)$ is isomorphic to $\widetilde{K}(S^2X)$. As a consequence of the last statement, in the last chapter, we show the degree-two periodicity of K-cohomology.

The arrangement of the paper is as follows. We start by giving the preliminaries needed to construct the K-cofunctor. Thus in the first chapter we give the definition of a vector bundle and describe operations on vector bundles, such as direct sums and tensor products. We also describe a way to construct vector bundles via clutching functions. We finish the first chapter by speaking informally about the universal vector bundle associated with the unitary group. In the second chapter, we define the rings $K(X)$ and $\widetilde{K}(X)$ for a compact Hausdorff space X. Here, we care only about the case of complex K-Theory. We describe the representation for $\widetilde{K}(X)$ in terms of a set of homotopy classes of maps from X to a special space BU, which is defined in chapter two. In the third chapter, we calculate $K(S^2)$ by using the techniques used by Atiyah and Bott in the proof of the periodicity theorem. Even though we can also compute $K(S^2)$ directly by using the knowledge of the homotopy groups of the unitary groups, $U(n)$'s, we focus on the Atiyah-Bott approach because it brings into a better understanding of the techniques that are essential to the proof of the complete periodicity theorem. Indeed, the general proof of the periodicity theorem is a parametrized version of the proof for the 2-sphere case. Finally, in the fourth chapter, we start by defining the suspension operation over pointed spaces and then constructing homotopy exact sequences with this kind of operation over compact spaces. Then, by using these exact homotopy sequences and the representation of the cofunctor $\widetilde{K}$, we are able to state the Bott periodicity theorem for the reduced $\widetilde{K}$-theory case. We finish this chapter with a discussion of the long exact sequence for K-theory.

Chapter 1

Generalities on Vector Bundles

In this first chapter we give the preliminaries needed to construct the K-cofunctor in the next chapter and eventually part of the machinery to be used to prove the periodicity theorem in the third chapter. In the first section we give the definition of a vector bundle and a description of how to obtain vector bundles via basic operations like direct sum and tensor products. In the second section we describe a way to construct vector bundles with base space a sphere via clutching functions.

1.1 Basic Definitions and results

We start by giving the definition of a vector bundle and describing the basic operations between vector bundles, namely, direct sums and tensor products. The last part of this section has to do with the construction of a vector bundle induced by a function, i.e., the pull-back of a vector bundle under a given function.

Definition 1.1.1. A k-dimensional vector bundle is a triple $\xi = (E, p, X)$, where E and X are topological spaces and $p : E \to X$ is a function such that the following conditions are satisfied

(a) For each $x \in X$, $p^{-1}(x)$ is a k-dimensional vector space.

(b) Local triviality condition: Each $x \in X$ has an open neighborhood and a homeomorphism $h : U \times \mathbb{C}^k \to p^{-1}(U)$ such that the restriction $x \times \mathbb{C}^k \to p^{-1}(x)$ is a vector space isomorphism.

The spaces E and X in the definition above are referred as the total space and the base space respectively. The vector space $p^{-1}(x)$ is called the fibre of the vector bundle at $x \in X$. Sometimes the space E of a vector bundle ξ, is denoted by $E(\xi)$.

In the next example we describe a vector bundle that we will use many times for our constructions.

Example 1.1.2. The canonical complex line bundle H is given by $(E, p_1, \mathbb{C}P^1)$, where $E = \{(\ell, \nu) \in \mathbb{C}P^1 \times \mathbb{C}^2 : \nu \in \ell\}$ and $p_1(\ell, \nu) = \ell$. In the above notation $\mathbb{C}P^1$ is the quotient of $\mathbb{C}^2 \backslash \{0\}$ under the equivalence relation $(z_0, z_1) \sim \lambda(z_0, z_1)$ for $\lambda \in \mathbb{C}\backslash\{0\}$, so $\ell \in \mathbb{C}P^1$ represents the line $\{\lambda(z_0, z_1) : \lambda \in \mathbb{C}\backslash\{0\}\}$.

Given two vector bundles ξ_1 and ξ_2, we can construct a new vector bundle $\xi_1 \oplus \xi_2$ which has as a fiber the direct sum of the fibers of ξ_1 and ξ_2. We make this clearer in the following definition.

Definition 1.1.3. The Whitney sum $\xi_1 \oplus \xi_2$ of two vector bundles $\xi_1 = (E_1, p_1, X)$ and $\xi_2 = (E_2, p_2, X)$ is the triple $(E_1 \oplus E_2, q, X)$ where

$$E_1 \oplus E_2 = \{(x, x') \in E_1 \times E_2 : p_1(x) = p_2(x')\}$$

and $q(x, x') = p_1(x) = p_2(x)$.

From the above definition it is clear that $q^{-1}(x) = p^{-1}(x) \oplus p^{-1}(x)$, therefore we have the following basic result.

Lemma 1.1.4. *The fibre of $\xi_1 \oplus \xi_2$ over $x \in X$ is the direct sum of the fibres of ξ_1 and ξ_2 over x.*

Similarly we can construct a vector bundle $\xi_1 \otimes \xi_2$ whose fiber is the tensor product of the fibers of ξ_1 and ξ_2.

Definition 1.1.5. The tensor product $\xi_1 \otimes \xi_2$ of two vector bundles $\xi_1 = (E_1, p_1, X)$ and $\xi_2 = (E_2, p_2, X)$ is the triple $(E_1 \otimes E_2, q, X)$ where $E_1 \otimes E_2$ is the disjoint union of $p_1^{-1}(x) \otimes p_2^{-1}(x)$ for all $x \in X$ and q is the canonical projection on each fibre.

It is clear from the definition that the fibre of $\xi_1 \otimes \xi_2$ over $x \in X$ is $p_1^{-1}(x) \otimes p_2^{-1}(x)$.

Given a vector bundle ξ over Y and a function $f : X \to Y$, we can construct another vector bundle $f^*(\xi)$ over X. This vector bundle is called the pullback of ξ under f and we describe this construction in the following definition.

Definition 1.1.6. Let $\xi = (E, p, Y)$ be a vector bundle and $f : X \to Y$ be a function. Then the pullback $f^*(\xi)$ of ξ under f is the triple (E', p_1, X) where $E' = \{(x, e) \in X \times E / f(x) = p(e)\}$ and p_1 is the projection on the first factor.

Definition 1.1.7. A morphism between two vector bundles $\xi = (E, p, X)$ and $\eta = (F, q, X)$ is a map $f : E \to F$, that commutes with the projections p and q, i.e., $qf = p$, and that has the restriction $f : p^{-1}(x) \to q^{-1}(x)$ a linear map for all $x \in X$.

If the restrictions $f : p^{-1}(x) \to q^{-1}(x)$ are isomorphisms for all $x \in X$, we say that it is an isomorphism of vector bundles.

Definition 1.1.8. A projection operator P for a vector bundle ξ is an endomorphism with $P^2 = P$

We state the following three lemmas and refer to the proofs in Husemoller[6] or Hatcher[5].

Lemma 1.1.9. *Let P be a projection operator for a vector bundle ξ. Then*

$$\xi = P(\xi) \oplus (1 - P)(\xi)$$

Lemma 1.1.10. *Let K be a closed subspace of a compact space X, and let ξ and η be two vector bundles over X. Then any isomorphism $\varphi : \xi|K \to \eta|K$ extends to an isomorphism $\theta : \xi|U \to \eta|U$ for some open neighborhood U of K.*

Lemma 1.1.11. *Let ξ be a vector bundle over $X \times [0, 1]$. Then the restrictions over $X \times \{0\}$ and $X \times \{1\}$ are isomorphic whenever X is compact.*

This lemma can be used to show that a vector bundle over a compact contractible space must be trivial, i.e, the cartesian product of the base space with a vector space.

Lemma 1.1.12. *Let ξ be a vector bundle over a compact Hausdorff space X. Then there exists a vector bundle ξ' over X such that $\xi \oplus \xi'$ is isomorphic to a trivial bundle.*

Remark 1.1.13. If the base space of a vector bundle is noncompact, lemma 1.1.12 can fail to be true. See Hatcher[5], Pg. 82, for an example.

1.2 Clutching Functions

Let $X = X_1 \cup X_2$ and $A = X_1 \cap X_2$, where all the spaces are compact and Hausdorff. Let $\xi_i = (E_i, p_i, X_i)$ be a vector bundle over X_i, $i = 1, 2$, and $\varphi : \xi_1|A \to \xi_2|A$ be an isomorphism of vector bundles. Then we define a vector bundle $\xi_1 \bigcup_\varphi \xi_2 = (E_1 \bigcup_\varphi E_2, p, X)$, where $E_1 \bigcup_\varphi E_2$ is the quotient of the disjoint union $E_1 \bigcup E_2$ obtained by identifying $e_1 \in \xi_1|A$ with $\varphi(e_1) \in \xi_2|A$. Identifying X with the corresponding quotient of $X_1 \bigcup X_2$ we obtain a natural projection $p : E_1 \bigcup_\varphi E_2 \to X$, and $p^{-1}(x)$ has a natural vector space structure for each $x \in X$. It remains to show that $\xi_1 \bigcup_\varphi \xi_2$ is locally trivial. Since

$$E_1 \bigcup_\varphi E_2|(X - A) = (E_1|X_1 - A) \bigcup (E_2|X_2 - A)$$

the local triviality at points $x \notin A$ follows from that of ξ_1 and ξ_2.

If $x \in A$, there is a closed neighborhood V_1 of x in X_1 over which ξ_1 is trivial, i.e., there is an isomorphism

$$\theta_1 : E_1|V_1 \to V_1 \times \mathbb{C}^n$$

Restricting to A we get an isomorphism

$$\theta_1 : E_1|V_1 \cap A \to (V_1 \cap A) \times \mathbb{C}^n$$

and by composition with the inverse of φ we get the following isomorphism

$$\theta_1 : E_2|V_1 \cap A \to (V_1 \cap A) \times \mathbb{C}^n$$

By lemma 1.1.10 we can extend this to an isomorphism

$$\theta_2 : E_2|V_2 \to V_2 \times \mathbb{C}^n$$

where V_2 is an open neighborhood of x in X_2. The pair θ_1, θ_1 defines naturally an isomorphism

$$\theta_1 \cup_\varphi \theta_2 : E_1 \cup_\varphi E_2|V_1 \cup V_2 \to (V_1 \cup V_2) \times \mathbb{C}^n$$

establishing the local triviality of $\xi_1 \bigcup_\varphi \xi_2$.

Summarizing all this we have

Proposition 1.2.1. *Let $X = X_1 \cup X_2$ and $A = X_1 \cap X_2$ where all the spaces are compact and Hausdorff. Let $\xi_i = (E_i, p_i, X_i)$ be a vector bundle over X_i, i=1,2, and $\varphi : \xi_1|A \to \xi_2|A$ be an isomorphism of vector bundles. Then there is a triple (ξ, μ_1, μ_2), where ξ is a vector bundle over X, $\mu_i : \xi_i \to \xi|X_i$ is an isomorphism for $i = 1, 2$, and $\mu_1\varphi = \mu_2$ over A. Moreover, if (η, ν_1, ν_2) is another triple such that η is a vector bundle over X, $\nu_i : \xi_i \to \eta|X_i$ is an isomorphism for $i = 1, 2$, $\nu_1\varphi = \nu_2$ over A, then there is an isomorphism*

$$\omega : \eta \to \xi$$

with $\mu_i = \omega\nu_i$ over X_i for $i = 1, 2$.

Note that the last part of proposition 1.2.1 claims that, up to isomorphism, there is a unique vector bundle that can be obtained by using the clutching construction described above.

Proof. Take ξ as $\xi_1 \bigcup_\varphi \xi_2$ and $\mu_1 : \xi_i \to \xi|X_1$ is defined naturally as $\mu_1(e) = p(e)$ for all $e \in E_1$, where $p : E_1 \bigcup_\varphi E_2 \to X$. Similarly we define μ_2. Clearly μ_1 and μ_2 are isomorphisms.

For the uniqueness statement, we define $\omega : \eta \to \xi$ as $\mu_i\nu_i^{-1} : \eta|X_i \to \xi|X_i$ on $E(\eta|X_i)$ for $i = 1, 2$. Since the total space of η is $E(\eta|X_1) \bigcup E(\eta|X_2)$ and the fact that each $E(\eta|X_i)$ is closed in $E(\eta)$, $\omega : \eta \to \xi$ is a well defined isomorphism. Clearly $\mu_i = \omega\nu_i$ over X_i for $i = 1, 2$. □

Definition 1.2.2. The map φ in proposition 1.2.1, is called the clutching function for $\xi_1 \bigcup_\varphi \xi_2$.

We'll say that the triple (ξ_1, φ, ξ_2) is called a clutching data over $(X; X_1, X_2)$.

Given a vector bundle ξ over X, we may always view ξ as obtained by clutching with the identity map on restriction over A, where $X = X_1 \cup X_2$ and $A = X_1 \cap X_2$.

Lemma 1.2.3. *If $\mathbb{I}_A : \xi_1|A \to \xi_2|A$ is the identity map, then $\xi_1 \bigcup_{\mathbb{I}_A} \xi_2 \cong \xi$*

Proof. This follows from the uniqueness part of proposition 1.2.1. □

Lemma 1.2.4. *Let (ξ_1, φ, ξ_2) and $(\xi_1', \varphi', \xi_2')$ be two clutching data over $(X; X_1, X_2)$. If $\beta : \xi_i \to \xi_i'$ are isomorphisms on X_i, i=1,2 such that $\varphi'\beta_1 = \beta_2\varphi$ over A, then*

$$\xi_1 \bigcup_{\varphi} \xi_2 \cong \xi_1' \bigcup_{\varphi'} \xi_2'$$

Proof. We define

$$\omega : \xi_1 \bigcup_{\varphi} \xi_2 \to \xi_1' \bigcup_{\varphi'} \xi_2'$$

by $\omega(e_1) = \beta_1(e_1)$ for $e_1 \in E_1$ and $\omega(e_2) = \beta_2(e_2)$ for $e_2 \in E_2$. Since $\varphi'\beta_1 = \beta_2\varphi$, ω is well defined on $\xi_1|A$ and ω is an isomorphism because β_1 and β_2 are isomorphisms. $\square$

We can also do the usual operations with clutching functions

Lemma 1.2.5. *Let (ξ_1, φ, ξ_2) and $(\xi_1', \varphi', \xi_2')$ be two clutching data over $(X; X_1, X_2)$. Then*

(a)

$$(\xi_1 \bigcup_{\varphi} \xi_2) \oplus (\xi_1' \bigcup_{\varphi'} \xi_2') \cong (\xi_1' \oplus \xi_1') \bigcup_{\varphi\oplus\varphi'} (\xi_2 \oplus \xi_2')$$

(b)

$$(\xi_1 \bigcup_{\varphi} \xi_2) \otimes (\xi_1' \bigcup_{\varphi'} \xi_2') \cong (\xi_1' \otimes \xi_1') \bigcup_{\varphi\otimes\varphi'} (\xi_2 \otimes \xi_2')$$

Proof. This follows from the uniqueness part of proposition 1.2.1. $\square$

1.3 Clutching Construction for vector bundles over S^2

We use the clutching construction described in the previous section to obtain vector bundles ξ with base space the sphere S^2. Write $S^2 = S^2_+ \cup S^2_-$, i.e., as the union of its upper and lower hemispheres S^2_+ and S^2_- , with $S^2_+ \cap S^2_- = S^1$. Let (ξ_1, φ, ξ_2) be a clutching data over $(S^2; S^2_+, S^2_-)$. Thus φ is an isomorphism from $\xi_1|S^1$ to $\xi_2|S^1$. Since S^2_+ and S^2_- are contractible, ξ_1 and ξ_2 are isomorphic to the trivial vector bundle $S^2_+ \times \mathbb{C}^n$ and $S^2_- \times \mathbb{C}^n$ respectively. So there are isomorphisms

$$h_1 : \xi_1 \to S^2_+ \times \mathbb{C}^n$$

and

$$h_2 : \xi_2 \to S^2_- \times \mathbb{C}^n$$

Then the restriction of the composition $h_1\varphi h_2^{-1}$ to $S^1 \times \mathbb{C}^n$ defines a function f from S^1 to $GL_n(\mathbb{C})$ defined by $f(z)(v) = h_1\varphi h_2^{-1}(z,v)$ for $z \in S^1$ and $v \in \mathbb{C}^n$. Thus in this case, the clutching construction described in the previous section is equivalent to constructing a vector bundle ξ over S^2 starting from a map $f : S^1 \to GL_n(\mathbb{C})$, taking $E(\xi)$ to be the quotient of the disjoint union $S^k_+ \times \mathbb{C}^n \bigcup S^k_- \times \mathbb{C}^n$ obtained by identifying $(z,v) \in \partial S^k_- \times \mathbb{C}^n$ with $(z, f(z)(v)) \in \partial S^k_+ \times \mathbb{C}^n$. Then we have a natural projection $p : E(\xi) \to S^2$ and the triple $(E(\xi), p, S^2)$ is an n-dimensional complex vector bundle over S^2.

Notation 1.3.1. The vector bundle $(E(\xi), p, S^2)$ obtained above is going to be denoted by

$$S^2_+ \times \mathbb{C}^n \bigcup_f S^2_- \times \mathbb{C}^n$$

and the function $f : S^1 \to GL_n(\mathbb{C})$ will be referred as the clutching function for $S^2_+ \times \mathbb{C}^n \bigcup_f S^2_- \times \mathbb{C}^n$.

Clearly $S^2_+ \times \mathbb{C}^n \bigcup_f S^2_- \times \mathbb{C}^n \cong S^2_+ \times \mathbb{C}^n \bigcup_\varphi S^2_- \times \mathbb{C}^n$.

Now let's see that any vector bundle over S^2 can be obtained by a clutching construction

Lemma 1.3.2. *Let ξ be a vector bundle over S^2. Then there is a clutching function $f : S^1 \to GL_n(\mathbb{C})$ such that*

$$\xi \cong S^2_+ \times \mathbb{C}^n \bigcup_f S^2_- \times \mathbb{C}^n$$

where S^2_+ and S^2_- are the upper and lower hemispheres of S^2 with $S^2_+ \cap S^2_- = S^1$.

Proof. Consider the restrictions $\xi|S^2_+$ and $\xi|S^2_-$. Since S^2_+ and S^2_- are contractible, $\xi|S^2_+$ and $\xi|S^2_-$ are isomorphic to the trivial vector bundles $S^2_+ \times \mathbb{C}^n$ and $S^2_- \times \mathbb{C}^n$ respectively. So there are isomorphisms

$$h_+ : \xi|S^2_+ \to S^2_+ \times \mathbb{C}^n$$

and

$$h_- : \xi|S^2_- \to S^2_- \times \mathbb{C}^n$$

Then the restriction of the composition $h_+h_-^{-1}$ to $S^1 \times \mathbb{C}^n$ defines a function from S^1 to $GL_n(\mathbb{C})$. Thus we take $f(z)(v) = h_+h_-^{-1}(z,v)$. $\square$

Notation 1.3.3. Let ξ be an n-dimensional vector bundle over S^2. We denote the cartesian products $S^2_+ \times \mathbb{C}^n$ by ξ_+ and $S^2_- \times \mathbb{C}^n$ by ξ_-. So if f is a clutching function for ξ, we are going to write

$$\xi \cong (\xi_+ \cup_f \xi_-)$$

In this setting we may ask what does the clutching function for the canonical line bundle H over $\mathbb{C}P^1 = S^2$ look like?

Example 1.3.4. The 1-dimensional complex projective space $\mathbb{C}P^1$ is the quotient of $\mathbb{C}^2 \backslash \{0\}$ under the equivalence relation $(z, w) \sim \lambda(z, w)$ for $\lambda \in \mathbb{C}\backslash\{0\}$. We denote the equivalence class of (z, w) by $[z, w]$. Equivalently, the space $\mathbb{C}P^1$ is the set of ratios $z/w \in \mathbb{C} \cup \{\infty\} = S^2$. Therefore we can write $\mathbb{C}P^1 = S^2 = S^2_+ \cup S^2_-$, where

$$S^2_+ = \{[1, z^{-1}] :\mid z^{-1} \mid\geq 1\} \ \text{ and } \ S^2_- = \{[z, 1] :\mid z \mid\leq 1\}$$

Then we have trivializations of H over these two hemispheres giving by $([z, 1], (z, 1)) \to [z, 1]$ and $([1, z^{-1}], (1, z^{-1})) \to [1, z^{-1}]$ respectively, and over their common boundary S^1 we pass from the $(z, 1)$ to $(1, z^{-1})$ by multiplying by z. Then $f : S^1 \to GL_1(\mathbb{C})$ defined by $f(z) = (z)$ is the clutching function for H.

Remark 1.3.5. In example 1.3.4, if we had taken $S^2_+ = \{[z, 1] :\mid z \mid\leq 1\}$ and $S^2_- = \{[1, z^{-1}] :\mid z^{-1} \mid\geq 1\}$, then the clutching function function would have been $f : S^1 \to GL_1(\mathbb{C})$ defined by $f(z) = (z^{-1})$.

Notation 1.3.6. From now on, we are going to write $H = (\xi_+ \cup_z \xi_-)$ and $H^{-1} = (\xi_+ \cup_{z^{-1}} \xi_-)$. This notation is justified by the fact that $H \otimes H^{-1} \cong \mathbb{I}$. We denote the n-fold tensor product of H with itself by H^n. Similarly the n-fold tensor product of H^{-1} with itself is denoted by H^{-n}.

What about the clutching function for H^n?

Example 1.3.7. By lemma 1.2.5, the clutching function for H^n is $f : S^1 \to GL_1(\mathbb{C})$ defined by $f(z) = (z^n)$ so we will write $H^n \cong (S^2_+ \cup_{z^n} S^2_-)$.

The isomorphism class of a vector bundle, depends only on the homotopy class of the clutching functions. More formally, we have the following

Lemma 1.3.8. *If $f : S^1 \to GL_n(\mathbb{C})$ and $g : S^1 \to GL_n(\mathbb{C})$ are homotopic, then*

$$(\xi_+ \cup_f \xi_-) \cong (\xi_+ \cup_g \xi_-)$$

Proof. Let $F : S^1 \times [0,1] \to GL_n\mathbb{C}$ be a homotopy from f to g. We construct a vector bundle $\eta = (E_F, p, S^2 \times [0,1])$ by using the same sort of clutching construction described in the previous section. Indeed, E_F is the quotient of the disjoint union $(S^2_+ \times [0,1] \times \mathbb{C}^n) \cup (S^2_- \times [0,1] \times \mathbb{C}^n)$ obtained by identifying $(z,t,v) \in \partial S^2_+ \times [0,1] \times \mathbb{C}^n$ with $(z,t,zv) \in \partial S^2_- \times [0,1] \times \mathbb{C}^n$ and p is the canonical projection over $S^2 \times [0,1]$. Clearly, the restriction of η to $S^2 \times \{0\}$ is isomorphic to $(\xi_+ \cup_f \xi_-)$ and the restriction of η to $S^2 \times \{1\}$ is isomorphic to $(\xi_+ \cup_g \xi_-)$. Therefore, by lemma 1.1.11, $(\xi_+ \cup_f \xi_-) \cong (\xi_+ \cup_g \xi_-)$. $\square$

Lemma 1.3.8 is very useful for establishing isomorphisms between vector bundles. Let's see how it works.

Lemma 1.3.9. *For the canonical line bundle H over $\mathbb{C}P^1$ we have*

$$(H \otimes H) \oplus \mathbb{I} \approx H \oplus H$$

where $\mathbb{I}$ is the trivial one-dimensional bundle.

Proof. Let $f : S^1 \to GL_2(\mathbb{C})$ and $g : S^1 \to GL_2(\mathbb{C})$ be clutching maps defined by

$$f(z) = \begin{bmatrix} z^2 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } g(z) = \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} \quad \text{for } z \in S^1$$

Then f and g are clutching maps for $(H \otimes H) \oplus \mathbb{I}$ and $H \oplus H$ respectively.

Let $F : S^1 \times [0,1] \to GL_n(\mathbb{C})$ defined by

$$F(z,t) = \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\pi t/2) & -\sin(\pi t/2) \\ \sin(\pi t/2) & \cos(\pi t/2) \end{bmatrix} \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\pi t/2) & \sin(\pi t/2) \\ -\sin(\pi t/2) & \cos(\pi t/2) \end{bmatrix}$$

Thus for $t = 0$, $F(z,0) = \begin{bmatrix} z^2 & 0 \\ 0 & 1 \end{bmatrix}$ and for $t = 1$, $F(z,1) = \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix}$.

Therefore F is a homotopy from f to g. By lemma 1.3.8, $(\xi_+ \cup_f \xi_-) \cong (\xi_+ \cup_g \xi_-)$ which gives the result. $\square$

1.4 Universal Bundles

There is a special k-dimensional vector bundle γ^n_k with the property that all k-dimensional vector bundles over paracompact base spaces are obtainable as pullbacks of this single bundle. Thus, vector bundles over a fixed base space are classified by homotopy classes of maps into the base space of γ_k^n. We briefly describe the construction of this vector bundle, see Hatcher[5] or Husemoller[6] for further details.

Let $V_k(\mathbb{C}^n)$ be the set of ordered k-tuples $(v_1, v_2, \cdots, v_k) \in \mathbb{C}^n$ with $\mid v_i \mid = 1$ and $v_i \cdot v_j = \delta_{ij}$, where $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ij} = 1$ if $i = j$. We denote by $< v_1, \cdots, v_k >$ the k-dimensional subspace of $\mathbb{C}^n$ generated by $v_1, v_2, \ldots, v_k \in \mathbb{C}^n$. Let $G_k(\mathbb{C}^n)$ be the set of k-dimensional subspaces of $\mathbb{C}^n$. Now consider the function $p : V_k(\mathbb{C}^n) \to G_k(\mathbb{C}^n)$ sending $(v_1, v_2, \cdots, v_k)$ to $< v_1, \cdots, v_k >$. Then $G_k(\mathbb{C}^n)$ is topologized by giving it the largest topology such that $p : V_k(\mathbb{C}^{k+m}) \to G_k(\mathbb{C}^{k+m})$ is continuous. Note that since every subspace of $\mathbb{C}^n$ has an orthonormal basis, p is surjective.

Let $E_k(\mathbb{C}^n) = \{(\ell, v) \in G_k(\mathbb{C}^n) \times \mathbb{C}^k : v \in \ell\}$ and $p_1 : E_k(\mathbb{C}^n) \to G_k(\mathbb{C}^n)$ defined by $p_1(\ell, v) = \ell$. Now the triple $\gamma_k^n = (E_k(\mathbb{C}^n), p_1, G_k(\mathbb{C}^n))$ is a vector bundle with fibre $\mathbb{C}^k$.

We denote the set of homotopy classes of maps $f : X \to Y$ by $[X, Y]$ and $Vect^k(X)$ denote the set of isomorphism classes of k-dimensional vector bundles over X. The following fact can be found in Husemoller[6], Pg 96.

Theorem 1.4.1. *Let X be a connected m-dimensional CW-complex. The map*

$$[X, G_k(\mathbb{C}^{k+m})] \to Vect^k(X),$$

sending $[f]$ to $f^*(\gamma_k^{k+m})$ is a bijection whenever $m \leq 2n$.

In other words, theorem 1.4.1 asserts that: For each k-dimensional vector bundle over a CW-complex of dimension n with $n \leq 2m$, there is a map $f : X \to G_k(\mathbb{C}^{k+m})$ such that $f^*(\gamma_k^{k+m})$ is isomorphic to ξ and this map f is unique up to homotopy. Now, because of this fact, γ_k^n is called the k-dimensional universal bundle.

Let X be a connected m-dimensional CW-complex. We can decompose any vector bundle over X into the Whitney sum of a trivial vector bundle and another vector bundle of lower dimension (see Husemoller[6] Pg. 112).

Theorem 1.4.2. *Each k-dimensional vector bundle ξ over a connected m-dimensional CW-complex X is isomorphic to $\eta \oplus (k - n)\mathbb{I}$ for some n-dimensional vector bundle η, where $m \leq 2n$.*

Chapter 2

The K and $\widetilde{K}$ Cofunctors

In this chapter we define the rings $K(X)$ and $\widetilde{K}(X)$ for a compact Hausdorff space X. There are many equivalent ways to define $K(X)$ and $\widetilde{K}(X)$; we have chosen the definition that is most useful for our purposes. Through this chapter, we are going to deal only with complex vector bundles unless otherwise specified and "=", applied to vector bundles, means isomorphism of vector bundles.

2.1 The K Cofunctor

For a compact Hausdorff space X, consider the set of formal differences $\xi - \eta$ of complex vector bundles over X. On this set we are going to define an equivalence relation " $\sim$ ". We say that $\xi_1 - \eta_1$ and $\xi_2 - \eta_2$ are equivalent if

$$\xi_1 + \eta_2 + n\mathbb{I} = \xi_2 + \eta_1 + n\mathbb{I}$$

where $n\mathbb{I}$ is the trivial n-dimensional vector bundle, for some n, and "+" denotes the Whitney sum as defined in Chapter 1. This is indeed an equivalence relation.

(a) Reflexivity: Clear.

(b) Symmetry: Clear.

(c) Transitivity: Let $\xi_1 - \eta_1 \sim \xi_2 - \eta_2$ and $\xi_2 - \eta_2 \sim \xi_3 - \eta_3$. Then there exit $n,\ m \geq 0$ such that

$$\xi_1 + \eta_2 + n\mathbb{I} = \xi_2 + \eta_1 + n\mathbb{I}$$

and

$$\xi_2 + \eta_3 + m\mathbb{I} = \xi_3 + \eta_2 + m\mathbb{I}$$

Therefore

$$\xi_1 + \eta_2 + \xi_2 + \eta_3 + n\mathbb{I} + m\mathbb{I} = \xi_2 + \eta_1 + \xi_3 + \eta_2 + n\mathbb{I} + m\mathbb{I}$$

By lemma 1.1.12, there are vector bundles ξ_2' and η_2' such that

$$\xi_2 + \xi_2' = r\mathbb{I} \text{ and } \eta_2 + \eta_2' = s\mathbb{I}.$$

This is where we need X to be a compact set. Therefore adding ξ_2' and η_2' to both sides in the last equation we have

$$\xi_1 - \eta_1 \sim \xi_3 - \eta_3$$

Let $K(X)$ be the set of all equivalence classes of the relation defined above. The set $K(X)$ has a natural commutative ring structure with the addition and multiplication given by

$$(\xi_1 - \eta_1) + (\xi_2 - \eta_2) = (\xi_1 + \xi_2) - (\eta_1 + \eta_2)$$

and

$$(\xi_1 - \eta_1) \otimes (\xi_2 - \eta_2) = (\xi_1 \otimes \xi_2) - (\xi_1 \otimes \eta_2) - (\eta_1 \otimes \xi_2) + (\eta_2 \otimes \eta_2)$$

Any function $f : X \to Y$ between compact Hausdorff spaces induces a ring homomorphism $f^* : K(Y) \to K(X)$ defined by

$$f^*(\xi - \eta) = f^*(\xi) - f^*(\eta),$$

where $f^*(\xi)$ is the pull-back of f.

Summarizing this we have the following

Theorem 2.1.1. *K is a cofunctor from the category of spaces and maps to the category of commutative rings and homomorphisms between rings.*

Example 2.1.2. Let $\mathbb{I}$ be the 1-dimensional trivial bundle over the single point "pt". Then $K(pt) = \{n\mathbb{I} - m\mathbb{I} : n, m \in \mathbb{Z}\}$ which is identified with the set $\{n - m : n, m \in \mathbb{Z}\} \cong \mathbb{Z}$. Therefore

$$K(pt) = \mathbb{Z}$$

2.2 The Cofunctor $\widetilde{K}$

In this section we consider topological compact spaces with base points to define the reduced K-theory of a pointed space $\widetilde{K}(X)$.

Definition 2.2.1. For a topological space X with base point $x_0 \in X$ consider the inclusion function $\imath : x_0 \hookrightarrow X$. This function induce a ring homomorphism

$$\imath^* : K(X) \to K(x_0)$$

Then we define $\widetilde{K}(X) = Ker(\imath^*)$.

Lemma 2.2.2. *For a pointed topological space, we have*

$$K(X) = \widetilde{K}(X) \oplus \mathbb{Z}$$

Proof. Consider the constant map $c : X \to x_0$. Then $c \circ i = Id$. Therefore $(c \circ i)^* = i^* \circ c^* = Id$. This implies that the exact sequence of abelian groups

$$0 \to \widetilde{K}(X) \hookrightarrow K(X) \xrightarrow{i^*} K(x_0) \to 0$$

is split. So we have, $K(X) = \widetilde{K}(X) \oplus K(x_0) = \widetilde{K}(X) \oplus \mathbb{Z}$. $\square$

Definition 2.2.3. Two vector bundles ξ and η over X are s-equivalent if there are n and m such that $\xi \oplus m\mathbb{I} \cong \eta \oplus n\mathbb{I}$.

The next theorem gives us a nice way to view the elements of $\widetilde{K}(X)$. In fact, they have the form $\xi - d(\xi)$, where $d(\xi)$ denotes the dimension of ξ.

Theorem 2.2.4. *The function $\alpha : Vect(X) \longrightarrow \widetilde{K}(X)$ defined by $\alpha(\xi) = \xi - d(\xi)$ is a surjection, and $\alpha(\xi) = \alpha(\eta)$ if and only if ξ and η are s-equivalent.*

Proof. Let's see first that α is surjective. Given $\xi - \eta$ in $\widetilde{K}(X)$, we have $d(\xi) = d(\eta)$. For η there is a vector bundle η' such that $\eta \oplus \eta' \cong m\mathbb{I}$. Since $\xi \oplus \eta \oplus \eta' \cong \xi \oplus \eta' \oplus \eta$, we have $\xi - \eta = \xi \oplus \eta' - \eta \oplus \eta' = \xi \oplus \eta' - m\mathbb{I}$, but since $d(\xi) = d(\eta)$, $m = d(\xi \oplus \eta')$. Therefore $\xi - \eta = \alpha(\xi \oplus \eta')$.

Now if $\alpha(\xi) = \alpha(\eta)$, then $\xi - d(\xi) = \eta - d(\eta)$. This implies that there is a bundle ζ such that $\xi \oplus m\mathbb{I} \oplus \zeta \cong \eta \oplus n\mathbb{I} \oplus \zeta$, where $n = d(\xi)$ and $m = d(\eta)$. Let ζ' be a vector bundle such that $\zeta \oplus \zeta' \cong q\mathbb{I}$. Then

$$\begin{aligned}
\xi \oplus (m+q)\mathbb{I} &\cong \xi \oplus m\mathbb{I} \oplus q\mathbb{I} \\
&\cong \xi \oplus m\mathbb{I} \oplus \zeta \oplus \zeta' \\
&\cong \eta \oplus n\mathbb{I} \oplus \zeta \oplus \zeta' \\
&\cong \eta \oplus n\mathbb{I} \oplus q\mathbb{I} \\
&\cong \eta \oplus (n+q)\mathbb{I}.
\end{aligned}$$

Therefore ξ and η are s-equivalent.
Conversely, if ξ and η are s-equivalent with $n = d(\xi)$ and $m = d(\eta)$. Then there are integers p and q such that $\xi \oplus p\mathbb{I} \cong \eta \oplus q\mathbb{I}$. Note that $n+p = m+q$. Then $\xi + p\mathbb{I} + (n+m)\mathbb{I} = \eta + q\mathbb{I} + (m+n)\mathbb{I}$. Thus $\alpha(\xi) = \alpha(\eta)$. $\square$

Corollary 2.2.5. *Any element of $\widetilde{K}(X)$ can be written as $\xi - n\mathbb{I}$, where n is the dimension ξ.*

2.3 Representation of $\widetilde{K}(X)$

Now we are going to use Theorem 1.4.1 to identify $\widetilde{K}(X)$ with a set of homotopy classes of maps from X to a special set that we will define in the process.

Let X be an n dimensional CW-complex.CW-complex we define We define the function

$$\phi_x : [X, G_n(\mathbb{C}^{2n})] \longrightarrow \widetilde{K}(X)$$

by $\phi_x([f]) = f^*(\gamma_n^{2n}) - n\mathbb{I}$.

Theorem 2.3.1. *Let X be a CW-complex of dimension $\leq m$, then*

$$\phi_x : [X, G_n(\mathbb{C}^{2n})] \to \widetilde{K}(X)$$

is a bijection for $m \leq 2n$.

Proof. By Theorem 1.4.2, every vector bundle over X is s-equivalent to an n-dimensional vector bundle where $m \leq 2n$. Now by Corollary 2.2.5, any element of $\widetilde{K}(X)$ can be written as $\xi - n\mathbb{I}$, where n is the dimension ξ. Now by Theorem 1.4.1, there is a function $f : X \to G_n(\mathbb{C}^{2n})$ such that $f^*(\gamma_n^{2n})$ is isomorphic to ξ. So, the function ϕ_x is surjective. Now if $\phi_x([f]) = \phi_x([g])$, then $f^*(\gamma_n^{2n}) \cong g^*(\gamma_n^{2n})$. Then again, by theorem 1.4.1, f and g are homotopic. Therefore we have $[f] = [g]$. Thus ϕ_x is a bijection for each X. $\square$

Definition 2.3.2. Let BU denote the $\bigcup_{1 \leq n} G_n(\mathbb{C}^{2n})$ with the inductive topology.

Theorem 2.3.3. *For each finite CW-complex X, there exists k such that the natural inclusion $G_n(\mathbb{C}^{2n}) \hookrightarrow BU$ induces a bijection*

$$[X, G_n(\mathbb{C}^{2n})] \to [X, BU] \quad \textit{for} \quad n \geq k.$$

Proof. For k with $\dim X \leq 2k$, the following bijections are induced by inclusions:

$$[X, G_k(\mathbb{C}^{2k})] \to \cdots \to [X, G_n(\mathbb{C}^{2n})] \cdots$$

Because X is compact, for every map $f : X \to BU$ we have $f(X) \subset G_m(\mathbb{C}^{2m})$ for some m with $k \leq m$. Therefore the function $[X, G_m(\mathbb{C}^{2m})] \to [X, BU]$ is surjective.

Because $X \times [0,1]$ is compact, the image of homotopy of maps $X \to BU$ lies in some $G_n(\mathbb{C}^{2n}) \subset BU$ for some n with $k \leq n$, and the function $[X, G_n(\mathbb{C}^{2n})] \longrightarrow [X, BU]$ is injective. □

Definition 2.3.4. On the category of finite CW-complexes, for q with $q \geq \dim X$, we define $\theta : [X, BU] \to \widetilde{K}(X)$ as the composition

$$[X, BU] \xrightarrow{\beta_q} [X, G_q(\mathbb{C}^{2q})] \xrightarrow{\phi_x} \widetilde{K}(X)$$

where β_q is the inverse of the bijection in theorem 2.3.3

For k as in theorem 2.3.3, the morphism θ is independent of q for $k \leq q$ (see Husemoller[6]).

Theorem 2.3.5. *Let X be a finite connected CW-complex. Then*

$$\theta : [X, BU] \to \widetilde{K}(X)$$

is a bijection.

Proof. By definition θ is a composition of bijections. □

2.4 The External Product

Let p_1 be the projection of $X \times Y$ onto X and p_2 be the projection of $X \times Y$ onto Y. Then the external product

$$\mu : K(X) \otimes K(Y) \to K(X \times Y)$$

is defined by $\mu(a \otimes b) = p_1^*(a) p_2^*(b)$. Note that $K(X) \otimes K(Y)$ has a ring structure with multiplication given by $(a \otimes b) \otimes (c \otimes d) = (ac \otimes bd)$.

Therefore

$$\begin{aligned}
\mu((a \otimes b) \otimes (c \otimes d)) &= \mu(ac \otimes bd) \\
&= p_1^*(ac)p_2^*(bd) \\
&= p_1^*(a)p_2^*(c)p_1^*(b)p_2^*(d) \\
&= p_1^*(a)p_2^*(b)p_1^*(c)p_2^*(d) \\
&= \mu(a \otimes b)\mu(c \otimes d)
\end{aligned}$$

Thus μ is a ring homomorphism.

Notation 2.4.1. In $K(X)$, $a * b = \mu(a \otimes b) = p_1^*(a)p_2^*(b)$.

Chapter 3

Bott Periodicity

In this section we are going to prove one of the most interesting results in K-theory, the Bott periodicity theorem. This theorem was first proven by Raoul Bott in 1950 by using elements of Morse theory. Here we present a different approach. This proof was given by Atiyah and Bott in [3]. Part of the organization of the proof given here has been taken from two books, Husemoller[6] and Hatcher[5].

The periodicity theorem asserts that there is an explicit isomorphism between $K(X) \otimes K(S^2)$ and $K(X \times S^2)$ for all compact Hausdorff spaces X.

We start by understanding how the techniques of the proof apply to a simple case, namely, when X is a single point, and then we will extend these ideas to the general case.

3.1 Some calculations

In this section we calculate $K(S^2)$ by using the same techniques used by Atiyah and Bott in the proof of the periodicity theorem. We can also compute $K(S^2)$ directly by using the knowledge of the homotopy groups of the unitary groups, $U(n)$'s. The main reason for the following approach is to understand the techniques of the periodicity theorem proof.

If H denotes the canonical line bundle over $S^2 = CP^1$, then according to Lemma 1.3.9 we have $(H \otimes H) \oplus \mathbb{I} \approx H \oplus H$ and in $K(S^2)$ this means $H^2 + \mathbb{I} = 2H$, so $H^2 - 2H + \mathbb{I} = 0$. Another way to write this is $(H - \mathbb{I})^2 = 0$, so we have a well-defined ring homomorphism

$$\mu : Z[H]/(H - \mathbb{I})^2 \to K(S^2)$$

given by $\mu(a+bH) = a\mathbb{I} + bH$, where $\mathbb{I}$ is the trivial bundle over S^2 and a and b are integers. The domain of μ is the quotient ring of the polynomial ring Z[H] by the ideal generated by $(H-\mathbb{I})^2$. Note that $Z[H]/(H-\mathbb{I})^2 = \{a + bH/a, b \in \mathbb{Z}\}$ is isomorphic (as abelian groups) to $\mathbb{Z} \oplus \mathbb{Z}$ with additive basis $1, H$.

Theorem 3.1.1. *The homomorphism*

$$\mu : Z[H]/(H-\mathbb{I})^2 \to K(S^2)$$

is an isomorphism of rings.

The following result which is immediate from Theorem 3.1.1 gives us a more familiar way to identify the algebraic structure of $K(S^2)$.

Corollary 3.1.2. *As abelian groups,* $K(S^2) = \mathbb{Z} \oplus \mathbb{Z}$.

The strategy of the proof will be to construct the inverse of the homomorphism μ, i.e., we wish to define a ring homomorphism

$$\nu : K(S^2) \to Z[H]/(H-\mathbb{I})^2$$

such that $\mu\nu$ =Identity and $\nu\mu$ =Identity.

In order to do that, we would like to be able to characterize all the elements of $K(S^2)$. In fact, we'll see that $K(S^2)$ is generated as a ring by the canonical line bundle H and the trivial bundle $\mathbb{I}$ over S^2, see Lemma 3.4.1.

In Section 1.2 we have seen that any vector bundle ξ over S^2 can be written as

$$\xi \cong (\xi_+ \cup_f \xi_-)$$

where ξ_+ is a trivialization of the restriction of ξ to the upper hemisphere S^2_+ and ξ_- is a trivialization of the restriction of ξ to the lower hemisphere S^2_- and $f : S^1 \to GL_n(\mathbb{C})$ is the clutching function for ξ; see Lemma 1.3.2.

The goal in the remainder of this section will be to reduce the function f of Lemma 1.3.2 to a simpler function, say a polynomial clutching function. The last expression will make sense after the following

Definition 3.1.3. A Laurent polynomial clutching function $\ell : S^1 \to GL_n(\mathbb{C})$ for a vector bundle ξ over S^2 is a clutching map of the form

$$\ell(z) = \sum_{|i| \leq n} A_i z^i$$

where $A_i : \mathbb{C}^n \to \mathbb{C}^n$ is a linear function and $z \in S^1 \subset \mathbb{C}$. A Laurent polynomial clutching function is called a polynomial clutching function if

$$\ell(z) = A_0 + A_1 z + \cdots + A_n z^n$$

and it is called a linear clutching function provided $\ell(z) = B + Az$.

The following result in essence is analogous to what is commonly done in real analysis when we want to approximate a function by a polynomial.

Theorem 3.1.4. *Every vector bundle ξ over S^2 is isomorphic to $(\xi_+ \bigcup_\ell \xi_-)$ for some Laurent Polynomial clutching function ℓ.*

Proof. We are going to show that there is a Laurent Polynomial clutching function ℓ homotopic to f and the result follows from Lemma 1.3.8.

Remark 3.1.5. Let $h : S^1 \to \mathbb{C}$ and consider its Fourier coefficients

$$a_k = \frac{1}{2\pi i} \int_{S^1} h(z) z^{-k} dz,$$

$$S_n(z) = \sum_{k=-n}^{n} a_k z^k \quad (z \in S^1), \quad \text{and} \quad \sigma_N(z) = \frac{1}{N} \sum_{n=0}^{N-1} S_n(z).$$

Then by Fejer's Theorem for Fourier series

$$| \sigma_N(z) |_\infty \leq | h(z) |_\infty$$

and

$$\lim_{N \to \infty} | \sigma_N(z) - h(z) |_\infty = 0$$

Note that these σ_N's have the form of a Laurent polynomial function.

Now, because the entries of $f(z)$ define functions $f_j : S^1 \to \mathbb{C}$, $j = 1, \cdots, n^2$, for a given $\epsilon \geq 0$ by Remark 3.1.5 there is a Laurent polynomial function $\ell_j : S^1 \to \mathbb{C}$ such that

$$| \ell_j(z) - f_j(z) |_\infty \leq \epsilon$$

Let $\mathfrak{M}_n$ be the set of all $n \times n$-matrices with complex entries. We define the polynomial function $\ell : S^1 \to \mathfrak{M}_n$, by setting the entries of $\ell(z)$ as those $\ell_j(z)$'s corresponding to the entries $f_j(z)$ above. Let V be the set of all continuous functions $g : S^1 \to \mathfrak{M}_n$. The set V is a vector space with a norm $\| g \| = \sup_{z\in S^1} \| g(z) \|_\infty$, where $\| g(z) \|_\infty$ is the matrix norm that takes the maximum of the absolute values of the entries of $g(z)$. Then the polynomials $\ell : S^1 \to \mathfrak{M}_n$ defined above are dense in V with this norm and the subspace $U = \{g \in V : \ g : S^1 \to GL_n(\mathbb{C})\}$ of V is open in the topology defined by the same norm because it is the preimage of $(0,\infty)$ under the continuous function $\varphi : V \to [0,\infty)$ defined by $\varphi(g) = \inf_{z\in S^1} | \det g(z) |$ (see remark 3.1.6 below for the continuity of φ).

Now, since $f \in U$, there is an open ball around f, $B_r(f) \subset V$ contained in U. Then by density, there is a Laurent Polynomial clutching function $\ell' \in B_r(f)$ approximating f, and it is homotopic to f via the linear homotopy $t\ell' + (1-t)f$.

Remark 3.1.6. To prove that φ is continuous at $g \in V$ we proceed in the following way: Consider a unit ball around g, i.e., $D_1(g) = \{h \in V : \| h - g \| < 1\}$.

Let $h \in D_1(g)$. Then

$$\varphi(h) = \inf_{z\in S^1} | \det h(z) | = | \det h(z_0) | \quad \text{for some} \quad z_0 \in S^1$$

Then

$$\begin{aligned} \varphi(g) &= \inf_{z\in S^1} | \det g(z) | \\ &\leq | \det g(z_0) | \\ &\leq | \det g(z_0) - \det h(z_0) | + | \det h(z_0) |, \end{aligned}$$

so

$$\varphi(g) - \varphi(h) \leq \sup_{z\in S^1} | \det g(z) - \det h(z) | \leq C \| g - h \|,$$

The last inequality is because the function $| \det(-) | : V \to [0,\infty)$ is Lipschitz on $D_1(g)$. This can be seen directly by using Hadamard's Inequality.

Finally, when $h \to g$, $\varphi(h) \to \varphi(g)$, therefore φ in continuous on V.

□

Note that any Laurent polynomial clutching function can be written, for some integer n, as $z^{-n}p$ for some polynomial clutching function p of degree $\leq 2n$, then by Lemma 1.2.5 and Example 1.3.7 we have the following:

Corollary 3.1.7. *Let ξ be a vector bundle over S^2, then there is a polynomial clutching function p of degree $\leq 2n$ such that,*

$$\begin{aligned}\xi &\cong (\xi_+ \bigcup\nolimits_{z^{-n}} \xi_-) \otimes (\xi_+ \bigcup\nolimits_p \xi_-) \\ &\cong H^{-n} \otimes (\xi_+ \cup_p \xi_-)\end{aligned}$$

3.2 Linearization

Because of Corollary 3.1.7, the next step of the proof is the analysis of polynomial clutching functions. We'll describe a linearization procedure for polynomial clutching functions.

Thus given a polynomial clutching function

$$P(z) = A_0 + A_1 z + \cdots + A_n z^n$$

of a complex vector bundle ξ over S^2, the matrix

$$L^{n+1}(P) = \begin{bmatrix} A_0 & A_1 & \cdots & A_{n-2} & A_{n-1} & A_n \\ -z & 1 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & -z & 1 & 0 \\ 0 & 0 & \cdots & 0 & -z & 1 \end{bmatrix}$$

defines a linear polynomial clutching function for $(n+1)\xi = \xi \oplus \cdots \oplus \xi$ by interpreting the (i, j) entry of the matrix as a linear function from the jth summand of $(n+1)\xi$ to the ith summand, with the entries 1 denoting the identity $\xi \to \xi$ and z denoting z times the identity, for $z \in S^1$.

Note that $L^{n+1}(P)$ is a square matrix of order $(n+1)m$, where m is the order of the matrix $P(z)$.

Example 3.2.1. Taking $P(z) = z^2$ in the above definition we have,

$$L^3(P) = \begin{bmatrix} 0 & 0 & 1 \\ -z & 1 & 0 \\ 0 & -z & 1 \end{bmatrix} = z \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus $L^3(P)$ is a linear clutching function for 3ξ.

Now by means of elementary row and column operations we can reduce $L^n(P)$ defined above to

$$P \oplus \mathbb{I}_n \;=\; \begin{bmatrix} p & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}$$

This reduction is done in the following way. In $L^n(P)$, add z times the last column to the previous column, then z times this column to the previous one, and so on. This produces a matrix with the polynomial P in the left upper corner and 0's below and above the diagonal except in the first row. After that for each $i = 1, \cdots n+1$, we subtract the appropriate multiple of the ith row from the first row to make all the entries in the first row 0 except for the first one, which is P. This algorithm is described in Hatcher[5]. The following result is a direct consequence of what we have just gotten above

Proposition 3.2.2. *If P is a polynomial clutching function of degree n for a vector bundle ξ over S^2, then*

$$\begin{aligned} \xi_+ \bigcup\nolimits_{L^{n+1}(P)} \xi_- &\cong (\xi_+ \bigcup\nolimits_P \xi_-) \oplus (\xi_+ \bigcup\nolimits_{I_n} \xi_-) \\ &\cong \xi \oplus n\mathbb{I} \end{aligned}$$

where I_n is the n-dimensional identity matrix and $\mathbb{I}$ is the 1-dimensional trivial bundle over S^2.

Applying Proposition 3.2.2 to P in Example 3.2.1, we have

Example 3.2.3.

$$\begin{aligned} \xi_+ \bigcup\nolimits_{L^3(z^2)} \xi_- &\cong (\xi_+ \bigcup\nolimits_{z^2} \xi_-) \oplus (\xi_+ \bigcup\nolimits_{I_2} \xi_-) \\ &\cong H^2 \oplus 2\mathbb{I} \end{aligned}$$

See Example 1.3.7 for the last part of the affirmation.

Given a polynomial clutching function P of degree n for a vector bundle ξ over S^2, we may consider P as a polynomial clutching function of degree $n+1$ for the same vector bundle ξ over S^2.

Proposition 3.2.4. *If P is a polynomial clutching function of degree n for a vector bundle ξ over S^2, then*

$$\begin{aligned}
\xi_+ \bigcup\nolimits_{L^{n+2}(P)} \xi_- &\cong (\xi_+ \bigcup\nolimits_{L^{n+1}(P)} \xi_-) \oplus (\xi_+ \bigcup\nolimits_{I_1} \xi_-) \\
&\cong (S^2_+ \bigcup\nolimits_P \xi_-) \oplus (\xi_+ \bigcup\nolimits_{I_{n+1}} \xi_-) \\
&\cong \xi \oplus (n+1)\mathbb{I}
\end{aligned}$$

where I_{n+1} is the $n+1$-dimensional identity matrix and $\mathbb{I}$ is the 1-dimensional trivial bundle over S^2.

Proof. By using the same algorithm described in the proof of proposition 3.2.2, we can verify the following reduction

$$L^{n+2}(P) = \begin{bmatrix} A_0 & A_1 & \cdots & A_{n-2} & A_{n-1} & A_n & 0 \\ -z & 1 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -z & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & -z & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & -z & 1 \end{bmatrix} \sim \begin{bmatrix} P & \vdots & \\ \cdots & \vdots & \cdots \\ & \vdots & \\ & \vdots & I_{n+1} \\ & \vdots & \\ & \vdots & \end{bmatrix}$$

In the last matrix, the entries of the empty boxes are all zeros. This prove the proposition. $\square$

Write P of Example 3.2.1 as $P = z^2 + 0z^3$, then by using Proposition 3.2.4 we have

Example 3.2.5.

$$\begin{aligned}
\xi_+ \bigcup\nolimits_{L^4(z^2)} \xi_- &\cong (\xi_+ \bigcup\nolimits_{z^2} \xi_-) \oplus (\xi_+ \bigcup\nolimits_{I_3} \xi_-) \\
&\cong H^2 \oplus 3\mathbb{I}
\end{aligned}$$

Given a polynomial clutching function P of degree n for a vector bundle ξ over S^2, the polynomial function zP is also a polynomial clutching function of degree $n+1$ for some vector bundle ξ' over S^2.

Proposition 3.2.6. *If p is a polynomial clutching function of degree n for a vector bundle ξ over S^2, then*

$$\begin{aligned} \xi_+ \bigcup\nolimits_{L^{n+2}(zP)} \xi_- &\cong (\xi_+ \bigcup\nolimits_z \xi_-) \oplus (\xi_+ \bigcup\nolimits_{L^{n+1}(P)} \xi_-) \\ &\cong (\xi_+ \bigcup\nolimits_P \xi_-) \oplus (\xi_+ \bigcup\nolimits_{I_{n+1}} \xi_-) \\ &\cong H \oplus (n+1)\mathbb{I} \end{aligned}$$

where I_n is the n-dimensional identity matrix and $\mathbb{I}$ is the 1-dimensional trivial bundle over S^2.

Proof. Again by using the algorithm described in the proof of proposition 3.2.2, we can verify the following reduction

$$L^{n+2}(P) = \begin{bmatrix} 0 & A_0 & A_1 & \cdots & A_{n-2} & A_{n-1} & A_n \\ -z & 1 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -z & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & -z & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & -z & 1 \end{bmatrix} \sim \left[\begin{array}{c|c} -z & \\ \cdots & \cdots \\ & \\ & L^{n+1}(P) \\ & \\ & \end{array}\right]$$

In the last matrix, the entries of the empty boxes are all zeros. This prove the proposition. $\square$

Now taking $P_1 = z$, $zP_1 = z^2 = P$, where P is the same as in Example 3.2.1, then by using Proposition 3.2.6 we have

Example 3.2.7.

$$\begin{aligned} \xi_+ \bigcup\nolimits_{L^3(zP_1)} \xi_- &\cong (\xi_+ \bigcup\nolimits_{-z} \xi_-) \oplus (\xi_+ \bigcup\nolimits_{L^2(P_1)} \xi_-) \\ &\cong H + H + \mathbb{I} \ \text{ by proposition 3.2.2} \\ &\cong 2H + \mathbb{I} \end{aligned}$$

Remark 3.2.8. By example 3.2.3 and example 3.2.7, we have

$$\xi_+ \bigcup_{L^3(z^2)} \xi_- \cong H^2 + 2\mathbb{I} \cong 2H + \mathbb{I}$$

Therefore

$$H^2 + \mathbb{I} \cong 2H$$

Note that we got this result directly in lemma 1.3.9.

3.3 Analysis of Linear Clutching Functions

In the previous section we have seen that any vector bundle over S^2 with a polynomial clutching function can be written in terms of vector bundles with linear clutching maps. Let $P(z) = Az + B$ be a linear clutching function of complex vector bundles ξ over S^2. We will see that there is an integer k such $\xi \cong kH + (n-k)\mathbb{I}$ in $K(S^2)$, where $n = Rank(P(z))$.

Definition 3.3.1. If $P(z) = Az + B$ is a linear clutching function of a complex vector bundle ξ over S^1, we define

$$Q = \frac{1}{2\pi i}\int_{S^1} (Az + B)^{-1} A dz$$

and

$$R = \frac{1}{2\pi i}\int_{S^1} A(Az + B)^{-1} dz$$

Thus Q and R are linear maps from $\mathbb{C}^n$ to $\mathbb{C}^n$ independent of z.

Remark 3.3.2. The linear clutching function $P(z) = Az + B$ is nonsingular for all $z \in S^1$. By compactness of S^1 and the continuity of P, $P(z)$ is nonsingular for all z with $1 - \epsilon <\mid z \mid< 1 + \epsilon$. Therefore we can write

$$Q = \frac{1}{2\pi i}\int_{|z|=r} (Az + B)^{-1} A dz$$

where $1 - \epsilon < r < 1 + \epsilon$.

We have a similar remark for R.

Example 3.3.3. For the linear clutching function $L^3(z^2)$ considered in Example 3.2.1 we have

$$L^3(P) = \begin{bmatrix} 0 & 0 & 1 \\ -z & 1 & 0 \\ 0 & -z & 1 \end{bmatrix} = z \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and its inverse is

$$(L^3(P))^{-1} = \begin{bmatrix} 1/z^2 & -1/z & -1/z^2 \\ 1/z & 0 & -1/z \\ 1 & 0 & 0 \end{bmatrix}$$

Therefore

$$\begin{aligned} Q &= \tfrac{1}{2\pi i}\int_{S^1}(L^3(z^2))^{-1}Adz \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} R &= \tfrac{1}{2\pi i}\int_{S^1} A(L^3(z^2))^{-1}dz \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

It can be easily verified that $Q^2 = Q$, $R^2 = R$, and $L^3(z^2)Q = RL^3(z^2)$.

Now let's see that it is true for any linear clutching function.

Proposition 3.3.4. *For all $z \in S^1$, we have $P(z)Q = RP(z)$, $Q^2 = Q$ and $R^2 = R$.*

Proof. For $z \neq w$, we have

$$\begin{aligned} \tfrac{1}{w-z}(Az+B)^{-1} + \tfrac{1}{z-w}(Aw+B)^{-1} &= (Aw+B)^{-1}(Aw+B)\tfrac{1}{w-z}(Az+B)^{-1} \\ &\quad + \tfrac{1}{z-w}(Aw+B)^{-1}(Az+B)(Az+B)^{-1} \\ &= (Aw+B)^{-1}\left[\tfrac{1}{w-z}(Aw+B)\right. \\ &\quad \left. + \tfrac{1}{z-w}(Az+B)\right](Az+B)^{-1} \\ &= (Aw+B)^{-1}A(Az+B)^{-1} \end{aligned}$$

Therefore we have the following identity

$$(Aw+B)^{-1}A(Az+B)^{-1} = \frac{1}{w-z}(Az+B)^{-1} + \frac{1}{z-w}(Aw+B)^{-1} \qquad (\alpha)$$

Similarly

$$(Az+B)^{-1}A(Aw+B)^{-1} = \frac{1}{w-z}(Az+B)^{-1} + \frac{1}{z-w}(Aw+B)^{-1} \qquad (\beta)$$

Therefore

$$(Aw+B)^{-1}A(Az+B)^{-1} = (Az+B)^{-1}A(Aw+B)^{-1} \qquad (\gamma)$$

Note that (γ) is true even when $z = w$. Now let's see that $P(z)Q = RP(z)$. By multiplying the relation (γ) by $(Az+B)$ on the left and right we get

$$(Az+B)(Aw+B)^{-1}A = A(Aw+B)^{-1}(Az+B)$$

Then we integrate

$$\begin{aligned} (Az+B)Q &= \tfrac{1}{2\pi i}\int_{S^1} A(Aw+B)^{-1}(Az+B)dw \\ &= R(Az+B) \end{aligned}$$

Now let's see that Q is a projection operator(see Def. 1.1.8). According to the remark 3.3.2 we can choose r_1 and r_2 with $1-\epsilon < r_1 < r_2 < 1+\epsilon$. Then by using the relation (β) we have

$$\begin{aligned} QQ &= \tfrac{1}{(2\pi i)^2}\int_{|z|=r_1}\int_{|w|=r_2}(Az+B)^{-1}A(Aw+B)^{-1}Adzdw \\ &= \tfrac{1}{(2\pi i)^2}\int_{|z|=r_1}\int_{|w|=r_2}\left[\tfrac{1}{w-z}(Az+B)^{-1}A + \tfrac{1}{z-w}(Aw+B)^{-1}A\right]dzdw \\ &= \tfrac{1}{(2\pi i)^2}\int_{|z|=r_1}(Az+B)^{-1}A\left[\int_{|w|=r_2}\tfrac{dw}{w-z}\right]dz \\ &\quad +\tfrac{1}{(2\pi i)^2}\int_{|w|=r_2}(Aw+B)^{-1}A\left[\int_{|z|=r_1}\tfrac{dz}{z-w}\right]dw \\ &= \tfrac{1}{2\pi i}\int_{|w|=r_2}(Aw+B)^{-1}Adw \\ &= Q \end{aligned}$$

because

$$\int_{|w|=r_2}\frac{dw}{w-z} = 0 \ \text{ for } \mid z \mid = r_1 > r_2$$

and

$$\int_{|z|=r_1} \frac{dz}{z-w} = 2\pi i \ \text{ for } \mid w \mid = r_2 < r_1$$

Similarly we have $R^2 = R$. □

Now since Q and R are projection operators, by Lemma 1.1.9, $Q(\xi)$, $R(\xi)$, $(\mathbb{I}-Q)(\xi)$,and $(\mathbb{I}-R)(\xi)$ are vector bundles and we have

$$\xi = Q(\xi) \oplus (\mathbb{I}-Q)(\xi)$$

and

$$\xi = R(\xi) \oplus (\mathbb{I}-R)(\xi)$$

Notation 3.3.5. We denote the bundle $Q(\xi)$ by ξ_+^Q, $R(\xi)$ by ξ_+^R, $(\mathbb{I}-Q)(\xi)$ by ξ_-^Q, and $(\mathbb{I}-R)(\xi)$ by ξ_-^R. The relation $P(z)Q = RP(z)$ guarantees that the following restrictions are defined

$$P_+(z) : \xi_+^Q \to \xi_+^R$$

$$P_-(z) : \xi_-^Q \to \xi_-^R$$

With the notations above we have

$$\xi = \xi_+^Q \oplus \xi_-^Q = \xi_+^R \oplus \xi_-^R$$

Proposition 3.3.6. *The restriction $P_+(z) : \xi_+^Q \to \xi_+^R$ is nonsingular for $\mid z \mid \geq 1$ and $P_-(z) : \xi_-^Q \to \xi_-^R$ is nonsingular for $\mid z \mid \leq 1$*

Proof. Let $v \in \mathbb{C}^n$. If $v \in Ker(P)$,i.e., $(Aw+B)v = 0$ for some $\mid w \mid \neq 1$, then $(Az+B)v = (z-w)Av$ and $(Az+B)^{-1}Av = (z-w)^{-1}v$ for $\mid z \mid = 1$. If we integrate over the circle $\mid z \mid = 1$, we get

$$Q(v) = \begin{cases} v & \text{for } \mid w \mid < 1 \\ 0 & \text{for } \mid w \mid > 1 \end{cases}$$

If $v \in Ker(Az+B)$ and $\mid w \mid < 1$, then $v \in \xi_+^0$ and $P_-(z)$ is a monomorphism for $\mid z \mid \leq 1$. If $v \in Ker(Aw+B)$ and $\mid w \mid > 1$, then $v \in \xi_-^0$ and $P_+(z)$ is a monomorphism for $\mid z \mid \geq 1$. For reasons of dimension, P_+ and P_- are isomorphisms for $\mid z \mid \geq 1$ or $\mid z \mid \leq 1$, respectively. □

Notation 3.3.7. Let P be a linear clutching function. We are going to write $P_+ = A_+z + B_+$ and $P_- = A_-z + B_-$, for the decomposition of P defined in 3.3.5.

Now let's see how it looks for the linear clutching function we worked out in Example 3.3.3.

Example 3.3.8. We have $L^3(z^2) = L^3(z^2)^+ + L^3(z^2)^-$, where $L^3(z^2)^+ = L^3(z^2)|Q(3\mathbb{I})$ and $L^3(z^2)^- = L^3(z^2)|(\mathbb{I} - Q)(3\mathbb{I})$. Therefore

$$L^3(p)^+ \quad = \quad \begin{bmatrix} 0 & 0 & 0 \\ -z & 1 & 0 \\ 0 & -z & 0 \end{bmatrix} \quad = \quad z \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \quad + \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$L^3(p)^- \quad = \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad = \quad z \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad + \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore

$$\begin{aligned} A^+ &= \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \\ B^+ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ A^- &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ B^- &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Clearly $L^3(z^2) \sim zA^+ + B^-$ and then we have

$$\begin{aligned} \xi_+ \bigcup\nolimits_{L^3(z^2)} \xi_- &= (\xi_+ \bigcup\nolimits_{zA^+} \xi_-) \oplus (\xi_+ \bigcup\nolimits_{B^-} \xi_-) \\ &\cong (\xi_+ \bigcup\nolimits_{zI_2} \xi_-) \oplus (\xi_+ \bigcup\nolimits_{I_1} \xi_-) \\ &\cong 2H \oplus \mathbb{I} \end{aligned}$$

Now let's see that it is again true for any linear clutching function.

Proposition 3.3.9. *Let P be a linear clutching function and let $P^t = P^t_+ + P^t_-$, where $P^t_+ = A^+z + tB^+$ and $P^t_- = tA^-z + B^-$ for $0 \le t \le 1$. Then P^t is a homotopy of linear clutching functions from $A^+z + B^-$ to P. Moreover*

$$\begin{aligned} \xi_+ \bigcup_P \xi_- &= (\xi_+ \bigcup_{zA^+} \xi_-) \oplus (\xi_+ \bigcup_{B^-} \xi_-) \\ &\cong (\xi_+ \bigcup_{zI_k} \xi_-) \oplus (\xi_+ \bigcup_{I_{n-k}} \xi_-) \\ &\cong kH \oplus (n-k)\mathbb{I} \end{aligned}$$

where $k = Rank(A^+)$ and n is the order of the matrix A^+.

Proof. For $t = 0$, we have $P^0 = A^+z + B^-$ and for $t = 1$ we have $P^1 = P$. By Proposition 3.3.6 P^t_+ and P^t_- are isomorphisms onto their images for all t with $0 \le t \le 1$. Then A^+z and B^- are nonsingular onto their images. Therefore

$$\xi_+ \bigcup_{zA^+} \xi_- \cong \xi_+ \bigcup_{zI_k} \xi_-$$

and

$$\xi_+ \bigcup_{B^-} \xi_- \cong \xi_+ \bigcup_{I_{n-k}} \xi_-$$

where $k = Rank(A^+)$ and n is the order of the matrix A^+. This completes the proof. □

eigenvalues

3.4 The inverse of the Periodicity Isomorphism for S^2

Finally we have the tools needed to define the homomorphism

$$\nu : K(S^2) \to Z[H]/(H-1)^2$$

such that $\mu\nu = \nu\mu = 1$.

The following result characterizes all the elements of $K(S^2)$.

Lemma 3.4.1. *Let ξ be a vector bundle over S^2. Then in $K(S^2)$ we have*

$$\xi = kH^{-n} + (2n+1)mH^{-n} - 2nH^{-n}$$

for some integers k, m and n.

Proof. Since any clutching function f of the vector bundle ξ over S^2 is homotopic to $z^{-n}P$, where P is a polynomial clutching function of degree $\leq 2n$, we have $\xi = \xi_+ \bigcup_{z^{-n}P} \xi_-$. Then

$$\begin{aligned}
\xi &= \xi_+ \bigcup\nolimits_{z^{-n}P} \xi_- \\
&= \xi_+ \bigcup\nolimits_{z^{-n}} \xi_- \otimes \xi_+ \bigcup\nolimits_{P} \xi_- \\
&= H^{-n} \otimes \left[\xi_+ \bigcup\nolimits_{L^{2n+1}(P)} \xi_- - 2n\mathbb{I}\right] \\
&= H^{-n} \otimes [kH + ((2n+1)m - k)\mathbb{I} - 2n\mathbb{I}] \\
&= kH^{1-n} + ((2n+1)m - k)H^{-n} - 2nH^{-n}
\end{aligned}$$

where $k = Rank(L^{2n+1}(P))^+$(see Proposition 3.3.9) and m is the order of the matrix $P(z)$.

$\square$

Since $H^2 + \mathbb{I} = 2H$, $H + H^{-1} = 2\mathbb{I}$. Therefore Lemma 3.4.1 asserts that $K(S^2)$ is generated by H and $\mathbb{I}$.

Finally, we present the proof of Proposition 3.1.1,

Proof of Proposition 3.1.1. By Lemma 3.4.1, the elements of $K(S^2)$ are linear combinations of $\mathbb{I}$ and H^n, so it is enough to define $\nu : K(S^2) \to Z[H]/(H-1)^2$ for these elements.

By using the relation $H^2 = 2H - 1$ in $K(S^2)$, we get

$$\begin{aligned}
H^3 &= 2H^2 - H \\
&= 2(2H-1) - H \\
&= 4H - 2 - H \\
&= 3H - 2
\end{aligned}$$

and

$$\begin{aligned}
H^4 &= 3H^2 - 2H \\
&= 3(2H-1) - 2H \\
&= 6H - 3 - 2H \\
&= 4H - 3
\end{aligned}$$

and if continue with this process we end up with

$$H^n = nH - (n-1) \text{ for } n \geq 0.$$

as can be proven by induction on n. So motivated by this calculation we define

$$\nu(H^n) = nH - (n-1) \text{ for } n \geq 0.$$

and $\nu(\mathbb{I}) = 1$.

ν is extended linearly to be a ring homomorphism. Now we show that $\nu\mu$ =Identity,

$$\nu\mu(a + bH) = \nu(a\mathbb{I} + bH) = a + bH$$

Finally,

$$\mu\nu(H^n) = \mu(nH - (n-1)) = nH - (n-1)\mathbb{I} = H^n$$

This proves that $\mu\nu$ =Identity. □

3.5 The Periodicity Theorem

Let X be a compact Hausdorff space. The proof of the periodicity theorem is a parametrized version of the proof we did for Proposition 3.1.1 in the previous section. Indeed we can view Proposition 3.1.1 as,

$$K(x) \otimes Z[H]/(H - \mathbb{I})^2 \cong K(x \times S^2) \text{ for } x \in X.$$

If we parametrize by $x \in X$ the constructions used in the proof of Proposition 3.1.1, we get an isomorphism μ from the composition

$$K(X) \otimes Z[H]/(H - \mathbb{I})^2 \to K(X) \otimes K(S^2) \to K(X \times S^2),$$

where the first homomorphism is given by Proposition 3.1.1 and the second one is the external product defined in the previous chapter. Therefore μ is given by $\mu(\xi \otimes H) = \xi * H$. Finally we state the periodicity theorem

Theorem 3.5.1. *The homomorphism*

$$\mu : K(X) \otimes Z[H]/(H - \mathbb{I})^2 \to K(X \times S^2)$$

is an isomorphism of rings for all compact Hausdorff spaces X.

The isomorphism μ in theorem 3.5.1 is called the periodicity isomorphism. The proof of Theorem 3.5.1 can be found in Atiyah and Bott[3].

Chapter 4

K-Cohomology

K-cohomology or just K-theory is the generalized cohomology theory defined by vector bundles. In other words, it is a cofunctor satisfying all of the Eilenberg-Steenrod axioms[1], [7], except for the dimension axiom. In this chapter we describe the generalized cohomology theory for complex K-theory.

We start by giving some preliminary definitions and results. In the first section we define the suspension operation over pointed spaces, an operation that is needed to construct exact homotopy sequences in the next section. In the third section, by using these homotopy sequences we will be able to state the Bott periodicity theorem for the reduced $\widetilde{K}$-theory case. In the last section, again by using the homotopy sequences in section 2, we'll get the long exact sequence for K-theory.

4.1 Suspensions

In this section we define the reduced suspension of a pointed topological space. Suspensions are very useful constructions in homotopy theory.

Definition 4.1.1. Let X be a pointed topological space and I the unit interval. We define the reduced suspension of X, SX, as the quotient space

$$SX = \frac{X \times I}{X \times \{0\} \cup X \times \{1\} \cup \{x_0\} \times I},$$

where x_0 is the base point.

Note that SX is a pointed space whose base point is the image of

$X \times \{0\} \cup X \times \{1\} \cup \{x_0\} \times I$, after it has been collapsed to a point in the above quotient.

If we denote by $x \wedge t$ the class of $(x,t) \in X \times I$ in the quotient above, then the base point is $\overline{x_0} = x_0 \wedge t = x \wedge 0$.

If $f : X \to Y$ is a pointed function, then it induces a pointed map

$$S(f) : SX \to SY$$

given by $S(f)(x \wedge t) = f(x) \wedge t$.

Notation 4.1.2. Let A be a closed subset of X with $x_0 \in A$. We are going to denote by j the constant map

$$j : \frac{X}{A} \to SA$$

sending X/A to $\overline{x_0}$.

4.2 Homotopy Sequences

In this section we will consider sequences of spaces constructed with quotient spaces, suspensions, and maps between them. Then applying $[-, BU]$, the functor of homotopy classes of maps into BU, we get exact sequences.

Definition 4.2.1. A sequence of homomorphisms and groups

$$G_1 \to G_2 \to \cdots G_n \to \cdots$$

is said to be exact if at each intermediate group G_i the kernel of the outgoing homomorphism equals the image of the incoming homomorphism.

As a first step we construct 3-term homotopy exact sequences. Consider the following sequence of maps and spaces

$$A \xrightarrow{i} X \xrightarrow{p} X/A$$

where i is the inclusion and p is the projection function. The homotopy sequence induced by this sequence of maps is exact.

Proposition 4.2.2. *Let (X, A) be a pointed topological pair with $i : A \hookrightarrow X$ the inclusion and $p : X \to X/A$ the canonical projection. Then the sequence*

$$[X/A, BU] \xrightarrow{p_\sharp} [X, BU] \xrightarrow{i_\sharp} [A, BU]$$

is exact, i.e.,

$$Im(p_\sharp) = Ker(i_\sharp) = \{[\varphi] \in [X, BU] : i_\sharp([\varphi]) = [\varphi \circ i] = [c]\},$$

where $c : A \to BU$ is the constant map.

Proof. Consider the following sequences of maps and spaces

$$\begin{array}{ccccc} A & \xrightarrow{i} & X & \xrightarrow{i_1} & X \cup CA \\ \| & & \| & & \approx\downarrow \\ A & \xrightarrow{i} & X & \xrightarrow{p} & X/A \end{array}$$

where CA is the cone of A defined by

$$CA = \frac{A \times I}{A \times \{1\} \cup \{x_0\} \times I},$$

and i_1 is the natural inclusion of X into $X \cup CA$, i.e., $i_1(x) = \overline{(x, 0)}$.

Note that the right corner downward map in the diagram above is a homotopy equivalence. Also note that $i_1 \circ i$ is nullhomotopic, because we have the homotopy $F : A \times I \to X \cup CA$ given by

$$F(a, t) = \overline{(a, 1 - t)}$$

which for $t = 0$ is constant and for $t = 1$ gives the point $\overline{(a, 0)} = i_1 \circ i(a)$. Thus

$$i_\sharp(i_1)_\sharp([\psi]) = [\psi \circ i_1 \circ i] = [c], \quad \forall\ [\psi] \in [X, BU].$$

Therefore $Im((i_1)_\sharp) \subset Ker(i_\sharp)$.

On the other hand, let $[\varphi] \in Ker(i_\sharp)$, so that $[\varphi \circ i] = [c]$. Note that φ is a map from X to BU.

Let $F : A \times I \to BU$ be the nullhomotopy of $\varphi \circ i : A \to BU$, i.e., $F(a, 0) = \varphi \circ i(a)$ and $F(a, 1) = c(a)$. So F defines a function

$$\psi' : CA \to BU$$

given by

$$\psi'(\overline{(a,t)}) = F(a,t)$$

Since $\psi'(\overline{(a,0)}) = F(a,0) = \varphi \circ i(a)$, we then can define a map

$$\psi : X \cup CA \to BU$$

by

$$\psi(\overline{(a,t)}) = \begin{cases} \psi'(\overline{(a,t)}) & \text{if } \overline{(a,t)} \in CA \\ \varphi(x) & \text{if } x \in X \end{cases} .$$

Consequently, $[\psi] \in [X \cup CA, BU]$ satisfies $(i_1)_\sharp([\psi]) = [\psi \circ i_1] = [\varphi]$, and so $Ker(i_\sharp) \subset Im((i_1)_\sharp)$.

□

Now Consider the following 5-term sequence of maps and spaces

$$A \xrightarrow{i} X \xrightarrow{p} X/A \xrightarrow{j} SA \xrightarrow{S(i)} SX \tag{4.2.1}$$

where j is the function described in notation 4.1.2 and $S(i)$ is described in definition 4.1.1.

Proposition 4.2.3. *Let (X, A) as in proposition 4.2.2. Then the sequence induced by equation 4.2.1,*

$$[SX, BU] \xrightarrow{S(i)_\sharp} [SA, BU] \xrightarrow{j_\sharp} [X/A, BU] \xrightarrow{p_\sharp} [X, BU] \xrightarrow{i_\sharp} [A, BU]$$

is exact.

Proof. First let's see that the homotopy sequence induced by the following sequence of maps

$$X \xrightarrow{p} X/A \xrightarrow{j} SA$$

is exact. We'll follow the same scheme of the proof of proposition 4.2.2.

Again, since $j \circ p$ is nullhomotopic, $Im(j_\sharp) \subset Ker(p_\sharp)$.

On the other hand, let $[\varphi] \in Ker(p_\sharp)$, then $[\varphi \circ p] = [c]$.

Let $F : X \times I \to BU$ be the nullhomotopy of $\varphi \circ p : X \to BU$. Define the function

$$\psi : SA \to BU$$

by

$$\psi(a \wedge t) = F(a,t).$$

Note that $\varphi = \psi \circ j$. We want to show that φ is homotopic to $\psi \circ j$. Consider $F' : X/A \times I \to BU$ defined by

$$F'([x]) = F(x, t).$$

Then F' is a homotopy between $\psi \circ j$ and φ. Therefore

$$j_\sharp([\psi]) = [\psi \circ j] = [\varphi]$$

Thus $Ker(j_\sharp) \subset Im(p_\sharp)$.

A similar proof applies to the following sequence of maps

$$X/A \xrightarrow{j} SA \xrightarrow{S(i)} SX$$

□

Extending the sequence 4.2.1, we get

$$A \xrightarrow{i} X \xrightarrow{p} X/A \xrightarrow{j} SA \xrightarrow{S(i)} SX \xrightarrow{S(p)} S(X/A) \xrightarrow{S(j)} S^2A \to \cdots \qquad (4.2.2)$$

Corollary 4.2.4. *The sequence induced by equation 4.2.2,*

$$\begin{aligned} \cdots \to [S^nX, BU] &\xrightarrow{S^n(i_\sharp)} [S^nA, BU] \xrightarrow{S^n(j_\sharp)} [S^n(X/A), BU] \xrightarrow{S^n(p_\sharp)} \\ &\to [S^{n-1}X, BU] \xrightarrow{S^{n-1}(i_\sharp)} [S^{n-1}A, BU] \xrightarrow{S^{n-1}(j_\sharp)} \cdots [SX, BU] \xrightarrow{S(i_\sharp)} \\ &\to [SA, BU] \xrightarrow{j_\sharp} [X/A, BU] \xrightarrow{p_\sharp} [X, BU] \xrightarrow{i_\sharp} [A, BU] \end{aligned}$$

is exact.

Proof. Apply successively proposition 4.2.3 to the following sequences of maps and spaces

$$X \xrightarrow{p} X/A \xrightarrow{j} SA \xrightarrow{S(i)} SX \xrightarrow{S(p)} S(X/A)$$

$$X/A \xrightarrow{j} SA \xrightarrow{S(i)} SX \xrightarrow{S(p)} S(X/A) \xrightarrow{S(j)} S^2A$$

and so on.

□

4.3 Bott periodicity for the reduced case

Now by using the exact sequence in proposition 4.2.2 and the representation for the cofunctor $\widetilde{K}$ described in Chapter 2, we'll be able to state the periodicity theorem for the reduced case.

We introduce the smash product of two spaces, a generalization of the reduced suspension defined in the first section, in order to describe better the 3-term exact sequence for the cofunctor $\widetilde{K}$.

Definition 4.3.1. Let X and Y be Hausdorff compact topological spaces with base points x_0 and y_0 respectively. We define the smash product $X \wedge Y$ of X and Y by

$$X \wedge Y = \frac{X \times Y}{X \vee Y}$$

where $X \vee Y = X \times \{y_0\} \cup Y \times \{x_0\} \subset X \times Y$.

Now consider the sequence

$$X \vee S^2 \xrightarrow{i} X \times S^2 \xrightarrow{p} X \wedge S^2$$

where i is the inclusion and p is the quotient map.

Then by proposition 4.2.2, we have the following exact sequence

$$0 \to [X \wedge S^2, BU] \xrightarrow{p_\sharp} [X \times S^2, BU] \xrightarrow{i_\sharp} [X \vee S^2, BU] \to 0$$

or equivalently (by theorem 2.3.5),

$$0 \to \widetilde{K}(X \wedge S^2) \to \widetilde{K}(X \times S^2) \to \widetilde{K}(X \vee S^2) \to 0$$

From this we get the following commutative diagram

$$\begin{array}{ccc} K(X) \otimes K(S^2) & = & (\widetilde{K}(X) \otimes \widetilde{K}(S^2)) \oplus \widetilde{K}(X) \oplus \widetilde{K}(S^2) \oplus \mathbb{Z} \\ \downarrow & & \downarrow \\ K(X \times S^2) & = & \widetilde{K}(X \wedge S^2) \oplus \widetilde{K}(X) \oplus \widetilde{K}(S^2) \oplus \mathbb{Z} \end{array}$$

where the vertical row on the right is the external product defined in Chapter 2.

This leads to the following result

Proposition 4.3.2. *The external product $K(X) \otimes K(S^2) \to K(X \times S^2)$ is an isomorphism if and only if $\widetilde{K}(X) \otimes \widetilde{K}(S^2) \to \widetilde{K}(X \times S^2)$ is an isomorphism.*

As a consequence of this and the periodicity theorem 3.5.1, we have the following version of Bott periodicity for the reduced case.

Corollary 4.3.3. *The homomorphism*

$$\beta : \widetilde{K}(X) \to \widetilde{K}(S^2 X), \quad \beta(a) = a * (H - 1)$$

is an isomorphism of rings for all Hausdorff compact spaces X.

Here $S^2 X$, the second iterated reduced suspension of X, is identified with the smash product $X \wedge S^2$, so that $\widetilde{K}(S^2 X) = \widetilde{K}(X \wedge S^2)$.

Corollary 4.3.4. $\widetilde{K}(S^{2n} X) \cong \widetilde{K}(X)$

4.4 Extending to a generalized cohomology theory

In this section we define the higher dimensional K- groups. First note that because of Theorem 2.3.5, the sequence in Corollary 4.2.4 can be written as

$$\begin{aligned} \cdots \to \widetilde{K}(S^2(X)) \to \widetilde{K}(S(X/X)) \to \widetilde{K}(SX) \to \\ \to \widetilde{K}(SA) \to \widetilde{K}(X/A) \to \widetilde{K}(X) \to \widetilde{K}(A) \end{aligned} \tag{4.4.1}$$

This suggests the following,

Definition 4.4.1. Let X be a pointed CW-complex. Then we define

$$\widetilde{K}^0(X) = \widetilde{K}(X) = [X, BU]$$

and

$$\widetilde{K}^{-n}(X) = \widetilde{K}(S^n X) \quad \text{for } n \geq 0.$$

If $A \subset X$ is closed, we define

$$\widetilde{K}^{-n}(X, A) = \widetilde{K}(X/A)$$

Now the sequence in equation 4.4.1 can be written as

$$\begin{aligned}\cdots \to \widetilde{K}^{-2}(A) \to \widetilde{K}^{-1}(X,A) \to \widetilde{K}^{-1}(X) \to \\ \to \widetilde{K}^{-1}(A) \to \widetilde{K}^{0}(X,A) \to \widetilde{K}^{0}(X) \to \widetilde{K}^{0}(A).\end{aligned} \tag{4.4.2}$$

This last sequence is known as the long exact sequence in K-theory for the pair (X, A).

As a consequence of the periodicity theorem, we have

Theorem 4.4.2. *Let X be a finite CW-complex, then*

$$\widetilde{K}^n(X) = \begin{cases} \widetilde{K}^0(X) & \textit{if } n \textit{ is even} \\ \widetilde{K}^{-1}(X) & \textit{if } n \textit{ is odd} \end{cases}$$

Proof. Because of corollary 4.3.4,

$$\widetilde{K}^{2n}(X) = \widetilde{K}(S^{2n}X) = \widetilde{K}(X) = \widetilde{K}^0(X)$$

Similarly

$$\widetilde{K}^{2n+1}(X) = \widetilde{K}(S^{2n+1}X) = \widetilde{K}(SX) = \widetilde{K}^{-1}(X)$$

□

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Vita

David Manuel Murrugarra Tomairo is currently a second year graduate student of the Mathematics Department at Virginia Tech. He graduated from San Marcos University (Lima-Peru) in 2004 with a bachelors degree in Pure Mathematics. His interest areas include algebra and algebraic topology.