Adaptive Control of Nonaffine Systems with Applications to Flight Control

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Thesis submitted to the Faculty of the
Virginia Polytechnic Institute and State University
in partial fulfillment of the requirements for the degree of

Master of Science
in
Aerospace Engineering

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May 5, 2006
Blacksburg, Virginia

Keywords: Flight Control, Adaptive Control, Nonaffine Systems

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(ABSTRACT)

Traditional flight control design is based on linearization of the equations of motion around a set of trim points and scheduling gains of linear (optimal) controllers around each of these points to meet performance specifications. For high angle of attack maneuvers and other aggressive flight regimes (required for fighter aircraft for example), the dynamic nonlinearities are dependent not only on the states of the system, but also on the control inputs. Hence, the conventional linearization-based logic cannot be straightforwardly extended to these flight regimes, and non-conventional approaches are required to extend the flight envelope beyond the one achievable by gain-scheduled controllers. Due to the nonlinear-in-control nature of the dynamical system in aggressive flight maneuvers, well-known dynamic inversion methods cannot be applied to determine the explicit form of the control law. Additionally, the aerodynamic uncertainties, typical for such regimes, are poorly modelled, and therefore there is a great need for adaptive control methods to compensate for dynamic instabilities. In this thesis, we present an adaptive control design method for both short-period and lateral/directional control of a fighter aircraft. The approach uses a specialized set of radial basis function approximators and Lyapunov-based adaptive laws to estimate the unknown nonlinearities. The adaptive controller is defined as a solution of fast dynamics, which verifies the assumptions of Tikhonov's theorem from singular perturbations theory. Simulations illustrate the theoretical findings.

Acknowledgments

First and foremost, I would like to thank God for giving me all the blessings in the world.

I would like to thank my advisor, Naira Hovakimyan, for believing in me and investing so much of her time and effort. If it were not for her passion and dedication, I would not be as successful as I am today.

Many thanks to Dr. Chengyu Cao for his patience, Dr. Eugene Lavretsky from the Boeing company for his suggestions and insights, Dr. Vijay Patel for sharing his guidance and experience, and Dr. Konda Reddy for his wisdom. I would like to thank my committee members Dr. Daniel Stillwell and Dr. Craig Woolsey for their time and commitment. Additionally, I thank my friends Imraan Faruque, Lili Ma, Nina Mahmoudian, Vahram Stepanyan, Laszlo Techy, and Jiang Wang for their help and support. I would also like to thank the department of Aerospace and Ocean Engineering and its faculty for transforming me into an engineer.

All my thanks to my best friend, Sam for his support and love. You mean everything to me. Thanks to my little pup, Kane, for big slobbery kisses and for putting up with my long hours and not messing in the house. Finally, I would like to thank my mom, Katherine for all her hard work and sacrifices. My opportunities in life are made possible because of you.

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Chapter 1

Introduction

For straight and level flight, the dynamic modes for conventional airplanes decouple into two independent sets: 1) longitudinal modes, called the short period and phugoid, and 2) lateral modes, called the roll, Dutch-Roll, and spiral. In this thesis, we are interested in examining the behavior of the short-period and Dutch-Roll systems.

1.1 Motivation

Research in adaptive control of nonaffine systems, in general, is motivated by the many emerging applications that employ novel actuation devices, like piezo-electric films or synthetic jets, which are typically nonlinearly coupled to the dynamics of the processes they are intended to control. Modelling for these applications varies from having accurate low frequency models in the case of structural control problems, to having no reasonable set of model equations in the case of active control of flows and combustion processes. From the perspective of flight control design, development of adaptive control methods for nonaffine systems has the potential of expanding the flight envelope to near-stall angles of attack flight regimes. This is especially relevant for design of flight control systems for fighter aircraft that are very much needed for military operations and homeland security. This thesis presents an adaptive augmentation of a baseline linear optimal (LQR) controller to encounter for uncertain control-dependent nonlinearities in aggressive flight regimes. Two specific systems are considered: short-period dynamics and Dutch-Roll dynamics.

1.2 Problem Definition

Consider the following system:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B(u(t) + f(x(t), u(t))), \quad x(0) = x_0 \\ y(t) &= C^\top x(t), \end{aligned} \tag{1.1}$$

where $x \in \mathbb{R}^n$ is the state of the system, $A \in \mathbb{R}^{n\times n}$ is the known system matrix, $B \in \mathbb{R}^{n\times m}$ is the input matrix, $C^\top$ is the output matrix, $y \in \mathbb{R}^m$ is the regulated output, $u \in \mathbb{R}^m$ is the control input, and $f \in \mathbb{R}^m$ is an *unknown* Lipshitz continuous nonlinear function dependent on both x and u. The control objective is to determine a control input u, such that $y(t)$ tracks a given set of bounded smooth reference inputs $r = [r_1, \ldots, r_m]^\top$, while all other error signals remain bounded.

1.3 Approach

The goal of this thesis is to design a controller for two nonlinear aircraft systems with uncertain dynamics that are nonlinear-in-control or *nonaffine*. First, a nominal system is designed using standard linear quadratic regulator (LQR) methods for the linearized system in the absence of uncertainties. This controller is then augmented with an adaptive element to compensate for modelling uncertainties. The adaptive laws for approximating system uncertainties are derived from Lyapunov-like analysis and the incremental adaptive control signal is sought as a solution of fast dynamics which verifies the assumptions of Tikhonov's theorem from singular perturbations theory for the closed-loop system.

1.4 Overview

This thesis is presented as follows. In chapter 3, we review some preliminaries from approximation theory and Tikhonov's theorem from singular perturbations theory, which are the key elements in the synthesis and analysis. In chapter 4, we formulate the single input short-period fighter aircraft dynamics at high angle of attack and propose a dynamic-inversion based controller to achieve the control objective. Simulations are given for this system that show the desired tracking performance. Next, we extend the methodology to the multi-input multi-output roll-yaw system and show simulations. A brief summary and some recommendations for future work are given in conclusion.

Chapter 2

Literature Review

2.1 Short Period Dynamics

The continuing development of flight control design methods in parallel to advances in nonlinear control theory have made controllability at high angles of attack near and beyond stall increasingly relevant. The main challenge encountered at high angles of attack, is that the short-period aircraft dynamics appear to be highly uncertain and nonlinear both in the states and in the control signal [6]. Hence, traditional linearization and gain scheduling methods are insufficient for modelling and controlling the system at high-angles of attack. To obtain a better control design and model, various methods have been explored and discussed. One such method is to re-write the system linearly by expanding the output and control vectors into a truncated Taylor series expansion in order to apply optimal control design [7]. Nonlinear control methods for affine-in-control systems using feedback linearization via dynamic inversion are found in [8] - [11]. The authors of [8] have utilized the natural time-scale separation between the faster pitch rate dynamics and the slower angle-of-attack dynamics and assumed that the control effectors have no effect on α so that α can be pseudo-controlled through the pitch rate. Without any modelling uncertainties, this controller was valid for an angle of attack range of up to 55°. In [9], two nonlinear controllers are implemented by separating the 6 DOF system dynamics into two time scales: angular rates which are fast, and angles of attack and sideslip, which are slow. The controller for the fast dynamics are found by inversion of the momentum equations, and the slow dynamics controller is obtained by inverting the force equations. The controller performance is tested in high-fidelity simulations, valid up to angles of attack of 30°. The problem with feedback linearization is that it has a low tolerance twoards modelling errors. Neural network approximations are introduced in [10] and [11] in order to compensate for modelling uncertainties. [10] uses an off-line neural network configuration to make an initial approximation of the nonlinear system dynamics and a second on-line neural network approximation to estimate the inversion error. In [11], the neural network approximation captures the nonlinear dynamics of the system for an angle of attack range of $\alpha \in [-5°, 21°]$, and the inverse dynamics are computed by differentiating the output with respect to time until a term appears that contains the input u. The authors of [12] differentiate the nonaffine-in-control system with respect to time to obtain a higher-order system, in which the derivative of the control input appears linearly. They then apply linear control methods and integrate to obtain the structure of the controller. This method results in a large control signal and would be susceptible to system saturation in realistic systems. Direct projection-based adaptive control is then easily implemented and the system is shown to be stable using a standard Lyapunov argument.

In the first part of this thesis, we apply the theory developed in [13] to a nonaffine fighter aircraft like the F-16 performing at or near the stall angle in the presence of system uncertainties. The method in [13] introduces a specialized set of radial basis functions (RBF) that retains the monotonic property with respect to control effectiveness of the original aircraft dynamics. It further defines the adaptive controller via fast dynamics and achieves time-scale separation between the system dynamics and the controller dynamics. The stability and tracking results follow from Tikhonov's theorem in singular perturbations theory.

2.2 Dutch-Roll Dynamics

The Dutch-Roll is a type of lightly damped oscillatory lateral/directional motion between sideslip, roll, and yaw that consists of an out-of-phase combination of "tail-wagging" and rocking from side to side that occurs when an aircraft is disturbed laterally from equilibrium flight. This type of motion occurs when a statically stable aircraft tries to re-establish lateral and directional equilibrium. Dutch roll is usually the most troublesome of the natural modes associated with the dynamics of an aircraft in free flight due to its sensitive stability balance, which could easily result in instability [16]. Hence, it is important to model the Dutch-Roll system as accurately as possible. Traditional Dutch-Roll approximations are made by linearizing the full nonlinear equations of motion so that the dynamic modes can be decoupled and evaluated [17], [18]. Additionally, the aerodynamic stability and control derivatives are taken from wind-tunnel data and the eigenvalues of the linearized equations can be numerically derived, so that a linear controller can be designed [15], [19] - [24]. Typically, these types of approximations neglect the rolling rate, bank angle, rolling momentum and aerodynamic coupling, which decreases the accuracy of the models [25]. An improved closed-loop form is derived in [16] using Taylor series expansion to approximate the roll-damping derivative. The effects of inertial, gyroscopic, and aerodynamic coupling are included for a linear model in [26]. The problem with linearization approaches, is that although they are accurate for low angles of attack, they becomes less than ideal for aggressive aircraft maneuvers and high angle of attack flight regimes when the nonlinear effects become significant enough to drive the system to instability.

In the second part of this thesis, we extend the method developed in [14] to uncertain nonaffine-in-control multi-input multi-output (MIMO) systems, which describe the lateral/directional dynamics of an F-16 at different angles of sideslip in the presence of system uncertainties.

Chapter 3

Mathematical Preliminaries

In this chapter, we recall some facts that will be used for derivation of main results. First, some results from nonlinear systems theory is given. Next, we summarize the main result from [2] on approximation of monotonic functions. Lastly, Tikhonov's theorem is stated from [1], which enables definition of the adaptive controller via fast dynamics.

3.1 Preliminaries on Nonlinear Systems Theory

Lemma 1 *([3], p.116) A time-varying linear system*

$$\dot{x} \;=\; A(t)x, \tag{3.1}$$

is globally exponentially stable if

1. *for any time $t \geq 0$, the eigenvalues of $A(t)$ have negative real parts, that is,*

$$\exists\beta > 0, \quad \forall t \geq 0,\ \lambda(A(t)) \leq -\beta, \tag{3.2}$$

   where $\lambda(A(t))$ denotes the eigenvalues of $A(t)$,

2. *the matrix $A(t)$ remains bounded:*

$$\int_0^\infty A^\top(t)A(t)dt < \infty. \tag{3.3}$$

Lemma 2 *(Comparison Lemma [1], p.102) Consider the scalar differential equation*

$$\dot{x} = f(t, x), \quad x(t_0) = x_0, \tag{3.4}$$

where $f(t, u)$ is continuous in t and locally Lipshitz in x, for all $t \geq 0$. Let $[t_0, T)$ (where T could be infinity) be the maximal interval of existence of the solution $x(t)$. Let $v(t)$ be a continuous function which satisfies the differential inequality

$$\dot{v}(t) \leq f(t, v(t)), \quad v(t_0) \leq x_0. \tag{3.5}$$

Then, $v(t) \leq x(t)$ for all $t \in [t_0, T)$.

Lemma 3 *([1], p.368) Consider the system*

$$\dot{z} \;=\; f(z + h(\alpha), \alpha) \triangleq g(z, \alpha) \tag{3.6}$$

and suppose $g(z, \alpha)$ is continuously differentiable and the Jacobian matrices $\frac{\partial g}{\partial z}$ and $\frac{\partial g}{\partial \alpha}$ satisfy

$$\left\| \frac{\partial g}{\partial z}(z, \alpha) \right\| \leq L_1, \quad \left\| \frac{\partial g}{\partial \alpha}(z, \alpha) \right\| \leq L_2 \|z\| \tag{3.7}$$

for all $(z, \alpha) \in D \times \Gamma$, where $\Gamma \subset \mathbb{R}^m$ and $D = \{z \in \mathbb{R}^n \mid \|z\| < r\}$. Let k_1, λ_1, and γ_0 be positive constants with $r_0 < \frac{r}{k_1}$, and define $D_0 = \{z \in \mathbb{R}^n \mid \|z\| < r_0\}$. Assume that the trajectories of the system satisfy

$$\|z(t)\| \leq k_1 \|z(0)\| e^{-\lambda_1 t}, \quad \forall \; z(0) \in D_0, \; \alpha \in \Gamma, \; t \geq 0. \tag{3.8}$$

Then, there is a function $V : D_0 \times \Gamma \to \mathbb{R}$ that satisfies

$$c_1 \|z\|^2 \;\leq\; V(z, \alpha) \leq c_2 \|z\|^2 \tag{3.9}$$

$$\frac{\partial V}{\partial z} g(z, \alpha) \;\leq\; -c_3 \|z\|^2 \tag{3.10}$$

$$\left\| \frac{\partial V}{\partial z} \right\| \;\leq\; c_4 \|z\| \tag{3.11}$$

$$\left\| \frac{\partial V}{\partial \alpha} \right\| \;\leq\; c_5 \|z\|^2 \tag{3.12}$$

for positive constants c_1, c_2, c_3, c_4, c_5 and all $z \in D, \alpha \in \Gamma$. Moreover, if all the assumptions hold globally (in z), then $V(z, \alpha)$ is defined and satisfies (3.9) through (3.12) on $\mathbb{R}^n \times \Gamma$.

3.2 Preliminaries on Approximation Theory

In this section, we recall the main result from [2]. To follow the notations in [2], let $\mathcal{N}$, $\mathbb{R}$, and $\mathbb{R}^r$ denote the set of natural numbers, the set of real numbers, and the set of real r-vectors, respectively. Let $\mathcal{L}^p(\mathbb{R}^r)$, $\mathcal{L}^\infty(\mathbb{R}^r)$, $\mathcal{C}(\mathbb{R}^r)$, $\mathcal{C}_c(\mathbb{R}^r)$, denote the usual spaces of $\mathbb{R}$-valued maps $f : \mathbb{R}^r \to \mathbb{R}$ such that f is p^{th} power integrable, essentially bounded, continuous, and continuous with compact support. Let further $\mathcal{L}^p_+(\mathbb{R}^r)$, $\mathcal{L}^\infty_+(\mathbb{R}^r)$, $\mathcal{C}_+(\mathbb{R}^r)$, $\mathcal{C}_{c+}(\mathbb{R}^r)$ be the spaces of positive valued maps $f : \mathbb{R}^r \to \mathbb{R}^+$ respectively. Since $\mathcal{C}_c(\mathbb{R}^r)$ is dense in $\mathcal{L}^p(\mathbb{R}^r)$ ([4], p. 69), then for every $f \in \mathcal{L}^p(\mathbb{R}^r)$ there exists an $f_c \in \mathcal{C}_c(\mathbb{R}^r)$ such that $||f_c - f||_p \leq \varepsilon$, where $\|\cdot\|_p$ denotes the p-norm. The main result from [2] is summarized via the following theorem:

Theorem 1 *Let $K : \mathbb{R}^r \to \mathbb{R}$ be an integrable bounded function such that K is continuous almost everywhere and $\int_{\mathbb{R}^r} K(x)dx \neq 0$. Then the family $\mathcal{S}_K$ of functions $q : \mathbb{R}^r \to \mathbb{R}$ defined as*

$$q(x) = \sum_{i=1}^{M} w_i K\left(\frac{x - z_i}{\delta}\right), \tag{3.13}$$

where $M \in \mathcal{N}$, $\delta > 0$, $w_i \in \mathbb{R}$, and $z_i \in \mathbb{R}^r$, is dense in $\mathcal{L}^p(\mathbb{R}^r)$ for every $p \in [1, \infty)$.

The proof of this theorem in [2] gives an explicit expression for the coefficients w_i in (3.13) as:

$$w_i = \frac{1}{\delta^r} f_c(\alpha_i) \left(\frac{2T}{n}\right)^r \frac{1}{\int_{\mathbb{R}^r} K(x)dx}, \tag{3.14}$$

where the set $\{\alpha_i \in \mathbb{R}^r : \ i = 1, ..., n^r\}$ consists of all points in $[-T, \ T]^r$ of the form

$$[-T + (2i_1T/n), ..., -T + (2i_rT/n)], \qquad i_1, i_2, .., i_r = 1, 2..., n,$$

in which T is a number such that $\sup(f_c) \subset [-T, \ T]^r$, where $\sup(\cdot)$ is defined as the supremum or least upper bound. This consequently implies that if $f \in \mathcal{L}^p_+(\mathbb{R}^r)$, then $f_c \in \mathcal{C}_{c+}(\mathbb{R}^r)$, and consequently the coefficients w_i in (3.14) are positive, i.e. $w_i > 0$. Let $\mathcal{S}_G$ be the subclass of functions from $\mathcal{S}_K$ with positive w_i's. Then $\mathcal{S}_G$ is dense in $\mathcal{L}^p_+(\mathbb{R}^r)$. We also notice that the Gaussians given by

$$\phi\left(\frac{x - x_{c_i}}{\sigma_i}\right) = \mathrm{e}^{-\frac{(x - x_{c_i})^2}{\sigma_i^2}},$$

where x_{c_i} is the center, while σ_i is the width parameter, represent one particular choice of K.

3.3 Preliminaries on Singular Perturbations Theory

Consider the problem of solving the system

$$\Sigma_0 : \begin{cases} \dot{x}(t) = f(t, x(t), u(t), \varepsilon), & x(0) = \xi(\varepsilon) \\ \varepsilon \dot{u}(t) = g(t, x(t), u(t), \varepsilon), & u(0) = \eta(\varepsilon), \end{cases} \tag{3.15}$$

where $\xi : \varepsilon \mapsto \xi(\varepsilon)$ and $\eta : \varepsilon \mapsto \eta(\varepsilon)$ are smooth. Additionally, assume that f and g are continuously differentiable in their arguments for $(t, x, u, \varepsilon) \in [0, \infty] \times D_x \times D_u \times [0, \varepsilon_0]$, where $D_x \subset \mathbb{R}^n$ and $D_u \subset \mathbb{R}^m$ are domains, $\varepsilon_0 > 0$. In addition, let Σ_0 be in *standard form*, that is,

$$0 = g(t, x, u, 0) \tag{3.16}$$

has $k \geq 1$ isolated real roots $u = h_i(t, x)$, $i \in \{1, \ldots, k\}$ for each $(t, x) \in [0, \infty] \times D_x$. Choosing a particular i and keeping it fixed, the subscript i is dropped. Let $\nu(t, x) = u - h(t, x)$.

In singular perturbations theory, the system given by

$$\Sigma_{00} : \dot{x}(t) \quad = \quad f(t, x(t), h(t, x(t)), 0), \quad x(0) = \xi(0), \tag{3.17}$$

is called the *reduced system*, and the system given by

$$\Sigma_b : \frac{d\nu}{d\tau} \quad = \quad g(t, x, \nu + h(t, x), 0), \quad \nu(0) = \eta_0 - h(0, \xi_0) \tag{3.18}$$

is called the *boundary layer system*, where $\eta_0 = \eta(0)$ and $\xi_0 = \xi(0)$, $(t, x) \in [0, \infty) \times D_x$ are treated as fixed parameters. The new time scale τ is related to the original time t via the relationship $\tau = \frac{t}{\varepsilon}$. The following result is due to Tikhonov [1], [5]:

Theorem 2 *Consider the singular perturbation system Σ_0 given in (3.15) and let $u = h(t, x)$ be an isolated root of (3.16). Assume that the following conditions are satisfied for all $[t, x, u - h(t, x), \epsilon] \in [0, \infty) \times D_x \times D_v \times [0, \epsilon_0]$ for some domains $D_x \subset \mathbb{R}^n$ and $D_v \subset \mathbb{R}^m$, which contain their respective origins:*

A1. On any compact subset of $D_x \times D_v$, the functions f, g, their first partial derivatives with respect to (x, u, ϵ), and the first partial derivative of g with respect to t are continuous and bounded, $h(t, x)$ and $\left[\frac{\partial g}{\partial u}(t, x, u, 0)\right]$ have bounded first derivatives with respect to their arguments, $\left[\frac{\partial f}{\partial x}(t, x, h(t, x))\right]$ is Lipschitz in x, uniformly in t, and the initial data given by ξ and η are smooth functions of ϵ.

A2. The origin is an exponentially stable equilibrium point of the reduced system Σ_{00} given by equation (3.17). There exists a Lyapunov function $V : [0,\infty) \times D_x \to [0,\infty)$ that satisfies

$$W_1(x) \leq V(t,x) \leq W_2(x)$$
$$\tfrac{\partial V}{\partial t}(t,x) + \tfrac{\partial V}{\partial x}(t,x) f(t,x,h(t,x),0) \leq -W_3(x)$$

for all $(t,x) \in [0,\infty) \times D_x$, where W_1, W_2, W_3 are continuous positive definite functions on D_x, and let c be a nonnegative number such that $\{x \in D_x \mid W_1(x) \leq c\}$ is a compact subset of D_x.

A3. The origin is an equilibrium point of the boundary layer system Σ_b given by equation (3.18), which is exponentially stable uniformly in (t,x).

Let $R_v \subset D_v$ denote the region of attraction of the autonomous system $\frac{dv}{d\tau} = g(0,\xi_0, v + h(0,\xi_0),0)$, and let Ω_v be a compact subset of R_v. Then for each compact set $\Omega_x \subset \{x \in D_x \mid W_2(x) \leq \rho c,\ 0 < \rho < 1\}$, there exists a positive constant ϵ_* such that for all $t \geq 0$, $\xi_0 \in \Omega_x$, $\eta_0 - h(0,\xi_0) \in \Omega_v$ and $0 < \epsilon < \epsilon_*$, Σ_0 has a unique solution x_ϵ on $[0,\infty)$ and

$$x_\epsilon(t) - x_{00}(t) = O(\epsilon)$$

holds uniformly for $t \in [0,\infty)$, where $x_{00}(t)$ denotes the solution of the reduced system Σ_{00} in (3.17).

The following Remarks of [1] will be useful for verifying Assumption A3:

Remark 1 *Verification of Assumption A3 can be done via a Lyapunov argument. If there exists a Lyapunov function $V : [0,\infty) \times D_x \times D_v$ that satisfies*

$$c_1\|\nu\|^2 \leq V(t,x,v) \leq c_2\|\nu\|^2 \tag{3.19}$$
$$\frac{\partial V}{\partial \nu} g(t,x,\nu + h(t,x)) \leq -c_3\|\nu\|^2, \tag{3.20}$$

for positive constants c_1, c_2, c_3, c_4 and for all $(t,x,v) \in [0,\infty) \times D_x \times D_v$, then Assumption A3 is satisfied.

Remark 2 *Assumption A3 can be locally verified by linearization. Let φ denote the map $\nu \mapsto g(t,\xi,\nu + h(t,\xi),\varepsilon)$. It can be shown that if there exists $\omega_0 > 0$ such that the Jacobian matrix $\frac{\partial \varphi}{\partial \nu}$ satisfies the eigenvalue condition*

$$\mathrm{Re}\left(\lambda\left[\frac{\partial \varphi}{\partial \nu}(t,x,h(t,x),0)\right]\right) \leq -\omega_0 < 0 \tag{3.21}$$

for all $(t, x) \in [0, \infty) \times D_x$*, then Assumption A3 is satisfied.*

Proof. The boundary layer system (3.18) can be written as a perturbation of its linearization at the equilibrium $\nu = 0$,

$$\frac{d\nu}{d\tau} = A(t,x)\nu + \underbrace{(g(t,x,\nu + h(t,x),0) - A(t,x)\nu)}_{\psi(t,x,\nu)}, \tag{3.22}$$

where

$$A(t,x) = \frac{\partial g}{\partial \nu}(t,x,h(t,x),0). \tag{3.23}$$

From Assumption A1 of Tikhonov's theorem, $\frac{\partial g}{\partial \nu}(t,x,\nu + h(t,x),0)$ and $g(t,x,\nu + h(t,x),0)$ are Lipshitz in ν so that the perturbation term $\psi(t,x,\nu)$ is also Lipshitz and satisfies

$$\|\psi(t,x,\nu)\|_p \leq k\|\nu\|_p^2\ , \quad \forall (t,x) \in [0,\infty) \times D_x, \tag{3.24}$$

where $\|\cdot\|_p$ denotes the p-norm. Assumption A1 implies that $A(t,x)$ is bounded for all time and (3.21) implies that the eigenvalues of $A(t,x)$ have negative real parts, so it can be concluded from Lemma 1 that the nominal system

$$\frac{d\nu}{d\tau} = A(t,x)\nu \tag{3.25}$$

is globally exponentially stable for all $(t,x) \in [0,\infty) \times D_x$ and the trajectories of the linear time-varying system (3.25) satisfy

$$\|v(t)\|_p \leq k_1\|v(0)\|_p\, e^{-\lambda_1 t}, \quad \forall\ v(0) \in D_\nu,\ x \in D_x,\ t \geq 0 \tag{3.26}$$

for positive constants k_1, λ_1.

The partial derivatives of $A(t,x)\nu$ with respect to its arguments are bounded as a result of Assumption A1 of Tikhonov's theorem and the time-varying nominal system (3.25) is globally exponentially stable. Hence, it follows then from applying Lemma 3, that there exists a Lyapunov function $V : [0,\infty) \times D_x \times D_\nu \to \mathbb{R}$ that satisfies the inequalities

$$\begin{aligned} c_1\|\nu\|_p^2 &\leq V(t,x,\nu) \leq c_2\|\nu\|_p^2 \\ \frac{\partial V}{\partial \nu}A(t,x)\nu &\leq -c_3\|\nu\|_p^2 \\ \left\|\frac{\partial V}{\partial \nu}(t,x,\nu)\right\|_p &\leq c_4\|\nu\|_p \end{aligned} \tag{3.27}$$

for some positive constants c_1, c_2, c_3, and c_4. Taking the τ derivative of the Lyapunov function V along the trajectory of (3.18) yields

$$\begin{aligned}\dot{V}(t,x,\nu) &= \left(\frac{\partial V}{\partial \nu}(t,x,\nu)\right) g(t,x,\nu+h(t,x),0) \\ &= \left(\frac{\partial V}{\partial \nu}(t,x,\nu)\right)(A(t,x)\nu+\psi(t,x,\nu)) \\ &\leq -c_3\|\nu\|_p^2 + c_4 k\|\nu\|_p^3 \\ &\leq -c_3\|\nu\|_p^2\end{aligned}$$

for $\|\nu\|_p \leq \frac{c_3}{c_4 k}$. This in turn implies

$$\dot{V}(t,x,\nu) \leq -\frac{c_3}{c_2}V(t,x,\nu), \tag{3.28}$$

and, consequently,

$$V(\tau) \leq V(0)\, e^{-\frac{c_3}{c_2}\tau}. \tag{3.29}$$

Using Lemma 2 (Comparison Lemma), it follows that

$$\|\nu(\tau)\|_p \leq \sqrt{\frac{c_2}{c_1}}\|\nu(0)\|_p\, e^{-\frac{c_3}{2c_2}\tau}, \quad \forall\ \|\nu(0)\| \leq \frac{c_3}{c_4 k}\sqrt{\frac{c_1}{c_2}},\ x\in D_x,\ \tau\geq 0 \tag{3.30}$$

for positive constants c_1, c_2, c_3, c_4, k. Thus, it can be concluded that Assumption A3 of Tikhonov's theorem on exponential stability of the time-varying boundary layer system at the origin can be verified locally by linearization.

□

Chapter 4

Adaptive Controller for Nonaffine Single Input System

4.1 Problem Formulation for Short Period Dynamics

Neglecting the influence of gravity and thrust, the short-period dynamics of a rigid aircraft performing high angle of attack maneuvers can be given as

$$\begin{cases} \dot{\alpha} \;=\; -\frac{L_\alpha(\alpha_0)}{V}\alpha + q - \frac{L_{\delta_e}(\alpha_0)}{V}\delta_e \\ \\ \dot{q} = M_0(\alpha) + M_q(\alpha)q + M_{\delta_e}(\alpha, \delta_e), \end{cases} \tag{4.1}$$

where q is the pitch rate, α is the angle of attack, α_0 is the trim angle, δ_e is the incremental elevator deflection, $L_\alpha(\alpha_0)$ is the known lift curve slope at α_0, $L_{\delta_e}(\alpha_0)$ is the known lift effectiveness due to elevator deflection, V is the trimmed airspeed, $M_q(\alpha)$ is the pitch damping, and $M_0(\alpha)$ and M_{δ_e} are the pitching moment components. From (4.1), it follows that the pitch dynamics depend on M_{δ_e}, which is nonlinear with respect to α and δ_e. Hence, the short-period dynamics of an aircraft at high angles of attack are nonlinear and nonaffine-in-control.

In general, the terms $M_0(\alpha), M_q(\alpha)$, and $M_{\delta_e}(\alpha, \delta_e)$ are unknown. However, some partial knowledge of aerodynamic stability and control derivatives are usually available from wind-tunnel experiments and theoretical predictions. Incorporating prior known data, the short-period dynamics of (4.1) can be rewritten as:

$$\begin{cases} \dot{\alpha} &= -\frac{L_\alpha(\alpha_0)}{V}\alpha + q - \frac{L_{\delta_e}(\alpha_0)}{V}\delta_e \\ \dot{q} &= M_\alpha(\alpha_0)\alpha + M_q(\alpha_0)q + M_{\delta_e}(\alpha_0,0)\delta_e + \\ & \underbrace{(M_0(\alpha) - M_\alpha(\alpha_0)\alpha)}_{\Delta M_0(\alpha)} + \underbrace{(M_q(\alpha) - M_q(\alpha_0))}_{\Delta M_q(\alpha)} q + \underbrace{(M_{\delta_e}(\alpha,\delta_e) - M_{\delta_e}(\alpha_0,0),\delta_e)}_{\Delta M_{\delta_e}(\alpha,\delta_e)}, \end{cases} \tag{4.2}$$

where it is assumed that $M_\alpha(\alpha_0), M_q(\alpha_0)$, and $M_{\delta_e}(\alpha_0, 0)$ are the known constant stability and control derivatives. Additionally, the lift derivative $L_{\delta_e}(\alpha_0)$ is known to be small with respect to airspeed V, so that for control design purposes, one can assume

$$\frac{L_{\delta_e}(\alpha_0)}{V} \approx 0, \tag{4.3}$$

which leads to the following model description:

$$\begin{cases} \dot{\alpha} = -\frac{L_\alpha}{V}\alpha + q \\ \dot{q} = M_\alpha\alpha + M_q q + M_{\delta_e}\delta_e + M_{\delta_e}\underbrace{(\Delta M_0(\alpha) + \Delta M_q(\alpha)q + \Delta M_{\delta_e}(\alpha,\delta_e))}_{f(\alpha,\delta_e)}. \end{cases} \tag{4.4}$$

This can be rewritten in state space form

$$\begin{aligned} \begin{bmatrix} \dot{\alpha}(t) \\ \dot{q}(t) \end{bmatrix} &= \underbrace{\begin{bmatrix} -\frac{L_\alpha}{V} & 1 \\ M_\alpha & M_q \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \alpha(t) \\ q(t) \end{bmatrix}}_{x(t)} + \underbrace{\begin{bmatrix} 0 \\ M_{\delta_e} \end{bmatrix}}_{b} (\delta_e(t) + f(\alpha(t), \delta_e(t))), \\ y(t) &= \underbrace{[1 \ \ 0]}_{c^\top} x(t), \end{aligned} \tag{4.5}$$

where $f(\alpha, \delta_e)$ is unknown to the controller. Based on wind-tunnel experiments and using curve-fitting methods, $f(\alpha, \delta_e)$ is known to have the structure

$$f(\alpha, \delta_e) \;=\; \left((1 - C_0)\,\mathrm{e}^{-\frac{\alpha^2}{2\sigma^2}} + C_0\right)(\tanh(\delta_e + h) + \tanh(\delta_e - h) + 0.01\delta_e), \tag{4.6}$$

which is well-defined for all $\alpha, \delta_e \in \mathbb{R}$ and continuously differentiable, where $\mathrm{e}^{-\frac{\alpha^2}{2\sigma^2}}$ is a Gaussian function with width σ and C_0, h are positive constants. A scaled plot demonstrating the structure of $f(\alpha, \delta_e)$ is shown in Figure 4.3. When the angle of attack is at the trim angle, then full elevator control effectiveness is obtained. If control is small, the control surface is within a boundary layer and does not produce an aerodynamic moment, so this is portrayed in the small elevator control induced moment term $0.01\delta_e$. As the control effort gets large, the surface stalls as the system reaches saturation and stops producing any additional moment which is approximated by the hyperbolic tangent functions. The control *objective* is to design elevator surface deflection

$$\delta_e(t) \;=\; \delta_{e_{nom}}(t) - \delta_{e_{ad}}(t) \tag{4.7}$$

so that the angle of attack $\alpha(t)$ tracks a commanded input $r^{cmd}(t)$, while all other signals in the system remain bounded. In (4.7), $\delta_{e_{nom}}(t)$ is the nominal control elevator deflection for the nominal (known) linear model, while $\delta_{e_{ad}}(t)$ is the incremental adaptive augmentation to compensate for unknown $f(\alpha(t), \delta_e(t))$.

4.2 Ideal Reference Model

To find the nominal control elevator for the short period aircraft dynamics (4.1), consider the nominal linear model of the longitudinal dynamics (4.5) in the absence of uncertainties $(f(\alpha(t), \delta_e(t)) = 0, \delta_{e_{ad}} = 0)$ and design an LQR controller to achieve the tracking objective: $\alpha(t) \to r^{cmd}(t)$ as $t \to \infty$ for a smooth $r^{cmd}(t)$. Towards that end, let

$$\begin{aligned} A_r &= \underbrace{\begin{bmatrix} -\frac{L_\alpha}{V} & 1 \\ M_\alpha & M_q \end{bmatrix}}_{A} - b \underbrace{\begin{bmatrix} k_{lqr_\alpha} & k_{lqr_q} \end{bmatrix}}_{k_{lqr}} \\ &= \begin{bmatrix} -\frac{L_\alpha}{V} & 1 \\ \\ \underbrace{M_\alpha - M_{\delta_e} k_{lqr_\alpha}}_{a^r_{2,1}} & \underbrace{M_q - M_{\delta_e} k_{lqr_q}}_{a^r_{2,2}} \end{bmatrix}, \end{aligned} \tag{4.8}$$

where the gain $k_{lqr} = -R^{-1} b^\top P$ is found by solving the corresponding Riccati equation for the unique positive definite symmetric matrix P:

$$Q + PA + A^\top P - PbR^{-1}b^\top P = 0, \tag{4.9}$$

in which Q is a positive definite matrix and R is a positive scalar control weight. This leads to the following form of the closed-loop reference system:

$$\begin{aligned} \begin{bmatrix} \dot{\alpha}_r(t) \\ \dot{q}_r(t) \end{bmatrix} &= A_r \begin{bmatrix} \alpha_r(t) \\ q_r(t) \end{bmatrix} + br(t), \qquad \alpha_r(0) = \alpha_{r_0}, \quad q_r(0) = q_{r_0}, \\ y_r(t) &= c^\top x_r(t). \end{aligned} \tag{4.10}$$

Due to the time-varying nature of the commanded reference input $r^{cmd}(t)$ and the definition of the reference model (4.10), setting $r(t) = r^{cmd}(t)$ will cause $y_r(t)$ to track $r^{cmd}(t)$ with bounded errors. To force $y_r(t)$ to track the commanded signal $r_{cmd}(t)$ asymptotically and guarantee zero steady state tracking error, define a new reference input $\tilde{r}^{cmd}(t)$ as

$$\tilde{r}^{cmd}(t) = \frac{1}{M_{\delta_e}} \left(\ddot{r}^{cmd}(t) - \dot{r}^{cmd}(t) \left(a^r_{2,2} - \frac{L_\alpha}{V} \right) - r^{cmd}(t) \left(a^r_{2,2} \frac{L_\alpha}{V} + a^r_{2,1} \right) \right). \tag{4.11}$$

The proof of this can be obtained by noticing that for the system (4.10), one can compute the transfer function $\frac{y_r(s)}{r(s)} = c(s\mathbb{I} - A_r)^{-1}\, b$, where $\mathbb{I}$ is the 2×2 identity matrix, s is the Laplace variable, and $y_r(s)$ and $r(s)$ are the Laplace transforms of $y_r(t)$ and $r(t)$ respectively, as

$$\frac{y_r(s)}{r(s)} = M_{\delta_e} \left(s^2 - s \left(a^r_{2,2} - \frac{L_\alpha}{V} \right) - \left(a^r_{2,2} \frac{L_\alpha}{V} + a^r_{2,1} \right) \right)^{-1}.$$

It is easy then to define

$$\tilde{r}^{cmd}(s) = \frac{1}{M_{\delta_e}} \left(s^2 - s \left(a^r_{2,2} - \frac{L_\alpha}{V} \right) - \left(a^r_{2,2} \frac{L_\alpha}{V} + a^r_{2,1} \right) \right) r^{cmd}(s), \tag{4.12}$$

whose representation in the time domain is given by (4.11), so that $y_r(s) = r^{cmd}(s)$, which results in asymptotic tracking $r^{cmd}(t)$. It follows then that the nominal system for the short-period aircraft dynamics (4.5) is

$$\begin{aligned} \begin{bmatrix} \dot{\alpha}_r(t) \\ \dot{q}_r(t) \end{bmatrix} &= A_r \underbrace{\begin{bmatrix} \alpha_r(t) \\ q_r(t) \end{bmatrix}}_{x_r(t)} + b\tilde{r}^{cmd}(t), \qquad \alpha_r(0) = \alpha_{r_0}, \quad q_r(0) = q_{r_0}, \\ y_r(t) &= c^\top x_r(t). \end{aligned} \tag{4.13}$$

The nominal elevator control input is then

$$\delta_{e_{nom}}(t) = -k_{lqr}^\top \begin{bmatrix} \alpha(t) \\ q(t) \end{bmatrix} + \tilde{r}^{cmd}(t), \tag{4.14}$$

which can be applied to the dynamics in (4.5) to yield

$$\begin{bmatrix} \dot{\alpha}(t) \\ \dot{q}(t) \end{bmatrix} = A_r \begin{bmatrix} \alpha(t) \\ q(t) \end{bmatrix} + b\tilde{r}^{cmd}(t) + b\left(-\delta_{e_{ad}}(t) + f(\,\alpha(t), \delta_{e_{nom}}(t) - \delta_{e_{ad}}(t)) \right). \tag{4.15}$$

4.3 RBF Approximation

In this section, we address approximation of monotonic nonlinear functions using the result from [2]. Towards that end, define

$$\begin{aligned} g(\alpha,\delta_e) &= \frac{\partial f}{\partial \delta_e}(\alpha,\delta_e) \\ &= \left((1-C_0)\ \mathrm{e}^{-\frac{\alpha^2}{2\sigma^2}} + C_0\right) \\ &\quad \left(2.001 - \tanh^2(\delta_e+h) - \tanh^2(\delta_e-h)\right) \\ &\geq 0.01C_0 \ > \ 0 \quad \text{for all } C_0 > 0. \end{aligned} \tag{4.16}$$

Using Leibnitz formula, $f(\alpha,\delta_e)$ can be expressed as

$$f(\alpha,\delta_e) = f(\alpha,0) + \int_0^{\delta_e} g(\alpha,\xi)\ d\xi, \tag{4.17}$$

where the integral term is strictly positive due to the positivity of $g(\alpha,\delta_e)$ defined in (4.16). Because $f(\alpha,0) \in \mathcal{C}_c(\mathbb{R})$ and $g(\alpha,\delta_e) \in \mathcal{C}_{c+}(\mathbb{R}^2)$, Theorem 1 can be applied to conclude that each component can be approximated by a linear combination of radial basis functions (RBFs) arbitrarily closely on a compact set. Consider a set of radial basis functions (RBF) $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by:

$$\phi(\alpha,\delta_e) = \phi_1(\alpha) + \phi_2(\alpha,\delta_e), \tag{4.18}$$

where

$$\phi_1(\alpha) \triangleq \theta^\top \Phi_1(\alpha) = \sum_{i=1}^{N} \theta_i\ \mathrm{e}^{-\frac{(\alpha-\alpha_{c_i})^\top(\alpha-\alpha_{c_i})}{\delta_i^2}}, \quad \delta_i > 0, \tag{4.19}$$

$$\phi_2(\alpha,\delta_e) \triangleq w^\top \Phi_2(\alpha,\delta_e) = \sum_{j=1}^{M} w_j \left(\int_0^{\delta_e} \mathrm{e}^{-\frac{(\chi-\chi_{c_j})^\top(\chi-\chi_{c_j})}{\rho_j^2}} d\xi \right), \tag{4.20}$$

$$\chi \triangleq [\alpha^\top\ \xi]^\top,\ \rho_j > 0,$$

and M, N are positive constants, $\phi_1(\alpha)$ is the vector of Gaussians dependent only on α, and $\phi_2(\alpha,\delta_e)$ is the vector of the integrals of the Gaussians dependent on both α and δ_e. Since $g(\alpha,\delta_e) > 0$, then following (3.14), the unknown coefficients w_i are positive. The vectors $\alpha_{c_i},\ i = 1,...,N,\ \chi_{c_j} = [\alpha_{c_j}^\top\ \ \delta_{e_{c_j}}]^\top,\ j = 1,...,M$ represent the fixed centers of the RBFs,

δ_i, ρ_j are the fixed widths of the basis functions respectively, and the unknown parameters are $\theta_i \in \mathbb{R}$ and $w_j \in \mathbb{R}^+$. It is easy to see from (4.18) - (4.20) that

$$\text{sgn}\left(\frac{\partial \phi}{\partial \delta_e}(\alpha, \delta_e)\right) = \text{sgn}\left(\frac{\partial \phi_2}{\partial \delta_e}(\alpha, \delta_e)\right) > 0. \tag{4.21}$$

Then,

$$\begin{aligned} f(\alpha, \delta_e) &= f(\alpha, 0) + \int_0^{\delta_e} g(\alpha, \xi)\ d\xi \\ &= W^\top \Phi(\alpha, \delta_e) + \varepsilon(\alpha, \delta_e), \qquad \text{for } (\alpha, \delta_e) \in \Omega_\alpha \times \Omega_{\delta_e}, \quad |\varepsilon(\alpha, \delta_e)| \leq \varepsilon^*, \end{aligned} \tag{4.22}$$

where $g(\alpha, \delta_e)$ is given in (4.16), $W^\top = [\theta^\top \ w^\top]^\top$, $\Phi(\alpha, \delta_e) = [\Phi_1^\top(\alpha) \quad \Phi_2^\top(\alpha, \delta_e)]^\top$, $w_i > 0$, $\varepsilon(\alpha, \delta_e)$ is the uniformly bounded approximation error, $\Omega_\alpha, \Omega_{\delta_e}$ are compact sets of $\mathbb{R}$, and ε^* is a constant.

4.4 State Predictor and Adaptive Law

A state predictor for the system (4.15) can be constructed as:

$$\begin{aligned} \begin{bmatrix} \dot{\hat{\alpha}}(t) \\ \dot{\hat{q}}(t) \end{bmatrix} &= A_r \begin{bmatrix} \hat{\alpha}(t) \\ \hat{q}(t) \end{bmatrix} + b\tilde{r}^{cmd}(t) + b\left(-\delta_{e_{ad}}(t) + \hat{W}^\top(t)\Phi(\alpha(t), \delta_e(t))\right), \\ \hat{\alpha}(0) &= \hat{\alpha}_0, \quad \hat{q}(0) = \hat{q}_0, \end{aligned} \tag{4.23}$$

where A_r is defined in (4.8), $\tilde{r}^{cmd}$ is given in (4.11), and $\hat{W}(t)$ is used to estimate the unknown constant vector W. The prediction error signal $e_s(t)$ is defined as

$$e_s(t) = \begin{bmatrix} \hat{\alpha}(t) - \alpha(t) \\ \hat{q}(t) - q(t) \end{bmatrix}. \tag{4.24}$$

Prediction error dynamics can be written as

$$\dot{e}_s(t) = A_r e_s(t) + b\left(\tilde{W}^\top(t)\Phi(\alpha(t), \delta_e(t)) - \varepsilon(\alpha(t), \delta_e(t))\right), \quad e_s(0) = e_{s_0}, \tag{4.25}$$

and $\tilde{W}(t) = \hat{W}(t) - W$. The following adaptive law

$$\dot{\hat{W}}(t) = \Gamma \ \text{Proj}\left(\hat{W}(t), -\Phi(\alpha(t), \delta_e(t)\ e_s^\top(t)\ P_0 B\right), \qquad \hat{W}(0) = W_0, \tag{4.26}$$

where Γ is a positive definite matrix of adaptation rates, Proj $(\cdot\ ,\ \cdot)$ denotes the projection operator, and P_0 is the solution to the Lyapunov equation $A_r^\top P_0 + P_0 A_r = -Q_0$ for a positive definite matrix Q_0, ensures that the prediction error dynamics in (4.25) and the parameter estimation error $\tilde{W}(t)$ are bounded independent of $\delta_e(t)$. Indeed, consider the following Lyapunov function candidate:

$$V(e_s(t), \tilde{W}(t)) \;=\; e_s^\top(t)P_0e_s(t) + \tilde{W}^\top(t)\Gamma^{-1}\tilde{W}(t). \tag{4.27}$$

Taking the time derivative along the trajectories (4.25), (4.26) gives

$$\begin{aligned}\dot{V}(t) &= -e_s^\top(t)Q_0e_s(t) + 2e_s^\top(t)P_0B\varepsilon(t)\\ &\leq -\|e_s\|\lambda_{\min}(Q_0)\|e_s\| + 2\|e_s\|\;\|P_0B\|\varepsilon^*,\end{aligned}$$

where $\lambda_{\min}(Q_0)$ denotes the minimum eigenvalue of Q_0. The Projection Operator ensures boundedness of parameter errors so that outside the compact set

$$\left\{\|e_s\| \leq 2\frac{\varepsilon^*\|P_0B\|}{\lambda_{\min}(Q_0)}\right\}\bigcap\left\{\|\tilde{W}\| \leq W^*\right\}, \tag{4.28}$$

where W^* is the maximum allowable upper-bound selected by the Projection operator and $\|\cdot\|$ indicates the 2-norm, the derivative of the Lyapunov function (4.27) is negative definite. Following standard invariant set arguments, it can be concluded that if the initial errors are within the largest level set of the Lyapunov function for which the RBF approximation has been defined, then the error dynamics (4.25) are ultimately bounded with respect to $e_s(t)$ and $\tilde{W}(t)$. Furthermore, this ultimate bound ζ can be any number larger than the value of the Lyapunov function on the minimum level set embracing the compact set in (4.28). That is,

$$\begin{aligned}\|\zeta\| &> \left\|V\left(2\frac{\varepsilon^*\|P_0B\|}{\lambda_{\min}(Q_0)}, W^*\right)\right\|\\ &= 4\frac{(\varepsilon^*)^2\|P_0B\|^2\lambda_{\max}(P_0)}{\lambda_{\min}^2(Q_0)} + (W^*)^2\|\Gamma^{-1}\|,\end{aligned}$$

where V is the Lyapunov function (4.27) and $\lambda_{\max}(P_0)$ denotes the maximum eigenvalue of P_0.

Remark 3 *Notice that since w_i are positive, the compact set in the application of the Projection operator can be selected so that $\hat{w}_i(t)$ remain positive for all $t \geq 0$, that is, $\hat{w}_i(t) > \hat{w}_{i_0} > 0$ for all $i = 1, ..., M$.*

4.5 Nonaffine Control Design

Define the tracking error between the predictor of (4.23) and the reference system (4.13) as

$$e(t) = \begin{bmatrix} \hat{\alpha}(t) - \alpha_r(t) \\ \hat{q}(t) - q_r(t) \end{bmatrix}. \tag{4.29}$$

The error dynamics are then:

$$\dot{e}(t) = A_r e(t) + b\left(-\delta_{e_{ad}}(t) + \hat{W}^\top(t)\Phi(\alpha(t), \delta_e(t))\right), \quad e(0) = e_0.$$

Dynamic inversion based control is to be determined from the solution of

$$\delta_{e_{ad}}(t) = \hat{W}^\top(t)\Phi(\alpha(t), \delta_{e_{nom}}(t) - \delta_{e_{ad}}(t), \tag{4.30}$$

so that the resulting closed-loop error dynamics are asymptotically stable, i.e. $\dot{e}(t) = A_r e(t)$. Due to the nonaffine nature of the longitudinal dynamics, (4.30) cannot be solved explicitly for $\delta_{e_{ad}}$. We construct fast dynamics to determine the solution for $\delta_{e_{ad}}$:

$$\epsilon\, \dot{\delta}_{e_{ad}} = -\text{sign}\left(\frac{\partial \mathbf{f}}{\partial \delta_{e_{ad}}}\right) \mathbf{f}(t, e, \delta_{e_{ad}}), \qquad \delta_{e_{ad}}(0) = \delta_{e_{ad_0}}, \tag{4.31}$$

where $0 < \epsilon \ll 1$ and

$$\mathbf{f}(t, e, \delta_{e_{ad}}) = \delta_{e_{ad}} - \hat{W}^\top(t)\Phi(\alpha(t), \delta_{e_{nom}}(t) - \delta_{e_{ad}}).$$

As an extension of Tikhonov's theorem, recall the main theorem of [13]:

Theorem 3 *Assume that the following conditions are satisfied for all* $[t, e, u - h(t, e), \epsilon] \in [0, \infty) \times \mathbb{R}^2 \times \mathbb{R} \times [0, \epsilon_0]$*:*

B1. On any compact subset of $\mathbb{R}^2 \times \mathbb{R}$*, the function* $\mathbf{f}$ *and its first partial derivative with respect to* (t, e, u) *are continuous and bounded,* $h(t, e)$ *and* $\frac{\partial \mathbf{f}}{\partial u}(t, e, u)$ *have bounded first derivatives with respect to their arguments, and* $\frac{\partial \mathbf{f}}{\partial e}$ *as a function of* $(t, e, h(t, e))$ *is Lipshitz in* e*, uniformly in* t*.*

B2. $(t, e, \nu) \mapsto \frac{\partial \mathbf{f}}{\partial u}(t, e, \nu + h(t, e))$ *is bounded below by some positive number for all* $(t, e) \in [0, \infty) \times \mathbb{R}^2$*.*

Then the origin of (4.31) is exponentially stable. Moreover, let Ω_ν be a compact subset of R_ν, where $R_\nu \subset \mathbb{R}$ denotes the region of attraction of the autonomous system

$$\frac{d\nu}{d\tau} = -\text{sign}\left(\frac{\partial \mathbf{f}}{\partial \delta_{e_{ad}}}\right)\mathbf{f}(0, e_0, \nu + h(0, e_0)). \tag{4.32}$$

The for each compact subset $\Omega_e \subset \mathbb{R}^2$, there exists a positive constant, ϵ and a $T > 0$ such that for all $t \geq 0$, $e_0 \in \Omega_e$, $u_0 - h(t, e_0) \in \Omega_{nu}$ and $0 < \epsilon < \epsilon^*$, the system of equations (4.23), (4.31 - 4.32) has a unique solution $\hat{\alpha}_\epsilon(t), \hat{q}_\epsilon(t)$ on $[0, \infty)$, and

$$\begin{bmatrix} \hat{\alpha}(t) \\ \hat{q}(t) \end{bmatrix} = \begin{bmatrix} \alpha_r(t) \\ q_r(t) \end{bmatrix} + O(\epsilon).$$

Let $\delta_{e_{ad}}(t) = h(t, e)$ be an isolated root of $\mathbf{f}(t, e, \delta_{e_{ad}}) = 0$, $\nu(t, \alpha) = \delta_{e_{ad}}(t) - h(t, \alpha)$, and $\tau = \frac{t}{\epsilon}$. The existence of such a root is guaranteed by noting that there must exist a point of intersection between $\hat{W}^\top(t)\Phi(\alpha(t), \delta_{e_{nom}}(t) - \delta_{e_{ad}}(t))$, a set of Gaussians that changes amplitude depending on the value of the positive multipliers $\hat{W}^\top(t)$, and $\delta_{e_{ad}}(t)$. An illustration of this is shown in Figure 4.1.

The *reduced system* for (4.29) is

$$\dot{e}(t) = A_r e(t), \quad e(0) = e_0,$$

and the *boundary layer* system is

$$\frac{d\nu}{d\tau} = -\text{sign}\left(\frac{\partial \mathbf{f}}{\partial \delta_{e_{ad}}}\right)\mathbf{f}(t, e, \nu + h(t, e)), \qquad \nu(0) = \delta_{e_{ad_0}} - h(t, e_0). \tag{4.33}$$

To apply Tikhonov's theorem of (Theorem 2), it is necessary to show that all of its assumptions are satisfied. A1 is satisfied by noting that the function $\mathbf{f}$ and its partial derivatives with respect to $(t, e, \delta_{e_{ad}})$ are continuous and bounded, $h(t, e)$ $\frac{\partial \mathbf{f}}{\partial \delta_{e_{ad}}}(t, e, \delta_{e_{ad}})$ has bounded first derivatives with respect to its arguments, and $\frac{\partial \mathbf{f}}{\partial e}(t, e, h(t, e))$ is Lipshitz in e, uniformly in t. Because the matrix A_r is Hurwitz by choice, the origin of the reduced system (4.33) is an exponentially stable equilibrium point and there exists a Lyapunov function that satisfies

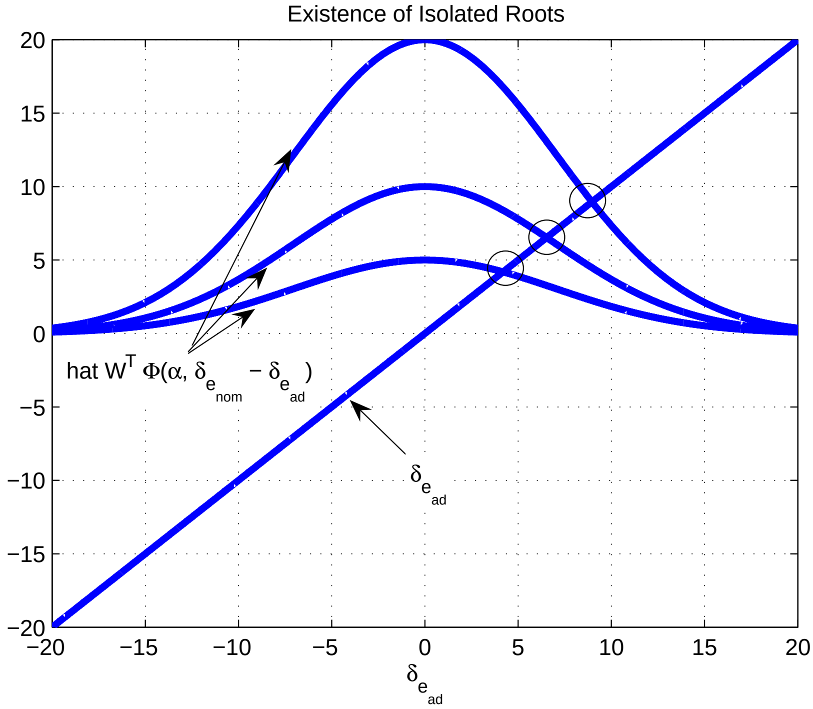

Figure 4.1: Intersection of the functions $\delta_{e_{ad}}(t)$ and $\hat{W}^{\top}(t)\Phi(\alpha(t), \delta_{e_{nom}}(t) - \delta_{e_{ad}}(t))$.

the inequalities of A2. To satisfy A3, note from (4.18) - (4.21) that

$$\begin{aligned}
\frac{\partial \mathbf{f}}{\partial \delta_{e_{ad}}} &= 1 - \frac{\partial \hat{W}^\top \Phi(\alpha, \delta_{e_{nom}} - \delta_{e_{ad}})}{\partial \delta_{e_{ad}}} \\
&= 1 - \frac{\partial \hat{w}^\top \Phi_2(\alpha, \delta_{e_{nom}} - \delta_{e_{ad}})}{\partial \delta_{e_{ad}}} \\
&= 1 - \frac{\partial \left(\sum_{j=1}^{M} \hat{w}_j \left(\int_0^{\delta_{e_{nom}} - \delta_{e_{ad}}} \mathrm{e}^{-\frac{(\chi - \chi_{c_j})^\top (\chi - \chi_{c_j})}{\rho_j^2}} d\xi \right) \right)}{\partial \delta_{e_{ad}}} \\
&= 1 + \sum_{j=1}^{M} \hat{w}_j \left(\mathrm{e}^{-\frac{(\chi - \chi_{c_j})^\top (\chi - \chi_{c_j})}{\rho_j^2}} \right) > b_0 > 0,
\end{aligned} \tag{4.34}$$

where $j = 1, \ldots, M$, $\hat{w}_j$'s are positive (Remark 3), and $M, \rho_j, \chi, \chi_{c_j}$ are introduced in (4.20). Hence, linearization of the boundary layer system (4.33) with respect to ν implies that the boundary layer system has locally exponentially stable origin. Following Remark 2, Assumption A3 is satisfied. The complete controller consists of (4.23), (4.26), (4.30), (4.31) and Tikhonov's theorem ensures that for the system of equations given by (4.23), (4.31), there exists a unique solution $\hat{\alpha}, \hat{q}$ that tracks α_r, q_r, and

$$\begin{bmatrix} \hat{\alpha}(t) \\ \hat{q}(t) \end{bmatrix} = \begin{bmatrix} \alpha_r(t) \\ q_r(t) \end{bmatrix} + O(\epsilon).$$

4.6 Simulations

Consider the short-period dynamics of an F-16 model trimmed at an airspeed of 502 ft/sec and angle of attack $\alpha = 2.11°$. From [15], the nominal system data are:

$$A = \begin{bmatrix} -1.0190 & 1 \\ 0.8223 & -1.0774 \end{bmatrix}, \qquad b = \begin{bmatrix} 0 \\ -0.1756 \end{bmatrix}, \tag{4.35}$$

which leads to open-loop system eigenvalues $\lambda_1 = -1.9554$, $\lambda_2 = -0.1409$. The commanded reference input of interest to track is

$$r^{cmd}(t) = -\frac{0.5}{1 + e^{t-8}} + \frac{1}{1 + e^{t-30}} - 0.5.$$

The Ricatti equation is solved with the following weighting matrices:

$$Q = \begin{bmatrix} 8 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad R = 0.01, \tag{4.36}$$

which leads to the following LQR gains: $k_{lqr} = [-16.8696 \quad -10.5911]$. The new reference input following (4.11) is defined as

$$\tilde{r}(t) = -5.6948\ddot{r}^{cmd}(t) - 22.5291\dot{r}^{cmd}(t) - 29.2296r^{cmd}(t), \tag{4.37}$$

so that the reference model for the nominal values of the aircraft system is

$$\dot{x}_r(t) = \underbrace{\begin{bmatrix} -1.0190 & 1 \\ -2.1400 & -2.9372 \end{bmatrix}}_{A_r} x_r(t) + b\tilde{r}(t),$$

having its closed loop eigenvalues at $\lambda_{lqr} = -1.9781 \pm 1.1045i$. The plot in Figure 4.4 shows the tracking result of the affine system in the absence of uncertainties for initial conditions $\alpha_{r_0} = 2.11°, q_{r_0} = 0°$. Next, consider when $f(\alpha(t), \delta_e(t))$ cannot be neglected and is unknown. For simulation purposes, the parameter σ is chosen as 0.15 so that the Gaussian has a width of 60° (see Figure 4.2), a range that contains both high and low angles of attack. The constant C_0 is set to 0.3 so that at high angles of attack, a maximum of 30% control effectiveness can be achieved, and the constant $h = 3$. Figure 4.3 demonstrates the complete structure of $f(\alpha, \delta_e)$. As seen from Figure 4.5, the performance of LQR is violated in the presence of uncertainties and adaptive control must be implemented to restore desired system behavior. The nonlinearity $|f(\alpha, \delta_e)| \leq 0.9$ causes a 6° loss of tracking in α and an 8° loss of tracking in q seen in Figure 4.6.

The state predictor is designed with 12 RBFs. Of these, 4 are $\Phi_1(\alpha)$-type Gaussians distributed over $\alpha \in [-30°, 30°]$ with the step size equal to 60° and the width $\delta = 1$. The remaining 8 $\Phi_2(\alpha, \delta_e)$-type basis functions were distributed over $\alpha \in [-30°, 30°]$, $\delta_{e_{ad}} \in [-30°, 30°]$ with step size 60° and width $\rho = 5$. The norm upper bound is $W^* = 10$, the lower bound for the positive widths w is 0.01, and the adaptation gain is $\Gamma = 0.2$. The following state predictor is implemented:

$$\begin{bmatrix} \dot{\hat{\alpha}}(t) \\ \dot{\hat{q}}(t) \end{bmatrix} = A_r \begin{bmatrix} \hat{\alpha}(t) \\ \hat{q}(t) \end{bmatrix} + b\left(-\delta_{e_{ad}}(t) + \hat{\theta}^\top(t)\Phi_1(\alpha(t)) + \hat{w}^\top(t)\Phi_2(\alpha(t), \delta_e(t))\right).$$

Then:

$$\dot{e}(t) = A_r e(t) + b\left(-\delta_{e_{ad}}(t) + \hat{\theta}^\top(t)\Phi_1(\alpha(t)) + \hat{w}^\top(t)\Phi_2(\alpha(t), \delta_e(t))\right).$$

To achieve the desired performance, the following equation needs to be solved for adaptive elevator control, $\delta_{e_{ad}}$:

$$\delta_{e_{ad}}(t) = \hat{\theta}^\top(t)\Phi_1(\alpha(t)) + \hat{w}^\top(t)\Phi_2(\alpha(t), \delta_{e_{nom}}(t) - \delta_{e_{ad}}(t)),$$

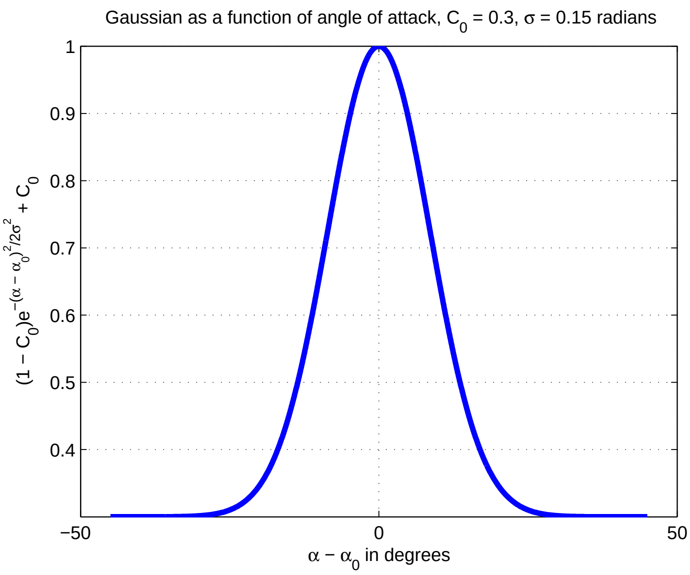

Figure 4.2: Gaussian as a function of α with an angle of attack range of 60°.

where $\delta_{nom}(t)$ is defined in (4.14). Obviously, this cannot be solved in terms of analytical functions.

The fast dynamics are designed as:

$$-0.05\dot{\delta}_{e_{ad}}(t) = \delta_{e_{ad}}(t) - \hat{\theta}^\top(t)\Phi_1(\alpha(t)) - \hat{w}^\top(t)\Phi_2(\alpha(t), \delta_{e_{nom}}(t) - \delta_{e_{ad}}(t)).$$

Initial conditions used for the simulations are:

$$\begin{aligned}[\alpha(0) \quad q(0)] &= [2.11^\circ \quad 0^\circ] \\ [\hat{\alpha}(0) \quad \hat{q}(0)] &= [2.11^\circ \quad 0^\circ].\end{aligned}$$

The plots are given in in Figures 4.7,4.8,4.9. Figure 4.7 shows the closed-loop tracking performance of the commanded reference input $r^{cmd}(t)$ by the estimator state $\hat{\alpha}(t)$, the reference state $\alpha_r(t)$ and the actual state $\alpha(t)$. Figure 4.8 shows the closed-loop tracking performance of the reference state $q_r(t)$ by the estimator state $\hat{q}(t)$ and the actual state $q(t)$, and Figure 4.9 shows the adaptive control effort $\delta_{e_{ad}}(t)$ which closely matches the unknown nonlinearity, $f(\alpha(t), \delta_e(t))$ as a function of time. We note the convergence ability of the predictor to the system states and the asymptotic tracking of $r^{cmd}(t)$ by α. All the states and control input remain in the domain of the RBF approximation.

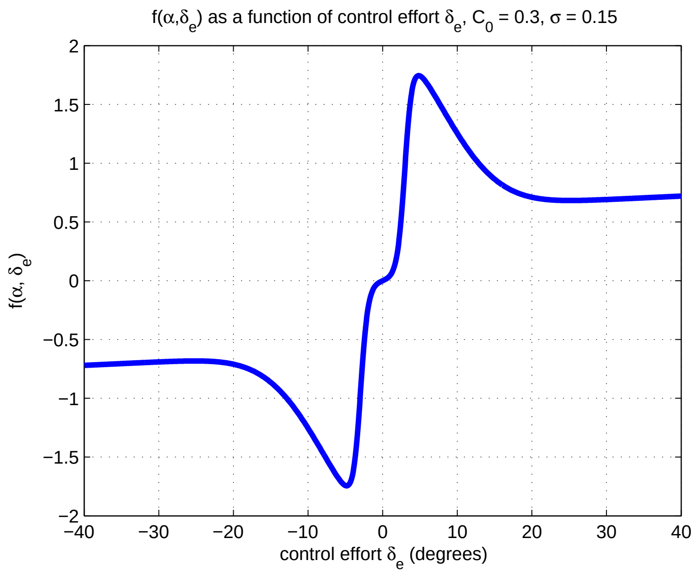

Figure 4.3: $f(\alpha, \delta_e)$ as a function of elevator control δ_e.

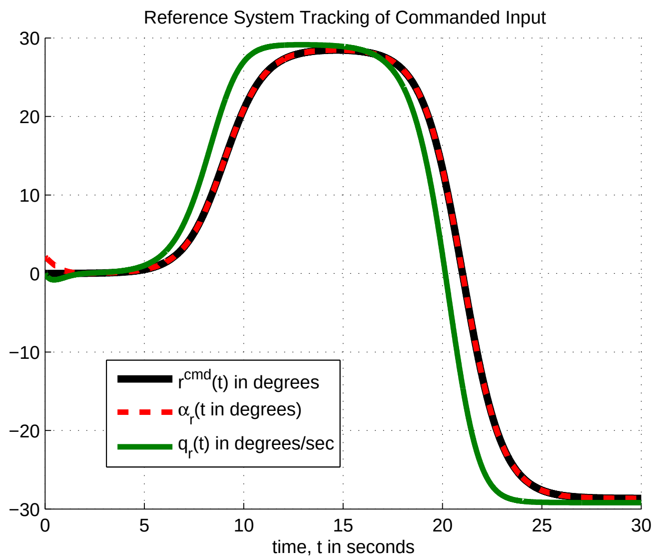

Figure 4.4: Short-period nominal system tracking with respect to defined commanded input $r^{cmd}(t)$.

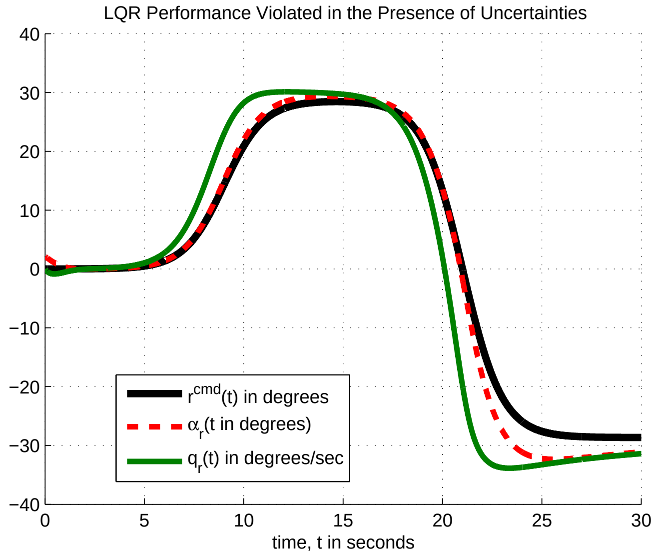

Figure 4.5: LQR performance violated in the presence of uncertainties.

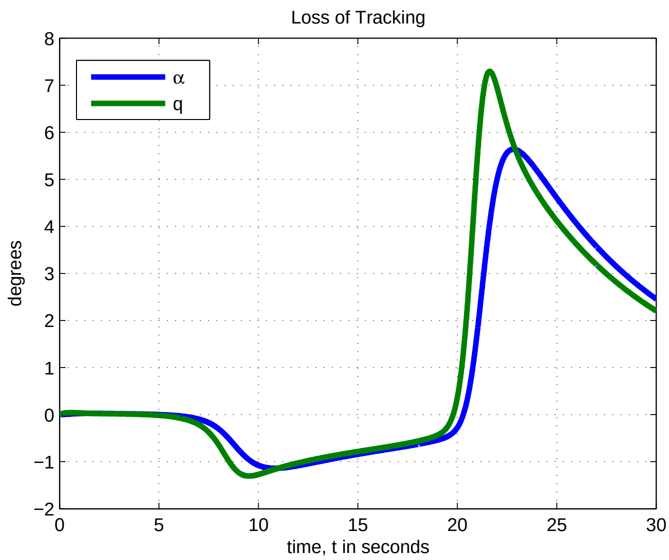

Figure 4.6: Loss of tracking of α and q caused by nonlinearity $f(\alpha, \delta_e)$ of order $O(0.9)$.

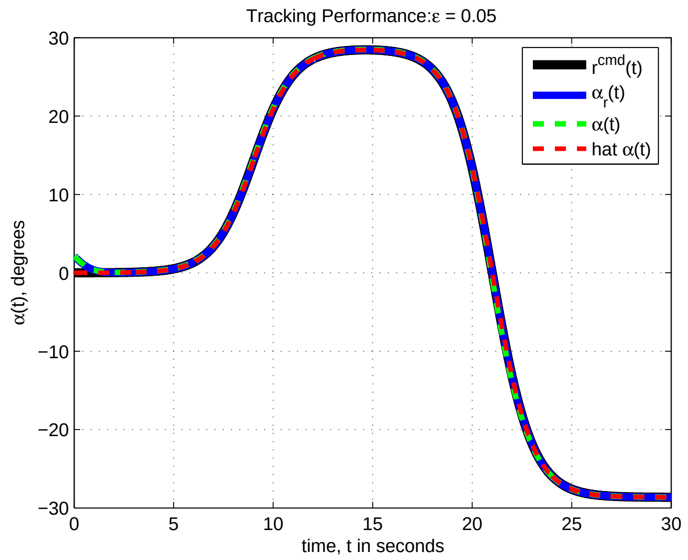

Figure 4.7: States $\hat{\alpha}(t), \alpha_r(t), \alpha(t)$ of short-period system.

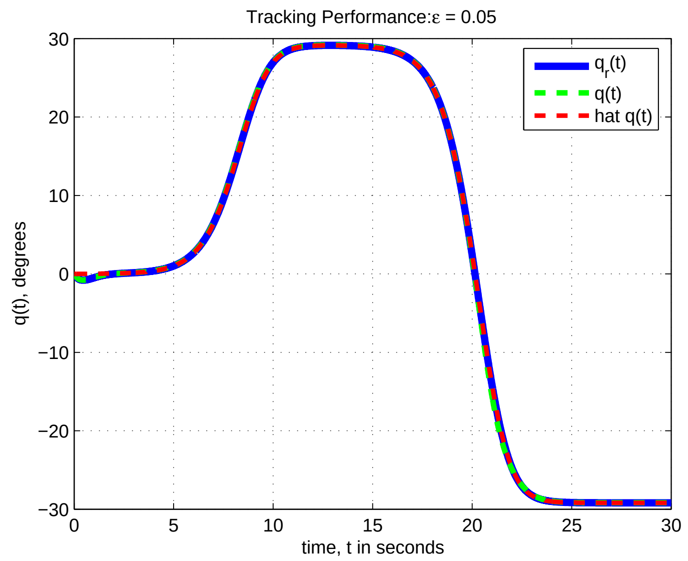

Figure 4.8: States $\hat{q}(t), q_r(t), q(t)$ of short-period system.

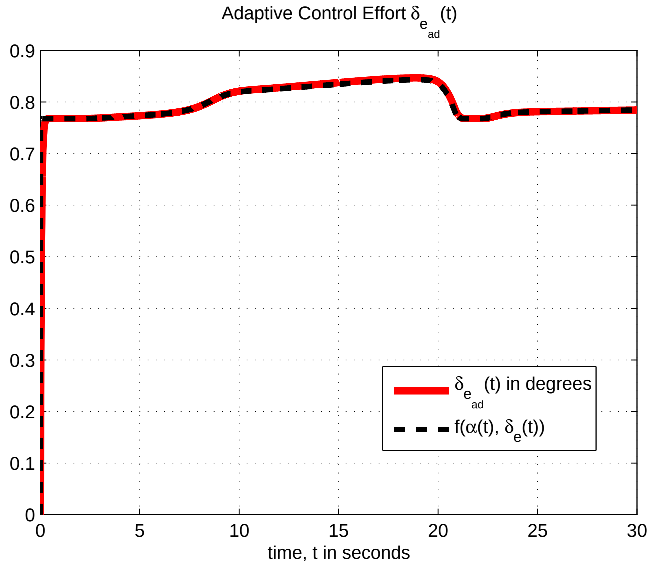

Figure 4.9: Adaptive elevator control, $\delta_{e_{ad}}(t)$of short-period system versus unknown nonlinearity $f(\alpha(t), \delta_e(t))$.

Chapter 5

Adaptive Controller for Nonaffine Multi-Input System

5.1 Problem Formulation for Dutch-Roll Dynamics

From the six degree-of-freedom flat-earth, body-axis aircraft model, the kinematic Euler roll rate equation is known to be

$$\dot{\phi} = p + \tan\theta(q\sin\phi + r\cos\phi), \tag{5.1}$$

where p, q, r are the body axis roll, pitch and yaw rates respectively, θ is the pitch angle, and ϕ is the roll angle. Denote the trimmed pitch angle as θ_0. For small roll angle ϕ, the following holds:

$$\dot{\phi} = p + r\tan\theta_0. \tag{5.2}$$

For a trimmed angle of attack α_0, one can write

$$\begin{aligned} p_s &= p\cos\alpha_0 + r\sin\alpha_0 \\ r_s &= r\cos\alpha_0 - p\sin\alpha_0, \end{aligned} \tag{5.3}$$

where p_s, r_s are the stability axis roll and yaw rates. Solving (5.3) for p, r and directly substituting into (5.2) yields

$$\dot{\phi} = (\cos\alpha_0 + \sin\alpha_0\tan\theta_0)p_s + (\cos\alpha_0\tan\theta_0 - \sin\alpha_0)r_s. \tag{5.4}$$

An additional relationship between flight path angle γ, pitch angle, and angle of attack at zero bank angle and zero angle of sideslip (AOS) is

$$\alpha_0 = \theta_0 - \gamma_0, \tag{5.5}$$

which can be substituted into (5.4) to get

$$\dot{\phi} = \frac{\cos\gamma_0}{\cos\theta_0} p_s + \frac{\sin\gamma_0}{\cos\theta_0} r_s. \tag{5.6}$$

The angle of sideslip dynamics in stability axis assuming small angles has the form

$$\dot{\beta} = \frac{1}{V}\left(Y_\beta \beta + Y_r r_s + Y_{\delta_a}\delta_a + Y_{\delta_r}\delta_r\right) + \left(\frac{g\cos\theta_0}{V}\right)\phi - r_s, \tag{5.7}$$

where V is the true airspeed, g is the gravitational constant, and δ_a, δ_r are aileron and rudder control. The terms $Y_\beta, Y_r Y_{\delta_a}, Y_{\delta_r}$ are aerodynamic stability and control derivatives that change slowly with time and can be adequately approximated by constants. Additionally, $Y_{\delta_a}, Y_{\delta_r}$ are known to be small with respect to true airspeed V, so that for control design purposes,

$$\frac{Y_{\delta_a}}{V} \approx \frac{Y_{\delta_r}}{V} \approx 0. \tag{5.8}$$

The full roll-yaw dynamics expressed in stability axis are

$$\begin{cases} \dot{\phi} = \frac{\cos\gamma_0}{\cos\theta_0} p_s + \frac{\sin\gamma_0}{\cos\theta_0} r_s \\ \dot{\beta} = \frac{Y_\beta}{V}\beta + \frac{Y_r}{V} r_s + \frac{g\cos\theta_0}{V}\phi - r_s \\ \dot{p}_s = L_\beta \beta + L_p p_s + L_r r_s + \delta_l(p_s, r_s) + L_{\delta_a}(\beta, \delta_a) + L_{\delta_r}(\beta, \delta_r) \\ \dot{r}_s = N_\beta \beta + N_p p_s + N_r r_s + \delta_n(p_s, r_s) + N_{\delta_a}(\beta, \delta_a) + N_{\delta_r}(\beta, \delta_r), \end{cases} \tag{5.9}$$

where δ_l and δ_n are incremental rolling moment and yawing moment that are generally *unknown* functions of the rates p_s, q_s. Additionally, $L_{\delta_a}, L_{\delta_r}, N_{\delta_a}, N_{\delta_r}$ are rolling and yaw moments due to aileron and rudder deflection that are also *unknown*. However, some partial knowledge is assumed from flight tests so that $L_{\delta_a}, L_{\delta_r}, N_{\delta_a}, N_{\delta_r}$ can be decomposed into a known (nominal) linear part and an unknown nonlinear part as follows

$$\begin{aligned} L_{\delta_a}(\beta, \delta_a) &= L_{\delta_{a_0}}\delta_a + L_{\delta_a}(\beta, \delta_a) \\ L_{\delta_r}(\beta, \delta_r) &= L_{\delta_{r_0}}\delta_r + L_{\delta_r}(\beta, \delta_r) \\ N_{\delta_a}(\beta, \delta_a) &= N_{\delta_{a_0}}\delta_a + N_{\delta_a}(\beta, \delta_a) \\ N_{\delta_r}(\beta, \delta_r) &= N_{\delta_{r_0}}\delta_r + N_{\delta_r}(\beta, \delta_r). \end{aligned} \tag{5.10}$$

Additionally, the nonlinear expressions $\delta_l(p_s, r_s) + L_{\delta_a}(\beta, \delta_a) + L_{\delta_r}(\beta, \delta_r)$ and $\delta_n(p_s, r_s) + N_{\delta_a}(\beta, \delta_a) + N_{\delta_r}(\beta, \delta_r)$ can be expressed as a linear combination of two unknown functions of $\beta, p_s, r_s, \delta_a, \delta_r$. Substituting (5.10), we get that

$$\begin{aligned}
\delta_l(p_s, r_s) + L_{\delta_a}(\beta, \delta_a) + L_{\delta_r}(\beta, \delta_r) &= L_{\delta_{a_0}}(\delta_a + f_1(\beta, p_s, r_s, \delta_a)) + L_{\delta_{r_0}}(\delta_r + f_2(\beta, p_s, r_s, \delta_r)) \\
\delta_n(p_s, r_s) + N_{\delta_a}(\beta, \delta_a) + N_{\delta_r}(\beta, \delta_r) &= N_{\delta_{a_0}}(\delta_a + f_1(\beta, p_s, r_s, \delta_a)) + N_{\delta_{r_0}}(\delta_r + f_2(\beta, p_s, r_s, \delta_r)).
\end{aligned}$$

Substituting this expression into (5.9) and re-writing the dynamics in state space form gives

$$\begin{aligned}
\begin{bmatrix} \dot\beta(t) \\ \dot\phi(t) \\ \dot p_s(t) \\ \dot r_s(t) \end{bmatrix} &= \underbrace{\begin{bmatrix} \frac{Y_\beta}{V} & \frac{g\cos\theta_0}{V} & \frac{Y_p}{V} & -(1-\frac{Y_r}{V}) \\ 0 & 0 & \frac{\cos\gamma_0}{\cos\theta_0} & \frac{\sin\gamma_0}{\cos\theta_0} \\ L_\beta & 0 & L_p & L_r \\ N_\beta & 0 & N_p & N_r \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \beta(t) \\ \phi(t) \\ p_s(t) \\ r_s(t) \end{bmatrix}}_{x(t)} + \\
&\quad \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ L_{\delta_{a_0}} & L_{\delta_{r_0}} \\ N_{\delta_{a_0}} & N_{\delta_{r_0}} \end{bmatrix}}_{B} \begin{bmatrix} \delta_a(t) + f_1(\beta(t), p_s(t), r_s(t), \delta_a(t)) \\ \delta_r(t) + f_2(\beta(t), p_s(t), r_s(t), \delta_r(t)) \end{bmatrix}, \\
y(t) &= \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_{C^\top} \begin{bmatrix} \beta(t) \\ \phi(t) \\ p_s(t) \\ r_s(t) \end{bmatrix}, \qquad \begin{matrix} \beta(0) = \beta_0, & \phi(0) = \phi_0 \\ p_s(0) = p_{s_0}, & r_s(0) = r_{s_0}, \end{matrix} \qquad (5.11)
\end{aligned}$$

where C is the output matrix that defines the outputs of interest for tracking. The controllability matrix has rank = 4, so the system is controllable. Additionally, the A matrix is Hurwitz but has slow eigenvalues that do not achieve desired system properties such as settling time and overshoot. Wind tunnel data shows that the nonlinear terms f_1 and f_2 have the structure

$$\begin{aligned}
f_1(\beta, p_s, r_s, \delta_a) &= \left((1-C_1)e^{-\frac{\beta^2}{2\sigma_1^2}} + C_1\right)(\tanh(\delta_a + h_1) + \tanh(\delta_a - h_1) + 0.001\delta_a) + \\
&\quad D_1\cos(A_1 p_s - \omega_1)\sin(A_2 r_s - \omega_2) + D_2 \qquad (5.12) \\
f_2(\beta, p_s, r_s, \delta_r) &= \left((1-C_2)e^{-\frac{\beta^2}{2\sigma_2^2}} + C_2\right)(\tanh(\delta_r + h_2) + \tanh(\delta_r - h_2) + 0.001\delta_r) + \\
&\quad D_3\cos(A_3 p_s - \omega_3)\sin(A_4 r_s - \omega_4) + D_4. \qquad (5.13)
\end{aligned}$$

The positive constants C_1, C_2 represent the percentage of control effectiveness available at high angles of sideslip and are always less than unity. When the system is at trim sideslip angle, full aileron and rudder effectiveness is obtained. The parameters σ_1, σ_2 define the width of the Gaussian functions $e^{-\frac{\beta^2}{2\sigma_1^2}}, e^{-\frac{\beta^2}{2\sigma_2^2}}$. An additional small aileron and rudder control induced moment term is introduced so that for small control efforts, the control surface is within a boundary layer and does not produce an aerodynamic moment. As the control effort increases, the surface stalls and fails to produce any additional moment as the system reaches saturation, which is approximated by the hyperbolic tangent functions where h_1, h_2 are constants. Lastly, the rolling and yaw moments can be locally approximated by sine functions where the amplitude and phase are designed based on wind-tunnel data analysis.

The control *objective* is to design a control signal

$$\delta(t) = \begin{bmatrix} \delta_a(t) \\ \delta_r(t) \end{bmatrix} = \begin{bmatrix} \delta_{a_{nom}}(t) + \delta_{a_{ad}}(t) \\ \delta_{r_{nom}}(t) + \delta_{r_{ad}}(t) \end{bmatrix} \tag{5.14}$$

such that the angles β, ϕ track a a smooth commanded reference input $R^{cmd}(t)$. First, a nominal controller is designed to achieve desired performance for the linearized system in the absence of uncertainties. Then, an adaptive element is augmented to the nominal controller to achieve the tracking objective in the presence of uncertainties.

5.2 Nominal Dutch-Roll Model

The linear affine system for the Dutch-Roll dynamics of (5.11) in the absence of uncertainties is

$$\begin{aligned} \begin{bmatrix} \dot{\beta}_r(t) \\ \dot{\phi}_r(t) \\ \dot{p}_{s_r}(t) \\ \dot{r}_{s_r}(t) \end{bmatrix} &= A_r \underbrace{\begin{bmatrix} \beta_r(t) \\ \phi_r(t) \\ p_{s_r}(t) \\ r_{s_r}(t) \end{bmatrix}}_{x_r(t)} + B\hat{R}(t), \\ y_r(t) &= C^\top \begin{bmatrix} \beta_r(t) \\ \phi_r(t) \\ p_{s_r}(t) \\ r_{s_r}(t) \end{bmatrix}, \end{aligned} \tag{5.15}$$

$$\beta_r(0) = \beta_{r_0}, \quad \phi_r(0) = \phi_{r_0}, \quad p_{s_r}(0) = p_{s_{r_0}}, \quad r_{s_r}(0) = r_{s_{r_0}},$$

where

$$A_r = A - B \underbrace{\begin{bmatrix} K_{lqr_{1,1}} & K_{lqr_{1,2}} & K_{lqr_{1,3}} & K_{lqr_{1,4}} \\ K_{lqr_{2,1}} & K_{lqr_{2,2}} & K_{lqr_{2,3}} & K_{lqr_{2,4}} \end{bmatrix}}_{K_{lqr}^{\top}}, \tag{5.16}$$

and the LQR gain $K_{lqr} = -R^{-1}B^{\top}P$ is found by solving the matrix Riccati equation for the unique 4×4 positive definite symmetric matrix P:

$$Q + PA + A^{\top}P - PBR^{-1}B^{\top}P = 0 \tag{5.17}$$

given positive definite symmetric control weights $Q \in \mathbb{R}^4$ and $R \in \mathbb{R}^2$.

Consider

$$\hat{R}(t) = k_g R^{cmd}(t), \tag{5.18}$$

where k_g is the feedforward gain chosen as:

$$k_g = -\frac{1}{C^{\top}A_r^{-1}B} \tag{5.19}$$

to achieve a unity DC gain between the commanded signal $R^{cmd}(t)$ and the system output $y_r(t)$ so that $y_r(t) \to R^{cmd}(t)$ as $t \to \infty$. Thus, it follows that the nominal controller is given by

$$\begin{bmatrix} \delta_{a_{nom}}(t) \\ \delta_{r_{nom}}(t) \end{bmatrix} = -K_{lqr}^{\top} \begin{bmatrix} \beta(t) \\ \phi(t) \\ p_s(t) \\ r_s(t) \end{bmatrix} + \hat{R}(t). \tag{5.20}$$

Substituting the nominal controller into the roll-yaw dynamics (5.11) gives

$$\begin{bmatrix} \dot{\beta}(t) \\ \dot{\phi}(t) \\ \dot{p_s}(t) \\ \dot{r_s}(t) \end{bmatrix} = A_r \begin{bmatrix} \beta(t) \\ \phi(t) \\ p_s(t) \\ r_s(t) \end{bmatrix} + B\hat{R}(t) + B \begin{bmatrix} \delta_{a_{ad}}(t) + f_1(\beta(t), p_s(t), r_s(t), \delta_a(t)) \\ \delta_{r_{ad}}(t) + f_2(\beta(t), p_s(t), r_s(t), \delta_r(t)) \end{bmatrix},$$

$$y(t) = C^{\top} \begin{bmatrix} \beta(t) \\ \phi(t) \\ p_s(t) \\ r_s(t) \end{bmatrix}, \quad \beta(0) = \beta_0, \quad \phi(0) = \phi_0, \quad p_s(0) = p_{s_0}, \quad r_s(0) = r_{s_0}. \tag{5.21}$$

5.3 Multi-Input RBF approximation

From (5.21), the unknown nonlinearities of the system are

$$\begin{bmatrix} f_1(\beta(t), p_s(t), r_s(t), \delta_a(t)) \\ f_2(\beta(t), p_s(t), r_s(t), \delta_r(t)) \end{bmatrix}, \tag{5.22}$$

which will be approximated using neural networks. Using the structure of f_1 and f_2 given in (5.4), (5.5), define the Jacobian matrix of (5.22) as

$$\begin{aligned} \mathbf{P}(\beta, \delta_a, \delta_r) &= \begin{bmatrix} \frac{\partial f_1}{\partial \delta_a}(\beta, \delta_a) & \frac{\partial f_1}{\partial \delta_r}(\beta, \delta_a) \\ \\ \frac{\partial f_2}{\partial \delta_a}(\beta, \delta_r) & \frac{\partial f_2}{\partial \delta_r}(\beta, \delta_r) \end{bmatrix} \\ &= \begin{bmatrix} \underbrace{\bar{C}_1(\beta)\ \tau_1(\delta_a)}_{P_1(\beta,\delta_a)} & 0 \\ \\ 0 & \underbrace{\bar{C}_2(\beta)\ \tau_2(\delta_r)}_{P_2(\beta,\delta_r)} \end{bmatrix}, \end{aligned} \tag{5.23}$$

where

$$\begin{aligned} \bar{C}_1(\beta) &= \left((1 - C_1)e^{-\frac{\beta^2}{2\sigma_1^2}} + C_1\right) \ \geq \ C_1 \ > \ 0, \\ \bar{C}_2(\beta) &= \left((1 - C_2)e^{-\frac{\beta^2}{2\sigma_2^2}} + C_2\right) \ \geq \ C_2 \ > \ 0, \\ \tau_1(\delta_a) &= 2.001 - \tanh^2(\delta_a + h_1) - \tanh^2(\delta_a - h_1) \ \geq \ 0.001, \\ \tau_2(\delta_r) &= 2.001 - \tanh^2(\delta_r + h_2) - \tanh^2(\delta_r - h_2) \ \geq \ 0.001, \end{aligned} \tag{5.24}$$

and C_1, C_2, h_1, h_2 are positive constants with C_1, C_2 less than one. For the roll-yaw dynamical system, the diagonal elements of the Jacobian matrix (5.23) are strictly positive and $\mathbf{P}$ is non-singular for every $t \geq 0$ and $\beta(t), \delta_a(t), \delta_r(t) \in \mathbb{R}$. Leibnitz Formula allows the functions f_1, f_2 to be represented as

$$\begin{aligned} f_1(\beta, \delta_a) &= f_1(\beta, 0) + \int_0^{\delta_a} P_1(\beta, \xi_1)\ d\xi_1 \\ f_2(\beta, \delta_r) &= f_2(\beta, 0) + \int_0^{\delta_r} P_2(\beta, \xi_2)\ d\xi_2 \end{aligned}$$

where the integrals must be positive by positivity of P_1 and P_2. Each component of f_1, f_2 can be approximated by a linear combination of radial basis functions (RBF) over a compact set with arbitrarily small approximation error. In the same manner as in the single-input short-period dynamics case, consider two sets of radial basis functions $\phi_1, \phi_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by

$$\phi_1(\beta, \delta_a) = \phi_{1,1}(\beta) + \phi_{1,2}(\beta, \delta_a) \tag{5.25}$$

$$\phi_2(\beta, \delta_r) = \phi_{2,1}(\beta) + \phi_{2,2,}(\beta, \delta_r) \tag{5.26}$$

with the following definitions:

$$\phi_{1,1}(\beta) \triangleq \theta_1^\top \Phi_{1,1}(\beta) = \sum_{i=1}^{M_1} \theta_{1,i}\ e^{-\frac{(\beta-\beta_{c_i})^\top(\beta-\beta_{c_i})}{\delta_{1,i}^2}}, \qquad \delta_{1,i} > 0 \tag{5.27}$$

$$\phi_{1,2}(\beta, \delta_a) \triangleq w_1^\top \Phi_{1,2}(\beta, \delta_a) = \sum_{j=1}^{N_1} w_{1,j} \left(\int_0^{\delta_a} e^{-\frac{(\chi_1-\chi_{1,c_j})^\top(\chi_1-\chi_{1,c_j})}{\rho_{1,j}^2}} d\xi_1 \right), \tag{5.28}$$

$$\chi_1 \triangleq [\beta \quad \xi_1]^\top, \rho_{1,j} > 0,$$

$$\phi_{2,1}(\beta) \triangleq \theta_2^\top \Phi_{2,1}(\beta) = \sum_{i=1}^{M_2} \theta_{2,i}\ e^{-\frac{(\beta-\beta_{c_i})^\top(\beta-\beta_{c_i})}{\delta_{2,i}^2}}, \qquad \delta_{2,i} > 0 \tag{5.29}$$

$$\phi_{2,2}(\beta, \delta_r) \triangleq w_2^\top \Phi_{2,2}(\beta, \delta_r) = \sum_{j=1}^{N_2} w_{2,j} \left(\int_0^{\delta_r} e^{-\frac{(\chi_2-\chi_{2,c_j})^\top(\chi_2-\chi_{2,c_j})}{\rho_{2,j}^2}} d\xi_2 \right), \tag{5.30}$$

$$\chi_2 \triangleq [\beta \quad \xi_2]^\top, \rho_{2,j} > 0,$$

where $M_1, M_2, N_1, N_2,$ are positive constants, the vectors

$$\chi_{1,c_k} \triangleq [\beta_{c_k} \quad \delta_{a_{c_k}}]^\top \qquad k \in \{1, \ldots, N_1\}$$

$$\chi_{2,c_l} \triangleq [\beta_{c_l} \quad \delta_{r_{c_l}}]^\top \qquad l \in \{1, \ldots, N_2\}$$

represent the fixed centers of the basis, $\delta_{1,i}, \delta_{2,j}, \sigma_{1,k}, \sigma_{2,l}$ are fixed width parameters, while $\theta_{1,i}, \theta_{2,k} \in \mathbb{R}$, $w_{1,k}, w_{2,l} \in \mathbb{R}^+$ are the unknown constant parameters. It is straightforward to verify from (5.27) - (5.30) that

$$\text{sgn}\left(\frac{\partial \phi_1}{\partial \delta_a}(\beta, \delta_a)\right) = \text{sgn}\left(\frac{\partial \phi_{1,2}}{\partial \delta_a}(\beta, \delta_a)\right) > 0$$

$$\text{sgn}\left(\frac{\partial \phi_2}{\partial \delta_r}(\beta, \delta_r)\right) = \text{sgn}\left(\frac{\partial \phi_{2,2}}{\partial \delta_r}(\beta, \delta_r)\right) > 0. \tag{5.31}$$

Thus, over a compact set of initial conditions of interest,

$$f_1(\beta, \delta_a) = W_1^\top \Phi_1(\beta, \delta_a) + \varepsilon_1(\beta, \delta_a), \quad |\varepsilon_1(\beta, \delta_a)| \le \varepsilon_1^*, \tag{5.32}$$

$$f_2(\beta, \delta_r) = W_2^\top \Phi_2(\beta, \delta_r) + \varepsilon_2(\beta, \delta_r), \quad |\varepsilon_2(\beta, \delta_r)| \le \varepsilon_2^*, \tag{5.33}$$

where

$$\begin{aligned}
W_1 &= [\theta_1^\top \; w_1^\top]^\top = [\theta_{1,1} \dots \theta_{1,M_1} \;\; w_{1,1} \dots w_{1,N_1}]^\top, \\
W_2 &= [\theta_2^\top \; w_2^\top]^\top = [\theta_{2,1} \dots \theta_{2,M_2} \;\; w_{2,1} \dots w_{2,N_2}]^\top, \\
\Phi_1(\beta, \delta_a) &= [\Phi_{1,1}^\top(\beta) \;\; \Phi_{1,2}^\top(\beta, \delta_a)]^\top, \\
\Phi_2(\beta, \delta_r) &= [\Phi_{2,1}^\top(\beta) \;\; \Phi_{2,2}^\top(\beta, \delta_r)]^\top
\end{aligned}$$

with constant coefficients $w_{1,1}, \dots, w_{1,N_1}, w_{2,1}, \dots, w_{2,N_2} > 0$ and $\varepsilon_1(\beta, \delta_a), \varepsilon_2(\beta, \delta_r)$ are the bounded approximation errors over the sets $(\beta, \delta_a) \in \Omega_\beta \times \Omega_{\delta_a}$, and $(\beta, \delta_r) \in \Omega_\beta \times \Omega_{\delta_r}$ respectively for compact sets $\Omega_\beta, \Omega_{\delta_a}, \Omega_{\delta_r} \in \mathbb{R}$.

5.4 Roll-Yaw Dynamics State Predictor

Consider the following state predictor using a series parallel model for the dynamics of (5.11):

$$\begin{bmatrix} \dot{\hat{\beta}}(t) \\ \dot{\hat{\phi}}(t) \\ \dot{\hat{p}}_s(t) \\ \dot{\hat{r}}_s(t) \end{bmatrix} = A \underbrace{\begin{bmatrix} \hat{\beta}(t) \\ \hat{\phi}(t) \\ \hat{p}_s(t) \\ \hat{r}_s(t) \end{bmatrix}}_{\hat{x}(t)} + B\hat{R}(t) + B \begin{bmatrix} \delta_{a_{ad}} + \hat{W}_1^\top(t)\Phi_1(\beta(t), \delta_a(t)) \\ \delta_{r_{ad}} + \hat{W}_2^\top(t)\Phi_2(\beta(t), \delta_r(t)) \end{bmatrix}, \tag{5.34}$$

$$\hat{\beta}(0) = \hat{\beta}_0, \quad , \hat{\phi}(0) = \hat{\phi}_0, \quad \hat{p}_s(0) = \hat{p}_{s_0}, \quad \hat{r}_s(0) = \hat{r}_{s_0},$$

where the Hurwitz matrix A_r is introduced in (5.16), $\hat{R}$ is given in (5.18), and $\hat{W}_1, \hat{W}_2$ are the estimates of the unknown constant vectors W_1, W_2. Define the prediction error signal

$$e_s(t) = \begin{bmatrix} \hat{\beta}(t) - \beta(t) \\ \hat{\phi}(t) - \phi(t) \\ \hat{p}_s(t) - p_s(t) \\ \hat{r}_s(t) - r_s(t) \end{bmatrix}. \tag{5.35}$$

The prediction error dynamics are:

$$\begin{aligned}
\dot{e}_s(t) &= A_r e_s(t) + B \begin{bmatrix} \tilde{W}_1^\top(t)\Phi_1(\beta(t), \delta_a(t)) - \varepsilon_1(\beta(t), \delta_a(t)) \\ \tilde{W}_2^\top(t)\Phi_2(\beta(t), \delta_r(t)) - \varepsilon_2(\beta(t), \delta_r(t)) \end{bmatrix}, \\
&= A_r e_s(t) + B_1 \left(\tilde{W}_1^\top(t)\Phi_1(\beta(t), \delta_a(t)) - \varepsilon_1(\beta(t), \delta_a(t)) \right) + \\
&\quad B_2 \left(\tilde{W}_2^\top(t)\Phi_2(\beta(t), \delta_r(t)) - \varepsilon_2(\beta(t), \delta_r(t)) \right),
\end{aligned} \tag{5.36}$$

with initial conditions $e_{s,1}(0) = \hat{\beta}(0) - \hat{\beta}_0, \quad e_{s,2}(0) = \hat{\phi}(0) - \hat{\phi}_0, \quad e_{s,3}(0) = \hat{p}_s(0) - \hat{p}_{s_0}, \quad e_{s,4}(0) = \hat{r}_s(0) - \hat{r}_{s_0}$, the parameter errors $\tilde{W}_1(t) = \hat{W}_1(t) - W_1(t), \tilde{W}_2(t) = \hat{W}_2(t) - W_2(t)$, and B_1, B_2 denoting the first and second columns of B respectively. That is,

$$B_1 = \begin{bmatrix} 0 \\ 0 \\ L_{\delta_{a_0}} \\ N_{\delta_{a_0}} \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 \\ 0 \\ L_{\delta_{r_0}} \\ N_{\delta_{r_0}} \end{bmatrix}. \tag{5.37}$$

Theorem 4 *The adaptive law*

$$\begin{aligned} \dot{\hat{W}}_1(t) &= \Gamma_1 \mathrm{Proj}\left(\hat{W}_1(t), -\Phi_1(\beta(t), \delta_a(t)) e_s^\top(t) P_0 B_1 \right), \quad \hat{W}_1(0) = \hat{W}_{1_0} \\ \dot{\hat{W}}_2(t) &= \Gamma_2 \mathrm{Proj}\left(\hat{W}_2(t), -\Phi_2(\beta(t), \delta_r(t)) e_s^\top(t) P_0 B_2 \right), \quad \hat{W}_2(0) = \hat{W}_{2_0}, \end{aligned} \tag{5.38}$$

where Proj $(\cdot,\cdot)$ denotes the Projection operator [2], $P_0 = P_0^\top > 0$ solves the Lyapunov equation $A_r^\top P_0 + P_0 A_r = -Q_0$ for a matrix $Q_0 > 0$, and $\Gamma_1^\top = \Gamma_1 > 0, \Gamma_2^\top = \Gamma_2 > 0$ are matrices of adaptation gains, ensures that the prediction error dynamics (5.36) are ultimately bounded with respect to $e_s(t), \tilde{W}_1(t)$, and $\tilde{W}_2(t)$.

Proof. Consider the following positive definite Lyapunov function candidate

$$V(e_s(t), \tilde{W}_1(t), \tilde{W}_2(t)) = e_s^\top(t) P_0 e_s(t) + \tilde{W}_1^\top(t) \Gamma_1^{-1} \tilde{W}_1(t) + \tilde{W}_2^\top(t) \Gamma_2^{-1} \tilde{W}_2(t). \tag{5.39}$$

Taking the time derivative yields

$$\begin{aligned} \dot{V}(e_s(t), \tilde{W}_1(t), \tilde{W}_2(t)) &= \dot{e}_s^\top(t) P_0 e_s(t) + e_s^\top(t) P_0 \dot{e}_s(t) + \dot{\tilde{W}}_1^\top(t) \Gamma_1^{-1} \tilde{W}_1(t) + \tilde{W}_1^\top(t) \Gamma_1^{-1} \dot{\tilde{W}}_1(t) + \\ &\quad \dot{\tilde{W}}_2^\top(t) \Gamma_2^{-1} \tilde{W}_2(t) + \tilde{W}_2^\top(t) \Gamma_2^{-1} \dot{\tilde{W}}_2(t) \\ &= e_s^\top(t) A_r^\top P_0 e_s(t) + e_s^\top(t) P_0 A_r e_s(t) + \\ &\quad \left(\Phi_1^\top(\beta(t), \delta_a(t)) \tilde{W}_1(t) - \varepsilon_1(\beta(t), \delta_a(t)) \right) B_1^\top P_0 e_s(t) + \\ &\quad \left(\Phi_2^\top(\beta(t), \delta_r(t)) \tilde{W}_2(t) - \varepsilon_2(\beta(t), \delta_r(t)) \right) B_2^\top P_0 e_s(t) + \\ &\quad e_s^\top(t) P_0 B_1 \left(\tilde{W}_1^\top(t) \Phi_1(\beta(t), \delta_a(t)) - \varepsilon_1(\beta(t), \delta_a(t)) \right) + \\ &\quad e_s^\top(t) P_0 B_2 \left(\tilde{W}_2^\top(t) \Phi_2(\beta(t), \delta_r(t)) - \varepsilon_2(\beta(t), \delta_r(t)) \right) + \\ &\quad \left(\Gamma_1 \mathrm{Proj}(\hat{W}_1(t), -\Phi_1(\beta(t), \delta_a(t)) e_s^\top(t) P_0 B_1) \right)^\top \Gamma_1^{-1} \tilde{W}_1(t) + \\ &\quad \tilde{W}_1^\top(t) \Gamma_1^{-1} \left(\Gamma_1 \mathrm{Proj}(\hat{W}_1(t), -\Phi_1(\beta(t), \delta_a(t)) e_s^\top(t) P_0 B_1) \right) + \\ &\quad \left(\Gamma_2 \mathrm{Proj}(\hat{W}_2(t), -\Phi_2(\beta(t), \delta_r(t)) e_s^\top(t) P_0 B_2) \right)^\top \Gamma_2^{-1} \tilde{W}_2(t) + \\ &\quad \tilde{W}_2^\top(t) \Gamma_2^{-1} \left(\Gamma_2 \mathrm{Proj}(\hat{W}_2(t), -\Phi_2(\beta(t), \delta_r(t)) e_s^\top(t) P_0 B_2) \right) \end{aligned}$$

$$
\begin{aligned}
&\leq -e_s^\top(t)Q_0e_s(t) - 2e_s^\top(t)P_0B_1\ \varepsilon_1(\beta(t),\delta_a(t)) - \\
&\quad 2e_s^\top(t)P_0B_2\ \varepsilon_2(\beta(t),\delta_r(t)) \\
&\leq -\|e_s(t)\|^2\lambda_{\min}(Q_0) + 2\|e_s(t)\|\ \|P_0B_1\|\varepsilon_1^* + 2\|e_s(t)\|\ \|P_0B_2\|\epsilon_2^* \\
&= -\|e_s(t)\|\left(\lambda_{\min}(Q_0)\|e_s(t)\| - 2\|P_0B_1\|\ \epsilon_1^* - 2\|P_0B_2\|\ \epsilon_2^*\right),
\end{aligned}
$$

where $\lambda_{\min}(Q_0)$ denotes the minimum eigenvalue of Q_0. It follows that outside the compact set

$$
\left\{\|e_s\| \leq 2\left(\frac{\varepsilon_1^*\|P_0B_1\| + \varepsilon_2^*\|P_0B_2\|}{\lambda_{\min}(Q_0)}\right)\right\}\bigcap\left\{\|\tilde{W}_1\| \leq W_1^*\right\}\bigcap\left\{\|\tilde{W}_2\| \leq W_2^*\right\}, \tag{5.40}
$$

where W_1^*, W_2^* are determined by the Projection operator and $\|\cdot\|$ denotes the 2-norm, the derivative of the Lyapunov function candidate, $\dot{V}(t)$ is negative. Applying well-known invariant set arguments, it can be concluded that if the initial errors are within the largest level set of the Lyapunov function (5.39), for which the RBF approximation has been defined, then the signals $e_s(t), \tilde{W}_1(t), \tilde{W}_2(t)$ are ultimately bounded and the ultimate bound ζ can be any number greater than the value of the Lyapunov function on the minimum level set embracing the compact set (5.40). That is,

$$
\begin{aligned}
\|\zeta\| &> \left\| V\left(2\left(\frac{\varepsilon_1^*\|P_0B_1\| + \varepsilon_2^*\|P_0B_2\|}{\lambda_{\min}(Q_0)}\right), W_1^*, W_2^*\right)\right\| \qquad (5.41)\\
&= 4\left(\frac{(\varepsilon_1^*\|P_0B_1\| + \varepsilon_2^*\|P_0B_2\|)^2}{\lambda_{\min}^2(Q_0)}\right)\|\lambda_{\max}(P_0)\| + (W_1^*)^2\|\Gamma_1^{-1}\|, (W_2^*)^2\|\Gamma_2^{-1}\|,
\end{aligned}
$$

where V is the Lyapunov function (5.39).

□

Remark 4 *Notice that boundedness of the prediction error $e_s(t)$ and the parameter errors $\tilde{W}_1(t), \tilde{W}_2(t)$ does not imply stability of the overall system. It must be proven additionally that the state predictor dynamics (5.34) achieves the tracking objective in the presence of feedback, implying that $\hat{\beta}(t), \hat{\phi}(t), \hat{p}_s(t), \hat{r}_s(t)$ remain bounded to lead to the system states $\beta(t), \phi(t), p_s(t), r_s(t)$ boundededness.*

Remark 5 *Notice that since w_1 and w_2 are positive, the compact set in the application of the Projection operator can be selected so that $\hat{w}_1(t), \hat{w}_2(t)$ remain positive for all $t \geq 0$, that is, $\hat{w}_1(t) > w_{1_0} > 0, \hat{w}_2(t) > w_{2_0} > 0$.*

5.5 MIMO Control Design

Let the tracking error signal between the predictor and the reference system be

$$e(t) = \begin{bmatrix} \hat{\beta}(t) - \beta_r(t) \\ \hat{\phi}(t) - \phi_r(t) \\ \hat{p}_s(t) - p_{s_r}(t) \\ \hat{r}_s(t) - r_{s_r}(t) \end{bmatrix}. \tag{5.42}$$

The open loop time-varying tracking error dynamics are:

$$\begin{aligned} \dot{e}(t) &= A_r e(t) + B \begin{bmatrix} \delta_{a_{ad}}(t) + \hat{W}_1^\top(t)\Phi_1(\beta(t), \delta_a(t)) \\ \delta_{r_{ad}}(t) + \hat{W}_2^\top(t)\Phi_2(\beta(t), \delta_r(t)) \end{bmatrix}, \\ e_1(0) &= \hat{\beta}_0 - \beta_{r_0}, \quad e_2(0) = \hat{\phi}_0 - \phi_{r_0}, \quad e_3(0) = \hat{p}_{s_0} - p_{s_{r_0}}, \quad e_4(0) = \hat{r}_{s_0} - r_{s_{r_0}}. \end{aligned} \tag{5.43}$$

Dynamic inversion based controller is to be determined from the solution of the following system of equations

$$\begin{bmatrix} \delta_{a_{ad}} \\ \delta_{r_{ad}} \end{bmatrix} = \begin{bmatrix} -\hat{W}_1^\top \Phi_1(\beta, \delta_{a_{nom}} + \delta_{a_{ad}}) \\ -\hat{W}_2^\top \Phi_2(\beta, \delta_{r_{nom}} + \delta_{r_{ad}}) \end{bmatrix}, \tag{5.44}$$

which would result in the stable closed-loop tracking error dynamics $\dot{e}(t) = A_r e(t)$. In general, (5.44) cannot be solved explicitly for $\delta_{a_{ad}}, \delta_{r_{ad}}$, so an approximation of the dynamic inversion controller is constructed via fast dynamics:

$$\begin{aligned} \epsilon \begin{bmatrix} \dot{\delta}_{a_{ad}}(t) \\ \dot{\delta}_{r_{ad}}(t) \end{bmatrix} &= -\bar{\mathbf{P}}(t, e(t), \delta_{a_{ad}}(t), \delta_{r_{ad}}(t))\mathbf{f}(t, e(t), \delta_{a_{ad}}(t), \delta_{r_{ad}}(t)), \quad 0 < \epsilon \ll 1, \\ & \delta_{a_{ad}}(0) = \delta_{a_{ad_0}}, \qquad \delta_{r_{ad}}(0) = \delta_{r_{ad_0}}, \end{aligned} \tag{5.45}$$

where

$$\mathbf{f}(t, e, \delta_{a_{ad}}, \delta_{r_{ad}}) = \begin{bmatrix} \delta_{a_{ad}} + \hat{W}_1^\top(t)\Phi_1(\beta(t), \delta_{a_{nom}}(t) + \delta_{a_{ad}}) \\ \delta_{r_{ad}} + \hat{W}_2^\top(t)\Phi_2(\beta(t), \delta_{r_{nom}}(t) + \delta_{r_{ad}}) \end{bmatrix} \tag{5.46}$$

and

$$\bar{\mathbf{P}}(t, e, \delta_{a_{ad}}, \delta_{r_{ad}}) = \begin{bmatrix} \hat{W}_1^\top(t)\frac{\partial \Phi_1}{\partial \delta_{a_{ad}}}(\beta(t), \delta_{a_{ad}}) + 1 & 0 \\ 0 & \hat{W}_2^\top(t)\frac{\partial \Phi_2}{\partial \delta_{r_{ad}}}(\beta(t), \delta_{r_{ad}}) + 1 \end{bmatrix} \tag{5.47}$$

is the Jacobian matrix of $\mathbf{f}(t, e, \delta_{a_{ad}}, \delta_{r_{ad}})$. To prove existence of a root for (5.46), recall Implicit Function Theorem of [1]:

Implicit Function Theorem *Assume that $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuously differentiable at each point (x, y) of an open set $S \subset \mathbb{R}^n \times \mathbb{R}^m$. Let (x_0, y_0) be a point in S for which $f(x_0, y_0) = 0$ and for which the Jacobian matrix $\frac{\partial f}{\partial x}(x_0, y_0)$ is nonsingular. Then there exist neighborhoods $u \subset \mathbb{R}^n$ of x_0 and $V \subset \mathbb{R}^m$ of y_0 such that for each $y \in V$, the equation $f(x, y) = 0$ has a unique solution $x \in U$. Moreover, this solution can be given as $x = g(y)$, where g is continuously differentiable at $y = y_0$.*

It is straightforward to verify from the definition of the RBF approximations (5.27) - (5.30) that the determinant of $\bar{\mathbf{P}}$ is

$$\begin{aligned} \det(\bar{\mathbf{P}}(t, \delta_{a_{ad}}, \delta_{r_{ad}})) &= \left(\hat{W}_1^\top(t) \frac{\partial \Phi_1}{\partial \delta_{a_{ad}}}(\beta(t), \delta_{a_{ad}}) + 1 \right) \left(\hat{W}_2^\top(t) \frac{\partial \Phi_2}{\partial \delta_{r_{ad}}}(\beta(t), \delta_{r_{ad}}) + 1 \right) \\ &= \left(\hat{w}_1^\top(t) \frac{\partial \Phi_{1,2}}{\partial \delta_{a_{ad}}}(\beta(t), \delta_{a_{ad}}) + 1 \right) \left(\hat{w}_2^\top(t) \frac{\partial \Phi_{2,2}}{\partial \delta_{r_{ad}}}(\beta(t), \delta_{r_{ad}}) + 1 \right). \end{aligned} \tag{5.48}$$

In the light of Remark 5, all the elements of $\hat{w}_1$ and $\hat{w}_2$ are positive and $\frac{\partial \Phi_{1,2}}{\partial \delta_{a_{ad}}}(t, e, \delta_{a_{ad}})$, $\frac{\partial \Phi_{2,2}}{\partial \delta_{r_{ad}}}(t, e, \delta_{r_{ad}})$ are positive-valued Gaussians, so the determinant of $\bar{\mathbf{P}}$ is always positive. Furthermore, the matrix $\bar{\mathbf{P}}^\top(t, e, \delta_{a_{ad_1}}, \delta_{r_{ad_1}})\bar{\mathbf{P}}(t, e, \delta_{a_{ad_2}}, \delta_{r_{ad_2}})$ is strictly positive definite, i.e. there exists $c_1 > 0$ such that

$$\xi^\top \bar{\mathbf{P}}^\top(t, e, \delta_{a_{ad_1}}, \delta_{r_{ad_1}})\bar{\mathbf{P}}(t, e, \delta_{a_{ad_2}}, \delta_{r_{ad_2}})\xi \ \geq \ 2c_1\|\xi\|^2, \quad \forall \xi \in \mathbb{R}^2. \tag{5.49}$$

The proof is straightforward by computing the left-hand side of (5.49) to obtain

$$\begin{aligned} \xi^\top \bar{\mathbf{P}}^\top(t, e, \delta_{a_{ad_1}}, \delta_{r_{ad_1}})\bar{\mathbf{P}}(t, e, \delta_{a_{ad_2}}, \delta_{r_{ad_2}})\xi \ &= \\ & \left(\hat{w}_1^\top(t) \frac{\partial \Phi_{1,2}}{\partial \delta_{a_{ad_1}}}(\beta(t), \delta_{a_{ad_1}}) + 1 \right) \left(\hat{w}_1^\top(t) \frac{\partial \Phi_{1,2}}{\partial \delta_{a_{ad_2}}}(\beta(t), \delta_{a_{ad_2}}) + 1 \right) \xi_1^2 + \\ & \left(\hat{w}_2^\top(t) \frac{\partial \Phi_{2,2}}{\partial \delta_{r_{ad_1}}}(\beta(t), \delta_{r_{ad_1}}) + 1 \right) \left(\hat{w}_2^\top(t) \frac{\partial \Phi_{2,2}}{\partial \delta_{r_{ad_2}}}(\beta(t), \delta_{r_{ad_2}}) + 1 \right) \xi_2^2 \\ & \geq \xi_1^2 + \xi_2^2, \end{aligned}$$

and it is easy to see that $c_1 = \frac{1}{2}$ is one choice of c_1 that satisfies the inequality of (5.49). Hence, we apply Implicit Function Theorem to guarantee the existence of a unique root within a neighborhood. Let $h(t, e, \delta_a, \delta_r)$ denote this isolated root of $\mathbf{f}(t, e, \delta_a, \delta_r)$ and define

$$\tau \ = \ \frac{t}{\epsilon}, \qquad v(t, \beta) \ = \ \begin{bmatrix} \delta_{a_{ad}}(t) \\ \delta_{r_{ad}}(t) \end{bmatrix} - h(t, \beta). \tag{5.50}$$

The reduced system for (5.42), (5.43), (5.45) is given by:

$$\dot{e}(t) = A_r e(t), \quad e(0) = e_0, \tag{5.51}$$

and the boundary layer system is

$$\frac{d\nu}{d\tau} = -\bar{\mathbf{P}}(t, e, \nu + h(t, e))\mathbf{f}(t, e, \nu + h(t, e)), \; v(0) \triangleq \nu_0 = \begin{bmatrix} \delta_{a_0} \\ \delta_{r_0} \end{bmatrix} - h(t, e_0). \tag{5.52}$$

The main result for the roll-yaw dynamics is stated as follows:

Theorem 5 *For the dynamics of (5.11), the boundary layer system (5.52) has exponentially stable origin. Moreover, the system given by (5.34), (5.45) has a unique solution*

$$\begin{bmatrix} \hat{\beta}(t) \\ \hat{\phi}(t) \\ \hat{p}_s(t) \\ \hat{r}_s(t) \end{bmatrix} = \begin{bmatrix} \beta_r(t) \\ \phi_r(t) \\ p_{s_r}(t) \\ r_{s_r}(t) \end{bmatrix} + O(\epsilon). \tag{5.53}$$

Proof. We must verify that the assumption in Tikhonov's theorem are satisfied. Verification of A1 is straightforward, since the function $\mathbf{f}$ and its first partial derivatives with respect to t are continuous and bounded, $h(t, e)$ and $\frac{\partial \mathbf{f}}{\partial \delta_a}(t, e, \nu + h(t, e))$, $\frac{\partial \mathbf{f}}{\partial \delta_r}(t, e, \nu + h(t, e))$ have bounded first derivatives with respect to their arguments, and $\frac{\partial \mathbf{f}}{\partial e}(t, e, \nu + h(t, e))$ is Lipshitz in e, uniformly in t. Since A_r is Hurwitz by definition, the origin is an exponentially stable equilibrium point for the reduced system given in (5.51), and, hence, A2 is satisfied. For verification of A3, consider the following Lyapunov function candidate for the boundary layer system (5.52):

$$V(t, e, \nu(\tau)) = \frac{1}{2}\mathbf{f}^\top(t, e, \nu(\tau) + h(t, e, z))\mathbf{f}(t, e, \nu(\tau) + h(t, e)). \tag{5.54}$$

Since $\mathbf{f}(t, e, h(t, e)) = 0$, then $V(t, e, 0) = 0$. Therefore

$$V(t, e, \nu) = \int_L \nabla V(t, e, s) ds, \tag{5.55}$$

where $\int_L$ assumes integration along the pass L from zero to ν and

$$\nabla V(t, e, \nu) = \mathbf{f}^\top(t, e, \nu + h(t, e, z))\bar{\mathbf{P}}(t, e, \nu + h(t, e)). \tag{5.56}$$

Since

$$\frac{\partial \mathbf{f}(t,e,\nu+h(t,e))}{\partial \nu} = \bar{\mathbf{P}}(t,e,\nu+h(t,e)), \tag{5.57}$$

then

$$\mathbf{f}(t,e,\nu+h(t,e)) = \int_L \bar{\mathbf{P}}(t,e,s+h(t,e))ds. \tag{5.58}$$

Hence, it follows from (5.55) that

$$V(t,e,\nu) = \int_L \left(\int_{L_1} \bar{\mathbf{P}}(t,e,s_1+h(t,e))ds_1\right)^\top \bar{\mathbf{P}}(t,e,s+h(t,e))\ ds,$$

and consequently,

$$V(t,e,\nu) = \int_L \int_{L_1} (ds_1)^\top \bar{\mathbf{P}}^\top(t,e,s_1+h(t,e))\bar{\mathbf{P}}(t,e,s+h(t,e))\ ds, \tag{5.59}$$

where $\int_L$,$\int_{L_1}$ assume integration along the pass L and L_1 from zero to ν, s, respectively. From (5.49), it follows then that

$$\begin{aligned} V(t,e,\nu) &\geq \int_L \int_{L_1} 2c_1(ds_1)^\top ds \\ &= \int_0^{||\nu||} \int_0^{\chi} 2c_1 d\xi d\chi. \end{aligned} \tag{5.60}$$

Therefore,

$$V(t,e,\nu) \geq c_1||\nu||^2, \tag{5.61}$$

where $c_1 = \frac{1}{2}$. Condition A1 implies that there exists positive c_2 such that $V(t,e,\nu) \leq c_2\|\nu\|^2$, which along with (5.61), verifies (3.19). Using the relationships in (5.52) and (5.54), one obtains

$$\dot{V}(t,e,\nu) = -\mathbf{f}^\top(t,e,\nu+h(t,e))\bar{\mathbf{P}}(t,e,\nu+h(t,e))\bar{\mathbf{P}}^\top(t,e,\nu+h(t,e))\mathbf{f}(t,e,\nu+h(t,e)),$$

which implies that $\dot{V}(t,e,\nu) \leq -2c_1V(t,e,\nu) \leq -2c_1^2\|\nu\|^2$, and this proves that (3.20) holds with $c_3 = c_1^2$. Hence, A3 holds, and the boundary layer (5.52) has exponentially stable origin. Following Tikhonov's theorem, we can conclude that there exists a unique solution, $\hat{\beta}, \hat{\phi}, \hat{p}_s, \hat{r}_s$ that tracks $\beta_r, \phi_r, p_{s_r}, r_{s_r}$ with an error of order $O(\epsilon)$.

□

5.6 Dutch-Roll Simulations

For our simulation we use the lateral/directional model of F-16 from [15] flying at sea level with an airspeed of 502 ft/s and angle of attack $\alpha = 2.11°$. The model matrices are given as

$$A = \begin{bmatrix} -0.3220 & 0.0640 & 0.0364 & -0.9917 \\ 0 & 0 & 1 & 0.0393 \\ -30.6490 & 0 & -3.6784 & 0.6646 \\ 8.5395 & 0 & -0.0254 & -0.4764 \\ & & & \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -0.7331 & 0.1315 \\ -0.0319 & -0.0620 \end{bmatrix},$$

and its open-loop eigenvalues are:

$$\begin{aligned} \lambda_1 &= -3.6153 \\ \lambda_{2,3} &= -0.4237 \pm 3.0635i \\ \lambda_4 &= -0.0142. \end{aligned}$$

Note that the Dutch-Roll system is stable, but that its eigenvalues are slow. Next we select our reference model by solving the Riccati equation with the following weighted matrices:

$$Q = \begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & 100 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 100 \end{bmatrix}, \qquad R = \begin{bmatrix} 1 & 0 \\ 0 & 0.1 \end{bmatrix}, \tag{5.62}$$

so that

$$k_{lqr} = \begin{bmatrix} 10.6901 & -9.5824 & -2.0328 & -6.1944 \\ -0.3982 & -0.2043 & -0.4170 & -27.0142 \end{bmatrix} \tag{5.63}$$

and

$$A_m = \begin{bmatrix} -0.3220 & 0.0640 & 0.0364 & -0.9917 \\ 0 & 0 & 1 & 0.0393 \\ -22.7599 & -6.9980 & -5.1138 & -0.3242 \\ 8.8560 & -0.3183 & -0.1161 & -2.3489 \end{bmatrix}.$$

The eigenvalues of the reference model are:

$$\begin{aligned} \lambda_{1,2} &= -2.6518 \pm 0.6710i \\ \lambda_{3,4} &= -1.2405 \pm 2.9696i. \end{aligned}$$

The commanded reference input of interest to track is

$$R^{cmd}(t) = \begin{bmatrix} 0.2\left(-\frac{0.5}{1+e^{t-8}} + \frac{1}{1+e^{t-30}} - 0.5\right) \\ 0.2\left(-\frac{0.5}{1+e^{t-8}} + \frac{1}{1+e^{t-30}} - 0.2\right) \end{bmatrix}.$$

In the absence of uncertainties, the nominal system behaves optimally using the LQR controller. Setting

$$k_g = \begin{bmatrix} -2.9031 & -9.9924 \\ 156.5907 & -2.4300 \end{bmatrix}, \tag{5.64}$$

the angles β, ϕ track R^{cmd} 5.1. Initial conditions used for the nominal system are $\beta_{r_0} = 0°, \phi_{r_0} = 0°, p_{s_{r_0}} = 0°/\text{sec}, r_{s_{r_0}} = 0°/\text{sec}$. To insert uncertainties into the roll-yaw dynamics, explicit expressions must be found for the nonlinearities $f_1(\beta, p_s, r_s, \delta_a, \delta_r)$ and $f_2(\beta, p_s, r_s, \delta_a, \delta_r)$ introduced in (5.4) - (5.5) for simulations purposes. Towards that end, the values of the constants $C_1, C_2, \sigma_1, \sigma_2$ and D_i, A_i, ω_i, for $i = \{1, 2, 3, 4\}$, are found by analyzing wind-tunnel data from [15] for an F-16 at angle of attack $\alpha = 0°$ (Table 5.1) and $\alpha = 5°$ (Table 5.2). The two data sets are averaged to obtain an approximation at $\alpha \approx 2.11°$ in order to remain consistent with the rest of the aircraft data. Using this averaged data set, curve-fitting methods can be applied to find approximations that are of the form (5.4), (5.5). From Figures 5.4 - 5.5, the approximations on $f_1(\beta, p_s, r_s, \delta_a, \delta_r)$ and $f_2(\beta, p_s, r_s, \delta_a, \delta_r)$ are shown along with the wind-tunnel data with the constant parameters chosen as

$$\begin{array}{llll} A_1 = 0.1 & A_2 = 0.1 & A_3 = 0.1 & A_4 = 0.1 \\ D_1 = 0.075 & D_2 = 0.0016 & D_3 = 0.45 & D_4 = 0 \\ \omega_1 = 1.5 & \omega_2 = 0 & \omega_3 = 1.5 & \omega_4 = 0 \\ C_1 = 0.3 & C_3 = 0.3 & h_1 = 7 & h_2 = 4 \\ \sigma_1 = 0.15 & \sigma_2 = 0.15. & & \end{array} \tag{5.65}$$

Physically, this means that at high sideslip angle, only 30% of the control is available to the system. The Gaussian parameters σ_1, σ_2 are chosen so that the Gaussian $e^{-\frac{\beta^2}{2\sigma^2}}$ has a width of 60°, a reasonable range to assume for high angle manuevers. The curve-fit approximation assumes $\beta \in [30°, 30°]$, $p_s \in [-360°, 360°]$, $r_s \in [-90°, 90°]$, $\delta_a \in [-21.5°, 21.5°]$, and $\delta_r \in [-30°, 30°]$, which is seen to be most accurate for sideslip angle range $\beta \in [-20°, 20°]$. Figures 5.2 shows that the performance of the LQR controller is violated when uncertainties are present in the system. The uncertainties f_1, f_2 of order $O(1)$ cause β to vary only slightly, but ϕ tracking is off by almost 30° (Figure 5.3). Hence, it is necessary to use adaptive control to compensate for these uncertainties to recover the optimal performance.

β	δ_l	δ_n	L_{δ_a}	L_{δ_r}	N_{δ_a}	N_{δ_r}
$-30°$	0	0	-0.053	0.014	-0.017	-0.052
$-20°$	-0.017	0.042	-0.053	0.014	-0.016	-0.043
$-10°$	-0.022	0.076	-0.052	0.011	-0.006	-0.038
$0°$	-0.023	0.106	-0.051	0.015	-0.010	-0.045
$10°$	-0.022	0.076	-0.048	0.013	-0.014	-0.041
$20°$	-0.017	0.042	-0.048	0.010	-0.004	-0.036
$30°$	0	0	-0.047	0.011	-0.003	-0.027

Table 5.1: Wind Tunnel Data of Aerodynamic and Control Derivatives at $\alpha = 0°$

β	δ_l	δ_n	L_{δ_a}	L_{δ_r}	N_{δ_a}	N_{δ_r}
$-30°$	0	0	-0.056	0.010	-0.013	-0.052
$-20°$	-0.024	0.042	-0.053	0.014	-0.016	-0.046
$-10°$	-0.034	0.074	-0.051	0.012	-0.006	-0.040
$0°$	-0.037	0.106	-0.052	0.014	-0.009	-0.045
$10°$	-0.034	0.074	-0.049	0.013	-0.012	-0.041
$20°$	-0.024	0.042	-0.047	0.011	-0.002	-0.036
$30°$	0	0	-0.045	0.011	-0.005	-0.028

Table 5.2: Wind Tunnel Data of Aerodynamic and Control Derivatives at $\alpha = 5°$

β	δ_l	δ_n	L_{δ_a}	L_{δ_r}	N_{δ_a}	N_{δ_r}
$-30°$	0	0	-0.055	0.012	-0.015	-0.052
$-20°$	-0.021	0.042	-0.053	0.014	-0.016	-0.045
$-10°$	-0.028	0.075	-0.052	0.012	-0.006	-0.039
$0°$	-0.030	0.106	-0.052	0.015	-0.001	-0.045
$10°$	-0.028	0.075	-0.049	0.013	-0.013	-0.041
$20°$	-0.021	0.042	-0.048	0.011	-0.003	-0.036
$30°$	0	0	-0.046	0.011	-0.004	-0.028

Table 5.3: Averaged Wind Tunnel Data of Aerodynamic and Control Derivatives to Approximate $\alpha \approx 2.11°$

The state predictor is designed with two sets of 24 RBFs. For each of the sets, 8 are $\Phi_{i,1}(\beta)$-type Gaussians distributed over $\beta \in [-30^\circ, 30^\circ]$ with the step size equal to 60° and width $\delta_1, \delta_2 = 1$ for $i = \{1, 2\}$. The remaining 16 were $\Phi_{1,2}(\beta, \delta_a), \Phi_{2,2}(\beta, \delta_r)$-type basis functions distributed over $\beta \in [-30^\circ, 30^\circ]$, $\delta_{a_{ad}} \in [-21.5^\circ, 21.5^\circ]$, and $\delta_{r_{ad}} \in [-30^\circ, 30^\circ]$ with width $\rho_1 = \rho_2 = 5$. The norm upper bound is $W_1^* = W_2^* = 10$, the lower bound for the positive widths w_1, w_2 is 0.01, and the adaptation gain $\Gamma_1 = \Gamma_2 = 0.2$. The state predictor is

$$\begin{bmatrix} \dot{\hat{\beta}}(t) \\ \dot{\hat{\phi}}(t) \\ \dot{\hat{p}}_s(t) \\ \dot{\hat{r}}_s(t) \end{bmatrix} = A_r \begin{bmatrix} \hat{\beta}(t) \\ \hat{\phi}(t) \\ \hat{p}_s(t) \\ \hat{r}_s(t) \end{bmatrix} + B \begin{bmatrix} \delta_a(t) + \hat{\theta}_1^\top(t)\Phi_{1,1}(\beta(t)) + \hat{w}_1^\top(t)\Phi_{1,2}(\beta(t), \delta_a(t)) \\ \delta_r(t) + \hat{\theta}_2^\top(t)\Phi_{2,1}(\beta(t)) + \hat{w}_2^\top(t)\Phi_{2,2}(\beta(t), \delta_r(t)) \end{bmatrix} \quad (5.66)$$

and the error dynamics are

$$\dot{e}(t) = A_r e(t) + B \begin{bmatrix} \delta_{a_{ad}}(t) + \hat{\theta}_1^\top(t)\Phi_{1,1}(\beta(t)) + \hat{w}_1^\top(t)\Phi_{1,2}(\beta(t), \delta_a(t)) \\ \delta_{r_{ad}}(t) + \hat{\theta}_2^\top(t)\Phi_{2,1}(\beta(t)) + \hat{w}_2^\top(t)\Phi_{2,2}(\beta(t), \delta_r(t)) \end{bmatrix}. \quad (5.67)$$

For desired performance, the following equation needs to be solved for adaptive aileron and rudder control, $\delta_{a_{ad}}, \delta_{r_{ad}}$:

$$\begin{bmatrix} \delta_{a_{ad}}(t) \\ \delta_{r_{ad}}(t) \end{bmatrix} = \begin{bmatrix} -\hat{\theta}_1^\top(t)\Phi_{1,1}(\beta(t)) + \hat{w}_1^\top(t)\Phi_{1,2}(\beta(t), \delta_{a_{nom}}(t) + \delta_{a_{ad}}) \\ -\hat{\theta}_2^\top(t)\Phi_{2,1}(\beta(t)) + \hat{w}_2^\top(t)\Phi_{2,2}(\beta(t), \delta_{r_{nom}}(t) + \delta_{r_{ad}}) \end{bmatrix}, \quad (5.68)$$

where $\delta_{a_{nom}}, \delta_{r_{nom}}$ has been defined in (5.20). The fast dynamics are designed as

$$-0.05 \begin{bmatrix} \dot{\delta}_{a_{ad}}(t) \\ \dot{\delta}_{r_{ad}}(t) \end{bmatrix} = \bar{\mathbf{P}}(t, e, \delta_{a_{ad}}(t), \delta_{r_{ad}}(t))\mathbf{f}(t, e, \delta_{a_{ad}}(t), \delta_{r_{ad}}(t)), \quad (5.69)$$

where

$$\bar{\mathbf{P}}(t, e, \delta_{a_{ad}}, \delta_{r_{ad}}) = \begin{bmatrix} \hat{w}_1^\top(t)\Phi_{1,2}(\beta(t), \delta_{a_{nom}}(t) + \delta_{a_{ad}}) + 1 & 0 \\ 0 & \hat{w}_2^\top(t)\Phi_{2,2}(\beta(t), \delta_{r_{nom}}(t) + \delta_{r_{ad}}) \end{bmatrix}$$

and

$$\mathbf{f}(t, e, \delta_{a_{ad}}, \delta_{r_{ad}}) = \begin{bmatrix} \delta_{a_{ad}}(t) + \hat{\theta}_1^\top(t)\Phi_{1,1}(\beta(t)) + \hat{w}_1^\top(t)\Phi_{1,2}(\beta(t), \delta_{a_{nom}}(t) + \delta_{a_{ad}}) \\ \delta_{r_{ad}}(t) + \hat{\theta}_2^\top(t)\Phi_{2,1}(\beta(t)) + \hat{w}_2^\top(t)\Phi_{2,2}(\beta(t), \delta_{r_{nom}}(t) + \delta_{r_{ad}}) \end{bmatrix}.$$

Initial conditions used for the simulations are

$$\begin{bmatrix} \beta_0 \\ \phi_0 \\ p_{s0} \\ r_{s0} \end{bmatrix} = \begin{bmatrix} 0^\circ \\ 0^\circ \\ 0^\circ/\text{sec} \\ 0^\circ/\text{sec} \end{bmatrix}, \quad \begin{bmatrix} \hat{\beta}_0 \\ \hat{\phi}_0 \\ \hat{p}_{s0} \\ \hat{r}_{s0} \end{bmatrix} = \begin{bmatrix} 0^\circ \\ 0^\circ \\ 0^\circ/\text{sec} \\ 0^\circ/\text{sec} \end{bmatrix}.$$

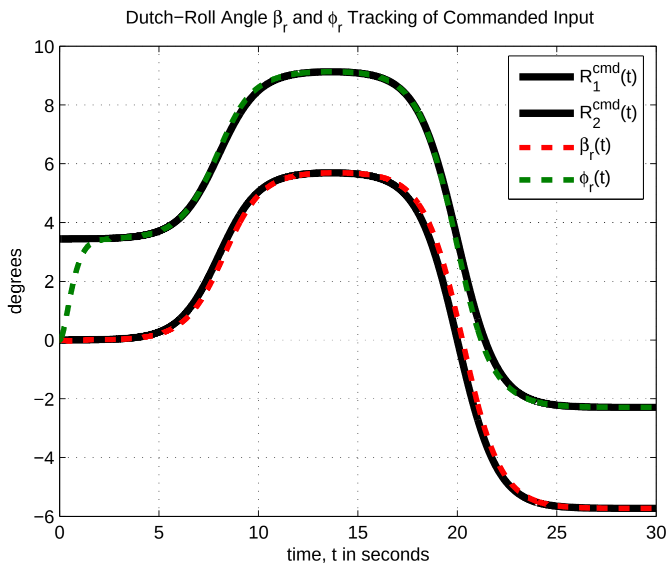

Figure 5.1: Dutch-roll nominal system $\beta_r(t)$ and $\phi_r(t)$ tracking of commanded input $R^{cmd}(t)$.

The plots in Figures 5.6 - 5.8 show the tracking performance with the augmented adaptive controller. Figure 5.9 shows the time history of the adaptive control signal and the unknown nonlinearities $f_1(\beta(t), \delta_a(t))$ and $f_2(\beta(t), \delta_r(t))$ as functions of time. Asymptotic tracking is achieved of the angles β, ϕ while the other states of the system p_s, r_s, remain bounded. Additionally, the approximations stay within their respective domains.

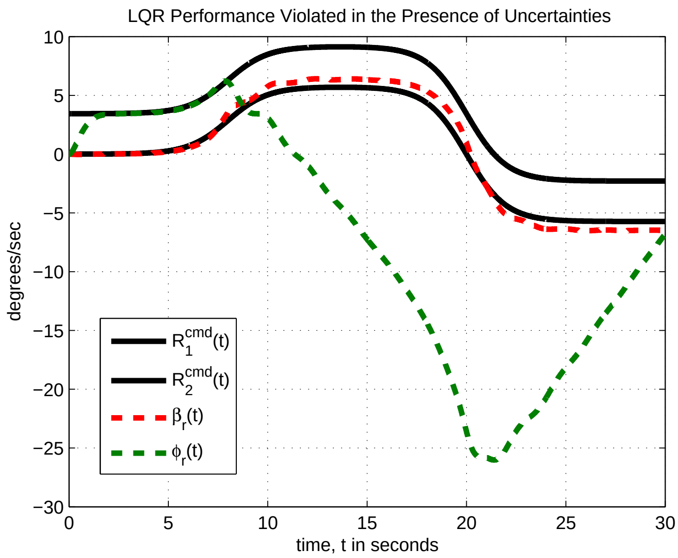

Figure 5.2: LQR performance violated for angle β_r, ϕ_r tracking.

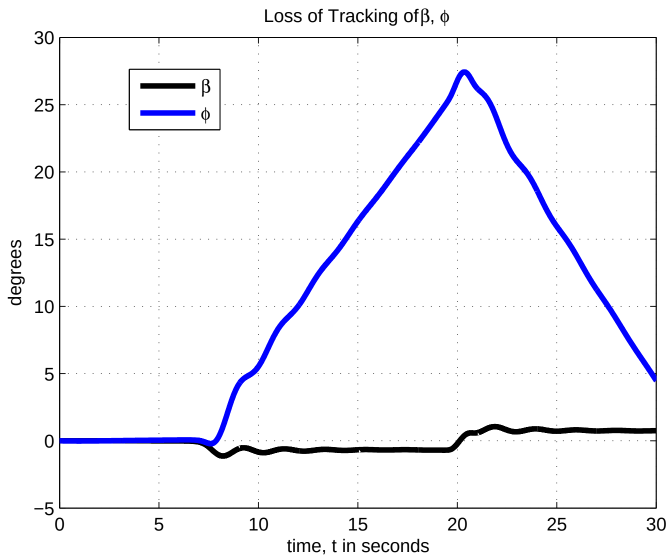

Figure 5.3: Loss of tracking of β and ϕ caused by nonlinearities $f_1(\beta, \delta_a)$ of order $O(1)$ and $f_2(\beta, \delta_r)$ of order $O(1.1)$.

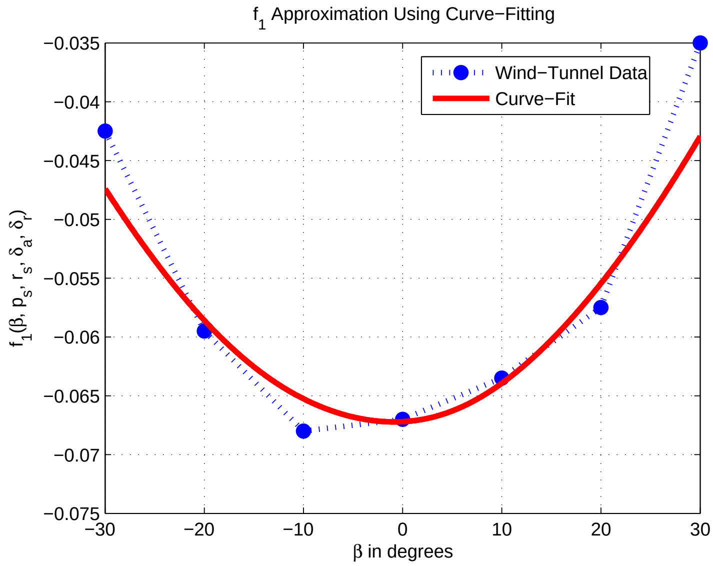

Figure 5.4: Curve-fitting for $f_l(\beta, p_s, r_s, \delta_a, \delta_r)$.

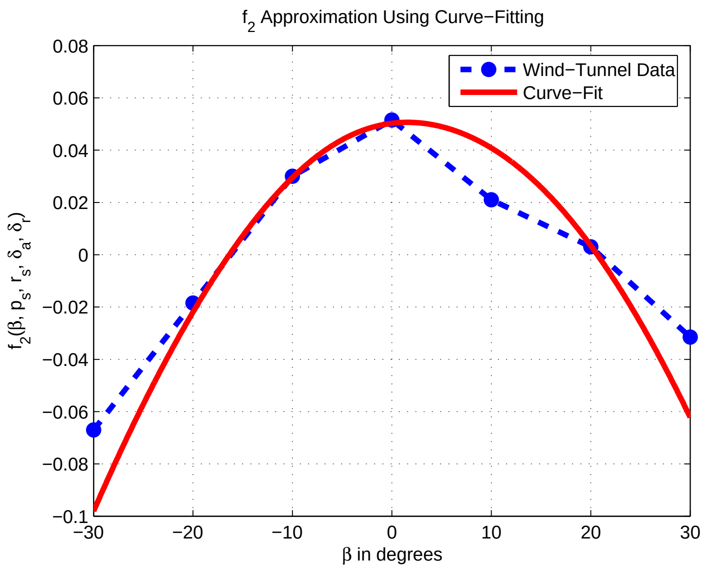

Figure 5.5: Curve-fitting for $f_2(\beta, p_s, r_s, \delta_a, \delta_r)$.

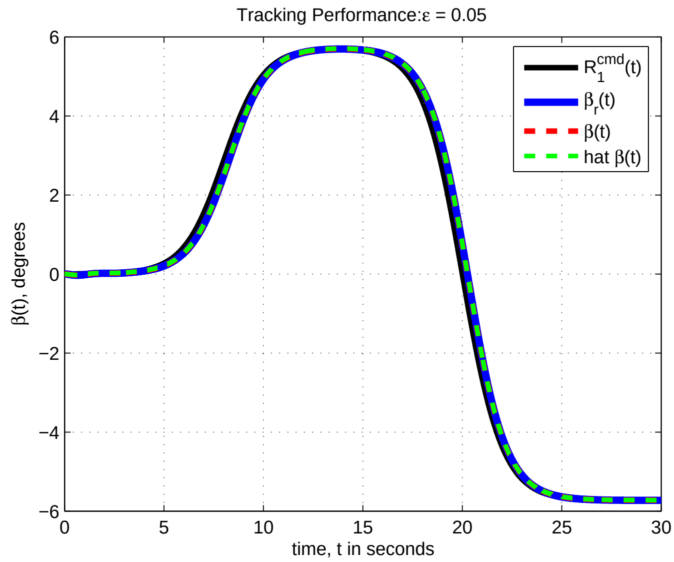

Figure 5.6: States $\hat{\beta}(t), \beta_r(t), \beta(t)$ of Dutch-Roll system.

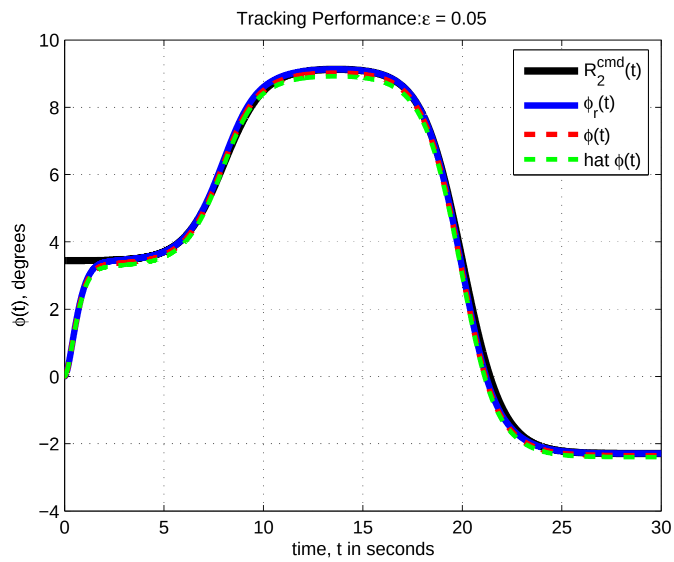

Figure 5.7: States $\hat{\phi}(t), \phi_r(t), \phi(t)$ of Dutch-Roll system.

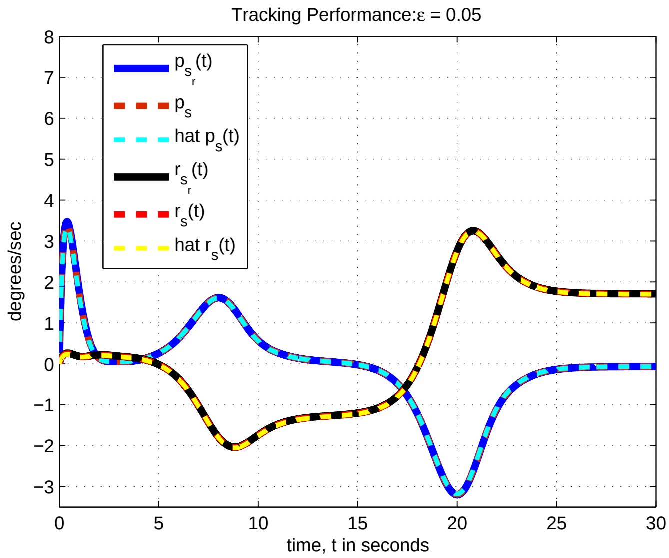

Figure 5.8: States $\hat{p}_s(t), p_{s_r}(t), p_s(t), \hat{r}_s(t), r_{s_r}(t), r_s(t)$ of Dutch-Roll system.

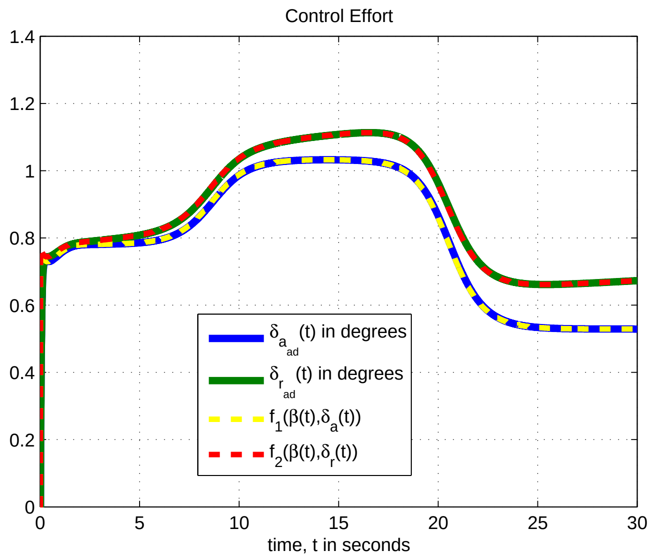

Figure 5.9: Adaptive control effort of Dutch-Roll System compared with uncertainties f_1 and f_2.

Chapter 6

Conclusion

6.1 Summary

For aircraft systems, uncertainties often arise within the system dynamics due to external and internal forces that are unknown to the controller. Additionally, for high-performance aircraft, the nonlinearities that result from the equations of motion cannot be neglected or linearized as in straight and level flight. Performance of traditional linear controllers are violated in the presence of these nonlinearities that are unknown unstructured functions of control input and states. This can lead to loss of tracking and instability. In order to account for unpredictable changes and unknown nonlinear dependencies in aircraft dynamics, functional approximations using basis functions can be used over a compact set to approximate these unknowns nonlinearities. An approximate dynamic inversion methodology has been presented for the nonaffine-in-control short-period and lateral/directional aircraft dynamics via time-scale separation where the control signal was defined as a solution of "fast" dynamics. For both systems, the form of the unknown functions allowed for the existence of an isolated root in control. Moreover, the derivatives with respect to control were strictly positive definite. Hence, we were able to verify the assumptions of Tikhonov's theorem for the closed-loop system.

6.2 Recommendations For Future Work

Development of general theory for adaptive control of multi-input nonaffine systems remains still to be solved. The particular case of Dutch-Roll dynamics, assumed in the paper, had a block-diagonal decoupled Jacobian with respect to two control inputs. This simplified verification of assumptions, required in [28], for multi-input nonaffine systems. In case of fully coupled dynamics, such verification for adaptive control design is not obvious and requires further investigation.

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