Appendix A

The Saddlepoint Approximation

In this Appendix the Saddlepoint Approximation is described. It is shown how the Moment Generating Functions (MGFs) can be used to calculate the error probabilities. MGFs often possess a mathematical form much simpler than the pdfs. In many problems it is relatively easy to obtain the MGF, but almost impossible to apply an inverse Laplace transform which converts the MGF to a pdf.

First, the upper tail $q_+(\alpha)$ of the probability density function *f(x)* is defined as [22,23]

$$q_+(\alpha) = \int_{\alpha}^{\infty} f(x)dx \tag{A.1}$$

and in a similar way the lower tail of the probability distribution $q_-(\alpha)$ is defined as

$$q_-(\alpha) = \int_{-\infty}^{\alpha} f(x)dx \tag{A.2}$$

An important point to notice is that

$$q_+(\alpha) + q_-(\alpha) = 1 \tag{A.3}$$

The Laplace transform of *f(x)* in terms of the MGF can be expressed as

$$\int_{-\infty}^{\infty} f(x)e^{-sx}dx = M(-s) \tag{A.4}$$

Thus, *f(x)* is equal to the inverse integral [23]

$$f(x) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} M(-s)e^{sx}ds \tag{A.5}$$

with c being the convergence region of the transform.

By using (A.1) and (A.4) and choosing the contour of integration such that c<0 to guarantee convergence of the integral the following expression is obtained for the upper tail [22]

$$q_+(\alpha) = \frac{-1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \frac{e^{s\alpha}}{s} M(-s) ds \tag{A.6}$$

By changing the integration variable from –s to s, the following expression is obtained

$$q_+(\alpha) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \frac{e^{-s\alpha}}{s} M(s) ds \quad \text{with } c > 0 \tag{A.7}$$

It should be noted that c is chosen such that it has to be the value of s for which the integrand is minimal. This point which is denoted as s_{ON} corresponds to a saddlepoint in the complex plane (thus the name Saddlepoint Approximation). A saddlepoint is a point that corresponds to a minimum in the real axis, and a maximum in the imaginary axis. The integrand of (A.7) for a typical MGF is illustrated in Figure A.1 [29,30].

The integrand can be expressed in terms of a "phase" function $\phi(s)$, which is given by

$$\phi(s) = \ln[M(s)] - s\alpha - \ln|s| \tag{A.8}$$

The function $\phi(s)$ is expanded in Taylor's series about the point $s = s_{ON}$

$$\phi(s) = \phi(s_{ON}) + \frac{1}{2}\phi''(s_{ON})(s - s_{ON})^2 + \ldots + (k!)^{-1}\phi^{(k)}(s_{ON})(s - s_{ON})^k + \ldots \tag{A.9}$$

The first derivative does not appear because s_{ON} is an extremum of $\phi(s)$.

Using (E7) and (E5) and calling $y = (s - s_{ON})$ we get the following saddlepoint approximation [19,22]

$$q_+(\alpha) \approx \frac{1}{2\pi}\exp[\phi(s_{ON})]\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2}\phi''(s_{ON})y^2\right]dy$$

$$= \frac{\exp[\phi(s_{ON})]}{\sqrt{2\pi\phi''(s_{ON})}} \qquad \text{(A.10)}$$

It should be noted that s_{ON} is the value of s that makes $\phi(s)$ minimum. It is obtained by taking the negative root of the equation

$$\phi'(s) = 0 \qquad \text{(A.11)}$$

For the lower tail, the process is analogous, and we get

$$q_-(\alpha) = \frac{-1}{2\pi j}\int_{c-j\infty}^{c+j\infty}\frac{\exp(-s\alpha)}{s}M(s)ds \quad \text{with } c < 0 \qquad \text{(A.12)}$$

By expanding $\phi(s)$ in a Taylor series and integrating, we get

$$q_-(\alpha) = \frac{\exp[\phi(s_{OFF})]}{\sqrt{2\pi\phi''(s_{OFF})}} \qquad \text{(A.13)}$$

with s_{OFF1} being the positive root of (E9).

An important point to be noted in this derivation is that to obtain the final forms of the upper and lower bounds only the first two terms from the Taylor series expansion were used. Even with this approximation the saddlepoint approximation method yields good results as it was shown in Chapter 2. If a better accuracy is needed higher-order terms can be included as shown in [23].

A.1. Error Probability

To obtain the error probability expression the symbol sent b_0 and the finite sequence $\tilde{B} = (b_{-L}, \ldots, b_{-1}, b_1, \ldots, b_L)$ of symbols surrounding b_0 have to be considered. The symbols $b_0 = 1$ and $b_0 = 0$ are assumed to be equally probable, and thus the conditional error given a sequence $\tilde{B}$ is [19]

$$P_e \,|_{\tilde{B}} = \frac{1}{2} P_r (Z < \alpha \,|_{\tilde{B}|b_0=1}) + \frac{1}{2} P_r (Z > \alpha \,|_{\tilde{B}|b_0=0})$$
$$= \frac{1}{2} \{ q_+(\alpha) + q_-(\alpha) \} \qquad \text{(A.14)}$$

By taking (A10) and (A13) we get

$$P_e \,|_{\tilde{B}} = \frac{1}{2} \left\{ \frac{\exp[\phi(s_0)]}{\sqrt{2\pi\phi''(s_0)}} + \frac{\exp[\phi(s_1)]}{\sqrt{2\pi\phi''(s_1)}} \right\} \qquad \text{(A.15)}$$

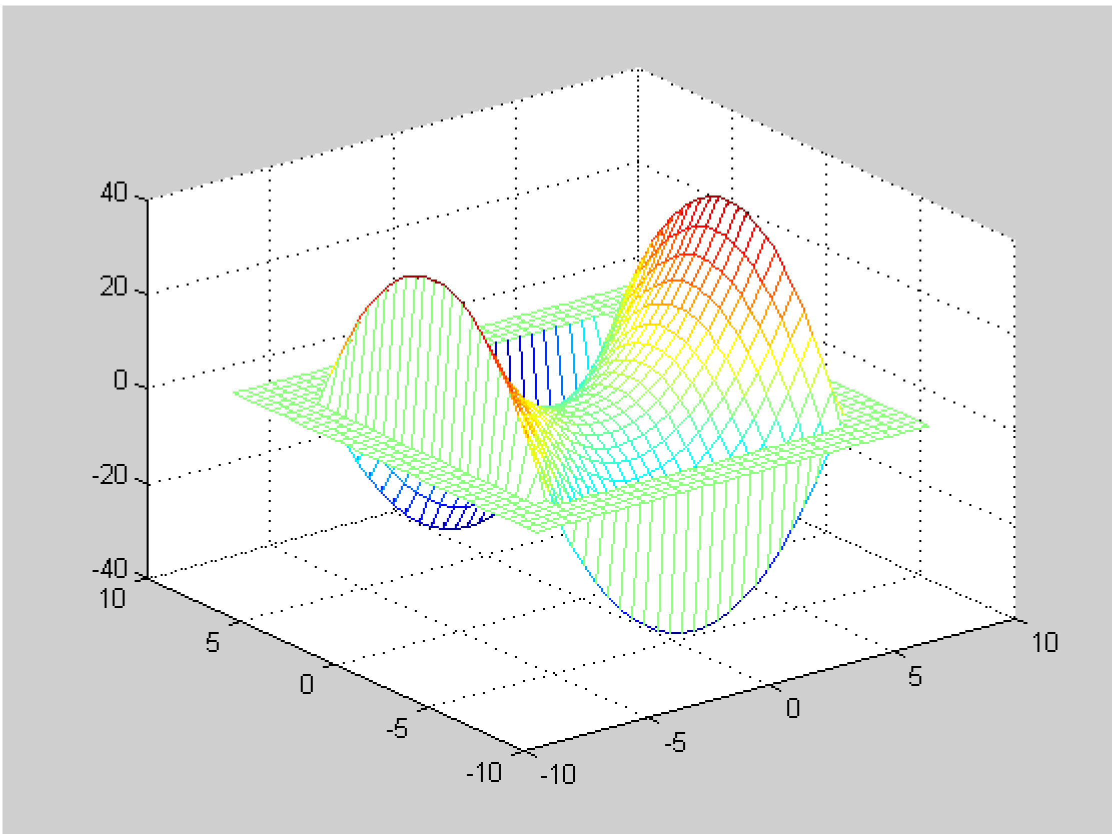

Fig A.1 *Arbitrary function with a saddlepoint.*