**Appendix B**

# The Chi-Square Distribution

## B.1. The Gamma Function

To define the chi-square distribution one has to first introduce the *Gamma function*, which can be denoted as [21]:

$$\Gamma(p) = \int_0^\infty x^{p-1} e^{-x} dx \quad , p > 0 \tag{B.1}$$

If we integrate by parts [25], making $e^{-x}dx = dv$ and $x^{p-1} = u$ we will obtain

$$\begin{aligned} \Gamma(p) &= -e^{-x} x^{p-1} |_0^\infty - \int_0^\infty [-e^{-x}(p-1)x^{p-2}dx] \\ &= 0 + (p-1)\int_0^\infty e^{-x} x^{p-2} dx \\ &= (p-1)\Gamma(p-1) \end{aligned} \tag{B.2}$$

By this way, we can demonstrate that the Gamma function obeys an interesting recurrence relation. If $p$ is a positive integer, then applying equation (B.2) repetitively we obtain [21]

$$\begin{aligned} \Gamma(p) &= (p-1)\Gamma(p-1) \\ &= (p-1)(p-2)\Gamma(p-2) = \ldots = (p-1)(p-2)\ldots\Gamma(1) \end{aligned} \tag{B.3}$$

But,

$$\Gamma(1) = \int_0^\infty e^{-x} dx = 1 \tag{B.4}$$

And thus we obtain

$$\Gamma(p) = (p-1)! \tag{B.5}$$

Another important relation for the Gamma function is [21,26]:

$$\Gamma\left(\frac{1}{2}\right) = \int_0^{\infty} x^{-1/2} e^{-x}\, dx = \sqrt{\pi} \tag{B.6}$$

## B.2. Derivation of the Chi-Square Distribution

A direct relation exists between a chi-square-distributed random variable and a gaussian random variable. The chi-square random variable is in a certain form a transformation of the gaussian random variable. If we have *X* as a gaussian random variable and we take the relation $Y=X^2$ then *Y* has a chi-square distribution with one degree of freedom [21].

If we define the random variable *Y* as

$$Y = aX^2 + b, \quad a > 0 \tag{B.7}$$

Then the pdf of *Y* in terms of the pdf of *X* can be expressed as [31]

$$f_Y(y) = \frac{f_X\left[\sqrt{(y-b)/a}\right]}{2a\sqrt{\left[(y-b)/a\right]}} + \frac{f_X\left[-\sqrt{(y-b)/a}\right]}{2a\sqrt{\left[(y-b)/a\right]}} \tag{B.8}$$

Using the results above we can now derive the pdf of a chi-square random variable with one degree of freedom. We will take *X* to be gaussian-distributed with zero mean and variance $\sigma^2$. As was mentioned previously we have $Y=X^2$ which implies that $a=1$ and $b=0$ in (B.7). Using (B.8) we obtain as the pdf of *Y* the following expression

$$f_Y(y) = \frac{1}{\sqrt{2\pi y}\sigma} e^{-y/2\sigma^2} \quad \text{, with } y \geq 0 \tag{B.9}$$

The characteristic function of $Y$ can be expressed as [31]

$$\begin{aligned} \psi_Y(j\omega) &= \int_{-\infty}^{\infty} e^{j\omega y} f_Y(y) dy \\ &= \frac{1}{(1 - j2\omega\sigma^2)^{1/2}} \end{aligned} \tag{B.10}$$

If we define now our random variable $Y$ as

$$Y = \sum_{i=1}^{n} X_i^2 \tag{B.11}$$

with the $X_i$, $i = 1,2,...,n$, being statistically independent and identically distributed gaussian random variables with zero mean and variance $\sigma^2$. Thus we obtain the characteristic function of $Y$ as [31]

$$\psi_Y(j\omega) = \frac{1}{(1 - j2\omega\sigma^2)^{n/2}} \tag{B.12}$$

Taking the inverse transform of (B.12) we get the pdf of $Y$ as

$$f_Y(y) = \frac{1}{\sigma^n 2^{n/2} \Gamma\left(\frac{1}{2}n\right)} y^{(n/2)-1} e^{-y/2\sigma^2} \quad \text{, for } y \geq 0 \tag{B.13}$$

This pdf is called a chi-square pdf with *n* degrees of freedom. Figures B.1 to B.4 illustrate this pdf, for purpose of illustration we assumed $\sigma^2 = 1$. An important point to notice is that when $n$=2, we obtain an exponential distribution.

## B.3. Moment Generating Function (MGF)

Let $X$ be a continuous random variable with probability density function (pdf) $f$. We will define the Moment Generating Function (MGF) as [32]

$$M_X(s) = \int_{-\infty}^{+\infty} e^{sx} f(x)dx \tag{B.14}$$

By comparing equations (B.10) and (B.14) it can be seen that the Moment Generating Function and the Characteristic Function are directly related. The Characteristic Function is obtained when the $s$ parameter in the MGF is substituted by $j\omega$ .

The Moment Generating Function has the following properties [21,32]:

$$\begin{aligned} M'(0) &= E(X) \\ M''(0) &= E(X^2) \\ &\vdots \\ M^{(n)}(0) &= E(X^n) \end{aligned}$$

Thus we obtain:

$$\sigma^2(X) = E(X^2) - [E(X)]^2 = M''(0) - [M'(0)]^2 \tag{B.15}$$

For the chi-square distribution with $n$ degrees of freedom, the MGF is given by[21]:

$$M_Y(s) = (1 - 2\sigma^2 s)^{-n/2} \tag{B.16}$$

The mean and variance of the chi-square distribution, which can be extracted from the MGF, are thus:

$$E(Y) = n\sigma^2 \tag{B.17}$$

$$\sigma_Y^2 = 2n\sigma^4 \tag{B.18}$$

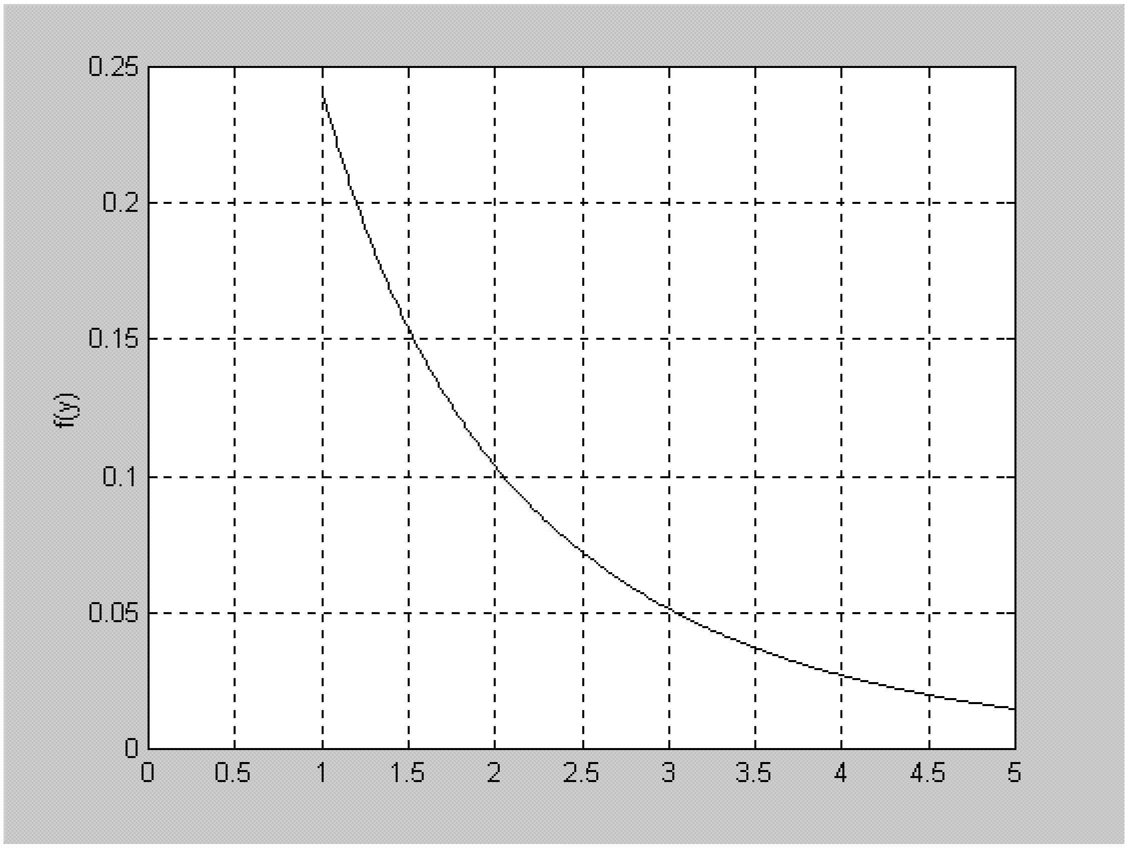


**Fig B.1** *Chi-square pdf for 1 degree of freedom*

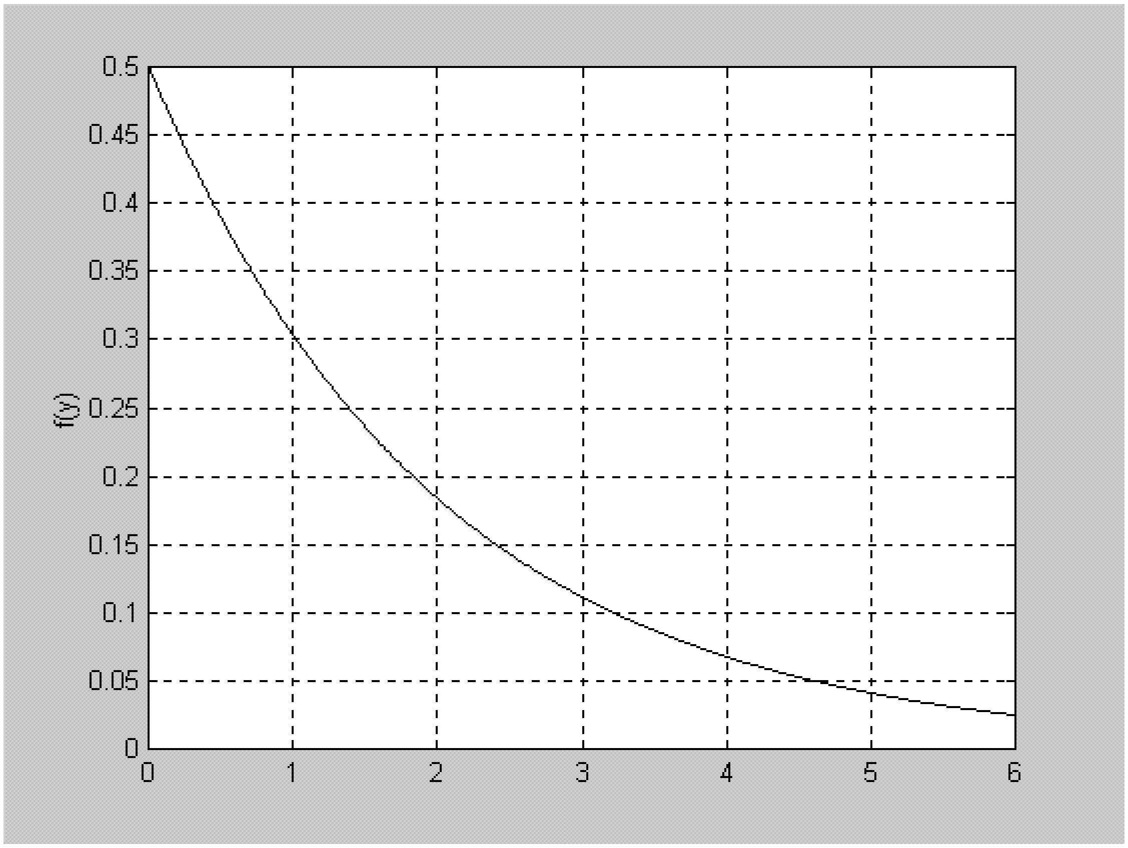


**Fig B.2** *Chi-square pdf for 2 degrees of freedom*

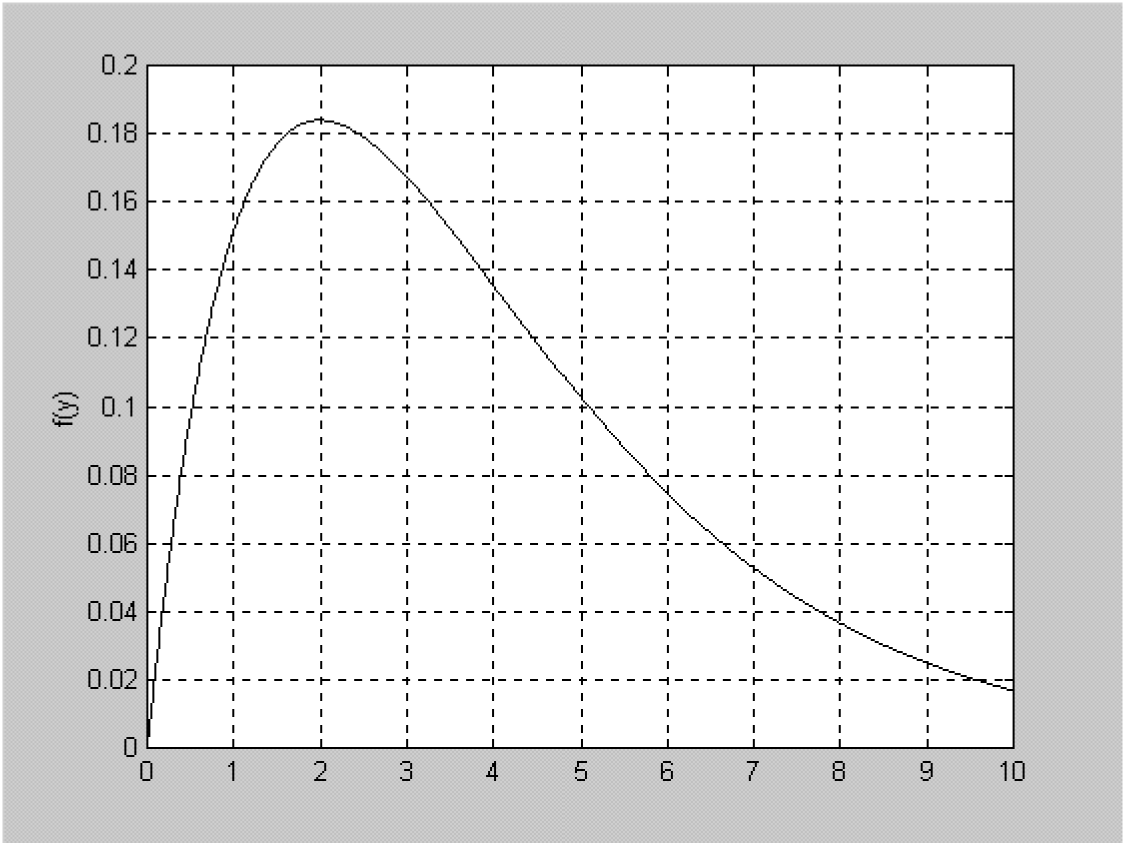


**Fig B.3** *Chi-square pdf for 4 degrees of freedom*

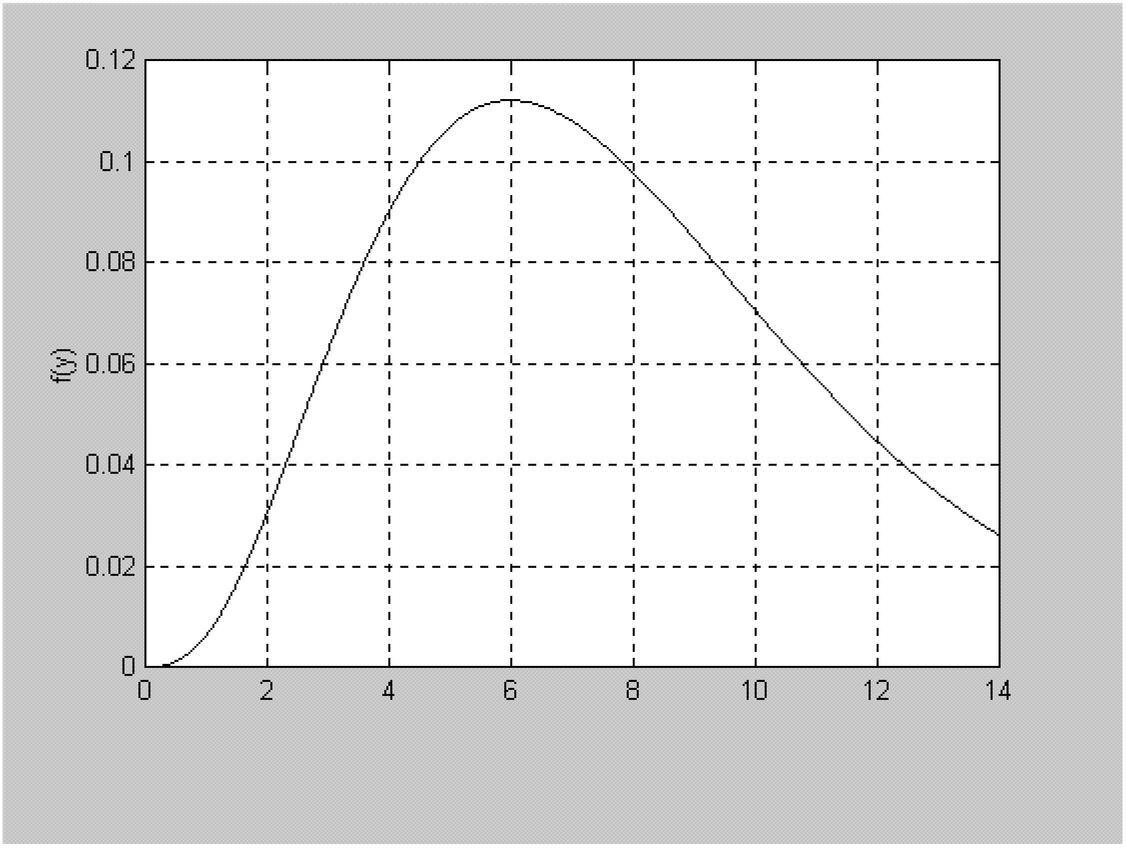


**Fig B.4** *Chi-square pdf for 8 degrees of freedom*