

# Analytical Solution of two Traction-Value Problems in Second-Order Elasticity with Live Loads

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## **(Abstract)**

We present a generalization of Signorini's method to the case of live loads which allows us to derive approximate solutions to some pure traction-value problems in finite elastostatics. The boundary value problems and the corresponding compatibility conditions are formulated in order to determine the displacement of the system up to the second-order approximation. In particular, we consider the case of homogeneous and isotropic elastic bodies and we solve the following two pure traction-value problems with live loads: (i) a sphere subjected to the action of a uniform pressure field; (ii) a hollow circular cylinder whose inner and outer surfaces are subjected to uniform pressures. Then, starting from these solutions, we suggest experiments to determine the second-order constitutive constants of the elastic body. Expressions of the second-order material constants in terms of displacements and Lamé coefficients are determined.

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# Introduction

## 1.1 Preliminary Considerations

The nonlinear Theory of Elasticity presents many interesting problems both from mathematical and physical points of view. The main issues are essentially related to the following topics.

1. Equations governing the equilibrium and the motion of an elastic system are not linear. This implies that, except for some cases of incompressible materials, it is not possible to obtain an analytical solution to these equations. Then, it is necessary to resort to procedures which allow us to determine *approximate solutions*, which are extremely useful in practical applications. It is also common to turn to numerical routines which are presently very effective thanks to the extraordinary developments in modern computers.
2. A further complication of the theory for live loads is that boundary conditions can be assigned only as a function of the unknown deformation.
3. Owing to the nonlinearity of the equations, the wave propagation problems are very hard to study and may present blow-up phenomena.
4. The determination of the response of a nonlinear elastic material is not an easy task. For homogeneous and isotropic solid bodies it reduces to finding the form of the specific strain energy. This situation is more complex than the one in the



linear Theory of Elasticity. In the latter case the response of an isotropic material is completely characterized by only two constants, the *Lamé coefficients*, which can be experimentally determined by a simple tension test.

In this thesis we deal with points 1, 2, and 4. The study of wave propagation in nonlinear materials represents the next step of the present work.

## 1.2 A Brief Historical Survey

In 1930, Signorini [1] suggested a perturbation method to find approximate solutions of boundary-value problems of finite elasticity in the presence of *dead loads* (loads independent of the deformation). This procedure is essentially an application of Poincaré's perturbation method (e.g., see [2] and [3]) to equations of finite elasticity. Furthermore, in [4] and [5] Signorini's method was used to investigate the uniqueness of solutions as well as the position of the classical linear theory with respect to the nonlinear theory. Later, Stoppelli [6-10] proved a local theorem of existence, uniqueness, and analytic dependence on a parameter for the solution to the traction-value problem, when the applied dead loads do not have an *axis of equilibrium*, and the existence and analyticity of solutions when the dead loads have an axis of equilibrium (see also Tolotti [11]). A discussion of Stoppelli's work can be found in [12-14]. In [15-19] Capriz and Podio-Guidugli investigate the compatibility of the linear and nonlinear elasticity theories and show that a very large class of traction-value problems can be solved by perturbation methods of Signorini's type. In particular, in [16], the authors provide a series expansion to construct an approximate solution

to the static balance equations of elastic bodies under dead loads and discuss the meaning and implication of the Fredholm-type conditions for the existence of such an expansion. Furthermore, in [18] and [19] traction-value problems of finite elasticity are analyzed in the presence of loads depending on the deformation (*live loads*). Although every realistic load depends on the deformation, the introduction of live loads leads to difficult mathematical problems. A crucial contribution in this framework, has been given by Valent in many papers which are collected in [20]. In this book, Valent proves theorems of existence, uniqueness, and analytic dependence on a parameter for boundary-value problems of place and traction in finite elastostatics with dead loads and some special types of live loads.

### **1.3 Detailed Summary of the Work**

This thesis is divided into four chapters.

The first chapter is devoted to the statement of the equilibrium problem in finite elastostatics. First, the balance equations are written in the Eulerian and the Lagrangian formulation for an arbitrary continuous system and then they are specialized to the case of elastic systems, both in the linear and the nonlinear framework.

The second chapter is concerned with some issues regarding the formulation of the equilibrium problem. We start from the classification of boundary-value problems of finite elastostatics and then discuss difficulties related to the pure traction-value problem. In particular, we analyze the concept of live loads and describe the resulting mathematical complications. Further, we discuss the nature of the global equilibrium conditions, observ-

ing that in the pure traction problem and in the presence of live loads they are essentially compatibility conditions for the data and the displacements.

The third chapter deals with the description of the classical Signorini's method for dead loads. In order to obtain conditions under which the perturbation method can be applied, the equilibrium equations, the boundary conditions and the corresponding compatibility conditions are written in a nondimensional form. Reference is made to the existence and the uniqueness results obtained by Stoppelli [6, 7, 9, 10] and Van Buren [21].

The last chapter collects results obtained in [22]. Starting from the results obtained by Valent in Chapter 6 of [20], we provide a generalization of Signorini's method to the case of live loads. In the framework of second-order elasticity theory we solve two traction-value problems with live loads and design four experiments which allow us to determine the second-order constitutive constants for the given material.

# Chapter 1

## Equilibrium of an Elastic System

### 1.1 Introduction

In this chapter *the local equilibrium equations* are written for a continuous system, both in the *Eulerian* and in the *Lagrangian* formulation. These equations are general relations, that is, they are valid for all continuous systems and do not depend on the material of the body. However, it is well known from experience that two bodies having the same geometric characteristics react differently when subjected to the same mechanical loads and thermal conditions. Thus it is necessary to introduce some criteria which allow us to distinguish between the macroscopic behaviors of different material bodies. The mathematical description of different material behaviors is the object of *the theory of constitutive equations*. Even if these equations represent the material constitution of the body, they must fulfill certain general principles, called *constitutive axioms*, which impose restrictions on their forms.

In Section 5 of the present chapter the constitutive equations of a *continuous elastic system* are derived. It is noted there that, although constitutive axioms impose severe restrictions on the form of constitutive equations, they still allow the theory wide margin of arbitrariness, which can be filled only by experimental data. This is due to the fact that the macroscopic behavior of a continuous body is related to its molecular structure. Since

this structure does not enter into the continuum description, the constitutive equation of a particular material must be determined experimentally. The experimental determination of the constitutive equations of an elastic material is an extremely complicated task. It can be simplified by exploiting the *symmetry* of the material. After having introduced the concept of *isotropy group of a material*, the constitutive equations of *homogeneous and isotropic* elastic bodies are exhibited. In the last Section these equations are specialized to the case of *linear elastic materials*.

## 1.2 Finite Deformations

We consider a three-dimensional continuous system  $S$  moving in an inertial reference frame  $\mathbf{I}$  in which is assigned a Cartesian coordinate system  $R \equiv (O, \mathbf{e}_i)$ ,  $i = 1, 2, 3$ , where  $O$  is the origin and  $\mathbf{e}_i$  the unit vectors. The region of space occupied by points of  $S$  at a certain time instant  $t$  is called the *configuration* of  $S$  at the instant  $t$  and denoted by  $C(t)$ . In order to determine the motion of  $S$ , it is necessary in the first place to "label" all points of  $S$  and then to follow them during the motion, assigning their position in  $R$  at every instant.

To this aim a *reference configuration*  $C_*$  is introduced, that is a possible configuration of  $S$ , and we call material or *Lagrangian* coordinates the coordinates  $(X_L)$ ,  $L = 1, 2, 3$ , in  $R$  of a particle  $p \in S$  in the configuration  $C_*$ .

The configuration  $C(t)$  of  $S$  at the instant  $t$  is the *actual* or the *present configuration* of  $S$  and the coordinates  $(x_i)$  in  $R$  of the particle  $p \in S$  in  $C$  are referred to as spatial or *Eulerian* coordinates. Any quantity  $\phi$  associated to the motion of  $S$  can be expressed

in either the Lagrangian or the Eulerian form depending on whether it is intended as a function of the variables  $(X_L)$  or  $(x_i)$ , in other words depending on whether it is assumed to be defined on  $C_*$  or on  $C$ .

We call *finite deformation* from  $C_*$  to  $C$  the vectorial function

$$\mathbf{x} = \mathbf{x}(\mathbf{X}) \quad (1.1)$$

which maps any point  $\mathbf{X} \in C_*$  onto the corresponding position in  $C$ , or equivalently, the three scalar functions

$$x_i = x_i(X_L) \quad i, L = 1, 2, 3. \quad (1.2)$$

The functions (1.2) are assumed to be

1. one-to-one, and
2. of class  $C^1$  together with their inverse.

The first assumption assures that the system neither fractures during the motion nor does a crack close; the second one translates into mathematical terms the basic property of the matter that two particles cannot simultaneously occupy the same place (*impenetrability principle*). In particular, we require that at any point  $\mathbf{X} \in C_*$  functions (1.2) fulfill the condition

$$J = \det \left( \frac{\partial x_i}{\partial X_L} \right) > 0,$$

in order to guarantee the right-hand orientation of the frame of references. In other words (1.2) are diffeomorphisms preserving the topological properties of the reference configura-

tion. Differentiation of (1.2) yields

$$dx_i = \frac{\partial x_i}{\partial X_L}(\mathbf{X}) dX_L, \quad (1.3)$$

where we have used Einstein's summation notation in which the summation over two repeated indices is understood. Equation (1.3) defines at any point  $\mathbf{X} \in C_*$  a linear transformation which maps an infinitesimal vector  $d\mathbf{X}$  coming from  $\mathbf{X}$  onto the corresponding infinitesimal vector  $d\mathbf{x}$  coming from  $\mathbf{x}(\mathbf{X})$ . This transformation is called *deformation gradient* at  $\mathbf{X}$  and it is represented in the reference frame  $\mathbf{R}$  by the matrix

$$\mathbf{F} = (F_{iL}) \equiv \left( \frac{\partial x_i}{\partial X_L} \right). \quad (1.4)$$

It is easy to realize that equation (1.3) contains all the information regarding the deformation of the volume element at  $\mathbf{X}$  when passing from  $C_*$  to  $C$ .

In the sequel we shall make use of the following formulae, which can be deduced from (1.3) and relate the corresponding infinitesimal surface elements  $d\sigma_*$  and  $d\sigma$ , as well as the infinitesimal volume elements  $dc_*$  and  $dc$  in  $C_*$  and  $C$  respectively (for their derivation see [23])

$$d\sigma = J(\mathbf{F}^{-1})^T d\sigma_*, \quad (1.5)$$

$$dc = Jdc_*. \quad (1.6)$$

From equation (1.5) we can write

$$|d\sigma|^2 = d\sigma_i d\sigma_i = J^2 |d\sigma_*|^2 (\mathbf{F}^{-1})_{Li} N_{*L} (\mathbf{F}^{-1})_{Kj} N_{*K}. \quad (1.7)$$

If we introduce the *right Cauchy-Green tensor*

$$\mathbf{C} = \mathbf{F}^T \mathbf{F}, \quad C_{LM} = \frac{\partial x_k}{\partial X_L} \frac{\partial x_k}{\partial X_M},$$

since

$$(\mathbf{F}^{-1})_{Li}(\mathbf{F}^{-1})_{Mi} = (\mathbf{C}^{-1})_{LM},$$

from (1.7) we obtain the following relation which will play an important role in the sequel

$$d\sigma = J\sqrt{\mathbf{N}_*\mathbf{C}^{-1}\mathbf{N}_*}d\sigma_*. \quad (1.8)$$

The deformation of  $S$  when passing from  $C_*$  to  $C$  can be described in a completely equivalent way by resorting to the *displacement field*  $\mathbf{u}(\mathbf{X})$  defined by the following relation

$$\mathbf{u}(\mathbf{X}) = \mathbf{x}(\mathbf{X}) - \mathbf{X}. \quad (1.9)$$

Introducing the *displacement gradient*  $\mathbf{H}$  by the definition

$$\mathbf{H} = \nabla\mathbf{u}, \quad H_{iL} = \frac{\partial u_i}{\partial X_L},$$

from (1.9) it follows that

$$\mathbf{H} = \mathbf{F} - \mathbf{1}, \quad (1.10)$$

where  $\mathbf{1}$  is the  $3 \times 3$  identity matrix.

### 1.3 Mass Conservation Equation

The mass of a continuous system  $S$  is assumed to be continuously distributed over the whole region  $C(t)$  occupied by  $S$  at the instant  $t$ . In other words, we postulate the existence of a function  $\rho(\mathbf{x}, t)$  of class  $C^1$ , called *mass density*, such that, if  $c_*$  is a part of  $S$  in  $C_*$  and  $c$  its image through the deformation  $\mathbf{x}(\mathbf{X})$ , the mass of  $c_*$  at time  $t$  is given by

$$m(c_*) = \int_c \rho(\mathbf{x}, t) dc. \quad (1.11)$$



Like every quantity associated with  $S$ , the mass density can be expressed in Lagrangian form. In this case it will be denoted by the symbol  $\rho_*$  to highlight that it is a function assigned on  $C_*$ , namely  $\rho_* = \rho_*(\mathbf{X}, t)$ . Hence, the following relation holds

$$m(c_*) = \int_c \rho(\mathbf{x}, t) dc = \int_{c_*} \rho_*(\mathbf{X}, t) dc_*,$$

from which, having in mind the rule of the variable change in the multiple integrals (equation (1.6)), it follows that

$$\int_{c_*} [\rho(\mathbf{x}(\mathbf{X}, t), t) J - \rho_*(\mathbf{X}, t)] dc_* = 0.$$

Since  $c_*$  is an arbitrary volume, at any point of  $c_*$  in which the integrand function is regular we obtain the following *local Lagrangian formulation of the mass conservation*

$$\rho J = \rho_*. \quad (1.12)$$

## 1.4 Eulerian Formulation of the Equilibrium Equation

In the Mechanics of the Continuous Systems loads acting on the material region  $c \subset C$  from its exterior  $c^e$  are divided into *mass forces* (or *body forces*), continuously distributed over the region  $c$  and *contact forces* (or *surface loads*) acting on the boundary  $\partial c$ . The resultant force  $\mathbf{R}$  and the resultant moment  $\mathbf{M}_0$  with respect to a fixed point  $O$  are given by

$$\mathbf{R}(c, c^e) = \int_c \rho \mathbf{b} dc + \int_{\partial c} \mathbf{t} d\sigma, \quad (1.13)$$

$$\mathbf{M}_0(c, c^e) = \int_c (\mathbf{x} - \mathbf{x}_0) \times \rho \mathbf{b} dc + \int_{\partial c} (\mathbf{x} - \mathbf{x}_0) \times \mathbf{t} d\sigma, \quad (1.14)$$

where  $\mathbf{x}_0$  is the position vector of  $O$ ,  $\rho$  is the mass density of  $S$  and the *specific mass force*  $\mathbf{b}$  and the *traction*  $\mathbf{t}$  are defined on  $c$  and  $\partial c$  respectively.

The first integral in (1.13) takes into account all forces acting over  $c$  from the exterior of  $S$ . These actions are assumed to be known "a priori". They may be gravitational, electromagnetic, thermal etc. and are expressed by the assigned specific force field  $\mathbf{b}(\mathbf{x}, t)$  acting on the whole volume  $c$  and whose values are not influenced by the motion of  $S$ .

On the contrary,  $\mathbf{t}$  represents a contact force field acting at the boundary  $\partial c$  of  $c$ ; the behavior of these forces is deeply related to the motion of  $S$  and therefore they are unknown quantities.

We remark that the previous assumptions, although natural, are extremely restrictive. First of all, the mass forces acting on  $c$  can originate from the region outside of  $c$  but not necessarily from that outside of  $S$ . However, in this case they cannot be assumed as known since they depend on the motion of the system. Further, the assumption regarding the contact forces implies that the action of all bodies contacting  $\partial c$  is equivalent to the vector  $\mathbf{t}d\sigma$ . If one takes into account phenomena related to the inner structure of the medium, then it can be assumed that these forces are better represented by a force  $\mathbf{t}d\sigma$  and a couple  $\mathbf{m}d\sigma$ . This assumption gives rise to a branch of Continuum Mechanics referred to as the *theory of Cosserat (or micropolar) Continua*<sup>1</sup>. Hence in a micropolar body, in addition to body forces, the existence of an independent body couple density  $\mathbf{m}$  is postulated. If it is assumed, as we do in the sequel, that  $\mathbf{m} = \mathbf{0}$ ,  $S$  is called a *simple continuum*.

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<sup>1</sup> For a detailed description of micropolar continua, see [24-27].

We postulate that the force acting upon the surface element  $d\sigma$  is related to the deformation of particles close to  $d\sigma$  only, and depends on the orientation of  $d\sigma$  only through its outward normal vector  $\mathbf{n}$ . This is known as *Cauchy's postulate* and can be expressed in mathematical terms as

$$\mathbf{t} = \mathbf{t}(\mathbf{x}, t, \mathbf{n}). \quad (1.15)$$

This is one of the fundamental assumptions in the Mechanics of Simple Continua.

We assume that at equilibrium for any arbitrary material volume  $c \subset C$  of  $S$  the following conditions hold

$$\mathbf{R}(c, c^e) = \mathbf{0}, \quad \mathbf{M}_0(c, c^e) = \mathbf{0},$$

which, from (1.13) and (1.14), become

$$\int_{\partial c} \mathbf{t} d\sigma + \int_c \rho \mathbf{b} dc = \mathbf{0}, \quad (1.16)$$

$$\int_{\partial c} (\mathbf{x} - \mathbf{x}_0) \times \mathbf{t} d\sigma + \int_c (\mathbf{x} - \mathbf{x}_0) \times \rho \mathbf{b} dc = \mathbf{0}. \quad (1.17)$$

A fundamental theorem due to Cauchy shows that, under proper regularity conditions,  $\mathbf{t}$  is a linear function of  $\mathbf{n}$ , i.e.,

$$\mathbf{t} = \mathbf{T}(\mathbf{x}, t) \cdot \mathbf{n}, \quad (1.18)$$

where  $\mathbf{T}$  is a second order tensorial field independent of  $\mathbf{n}$  known as the *Cauchy stress tensor*<sup>2</sup>. Equation (1.18) allows us to write the *local* form of equation (1.16). In fact,

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<sup>2</sup> For a detailed treatment of the Cauchy theorem see [23].

applying the Gauss theorem we can write the first integral as

$$\int_{\partial c} \mathbf{t} d\sigma = \int_{\partial c} \mathbf{T} \mathbf{n} d\sigma = \int_c \nabla_{\mathbf{x}} \cdot \mathbf{T} dc, \quad (1.19)$$

where  $\nabla_{\mathbf{x}} \cdot \mathbf{T}$  is a vector whose components in  $R$  are  $\frac{\partial T_{ij}}{\partial x_j}$ . Substituting equation (1.19) into (1.16) we obtain

$$\int_c (\rho \mathbf{b} + \nabla_{\mathbf{x}} \cdot \mathbf{T}) dc = \mathbf{0}.$$

Since the integration domain is arbitrary, at all points of  $c$  where the integrand functions are regular, the relation written above implies the following local form of the equilibrium of  $S$

$$\rho \mathbf{b} + \nabla_{\mathbf{x}} \cdot \mathbf{T} = \mathbf{0}. \quad (1.20)$$

In order to derive the *local* equation corresponding to equation (1.17), we remark that by virtue of the Cauchy theorem, the second integral can be written as

$$\begin{aligned} \int_{\partial c} \epsilon_{ijk} (x_j - x_{0j}) t_k &= \int_{\partial c} \epsilon_{ijk} (x_j - x_{0j}) T_{kh} n_h = \\ &= \int_c \epsilon_{ijk} T_{kj} dc + \int_c \epsilon_{ijk} (x_j - x_{0j}) \frac{\partial T_{kh}}{\partial x_h} dc \end{aligned}$$

where  $\epsilon_{ijk}$  is the *Levi-Civita* symbol. Then, substituting into (1.17) and taking into account (1.20) we obtain

$$\int_c \epsilon_{ijk} T_{kj} dc = 0.$$

Since  $c$  is an arbitrary material volume, it follows that

$$\epsilon_{ijk} T_{kj} = 0,$$

which implies that the Cauchy stress tensor is symmetric

$$\mathbf{T} = \mathbf{T}^T. \quad (1.21)$$

We conclude that the local equilibrium equation of a continuous body in the Eulerian form is given by equation (1.20) in which  $\mathbf{T}$  is a symmetric tensor.

## 1.5 Lagrangian Formulation of the Equilibrium Equation

In the previous section the fields  $\rho$  and  $\mathbf{T}$  have been written in Eulerian form. However there are many physical problems of practical interest in which it is more convenient to express these fields as functions of  $\mathbf{X}$  in  $C_*$ .

From the mass conservation equation in Lagrangian form,  $\rho_* = \rho J$ , the following identity follows

$$\int_c \rho \mathbf{b} dc = \int_{c_*} \rho_* \mathbf{b} dc_* \quad (1.22)$$

for any region  $c_* \subset C_*$ , where  $c$  is the material volume corresponding to  $c_*$  in the actual configuration  $C$ .

We define the *first Piola-Kirchhoff stress tensor*  $\mathbf{T}_*$  by the condition

$$\int_{\partial c_*} \mathbf{T}_* \cdot \mathbf{N}_* d\sigma_* = \int_{\partial c} \mathbf{T} \cdot \mathbf{n} d\sigma, \quad (1.23)$$

where  $\mathbf{N}_*$  is the unit outward normal vector to the surface element  $d\sigma_*$  of  $\partial c_*$ . Recalling equation (1.5), i. e.,

$$d\sigma n_i = J (F^{-1})_{Li} N_{*L} d\sigma_*,$$

equation (1.23) becomes

$$\int_{\partial c_*} (T_{*iL} - T_{ij} J (F^{-1})_{Lj}) \mathbf{N}_{*L} d\sigma_* = 0, \quad \forall c_* \subset C_*,$$

from which we have

$$\mathbf{T}_* = J\mathbf{T}(\mathbf{F}^{-1})^T. \quad (1.24)$$

Equation (1.24) expresses the relation between Cauchy's stress tensor  $\mathbf{T}$  and the first Piola-Kirchhoff stress tensor  $\mathbf{T}_*$ . From (1.22) and (1.23) it follows that the integral equilibrium equation of  $S$  can be put in the form

$$\int_{\partial c_*} \mathbf{T}_* \cdot \mathbf{N}_* d\sigma_* + \int_{c_*} \rho_* \mathbf{b} dc_* = \mathbf{0}, \quad \forall c_* \subset C_*. \quad (1.25)$$

By employing the same arguments as those used in the previous section we conclude from (1.25) that, if the involved fields are regular, then the local equilibrium equation in Lagrangian form is

$$\nabla_{\mathbf{X}} \cdot \mathbf{T}_* + \rho_* \mathbf{b} = \mathbf{0}, \quad (1.26)$$

where  $\nabla_{\mathbf{X}} \cdot \mathbf{T}_*$  is a vector whose  $i$ -th component in the reference frame  $R$  is given by  $\frac{\partial T_{*iL}}{\partial X_L}$ .

Equation (1.26) presents the following two advantages over equation (1.20):

1. the mass density  $\rho_*$  is a known function of the point  $\mathbf{X}$ ;
2. the involved fields and the function  $\mathbf{x}(\mathbf{X})$  are defined in the known and fixed region  $C_*$ .

## 1.6 Elastic Bodies

The local equilibrium equations are general relations, that is, they are valid for all continuous systems and do not depend on the material of the body. However, it is well known from experience that two bodies having the same geometric characteristics may react differently

when subjected to the same mechanical loads and thermal conditions. This means that for a continuous system the knowledge of the equilibrium conditions and of the external forces acting upon the body is not sufficient to determine the deformation field.

The above considerations can be expressed in a more formal way by noting that equation (1.26) constitutes a system of three partial differential equations in the 6 unknown components of  $\mathbf{T}_*(\mathbf{X})$ <sup>3</sup>. This means that the equilibrium equations do not form a closed set of field equations, so we add additional relations connecting the stress tensor  $\mathbf{T}_*$  to the the deformation  $\mathbf{x}(\mathbf{X})$  or the displacement  $\mathbf{u}(\mathbf{X})$ . Thus it is necessary to introduce criteria which allow us to distinguish between macroscopic behaviors of different material bodies. These relations are called *constitutive equations* because they translate in mathematical terms the material constitution of the body. In order to get these equations one can start from the assumption that the macroscopic response of a body depends on its molecular structure. This means that in principle the response functions can be obtained from Statistical Mechanics in terms of the average of microscopic quantities. Such an investigation is very stimulating both from theoretical and practical points of view. In this way, we can improve material properties and create new materials that respond to technological demands. But this approach is not straightforward if applied to complex materials, which are of interest in applications. In Continuum Mechanics the basic assumption of a continuously distributed matter cancels its discrete structure, so constitutive equations are determined experimentally. Even if these equations translate the material constitution of the body, they have to fulfill certain general principles, called *constitutive axioms*, which

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<sup>3</sup> Because  $\mathbf{T}^T = \mathbf{T}$ , from (1.24) it can be seen that only 6 components of the first Piola-Kirchhoff stress tensor are independent.

impose restrictions on their form. For an exhaustive study of this subject see pages 134-155 of [13].

Here we only remark that constitutive equations are assumed to be *objective*, i.e., we postulate that the material behavior is independent of the observer (*principle of material frame indifference*) and further that in any evolution of the system they satisfy the second law of thermodynamics (*principle of dissipation of Coleman and Noll*)<sup>4</sup>.

A material point  $\mathbf{X}$  in a continuous system  $S$  is *elastic* if the Cauchy stress tensor  $\mathbf{T}$  at the point  $\mathbf{X}$  of  $S$  depends on the deformation which  $S$  has experienced in the neighborhood of  $\mathbf{X}$  when passing from the reference configuration  $C_*$  to the actual configuration  $C$ . Having in mind that the deformation at  $\mathbf{X} \in C_*$  is completely described by the deformation gradient  $\mathbf{F}$ , the material point  $\mathbf{X}$  is said to be *elastic* if

$$\mathbf{T}(\mathbf{X}, t) = \mathbf{f}(\mathbf{F}, \mathbf{X}, t). \quad (1.27)$$

It can be shown (see [13]) that equation (1.27) satisfies the principle of dissipation and the material frame indifference principle if the Cauchy stress tensor has the following form

$$\mathbf{T} = \rho \frac{\partial \psi}{\partial \mathbf{F}} \mathbf{F}^T, \quad (1.28)$$

where  $\psi = \psi(\mathbf{F})$  is called the *specific strain energy*.

An elastic material point  $\mathbf{X}$  is *incompressible* if it can undergo only volume preserving deformations, i.e.,

$$\det \mathbf{C} = 1. \quad (1.29)$$

---

<sup>4</sup> See the fundamental memoir [28].



For an incompressible material equation (1.28) becomes

$$\mathbf{T} = -p\mathbf{1} + \rho \frac{\partial \psi}{\partial \mathbf{F}} \mathbf{F}^T, \quad (1.30)$$

where the *pressure*  $p(\mathbf{x}) = \tilde{p}(\mathbf{X})$  is an unknown function. In other words, the constitutive equation for an incompressible elastic material contains a pressure which is unknown in the problem. This is similar to the situation in the Dynamics of Rigid Bodies. There, the assumption of material rigidity makes all components of stress tensor unknown.

From equation (1.28), having in mind the relation (1.24) and the mass conservation equation in Lagrangian form (1.12), we obtain the following expression for the first Piola-Kirchhoff stress tensor for an unconstrained elastic medium

$$\mathbf{T}_* = \rho_* \frac{\partial \psi}{\partial \mathbf{F}}. \quad (1.31)$$

We conclude that the Lagrangian equilibrium equations for an elastic system are given by (1.26) in which the first Piola-Kirchhoff stress tensor is given by (1.31).

We point out that although constitutive axioms impose severe restrictions on the constitutive equation (1.27), they still allow the theory a wide margin of arbitrariness, which can be filled only by experimental data. The experimental determination of the constitutive equations of an elastic material is an extremely complicated task. From equations (1.28) and (1.31) it can be seen that it consists in the determination of the specific strain energy function  $\psi(\mathbf{F})$ . On the other hand the symmetry properties of the material can simplify it to some extent. We recall that the *material symmetry group* or simply the *isotropy group* is the group of unimodular transformations (i.e. whose determinant equals 1) of the material

coordinates under which the constitutive equations are invariant.<sup>5</sup> In this sense, the symmetry group of an *isotropic* material, i.e., a material whose properties are independent of the particular direction, is the whole group of orthogonal transformations. In mathematical terms, for any orthogonal matrix  $\mathbf{Q}$  the function  $\mathbf{f}(\mathbf{F})$  must satisfy

$$\mathbf{f}(\mathbf{F}) = \mathbf{f}(\mathbf{F}\mathbf{Q}).$$

It can be shown (see [23]) that an elastic material point, whose constitutive equations satisfy the objectivity principle and the second principle of thermodynamics, is an isotropic solid if and only if

$$\psi = \psi(I, II, III), \quad \mathbf{T} = \varphi_0 \mathbf{1} + \varphi_1 \mathbf{B} + \varphi_2 \mathbf{B}^2, \quad (1.32)$$

where  $I, II, III$ , are the three principal invariants of the *left Cauchy-Green tensor*  $\mathbf{B} = \mathbf{F}\mathbf{F}^T$  and functions  $\varphi_i$ ,  $i = 0, 1, 2$ , are defined as follows

$$\begin{aligned} \varphi_0 &= 2\rho III \frac{\partial \psi}{\partial III}, \\ \varphi_1 &= 2\rho \left( \frac{\partial \psi}{\partial I} + I \frac{\partial \psi}{\partial II} \right), \\ \varphi_2 &= -2\rho \frac{\partial \psi}{\partial II}. \end{aligned} \quad (1.33)$$

A simple application of the *Cayley-Hamilton theorem* allows us to obtain the following form of (1.32)<sub>2</sub>

$$\mathbf{T} = f_0 \mathbf{1} + f_1 \mathbf{B} + f_2 \mathbf{B}^{-1}, \quad (1.34)$$

---

<sup>5</sup> This form of the notion of symmetry group was introduced by W. Noll [29].

where  $f_i$ ,  $i = 0, 1, 2$ , are related to the  $\varphi_i$  by the following relations:<sup>6</sup>

$$\begin{aligned} f_0 &= \varphi_0 - II\varphi_2, \\ f_1 &= \varphi_1 - I\varphi_2, \\ f_2 &= III\varphi_2. \end{aligned} \tag{1.35}$$

Using (1.33) in (1.35), we obtain

$$\begin{aligned} f_0 &= 2\rho \left( II \frac{\partial \psi}{\partial II} + III \frac{\partial \psi}{\partial III} \right), \\ f_1 &= 2\rho \frac{\partial \psi}{\partial I}, \\ f_2 &= -2\rho III \frac{\partial \psi}{\partial II}. \end{aligned} \tag{1.36}$$

For an incompressible isotropic solid, we have

$$\psi = \psi(I, II), \quad \mathbf{T} = -p\mathbf{1} + 2\rho \frac{\partial \psi}{\partial I} \mathbf{B} - 2\rho \frac{\partial \psi}{\partial II} \mathbf{B}^{-1}. \tag{1.37}$$

For the proof of these results and a detailed description of the symmetry properties of a continuum system we refer the reader to [13] and [23].

---

<sup>6</sup> The Cayley-Hamilton theorem states that an  $n \times n$  square matrix  $\mathbf{A}$  satisfies the following identity

$$(-\mathbf{A})^n + I_1(-\mathbf{A})^{n-1} + \dots + I_{n-1}(-\mathbf{A}) + I_n \mathbf{1} = \mathbf{0},$$

where  $I_1, I_2, \dots, I_n$  are scalar functions depending on the components of  $\mathbf{A}$  and are called *principal invariants* of  $\mathbf{A}$ . For  $n = 3$  we have

$$\mathbf{A}^3 - I_A \mathbf{A}^2 + II_A \mathbf{A} - III_A \mathbf{1} = \mathbf{0},$$

where

$$\begin{aligned} I_A &= A_{kk} = \text{tr} \mathbf{A}, \\ II_A &= \frac{1}{2}(A_{kk}A_{ll} - A_{kl}A_{lk}) = \frac{1}{2}(\text{tr} \mathbf{A})^2 - \frac{1}{2}\text{tr} \mathbf{A}^2, \\ III_A &= \det \mathbf{A}. \end{aligned}$$

In particular for the left Cauchy-Green tensor we obtain

$$\mathbf{B}^3 - I\mathbf{B}^2 + II\mathbf{B} - III\mathbf{1} = \mathbf{0}.$$

Multiplying the preceding equation by  $\mathbf{B}^{-1}$  and solving for  $\mathbf{B}^2$  we get

$$\mathbf{B}^2 = I\mathbf{B} - III + IIIB^{-1},$$

which, substituted into (1.32)<sub>2</sub> gives (1.34).

## 1.7 Linear Elastic Bodies

The deformation  $\mathbf{x}(\mathbf{X})$  is said to be *infinitesimal* if components of the displacement field (1.9) and those of the displacement gradient (1.10) are *first order quantities*, that is, if their powers or products can be neglected as compared to the quantities themselves. When the deformation  $\mathbf{x}(\mathbf{X})$  from  $C_*$  to  $C$  is infinitesimal and  $C_*$  represents a *natural state* (that is  $\mathbf{T} = \mathbf{0}$  in  $C_*$ ), from (1.32) we derive the constitutive equation for an *isotropic linear elastic* material. We note that for an infinitesimal deformation we obtain the following expressions for  $\mathbf{B}$  and  $\mathbf{B}^2$ :

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T = (\mathbf{1} + \mathbf{H})(\mathbf{1} + \mathbf{H}^T) = \mathbf{1} + 2\mathbf{E} + \dots, \quad (1.38)$$

$$\mathbf{B}^2 = \mathbf{1} + 4\mathbf{E} + \dots,$$

where the symmetric tensor

$$\mathbf{E} = \frac{1}{2}(\mathbf{H} + \mathbf{H}^T) \quad (1.39)$$

is known as the *infinitesimal strain tensor* and plays a fundamental role in the theory of infinitesimal deformations. If we assume that functions  $\varphi_i, i = 0, 1, 2$ , can be approximated by their Taylor series expansions in the invariants of  $\mathbf{B}$  in the neighborhood of  $\mathbf{B} = \mathbf{1}$  (absence of deformation), up to first order terms in  $\mathbf{H}$ , we obtain

$$\varphi_i = a_i + b_i(I - 3) = a_i + 2b_i I_E, \quad (1.40)$$

where  $I_E$  is the first principal invariant of  $\mathbf{E}$ . The condition  $\mathbf{T} = \mathbf{0}$  for  $\mathbf{B} = \mathbf{1}$  requires that constants  $a_i$  and  $b_i$  satisfy the relation

$$a_1 + a_2 + a_3 = 0.$$

Hence, for small deformations of an elastic isotropic material, the stress tensor becomes

$$\mathbf{T} = \lambda I_E \mathbf{1} + 2\mu \mathbf{E}, \quad (1.41)$$

where coefficients  $\lambda$  and  $\mu$ , which depend in a proper way on constants  $a_i$  and  $b_i$ , are called *Lamé coefficients* of the elastic material.

The Cauchy stress tensor for infinitesimal deformations of an anisotropic elastic material can be written as

$$\mathbf{T} = \mathbb{C}\mathbf{E}, \quad (1.42)$$

where the *elasticity tensor*  $\mathbb{C}$  is a fourth order tensor characterized by the following symmetries

$$\mathbb{C}_{ijhk} = \mathbb{C}_{ijkh} = \mathbb{C}_{jihk}.$$

These properties can be proved starting from the symmetry properties of tensors  $\mathbf{T}$  and  $\mathbf{E}$ :

$$T_{ij} = \mathbb{C}_{ijhk} E_{hk} = \mathbb{C}_{ijkh} E_{kh} = \mathbb{C}_{jihk} E_{hk} = T_{ji} = \mathbb{C}_{jihk} E_{hk}.$$

In particular, for an isotropic linear elastic medium, from (1.41) and (1.42) it follows that the elasticity tensor becomes

$$\mathbb{C}_{ijhk} = \lambda \delta_{ij} \delta_{hk} + \mu (\delta_{ih} \delta_{jk} + \delta_{jh} \delta_{ik}).$$

In the Theory of Elasticity it is useful to consider the *Green-Saint Venant strain tensor*

$$\mathbf{G} = \mathbf{E} + \frac{1}{2} \mathbf{H}^T \mathbf{H}. \quad (1.43)$$

It is easy to see that in an infinitesimal deformation

$$\mathbf{G} \simeq \mathbf{E}, \quad (1.44)$$

where the notation  $\simeq$  means that quantities on the left and the right-hand side of (1.44) differ by an order greater than  $|\mathbf{u}|$  and  $|\mathbf{H}|$ . The following relations between the invariants of  $\mathbf{G}$  and  $\mathbf{B}$  hold:

$$\begin{aligned}2I_{\mathbf{G}} &= I - 3, \\4II_{\mathbf{G}} &= II - 2I + 3, \\8III_{\mathbf{G}} &= III - II + I - 1,\end{aligned}\tag{1.45}$$

and

$$\begin{aligned}I &= 2I_{\mathbf{G}} + 3, \\II &= 4II_{\mathbf{G}} + 4I_{\mathbf{G}} + 3, \\III &= 8III_{\mathbf{G}} + 4II_{\mathbf{G}} + 2I_{\mathbf{G}} + 1.\end{aligned}\tag{1.46}$$

# Chapter 2

## Nonlinear Elastostatics

### 2.1 Introduction

In this chapter we formulate the boundary-value problems of finite elastostatics. We study in depth the meaning and the difficulties related to the assignment of the traction boundary conditions. In fact, in the case of the pure traction-value problem the boundary condition depends on the unknown deformation. In order to describe this, we discuss two examples. The first example is concerned with an elastic system at equilibrium in an uniform pressure field acting upon its boundary. The second example deals with an elastic system subjected to the action of elastic surface forces.

A further complication is represented by the fact that, for the case of pure traction-value problem, the global equilibrium conditions depend also on the unknown deformation.

### 2.2 The Local Equilibrium Equations

Let  $S$  be an elastic system at equilibrium in a reference configuration  $C_*$ . Let us assume that under the action of body forces and surface tractions acting on a part or on the whole boundary  $\partial S$ ,  $S$  reaches a new equilibrium configuration  $C$ . The task of an elastostatic problem is to determine the finite deformation  $\mathbf{x}(\mathbf{X})$ , or equivalently the displacement

$\mathbf{u}(\mathbf{X})$ , which  $S$  undergoes when passing from  $C_*$  to  $C$ , under the action of the aforesaid forces.

Since the function  $\mathbf{x} = \mathbf{x}(\mathbf{X})$  is defined on the reference configuration  $C_*$ , it is necessary to resort to the Lagrangian equilibrium equations. Also the traction acting upon the boundary  $\partial C$  of the equilibrium configuration  $C$  can not be assigned since  $\partial C$  appears among the unknowns of the problem. At the equilibrium the following equations hold (see (1.24)):

$$\begin{cases} \nabla_{\mathbf{x}} \cdot \mathbf{T}_* + \rho_* \mathbf{b} = \mathbf{0}, & \forall \mathbf{X} \in C_*, \\ \mathbf{T}_* \cdot \mathbf{N}_* = \mathbf{t}_*, & \forall \mathbf{X} \in \partial C'_*, \end{cases} \quad (2.47)$$

where  $\rho_*$  is the mass density in the reference configuration,  $\mathbf{T}_*$  the first Piola-Kirchhoff stress tensor and  $\mathbf{N}_*$  the outward unit vector normal to the part  $\partial C'_*$  of  $\partial C_*$  on which the surface forces with density  $\mathbf{t}_*$  act. (If  $\partial C'_* \neq \partial C_*$ , then boundary conditions on  $\partial C_* - \partial C'_*$  need to be given).

For an elastic material, by virtue of (1.28), we have

$$\mathbf{T} = \rho \frac{\partial \psi}{\partial \mathbf{F}} \mathbf{F}^T, \quad (2.48)$$

from which, remembering the definition (1.24) of the first Piola-Kirchhoff stress tensor, the Lagrangian formulation of the mass equation  $\rho J = \rho_*$  and the definition (1.10) of the displacement gradient  $\mathbf{H}$ , we obtain

$$\mathbf{T}_* = \rho_* \frac{\partial \psi(\mathbf{F})}{\partial \mathbf{F}} = \rho_* \frac{\partial \tilde{\psi}(\mathbf{H})}{\partial \mathbf{H}}. \quad (2.49)$$

Substitution of (2.49) into (2.47)<sub>1</sub> yields the following system of three *non linear* second order partial differential equations in the three scalar unknown functions  $x_j(\mathbf{X})$  which are



regular in the domain  $C_*$ :

$$A_{LMij}(\mathbf{X}, \mathbf{H}) \frac{\partial^2 u_j}{\partial X_L \partial X_M} + r_i(\mathbf{X}, \mathbf{H}) + \rho_* b_i(\mathbf{X}) = 0, \quad i = 1, 2, 3, \quad (2.50)$$

where

$$A_{LMij} = \frac{\partial T_{*iL}}{\partial H_{jM}} = \rho_* \frac{\partial^2 \psi}{\partial H_{iL} \partial H_{jM}}, \quad r_i = \left( \frac{\partial T_{*iL}}{\partial X_L} \right). \quad (2.51)$$

In particular, when the body is homogeneous in the reference configuration, the coefficients  $A_{LMij}$  do not depend explicitly on  $\mathbf{X}$  and  $r_i = 0, \forall i$ .

The main goal of the non linear elastostatics is to find the finite deformation  $\mathbf{x}(\mathbf{X})$  of  $S$  from equations (2.50), together with boundary conditions (2.47)<sub>2</sub>.

## 2.3 Some Considerations about Boundary Conditions

In the boundary-value problem (2.47) there is no mention of conditions on the part  $\partial C_*'' = \partial C - \partial C_*'$  of the boundary of  $S$  on which no surfaces forces are applied. Here, we assume that this part is fixed or deformed in a known way by virtue of suitable constraints. That is

$$\mathbf{x}(\mathbf{X}) = \mathbf{x}_0(\mathbf{X}), \quad \forall \mathbf{X} \in \partial C_*''. \quad (2.52)$$

If  $\partial C_*' = \phi$ , the boundary-value problem (2.47) is a *pure displacement problem*; if  $\partial C_*'' = \phi$  we deal with a *pure traction-value problem*, and finally the problem is said to be *mixed* if  $\partial C_*''' \subset \partial C_*$ .

The following remarks can be made concerning the boundary-value problems.

1. In a boundary-value problem the data are usually *assigned* functions on the boundary of the domain which is to be determined as a part of the solution. For instance, in the

*Dirichlet* problem for the *Laplace* equation on the domain  $\Omega$ , we assign the value of the unknown function on the boundary of  $\Omega$ . In the *Neumann* problem the value of the normal derivative of the unknown function is given. Finally, in the mixed problem we assign the value of the unknown function on a part of  $\partial\Omega$  and the value of the normal derivative on the remaining part of  $\partial\Omega$ . However, the assignment of the surface traction on the boundary in a non linear elastostatic problem is not as easy as it may appear.

Let us consider, for example, an elastic system  $S$  at equilibrium in the absence of body forces (i.e.,  $\mathbf{b} = \mathbf{0}$ ) under a uniform pressure  $p_0$  acting on the boundary  $\partial C$  of the actual configuration. The corresponding Eulerian boundary-value problem is (see (1.20))

$$\begin{cases} \nabla_{\mathbf{x}} \cdot \mathbf{T} = \mathbf{0}, & \forall \mathbf{x} \in C, \\ \mathbf{T} \cdot \mathbf{N} = -p_0 \mathbf{N}, & \forall \mathbf{x} \in \partial C'. \end{cases} \quad (2.53)$$

This boundary-value problem can be formulated in the Lagrangian form. Having in mind equation (1.5) and the definition of the first Piola-Kirchhoff stress tensor we obtain

$$T_{ij}N_j = T_{ij}J(\mathbf{F}^{-1})_{Lj}N_{*L}\frac{d\sigma_*}{d\sigma} = T_{*iL}N_{*L}\frac{d\sigma_*}{d\sigma}.$$

From (1.5) and (2.53)<sub>2</sub>, it follows that

$$t_i = T_{ij}N_j = -p_0N_i = -p_0J(\mathbf{F}^{-1})_{Li}N_{*L}\frac{d\sigma_*}{d\sigma}.$$

Comparing the two preceding relations we obtain

$$T_{*iL}N_{*L} = -p_0J(\mathbf{F}^{-1})_{Li}N_{*L}.$$

Thus the Lagrangian formulation of the problem (2.53) becomes

$$\begin{cases} \nabla_{\mathbf{x}} \cdot \mathbf{T}_* = \mathbf{0}, & \forall \mathbf{X} \in C_*, \\ \mathbf{T}_* \cdot \mathbf{N}_* = -p_0 J (\mathbf{F}^{-1})^T \mathbf{N}_* \equiv \mathbf{t}_*, & \forall \mathbf{X} \in \partial C'_*. \end{cases} \quad (2.54)$$

It is easy to realize that  $\mathbf{t}_*$  is not a known function of the point  $\mathbf{X} \in \partial C_*$ . In fact it contains the unknown deformation  $\mathbf{x}(\mathbf{X})$  through its gradient  $\mathbf{F}$ . In other words, the function  $\mathbf{t}_*(\mathbf{X})$  can not be given.

Similarly, let us consider an elastic system  $S$  subjected to the action of the elastic forces  $\mathbf{t}(\mathbf{x}) = -kh(\mathbf{x})\mathbf{i}$  (where  $h(\mathbf{x})$  is the elongation of a linear spring at the point  $\mathbf{X}$ ,  $k$  is a constant, and  $\mathbf{i}$  is a unit vector) acting on the part  $\partial C'$  of the boundary  $\partial C$  of its present equilibrium configuration. Because of (1.8) and having in mind that the relation between  $\mathbf{t}_*$  and  $\mathbf{t}$  is

$$\mathbf{t}_* d\sigma_* = \mathbf{t} d\sigma,$$

we obtain the following relation between  $\mathbf{t}_*$  and the actual stress vector  $\mathbf{t}$

$$\mathbf{t}_* = J \mathbf{t} \sqrt{\mathbf{N}_* \mathbf{C}^{-1} \mathbf{N}_*}. \quad (2.55)$$

Hence, the data to assign on the part  $\partial C'_*$  corresponding to the part  $\partial C'$  is

$$\mathbf{t}_*(\mathbf{F}, \mathbf{X}) = \mathbf{t} \frac{d\sigma}{d\sigma_*} = -J \sqrt{\mathbf{N}_* \mathbf{C}^{-1} \mathbf{N}_*} kh(\mathbf{x}(\mathbf{X}))\mathbf{i}, \quad (2.56)$$

which depends on the unknown deformation and, consequently, can not be assigned.

These two examples together with the fact that equations (2.50) are essentially non linear make the boundary-value problems of elastostatics very complicated. Hence, it is natural to attempt to simplify the problems described above by limiting the analysis to those problems in which boundary conditions can be assigned. All loads, which in  $C_*$  depend on the deformation are called *live loads*; differently, loads which can

be given as known functions of  $\mathbf{X} \in \partial C_*$  are called *dead loads*. These dead loads have been widely studied in the literature. However, they are very difficult to realize in practice. As a matter of the fact, from the condition

$$\mathbf{t}_*(\mathbf{X}) = \frac{d\sigma}{d\sigma_*} \mathbf{t}$$

it follows that the traction  $\mathbf{t}$  acting upon  $\partial C'$  has to be assigned in such a way that  $\mathbf{t}_*$  be independent of the deformation but depend only on  $\mathbf{X}$ .

2. Equations (2.47) represent *necessary conditions* for the equilibrium of  $S$ . Therefore, together with (2.47), we have to consider the *global equilibrium conditions* which express the vanishing of the resultant force and the resultant moment with respect to a point  $O$  of all forces acting on  $S$ . In order to write them, we denote by  $\Phi$  the reaction due to constraints which are necessary to realize the displacement (2.52). Then the following global equilibrium conditions must hold

$$\begin{aligned} \int_{C_*} \rho_* \mathbf{b} dc_* + \int_{\partial C'_*} \mathbf{t}_* d\sigma_* + \int_{\partial C''_*} \Phi d\sigma_* &= \mathbf{0}, \\ \int_{C_*} \rho_* \mathbf{r} \times \mathbf{b} dc_* + \int_{\partial C'_*} \mathbf{r} \times \mathbf{t}_* d\sigma_* + \int_{\partial C''_*} \mathbf{r} \times \Phi d\sigma_* &= \mathbf{0}. \end{aligned} \quad (2.57)$$

It is plain that for a mixed-value and a displacement-value problem the constraints have to satisfy these conditions. On the contrary, for the traction-value problem conditions (2.57) reduce to the following

$$\begin{aligned} \int_{C_*} \rho_* \mathbf{b} dc_* + \int_{\partial C'_*} \mathbf{t}_* d\sigma_* &= \mathbf{0}, \\ \int_{C_*} \rho_* \mathbf{r} \times \mathbf{b} dc_* + \int_{\partial C'_*} \mathbf{r} \times \mathbf{t}_* d\sigma_* &= \mathbf{0}. \end{aligned} \quad (2.58)$$

Since  $\mathbf{t}_*$  depends on  $\mathbf{x}(\mathbf{X})$  and  $\mathbf{F}$  on  $\partial C_*$  and owing to the presence of  $\mathbf{r} = (\mathbf{x}(\mathbf{X}) - \mathbf{x}_0)$  in (2.58), it is not possible to establish if the data satisfy these conditions unless we know the deformation corresponding to the force system  $(\mathbf{t}_*, \mathbf{b})$ . For this reason,

Signorini [1] suggested to regard (2.58) as *compatibility conditions*, i.e., if there exists no deformation satisfying equations (2.47) and (2.52), then the traction boundary-value problem has no solution. In the particular case of dead loads, condition (2.58)<sub>1</sub> is an "a priori" restriction upon the traction data, while (2.58)<sub>2</sub> still remains a compatibility condition due to the presence of  $\mathbf{r}$ .

# Chapter 3

## Signorini's Method for Dead Loads

### 3.1 Dimensional Analysis of the Equilibrium Equations

In 1930, Signorini [1] suggested a perturbation method to find approximate solutions of boundary-value problems of finite elasticity in the presence of *dead loads*. First the elastic system  $S$  is assumed to be in equilibrium, in the absence of forces, in a homogeneous, isotropic and unstressed configuration  $C_*$ . Subsequently, under the action of a system of mass forces  $\mathbf{b}$  and surface tractions  $\mathbf{t}$ , the continuum  $S$  deforms until it assumes a new equilibrium configuration  $C$ . The basic assumption of Signorini's perturbation method is that the response of  $S$  to applied loads is not so different from its response if the system behaved like a *linear elastic material*. Consequently, the first step in order to check if the method can be applied is to write equations (2.47) in a nondimensional form. To this aim, we introduce the following *reference quantities*

$$\tilde{T}, \quad l, \quad L, \quad \tilde{b}, \quad \tilde{t}, \quad \tilde{\rho}, \quad (3.59)$$

where  $\tilde{T}$  has the dimension of stress and describes the internal state of the system in  $C_*$ ;  $l$  and  $L$  are lengths which represent measures for the displacements and the characteristic dimensions of the body, respectively;  $\tilde{b}$  and  $\tilde{t}$  are reference mass and surface forces, and finally  $\tilde{\rho}$  has the dimension of mass density. If we continue to use the same notations for

the nondimensional quantities, the pure traction boundary-value problem (2.47) becomes

$$\begin{aligned}\frac{\tilde{T}}{L}\nabla_* \cdot \mathbf{T}_* &= -\tilde{\rho}\rho_*\tilde{b}\mathbf{b}, \\ \tilde{T}\mathbf{T}_* \cdot \mathbf{N}_* &= \tilde{t}\mathbf{t}.\end{aligned}\tag{3.60}$$

On the other hand, if we use the Cauchy stress tensor of linear elasticity as a measure of the state of stress of the body, from the constitutive equation (1.41)

$$\mathbf{T} = \lambda I_E \mathbf{1} + 2\mu \mathbf{E},$$

in which  $I_E$  is the trace of the infinitesimal deformation tensor  $\mathbf{E}$  and  $\lambda$  and  $\mu$  are the Lamé coefficients, it follows that

$$\tilde{T} \simeq \Gamma \frac{l}{L} \equiv \alpha \Gamma,$$

where  $\Gamma = \max\{\lambda, \mu\}$  and  $\alpha \equiv \frac{l}{L}$ . Substituting into (3.60) the preceding expression of  $\tilde{T}$ , we obtain the following system

$$\begin{aligned}\nabla_* \cdot \mathbf{T}_* &= -\frac{L\tilde{b}\tilde{\rho}}{\alpha\Gamma}\rho_*\mathbf{b}, \\ \mathbf{T}_* \cdot \mathbf{N}_* &= \frac{\tilde{t}}{\alpha\Gamma}\mathbf{t},\end{aligned}\tag{3.61}$$

where all quantities are nondimensional. It is now easy to understand that if we want  $C_*$  to be the unperturbed state of  $S$ , we must have

$$\epsilon \equiv \frac{L\tilde{b}\tilde{\rho}}{\alpha\Gamma} \simeq \frac{\tilde{t}}{\alpha\Gamma} \ll 1.\tag{3.62}$$

This relation allows us to estimate the order of magnitude of the body forces and surface tractions acting upon  $S$  starting from the material of the body ( $\Gamma$ ), its dimensions ( $L$ ) and the relative magnitude of displacements ( $\alpha = l/L$ ).

In terms of reference and nondimensional quantities, the compatibility condition (2.58)<sub>1</sub> can be written as

$$\tilde{\rho} \tilde{b} L^3 \int_{C_*} \rho_* \mathbf{b} dc_* + \tilde{t} L^2 \int_{\partial C_*} \mathbf{t}_* d\sigma_* = \mathbf{0},$$

which, dividing by  $\alpha \Gamma L^2$  and having in mind the value of the small parameter  $\epsilon$  given by (3.62) becomes

$$\int_{C_*} \rho_* \epsilon \mathbf{b} dc_* + \int_{\partial C_*} \epsilon \mathbf{t}_* d\sigma_* = \mathbf{0}. \quad (3.63)$$

In the same way, for the second compatibility condition (2.58)<sub>2</sub> we have

$$\tilde{\rho} \tilde{b} l L^3 \int_{C_*} \rho_* \mathbf{r} \times \mathbf{b} dc_* + \tilde{t} l L^2 \int_{\partial C_*} \mathbf{r} \times \mathbf{t}_* d\sigma_* = \mathbf{0},$$

and, dividing by  $\alpha \Gamma l L^2$ , we obtain

$$\int_{C_*} \rho_* \mathbf{r} \times \epsilon \mathbf{b} dc_* + \int_{\partial C_*} \mathbf{r} \times \epsilon \mathbf{t}_* d\sigma_* = \mathbf{0}. \quad (3.64)$$

## 3.2 Signorini's Method for Dead Loads

We note that the use of an approximation procedure makes no sense unless we have ensured that there exists at least a solution to the equilibrium problem. Consequently, the existence and uniqueness results play a crucial role within the boundary-value problems of finite elasticity. Unfortunately, the known existence and uniqueness theorems for solutions to the partial differential equations are not general enough to include the boundary-value problems we have described above. Van Buren [21] proved an existence and uniqueness theorem for the solution of the mixed problem involving dead loads. This result starts from



the Banach-Caccioppoli theorem on the inverse functions and is *local*, i.e., it is valid for finite deformations which are not far from the linear deformations. Furthermore, it requires that an existence and uniqueness theorem hold for the corresponding linear problem. For a mixed boundary-value problem Van Buren showed that if the first Piola-Kirchhoff stress tensor depends analytically on the displacement gradient and the applied body and surface loads are analytical functions of the perturbation parameter  $\epsilon$  then there exists a unique solution of the problem which is an analytical function of  $\epsilon$ . Hence, provided that  $\epsilon \ll 1$ , the perturbation method can be applied if the following main hypotheses are satisfied:

1. The first Piola-Kirchhoff stress tensor  $\mathbf{T}_* = J\mathbf{T}(\mathbf{F}^{-1})^T$  depends analytically on the displacement gradient  $\mathbf{H}$ ;
2.  $\mathbf{b}(\epsilon, \mathbf{X})$  and  $\mathbf{t}_*(\epsilon, \mathbf{X})$  are analytical functions of  $\epsilon$ .

In order to derive a sequence of linear problems to be solved, we write

$$\mathbf{T}_* = \mathbf{A}(\mathbf{H}) = \sum_{n=1}^{\infty} \mathbf{A}_n(\mathbf{H}), \quad \mathbf{A}(\mathbf{0}) = \mathbf{0}, \quad (3.65)$$

where functions  $\mathbf{A}_n(\mathbf{H})$  are homogeneous polynomials of degree  $n$  in  $\mathbf{H}$ <sup>7</sup>, and further

$$\mathbf{b} = \sum_{n=1}^{\infty} \epsilon^n \mathbf{b}_n, \quad \mathbf{t}_* = \sum_{n=1}^{\infty} \epsilon^n \mathbf{t}_{*n}, \quad (3.67)$$

$$\mathbf{u} = \sum_{n=1}^{\infty} \epsilon^n \mathbf{u}_n, \quad (3.68)$$

---

<sup>7</sup> In fact it is sufficient to note that

$$T_{*iL} = A_{iL}(\mathbf{H}) = C_{(1)iLjM} H_{jM} + C_{(2)iLjMhN} H_{jM} H_{hN} + \dots \quad (3.66)$$

where series (3.65), (3.67), and (3.68) are absolutely and uniformly convergent in a proper radius of convergence.

Assuming that  $\mathbf{H}_n = \nabla \mathbf{u}_n$ , from (3.68) it follows that

$$\mathbf{H} = \sum_{n=1}^{\infty} \epsilon^n \mathbf{H}_n. \quad (3.69)$$

Substituting from (3.69) into (3.65), we have

$$\begin{aligned} T_{*iL} &= C_{(1)iLjM} (\epsilon H_{1jM} + \epsilon^2 H_{2jM} + \dots) + \\ &+ C_{(2)iLjMhN} (\epsilon H_{1jM} + \epsilon^2 H_{2jM} + \dots) (\epsilon H_{1hN} + \epsilon^2 H_{2hN} + \dots) + \dots \\ &= \epsilon C_{(1)iLjM} H_{1jM} + \epsilon^2 (C_{(1)iLjM} H_{2jM} + C_{(2)iLjMhN} H_{1jM} H_{1hN}) + \dots \end{aligned}$$

or equivalently

$$\mathbf{T}_* = \sum_{n=1}^{\infty} \epsilon^n (\mathbf{C}_{(1)} \mathbf{H}_n + \mathbf{B}_n (\mathbf{H}_1, \dots, \mathbf{H}_{n-1})), \quad (3.70)$$

where  $\mathbf{C}_{(1)}$  is a fourth order tensor and  $\mathbf{B}_n (\mathbf{H}_1, \dots, \mathbf{H}_{n-1})$ ,  $n = 2, 3, \dots$ , are homogeneous polynomial of degree  $n$  in variables  $\mathbf{H}_1, \dots, \mathbf{H}_{n-1}$ , while  $\mathbf{B}_1 = \mathbf{0}$ . From (3.66) it follows also that  $\mathbf{C}_{(1)}$  has to be identified with the tensor  $\mathbb{C}$  of the linear theory of elasticity (see equation (1.42)), so that (3.70) becomes

$$\mathbf{T}_* = \sum_{n=1}^{\infty} \epsilon^n (\mathbb{C} \cdot \mathbf{E}_n + \mathbf{B}_n (\mathbf{H}_1, \dots, \mathbf{H}_{n-1})), \quad (3.71)$$

where  $\mathbf{E}_n = \frac{(\mathbf{H}_n + \mathbf{H}_n^T)}{2}$ .

Substituting from (3.67) and (3.71) into (2.47) with homogeneous displacement boundary condition, we obtain

$$\begin{cases} \nabla \cdot (\mathbb{C} \cdot \mathbf{E}_n) + \rho_* \hat{\mathbf{b}}_n = \mathbf{0}, & \text{on } C_*, \\ (\mathbb{C} \cdot \mathbf{E}_n) \cdot \mathbf{N}_* = \hat{\mathbf{t}}_{*n}, & \text{on } \partial C'_*, \\ \mathbf{u}_n = \mathbf{0}, & \text{on } \partial C''_*, \quad n = 1, 2, \dots \end{cases} \quad (3.72)$$

where the following definitions have been made

$$\begin{cases} \rho_* \hat{\mathbf{b}}_n \equiv \rho_* \mathbf{b}_n + \nabla \cdot \mathbf{B}_n (\mathbf{H}_1, \dots, \mathbf{H}_{n-1}), \\ \hat{\mathbf{t}}_{*n} \equiv \mathbf{t}_{*n} - \mathbf{B}_n (\mathbf{H}_1, \dots, \mathbf{H}_{n-1}) \cdot \mathbf{N}_*. \end{cases} \quad (3.73)$$

When  $n = 1$ ,  $\hat{\mathbf{b}}_1 = \mathbf{b}_1$ ,  $\hat{\mathbf{t}}_{*1} = \mathbf{t}_1$  and equations (3.72) coincide with those of the mixed boundary-value problem in the linear theory of elasticity. More generally, if we assume that fields  $\mathbf{u}_1, \dots, \mathbf{u}_{m-1}$  are solutions to the problems (3.72) for  $n = 1, \dots, m-1$ ; then, equations (3.72) written for  $n = m$  define a new mixed boundary-value problem *for the same material and for the same domain  $C_*$* , but with loads  $\hat{\mathbf{b}}_m, \hat{\mathbf{t}}_{*m}$  depending in a known way on  $\mathbf{u}_1, \dots, \mathbf{u}_{m-1}$ . In other words, the determination of the  $m$ -th term of series (3.68) reduces to the solution of  $m$  mixed boundary-value problems for the same body in  $C_*$ , but with different loads. The great advantage of using Signorini's method lies in the fact that it allows us to pass from a non linear problem to a set of *linear* problems.

Van Buren's theorem can not be directly extended to the case of the pure traction-value problem for the following two reasons. First, the linear and the nonlinear pure traction boundary-value problems admit at least a solution if the applied loads are balanced, i.e.,

$$\int_{C_*} \rho_* \mathbf{b} dC_* + \int_{\partial C_*} \mathbf{t}_* d\sigma_* = \mathbf{0}, \quad (3.74)$$

$$\int_{C_*} \mathbf{r} \times \mathbf{b} \rho_* dC_* + \int_{\partial C_*} \mathbf{r} \times \mathbf{t}_* d\sigma_* = \mathbf{0}, \quad \mathbf{r} = \mathbf{r}_* + \mathbf{u}. \quad (3.75)$$

Second, for the linear problem there does not exist a uniqueness theorem since the solution is determined to within an arbitrary infinitesimal rigid displacement (see Theorem 10.4 of [23]). This means that in order to obtain a unique solution to the linear pure traction-value problem, we have to add further conditions to the displacement. For instance, we may require that

$$\mathbf{u}(\mathbf{0}) = \mathbf{0}, \quad \mathbf{H}(\mathbf{0}) = \mathbf{H}^T(\mathbf{0}), \quad (3.76)$$

which exclude the translation and the infinitesimal rigid rotation respectively. Concerning Signorini's method for the pure traction-value problem with dead loads, Stoppelli [6-10] proved a local existence and uniqueness theorem, and analytic dependence of the solution on a parameter, when the applied dead loads do not have an *axis of equilibrium*, and the existence and analyticity of solutions when the dead loads have an axis of equilibrium (see also Tolotti [11]). As Van Buren's theorem, Stoppelli's theorem is an application of Banach-Caccioppoli theorem on the inverse functions to the pure traction-value problem and is local. A discussion of Stoppelli's papers can be found in [12-14], and [33].

In [4] and [5] Signorini's method was used to investigate the uniqueness of the above solutions as well as the position of the classical linear theory with respect to the nonlinear theory. Later, Capriz and Podio-Guidugli [15-19] investigated the compatibility of the linear and the nonlinear elasticity theories and showed that a very large class of traction-value problems can be solved by perturbation methods of Signorini's type.

# Chapter 4

## Signorini's Method for Traction-Value Problems with Live Loads

### 4.1 Introduction

Starting from the fundamental paper [1], the research relative to pure traction-value problems (see [4-17]) has essentially been developed only in the presence of dead loads  $\mathbf{b} = \mathbf{b}(\mathbf{X})$  and  $\mathbf{t}_* = \mathbf{t}_*(\mathbf{X})$ . Nevertheless, it is easy to realize that the physically meaningful loads depend on the deformation. Further the hypothesis of live loads in the pure traction-value problems introduces the following mathematical difficulties (see Section 2.3):

1. The boundary conditions depend on the unknown deformation.
2. The global equilibrium conditions represent compatibility conditions for the data and the displacement and can not be verified a priori.

Only since the 80's the research in finite elastostatics has been devoted to pure traction-value problems with live loads. In fact, in [18], [19], and [30] Signorini's method has been extended to live traction-value problems by Grioli, Capriz and Podio-Guidugli. In particular, in [30] Grioli studied an equilibrium problem for a heavy solid immersed in a homogeneous incompressible fluid and, later, in [31] and [32] Grioli provided a new perturbation procedure for the pure traction boundary-value problems introducing a convenient

constitutive parameter. In [19], starting from two problems suggested by Grioli, Capriz and Podio-Guidugli generalized the results of [18] and presented a perturbation method for the pure traction-value problems with live loads for *almost rigid* hyperelastic bodies. An important contribution in the framework of non linear elasticity with live loads has been made by Valent in many papers which are collected in [20]. In this book, Valent extended Stoppelli's theorem to some pure traction-value problems with live loads. In particular, Valent proved some local theorems of existence, uniqueness and analytic dependence on a parameter which allow us to use Signorini's method for pure traction-value problem in which the prescribed surface traction is parallel to the normal to the boundary of the body. More precisely, within this class Valent considered the traction-value problem for loads invariant under translations and rotations, and the case of a heavy body submerged in a homogeneous liquid.

In this chapter, starting from the results of Valent [Chapter 6, 20], we study traction-value problems for a body subjected to a uniform pressure. Besides the introduction, this chapter is divided into five sections. In the second section, we provide a generalization of Signorini's method to the case of live loads. Furthermore, we formulate the boundary-value problems and the corresponding compatibility conditions (global equilibrium conditions) in order to determine the displacement of the system up to the second-order approximation. In the third section, we solve two live traction-value problems for an elastic continuous system of simple geometry (sphere and hollow cylinder) in a uniform pressure field. The obtained solutions allow us to propose four experiments for determining the second-order constitu-

tive constants for the given material. Finally, in the last section we present expressions of the second-order material constants.

## 4.2 A Generalization of Signorini's Method to Live Loads

### 4.2.1 Dimensional Analysis

For the sake of clarity, here we reproduce briefly the same arguments as used earlier to describe the dimensional analysis for dead loads. We assume that the elastic system  $S$  is in equilibrium in the absence of forces in a homogeneous, isotropic and unstressed configuration  $C_*$ . Then, by virtue of the action of a system of mass forces  $\mathbf{b}$  and surface tractions  $\mathbf{t}$ ,  $S$  assumes a new equilibrium configuration  $C$ . We assume that the response of  $S$  is close to the linear elastic response. Consequently, the first step in order to check if Signorini's method can be applied is to write equations (2.47) in a nondimensional form. In order to do that, we introduce the following reference quantities (see section 3.1)

$$\tilde{T}, \quad l, \quad L, \quad \tilde{b}, \quad \tilde{t}, \quad \tilde{\rho}, \quad (4.77)$$

and continue to use the same notations for the nondimensional quantities, the pure traction boundary-value problem (2.47) and compatibility conditions (2.58) (see equations (3.63) and (3.64)). We write them as

$$\begin{cases} \nabla_* \cdot \mathbf{T}_* = -\epsilon \mathbf{b} & \text{in } C_*, \\ \mathbf{T}_* \cdot \mathbf{N}_* = \epsilon \mathbf{t}_* & \text{on } \partial C_*, \end{cases} \quad (4.78)$$

$$\begin{cases} \int_{C_*} \rho_* \epsilon \mathbf{b} dc_* + \int_{\partial C_*} \epsilon \mathbf{t}_* d\sigma_* = \mathbf{0}, \\ \int_{C_*} \rho_* \mathbf{r} \times \epsilon \mathbf{b} dc_* + \int_{\partial C_*} \mathbf{r} \times \epsilon \mathbf{t}_* d\sigma_* = \mathbf{0}, \end{cases} \quad (4.79)$$

where, adopting the Cauchy stress tensor  $\mathbf{T}$  of the linear theory to describe the stress state, the perturbation parameter  $\epsilon$  is given by

$$\epsilon \equiv \frac{L}{\alpha \Gamma} \tilde{\rho} \tilde{\mathbf{b}} \simeq \frac{\tilde{t}}{\alpha \Gamma} \quad (4.80)$$

in which  $\Gamma = \max\{\lambda, \mu\}$ ,  $\lambda$  and  $\mu$  are the Lamé coefficients, and  $\alpha = l/L$ .

Provided that  $\epsilon \ll 1$ , the perturbation method can be applied if the following hypotheses are satisfied:

1. The first Piola-Kirchhoff stress tensor  $\mathbf{T}_*$  depends analytically on the displacement gradient  $\mathbf{H}$ ;
2.  $\mathbf{b}(\epsilon, \mathbf{X}, \mathbf{u}, \mathbf{H})$  and  $\mathbf{t}_*(\epsilon, \mathbf{X}, \mathbf{u}, \mathbf{H})$  are analytical functions of  $\epsilon$ .

Note that in this case the body force and surface tractions depend on the displacement field and its gradient.

Then, if local theorems of existence and uniqueness for the problem (4.78) hold and the solution is an analytic function of  $\epsilon$  which satisfies the compatibility conditions (4.79) together with the assigned loads, problem (4.78)-(4.79) reduces to solving *a set of linear pure traction boundary-value problems with dead loads*.



### 4.2.2 Signorini's Method for Live Traction-Value Problem

In view of the applications of Section 4, the aforesaid perturbation method will be used to solve *up to the second order* of approximation the following problem

$$\begin{cases} \nabla_* \cdot \mathbf{T}_* = \mathbf{0} & \text{in } C_*, \\ \mathbf{T}_* \cdot \mathbf{N}_* = \epsilon \mathbf{t}_* & \text{on } \partial C_*, \end{cases} \quad (4.81)$$

$$\int_{\partial C_*} \epsilon \mathbf{t}_* d\sigma_* = \mathbf{0}, \quad \int_{\partial C_*} \mathbf{r} \times \epsilon \mathbf{t}_* d\sigma_* = \mathbf{0}, \quad (4.82)$$

where

$$\mathbf{t}_* = J \mathbf{t} \sqrt{\mathbf{N}_* \cdot \mathbf{C}^{-1} \mathbf{N}_*}, \quad (4.83)$$

(see equation (2.55)).

The first Piola-Kirchhoff stress tensor  $\mathbf{T}_*$  of an elastic, isotropic and homogeneous body up to second order in  $\mathbf{H}$  is given by (see [13])

$$\begin{aligned} \mathbf{T}_* &= \lambda I_{\mathbf{E}} \mathbf{I} + 2\mu \mathbf{E} + \left[ \frac{\lambda}{2} (I_{\mathbf{H}\mathbf{H}^T} + 2I_{\mathbf{E}}^2) + \beta_1 I_{\mathbf{E}}^2 + \beta_2 II_{\mathbf{E}} \right] \mathbf{I} \\ &\quad + \beta_3 I_{\mathbf{E}} \mathbf{E} + \beta_4 \mathbf{E}^2 - \lambda I_{\mathbf{E}} \mathbf{H}^T - \mu (\mathbf{H}^T)^2, \end{aligned} \quad (4.84)$$

where  $\mathbf{E} = \frac{1}{2} (\mathbf{H} + \mathbf{H}^T)$  is the *infinitesimal strain tensor*,  $\beta_i$ ,  $i = 1, \dots, 4$ , are the *second order constitutive constants*, and  $I_{\mathbf{E}}$  and  $II_{\mathbf{E}}$  denote the first and the second principal invariants of the tensor  $\mathbf{E}$ , respectively.

The series expansion of the displacement  $\mathbf{u}$  up to second-order terms in the small parameter  $\epsilon$  is given by

$$\mathbf{u} = \epsilon \mathbf{u}_1 + \epsilon^2 \mathbf{u}_2. \quad (4.85)$$

From (4.85) we can easily deduce the following relations:

$$\begin{aligned}
\mathbf{H} &= \epsilon \mathbf{H}_1 + \epsilon^2 \mathbf{H}_2, \quad \mathbf{E} = \epsilon \mathbf{E}_1 + \epsilon^2 \mathbf{E}_2, \quad I_{\mathbf{E}} = \epsilon I_{\mathbf{E}_1} + \epsilon^2 I_{\mathbf{E}_2}, \\
I_{\mathbf{H}\mathbf{H}^T} &= \epsilon^2 I_{\mathbf{H}_1\mathbf{H}_1^T}, \quad I_{\mathbf{E}}^2 = \epsilon^2 I_{\mathbf{E}_1}^2, \quad II_{\mathbf{E}} = \epsilon^2 II_{\mathbf{E}_1}, \quad I_{\mathbf{E}}\mathbf{E} = \epsilon^2 I_{\mathbf{E}_1}\mathbf{E}_1, \\
\mathbf{E}^2 &= \epsilon^2 \mathbf{E}_1^2, \quad I_{\mathbf{E}}\mathbf{H}^T = \epsilon^2 I_{\mathbf{E}_1}\mathbf{H}_1^T, \quad (\mathbf{H}^T)^2 = \epsilon^2 (\mathbf{H}_1^T)^2.
\end{aligned} \tag{4.86}$$

Substituting the second-order expansions (4.86) into (4.84) we obtain

$$\mathbf{T}_* = \epsilon \mathbf{T}_{*1} + \epsilon^2 (\mathbf{T}_{*2} + \mathbf{B}_{*1}), \tag{4.87}$$

where

$$\mathbf{T}_{*i} = \lambda I_{\mathbf{E}_i} \mathbf{I} + 2\mu \mathbf{E}_i \quad i = 1, 2, \tag{4.88}$$

$$\begin{aligned}
\mathbf{B}_{*1} &= \left[ \frac{\lambda}{2} \left( I_{\mathbf{H}_1\mathbf{H}_1^T} + 2I_{\mathbf{E}_1}^2 \right) + \beta_1 I_{\mathbf{E}_1}^2 + \beta_2 II_{\mathbf{E}_1} \right] \mathbf{I} + \beta_3 I_{\mathbf{E}_1} \mathbf{E}_1 \\
&\quad + \beta_4 \mathbf{E}_1^2 - \lambda I_{\mathbf{E}_1} \mathbf{H}_1^T - \mu (\mathbf{H}_1^T)^2.
\end{aligned} \tag{4.89}$$

We now derive the form of the traction (4.83) up to second order terms. First, from (4.85)

we get the following expression for  $\mathbf{t} = \mathbf{t}(\mathbf{X}, \mathbf{u}(\epsilon), \mathbf{H}(\epsilon))$

$$\epsilon \mathbf{t} = \epsilon \mathbf{t}_1(\mathbf{X}) + \epsilon^2 [(\nabla_{\mathbf{u}} \mathbf{t})_0 \mathbf{u}_1 + (\nabla_{\mathbf{H}} \mathbf{t})_0 \mathbf{H}_1]. \tag{4.90}$$

Furthermore, for a matrix  $\mathbf{A}$  written as

$$\mathbf{A} = \mathbf{1} + \mathbf{S},$$

$$\det \mathbf{A} = 1 + I_{\mathbf{S}} + II_{\mathbf{S}} + III_{\mathbf{S}}, \tag{4.91}$$

$$\mathbf{A}^{-1} = \mathbf{1} - \mathbf{S} + \mathbf{S}^2 + 0(\mathbf{S}^2). \tag{4.92}$$

Therefore, for

$$\mathbf{F} = \mathbf{1} + \mathbf{H},$$

from (4.91) and (4.92), to within an error of third order in the components of  $\mathbf{H}$ , we get

$$J = \det \mathbf{F} \simeq 1 + I_{\mathbf{H}} + II_{\mathbf{H}} = 1 + I_{\mathbf{H}} + \frac{1}{2} [I_{\mathbf{H}}^2 - I_{\mathbf{H}^2}], \quad (4.93)$$

$$\mathbf{F}^{-1} = \mathbf{1} - \mathbf{H} + \mathbf{H}^2 + o(\mathbf{H}^2), \quad (4.94)$$

$$\begin{aligned} \mathbf{C}^{-1} &= (\mathbf{F}^T \mathbf{F})^{-1} = \mathbf{F}^{-1} (\mathbf{F}^T)^{-1} \simeq (\mathbf{1} - \mathbf{H} + \mathbf{H}^2) (\mathbf{1} - \mathbf{H}^T + (\mathbf{H}^T)^2) \\ &\simeq \mathbf{1} - (\mathbf{H} + \mathbf{H}^T) + (\mathbf{H}^T)^2 + \mathbf{H}\mathbf{H}^T + \mathbf{H}^2. \end{aligned} \quad (4.95)$$

From (4.86)<sub>1</sub> it follows that

$$\mathbf{H}^2 \simeq \epsilon^2 \mathbf{H}_1^2 \quad (4.96)$$

$$I_{\mathbf{H}} = \epsilon I_{\mathbf{H}_1} + \epsilon^2 I_{\mathbf{H}_2}, \quad I_{\mathbf{H}^2} = \epsilon^2 I_{\mathbf{H}_1^2}. \quad (4.97)$$

Substituting relations (4.97) into (4.93) we obtain

$$\begin{aligned} J &\simeq 1 + \epsilon I_{\mathbf{H}_1} + \epsilon^2 I_{\mathbf{H}_2} + \frac{1}{2} [(\epsilon I_{\mathbf{H}_1} + \epsilon^2 I_{\mathbf{H}_2})^2 - \epsilon^2 I_{\mathbf{H}_1^2}] \\ &\simeq 1 + \epsilon I_{\mathbf{H}_1} + \epsilon^2 I_{\mathbf{H}_2} + \frac{1}{2} \epsilon^2 (I_{\mathbf{H}_1}^2 - I_{\mathbf{H}_1^2}) \end{aligned}$$

and, since

$$II_{\mathbf{H}_1} = \frac{1}{2} (I_{\mathbf{H}_1}^2 - I_{\mathbf{H}_1^2}),$$

we finally come to the following second-order expression for  $J$

$$J = 1 + \epsilon I_{\mathbf{H}_1} + \epsilon^2 [I_{\mathbf{H}_2} + II_{\mathbf{H}_1}]. \quad (4.98)$$

By using equations (4.86)<sub>1</sub>, (4.86)<sub>10</sub>, and (4.96) we get

$$\begin{aligned}
\mathbf{C}^{-1} &= \mathbf{1} - (\epsilon \mathbf{H}_1 + \epsilon^2 \mathbf{H}_2 + \epsilon \mathbf{H}_1^T + \epsilon^2 \mathbf{H}_2^T) + (\epsilon \mathbf{H}_1^T + \epsilon^2 \mathbf{H}_2^T)^2 + \\
&\quad (\epsilon \mathbf{H}_1 + \epsilon^2 \mathbf{H}_2) (\epsilon \mathbf{H}_1^T + \epsilon^2 \mathbf{H}_2^T) + (\epsilon \mathbf{H}_1 + \epsilon^2 \mathbf{H}_2)^2 \\
&\simeq \mathbf{1} - (\epsilon \mathbf{H}_1 + \epsilon^2 \mathbf{H}_2 + \epsilon \mathbf{H}_1^T + \epsilon^2 \mathbf{H}_2^T) + (\epsilon \mathbf{H}_1^T)^2 + \epsilon^2 \mathbf{H}_1 \mathbf{H}_1^T + \epsilon^2 \mathbf{H}_1^2 \\
&\simeq \mathbf{1} - \epsilon (\mathbf{H}_1 + \mathbf{H}_1^T) + \epsilon^2 (\mathbf{H}_1^2 + (\mathbf{H}_1^T)^2 + \mathbf{H}_1 \mathbf{H}_1^T - \mathbf{H}_2 - \mathbf{H}_2^T),
\end{aligned}$$

and remembering that

$$2\mathbf{E}_i = (\mathbf{H}_i + \mathbf{H}_i^T), \quad i = 1, 2,$$

we have

$$\mathbf{C}^{-1} \simeq \mathbf{1} - 2\epsilon \mathbf{E}_1 - \epsilon^2 (2\mathbf{E}_2 - \mathbf{H}_1^2 - \mathbf{H}_1 \mathbf{H}_1^T - (\mathbf{H}_1^T)^2). \quad (4.99)$$

From (4.99) it follows that

$$\mathbf{N}_* \cdot \mathbf{C}^{-1} \mathbf{N}_* = 1 - 2\epsilon \mathbf{N}_* \cdot \mathbf{E}_1 \mathbf{N}_* - \epsilon^2 \mathbf{N}_* \cdot (2\mathbf{E}_2 - \mathbf{H}_1^2 - \mathbf{H}_1 \mathbf{H}_1^T - (\mathbf{H}_1^T)^2) \mathbf{N}_*.$$

Using the Taylor series expansion

$$\sqrt{1 - 2a\epsilon - b\epsilon^2} = 1 - \epsilon a - \left(\frac{a^2 + b}{2}\right) \epsilon^2 + o(\epsilon^2),$$

and introducing the following constants

$$a = \mathbf{N}_* \cdot \mathbf{E}_1 \mathbf{N}_*$$

$$b = \mathbf{N}_* \cdot (2\mathbf{E}_2 - \mathbf{H}_1^2 - \mathbf{H}_1 \mathbf{H}_1^T - (\mathbf{H}_1^T)^2) \mathbf{N}_*,$$

we obtain

$$\begin{aligned}
\sqrt{\mathbf{N}_* \cdot \mathbf{C}^{-1} \mathbf{N}_*} &= \sqrt{1 - 2a\epsilon - \epsilon^2 b} \\
&\simeq 1 - \epsilon a - \left(\frac{a^2 + b}{2}\right) \epsilon^2.
\end{aligned} \quad (4.100)$$

Thus from equations (4.90), (4.98), and (4.100) the relation (4.83) can be written as

$$\begin{aligned}\epsilon \mathbf{t}_* &= J \epsilon \mathbf{t} \sqrt{\mathbf{N}_* \cdot \mathbf{C}^{-1} \mathbf{N}_*} \simeq (1 + \epsilon I_{\mathbf{H}_1} + \epsilon^2 [I_{\mathbf{H}_2} + II_{\mathbf{H}_1}]) \\ &\quad (\epsilon \mathbf{t}_1(\mathbf{X}) + \epsilon^2 \mathbf{f}) \left[ 1 - \epsilon a - \left( \frac{a^2 + b}{2} \right) \epsilon^2 \right],\end{aligned}$$

where

$$\mathbf{f} = (\nabla_{\mathbf{u}} \mathbf{t})_{\mathbf{0}} \mathbf{u}_1 + (\nabla_{\mathbf{H}} \mathbf{t})_{\mathbf{0}} \mathbf{H}_1.$$

Retaining only the terms up to second order in  $\epsilon$  we obtain

$$\begin{aligned}\epsilon \mathbf{t}_* &= (\epsilon \mathbf{t}_1 + \epsilon^2 \mathbf{f} + \epsilon^2 I_{\mathbf{H}_1} \mathbf{t}_1) \left[ 1 - \epsilon a - \left( \frac{a^2 + b}{2} \right) \epsilon^2 \right] \\ &\simeq \epsilon \mathbf{t}_1 - \epsilon^2 \mathbf{t}_1 a + \epsilon^2 \mathbf{f} + \epsilon^2 I_{\mathbf{H}_1} \mathbf{t}_1 \\ &= \epsilon \mathbf{t}_1 + \epsilon^2 [I_{\mathbf{H}_1} \mathbf{t}_1 - \mathbf{t}_1 \mathbf{N}_* \cdot \mathbf{E}_1 \mathbf{N}_* + (\nabla_{\mathbf{u}} \mathbf{t})_{\mathbf{0}} \mathbf{u}_1 + (\nabla_{\mathbf{H}} \mathbf{t})_{\mathbf{0}} \mathbf{H}_1].\end{aligned}$$

Thus, the the traction (4.83) assumes the following form up to second order terms

$$\epsilon \mathbf{t}_* = \epsilon \mathbf{t}_{*1} + \epsilon^2 \mathbf{t}_{*2}, \quad (4.101)$$

where

$$\mathbf{t}_{*1} = \mathbf{t}_1, \quad (4.102)$$

$$\mathbf{t}_{*2} = I_{\mathbf{H}_1} \mathbf{t}_1 - \mathbf{t}_1 \mathbf{N}_* \cdot \mathbf{E}_1 \mathbf{N}_* + (\nabla_{\mathbf{u}} \mathbf{t})_{\mathbf{0}} \mathbf{u}_1 + (\nabla_{\mathbf{H}} \mathbf{t})_{\mathbf{0}} \mathbf{H}_1.$$

Finally, because of equations (4.87) and (4.101), problem (4.81) reduces to solving the following two *linear boundary-value problems with dead loads*

$$\begin{cases} \nabla_* \cdot \mathbf{T}_{*1} = \mathbf{0} & \text{in } C_*, \\ \mathbf{T}_{*1} \cdot \mathbf{N}_* = \mathbf{t}_{*1} & \text{on } \partial C_*, \end{cases} \quad (4.103)$$

$$\begin{cases} \nabla_* \cdot (\mathbf{T}_{*2} + \mathbf{B}_{*1}) = \mathbf{0} & \text{in } C_*, \\ (\mathbf{T}_{*2} + \mathbf{B}_{*1}) \cdot \mathbf{N}_* = \mathbf{t}_{*2} & \text{on } \partial C_*, \end{cases} \quad (4.104)$$

respectively with the following compatibility conditions

$$\int_{\partial C_*} \mathbf{t}_{*1} d\sigma_* = \mathbf{0}, \quad (4.105)$$

$$\int_{\partial C_*} (\mathbf{X} + \mathbf{u}_1) \times \mathbf{t}_{*1} d\sigma_* = \mathbf{0},$$

$$\int_{\partial C_*} \mathbf{t}_{*2} d\sigma_* = \mathbf{0}, \quad (4.106)$$

$$\int_{\partial C_*} [(\mathbf{X} + \mathbf{u}_1) \times \mathbf{t}_{*2} + \mathbf{u}_2 \times \mathbf{t}_{*1}] d\sigma_* = \mathbf{0}.$$

Here moments are taken with respect to the origin of the coordinate system.

In the next section, we solve the problem (4.81)-(4.82) only when  $S$  is subjected to a *uniform pressure*. Furthermore, since the uniform pressure fields are live loads which are invariant under translations, the theorems of existence, uniqueness, and continuous dependence on a parameter, proved in [20], hold for the problem (4.81)-(4.82).

Equations (4.105) are compatibility conditions imposing restrictions both on the applied forces and on the corresponding displacement which is solution to the problem (4.103).

In other words, in (4.105) the following relations

$$\int_{\partial C_*} \mathbf{t}_{*1} d\sigma_* = \mathbf{0}, \quad \int_{\partial C_*} \mathbf{X} \times \mathbf{t}_{*1} d\sigma_* = \mathbf{0},$$

representing a restriction on the applied forces, have to be verified a priori. However, the condition which guarantees the physical meaning of the displacement  $\mathbf{u}_1$

$$\int_{\partial C_*} \mathbf{u}_1 \times \mathbf{t}_{*1} d\sigma_* = \mathbf{0},$$

can only be verified a posteriori.

The inspection of the compatibility conditions (4.106) can only be made a posteriori.

### 4.3 Two Live Traction-Value Problems

In this section problems (4.103)-(4.106) are solved for the case of a continuum with spherical and cylindrical geometry. Then, starting from the obtained solutions, some experimental procedures are suggested to determine the constitutive constants  $\beta_i$  of the body.

#### 4.3.1 The First Traction-Value Problem: Sphere

Let  $S$  be a sphere of radius  $R$  made of an elastic, homogeneous, and isotropic material. Suppose  $S$  be at equilibrium in a current configuration  $C$  under the action of a uniform pressure field

$$\mathbf{t} = -p_0\mathbf{N}, \quad (4.107)$$

where  $p_0$  is a positive constant and  $\mathbf{N}$  is the unit outward normal vector to the boundary  $\partial C$  of  $S$  (see Figure 4.1).

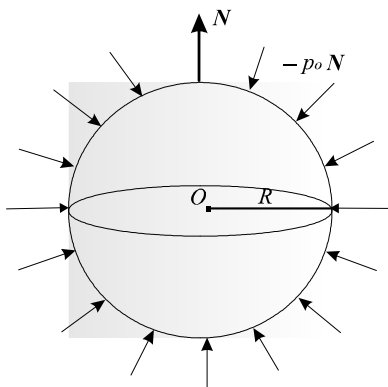


Fig. 4.1. A sphere loaded by a uniform pressure.

In order to write the Lagrangian equilibrium equations in a nondimensional form, it is useful to introduce the following characteristic quantities (see (4.77))

$$\begin{aligned} \tilde{t} &= \tilde{p}, \quad l = \tilde{u}(R), \\ \tilde{T} &= \max\{\lambda, \mu\} \equiv \Gamma, \quad L = R, \end{aligned} \quad (4.108)$$

where  $\tilde{p}$  is a pressure which induces a linear response and  $\tilde{u}(R)$  is the displacement of the boundary of  $S$  in the linear approximation.

From (4.80) and (4.108), the perturbation parameter  $\epsilon$  is given by

$$\epsilon = \frac{\tilde{p}}{\alpha\Gamma}. \quad (4.109)$$

Owing to spherical symmetry, we search for the solution of pure traction-value problem (4.78)-(4.79) in the following form

$$\mathbf{u}(r) = [\epsilon u_1(r) + \epsilon^2 u_2(r)] \mathbf{a}_r, \quad (4.110)$$

where  $\mathbf{a}_r$  is the radial unit vector of the physical basis associated with the spherical coordinates  $\{r, \varphi, \theta\}$ .

The first-order boundary-value problem (4.103) is

$$\begin{cases} \nabla_* \cdot \mathbf{T}_{*1} = \mathbf{0} & \text{in } C_*, \\ \mathbf{T}_{*1} \cdot \mathbf{N}_* = \mathbf{t}_{*1} & \text{on } \partial C_*. \end{cases} \quad (4.111)$$

In order to write the problem (4.111) in spherical coordinates we start by writing the metric tensor  $(g_{ij})$  associated with the natural (or holonomic) basis  $(\mathbf{e}_i)$ .

$$(g_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 \sin^2 \theta & 0 \\ 0 & 0 & r^2 \end{pmatrix}, \quad (g^{ij}) = (g_{ij})^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2 \sin^2 \theta} & 0 \\ 0 & 0 & \frac{1}{r^2} \end{pmatrix}. \quad (4.112)$$



First, we recall that the relation between the unit vectors of the natural basis ( $\mathbf{e}_i$ ) and the unit vectors ( $\mathbf{a}_i$ ) of the physical basis is<sup>8</sup>

$$\mathbf{a}_i = \frac{1}{\sqrt{g_{ii}}} \mathbf{e}_i \quad (\text{no sum over } i), \quad (4.113)$$

while the relation between the reciprocal basis is

$$\mathbf{a}^i = \sqrt{g_{ii}} \mathbf{e}^i \quad (\text{no sum over } i). \quad (4.114)$$

The covariant components of the first-order displacement gradient  $\mathbf{H}_1$  in the basis  $\mathbf{e}^i \otimes \mathbf{e}^j$  are given by

$$(\mathbf{H}_1)_{ij} = \frac{\partial u_i}{\partial x_j} - \Gamma_{ij}^h u_h \quad (4.115)$$

where the *Christoffel symbols*  $\Gamma_{ij}^h$  are related to the metric coefficients  $g_{ij}$  and  $g^{ij}$  by the following formulae

$$\Gamma_{jh}^l = \frac{1}{2} g^{li} (g_{ij,h} + g_{hi,j} - g_{jh,i}). \quad (4.116)$$

From (4.112) it follows that the non-zero Christoffel symbols are

$$\Gamma_{31}^3 = \Gamma_{13}^3 = \Gamma_{21}^2 = \Gamma_{12}^2 = \frac{1}{r}, \quad \Gamma_{22}^1 = -r \sin^2 \theta, \quad \Gamma_{33}^1 = -r, \quad (4.117)$$

$$\Gamma_{23}^2 = \Gamma_{32}^2 = \cot \theta, \quad \Gamma_{22}^3 = -\sin \theta \cos \theta. \quad (4.118)$$

It is now an easy task to compute from (4.115), (4.117), and (4.118) the covariant components of the first-order displacement gradient  $\mathbf{H}_1$  in the basis  $\mathbf{e}^i \otimes \mathbf{e}^j$

$$(\mathbf{H}_1)_{11} = u_1', \quad (\mathbf{H}_1)_{22} = r \sin^2 \theta u_1, \quad (\mathbf{H}_1)_{33} = r u_1,$$

$$(\mathbf{H}_1)_{ij} = 0, \quad i \neq j,$$

<sup>8</sup> Note that from (4.112) and (4.113) it follows that

$$\mathbf{a}_1 = \mathbf{e}_1, \quad \mathbf{a}_2 = \frac{\mathbf{e}_2}{r \sin \theta}, \quad \mathbf{a}_3 = \frac{\mathbf{e}_3}{r}.$$

where  $u'_1 = \frac{du_1}{dr}$ . The non-zero contravariant components of  $\mathbf{H}_1$  are given by the relation

$$(\mathbf{H}_1)^{ij} = g^{ik} g^{jh} (\mathbf{H}_1)_{kh}$$

and, from (4.112), they assume the following values

$$(\mathbf{H}_1)^{11} = u'_1, \quad (\mathbf{H}_1)^{22} = \frac{u_1}{r^3 \sin^2 \theta}, \quad (\mathbf{H}_1)^{33} = \frac{u_1}{r^3}.$$

Hence, we can write

$$\mathbf{H}_1 = u'_1 \mathbf{e}_1 \otimes \mathbf{e}_1 + \frac{u_1}{r^3 \sin^2 \theta} \mathbf{e}_2 \otimes \mathbf{e}_2 + \frac{u_1}{r^3} \mathbf{e}_3 \otimes \mathbf{e}_3$$

and, from (4.113), we finally obtain

$$\mathbf{H}_1 = u'_1 \mathbf{a}_1 \otimes \mathbf{a}_1 + \frac{u_1}{r} \mathbf{a}_2 \otimes \mathbf{a}_2 + \frac{u_1}{r} \mathbf{a}_3 \otimes \mathbf{a}_3,$$

or in matrix notation

$$\mathbf{H}_1 = \begin{pmatrix} u'_1 & 0 & 0 \\ 0 & \frac{u_1}{r} & 0 \\ 0 & 0 & \frac{u_1}{r} \end{pmatrix}. \quad (4.119)$$

From (4.119) we have

$$I_{\mathbf{H}_1} = u'_1 + 2\frac{u_1}{r}, \quad \mathbf{E}_1 = \mathbf{H}_1,$$

and from (4.88) we obtain

$$\mathbf{T}_{*1} = \begin{pmatrix} (\lambda + 2\mu)u'_1 + 2\lambda\frac{u_1}{r} & 0 & 0 \\ 0 & \lambda u'_1 + 2(\lambda + \mu)\frac{u_1}{r} & 0 \\ 0 & 0 & \lambda u'_1 + 2(\lambda + \mu)\frac{u_1}{r} \end{pmatrix}. \quad (4.120)$$

We recall that the physical components of the divergence of a symmetric tensor field  $\mathbf{T}$  in spherical coordinates are given by (see [33])

$$\begin{aligned}
(\nabla \cdot \mathbf{T})_r &= \frac{\partial T_{rr}}{\partial r} + \frac{1}{r} \frac{\partial T_{r\theta}}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial T_{r\varphi}}{\partial \varphi} + \\
&\quad \frac{1}{r} (2T_{rr} - T_{\theta\theta} - T_{\varphi\varphi} + \cot \theta T_{r\theta}), \\
(\nabla \cdot \mathbf{T})_\theta &= \frac{\partial T_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial T_{\theta\varphi}}{\partial \varphi} + \\
&\quad \frac{1}{r} [3T_{r\theta} + \cot \theta (T_{\theta\theta} - T_{\varphi\varphi})], \\
(\nabla \cdot \mathbf{T})_\varphi &= \frac{\partial T_{r\varphi}}{\partial r} + \frac{1}{r} \frac{\partial T_{\theta\varphi}}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial T_{\varphi\varphi}}{\partial \varphi} + \\
&\quad \frac{1}{r} (3T_{r\varphi} + 2 \cot \theta T_{\theta\varphi}).
\end{aligned} \tag{4.121}$$

It can be easily seen from (4.120) that the relations (4.121) assume the following form

$$\begin{aligned}
(\nabla \cdot \mathbf{T}_{*1})_r &= \frac{d(\mathbf{T}_{*1})_{rr}}{dr} + \frac{1}{r} [2(\mathbf{T}_{*1})_{rr} - (\mathbf{T}_{*1})_{\theta\theta} - (\mathbf{T}_{*1})_{\varphi\varphi}] \\
&= (\lambda + 2\mu) \left( u_1'' + \frac{2}{r} u_1' - \frac{2}{r^2} u_1 \right), \\
(\nabla \cdot \mathbf{T}_{*1})_\theta &= 0, \quad (\nabla \cdot \mathbf{T}_{*1})_\varphi = 0,
\end{aligned} \tag{4.122}$$

and, having in mind that  $\mathbf{N}_* = (1, 0, 0)^T$ ,

$$\mathbf{T}_{*1} \cdot \mathbf{N}_* = (\lambda + 2\mu) u_1' + \frac{2\lambda}{r} u_1. \tag{4.123}$$

Furthermore, from (4.107) and (4.102)<sub>1</sub> it follows that

$$\mathbf{t}_{*1} = -p_0 \mathbf{N}_*. \tag{4.124}$$

Hence, from (4.122) and (4.123), and (4.124), the boundary-value problem (4.111) becomes

$$\begin{cases} r^2 u_1'' + 2r u_1' - 2u_1 = 0, \\ \left[ (\lambda + 2\mu) u_1' + \frac{2\lambda}{r} u_1 \right]_{r=R} = -p_0. \end{cases} \tag{4.125}$$

The general integral of the differential equation (4.125)<sub>1</sub> is

$$u_1(r) = \bar{A}r + \frac{\bar{B}}{r}. \quad (4.126)$$

Substituting from (4.126) into the boundary condition (4.125)<sub>2</sub> and due to the spherical symmetry of the problem we must have

$$u(0) = 0,$$

by routine calculations we get

$$\bar{A} = -\frac{p_0}{3\lambda + 2\mu}, \quad \bar{B} = 0.$$

Thus, the solution of system (4.125) is

$$u_1(r) = -\frac{p_0}{3\lambda + 2\mu}r. \quad (4.127)$$

On the other hand,  $\mathbf{t}_{*1} = -p_0\mathbf{N}_*$ , so that from (4.127) we easily verify that the first-order compatibility conditions (4.105) are satisfied.

The second-order boundary-value problem is

$$\begin{cases} \nabla_* \cdot (\mathbf{T}_{*2} + \mathbf{B}_{*1}) = \mathbf{0} & \text{in } C_*, \\ (\mathbf{T}_{*2} + \mathbf{B}_{*1}) \cdot \mathbf{N}_* = \mathbf{t}_{*2} & \text{on } \partial C_*. \end{cases} \quad (4.128)$$

From the first-order displacement (4.127) we obtain

$$\mathbf{H}_1 = \mathbf{E}_1 = \bar{A}\mathbf{1}, \quad \mathbf{E}_1^2 = (\mathbf{H}_1^T)^2 = \bar{A}^2\mathbf{1}, \quad (4.129)$$

$$I_{\mathbf{H}_1} = I_{\mathbf{E}_1} = 3\bar{A}, \quad I_{\mathbf{H}_1\mathbf{H}_1^T} = 3\bar{A}^2, \quad II_{\mathbf{E}_1} = 3\bar{A}^2.$$

Substitution of expressions (4.129) into (4.89) and (4.102)<sub>2</sub> yields

$$\begin{aligned} \mathbf{B}_{*1} &= \Lambda\bar{A}^2\mathbf{1}, \\ \mathbf{t}_{*2} &= -2p_0\bar{A}\mathbf{N}_*, \end{aligned} \quad (4.130)$$

where

$$\Lambda = \frac{15}{2}\lambda - \mu + 9\beta_1 + 3\beta_2 + 3\beta_3 + \beta_4.$$

Since  $\Lambda$  and  $\bar{A}$  are constants, equation (4.130)<sub>1</sub> implies that

$$\nabla_* \cdot \mathbf{B}_{*1} = \mathbf{0}. \quad (4.131)$$

Using the same arguments as those used to derive the expression of the divergence of the first-order stress tensor we obtain

$$\begin{aligned} (\nabla \cdot \mathbf{T}_{*2})_r &= \frac{\partial(\mathbf{T}_{*2})_{rr}}{\partial r} + \frac{1}{r}[2(\mathbf{T}_{*2})_{rr} - (\mathbf{T}_{*2})_{\theta\theta} - (\mathbf{T}_{*2})_{\varphi\varphi}] \\ &= (\lambda + 2\mu) \left( u_2'' + \frac{2}{r}u_2' - \frac{2}{r^2}u_2 \right), \\ (\nabla \cdot \mathbf{T}_{*2})_\theta &= 0, \quad (\nabla \cdot \mathbf{T}_{*2})_\varphi = 0. \end{aligned} \quad (4.132)$$

Furthermore

$$\mathbf{T}_{*2} \cdot \mathbf{N}_* = (\lambda + 2\mu) u_2' + \frac{2\lambda}{r} u_2. \quad (4.133)$$

Therefore, from (4.132), (4.131) and (4.130), the second-order boundary-value problem (4.104) becomes

$$\begin{cases} r^2 u_2'' + 2r u_2' - 2u_2 = 0, \\ \left[ (\lambda + 2\mu) u_2' + \frac{2\lambda}{r} u_2 \right]_{r=R} + \Lambda \bar{A}^2 = -2p_0 \bar{A}, \end{cases} \quad (4.134)$$

and it admits the following solution

$$u_2(r) = \left( 2\bar{A}^2 + \frac{\Lambda \bar{A}^3}{p_0} \right) r. \quad (4.135)$$

From (4.130)<sub>2</sub> and (4.135), it is easy to verify that the second-order compatibility conditions (4.106) are also satisfied.

In conclusion, the pure second-order traction-value problem has the solution

$$\mathbf{u} = \left[ \bar{A}\epsilon + \left( 2\bar{A}^2 + \frac{\Lambda \bar{A}^3}{p_0} \right) \epsilon^2 \right] r \mathbf{a}_r. \quad (4.136)$$

It is influenced by the second-order elastic constants through the presence of  $\Lambda$  in (4.136). Even if all the second-order elastic constants vanish, the second-order displacement field is non-zero.

### 4.3.2 The Second Traction-Value Problem: Hollow Cylinder

We consider an infinite hollow cylinder  $S$  made of an elastic, homogeneous and isotropic material. Let  $R_i$  and  $R_e$  be the internal and the external radii, respectively. Let  $S$  be at equilibrium in the current configuration  $C$  under the action of the uniform pressure field

$$\mathbf{t} = \begin{cases} -p_i \mathbf{N}_i & \text{on } \partial C_i, \\ -p_e \mathbf{N}_e & \text{on } \partial C_e, \end{cases} \quad (4.137)$$

where  $p_i$  and  $p_e$  are positive constants,  $\mathbf{N}_i$  and  $\mathbf{N}_e$  are the unit outward normal vectors to  $\partial C_i$  and  $\partial C_e$ , respectively (see Figure 4.2).

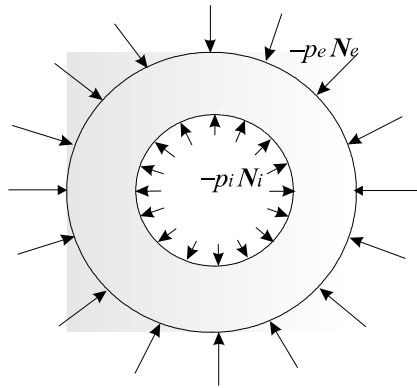


Fig. 4.2. Schematic sketch of a circular hollow cylinder subjected to pressure on the inner and outer surfaces.

Adopting the same arguments as in the previous section, assuming that  $L = R_e$ ,  $\tilde{t} = \hat{p}$  and  $l = \tilde{u}(R_e)$ , where  $\tilde{u}(r)$  is an infinitesimal displacement, the parameter  $\epsilon$  can be written

as

$$\epsilon = \frac{\hat{p}}{\alpha\Gamma}.$$

Adopting the cylindrical coordinates  $\{r, \varphi, z\}$ , the following first-order pure traction-value problem has to be solved

$$\begin{cases} \nabla_* \cdot \mathbf{T}_{*1} = \mathbf{0} & \text{in } C_*, \\ \mathbf{T}_{*1} \cdot \mathbf{N}_{*i} = \mathbf{t}_{*1}^{(i)} & \text{on } \partial C_{*i}, \\ \mathbf{T}_{*1} \cdot \mathbf{N}_{*e} = \mathbf{t}_{*1}^{(e)} & \text{on } \partial C_{*e}, \end{cases} \quad (4.138)$$

for the unknown displacement  $\mathbf{u}_1(r) = u_1(r) \mathbf{a}_r$ , where  $\mathbf{a}_r$  is the radial unit vector of the physical basis  $\{\mathbf{a}_r, \mathbf{a}_\varphi, \mathbf{a}_z\}$  associated with the cylindrical coordinates. In order to write the problem (4.138) in cylindrical coordinates we start by writing the metric tensor  $(g_{ij})$  associated with the natural basis  $(\mathbf{e}_r, \mathbf{e}_\varphi, \mathbf{e}_z)$ <sup>9</sup>

$$(g_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (g^{ij}) = (g_{ij})^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (4.139)$$

From (4.139) and (4.116) it follows that the non-zero Christoffel symbols are

$$\Gamma_{22}^1 = -r, \quad \Gamma_{12}^2 = \Gamma_{21}^2 = \frac{1}{r}. \quad (4.140)$$

It is now easy to see from (4.115) and (4.140) that the non-zero covariant components of the first-order displacement gradient  $\mathbf{H}_1$  in the basis  $\mathbf{e}^i \otimes \mathbf{e}^j$  are

$$(\mathbf{H}_1)_{11} = u_1', \quad (\mathbf{H}_1)_{22} = ru_1.$$

The corresponding contravariant components of  $\mathbf{H}_1$  are given by the relation

$$(\mathbf{H}_1)^{ij} = g^{ik} g^{jh} (\mathbf{H}_1)_{kh}$$

<sup>9</sup> Note that from (4.139) and (4.113) it follows that

$$\mathbf{a}_1 = \mathbf{e}_1, \quad \mathbf{a}_2 = \frac{\mathbf{e}_2}{r}, \quad \mathbf{a}_3 = \mathbf{e}_3.$$

and, from (4.139), they take the following values

$$(\mathbf{H}_1)^{11} = u'_1, \quad (\mathbf{H}_1)^{22} = \frac{u_1}{r^3}.$$

Hence, we can write

$$\mathbf{H}_1 = u'_1 \mathbf{e}_1 \otimes \mathbf{e}_1 + \frac{u_1}{r^3} \mathbf{e}_2 \otimes \mathbf{e}_2,$$

and, from (4.113), we obtain

$$\mathbf{H}_1 = u'_1 \mathbf{a}_1 \otimes \mathbf{a}_1 + \frac{u_1}{r} \mathbf{a}_2 \otimes \mathbf{a}_2,$$

or in matrix notation

$$\mathbf{H}_1 = \begin{pmatrix} u'_1 & 0 & 0 \\ 0 & \frac{u_1}{r} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (4.141)$$

From (4.141) we have

$$I_{\mathbf{H}_1} = u'_1 + \frac{u_1}{r}, \quad \mathbf{E}_1 = \mathbf{H}_1,$$

and from (4.88) we get

$$\mathbf{T}_{*1} = \begin{pmatrix} (\lambda + 2\mu)u'_1 + \lambda \frac{u_1}{r} & 0 & 0 \\ 0 & \lambda u'_1 + (\lambda + 2\mu) \frac{u_1}{r} & 0 \\ 0 & 0 & \lambda \left( u'_1 + \frac{u_1}{r} \right) \end{pmatrix}. \quad (4.142)$$

We recall that the physical components of the divergence of a symmetric tensor field  $\mathbf{T}$  in cylindrical coordinates are given by (see [33])

$$\begin{aligned} (\nabla \cdot \mathbf{T})_r &= \frac{\partial T_{rr}}{\partial r} + \frac{1}{r} \frac{\partial T_{r\theta}}{\partial \theta} + \frac{\partial T_{rz}}{\partial z} + \frac{T_{rr} - T_{\theta\theta}}{r}, \\ (\nabla \cdot \mathbf{T})_\theta &= \frac{\partial T_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{\partial T_{\theta z}}{\partial z} + \frac{2}{r} T_{r\theta}, \\ (\nabla \cdot \mathbf{T})_z &= \frac{\partial T_{rz}}{\partial r} + \frac{1}{r} \frac{\partial T_{\theta z}}{\partial \theta} + \frac{\partial T_{zz}}{\partial z} + \frac{1}{r} T_{rz}. \end{aligned} \quad (4.143)$$



It can be easily seen from (4.142) that quantities in (4.143) assume the following values

$$\begin{aligned}
 (\nabla \cdot \mathbf{T}_{*1})_r &= \frac{\partial(\mathbf{T}_{*1})_{rr}}{\partial r} + \frac{(\mathbf{T}_{*1})_{rr} - (\mathbf{T}_{*1})_{\theta\theta}}{r} \\
 &= (\lambda + 2\mu) \left( u_1'' + \frac{u_1'}{r} - \frac{u_1}{r^2} \right), \\
 (\nabla \cdot \mathbf{T}_{*1})_\theta &= 0, \quad (\nabla \cdot \mathbf{T}_{*1})_z = 0,
 \end{aligned} \tag{4.144}$$

and, since

$$\mathbf{N}_{*i} = (-1, 0, 0)^T, \quad \mathbf{N}_{*e} = (1, 0, 0)^T, \tag{4.145}$$

we have

$$\begin{aligned}
 \mathbf{T}_{*1} \cdot \mathbf{N}_{*i} &= - \left[ (\lambda + 2\mu) u_1' + \frac{\lambda}{r} u_1 \right], \\
 \mathbf{T}_{*1} \cdot \mathbf{N}_{*e} &= (\lambda + 2\mu) u_1' + \frac{\lambda}{r} u_1.
 \end{aligned} \tag{4.146}$$

Furthermore from (4.137) and (4.102)<sub>1</sub> we have

$$\mathbf{t}_{*1}^{(i)} = -p_i \mathbf{N}_{*i}, \quad \mathbf{t}_{*1}^{(e)} = -p_e \mathbf{N}_{*e}. \tag{4.147}$$

Hence, from (4.144), (4.146) and (4.147), the boundary-value problem (4.138) becomes

$$\begin{cases} r^2 u_1'' + r u_1' - u_1 = 0, \\ \left[ (\lambda + 2\mu) u_1' + \frac{\lambda}{r} u_1 \right]_{r=R_i} = -p_i, \\ \left[ (\lambda + 2\mu) u_1' + \frac{\lambda}{r} u_1 \right]_{r=R_e} = -p_e. \end{cases} \tag{4.148}$$

The general solution of (4.148) is

$$u_1(r) = Ar + \frac{B}{r}. \tag{4.149}$$

Substituting (4.149) into boundary conditions (4.148)<sub>2,3</sub>, we obtain following values of constants  $A$  and  $B$

$$A = \frac{R_e^2 p_e - R_i^2 p_i}{2(\lambda + \mu)(R_i^2 - R_e^2)}, \quad B = \frac{R_e^2 R_i^2 (p_e - p_i)}{2\mu(R_i^2 - R_e^2)}. \quad (4.150)$$

The second-order boundary-value problem is

$$\begin{cases} \nabla_* \cdot (\mathbf{T}_{*2} + \mathbf{B}_{*1}) = \mathbf{0} & \text{in } C_*, \\ (\mathbf{T}_{*2} + \mathbf{B}_{*1}) \cdot \mathbf{N}_{*i} = \mathbf{t}_{*2}^{(i)} & \text{on } \partial C_{*i}, \\ (\mathbf{T}_{*2} + \mathbf{B}_{*1}) \cdot \mathbf{N}_{*e} = \mathbf{t}_{*2}^{(e)} & \text{on } \partial C_{*e}, \end{cases} \quad (4.151)$$

Adopting the same arguments that brought us to the expression of the first-order stress tensor (4.142), we obtain

$$\mathbf{T}_{*2} = \begin{pmatrix} (\lambda + 2\mu)u'_2 + \lambda \frac{u_2}{r} & 0 & 0 \\ 0 & \lambda u'_2 + (\lambda + 2\mu) \frac{u_2}{r} & 0 \\ 0 & 0 & \lambda \left( u'_2 + \frac{u_2}{r} \right) \end{pmatrix}, \quad (4.152)$$

from which we get

$$\begin{aligned} (\nabla \cdot \mathbf{T}_{*2})_r &= \frac{\partial(\mathbf{T}_{*2})_{rr}}{\partial r} + \frac{(\mathbf{T}_{*2})_{rr} - (\mathbf{T}_{*2})_{\theta\theta}}{r} \\ &= (\lambda + 2\mu) \left( u''_2 + \frac{u'_2}{r} - \frac{u_2}{r^2} \right), \\ (\nabla \cdot \mathbf{T}_{*2})_\theta &= 0, \quad (\nabla \cdot \mathbf{T}_{*2})_z = 0. \end{aligned} \quad (4.153)$$

From the first-order displacement (4.149) we obtain

$$I_{\mathbf{H}_1} = I_{\mathbf{E}_1} = 2A, \quad I_{\mathbf{H}_1 \mathbf{H}_1^T} = 2 \left( A^2 + \frac{B^2}{r^4} \right), \quad II_{\mathbf{E}_1} = A^4 - \frac{B^2}{r^4}. \quad (4.154)$$

Substituting relations (4.154) into (4.89) we obtain

$$\begin{aligned}
(\mathbf{B}_{*1})_{11} &= A^2(3\lambda - \mu) + \frac{2AB(\lambda + \mu)}{r^2} + \frac{B^2(\lambda - \mu)}{r^4} \\
&\quad + 4A^2\beta_1 + \left(A^2 - \frac{B^2}{r^4}\right)\beta_2 + \left(2A^2 - \frac{2AB}{r^2}\right)\beta_3 + \left(A - \frac{B}{r^2}\right)^2\beta_4, \\
(\mathbf{B}_{*1})_{22} &= A^2(3\lambda - \mu) - \frac{2AB(\lambda + \mu)}{r^2} + \frac{B^2(\lambda - \mu)}{r^4} \\
&\quad + 4A^2\beta_1 + \left(A^2 - \frac{B^2}{r^4}\right)\beta_2 + \left(2A^2 + \frac{2AB}{r^2}\right)\beta_3 + \left(A + \frac{B}{r^2}\right)^2\beta_4, \\
(\mathbf{B}_{*1})_{33} &= 5A^2\lambda + \frac{B^2\lambda}{r^4} + 4A^2\beta_1 + \left(A^2 - \frac{B^2}{r^4}\right)\beta_2, \\
(\mathbf{B}_{*1})_{ij} &= 0, \quad i \neq j.
\end{aligned} \tag{4.155}$$

It is now easy to compute from (4.143) components of the divergence of  $\mathbf{B}_{*1}$

$$\begin{aligned}
(\nabla \cdot \mathbf{B}_{*1})_r &= \frac{\partial(\mathbf{B}_{*1})_{rr}}{\partial r} + \frac{(\mathbf{B}_{*1})_{rr} - (\mathbf{B}_{*1})_{\theta\theta}}{r} \\
&= \frac{4B^2(\mu - \lambda + \beta_2 - \beta_4)}{r^5}, \\
(\nabla \cdot \mathbf{B}_{*1})_\theta &= 0, \quad (\nabla \cdot \mathbf{B}_{*1})_z = 0.
\end{aligned} \tag{4.156}$$

In order to write boundary conditions (4.151)<sub>2,3</sub> we start by noting that (4.152), (4.155), and (4.145) give

$$(\mathbf{T}_{*2} + \mathbf{B}_{*1}) \cdot \mathbf{N}_{*i} = - \left[ (\lambda + 2\mu) u'_2 + \frac{\lambda}{r} u_2 + (\mathbf{B}_{*1})_{11} \right] \mathbf{a}_r, \tag{4.157}$$

$$(\mathbf{T}_{*2} + \mathbf{B}_{*1}) \cdot \mathbf{N}_{*e} = \left[ (\lambda + 2\mu) u'_2 + \frac{\lambda}{r} u_2 + (\mathbf{B}_{*1})_{11} \right] \mathbf{a}_r. \tag{4.158}$$

Further, from (4.137) and (4.102)<sub>2</sub> it follows that

$$\mathbf{t}_{*2}^{(i)} = p_i (\mathbf{H}_1^T - I_{\mathbf{H}_1} \mathbf{1}) \mathbf{N}_{*i},$$

$$\mathbf{t}_{*2}^{(e)} = p_e (\mathbf{H}_1^T - I_{\mathbf{H}_1} \mathbf{1}) \mathbf{N}_{*e},$$

which, recalling (4.154) and (4.145), become

$$\mathbf{t}_{*2}^{(i)} = \left( A + \frac{B}{r^2} \right) \mathbf{a}_r, \quad \mathbf{t}_{*2}^{(e)} = - \left( A + \frac{B}{r^2} \right) \mathbf{a}_r. \quad (4.159)$$

Hence, from (4.153), (4.156), (4.157), (4.158), and (4.159) the second-order boundary-value problem (4.151) becomes

$$\begin{cases} r^2 u_2'' + r u_2' - u_2 = - \frac{4B^2 (\mu - \lambda + \beta_2 - \beta_4)}{(\lambda + 2\mu) r^3}, \\ \left[ -(\lambda + 2\mu) u_2' - \frac{\lambda}{r} u_2 - (\mathbf{B}_{*1})_{11} \right]_{r=R_i} = D_i p_i, \\ \left[ (\lambda + 2\mu) u_2' + \frac{\lambda}{r} u_2 + (\mathbf{B}_{*1})_{11} \right]_{r=R_e} = -D_e p_e, \end{cases} \quad (4.160)$$

where

$$D_i = A + \frac{B}{R_i^2}, \quad D_e = A + \frac{B}{R_e^2}. \quad (4.161)$$

It will be useful in the sequel to write  $(\mathbf{B}_{*1})_{11}$  as follows (see (4.155))

$$(\mathbf{B}_{*1})_{11} = \Lambda_0 + \sum_{i=1}^4 \Lambda_i \beta_i, \quad (4.162)$$

where

$$\begin{aligned} \Lambda_0 &= A^2 (3\lambda - \mu) + \frac{2AB(\lambda + \mu)}{r^2} + \frac{B^2(\lambda - \mu)}{r^4}, \\ \Lambda_1 &= 4A^2, \quad \Lambda_2 = A^2 - \frac{B^2}{r^4}, \quad \Lambda_3 = 2A^2 - \frac{2AB}{r^2}, \\ \Lambda_4 &= A^2 + \frac{B^2}{r^4} - \frac{2AB}{r^2} = \Lambda_3 - \Lambda_2. \end{aligned} \quad (4.163)$$

The problem (4.160) has the following solution

$$u_2(r) = rC_1 + \frac{C_2}{r} + \frac{C_3}{r^3}, \quad (4.164)$$

where

$$C_3 = \frac{B^2 (\lambda - \mu - \beta_2 + \beta_4)}{2(\lambda + 2\mu)}$$

and the integration constants  $C_1$  and  $C_2$  have to be determined from the boundary conditions (4.160)<sub>2,3</sub>.

In anticipation of the developments in the next section, it is useful to write constants  $C_h$ ,  $h = 1, 2, 3$ , in such a way as to highlight their dependence on the second-order material moduli. In particular, it is possible to write them as

$$C_h = A_{h0} + \sum_{j=1}^4 A_{hj}\beta_j, \quad h = 1, 2, 3, \quad (4.165)$$

where for  $h = 1, 2$ ,

$$\begin{cases} A_{h0} = \hat{g}_{h0} - (\lambda - \mu)f_h - h_h, & A_{h1} = \hat{g}_{h1}, & A_{h2} = \hat{g}_{h2} + f_h, \\ A_{h3} = \hat{g}_{h3}, & A_{h4} = \hat{g}_{h4} - f_h, \end{cases} \quad (4.166)$$

and

$$A_{30} = \frac{B^2(\lambda - \mu)}{2(\lambda + 2\mu)}, \quad A_{31} = A_{33} = 0, \quad A_{32} = -\frac{B^2}{2(\lambda + 2\mu)}, \quad A_{34} = -A_{32}, \quad (4.167)$$

where we have introduced the notations

$$\hat{g}_{1j} = \frac{R_i^2 \Lambda_j(R_i) - R_e^2 \Lambda_j(R_e)}{2(\lambda + \mu)(R_e^2 - R_i^2)}, \quad \hat{g}_{2j} = \frac{R_i^2 R_e^2 (\Lambda_j(R_i) - \Lambda_j(R_e))}{2\mu(R_e^2 - R_i^2)}, \quad j = 0, \dots, 4, \quad (4.168)$$

$$f_1 = \frac{B^2(\lambda + 3\mu)}{2(\lambda + \mu)(\lambda + 2\mu)R_e^2 R_i^2}, \quad h_1 = \frac{-A(R_i^2 p_i - R_e^2 p_e) + B(p_e - p_i)}{2(\lambda + \mu)(R_e^2 - R_i^2)}, \quad (4.169)$$

$$f_2 = \frac{B^2(\lambda + 3\mu)(R_i^2 + R_e^2)}{2\mu(\lambda + 2\mu)R_e^2 R_i^2}, \quad h_2 = \frac{AR_i^2 R_e^2 (p_e - p_i) + B(R_i^2 p_e - R_e^2 p_i)}{2\mu(R_e^2 - R_i^2)}.$$

Since  $\Lambda_1 = 4A^2$ , (4.166) gives  $A_{21} = 0$ .

Finally, it can be seen that, owing to the symmetry of the problem and the loads acting on  $S$ , the compatibility conditions (4.105) and (4.106) are evidently satisfied.

For a solid cylinder with pressure applied on the outer surface, we must have

$$B = 0, \quad C_2 = 0, \quad C_3 = 0.$$

Thus the second-order traction boundary-value problem has the solution

$$\mathbf{u}(\mathbf{r}) = (\epsilon A + \epsilon^2 C_1) r \mathbf{a}_1.$$

## 4.4 The Experimental Procedures

The analyses presented in the previous two sections suggest experiments to determine the second-order constitutive constants  $\beta_i$ . Consider a homogeneous and isotropic elastic material  $S$ , whose Lamé coefficients are  $\lambda$  and  $\mu$ . Let  $S_s$  and  $S_c$  be two specimens of  $S$  of spherical and cylindrical geometry, respectively, and assume that geometrical characteristics and forces acting upon  $S_s$  and  $S_c$  be those described in the previous subsections (see Figures 1 and 2). Then, the displacement fields for  $S_s$  and  $S_c$  are given by (4.136), (4.149), and (4.164), which can be written in the following dimensional form

$$\mathbf{u}_s = \left[ \bar{A} + 2\bar{A}^2 + \frac{\Lambda \bar{A}^3}{p_0} \right] r \mathbf{a}_r, \quad (4.170)$$

$$\mathbf{u}_c = \left[ (A + C_1) r + \frac{(B + C_2)}{r} + \frac{C_3}{r^3} \right] \mathbf{a}_r. \quad (4.171)$$

We propose the following experiments:

1. For a sphere  $S_s$  subjected to a uniform pressure  $\mathbf{t} = -p_0 \mathbf{N}$ , we experimentally measure the displacement  $u_s(R)$  of the external surface. Then, (4.170) provides one equation in the unknowns  $\beta_i$ :

$$\left[ \bar{A} + \left( 2\bar{A}^2 + \frac{\Lambda \bar{A}^3}{p_0} \right) \right] R = u_s(R). \quad (4.172)$$

2. Consider a very long hollow cylinder  $S_c$  and assume that  $p_i = 0$  and  $p_e = \pi_1$ . Then, if we denote by  $u_{c1}(R_e)$  the experimentally measured displacement of the outer surface, from (4.171) we get the following equation in the unknowns  $\beta_i$ :

$$\left(A^{(1)} + C_1^{(1)}\right) R_e + \frac{\left(B^{(1)} + C_2^{(1)}\right)}{R_e} + \frac{C_3^{(1)}}{R_e^3} = u_{c1}(R_e), \quad (4.173)$$

where  $A^{(1)} = A|_{(p_i=0, p_e=\pi_1)}$ ,  $B^{(1)} = B|_{(p_i=0, p_e=\pi_1)}$ , and  $C_i^{(1)} = C_i|_{(p_i=0, p_e=\pi_1)}$ , for  $i = 1, 2, 3$ .

3. Let the hollow cylinder  $S_c$  be subjected to the pressures  $p_i = \pi_2$  and  $p_e = 0$ . Then, instead of (4.173), we obtain

$$\left(A^{(2)} + C_1^{(2)}\right) R_e + \frac{\left(B^{(2)} + C_2^{(2)}\right)}{R_e} + \frac{C_3^{(2)}}{R_e^3} = u_{c2}(R_e), \quad (4.174)$$

where  $A^{(2)} = A|_{(p_i=\pi_2, p_e=0)}$ ,  $B^{(2)} = B|_{(p_i=\pi_2, p_e=0)}$ , and  $C_i^{(2)} = C_i|_{(p_i=\pi_2, p_e=0)}$ , for  $i = 1, 2, 3$ .

4. Finally, let  $u_{c3}(R_e)$  be the displacement corresponding to the pressures  $p_i = p_e = \pi_3$ .

Then

$$\left(A^{(3)} + C_1^{(3)}\right) R_e + \frac{\left(B^{(3)} + C_2^{(3)}\right)}{R_e} + \frac{C_3^{(3)}}{R_e^3} = u_{c3}(R_e), \quad (4.175)$$

where  $A^{(3)} = A|_{(p_i=\pi_3, p_e=\pi_3)}$ ,  $B^{(3)} = B|_{(p_i=\pi_3, p_e=\pi_3)}$ , and  $C_i^{(3)} = C_i|_{(p_i=\pi_3, p_e=\pi_3)}$ , for  $i = 1, 2, 3$ .

It is now easy to verify that (4.172)-(4.175) provide an algebraic system of four equations in the unknowns  $\beta_1, \dots, \beta_4$  which has a unique solution. In fact, the determinant of

the coefficient matrix  $\mathcal{A}$  given by

$$\det \mathcal{A} = \frac{RR_i^4 R_e^5 p_0^2 \pi_1^2 \pi_2^2 \pi_3^2}{64\mu^4 (\lambda + \mu)^5 (3\lambda + 2\mu)^3 (R_e^2 - R_i^2)^3},$$

is non-zero.

In particular, the solution of the system (4.172)-(4.175) is

$$\begin{aligned} \beta_1 = & 2 \frac{\mu^2 (\lambda + \mu) (R_e^4 - R_i^4)}{R_i^2 R_e^3 \pi_1^2} u_{c1}(R_e) - 4 \frac{\mu^2 (\lambda + \mu) (R_i^2 - R_e^2)}{R_i^2 R_e \pi_2^2} u_{c2}(R_e) - \quad (4.176) \\ & 2 \left[ \frac{(\lambda + \mu)^3}{R_e \pi_3^2} + 2 \frac{\mu^2 (\lambda + \mu)}{R_e \pi_3^2} + \frac{\mu^2 (\lambda + \mu) R_e}{R_i^2 \pi_3^2} \right] u_{c3}(R_e) + \\ & \frac{3}{2} \mu + \frac{(\lambda + \mu) R_i^2}{2R_e^2} + 2 \frac{\mu (\lambda + 2\mu)}{\pi_2} - \frac{(\lambda + \mu)^2}{\pi_3} + 2\mu^2 \left( \frac{1}{\pi_1} - \frac{1}{\pi_3} \right) - \\ & \frac{\mu^2 R_e^2}{R_i^2 \pi_3} + \frac{\lambda \mu}{\pi_1} + \frac{\mu (\lambda + \mu) R_i^2}{R_e^2 \pi_1} + \frac{\mu^2 R_e^2}{R_i^2 \pi_1}, \end{aligned}$$

$$\begin{aligned} \beta_2 = & -2 \left[ \frac{\mu^2 (\lambda + \mu) (R_e^4 - R_i^4)}{R_i^2 R_e^3 \pi_1^2} + 2 \frac{\mu^2 (\lambda + \mu) (R_e^2 - R_i^2)}{R_i^2 R_e \pi_1^2} \right] u_{c1}(R_e) + \\ & 8 \frac{\mu^2 (\lambda + \mu) (R_e^2 - R_i^2)}{R_i^2 R_e \pi_2^2} u_{c2}(R_e) + \\ & 2 \left[ 9 \frac{(\lambda + \mu)^3}{R_e \pi_3^2} + 4 \frac{\mu^2 (\lambda + \mu)}{R_e \pi_3^2} + 3 \frac{\mu^2 (\lambda + \mu) R_e}{R_i^2 \pi_3^2} \right] u_{c3}(R_e) - \frac{(3\lambda + 2\mu)^3}{Rp_0^2} u_s(R) + \\ & \frac{\lambda}{2} - \frac{3}{2} \mu - \frac{(\lambda + \mu) R_i^2}{2R_e^2} - \frac{(3\lambda + 2\mu)^2}{p_0} - 4 \frac{\mu (\lambda + 2\mu)}{\pi_2} + 9 \frac{(\lambda + \mu)^2}{\pi_3} - \\ & 4\mu^2 \left( \frac{1}{\pi_1} - \frac{1}{\pi_3} \right) + 3 \frac{\mu^2 R_e^2}{R_i^2 \pi_3} - 3 \frac{\lambda \mu}{\pi_1} - \frac{\mu (\lambda + \mu) R_i^2}{R_e^2 \pi_1} - 3 \frac{\mu^2 R_e^2}{R_i^2 \pi_1}, \quad (4.177) \end{aligned}$$



$$\begin{aligned}
\beta_3 = & 2 \left[ -3 \frac{\mu^2 (\lambda + \mu) (R_e^4 - R_i^4)}{R_i^2 R_e^3 \pi_1^2} + 4 \frac{\mu^2 (\lambda + \mu) (R_e^2 - R_i^2)}{R_i^2 R_e \pi_1^2} \right] u_{c1} (R_e) + \\
& 4 \frac{\mu^2 (\lambda + \mu) (R_e^2 - R_i^2)}{R_i^2 R_e \pi_2^2} u_{c2} (R_e) + \\
& 2 \left[ -9 \frac{(\lambda + \mu)^3}{R_e \pi_3^2} + 2 \frac{\mu^2 (\lambda + \mu)}{R_e \pi_3^2} - \frac{\mu^2 (\lambda + \mu) R_e}{R_i^2 \pi_3^2} \right] u_{c3} (R_e) + \\
& \frac{(3\lambda + 2\mu)^3}{R p_0^2} u_s (R) - \frac{3\lambda + 5\mu}{2} - 3 \frac{(\lambda + \mu) R_i^2}{2 R_e^2} + \frac{(3\lambda + 2\mu)^2}{p_0} - 2 \frac{\mu (\lambda + 2\mu)}{\pi_2} - \\
& 9 \frac{(\lambda + \mu)^2}{\pi_3} - 2\mu^2 \left( \frac{1}{\pi_1} - \frac{1}{\pi_3} \right) - \frac{\mu^2 R_e^2}{R_i^2 \pi_3} + \frac{\lambda \mu}{\pi_1} - 3 \frac{\mu (\lambda + \mu) R_i^2}{R_e^2 \pi_1} + \frac{\mu^2 R_e^2}{R_i^2 \pi_1},
\end{aligned} \tag{4.178}$$

$$\begin{aligned}
\beta_4 = & 6 \left( \frac{\mu^2 (\lambda + \mu) (R_e^4 - R_i^4)}{R_i^2 R_e^3 \pi_1^2} - 2 \frac{\mu^2 (\lambda + \mu) (R_e^2 - R_i^2)}{R_i^2 R_e \pi_1^2} \right) u_{c1} (R_e) + \\
& 6 \left[ 3 \frac{(\lambda + \mu)^3}{R_e \pi_3^2} + \frac{\mu^2 (\lambda + \mu) R_e}{R_i^2 \pi_3^2} \right] u_{c3} (R_e) - \frac{(3\lambda + 2\mu)^3}{R p_0^2} u_s (R) + \frac{3\lambda + 7\mu}{2} + \\
& 3 \frac{(\lambda + \mu) R_i^2}{2 R_e^2} - \frac{(3\lambda + 2\mu)^2}{p_0} + 9 \frac{(\lambda + \mu)^2}{\pi_3} + 3 \frac{\mu^2 R_e^2}{R_i^2 \pi_3} - 3 \frac{\lambda \mu}{\pi_1} + 3 \frac{\mu (\lambda + \mu) R_i^2}{R_e^2 \pi_1} - 3 \frac{\mu^2 R_e^2}{R_i^2 \pi_1}.
\end{aligned} \tag{4.179}$$

It must be noted that expressions (4.176)-(4.179) of the second-order constitutive constants just appear to be complicated. Indeed, once Lamé coefficients of the material  $S$  together with its geometry and the forces acting on it are known, (4.176)-(4.179) only depend on the displacements  $u_{c1} (R_e)$ ,  $u_{c2} (R_e)$ ,  $u_{c3} (R_e)$ , and  $u_s (R)$  which are measured in the experiments. Thus the four second-order elastic constants can be evaluated. Note that loads applied must be such that  $\epsilon \ll 1$ , otherwise the response of  $S$  to applied loads may not be governed by the second-order elasticity theory used to derive equations (4.176)-(4.179).

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## **Vita**

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