5.0 Curve and Surface Theory

5.1 Parametric Representation of Curves

Consider the parametric representation of a curve as a vector $P(t)$:

$$P(t) = [x(t) \quad y(t) \quad z(t)] \tag{5.1}$$

The derivative of such a vector evaluated at $t = t_0$ is given by

$$P'(t_0) = [x'(t_0) \quad y'(t_0) \quad z'(t_0)] \tag{5.2}$$

The curve is described as regular if the condition $P'(t_0) \neq 0$ is satisfied. Any point at which the condition is not satisfied is called a singular point. If the derivatives of all points on the curve to the order r exist and if the curve is regular throughout, then the curve is said to be C^r.

Figure 5.1 shows a plane passing through the point $P(t_0)$ on the curve and perpendicular to the tangent line to the curve at that point. This plane is called the *normal plane.* Again considering the point $P(t_0)$ and two points very close to it, say $P(t_0 + \Delta_1 t)$ and $P(t_0 + \Delta_2 t)$, the plane approached as the limits ý$_1t$ and ý$_2t$ independently approach zero is called the *osculating plane*. Intuitively, the osculating plane can be best described as the plane that best fits the curve at the point $P(t_0)$.

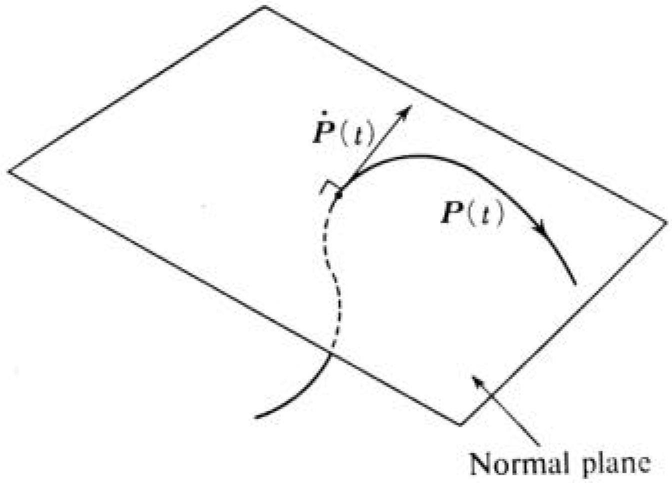

Figure 5.1: Parametric curve showing the tangent at a point along with the Normal plane at that point.

The line that lies in the osculating plane, passes through the point $P(t_0)$ and is perpendicular to the tangent vector P¢(t_0), is called the *principal normal*. Another line of interest to the designer is the *binormal*. The binormal is the line that passes through the point $P(t_0)$ and is perpendicular to the osculating plane. The plane formed by the tangent and the binormal is called the *rectifying plane*. The figure 5.2 shows the relationship between the entities described above.

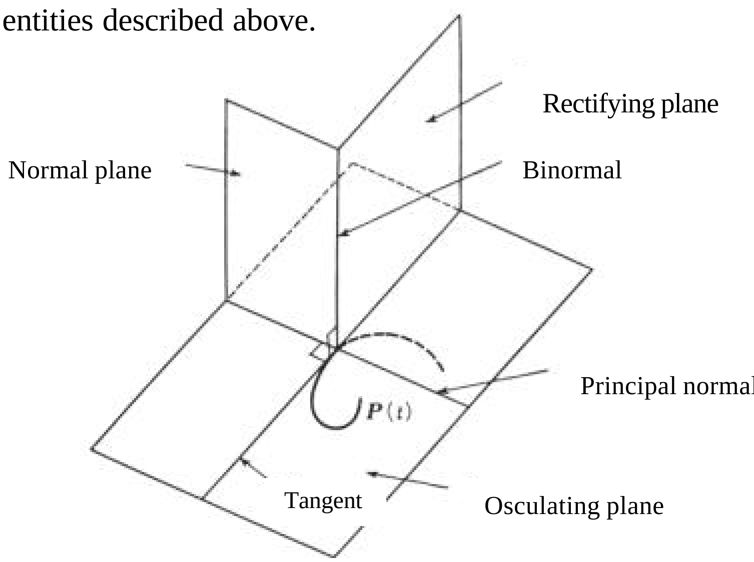

Figure 5.2: Figure showing the relationship between tangent, normal, binormal, osculating plane, normal plane, and the rectifying plane.

5.2 Curvature and Torsion

A curve can be parameterized in terms of its length, *s*. From the definition of the derivative, the second derivative of such a curve *P(s)* is given by

$$P''(s_0) = \lim_{\Delta s \to 0} \frac{P'(s_0+\Delta s) - P'(s_0)}{\Delta s} \tag{5.3}$$

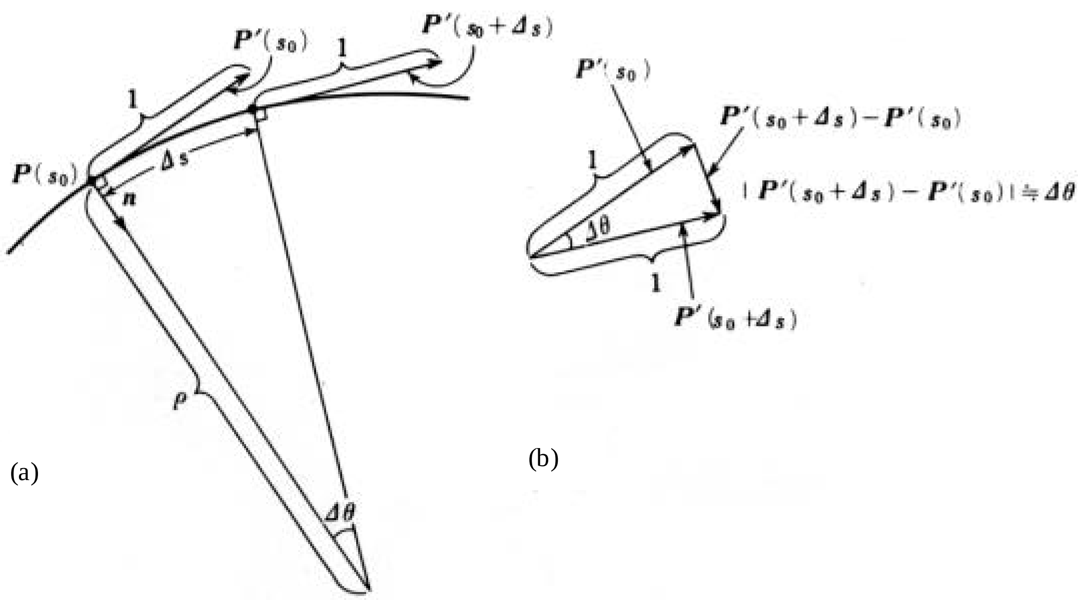

Figure 5.3: (a) Curve parameterized in terms of its length showing the geometric relation between the center of curvature and the tangents. (b) Calculations shown for curvature computation.

As can be seen in Fgure 5.3 (a), as the limit $\Delta s \to 0$, the numerator $P'(s_0 + \Delta s) - P'(s_0)$ becomes perpendicular to the tangent and points towards the center of curvature of the curve. The magnitude of the numerator can be calculated using figure 5.3 (b) and is equal to $\Delta\theta$. Therefore, the magnitude of $P''(s_0)$ is:

$$\left|P''(s_0)\right| = \lim_{\Delta s \to 0} \frac{\Delta q}{\Delta s} = \lim_{\Delta s \to 0} \frac{\frac{\Delta s}{r}}{\Delta s} = \frac{1}{r} \equiv k \qquad (5.4)$$

$$P'' = \frac{1}{r}\, n \equiv k n \qquad (5.5)$$

Here, **n** is the unit vector pointing towards the center of curvature. ρ is the radius of curvature and κ is the curvature. The curvature is given by:

$$k = \frac{\left|P''(t) \times P''(t)\right|}{\left|P'(t)\right|^3} \qquad (5.6)$$

Curvature is the rate of turning of the unit tangent vector with respect to the length of the curve *s*. It is a measure of how rapidly or slowly the curve is turning. A point on the curve at which $k = 0$ is called a *point of inflection*.

Any point on a curve can be described easily in terms of a special local Cartesian system that has its axes as **t**, **n**, **b**. The vector **t** is called the *tangent vector*, **n** is called the *principal normal vector*, and **b** is called the *unit binormal vector*. The unit binormal vector is expressed as **b** = **t** × **n**. Figure 5.4 shows the relation between the three vectors. This coordinate system is called the *Frenet frame* and its orientation in three dimensional space changes with the parameter value. As described earlier, the plane passing through the point *P(t)* and defined by the vectors **t**, **n** is called the osculating plane.

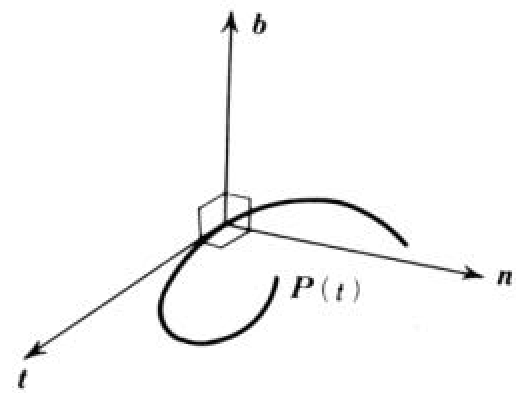

Figure 5.4: Relation between the unit tangent vector, unit principal vector, and unit binormal vector.

Torsion is a measure of the rotation of the osculating plane with respect to the length of the curve. It basically indicates whether the curve is twisting rapidly or not. If *Df* be the angle between the osculating planes at two points adjacent to each other, say $P(s_0)$ and $P(s_0 + Ds)$, the torsion of that curve is defined as:

$$t = \lim_{\Delta s \to 0} \frac{\Delta f}{\Delta s} \tag{5.7}$$

Mathematically, torsion is expressed in terms of the derivatives of *P(t)*.

$$t = \frac{(P'(t) \times P'(t)) \cdot P'''(t)}{((P'(t) \times P''(t))^2} \tag{5.8}$$

Just as curvature measures the failure of a curve to be a straight line, torsion measures the inability of a curve to lie in a plane. In fact, torsion is also referred to as the second curvature of a curve.

In this research, radius of curvature of points along isoparametric curves on a given surface are calculated using the formulation described earlier in this section. The difference in curvature values is calculated for every point and the data reduction scheme discussed earlier is used to elicit useful information.

5.3 Parametric Representation of Surfaces

Let a curved surface be defined in terms of two parameters *u* and *v* such that:

$$P\,(u,v) = [x(u,v)\; y(u,v)\; z(u,v)] \tag{5.9}$$

If this surfaces satisfies the conditions that the three jacobians J_x, J_y, and J_z are not all simultaneously zero and that the derivatives of *P(u,v)* with respect to x, y, and z are continuous and exist up to the r^{th} order then the surface *P(u,v)* is said to be C^r. The jacobians are defined as:

$$J_x \equiv \frac{\partial (y,z)}{\partial (u,v)} \qquad J_y \equiv \frac{\partial (z,x)}{\partial (u,v)} \qquad J_z \equiv \frac{\partial (x,y)}{\partial (u,v)} \tag{5.10}$$

$$\text{where, } \frac{\partial (y,z)}{\partial (u,v)} = \begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} \tag{5.11}$$

Also, the condition that the jacobians not be all equal to zero simultaneously can be expressed mathematically as:

$$J_x^2 + J_y^2 + J_z^2 \neq 0 \tag{5.12}$$

5.4 Surface Normals

The normal vector **n** for a surface is computed from the cross product of any two linearly independent vectors that are tangent to the surface at that point. The parametric representation of a surface *P(u,v)* describes a curve on the surface if the surface is evaluated at a constant value for one of the parameters, say $v = v_0$. Such a curve will be an isoparametric curve along the *u* parametric direction. Similarly, a curve can be described in the other parametric direction. At any point $P(u_0, v_0)$ on the surface, the tangent vector in the *u* parametric direction is:

$$P_u = \left[\frac{\partial x(u,v_0)}{\partial u} \quad \frac{\partial x(u,v_0)}{\partial u} \quad \frac{\partial x(u,v_0)}{\partial u} \right]_{u=u0} \tag{5.13}$$

Similarly, the tangent at the same point in the *v* parametric direction is:

$$P_v = \left[\frac{\partial x(u_0,v)}{\partial v} \quad \frac{\partial x(u_0,v)}{\partial v} \quad \frac{\partial x(u_0,v)}{\partial v} \right]_{v=v0} \tag{5.14}$$

Figure 5.5 shows the tangent vectors in the two parametric directions for a curved surface. The condition of the jacobians not all being zero ensures that the two tangent vectors at any point are linearly independent of each other. The normal vector can be expressed at a point as:

$$\boldsymbol{n} = \frac{P_u \times P_v}{\sqrt{(P_u \times P_v)^2}} \tag{5.15}$$

where **n** is the unit normal vector.

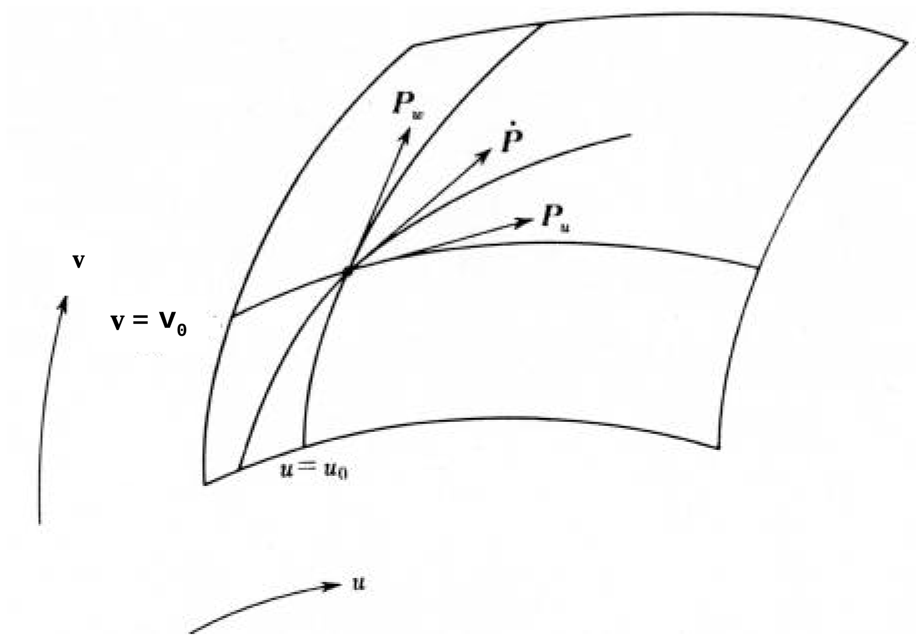

Figure 5.5: Figure showing the tangent vectors in the two parametric directions on a surface.

The surface normal data sampled at a number of points on the surface is abstracted in a manner similar to that of the position error data. The error at every point was calculated and a linear least squares fit was performed for the data along each isoparametric curve. The resulting ordinate intercept values were then plotted against the parameter values. With surface normals, the procedure remains the same, however instead of the position error, the angle between the surface normals is used for the linear least squares fit. The resulting condensation adds to the information provided by the position error plot. Figures 5.6 and 5.7 show the condensate of the position error plot and the corresponding surface normals error plot. It can be easily seen that even though the errors at both the places are comparable in magnitude, the orientation of the matching surface at the first defect is distorted so that angle between the surface normals are much larger than the second one.

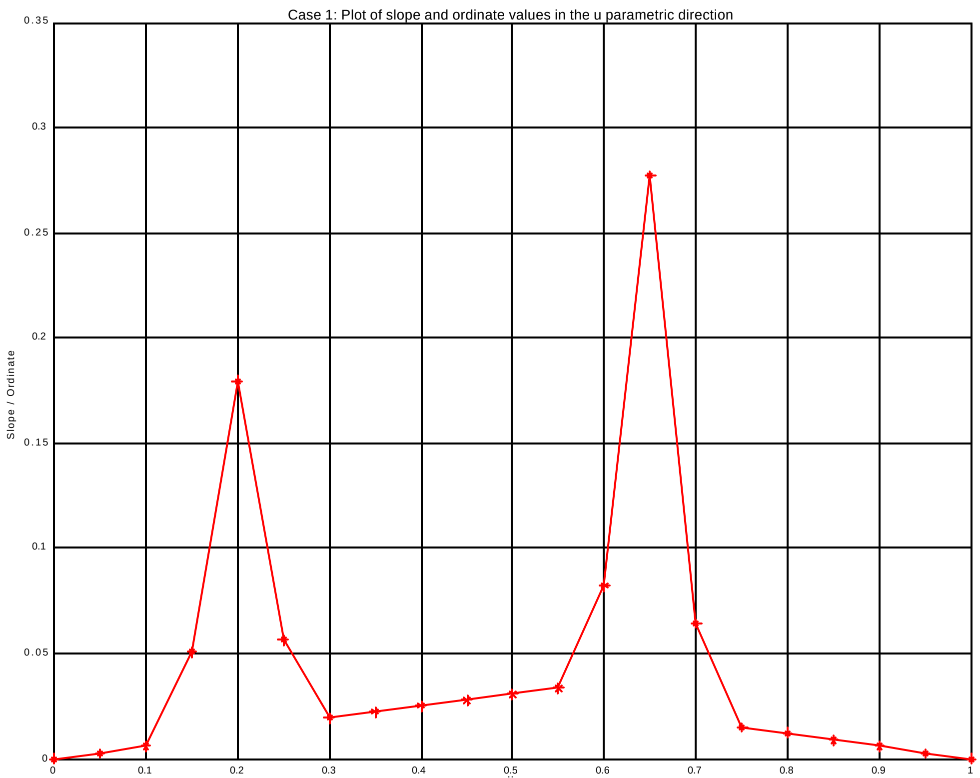

Figure 5.6: Ordinate plot visualizing position error in the *u* parametric direction for a particular surface.

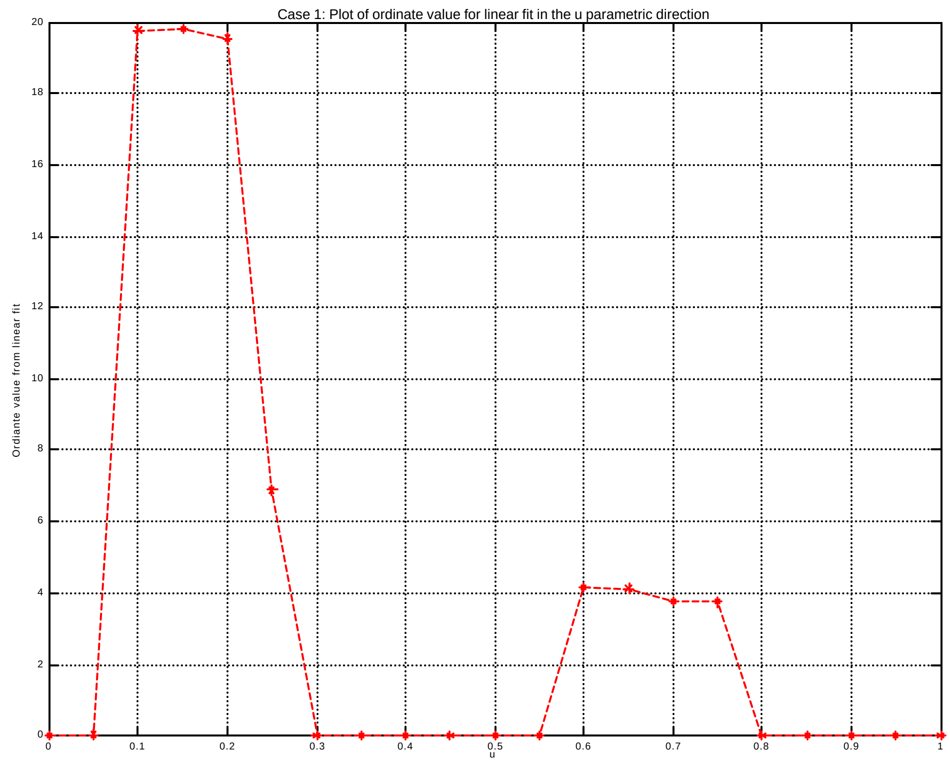

Figure 5.7: Ordinate plot visualizing angle between normals of the original and matching surface in the same direction.