

## 3.0 Iterative Image Reconstruction

Mostly, the problem of image restoration is that the representative information about the original image is only known partially or that only a distorted version is received. The goal of image restoration is to find a good approximation to the original signal based on the available information.

### 3.1 Mathematics Review

Consider a Hilbert space  $\mathbf{H}$  with elements  $f, g, h$ , and closed linear manifolds  $\mathbf{M}$  and  $\mathbf{N}$ . Let  $f = g + h$ ,  $g \in \mathbf{M}$  and  $h \in \mathbf{M}^\wedge$ , the orthogonal complement of  $\mathbf{M}$ . Youla [Youla 1978] points out that image restoration can be perceived as reconstructing  $f \in \mathbf{N}$  from  $g$ , its projection onto  $\mathbf{M}$ . Let us define  $P_M$  and  $P_N$  as projection operators onto the manifold  $\mathbf{M}$  and  $\mathbf{N}$  respectively. The projection onto  $\mathbf{M}$  can be written as

$$g = P_M f \tag{3.1}$$

Due to the fact that, in general, the inverse of  $P_M$  does not exist,  $f$  cannot be uniquely determined from  $g$ . Let  $f_1 = g + h_1$  and  $f_2 = g + h_2$ , where  $h_1, h_2 \in \mathbf{M}^\wedge$  and  $h_2 \neq h_1$ . The projections of both  $f_1$  and  $f_2$  on  $\mathbf{M}$  are exactly  $g$ . Since  $f_1 \neq f_2$ , this example illustrates the nonuniqueness of  $f$ .

Let  $Q_M$  be a projection operator onto  $\mathbf{M}^\wedge$ . Since  $P_N f = f$ , (3.1) can then be rewritten as

$$\begin{aligned}
g &= (1-Q_M) f \\
&= (1 - Q_M) P_N f, \\
g &= f - Q_M P_N f
\end{aligned} \tag{3.12}$$

From (3.2), we can now rewrite  $f$  as follows

$$f = g + Q_M P_N f. \tag{3.3}$$

Generally, only  $f_k$ , an approximate version of  $f$ , is known during the reconstruction.

With a proper projection operator  $P_N$  in (3.3),  $f_k$  is constrained to obtain a new approximate reconstruction  $f_{k+1}$ . The process can be repeated, which, consequently, leads to a recursive reconstruction process.

$$f_{k+1} = g + Q_M P_N f_k \tag{3.4}$$

$$= T f_k \tag{3.5}$$

where  $k=1 \rightarrow \infty$  and  $f_1$  is  $g$ . (3.4) indicates that  $f_{k+1}$  can be built up by first projecting  $f_k$  onto  $\mathbf{N}$ . Secondly project the result onto  $\hat{\mathbf{M}}$  and then add  $g$  to the composite projection.

The iteration algorithm is not very useful without a proof of its convergence. A sequence  $\{f_k\}$  in an infinite dimensional Hilbert space is said to be strongly convergent to  $f$  if

$$\lim_{k \rightarrow \infty} \| f_k - f \| = 0 \quad (3.6)$$

and weakly convergent if

$$\lim_{k \rightarrow \infty} (f_k, g) = (f, g). \quad (3.7)$$

A set  $S$  is said to be closed if whenever a sequence  $\{f_i\}$  of elements of  $S$  converges to  $f \in S$  [Moschovakis 1994]. Ideally, we need strongly convergent sequences to make sure that the iteration approaches a limit point  $f$ . However, for a closed and finite dimensional space, weak convergence implies strong convergence [Moose 1994].

Convexity of sets plays an important role in this thesis. Let set  $S$  contain  $x$  and  $y$ .  $S$  is convex if all points in the line segment joining  $x$  and  $y$ , expressed in  $\mu x + (1-\mu)y$  where  $0 \leq \mu \leq 1$ , are in the set  $S$  [Valentine 1964].

Consider a mapping operator  $T: S \rightarrow S$ .  $f \in S$  is said to be a fixed point of  $T$  if and only if

$$f = Tf. \quad (3.8)$$

For instance, let  $P_M$  be an orthogonal projection onto a subspace  $M$ . Any vectors in the subspace are fixed points of  $P$ . Comparing (3.5) with (3.8), we need, in image reconstruction, a unique fixed point for the mapping operator  $T$ .

To be useful, the mapping operator  $T$ , for each iteration, should bring the result closer to  $f$ . This statement can be written as

$$\| Tf - Tf_k \| = \| f - Tf_k \| \leq \alpha \| f - f_k \| \text{ for } 0 \leq \alpha < 1. \quad (3.9)$$

This property is called contraction. If the expression includes  $\alpha=1$ , the distance between  $f$  and  $Tf_k$  may not be less than the distance between  $f$  and the original  $f_k$ . The relaxed characteristic is called nonexpansive.

Roughly speaking, contraction mappings force the distance between  $f$  and  $f_k$  closer to zero. There exists, consequently, a unique fixed point for the mapping operator. Unlike for contraction mappings, there is possibly an infinite number of fixed points in nonexpansive mappings. A projection operator is an example of a nonexpansive mapping since the distance of the images never increases.

In the next section the subspace  $N$ , where the solution lies, is described by a set of convex sets. The restoration algorithm that utilizes the recursive mapping is discussed afterward.

### **3.2 Projection onto Convex Sets (POCS)**

The purpose of an iterative restoration algorithm is to search for a fixed point that is an acceptable approximation of the original signal  $f$ . The iterative restoration algorithm requires at least a nonexpansive mapping. Schafer et al. [Schafer 1981] model a general relation of the original signal  $f$  and the observed signal  $y$  as

$$f = Cf + \lambda(y - Df) \quad (3.10)$$

However, the restoration algorithm requires both a distortion operator  $D$ , and a constraint operator  $C$ .

Youla [Youla 1982] derives a reconstruction algorithm, named projection onto convex sets (POCS), which allows an arbitrary amount of a priori information to be incorporated as constraints. The convexity of the sets is required to prevent the occurrence of divergence.

Let  $f$  be the original signal that is restricted to lie in closed and convex sets,  $C_1, C_2, \dots, C_M$ . Then  $f$  does lie in the intersection of the closed and convex sets.

$$f \in C_0 = \bigcap_{i=1}^M C_i \quad (3.11)$$

As a result, projection of a distorted signal onto  $C_0$  results in a new signal for which the distance to the original is reduced or at least not increased. However, projecting from a Hilbert space directly to  $C_0$  is, in general, more complicated than projection from a Hilbert space to the individual set  $C_i$ . To avoid the complication, the initial signal is projected onto the first individual set  $C_1$ . The result, which now does lie on  $C_1$ , is projected onto the next set,  $C_2$ . The process keeps going from  $C_i$  to  $C_{i+1}$ , until finally  $C_M$ . Let us call projecting from  $C_1$  to  $C_M$  as a projecting sequence. The last projection onto  $C_M$  does not guarantee that it lies in  $C_0$ . However, we can repeat the process by projecting the resulting signal, which is now on  $C_M$ , onto  $C_1$  and then starting a new sequence of projections. The

recursive process hopefully gives a signal in  $C_0$ . The complexity of the composite projection in this algorithm is the complexity of the most complicated projection onto the individual sets  $C_i$ .

Let  $P_i: \mathbf{H} \rightarrow C_i$ , be a projection operator onto  $C_i$  and  $T = P_M P_{M-1} \dots P_1$  be a composite projection onto  $C_0$ . Since  $C_0$  is the intersection of all  $C_i$ , any points in  $C_0$  lie in all  $C_i$ . As a result, the points in  $C_0$  are also the fixed points of all individual projection operators  $P_i$ . Therefore, any points in  $C_0$  are fixed points of the composite projection  $T$ .

Due to the reduced complexity, the composite projection is used to iteratively generate a sequence  $\{f_k\}$  where  $f_k$  is derived from

$$f_k = T^k f_0 \quad (3.12)$$

The sequence  $\{f_k\}$ , for a finite dimensional space, strongly converges to a fixed point in  $C_0$ . Because of the individual projection operators, the mapping operator  $T$  is nonexpansive and the fixed points are not unique. The final solution depends on the ordering of the  $P_i$  and, also of course, on the initial point  $f_0$  [Moose, 1994].

### 3.3 MBR Constraint

In general, a compressed image describes only partially what the original image looks like. If some characteristics of the original signal are known, we can use these to constrain the restored image to move it closer to the original signal. Fortunately, such some characteristics do not incur much extra cost. For example, the signals that represent the

intensity of image pixels are never negative valued. In case of the MBR compressed signal, the reconstructed signal is forced to have the same MBR coefficients as the original.

### 3.3.1 MBR Constraint Definition and MBR Projection Operator

Considering the MBR representation, the original signal  $f$  can be decomposed as

$$f = P_M f + P_M^\perp f \quad (3.13)$$

The projection onto the subspace  $\mathbf{M}$  of  $f$  is then, in the absence of noise, the observed signal  $y$  and the orthogonal projection  $P_M^\perp f$  is the residual. During the restoration process,  $f_k$  must have  $y$  as its projection onto the subspace  $\mathbf{M}$ . This idea brings out a constraint that requires no extra a priori information other than the observed signal. The MBR constraint can be mathematically expressed as

$$C_{\text{MBR}} = \{ f: y = P_M f \} \quad (3.14)$$

Let  $\mathbf{B}$  be a subspace such that the projection of any  $b \in \mathbf{B}$  onto  $\mathbf{M}$  is  $y$ . Of course,  $y$  is in both subspaces  $\mathbf{B}$  and  $\mathbf{M}$ . Let  $P_B g$  be the projection of  $g$  onto  $\mathbf{B}$ . The projection onto the subspace  $\mathbf{M}$  of  $P_B g$ ,  $P_M(P_B g)$ , is  $y$ . As a result,  $P_B g$ , satisfies the constraint defined in (3.14). Since the basis vectors are known, the projection operator onto the subspace  $\mathbf{M}$ ,  $P_M$ , can be found. It is beneficial to derive  $P_B$  from  $P_M$ .

By the orthogonal projection theorem [Brogan 1985],  $g$  can be decomposed into the sum of  $P_M g$  and  $P_M^\perp g$ , and the projection of  $P_M^\perp g$  onto the orthogonal subspace of  $\mathbf{M}$  is

zero. Consequently, the projection onto  $\mathbf{M}$  of the signal formed by adding  $\mathbf{P}_M^\perp \mathbf{g}$  to  $\mathbf{y}$ , is still  $\mathbf{y}$ . As a consequence,  $\mathbf{y} + \mathbf{P}_M^\perp \mathbf{g}$  satisfies the condition of  $C_{\text{MBR}}$ . Therefore,

$$\begin{aligned}
 \mathbf{P}_{\text{MBR}} \mathbf{g} &= \mathbf{P}_B \mathbf{g} \\
 &= \mathbf{y} + \mathbf{P}_M^\perp \mathbf{g} \\
 &= \mathbf{y} + (\mathbf{I} - \mathbf{P}_M) \mathbf{g} \\
 &= \mathbf{y} + \mathbf{g} - \mathbf{P}_M \mathbf{g}
 \end{aligned} \tag{3.15}$$

The projection process is depicted geometrically in Figure 3.1.

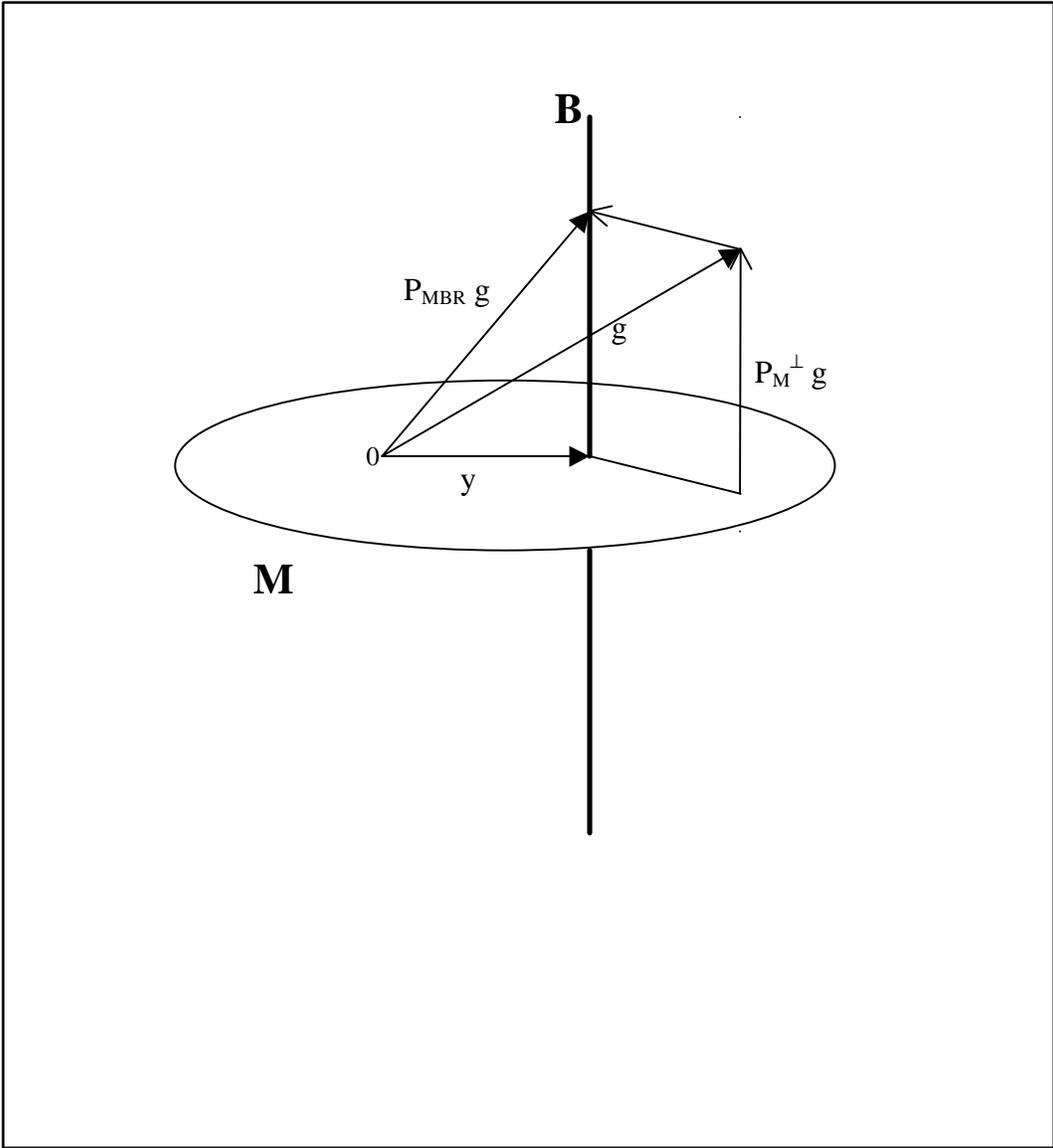
To constrain  $\mathbf{g}$  to  $C_{\text{MBR}}$ , from (3.15), projection onto  $\mathbf{M}$ ,  $\mathbf{P}_M$ , is needed. Let  $\mathbf{A}$  be a matrix composed columnwise of the MBR participant basis vectors. The operator  $\mathbf{P}_M$  can be written in terms of  $\mathbf{A}$  as  $\mathbf{P}_M = \mathbf{A}(\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}$ . However, the direct approach is quite complicated. Safar [Safar 1988] introduced a practical method to implement the projection as follows.

The projection onto  $\mathbf{M}$  of  $\mathbf{g}$  can be illustrated as

$$\mathbf{P}_M \mathbf{g} = \mathbf{A} \mathbf{w}, \tag{3.16}$$

where  $\mathbf{w}$  satisfies the linear equation

$$(\mathbf{A}^T \mathbf{A}) \mathbf{w} = \mathbf{A}^T \mathbf{g} \tag{3.17}$$



**Figure 3.1** The MBR constraint.

Since  $A$  consists columnwise of basis vectors of fast transforms,  $A^T g$  can be efficiently obtained by picking the transform coefficients that correspond to the participant basis vectors. Moreover, some of the columns of  $A$  are from the same orthonormal transform. The orthonormality causes  $A^T A$  to contain identity matrix partitions. Assuming two transforms are involved in the MBR representation, (3.17) can be partitioned as

$$\begin{bmatrix} I_1 & D \\ D^T & I_2 \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} A^{T1} g \\ A^{T2} g \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \quad (3.18)$$

where each element in  $D$  is the inner product of a  $T_1$  basis vector and a  $T_2$  basis vector. Based on Safar's method, the partitioned matrix leads to

$$w_1 + Dw_2 = b_1 \quad (3.19)$$

$$D^T w_1 + w_2 = b_2 \quad (3.20)$$

Substituting  $w_1$  in (3.20) from (3.19), we obtain

$$D^T(b_1 - Dw_2) + w_2 = b_2$$

We can then solve for  $w_2$  from:

$$(I - D^T D)w_2 = b_2 - D^T b_1 \quad (3.21)$$

So far, the linear system of equations is simplified by reduction in dimension. Therefore the computational cost of using (3.21) is significantly less than that of the direct projection method. Obviously,  $(I-D^T D)$  is symmetric and invertible so that the solution of  $w_2$  is unique. This reduced linear system of equations can be solved simply by several matrix manipulations.

### 3.3.2 MBR Soft Constraint

In general noise corrupts the compressed signal  $y$ . Thus only an approximate version of the compressed signal, say  $y'$ , is observed [Beex 1983]. Even in the case of a noiseless channel, coefficient quantization causes a noisy signal to be observed. Particularly, when the MBR coefficients are estimated by RRP, the coefficients do not exactly represent the projection onto the MBR vectors. As a result, the constraint defined in (3.14) still strictly constrains the reconstructed signal to a noisy prototype signal.

Assuming the noise characteristics are partially known, (3.14) may be softened to

$$C'_{MBR} = \{f: \|y' - P_M f\| < e\} \quad (3.22)$$

where  $e$  represents an acceptable tolerance, derived from the noise characteristics. Note that  $e$  for  $C_{MBR}$  is 0.

Both  $y'$  and  $f_M = P_M f$  can be decomposed into the components of each participant transform.

$$y' = \sum_{i=1}^K y'_{Ti} \quad (3.23)$$

$$f_M = \sum_{i=1}^K f_{M,T_i} \quad (3.24)$$

Since  $y'$  and  $f_M$  can be decomposed into the participant transform components, the constraint in (3.22) might be applied to each participant transform  $T_i$  separately.

$$C'_{MBR} = \{f: \|y'_{T_i} - f_{M,T_i}\| < e_i, i=1,2,\dots,K\} \quad (3.25)$$

With the decomposition in (3.23) and (3.24), the projection operator in (3.15) can be rewritten as

$$\begin{aligned} P_{MBR} g &= g + \sum_{i=1}^K y_{T_i} - \sum_{i=1}^K g_{M,T_i} \\ &= g + \sum_{i=1}^K (y_{T_i} - g_{M,T_i}) \end{aligned} \quad (3.26)$$

The soft projection operator is relaxed to [Safar 1988]

$$P'_{MBR} g = g + \sum_{i=1}^K s_i (y'_{T_i} - g_{M,T_i}), \quad i \in [1,k] \quad (3.27)$$

where

$$s_i = \begin{cases} 1 - \frac{e_i}{\|y'_{T_i} - g_{M,T_i}\|} & \|y'_{T_i} - g_{M,T_i}\| \geq e_i \\ 0 & \text{otherwise} \end{cases}$$

If  $e_i = 0$ , the projection  $P'_{\text{MBR}}$  is exactly the same as  $P_{\text{MBR}}$ . Unless  $e_i = 0$ , the projection onto  $\mathbf{M}$  of  $P'_{\text{MBR}}\mathbf{g}$  is no longer  $\mathbf{y}'$ ; it becomes  $\mathbf{y}' + (1-s_i)\mathbf{g}_M$  instead. In addition,  $P'_{\text{MBR}}$  does nothing if all distances between  $\mathbf{y}'_{T_i}$  and  $\mathbf{g}_{M,T_i}$  are less than the thresholds  $e_i$ .

### 3.4 Local Constraint Definitions and Projection Operators

The MBR compressor separates the original signal  $\mathbf{x}$  into two orthogonal signals.  $\mathbf{x}_M$  lies in the subspace of the transmitted coefficients, and the residual signal  $\mathbf{e}_x$  lies in the orthogonal subspace. At the receiver end, the compressed signal has lost information about the error signal  $\mathbf{e}_x$ . As a result, one way to recover the MBR distorted signal is to take additional information on the residual signal  $\mathbf{e}_x$  and then transmit it along with the coefficients. The extra information describes features of the original signal (MBR plus error) and consequently introduces sets of constraints that can be used in the iterative reconstruction process. To produce convergence, constraints must be convex. Nevertheless, using non-convex constraints, together with convex ones, can be useful to increase the rate of convergence.

In this thesis, the whole image is split into subimages called blocks. Each block is then transformed using MBR. A priori information might be found from the residual signal for each block locally or can also be prepared from the entire image. The latter is important in removing blocking artifacts.

Moose [Moose 1994] defines and tests several constraints in his thesis. Mostly he investigated constraints that were local to each subimage block. The constraints and projections are grouped and briefly reviewed.

### 3.4.1 Constraints on Boundary: Positivity and Extremum Bound Constraint

Since the intensity of digital images is greater or equal to zero, for any iterations, the image pixel must be maintained to a positive value. No a priori information is needed in this constraint. The positivity constraint is applied directly to the reconstructed signal. For a signal

$$f = [f_1, f_2, f_3, \dots, f_{N-1}]^T,$$

the positivity constraint can be defined by

$$C_{\text{pos}} = \{f : f_i \geq 0 \quad \forall i \in [0, 1, \dots, N-1]\} \quad (3.28)$$

and the element-wise projection operator is

$$P_{\text{pos}}g_i = \begin{cases} 0 & g_i < 0 \\ g_i & g_i \geq 0 \end{cases} \quad (3.29)$$

Even though the probability density of the error signal  $e_x$  is Laplacian, which has no bound, the error signal is most likely to have a value in the image range. Consequently, the error signal can be reasonably bounded to  $[-256, 255]$ . Likewise, the constraint for the extreme bound on the error signal  $e_f = [e_{f,1} \ e_{f,2} \ \dots \ e_{f,N-1}]^T$  can be written as

$$C_{\text{EB}} = \{e_f : b \leq e_{f,i} \leq a \quad \forall i \in [0, 1, \dots, N-1]\} \quad (3.30)$$

and the associated projection is

$$P_{EB}g_i = \begin{cases} a & g_i > a \\ g_i & a \leq g_i \leq b \\ b & g_i < b \end{cases} \quad (3.31)$$

where a and b are the maximum and minimum value of g respectively.

### 3.4.2 Constraints Associated with Sign :

#### (1) Sign Constraint

The sign of a signal f is defined element-wise by

$$\text{sign}(f_i) = \begin{cases} -1 & f_i < 0 \\ +1 & f_i \geq 0 \end{cases} \quad (3.32)$$

Let set  $S_j = [s_0, s_1, \dots, s_{j-1}]$  contain the sign of the first j elements. The sign constraint is defined by,

$$C_{S,J} = \{ f: \text{sign}(f_i) = s_i \quad \forall i < J \}. \quad (3.33)$$

The projection onto  $C_{S,J}$  is computed element-wise as,

$$P_{S,J} g_j = \begin{cases} s_j |g_j| & j \leq J \\ g_j & \textit{elsewhere} \end{cases} \quad (3.34)$$

The sign constraint is simple since only two possible values of the sign function are of concern. For an N-tuple signal, J can be up to N. The only a priori information is  $S_J$ , the set of desired signs.  $S_J$ , however, provides only the signs of the first J locations. It leads to redundancy if the signs of the signal components do not change often. In the case where the signs hardly change, specifying locations where the changes occur is more efficient than keeping sign information about all signal locations.

In stead of choosing the first J elements, an alternative way is to uniformly spread out the sign locations to the entire of the N-dimensional tuple. Let  $S_{mJ} = [sm_0, sm_1, sm_2, \dots, sm_{J-1}]$  be a vector containing the sign of elements which locate at the multiple of  $N/J$ , e.g., the sign of element at location  $iN/J$  is mapped to  $sm_i$  for  $i=0,1,\dots,J-1$ . The modified sign constraint can be written as

$$C_{S_{m,J}} = \{f: \text{sign}(f_{iN/J}) = sm_i \quad \forall i=0,1,\dots,J-1\} \quad (3.35)$$

The associated projection onto  $C_{S_{m,J}}$  is then,

$$P_{S_{m,J}} g_j = \begin{cases} sm_i & j=0, \frac{N}{J}, \frac{2N}{J}, \dots, N - \frac{N}{J} \quad \text{where} \quad i = \frac{j}{N/J} \\ g_j & \text{elsewhere} \end{cases} \quad (3.36)$$

The modified version of the sign constraint uniformly samples the signs of the signal

f. One advantage of the modified version is that the information about signs is distributed

on the projected signal uniformly. The redundancy caused by the unchanged signs of the consecutive elements does not obviously appear.

## (2) Zero-Crossing Constraint

A zero-crossing occurs when the signs of two adjacent elements change from negative to positive or vice versa. Let us assume  $f_i$  and the adjacent element  $f_{i+1}$  are to be constrained. The **zero-crossing constraint** is defined by,

$$C_Z(i) = \{f: \text{sign}(f_i) = s_i ; \text{sign}(f_{i+1}) = -s_i\} \quad (3.37)$$

where  $s_i$  is the desired sign. The corresponding projection operator onto the zero-crossing set of the N-tuple signal  $g$  can be written as,

$$P_Z(i) g_j = \begin{cases} s_i |g_j| & j = i \\ -s_i |g_j| & j = i + 1 \\ g_j & \text{elsewhere} \end{cases} \quad (3.38)$$

The maximum number of zero-crossings of an N-tuple signal is N-1. Generally, only some selected zero-crossings are to be included as side information. Moose [Moose 1994] states two criteria used to choose the zero crossing locations. The first criterion utilizes the zero-crossing location in which the intensity difference between the two signal elements that form the zero-crossing is the largest. The other criterion is the location that provides the smallest intensity distance.

## (3) Spike Constraint

A more complicated constraint that relates to the sign change is the spike constraint. A spike is defined over three consecutive elements. Vector element  $f_i$  is said to be a spike if the sign of  $f_i$  is different from the sign of  $f_{i-1}$  and  $f_{i+1}$ . In addition, in order to look like a spike, the magnitude of  $f_i$  must be larger than a threshold  $K$ . Consequently, the **spike constraint** at the  $i^{\text{th}}$  location is:

$$C_{\text{SPK}} = \{f: \text{sign}(f_{i-1}) = \text{sign}(f_{i+1}) \neq \text{sign}(f_i) = s_i \text{ and } |f_i| \geq K\} \quad (3.39)$$

A priori information required by the associated projection operator consists of  $K$ , location  $i$ , and its sign  $s_i$ . The element-wise projection is then

$$P_{\text{SPK}}(i) g_j = \begin{cases} -s_i |g_j - 1| & j = i - 1 \\ s_i \mathbf{c} & j = i \\ -s_i |g_j + 1| & j = i + 1 \\ g_j & \text{elsewhere} \end{cases} \quad (3.40)$$

where  $\mathbf{c}$  is  $\max\{|g_i|, K\}$ . The sign  $s_i$  forces the projection to have a spike at location  $j$ . The magnitude of the spike  $\mathbf{c}$  is selected from the larger of  $|g_i|$  and the threshold  $K$ .

Since a typical signal consists of many zero-crossings and spikes, we might use more than one of these to constrain the reconstructed signal. The spike constraint corrects the signs of  $g$  and increases, if it is less than the threshold  $K$ , the magnitude of the element where the spike occurs. If  $J$  spikes are to be invoked, the  $J$  largest magnitude spikes are reasonable choices. The  $J$  largest magnitude spikes will add the most energy to the reconstructed signal.

For the zero-crossing constraint, Moose [Moose 1994] introduces two criteria used to determine the  $J$  most important zero-crossings. Based on the distance between the intensity values of the two elements that form zero-crossings, the first criterion is to choose the first  $J$  zero-crossings that have the largest distances. The other criterion is in the opposite way—choosing  $J$  zero-crossings that have the smallest intensity distances.

Invoking  $J$  zero-crossing locations that satisfy one of those criteria, the composite constraint and projection operator are assigned as follows

$$C_{ZJ} = \bigcap_1^J C_Z \quad (3.41)$$

and

$$P_{ZJ} = \prod_1^J P_Z \quad (3.42)$$

respectively. Each zero-crossing constraint enforces one zero-crossing into the reconstructed signal. Therefore,  $J$  zero-crossings are enforced by the composite constraint.

According to the zero-crossing constraint defined by (3.37), a zero-crossing at location  $i$  is used to correct the sign of the two signal elements  $f_i$  and  $f_{i+1}$ . Similarly, the information of having a zero-crossing at location  $i+1$  correct the signs of  $f_{i+1}$  and  $f_{i+2}$ . If both zero-crossings are involved,  $f_{i+1}$  is corrected twice. A redundancy occurs. Thus, storing another zero-crossing location is more noteworthy.

Likewise, one spike location, by (3.39), enforces three consecutive signal elements, e.g., a spike at locations  $i$  constraint  $f_{i-1}$ ,  $f_i$  and  $f_{i+1}$ . In order to avoid the redundancy, if spike location  $i$  is already stored, the next beneficial spike location to be stored is  $i+3$ . The

composite spike constraint and the associated projection can be defined similarly to (3.41) and (3.42).

### 3.4.3 Constraint on Adjacent Intensity Differences

The idea of this constraint is that the locations where the intensity of the signal increases or decreases suddenly are detected and used to constrain the reconstructed signal. This kind of a priori information preserves information on the edge of a block.

To generalize both increase and decrease cases, let  $e = [e_0, e_1, \dots, e_{N-1}]^T$  be an N-tuple signal. Let

$$d_{ei} = e_{i+1} - e_i \quad \text{for } i = 0, 1, \dots, N-1. \quad (3.43)$$

Define sets of indices  $I_{MI}$  and  $I_{MD}$  as

$$I_{MI} = \{i: d_i \geq K_{MI}\} \quad (3.44)$$

and

$$I_{MD} = \{i: d_i \leq K_{MD}\} \quad (3.45)$$

where  $K_{MI}$  and  $K_{MD}$  are threshold levels for sudden increases and decreases in signal level respectively.

**The Minimum increase constraint**  $C_{MI}$  is defined by

$$C_{MI} = \{f: f_{i+1} - f_i \geq K_{MI}, \forall i \in I_{MI}\} \quad (3.46)$$

$C_{MI}$  enforces values of  $d_i$  to be greater than or equal to  $K_{MI}$ . The element-wise projection onto this set is then defined as

$$\left. \begin{aligned} P_{MI} f &= \frac{f_{i+1} + f_i}{2} - \frac{K_{MI}}{2} \\ P_{MI} f_{i+1} &= \frac{f_{i+1} + f_i}{2} + \frac{K_{MI}}{2} \\ P_{MI} f_i &= f_i \end{aligned} \right\} \begin{array}{l} i \in I_{MI} \quad \text{and} \quad f_{i+1} - f_i > K_{MI} \\ \text{otherwise} \end{array} \quad (3.47)$$

Similarly  $C_{MD}$ , the **minimum decrease constraint** is defined by

$$C_{MD} = \{f: f_{i+1} - f_i \leq K_{MD}, \forall i \in I_{MD}\} \quad (3.48)$$

and the related projection operator is

$$\left. \begin{aligned} P_{MD} f &= \frac{f_{i+1} + f_i}{2} - \frac{K_{MD}}{2} \\ P_{MD} f_{i+1} &= \frac{f_{i+1} + f_i}{2} + \frac{K_{MD}}{2} \\ P_{MD} f_i &= f_i \end{aligned} \right\} \begin{array}{l} i \in I_{MD} \quad \text{and} \quad f_{i+1} - f_i > K_{MD} \\ \text{otherwise} \end{array} \quad (3.49)$$

$K_{MI}$  and  $K_{MD}$  might be found from the maximum and minimum value of  $d_i$  respectively. The maximum value of  $d_i$  is typically positive but this is not always the case. Assume the signal  $e$  to be a monotone decreasing signal;  $d_i$  are all negative.  $K_{MI}$  is now  $d_i$  which has the smallest absolute value.  $C_{MI}$  might not be helpful in this particular situation.

Similarly, monotone increasing signal and the minimum decreasing constraint are the same circumstance. However, the projections defined in (3.47) and (3.49) are still usable.