SOLUTION OF ST.-VENANT'S AND ALMANSI-MICHELL'S PROBLEMS

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(ABSTRACT)

We use the semi-inverse method to solve a St. Venant and an Almansi-Michell problem for a prismatic body made of a homogeneous and isotropic elastic material that is stress free in the reference configuration. In the St. Venant problem, only the end faces of the prismatic body are loaded by a set of self-equilibrated forces. In the Almansi-Michell problem self equilibrated surface tractions are also applied on the mantle of the body. The St. Venant problem is also analyzed for the following two cases: (i) the reference configuration is subjected to a hydrostatic pressure, and (ii) stress-strain relations contain terms that are quadratic in displacement gradients. The Signorini method is also used to analyze the St. Venant problem. Both for the St. Venant and the Almansi-Michell problems, the solution of the three dimensional problem is reduced to that of solving a sequence of two dimensional problems. For the St. Venant problem involving a second-order elastic material, the first order deformation is assumed to be an infinitesi-

mal twist. In the solution of the Almansi-Michell problem, surface tractions on the mantle of the cylindrical body are expressed as a polynomial in the axial coordinate. When solving the problem by the semi-inverse method, displacements are also expressed as a polynomial in the axial coordinate. An explicit solution is obtained for a hollow circular cylindrical body with surface tractions on the mantle given by an affine function of the axial coordinate

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Chapter 1

Introduction

Our goal is to find the deformed shape of an elastic cylindrical body when certain loads are applied to its boundary[1]. We can start from the basic properties of structure of the matter like atoms; thus use the Quantum Mechanics approach[2]. However, in most applications we do not need this level of precision.

Accordingly, we neglect the microstructure of matter and regard the body as a three-dimensional continua[3]. We make other simplifying assumptions in order to solve the problem; the validity of these assumptions is verified *a posteriori* through a comparison of computed results with those observed experimentally.

The first simplifying assumption is on the shape of the body. We consider prismatic body of uniform cross-section loaded either only at the end faces and/or on the mantle by tractions that can be expressed as a polynomial in the axial co-ordinate[4].

The material of the body is assumed to be elastic[5].

[1] Usually known as "The Problem of Saint-Venant". Many scientists have worked upon it; *e. g.* see [14], [17], [16], [18], [19], [20], [27], [28], [29], [30], [36], [39] and [40].

[2] A Quantum Mechanic approach for this kind of problem does not exist; however one could start from the principles of Quantum and Statistical Mechanics ([13], [42]).

[3] The theory of Continuum Mechanics is avaliable in many texts: [21], [22], [35] and [38].

[4] Usually known as the Almansi-Michell's problem. The few references found are: [2], [3] [10], [11] and [12].

[5] Many texts concerning the theory of elasticity are available: [23], [34], [41] and [43].

The first problem is usually referred to as the St.-Venant's Problem and the second one as Almansi-Michell's. It is difficult to solve these problems in complete generality. Accordingly, we assume that the deformations of the body are infinitesimal and the stresses depend upon strains at most quadratically.

We study four problems and in one of them we assume that the reference configuration is not stress free. These problems are summarized in the following table.

List of Problems studied:

Problem	Mantle traction-free	Ref. Conf. stress-free	Order[6]
St.-Venant	Yes	Yes	1
St.-Venant	Yes	No	1
St.-Venant	Yes	Yes	2
Alm.-Michell	No	Yes	1

We will use the semi-inverse method to solve these problems. That is, a part of the solution is assumed *a priori* and the remaining unknowns are found by satisfying the equilibrium equations and the boundary conditions.

[6] The order refers to both the Balance and Constitutive Equations.

Suggested future work

We have analyzed the case when the state of stress in the reference configuration is that of hydrostatic pressure. The next step is to consider a general state of stress in the reference configuration. An other extension of the present work is the consideration of body forces in the solution of the Almansi-Michell's problem. The third possibility is to use the complete Almansi-Michell solution to find the general non-linear solution for the St.-Venant's Problem.

Chapter 2

Theory of Elasticity

Theory of deformation for a continuous media

Deformations of line, area, and volume elements

Let $\mathfrak{S}$ be a continuous body. Consider the reference configuration $\mathfrak{C}_*$. The goal of this section is to describe the deformation of $\mathfrak{S}$ from the reference configuration $\mathfrak{C}_*$ to the present configuration $\mathfrak{C}$. Let $\mathbf{X}$ be a generic point in the reference configuration $\mathfrak{C}_*$ and $\mathbf{x}$ its position in the present configuration $\mathfrak{C}$. The finite deformation from $\mathfrak{C}_*$ to $\mathfrak{C}$ is described by the following vector valued function:

$$\mathbf{x} = \mathbf{x}(\mathbf{X}), \tag{2.1}$$

which associates to each point $\mathbf{X} \in \mathfrak{C}_*$ its corresponding position in $\mathfrak{C}$.

For the sake of simplicity, we do not study tearing and cavitation; so $\mathbf{x}$ is assumed to be a single valued function of $\mathbf{X}$. This means that a point does not deform into more then one point. Besides we consider the impenetrability of bodies and so assume the injective property for $\mathbf{x}$; i.e., two points do not deform into one. The function $\mathbf{x}$ is thus invertible. Furthermore we require that $\mathbf{x} \in C^2(\mathfrak{C}_*)$, and

$$J \equiv \det(\mathbf{F}) \equiv \det\left(\frac{\partial \mathbf{x}}{\partial \mathbf{X}}\right) > 0. \tag{2.2}$$

Taking the differential of both sides of (2.1), we get

$$\mathbf{dx} = \mathbf{FdX}, \quad \mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}, \tag{2.3}$$

where $\mathbf{F}$ is a linear transformation called the deformation gradient defined $\forall \mathbf{X} \in \mathfrak{C}_*$. It associates to each infinitesimal vector $\mathbf{dX}$ at $\mathbf{X} \in \mathfrak{C}_*$ the corresponding $\mathbf{dx}$ at $\mathbf{x} \in \mathfrak{C}$.

The polar decomposition theorem of Cauchy applied to the matrix $\mathbf{F}$ gives

$$\mathbf{F} = \mathbf{RU} \qquad , \qquad \mathbf{F} = \mathbf{VR}; \tag{2.4}$$

where $\mathbf{R}$ is a rotation and $\mathbf{U} = \mathbf{U}^T$ and $\mathbf{V} = \mathbf{V}^T$ describe the stretch. $\mathbf{R}$ is an orthogonal matrix and $\mathbf{U}$ and $\mathbf{V}$ are symmetric and positive definite matrices.

The right and left Cauchy-Green tensors are defined as

$$\begin{cases} \mathbf{C} \equiv \mathbf{F}^T\mathbf{F} = \mathbf{U}^T\mathbf{R}^T\mathbf{RU} = \mathbf{U}^2, \\ \mathbf{B} = \mathbf{FF}^T = \mathbf{VRR^TV}^T = \mathbf{V}^2. \end{cases} \tag{2.5}$$

Proposition 1 *The tensors* $\mathbf{B}$ *and* $\mathbf{C}$ *have the same eigenvalues; the eigenvalues are real and positive.*

The deformation field and the deformation gradient can be described equivalently by the displacement field

$$\mathbf{u}(\mathbf{X}) \equiv \mathbf{x}(\mathbf{X}) - \mathbf{X}, \tag{2.6}$$

and the displacement gradient

$$\mathbf{H} \equiv \frac{\partial \mathbf{u}}{\partial \mathbf{X}}. \tag{2.7}$$

From (2.6) we have:

$$\mathbf{F} = \mathbf{I} + \mathbf{H}. \tag{2.8}$$

The Green-Saint-Venant strain tensor is defined as

$$\mathbf{G} \equiv \frac{1}{2}\left(\mathbf{C} - \mathbf{I}\right). \tag{2.9}$$

$\mathbf{G}$ is symmetric and in terms of the displacement field it is given by

$$\mathbf{G} = \frac{1}{2}\left(\mathbf{H} + \mathbf{H}^T + \mathbf{H}^T\mathbf{H}\right). \tag{2.10}$$

For the description of the deformation of a continuous body it is important to know how the elements of length, surface and volume change. Let $\mathbf{dX} = dl_*\mathbf{M}$ be an infinitesimal vector at the point $\mathbf{X}$; $\mathbf{M}$ is the unit vector along $\mathbf{dX}$. Let the corresponding infinitesimal vector at $\mathbf{x}$ be $\mathbf{dx} = dl\ \mathbf{m}$; $\mathbf{m}$ is the unit vector along $\mathbf{dx}$. Then

$$dl = \sqrt{\mathbf{m} \cdot \mathbf{Cm}}\, dl_*. \tag{2.11}$$

Let $\mathbf{dS}^* = \mathbf{N}\ dS^*$ be the infinitesimal element of area at $\mathbf{X}$; $\mathbf{N}$ is the outward unit normal. Let the corresponding infinitesimal element of area at $\mathbf{x}$ be $\mathbf{dS} = \mathbf{n}dS$; $\mathbf{n}$ is its outward unit normal. Then

$$\mathbf{dS} = J\mathbf{F}^{-T}\mathbf{dS}_* \tag{2.12}$$

Finally let dc_* be the infinitesimal volume at $\mathbf{X}$, and dc be the corresponding infinitesimal volume at $\mathbf{x}$. Then

$$dc = J dc_* \tag{2.13}$$

From (2.12) we have

$$\begin{aligned} dS^2 = |\mathbf{dS}|^2 = (\mathbf{dS})^T (\mathbf{dS}) &= \mathbf{dS}_*^T \mathbf{F}^{-1} J J \mathbf{F}^{-T} \mathbf{dS}_*, \\ &= dS_* \mathbf{N}^T \mathbf{F}^{-1} J J \mathbf{F}^{-T} \mathbf{N} dS_* = J^2 dS_*^2 \mathbf{N}^T \mathbf{F}^{-1} \mathbf{F}^{-T} \mathbf{N}, \\ &= J^2 dS_*^2 \mathbf{N}^T \mathbf{C}^{-1} \mathbf{N}; \end{aligned} \tag{2.14}$$

and so

$$dS = J\sqrt{\mathbf{N}^T \mathbf{C}^{-1} \mathbf{N}} dS_*. \tag{2.15}$$

Let the first deformation deform the body $\mathfrak{S}$ from the reference configuration $\mathfrak{C}_*$ into the intermediate configuration $\mathfrak{C}^1$; the corresponding deformation field, the deformation gradient and the displacement gradient are

$$\begin{aligned} \mathbf{x}_1 &= \mathbf{x}_1(\mathbf{X}) \equiv \mathbf{X} + \mathbf{u}_1(\mathbf{X}), \\ \mathbf{F}_1 &\equiv \frac{\partial \mathbf{x}_1}{\partial \mathbf{X}}, \\ \mathbf{H}_1 &\equiv \frac{\partial \mathbf{u}_1}{\partial \mathbf{X}}. \end{aligned} \tag{2.16}$$

Thus

$$\begin{aligned} \mathbf{C}_1 &\equiv \mathbf{F}_1^T \mathbf{F}_1 = \mathbf{I} + \mathbf{H}_1 + \mathbf{H}_1^T + \mathbf{H}_1^T \mathbf{H}_1, \\ \mathbf{G}_1 &\equiv \tfrac{1}{2}(\mathbf{C}_1 - \mathbf{I}) = \tfrac{1}{2}\left(\mathbf{H}_1 + \mathbf{H}_1^T + \mathbf{H}_1^T \mathbf{H}_1\right). \end{aligned} \tag{2.17}$$

The second deformation deforms the system $\mathfrak{S}$ from the intermediate configuration $\mathfrak{C}^1$ into the present configuration $\mathfrak{C} = \mathfrak{C}^2$. We can write the present position in two ways: $\mathbf{x} = \mathbf{x}(\mathbf{X})$ or equivalently $\mathbf{x}_2 = \mathbf{x}_2(\mathbf{x}_1)$; so we have:

$$\mathbf{x} = \mathbf{x}(\mathbf{X}) = \mathbf{x}_2 = \mathbf{x}_2(\mathbf{x}_1). \tag{2.18}$$

Accordingly

$$\begin{cases} \mathbf{u}(\mathbf{X}) = \mathbf{x}(\mathbf{X}) - \mathbf{X}, \\ \mathbf{H} \equiv \dfrac{\partial \mathbf{u}}{\partial \mathbf{X}}; \end{cases} \tag{2.19}$$

or

$$\begin{cases} \mathbf{u}_2(\mathbf{x}_1) = \mathbf{x}_2(\mathbf{x}_1) - \mathbf{x}_1, \\ \mathbf{H}_2 \equiv \dfrac{\partial \mathbf{u}_2}{\partial \mathbf{x}_1}. \end{cases} \tag{2.20}$$

However it is important to note that displacement fields $\mathbf{u}$ and $\mathbf{u}_2$ are not the same. Analogous to (2.17) we have

$$\begin{cases} \mathbf{C}_2 \equiv \mathbf{I} + \mathbf{H}_2 + \mathbf{H}_2^T + \mathbf{H}_2^T\mathbf{H}_2, \\ \mathbf{G}_2 \equiv \frac{1}{2}(\mathbf{C}_2 - \mathbf{I}) = \frac{1}{2}\left(\mathbf{H}_2 + \mathbf{H}_2^T + \mathbf{H}_2^T\mathbf{H}_2\right). \end{cases} \tag{2.21}$$

The deformation gradient for the total deformation from $\mathfrak{C}_*$ to $\mathfrak{C}$ is given by

$$\begin{aligned} \mathbf{F} &= \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \frac{\partial \mathbf{x}_2}{\partial \mathbf{x}_1}\frac{\partial \mathbf{x}_1}{\partial \mathbf{X}} = \left(\frac{\partial \mathbf{x}_1}{\partial \mathbf{x}_1} + \frac{\partial \mathbf{u}_2}{\partial \mathbf{x}_1}\right)\frac{\partial \mathbf{x}_1}{\partial \mathbf{X}}, \\ &= (\mathbf{I} + \mathbf{H}_2)\,\mathbf{F}_1 = \mathbf{F}_1 + \mathbf{H}_2\mathbf{F}_1. \end{aligned} \tag{2.22}$$

From (2.9), $(2.5)_1$ and (2.22) we obtain

$$\mathbf{G} = \mathbf{G}_1 + \mathbf{F}_1^T\mathbf{G}_2\mathbf{F}_1 \tag{2.23}$$

for Green-Saint-Venant strain tensor.

Infinitesimal deformations

A deformation is said to be infinitesimal if the components of $\mathbf{u}$ and of $\mathbf{H}$ are small so that $|\mathbf{u}|^2$ and $|\mathbf{H}|^2$ are negligible as compared to $|\mathbf{u}|$ and $|\mathbf{H}|$ respectively.

Note that:

$$\lim_{\mathbf{u}\to\mathbf{0}} \{\mathbf{F}, \mathbf{C}, \mathbf{B}\} = \{\mathbf{I}, \mathbf{I}, \mathbf{I}\}, \tag{2.24}$$

and

$$\lim_{\mathbf{u}\to\mathbf{0}} \{\mathbf{H}, \mathbf{G}\} = \{\mathbf{0}, \mathbf{0}\}. \tag{2.25}$$

For two successive infinitesimal deformations we have from (2.23)

$$\mathbf{G} \backsimeq \mathbf{G}_1 + \mathbf{G}_2. \tag{2.26}$$

If we define the infinitesimal strain $\mathbf{E}$ by

$$\mathbf{E} = Sym\,(\mathbf{H}), \tag{2.27}$$

we obtain

$$\mathbf{G} \backsimeq \mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 \tag{2.28}$$

from (2.10) and (2.26). It can be proved that

$$\begin{cases} \mathbf{F} = \mathbf{I} + \mathbf{H}, \\ \mathbf{C} = \mathbf{I} + 2\mathbf{H}, \\ J = \det(\mathbf{I} + \mathbf{H}) = 1 + tr\,(\mathbf{H}). \end{cases} \tag{2.29}$$

Equilibrium laws

Assumptions on Forces

Let $c \subset \mathfrak{C}$ be a subregion of the body in the present configuration. In order to simplify the problem, in continuum mechanics it is usual to make some assumptions on forces acting upon c. The forces acting on c are divided into two kinds: body forces and contact forces. The body forces are distributed within the region c and the contact forces upon its boundary ∂c. We can thus write the total force and the total momentum with respect to the pole O as

$$\begin{aligned}\mathbf{F}\left(c, c^{e}\right) &= \int_{c} \rho\, \mathbf{b}\, dc + \int_{\partial c} \mathbf{t}\, dS\,, \\ \mathbf{M}_{O}\left(c, c^{e}\right) &= \int_{c} \left(\mathbf{x}-\mathbf{x}_{O}\right) \times \rho\, \mathbf{b}\, dc + \int_{\partial c} \left(\mathbf{x}-\mathbf{x}_{O}\right) \times \mathbf{t}\, dS\,,\end{aligned} \tag{2.30}$$

where $\mathbf{x}_O$ is the position vector of O and ρ is the mass density in the present configuration; $\mathbf{b}$ and $\mathbf{t}$ are respectively the specific body force defined in c and the contact force per unit surface area defined on ∂c. $\mathbf{F}$ and $\mathbf{M}_O$ depend not only upon the region c but also upon its external world c^e. We suppose that the body force can not depend upon changes in the positions of particles of $\mathfrak{C}$ that belong to c^e i.e. that are outside c. We thus neglect, for example, the mutual attractive or repulsive force that particles feel because of other particles of the same body. The functional dependence of $\mathbf{b}$ is: $\mathbf{b} = \mathbf{b}(\mathbf{x})$. For the density of contact force, we adopt Cauchy's hypothesis; $\mathbf{t} = \mathbf{t}(\mathbf{x}, \mathbf{n})$, where $\mathbf{n}$ is the outward unit normal to ∂c at $\mathbf{x}$. We state without proof

Theorem 2 (The fundamental theorem of Cauchy.) *If* $\mathbf{t}$ *is regular enough , then it is a linear function of* $\mathbf{n}$. *That is,*

$$\mathbf{t}(\mathbf{x}, \mathbf{n}) = \mathbf{T}(\mathbf{x})\, \mathbf{n}. \tag{2.31}$$

The Conservation of Mass

The mass $m(c_*)$ of a generic volume $c_* \subset \mathfrak{C}_*$ in the reference configuration is given by

$$m(c_*) = \int_{c_*} \rho_*(\mathbf{X})\ dc_*. \tag{2.32}$$

We assume that the mass, in the present configuration, of the corresponding part c is the same. Thus

$$m(c) = \int_c \rho(\mathbf{x})\, dc = m(c_*)\,. \tag{2.33}$$

We have:

$$\int_c \rho(\mathbf{x})\, dc = \int_{c_*} \rho(\mathbf{x}(\mathbf{X}))\ J\, dc_* = \int_{c_*} \rho_*(\mathbf{X})\ dc_* \tag{2.34}$$

and the relation between the mass densities in the present and the reference configurations is

$$J\,\rho(\mathbf{x}) = \rho_*(\mathbf{X}(\mathbf{x}))\,, \tag{2.35}$$

or

$$J\,\rho(\mathbf{x}(\mathbf{X})) = \rho_*(\mathbf{X})\,. \tag{2.36}$$

Balance of Linear Momentum

We study only static deformations of a body, and assume that in the present configuration the total force on a generic part $c \subset \mathfrak{C}$ is null:

$$\mathbf{F}(c, c^e) = \int_c \rho(\mathbf{x})\,\mathbf{b}(\mathbf{x})\, dc + \int_{\partial c} \mathbf{t}(\mathbf{x}, \mathbf{n})\, dS = \mathbf{0}, \qquad \forall c \subset \mathfrak{C}. \tag{2.37}$$

Using the fundamental theorem of Cauchy (2.31) we have

$$\int_c \rho\left(\mathbf{x}\right)\mathbf{b}\left(\mathbf{x}\right)dc + \int_{\partial c} \mathbf{T}\left(\mathbf{x}\right)\mathbf{n}dS = \mathbf{0}, \qquad \forall c \subset \mathfrak{C}. \tag{2.38}$$

Using the Divergence theorem, we get the local form

$$Div\mathbf{T}\left(\mathbf{x}\right) + \rho\left(\mathbf{x}\right)\mathbf{b}\left(\mathbf{x}\right) = \mathbf{0}, \qquad \forall\, \mathbf{x} \in \mathfrak{C}, \tag{2.39}$$

of the equilibrium equation in the Eulerian description of deformation.

Note that this is a system of partial differential equations and in order to solve them we also need boundary conditions.

At every point $\mathbf{x} \in \partial\mathfrak{C}$, either the surface traction $\mathbf{t}$

$$\mathbf{t}\left(\mathbf{x}, \mathbf{n}\right) = \mathbf{T}\left(\mathbf{x}\right)\mathbf{n}, \tag{2.40}$$

or the displacement $\mathbf{u}$,

$$\mathbf{u}\left(\mathbf{x}\right) = \bar{\mathbf{u}}\left(\mathbf{x}\right) \tag{2.41}$$

or a combination of $\mathbf{t}$ and $\bar{\mathbf{u}}$ is prescribed. If we have only surface traction conditions

$$\begin{cases} Div\mathbf{T}\left(\mathbf{x}\right) + \rho\left(\mathbf{x}\right)\mathbf{b}\left(\mathbf{x}\right) = \mathbf{0}, & \forall\, \mathbf{x} \in \mathfrak{C}, \\ \mathbf{t}\left(\mathbf{x}, \mathbf{n}\right) = \mathbf{T}\left(\mathbf{x}\right)\mathbf{n}, & \forall\, \mathbf{x} \in \partial\mathfrak{C}. \end{cases} \tag{2.42}$$

We transform the integral in (2.38) over c to that on the region c_* in the reference configuration that corresponds to c. Using (2.35) and (2.12), we get

$$\int_{c_*} \rho_*\left(\mathbf{X}\right)\mathbf{b}\left(\mathbf{x}\left(\mathbf{X}\right)\right)dc_* + \int_{\partial c_*} \mathbf{T}\left(\mathbf{x}\left(\mathbf{X}\right)\right)J\mathbf{F}^{-T}\mathbf{N}dS^* = \mathbf{0}. \tag{2.43}$$

In terms of the first Piola-Kirchhoff stress tensor

$$\mathbf{T}_* (\mathbf{X}) = J\mathbf{T} (\mathbf{x} (\mathbf{X})) \mathbf{F}^{-T}, \tag{2.44}$$

(2.43) becomes

$$\int_{c_*} \rho_* \mathbf{b} \, dc_* + \int_{\partial c_*} \mathbf{T}_* \mathbf{N} dS^* = \mathbf{0}, \quad \forall c_* \subset \mathfrak{C}_*. \tag{2.45}$$

Using the Divergence theorem we write its local form and the traction boundary condition as

$$\begin{aligned} Div\mathbf{T}_* + \rho_* \mathbf{b} = \mathbf{0}, \quad \forall \mathbf{X} \in \mathfrak{C}_*, \\ \mathbf{T}_* \mathbf{N}_* = \mathbf{t}_*, \quad \forall \mathbf{X} \in \partial \mathfrak{C}_*. \end{aligned} \tag{2.46}$$

The Lagrangian form (2.46) of the equilibrium equations has the following advantages over the Eulerian form (2.42). The mass density ρ_* is known and fields are defined on the known shape $\mathfrak{C}_*$ of the body.

Balance of Momentum of Momentum

We assume that in the present configuration, the total momentum of forces acting on a generic part $c \subset \mathfrak{C}$ is null. Using the fundamental theorem of Cauchy we have

$$\mathbf{M}_O (c, c^e) = \int_c (\mathbf{x} - \mathbf{x}_O) \times \rho (\mathbf{x}) \, \mathbf{b} (\mathbf{x}) dc + \int_{\partial c} (\mathbf{x} - \mathbf{x}_O) \times \mathbf{T} (\mathbf{x}) \, \mathbf{n} dS = \mathbf{0}.$$

Applying the Divergence theorem we get the local form of the balance of momentum of momentum in the Eulerian form:

$$Div \left[(\mathbf{x} - \mathbf{x}_O) \times \mathbf{T} (\mathbf{x}) \right] + (\mathbf{x} - \mathbf{x}_O) \times \rho (\mathbf{x}) \, \mathbf{b} (\mathbf{x}) = \mathbf{0}, \qquad \forall \mathbf{x} \quad \mathfrak{C}. \tag{2.47}$$

In a rectangular Cartesian co-ordinate system $(\mathbf{x}_O, \mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3)$,

$$\left(Div\left[\left(\mathbf{x}-\mathbf{x}_O\right)\times\mathbf{T}\left(\mathbf{x}\right)\right]\right)_i = \varepsilon_{ihk}T_{hk} + \left(\left(\mathbf{x}-\mathbf{x}_O\right)\times\left(Div\mathbf{T}\right)\right)_i,$$

where ε_{ihk} is the permutation tensor. Because of the balance of linear momentum, (2.47) is equivalent to

$$\varepsilon_{ihk}T_{hk} = 0, \qquad i = 1, 2, 3.$$

The skew symmetry of $\boldsymbol{\varepsilon}$ implies the symmetry of the Cauchy stress tensor,

$$\mathbf{T} = \mathbf{T}^T. \tag{2.48}$$

Using (2.44), (2.48) becomes

$$\mathbf{T}_*\mathbf{F}^T = \mathbf{F}\mathbf{T}_*^T.$$

Constitutive Equations

Different materials react in different ways to the same set of forces. In this section we characterize an elastic material by equations called "Constitutive Equations".

An unconstrained elastic body is characterized by the following constitutive equation:

$$\mathbf{T} = \mathbf{T}\left(\mathbf{X}\right) = \mathbf{T}\left(\mathbf{X}, \mathbf{F}\left(\mathbf{X}\right)\right), \qquad \forall\mathbf{X}\in\mathfrak{C}_*, \tag{2.49}$$

where $\mathbf{T}\left(\mathbf{X}, \mathbf{F}\left(\mathbf{X}\right)\right)$ is a C^1 function. A body is said to be homogeneous if in the reference configuration $\mathfrak{C}_*$ the mass density does not depend upon the material point $\mathbf{X}$, i.e.,

$$\rho_* = const, \qquad \forall \mathbf{X} \in \mathfrak{C}_*,$$

and the function $\mathbf{T}$ on the right-hand side of the (2.49) does not depend explicitly upon $\mathbf{X}$. Note that for a homogeneous elastic body,

$$\mathbf{T}_* = \mathbf{T}_* \left(\mathbf{F} \right). \tag{2.50}$$

An unconstrained material point is called hyperelastic if there exists a function $\Psi \left(\mathbf{F} \right)$, called the specific energy, such that

$$\mathbf{T}_* \left(\mathbf{F} \right) = \frac{\partial \Psi}{\partial \mathbf{F}}, \tag{2.51}$$

or

$$\mathbf{T} = \frac{\rho}{\rho_*} \frac{\partial \Psi}{\partial \mathbf{F}} \mathbf{F}^T. \tag{2.52}$$

The principle of Material Frame indifference requires that

$$\Psi = \Psi \left(\mathbf{G} \right). \tag{2.53}$$

Thus

$$\mathbf{T}_* = \mathbf{F} \frac{\partial \Psi}{\partial \mathbf{G}} \tag{2.54}$$

and

$$\mathbf{T} \left(\mathbf{F} \right) = \frac{\rho}{\rho_*} \mathbf{F} \frac{\partial \Psi}{\partial \mathbf{G}} \mathbf{F}^T = \frac{\rho}{\rho_*} \left(\mathbf{I} + \mathbf{H} \right) \frac{\partial \Psi}{\partial \mathbf{G}} \left(\mathbf{I} + \mathbf{H}^T \right). \tag{2.55}$$

It can be proved that for an isotropic material

$$\Psi = \Psi \left(\mathbf{G} \right) = \Psi \left(I_{\mathbf{G}}, II_{\mathbf{G}}, III_{\mathbf{G}} \right),$$

where $I_{\mathbf{G}}$,$II_{\mathbf{G}}$ and $III_{\mathbf{G}}$ are respectively the first, the second and the third principal invariants of $\mathbf{G}$.

Assuming that Ψ is an analytical function of $\mathbf{G}$, we obtain

$$\Psi(\mathbf{G}) = \Psi(\mathbf{0}) + \frac{\partial \Psi}{\partial \mathbf{G}} \mid_{\mathbf{G}=\mathbf{0}} \mathbf{G} + \frac{1}{2}\frac{\partial^2 \Psi}{\partial \mathbf{G}^2} \mid_{\mathbf{G}=\mathbf{0}} \mathbf{G}^2 + ...,$$

and therefore

$$\frac{\partial \Psi}{\partial \mathbf{G}} = \frac{\partial \Psi}{\partial \mathbf{G}} \mid_{\mathbf{G}=\mathbf{0}} + \frac{\partial^2 \Psi}{\partial \mathbf{G}^2} \mid_{\mathbf{G}=\mathbf{0}} \mathbf{G} + ... \quad .$$

From (2.35) and $(2.29)_2$ we have

$$\frac{\rho}{\rho_*} = \frac{1}{J} = 1 - tr(\mathbf{H}) + ... \quad , \tag{2.56}$$

and (2.55) can be written as

$$\mathbf{T} = (1 - tr(\mathbf{H}))(\mathbf{I} + \mathbf{H}) \left[\frac{\partial \Psi}{\partial \mathbf{G}} \mid_{\mathbf{G}=\mathbf{0}} + \frac{\partial^2 \Psi}{\partial \mathbf{G}^2} \mid_{\mathbf{G}=\mathbf{0}} \mathbf{G} \right] \left(\mathbf{I} + \mathbf{H}^T \right) + ... \quad . \tag{2.57}$$

We set

$$\begin{aligned} \mathbf{T}_0 &= \frac{\partial \Psi}{\partial \mathbf{G}} \mid_{\mathbf{G}=\mathbf{0}}, \\ \mathbb{C} &= \frac{\partial^2 \Psi}{\partial \mathbf{G}^2} \mid_{\mathbf{G}=\mathbf{0}}, \end{aligned} \tag{2.58}$$

where $\mathbf{T}_0$ equals the stress in the reference configuration, and $\mathbb{C} = \mathbb{C}^T$ is the fourth-order elasticity tensor. In rectangular Cartesian coordinates, $\mathbb{C}$ satisfies

$$\mathbb{C}_{ijkl} = \mathbb{C}_{klij} = \mathbb{C}_{ijlk}.$$

Equation (2.57) now becomes

$$\mathbf{T} = (1 - tr(\mathbf{H}))(\mathbf{I} + \mathbf{H})[\mathbf{T}_0 + \mathbb{C}\mathbf{G}] \left(\mathbf{I} + \mathbf{H}^T \right) + ... \quad . \tag{2.59}$$

For infinitesimal deformations, we neglect terms in $\mathbf{H}$ of order greater than one. Equation (2.59) becomes

$$\mathbf{T} = \mathbf{T}_0 + \mathbf{H}\mathbf{T}_0 + \mathbf{T}_0\mathbf{H}^T + \mathbb{C}\mathbf{E} - \mathbf{T}_0 tr\left(\mathbf{H}\right). \tag{2.60}$$

If $\mathbf{T}_0 = \mathbf{0}$ the reference configuration is called natural and we have

$$\mathbf{T} = \mathbb{C}\mathbf{E}. \tag{2.61}$$

The values of the components of the elasticity tensor $\mathbb{C}$ depend upon the choice of the reference configuration. For infinitesimal deformations, because of (2.44), (2.56), (2.60) and (2.8), the first Piola-Kirchhoff stress tensor is given by

$$\begin{aligned}\mathbf{T}_* = J\mathbf{T}\mathbf{F}^{-T} &\simeq \left(1 + tr\left(\mathbf{H}\right)\right)\left(\mathbf{T}_0 + \mathbf{H}\mathbf{T}_0 + \mathbf{T}_0\mathbf{H}^T + \mathbb{C}\mathbf{E} - \mathbf{T}_0 tr\left(\mathbf{H}\right)\right)\left(\mathbf{I} - \mathbf{H}^T\right), \\ &\simeq \mathbf{T}_0 + \mathbf{H}\mathbf{T}_0 + \mathbb{C}\mathbf{E}. \end{aligned} \tag{2.62}$$

For a stress free reference configuration, $\mathbf{T}_* = \mathbf{T}$ in linear elasticity. When $|\mathbf{T}_0| << |\mathbb{C}|$, then $\mathbf{H}\mathbf{T}_0$ is negligible as compared to the other two terms on the right-hand side of (2.62) and we obtain

$$\mathbf{T}_* = \mathbf{T}_0 + \mathbb{C}\mathbf{E}.$$

For the specific energy of deformation given by

$$\Psi\left(\mathbf{G}\right) = \frac{\lambda + 2\mu}{2} I_{\mathbf{G}}^2 - 2\mu\, II_{\mathbf{G}}, \tag{2.63}$$

we conclude from (2.55) that

$$\mathbf{T} = \lambda\left(tr\mathbf{E}\right)\mathbf{I} + 2\mu\mathbf{E}. \tag{2.64}$$

For an infinitesimal deformation with displacement gradient $\mathbf{H}_2$ superimposed upon a finite deformation of deformation gradient $\mathbf{F}_1$, we have from (2.23) and (2.53),

$$\Psi = \Psi(\mathbf{G}) = \Psi\left(\mathbf{G}_1 + \mathbf{F}_1^T \mathbf{E}_2 \mathbf{F}_1\right),$$

$$= \Psi(\mathbf{G}_1) + \frac{\partial \Psi}{\partial \mathbf{G}}\,|_{\mathbf{G}=\mathbf{G}_1} (\mathbf{G} - \mathbf{G}_1) + \frac{1}{2}\frac{\partial^2 \Psi}{\partial \mathbf{G}^2}\,|_{\mathbf{G}=\mathbf{G}_1} (\mathbf{G} - \mathbf{G}_1)^2 + ... \quad .$$

Neglecting terms of order greater than one in $\mathbf{E}_2$, we get

$$\frac{\partial \Psi}{\partial \mathbf{G}} = \frac{\partial \Psi}{\partial \mathbf{G}}\,|_{\mathbf{G}=\mathbf{G}_1} + \frac{\partial^2 \Psi}{\partial \mathbf{G}^2}\,|_{\mathbf{G}=\mathbf{G}_1} (\mathbf{G} - \mathbf{G}_1), \tag{2.65}$$

and

$$\frac{\rho}{\rho_*} = \frac{\rho}{\rho_1}\frac{\rho_1}{\rho_*} = \frac{\rho}{\rho_1}\left(1 - tr(\mathbf{H}_2)\right). \tag{2.66}$$

From (2.55), (2.66), (2.22) and (2.65) we conclude that

$$\mathbf{T} = \frac{\rho}{\rho_1}\left(1 - tr(\mathbf{H}_2)\right)(\mathbf{F}_1 + \mathbf{H}_2\mathbf{F}_1)$$
$$\left[\frac{\partial \Psi}{\partial \mathbf{G}}\,|_{\mathbf{G}=\mathbf{G}_1} + \frac{\partial^2 \Psi}{\partial \mathbf{G}^2}\,|_{\mathbf{G}=\mathbf{G}_1} (\mathbf{G} - \mathbf{G}_1)\right]\left(\mathbf{F}_1^T + \mathbf{F}_1^T\mathbf{H}_2^T\right).$$

Expanding these terms and recalling that the Cauchy stress tensor in the intermediate configuration is given by

$$\mathbf{T}_1 = \frac{\rho_1}{\rho_*}\mathbf{F}_1 \frac{\partial \Psi}{\partial \mathbf{G}_1}\mathbf{F}_1^T,$$

we get

$$\mathbf{T} = \mathbf{T}_1 + \mathbf{T}_1\mathbf{H}_2^T + \frac{\rho_1}{\rho_*}\mathbf{F}_1\left[\frac{\partial^2 \Psi}{\partial \mathbf{G}_1^2} : \mathbf{F}_1\mathbf{E}_2\mathbf{F}_1^T\right]\mathbf{F}_1^T + \mathbf{H}_2\mathbf{T}_1 - \mathbf{T}_1 tr(\mathbf{H}_2). \tag{2.67}$$

For the specific deformation energy given by (2.63), (2.67) becomes

$$\mathbf{T} = \mathbf{T}_1 + \mathbf{T}_1\mathbf{H}_2^T + \mathbb{C}\mathbf{E}_2 + \mathbf{H}_2\mathbf{T}_1 - \mathbf{T}_1 tr(\mathbf{H}_2),$$

where in rectangular Cartesian coordinates[1],

$$C_{ijkl} = \frac{\rho_1}{\rho_*}\left(2\mu B^1_{il}B^1_{kj} + \lambda B^1_{ij}B^1_{kl}\right).$$

When considering second order effects in the constitutive relation for infinitesimal deformations, we expand the specific deformation energy until the third order terms in $\mathbf{G}$. That is,

$$\begin{aligned}\Psi \simeq & (\Psi) + \left(\frac{\partial\Psi}{\partial I_{\mathbf{G}}}\right) I_{\mathbf{G}} + \left(\frac{\partial\Psi}{\partial II_{\mathbf{G}}}\right) II_{\mathbf{G}} + \left(\frac{\partial\Psi}{\partial III_{\mathbf{G}}}\right) III_{\mathbf{G}} + \frac{1}{2}\left(\frac{\partial^2\Psi}{\partial I^2_{\mathbf{G}}}\right) I^2_{\mathbf{G}} \\ & + \frac{1}{2}\left(\frac{\partial^2\Psi}{\partial II_{\mathbf{G}}\partial I_{\mathbf{G}}}\right) I_{\mathbf{G}} II_{\mathbf{G}} + \frac{1}{6}\left(\frac{\partial^3\Psi}{\partial I^3_{\mathbf{G}}}\right) I^3_{\mathbf{G}}, \end{aligned} \tag{2.68}$$

where terms in () are evaluated at $\mathbf{G} = \mathbf{0}$.

Substitution from (2.68) into (2.54) and retaining terms up to and including $\mathbf{H}^2$ yields

$$\begin{aligned}\mathbf{T}_*(\mathbf{H}) \simeq & \lambda I_{\mathbf{E}}\mathbf{I} + 2\mu\mathbf{E} \\ & + \left(\frac{\lambda}{2}\left(I_{\mathbf{H}\mathbf{H}^T} + 2I^2_{\mathbf{E}}\right) + \mu\alpha_3 I^2_{\mathbf{E}} + \mu\alpha_4 II_{\mathbf{E}}\right)\mathbf{I} \\ & + \mu\left(2+\alpha_5\right) I_{\mathbf{E}}\mathbf{E} - \lambda I_{\mathbf{E}}\mathbf{H}^T - \mu\left(\mathbf{H}^T\right)^2 + \mu\alpha_6\mathbf{E}^2; \end{aligned} \tag{2.69}$$

where α_3, α_4, α_5 and α_6, are the second order elasticities for the isotropic material, and the reference configuration has been assumed to be stress free.

[1] B^1_{ij} are the components in rectangular coordinates of the left Cauchy-Green stress tensor $\mathbf{B}_1$ defined in $(2.5)_2$.

Equations of elastostatics

The equilibrium equations (2.46) with the constitutive equation (2.50) give the following partial differential equation for a homogeneous elastic body.

$$\frac{\partial T_{*iL}}{\partial X_L} + b_i\left(\mathbf{x}\left(\mathbf{X}\right)\right) = \frac{\partial T_{*iL}}{\partial F_{jM}}\frac{\partial F_{jM}}{\partial X_L} + b_i\left(\mathbf{x}\left(\mathbf{X}\right)\right) = 0. \qquad (2.70)$$

Here we have used rectangular Cartesian coordinates. With the definitions

$$A_{ijLM}\left(\mathbf{F}\right) = \frac{\partial T_{*iL}}{\partial F_{jM}}, \qquad b_i^*\left(\mathbf{X}\right) = b_i\left(\mathbf{x}\left(\mathbf{X}\right)\right),$$

we write (2.70) as

$$A_{ijLM}\left(\mathbf{F}\right)\frac{\partial^2 x_j}{\partial X_L \partial X_M} + \rho_* b_i^*\left(\mathbf{X}\right) = 0. \qquad (2.71)$$

This gives a system of 3 partial differential equations in the 3 unknown deformation fields $x_i\left(\mathbf{X}\right)$ defined on the regular and known domain $\mathfrak{C}_*$.

The relevant boundary conditions is $(2.46)_2$. More general boundary condition that account for the interaction between the body and its surroundings have been discussed by Batra [4].

When a system of linear springs (fig. 2.1) with spring constant k act on a part of the boundary and springs are constrained to stay vertical, then

$$\mathbf{t}(\mathbf{x},\mathbf{n}) = -ku_2(\mathbf{x})\mathbf{e}_2.$$

However when prescribed surface traction on $\partial\mathfrak{C}_*$ are such that they do not depend upon the deformation of $\partial\mathfrak{C}_*$, then they are called dead loads.

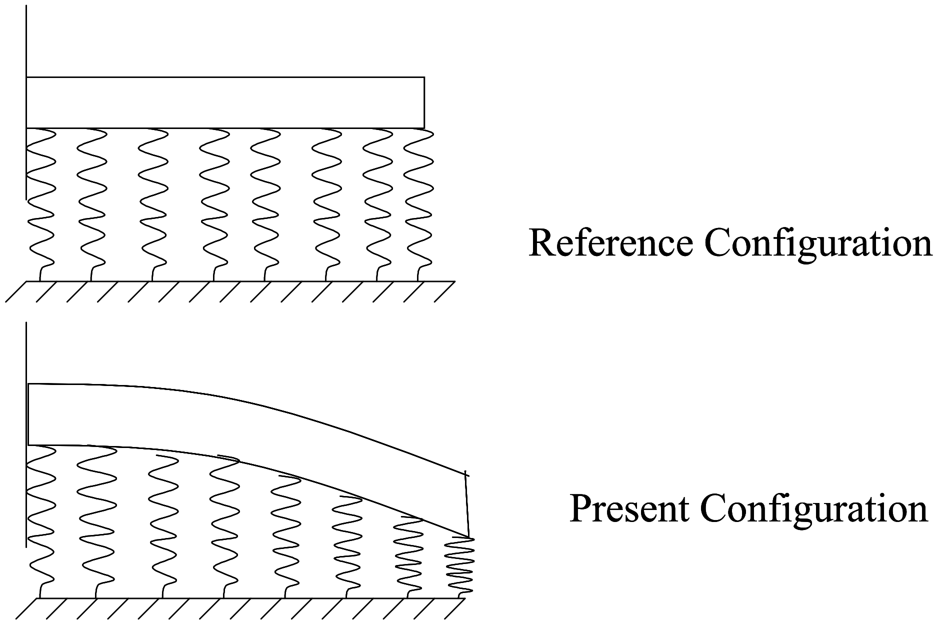

Figure 2.1: A system of linear springs

Traction boundary value problems involving dead load are easier to analyze than those in which surface tractions depend upon the deformation of the surface of the body. An exception may be the pressure loading.

A boundary value problem for a homogenous body, that is stress free in the reference configuration, in the Lagrangian description of motion can be written as

$$\begin{aligned} & A_{ijLM}\left(\mathbf{F}\right)\frac{\partial^2 x_j}{\partial X_L \partial X_M} + \rho_* b_i\left(\mathbf{X}\right) = 0, && \forall \mathbf{X} \in \mathfrak{C}_*, \\ & \mathbf{x}\left(\mathbf{X}\right) = \bar{\mathbf{x}}\left(\mathbf{X}\right), && \forall \mathbf{X} \in \partial\mathfrak{C}'_*, \\ & \mathbf{T}_*\left(\mathbf{F}\right)\mathbf{N}_* = \mathbf{t}_*\left(\mathbf{X},\mathbf{x},\mathbf{F}\right), && \forall \mathbf{X} \in \partial\mathfrak{C}''_*, \end{aligned} \tag{2.72}$$

where $\partial\mathfrak{C}_* = \partial\mathfrak{C}'_* \cup \partial\mathfrak{C}''_*$ and $\partial\mathfrak{C}'_* \cap \partial\mathfrak{C}''_* = \varnothing$

If $\partial\mathfrak{C}'_* = \emptyset$ then the problem is said to be a traction-value problem; if $\partial\mathfrak{C}''_* = \emptyset$ then the problem is said to be a place-value problem. In other cases the problem is a mixed boundary-value problem. If $\mathbf{t}_* = \mathbf{t}_*(\mathbf{X})$ then the load is called dead load. $\bar{\mathbf{x}}(\mathbf{X})$ is an assigned function of $\mathbf{X}$.

We study only static problems and are not interested in its rigid motion. If final positions of three non-colinear points are prescribed, then the rigid motion is not possible. For a traction value problem, the applied traction must be such that the body is in equilibrium. That is

$$\begin{aligned} \int_{\partial\mathfrak{C}} \mathbf{Tn} dS + \int_{\mathfrak{C}} \rho(\mathbf{x})\, \mathbf{b}(\mathbf{x})\, dc = \mathbf{0}, \\ \int_{\partial\mathfrak{C}} (\mathbf{x} - \mathbf{x}_O) \times \mathbf{Tn} dS + \int_{\mathfrak{C}} (\mathbf{x} - \mathbf{x}_O) \times \rho(\mathbf{x})\, \mathbf{b}(\mathbf{x})\, dc = \mathbf{0}, \end{aligned} \tag{2.73}$$

or

$$\begin{aligned} \int_{\partial\mathfrak{C}_*} \mathbf{T}_*(\mathbf{F})\, \mathbf{N}_* dS^* + \int_{\mathfrak{C}_*} \rho_* \mathbf{b} dc_* = \mathbf{0}, \\ \int_{\partial\mathfrak{C}_*} (\mathbf{x} - \mathbf{x}_O) \times \mathbf{T}_*(\mathbf{F})\, \mathbf{N}_* dS^* + \int_{\mathfrak{C}_*} (\mathbf{x} - \mathbf{x}_O) \times \rho_* \mathbf{b} dc_* = \mathbf{0}. \end{aligned} \tag{2.74}$$

We see that it is not possible to verify *a priori* if loads $(\mathbf{t}_*, \mathbf{b})$ satisfy (2.74) because the dependence of $\mathbf{T}_*$ on $\mathbf{F}$ is unknown. Even for dead loads we have the same problem because of the presence of $\mathbf{x}$ in $(2.74)_2$. For this reason (2.74) is used as a compatibility condition.

Truesdell and Noll [38] have given examples illustrating the non uniqueness of solutions in non-linear elasticity.

We now consider the case when the applied surface tractions and body forces depend continuously upon a small parameter ε. We hypothesize that displacements resulting from the application of these loads to an elastic body also depend continuously upon ε. We further assume that

$$\begin{aligned}
&\mathbf{b}_{\varepsilon}\left(\mathbf{X},\mathbf{u}_{\varepsilon}\right)=\varepsilon\mathbf{b}_{1}\left(\mathbf{X},\mathbf{u}_{\varepsilon}\right), && \forall\mathbf{X}\in\mathfrak{C}_{*},\\
&\mathbf{u}_{\varepsilon}\left(\mathbf{X}\right)=\varepsilon\tilde{\mathbf{u}}_{1}\left(\mathbf{X}\right), && \forall\mathbf{X}\in\partial\mathfrak{C}_{*}^{\prime},\\
&\mathbf{T}_{*\varepsilon}\mathbf{N}_{*}=\mathbf{t}_{*\varepsilon}\left(\mathbf{X},\mathbf{u}_{\varepsilon},\mathbf{H}_{\varepsilon}\right)=\varepsilon\mathbf{t}_{*1}\left(\mathbf{X},\mathbf{u}_{\varepsilon},\mathbf{H}_{\varepsilon}\right), && \forall\mathbf{X}\in\partial\mathfrak{C}_{*}^{\prime\prime}.
\end{aligned}\tag{2.75}$$

Thus, $\forall\mathbf{X}\in\mathfrak{C}_{*}$

$$\lim_{\varepsilon\to 0}\mathbf{u}_{\varepsilon}\left(\mathbf{X}\right)=\mathbf{0},$$

and

$$\begin{aligned}
\mathbf{u}_{\varepsilon}\left(\mathbf{X}\right)&=\varepsilon\mathbf{u}_{1}\left(\mathbf{X}\right)+o\left(\varepsilon\right),\\
\mathbf{H}_{\varepsilon}\left(\mathbf{X}\right)&=\varepsilon\mathbf{H}_{1}\left(\mathbf{X}\right)+o\left(\varepsilon\right),\\
\mathbf{E}_{\varepsilon}\left(\mathbf{X}\right)&=\varepsilon\mathbf{E}_{1}\left(\mathbf{X}\right)+o\left(\varepsilon\right),\\
\mathbf{F}_{\varepsilon}&=\mathbf{I}+\varepsilon\mathbf{H}_{1}\left(\mathbf{X}\right)+o\left(\varepsilon\right),\\
\mathbf{C}_{\varepsilon}&=\mathbf{I}+2\varepsilon\mathbf{E}_{1}^{T}+o\left(\varepsilon\right),\\
\mathbf{C}_{\varepsilon}^{-1}&=\mathbf{I}-2\varepsilon\mathbf{E}_{1}^{T}+o\left(\varepsilon\right),\\
\mathbf{T}_{\varepsilon}&=\mathbf{T}_{\varepsilon}\left(\mathbf{H}_{\varepsilon}\right)=\varepsilon\mathbb{C}\mathbf{E}_{1}+o\left(\varepsilon\right),\\
J_{\varepsilon}&=1+\varepsilon tr\left(\mathbf{H}_{1}\right)+o\left(\varepsilon\right).
\end{aligned}\tag{2.76}$$

From (2.44), the first Piola-Kirchhoff stress tensor is given by

$$\mathbf{T}_{*\varepsilon}=J_{\varepsilon}\mathbf{T}_{\varepsilon}\mathbf{F}_{\varepsilon}^{-T}=\left(1+\varepsilon tr\left(\mathbf{H}_{1}\right)\right)\left(\varepsilon\mathbb{C}\mathbf{E}_{1}\right)\left(1+\varepsilon\mathbf{H}_{1}^{-T}\right)+o\left(\varepsilon\right),\tag{2.77}$$

$$=\varepsilon\mathbb{C}\mathbf{E}_{1}+o\left(\varepsilon\right)=\mathbf{T}_{\varepsilon}.$$

Thus $\mathbf{T}_{*\varepsilon}$ equals the Cauchy stress tensor $\mathbf{T}_{\varepsilon}$.

Let

$$\mathbf{t}_{\varepsilon}\left(\mathbf{x},\mathbf{u}_{\varepsilon},\mathbf{H}_{\varepsilon}\right)=\varepsilon\mathbf{t}_{1}\left(\mathbf{x},\mathbf{u}_{\varepsilon},\mathbf{H}_{\varepsilon}\right)+o\left(\varepsilon\right),$$

be the load in the present configuration. Since the force acting on the same set of material particles has to be the same, therefore

$$\mathbf{t}_{*\varepsilon}\left(\mathbf{x},\mathbf{u}_{\varepsilon},\mathbf{H}_{\varepsilon}\right)dS^{*}=\mathbf{t}_{\varepsilon}\left(\mathbf{x},\mathbf{u}_{\varepsilon},\mathbf{H}_{\varepsilon}\right)dS.$$

Using (2.15), $(2.29)_{3,2}$, we obtain

$$\mathbf{t}_{*\varepsilon}\left(\mathbf{X},\mathbf{u}_{\varepsilon},\mathbf{H}_{\varepsilon}\right)=J_{\varepsilon}\mathbf{t}_{\varepsilon}\left(\mathbf{x},\mathbf{u}_{\varepsilon},\mathbf{H}_{\varepsilon}\right)\sqrt{\mathbf{N}^{T}\mathbf{C}_{\varepsilon}^{-1}\mathbf{N}},$$

$$=\left[1+\varepsilon tr\left(\mathbf{H}_{1}\right)\right]\mathbf{t}_{\varepsilon}\left(\mathbf{x},\mathbf{u}_{\varepsilon},\mathbf{H}_{\varepsilon}\right)\left[1-\varepsilon\mathbf{N}^{T}\mathbf{E}_{1}\left(\mathbf{X}\right)\mathbf{N}\right]+o\left(\varepsilon\right),$$

$$=\mathbf{t}_{\varepsilon}\left(\mathbf{x},\mathbf{u}_{\varepsilon},\mathbf{H}_{\varepsilon}\right)+o\left(\varepsilon\right).$$

Expanding $\mathbf{t}_{\varepsilon}\left(\mathbf{x},\mathbf{u}_{\varepsilon},\mathbf{H}_{\varepsilon}\right)$ in Taylor series around the point $(\mathbf{X},\mathbf{0},\mathbf{0})$, we get

$$\mathbf{t}_{\varepsilon}\left(\mathbf{x},\mathbf{u}_{\varepsilon},\mathbf{H}_{\varepsilon}\right)=\mathbf{t}_{\varepsilon}\left(\mathbf{X},\mathbf{0},\mathbf{0}\right)$$

$$+\left\{\left[\frac{\partial\mathbf{t}_{\varepsilon}}{\partial\mathbf{x}}\right]_{(\mathbf{X},\mathbf{0},\mathbf{0})}\left(\mathbf{x}-\mathbf{X}\right)+\left[\frac{\partial\mathbf{t}_{\varepsilon}}{\partial\mathbf{u}_{\varepsilon}}\right]_{(\mathbf{X},\mathbf{0},\mathbf{0})}\mathbf{u}_{\varepsilon}+\left[\frac{\partial\mathbf{t}_{\varepsilon}}{\partial\mathbf{H}_{\varepsilon}}\right]_{(\mathbf{X},\mathbf{0},\mathbf{0})}\mathbf{H}_{\varepsilon}\right\}$$

$$+o\left(\left\|\left(\mathbf{x},\mathbf{u}_{\varepsilon},\mathbf{H}_{\varepsilon}\right)-\left(\mathbf{X},\mathbf{0},\mathbf{0}\right)\right\|\right)=\varepsilon\mathbf{t}_{1}\left(\mathbf{X},\mathbf{0},\mathbf{0}\right)$$

$$+\varepsilon^{2}\left\{\left[\frac{\partial\mathbf{t}_{1}}{\partial\mathbf{x}}\right]_{(\mathbf{X},\mathbf{0},\mathbf{0})}\mathbf{u}_{1}\left(\mathbf{X}\right)+\left[\frac{\partial\mathbf{t}_{1}}{\partial\mathbf{u}_{\varepsilon}}\right]_{(\mathbf{X},\mathbf{0},\mathbf{0})}\mathbf{u}_{1}\left(\mathbf{X}\right)+\left[\frac{\partial\mathbf{t}_{1}}{\partial\mathbf{H}_{\varepsilon}}\right]_{(\mathbf{X},\mathbf{0},\mathbf{0})}\mathbf{H}_{1}\left(\mathbf{X}\right)\right\}+o\left(\varepsilon^{2}\right).$$

Thus

$$\mathbf{t}_{*\varepsilon}\left(\mathbf{X},\mathbf{u}_{\varepsilon},\mathbf{H}_{\varepsilon}\right)=\varepsilon\mathbf{t}_{1}\left(\mathbf{X},\mathbf{0},\mathbf{0}\right)+o\left(\varepsilon\right)=\varepsilon\tilde{\mathbf{t}}_{1}\left(\mathbf{X}\right)+o\left(\varepsilon\right). \tag{2.78}$$

Similarly, for the body force we have

$$\mathbf{b}_{\varepsilon}\left(\mathbf{X},\mathbf{u}_{\varepsilon}\right)=\mathbf{b}_{\varepsilon}\left(\mathbf{X},\mathbf{0}\right)+\left[\frac{\partial\mathbf{b}_{\varepsilon}}{\partial\mathbf{u}_{\varepsilon}}\right]_{(\mathbf{X},\mathbf{0})}\mathbf{u}_{\varepsilon}+o\left(\left\|\left(\mathbf{X},\mathbf{u}_{\varepsilon}\right)-\left(\mathbf{X},\mathbf{0}\right)\right\|\right), \tag{2.79}$$

$$=\varepsilon\mathbf{b}_{1}\left(\mathbf{X},\mathbf{0}\right)+\varepsilon^{2}\left[\frac{\partial\mathbf{b}_{1}}{\partial\mathbf{u}_{\varepsilon}}\right]_{(\mathbf{X},\mathbf{0})}\mathbf{u}_{1}+o\left(\varepsilon^{2}\right)=\varepsilon\mathbf{b}_{1}\left(\mathbf{X},\mathbf{0}\right)+o\left(\varepsilon\right)=\varepsilon\tilde{\mathbf{b}}_{1}\left(\mathbf{X}\right)+o\left(\varepsilon\right).$$

We conclude from (2.78) and (2.79) that in Linear Elasticity only dead loads can be considered.

From (2.46), (2.77), (2.78), (2.79) and (2.61), we conclude that

$$\varepsilon Div\left(\mathbb{C}\mathbf{E}_1\right) + \varepsilon\rho_*\tilde{\mathbf{b}}_1\left(\mathbf{X}\right) + o(\varepsilon) = \mathbf{0}, \qquad \forall\mathbf{X}\in\mathfrak{C}_*,$$

$$\varepsilon\mathbf{u}_1\left(\mathbf{X}\right) + o(\varepsilon) = \varepsilon\tilde{\mathbf{u}}_1\left(\mathbf{X}\right), \qquad \forall\mathbf{X}\in\partial\mathfrak{C}'_*,$$

$$\varepsilon\left(\mathbb{C}\mathbf{E}_1\right)\mathbf{N}_* + o(\varepsilon) = \varepsilon\tilde{\mathbf{t}}_1\left(\mathbf{X}\right), \qquad \forall\mathbf{X}\in\partial\mathfrak{C}''_*.$$

We divide by ε, take the limit as $\varepsilon\to 0$, and arrive at the following equations governing static deformations of a linear elastic body that is stress-free in the reference configuration.

$$\begin{aligned} &Div\left(\mathbb{C}\mathbf{E}_1\right) + \rho_*\tilde{\mathbf{b}}_1 = \mathbf{0}, && \forall\mathbf{X}\in\mathfrak{C}_*,\\ &\mathbf{u}_1 = \tilde{\mathbf{u}}_1, && \forall\mathbf{X}\in\partial\mathfrak{C}'_*,\\ &\left(\mathbb{C}\mathbf{E}_1\right)\mathbf{N}_* = \tilde{\mathbf{t}}_1, && \forall\mathbf{X}\in\partial\mathfrak{C}''_*. \end{aligned} \tag{2.80}$$

For a traction-value problem, loads must satisfy the following equations for the problem to have a solution.

$$\begin{aligned} &\int_{\partial\mathfrak{C}_*}\tilde{\mathbf{t}}_1 dS^* + \int_{\mathfrak{C}_*}\rho_*\tilde{\mathbf{b}}_1 dc_* = \mathbf{0},\\ &\int_{\partial\mathfrak{C}_*}\left(\mathbf{X}-\mathbf{X}_O\right)\times\tilde{\mathbf{t}}_1 dS^* + \int_{\mathfrak{C}_*}\left(\mathbf{X}-\mathbf{X}_O\right)\times\rho_*\tilde{\mathbf{b}}_1 dc_* = \mathbf{0}. \end{aligned} \tag{2.81}$$

Signorini's Method

Non-linear problems that satisfy the following conditions can be solved by the Signorini method.

1. The first Piola-Kirchhoff stress tensor $\mathbf{T}_*$ depends analytically upon the

displacement field $\mathbf{H}$:

$$\mathbf{T}_* = \mathbf{A}(\mathbf{H}) = \sum_{n=1}^{\infty} \mathbf{A}^{(n)}(\mathbf{H}), \quad \mathbf{A}(\mathbf{0}) = \mathbf{0},$$

where $\mathbf{A}_n(\mathbf{H})$ is a homogeneous polynomial of degree n in $\mathbf{H}$. In component form,

$$\begin{aligned} T_{*iL} &= A_{iL}(\mathbf{H}) = A_{iL}^{(1)}(\mathbf{H}) + A_{iL}^{(2)}(\mathbf{H}) + A_{iL}^{(3)}(\mathbf{H}) + ..., \\ &= \mathbb{C}_{ijLM}^{(1)} H_{jM} + \mathbb{C}_{ijhLMN}^{(2)} H_{jM} H_{hN} + \mathbb{C}_{ijhkLMNO}^{(3)} H_{jM} H_{hN} H_{kO} + ... \quad . \end{aligned} \tag{2.82}$$

2. The surface traction is a known function of the position $\mathbf{X}$ in the reference configuration, so the loads are dead:

$$\mathbf{t}_* = \mathbf{t}_*(\mathbf{X}).$$

3. There exists a non-dimensional parameter ε such that loads $(\mathbf{b}, \mathbf{t}_*)$ are analytical functions of ε. That is,

$$\mathbf{b}(\varepsilon, \mathbf{X}) = \sum_{n=1}^{\infty} \varepsilon^n \mathbf{b}_n(\mathbf{X}) \quad , \quad \mathbf{t}_*(\varepsilon, \mathbf{X}) = \sum_{n=1}^{\infty} \varepsilon^n \mathbf{t}_{*n}(\mathbf{X}).$$

Under these hypotheses, Poincarre‘, as cited by Romano [35], has demonstrated that the displacement field is an analytical function of ε and can be represented as

$$\mathbf{u}(\varepsilon, \mathbf{X}) = \sum_{n=1}^{\infty} \varepsilon^n \mathbf{u}_n(\mathbf{X}). \tag{2.83}$$

Thus

$$\mathbf{H} = \sum_{n=1}^{\infty} \varepsilon^n \mathbf{H}_{(n)}. \tag{2.84}$$

Substituting (2.84) into (2.82), we obtain

$$T_{*iL} = \mathbb{C}^{(1)}_{ijLM}\left(\varepsilon H_{(1)jM} + \varepsilon^2 H_{(2)jM} + ...\right)$$
$$+ \mathbb{C}^{(2)}_{ijhLMN}\left(\varepsilon H_{(1)jM} + \varepsilon^2 H_{(2)jM} + ...\right)\left(\varepsilon H_{(1)hN} + \varepsilon^2 H_{(2)hN} + ...\right) + ... \quad ,$$

which can be written as

$$\mathbf{T}_* = \sum_{n=1}^{\infty} \varepsilon^n \left(\mathbb{C}^{(1)}\mathbf{H}_{(n)} + \mathbf{B}_n\left(\mathbf{H}_1, ..., \mathbf{H}_{n-1}\right)\right), \tag{2.85}$$

where

$$\mathbf{B}_1 = \mathbf{0},$$

and

$$\mathbb{C}^{(1)} = \mathbb{C}$$

is the fourth-order tensor of linear elasticity. Because of the symmetry properties of this tensor, (2.85) can be written as

$$\mathbf{T}_* = \sum_{n=1}^{\infty} \varepsilon^n \left(\mathbb{C}\mathbf{E}_{(n)} + \mathbf{B}_n\left(\mathbf{H}_{(1)}, ..., \mathbf{H}_{(n-1)}\right)\right), \tag{2.86}$$

where

$$\mathbf{E}_{(n)} = Sym\left[\mathbf{H}_{(n)}\right] = \frac{1}{2}\left(\mathbf{H}_{(n)} + \mathbf{H}^T_{(n)}\right).$$

Substitution from (2.86) into $(2.46)_1$ and $(2.72)_{2,3}$ yields the following equations for a mixed boundary value problem.

$$\begin{aligned} &Div\mathbb{C}\mathbf{E}_n + \rho_*\hat{\mathbf{b}}_n = 0, && \forall\mathbf{X} \in \mathfrak{C}_*, \\ &\mathbf{u}_n = \bar{\mathbf{u}}_n, && \forall\mathbf{X} \in \partial\mathfrak{C}'_*, \\ &\left(\mathbb{C}\mathbf{E}_n\right)\mathbf{N}_* = \hat{\mathbf{t}}_{*n}, && \forall\mathbf{X} \in \partial\mathfrak{C}''_*, \end{aligned} \tag{2.87}$$

where

$$\rho_*\hat{\mathbf{b}}_n \equiv \rho_*\mathbf{b}_n + Div\mathbf{B}_n\left(\mathbf{H}_{(1)}, ..., \mathbf{H}_{(n-1)}\right),$$
$$\hat{\mathbf{t}}_{*n} = \mathbf{t}_{*n} - \mathbf{B}_n\left(\mathbf{H}_{(1)}, ..., \mathbf{H}_{(n-1)}\right)\mathbf{N}_*. \tag{2.88}$$

For $n = 1$, we have a linear problem which can be solved. Then we can solve the linear problem for $n = 2$. Note that loads for this linear problem depend upon the solution of the problem for $n = 1$. We can continue this procedure for $n = 3, 4, ...$. In order for the n-th order problem to have a solution, loads (2.88) must satisfy the overall balance of forces and moments given by (2.74).

There are two ways to accomplish this. We first note that the loads are dead. Thus $(2.74)_2$ is a compatibility condition because of the presence of $\mathbf{x}$ in it. Equation $(2.74)_1$ can be considered as *a priori* restriction on the loads. We can either use Da Silva's theorem, as done by Romano [35], or follow Green and Adkins [21] and set at $\mathbf{X} = \mathbf{0}$:

$$\mathbf{u} = \mathbf{0}, \qquad \mathbf{H} = \mathbf{H}^T. \tag{2.89}$$

Green and Adkins have demonstrated that (2.89) can be substituted for the compatibility condition (2.74).

Semi-inverse method

Techniques to solve a given boundary-value problem can be classified into the following three categories.

1. Direct method.

2. Inverse method.

3. Semi-inverse method.

In the direct method, governing equations are integrated without any *a priori* assumptions on the displacement field, and the constants of integration are determined from the boundary conditions. In the other two methods, either all or a part of the solution is assumed and loads required to produce the solution are computed. If the computed loads equal the prescribed ones, we have a solution of the given boundary- value problem; otherwise, we need to redo the problem. In the semi-inverse method, a part of solution is presumed and unknowns in it are determined by satisfying the pertinent field equations and boundary conditions. Each of these three methods is suitable for analyzing a certain class of problems.

We often invoke the St.-Venant principle to find an approximate solution of a traction-value problem; it may be stated as follows.

If on a given portion of a body or on a part of its surface, small with respect to a typical dimension of the body, act respectively a system of body or contact forces and the body is in equilibrium, then in the region that is far from that where forces act, the deformation and the state of stress are determined from the resultant forces and the resultant moment of these forces. This means that the details of the distribution of these forces are important only in the neighborhood of the region of application.

This principle was enunciated by Saint Venant for a cylindrically-body. Boussinesq extended it to the general three dimensional case.

Mathematically precise versions of this principle have been given by Knowles and Sternberg [31], Toupin [37] and others; Batra ([6], [7] and [8]) proved it even for a helical body ([5] and [9]). Estimates of the rate of decay of the solution with the distance from the loaded surface have been given by Horgan ([24] and [26]).

This principle is important in applications because it enables us to substitute a particular distribution of forces by one statically equivalent to it and simpler to use.

We will use the St.-Venant principle in solving the St.-Venant problem in the sense that displacement fields within a cylindrical body loaded only at the end faces will be expressed in terms of the resultant of these loads. In the solution of the problem, we will assume that the displacement field is a polynomial in the axial co-ordinate.

Chapter 3

St.-Venant's Problem

Governing equations

We analyze infinitesimal deformations of a homogeneous elastic cylindrical body like the one shown in Fig. 3.1. The cross-section of the body is arbitrary except that we require it to be smooth enough so that the divergence theorem in the plane is applicable.

We point out that the Euclidean space $\mathcal{E}$ is not vectorial, so we introduce a fixed point $O \in \mathcal{E}$ and measure positions of points in $\mathcal{E}$ relative to O; the point O is called the origin. Thus points become vectors belonging to the space of translations $\mathfrak{U} \equiv V\mathcal{E}$. We introduce the unit vector $\mathbf{e} \in \mathfrak{U}$ to characterize the direction of generators of a cylinder; the axis $\mathcal{W}$ of the cylinder is defined as follows[1]:

$$\mathcal{W} \equiv span\,\{\mathbf{e}\}\,. \tag{3.1}$$

The vector $\mathbf{e}$ is called the axis of cylinder. The sections $\mathfrak{D}$ of the cylinder

[1] The *span*

$$S \equiv span\,\{v_1, v_2, ..., v_n\}$$

of a set of vectors $\{v_1, v_2, ..., v_n\}$ is the vectorial space of all linear combinations of vectors $v_1, v_2, ..., v_n$. That is,

$$a \in S \Leftrightarrow \exists\,(a_1, a_2, ..., a_n) \in \mathbb{R}^n \diagup a = a_i v_i.$$

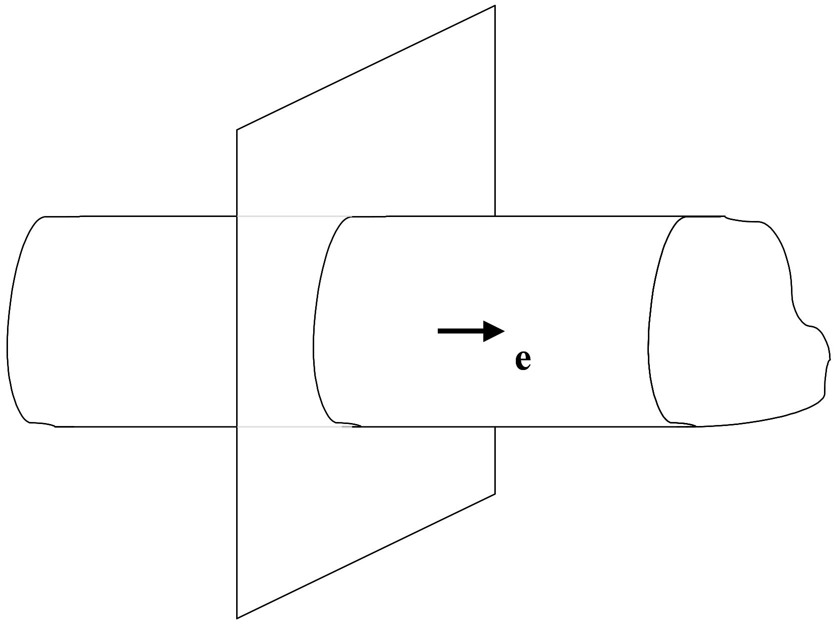

Figure 3.1: A cylindrical body

belong to the space orthogonal to $\mathcal{W}$ that is denoted by $\mathcal{V}$:

$$\mathfrak{D} \subset \mathcal{V} \equiv \mathcal{W}^{\perp} = \{\mathbf{e}\}^{\perp}. \tag{3.2}$$

So the space of translation can be written as[2]

$$\mathfrak{U} = \mathcal{W} \perp \mathcal{V}. \tag{3.3}$$

$\mathfrak{D}$ represents all sections of the cylinder. Often it is necessary to specify only one particular section which can be done by giving the distance of the section from the origin O. The section distant Z from O is denoted by $\mathfrak{D} \times \{Z\}$, or $\mathfrak{D}_Z$. To simplify the notation we locate the origin in the bottom section of the cylinder.

[2] $\perp$ is the direct sum $\oplus$ when the vectorial spaces being summed are orthogonal.

We can define the cylinder as

$$\mathfrak{C}_* \equiv \left\{ \mathbf{X} = \hat{\mathbf{X}} + Z\mathbf{e} \quad : \quad \hat{\mathbf{X}} \in \mathfrak{D} \quad , \quad Z \in [0, L] \right\}, \tag{3.4}$$

and write

$$\begin{aligned} &\mathfrak{C}_* = \mathfrak{D} \times [0, L], \\ &\partial \mathfrak{C}_* = (\partial \mathfrak{D} \times [0, L]) \cup (\mathfrak{D} \times \{0\}) \cup (\mathfrak{D} \times \{L\}) \, . \end{aligned} \tag{3.5}$$

We require that $\mathfrak{D}$ is closed, linearly connected and

$$\mathfrak{D} = \overline{(\mathfrak{D}^0) \setminus \left(\bigcup_{\lambda=1}^{\Lambda} \mathfrak{D}^\lambda \right)},$$

where [3] $\mathfrak{D}^0$ is the smallest simply connected set that contains $\mathfrak{D}$,

$$\begin{cases} \forall \lambda \in \{1, ..., \Lambda\}, \quad \mathfrak{D}^\lambda \subset \{(\mathfrak{D}^0) \setminus (\partial \mathfrak{D}^0)\}, \\ \forall \lambda, \mu \in \{1, ..., \Lambda\} \quad \textit{and} \quad \lambda \neq \mu, \quad \mathfrak{D}^\lambda \cap \mathfrak{D}^\mu = \varnothing. \end{cases} \tag{3.6}$$

Thus a cross-section can have at most Λ simply connected holes.

The boundary of $\mathfrak{D}$ is a one-dimensional set, its connected parts are the external boundary $\partial \mathfrak{D}^0$ and the boundaries of the holes $\partial \mathfrak{D}^\lambda$; so we write

$$\partial \mathfrak{D} = \bigcup_{\lambda=0}^{\Lambda} \partial \mathfrak{D}^\lambda. \tag{3.7}$$

In order to evaluate a line integral it is important to fix the orientation of the boundary $\partial \mathfrak{D}$. Let $\mathbf{e}_t$ be a unit tangent vector of $\partial \mathfrak{D}$ that is coherent with the orientation, and the vector $\mathbf{N}$ be an outward unit normal. The orientation of $\partial \mathfrak{D}$ is taken such that $(\mathbf{e}_t, -\mathbf{N}, \mathbf{e})$ follows the right-hand rule. Hence the orientation of $\partial \mathfrak{D}^0$ is anticlockwise and that of $\partial \mathfrak{D}^\lambda$ ($\lambda = 1, ..., \Lambda$) clockwise; these are shown in Fig. 3.2.

[3] $\setminus$ is the usual symmetric difference.

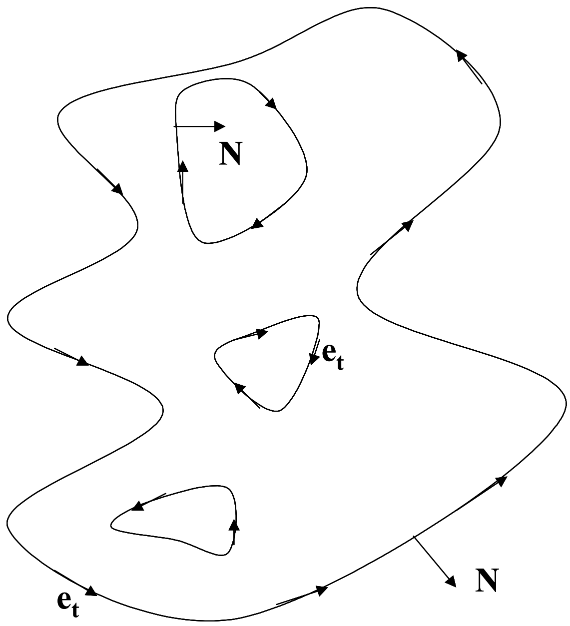

Figure 3.2: Positive direction of paths on the boundary and direction of the outward normal

The St.-Venant problem involves finding the displacement field in a cylindrical body loaded only at the end faces $\mathfrak{D}_0$ and $\mathfrak{D}_L$. We first consider infinitesimal deformations of a stress-free reference configuration, and assume that the body is made of a homogeneous and isotropic linear elastic material that obeys the constitutive equation

$$\mathbf{T}_* = \lambda(tr\mathbf{E})\mathbf{1} + 2\mu\mathbf{E}, \qquad \forall\mathbf{X} \in \mathfrak{C}_*. \tag{3.8}$$

Let

$$\begin{aligned} &\mathbf{u} = w\mathbf{e} + \mathbf{v}, && \forall\mathbf{X} \in \mathfrak{C}_*, \\ &\mathbf{T}_* = \sigma\mathbf{e}\otimes\mathbf{e} + \mathbf{t}\otimes\mathbf{e} + \mathbf{e}\otimes\tilde{\mathbf{t}} + \hat{\mathbf{T}}, && \forall\mathbf{X} \in \mathfrak{C}_*, \\ &\mathbf{E} = SymGrad\mathbf{u} = \varepsilon\mathbf{e}\otimes\mathbf{e} + \boldsymbol{\gamma}\otimes\mathbf{e} + \mathbf{e}\otimes\boldsymbol{\gamma} + \hat{\mathbf{E}}, && \forall\mathbf{X} \in \mathfrak{C}_*. \end{aligned} \tag{3.9}$$

Because of (3.8)

$$\tilde{\mathbf{t}} = \mathbf{t}, \qquad \forall\mathbf{X} \in \mathfrak{C}_*.$$

Here w and $\mathbf{v}$ are, respectively, the axial and the in-plane components of the displacement field $\mathbf{u}$; σ and $\mathbf{t}$ are respectively the axial and the transverse shear components of the first Piola-Kirchhoff stress tensor $\mathbf{T}_*$; $\hat{\mathbf{T}}$ is the in-plane part of $\mathbf{T}_*$; ε and $\boldsymbol{\gamma}$ are, respectively, the axial and the transverse shear components of the infinitesimal strain tensor $\mathbf{E}$; $\hat{\mathbf{E}}$ is the in-plane part of $\mathbf{E}$. This decomposition can be represented in the matrix notation as

$$\mathbf{T}_* = \begin{pmatrix} \hat{\mathbf{T}} & \mathbf{t} \\ \mathbf{t} & \sigma \end{pmatrix}, \qquad \mathbf{E} = \begin{pmatrix} \hat{\mathbf{E}} & \boldsymbol{\gamma} \\ \boldsymbol{\gamma} & \varepsilon \end{pmatrix}.$$

In rectangular Cartesian coordinates with X_1 and X_2 axes in $\mathfrak{D}_0$, we have

$$\mathbf{T}_* = \begin{pmatrix} \hat{T}_{11} & \hat{T}_{12} & t_1 \\ \hat{T}_{12} & \hat{T}_{22} & t_2 \\ t_1 & t_2 & \sigma \end{pmatrix}, \qquad \mathbf{E} = \begin{pmatrix} \hat{E}_{11} & \hat{E}_{12} & \gamma_1 \\ \hat{E}_{12} & \hat{E}_{22} & \gamma_2 \\ \gamma_1 & \gamma_2 & \varepsilon \end{pmatrix}.$$

The components of the infinitesimal strain tensor are related to the displacement field by

$$\begin{aligned} &\varepsilon = w', \\ &\boldsymbol{\gamma} = \tfrac{\mathbf{1}}{\mathbf{2}}\left(\mathbf{v}' + grad w\right), \\ &\hat{\mathbf{E}} = Symgrad\mathbf{v}, \end{aligned} \tag{3.10}$$

where a prime denotes differentiation with respect to Z, and $grad$ is the 2-dimensional gradient operator. These can be easily verified by noting that

$$Grad\mathbf{u} = w'\mathbf{e} \otimes \mathbf{e} + \mathbf{v}' \otimes \mathbf{e} + \mathbf{e} \otimes (grad w) + grad\mathbf{v}, \qquad \forall \mathbf{X} \in \mathfrak{C}_*.$$

Thus

$$\begin{aligned} \mathbf{E} = SymGrad\mathbf{u} = w'\mathbf{e} \otimes \mathbf{e} + \mathbf{e} \otimes \frac{\mathbf{1}}{\mathbf{2}}\left(\mathbf{v}' + grad w\right) \\ + \frac{\mathbf{1}}{\mathbf{2}}\left(\mathbf{v}' + grad w\right) \otimes \mathbf{e} + Symgrad\mathbf{v}, \qquad \forall \mathbf{X} \in \mathfrak{C}_*. \end{aligned} \tag{3.11}$$

A comparison of (3.9) and (3.11) gives (3.10). Equations (3.10) and (3.11) give

$$\begin{cases} tr\mathbf{E} = tr\hat{\mathbf{E}} + \varepsilon, \\ tr\hat{\mathbf{E}} = tr\left(Symgrad\mathbf{v}\right) = div\mathbf{v}, \\ tr\mathbf{E} = w' + div\mathbf{v}. \end{cases} \tag{3.12}$$

Substitution from $(3.9)_2$ and $(3.9)_3$ into (3.8) yields

$$\begin{cases} \sigma = (\lambda + 2\mu)\, w' + \lambda div\mathbf{v}, \\ \mathbf{t} = 2\mu\boldsymbol{\gamma} = \mu\,(\mathbf{v}' + gradw)\,, \\ \hat{\mathbf{T}} = \lambda\,(w' + div\mathbf{v})\,\hat{\mathbf{1}} + \mu\left[(grad\mathbf{v}) + (grad\mathbf{v})^T\right]. \end{cases} \tag{3.13}$$

From (6.18) and $(3.9)_2$ we have

$$Div\mathbf{T}_* = (\sigma' + div\mathbf{t})\,\mathbf{e} + \mathbf{t}' + div\hat{\mathbf{T}}.$$

Therefore the balance of linear momentum $(2.46)_1$ in the absence of body forces can be written as

$$\begin{cases} \sigma' + div\mathbf{t} = 0, & \forall\mathbf{X} \in \mathfrak{C}_*, \\ \mathbf{t}' + div\hat{\mathbf{T}} = \mathbf{0}, & \forall\mathbf{X} \in \mathfrak{C}_*. \end{cases} \tag{3.14}$$

We now substitute from (3.13) into (3.14) to obtain the following field equations for the determination of displacements $\mathbf{v}$ and w.

$$\begin{cases} (\lambda + 2\mu)w'' + (\lambda + \mu)\, div\mathbf{v}' + \mu \triangle_R w = 0, & \forall\mathbf{X} \in \mathfrak{C}_*, \\ div\left(2\mu Symgrad\mathbf{v} + \lambda\hat{\mathbf{1}}div\mathbf{v}\right) + (\lambda + \mu)\, gradw' + \mu\mathbf{v}' = \mathbf{0}, & \forall\mathbf{X} \in \mathfrak{C}_*, \end{cases} \tag{3.15}$$

or

$$\begin{cases} (\lambda + 2\mu)w'' + (\lambda + \mu)\, div\mathbf{v}' + \mu \triangle_R w = 0, & \forall\mathbf{X} \in \mathfrak{C}_*, \\ \mu\mathbf{v}' + (\lambda + \mu)\, gradw' + \mu \triangle_R \mathbf{v} + (\lambda + \mu)\, graddiv\mathbf{v} = \mathbf{0}, & \forall\mathbf{X} \in \mathfrak{C}_*. \end{cases}$$

The condition of zero surface tractions on the mantle can be written as

$$\begin{aligned} \hat{\mathbf{T}}\mathbf{N} = \mathbf{0}, &\qquad \forall\mathbf{X} \in \partial\mathfrak{D} \times [0, L]\,, \\ \mathbf{t} \cdot \mathbf{N} = 0, &\qquad \forall\mathbf{X} \in \partial\mathfrak{D} \times [0, L]\,. \end{aligned} \tag{3.16}$$

In terms of the displacement field, it becomes

$$\begin{aligned} &\lambda(w' + div\mathbf{v})\hat{\mathbf{1}}\mathbf{N} \\ &\quad +\mu((grad\mathbf{v}) + (grad\mathbf{v})^T)\mathbf{N} = \mathbf{0}, \qquad \forall \mathbf{X} \in \partial\mathfrak{D} \times [0, L]\,, \\ &(\mathbf{v}' + gradw) \cdot \mathbf{N} = 0, \qquad \forall \mathbf{X} \in \partial\mathfrak{D} \times [0, L]\,. \end{aligned} \tag{3.17}$$

In the semi-inverse method used to solve the problem, the loads on the end faces are not specified but are determined by the solution. However, they must satisfy the global equilibrium equations:

$$\begin{aligned} \int_{\partial\mathfrak{C}} \mathbf{T}\mathbf{n} dS = \mathbf{0}, \\ \int_{\partial\mathfrak{C}} \mathbf{x} \times \mathbf{T}\mathbf{n} dS = \mathbf{0}. \end{aligned} \tag{3.18}$$

We now consider the equilibrium of the portion $\mathfrak{B}_z$ of the cylinder $\mathfrak{C}$ that is the image in the present configuration of the portion

$$\mathfrak{B}_{*Z} = \left\{ \mathbf{X} = \left(\hat{\mathbf{X}}, \tilde{Z} \right) \in \mathfrak{C}_* \quad / \quad \forall \hat{\mathbf{X}} \in \mathfrak{D} \quad , \quad \forall \tilde{Z} \in [0, Z] \right\}$$

of the cylinder $\mathfrak{C}_*$ in the reference configuration between the bases $\mathfrak{D}(0)$ and $\mathfrak{D}(Z)$.

Global equilibrium equations for $\mathfrak{B}_{*Z}$ are

$$\begin{aligned} \int_{\mathfrak{B}_{*Z}} \mathbf{T}_* \mathbf{N} dS_* = \mathbf{0}, \\ \int_{\mathfrak{B}_{*Z}} (\mathbf{X} + \mathbf{u}) \times \mathbf{T}_* \mathbf{N} dS_* = \mathbf{0}. \end{aligned}$$

In the linear theory, these become

$$\begin{aligned} \int_{\mathfrak{B}_{*Z}} \mathbf{T}_* \mathbf{N} dS_* = \mathbf{0}, \\ \int_{\mathfrak{B}_{*Z}} \mathbf{X} \times \mathbf{T}_* \mathbf{N} dS_* = \mathbf{0}. \end{aligned} \tag{3.19}$$

Let

$$\begin{cases} \mathbf{f}(Z) \equiv \int_{\mathfrak{D}(Z)} \mathbf{T}_* \mathbf{e} dS_*, \\ \mathbf{m}(Z) \equiv \int_{\mathfrak{D}(Z)} \hat{\mathbf{X}} \times \mathbf{T}_* \mathbf{e} dS_*, \end{cases} \tag{3.20}$$

be the resultant force and the resultant moment with respect to $\left(\hat{\mathbf{O}}, Z\right)$ of forces acting on the section $\mathfrak{D}_Z$.

We set

$$\begin{aligned} \mathbf{f}(Z) &= \mathbf{f}_0 = \mathbf{f}_0^p + f_0^a \mathbf{e}, && \forall Z \in [0, L], \\ \mathbf{m}(Z) &= \mathbf{m}_0 + Z\mathbf{m}_1 = \mathbf{m}^p(Z) + m^a \mathbf{e}, && \forall Z \in [0, L], \end{aligned} \tag{3.21}$$

where

$$\begin{aligned} \mathbf{m}^p(Z) &= \mathbf{m}_0^p + Z\mathbf{m}_1, \\ \mathbf{m}_1 &= - * \mathbf{f}_0^p, \\ m^a &= m_0^a, \end{aligned} \tag{3.22}$$

and

$$\begin{aligned} \mathbf{f}_0^p &= \int_{\mathfrak{D}(0)} \mathbf{t} dS_*, \\ f_0^a &= \int_{\mathfrak{D}(0)} \sigma dS_*, \\ \mathbf{m}_0^p &= -\int_{\mathfrak{D}(0)} \left(*\hat{\mathbf{X}}\right) \sigma dS, \\ m_0^a &= \left(\int_{\mathfrak{D}(0)} \hat{\mathbf{X}} \times \mathbf{t} dS\right) \cdot \mathbf{e}. \end{aligned} \tag{3.23}$$

In order to prove (3.21), we recall that

$$\begin{aligned} \mathbf{N} &= -\mathbf{e}, && \forall \mathbf{X} \in \mathfrak{D} \times \{0\}, \\ \mathbf{N} &= \mathbf{e}, && \forall \mathbf{X} \in \mathfrak{D} \times \{L\}. \end{aligned} \tag{3.24}$$

From $(3.9)_2$, we get

$$\mathbf{T}_* \mathbf{e} = \mathbf{t} + \sigma \mathbf{e}, \qquad \forall \mathbf{X} \in \mathfrak{D}.$$

Since $\mathfrak{B}_{*Z}$ is in equilibrium and the mantle is traction free, therefore

$$\int_{\mathfrak{D}(Z)} \mathbf{T}_* \mathbf{e} dS_* = \int_{\mathfrak{D}(0)} \mathbf{T}_* \mathbf{e} dS_*,$$

and $\forall Z \in [0, L]$

$$\mathbf{f}\left(Z\right) = \int_{\mathfrak{D}(Z)} \mathbf{T}_* \mathbf{e} dS_* = \int_{\mathfrak{D}(0)} \mathbf{T}_* \mathbf{e} dS_* = \int_{\mathfrak{D}(0)} \mathbf{t} dS_* + \int_{\mathfrak{D}(0)} \sigma \mathbf{e} dS_*,$$

or

$$\mathbf{f}_0 = \mathbf{f}_0^p + f_0^a \mathbf{e}.$$

In the same way $(3.19)_2$ gives

$$-\int_{\mathfrak{D}(0)} \hat{\mathbf{X}} \times \mathbf{T}_* \mathbf{e} dS_* + \int_{\mathfrak{D}(Z)} \left(\hat{\mathbf{X}} + Z\mathbf{e}\right) \times \mathbf{T}_* \mathbf{e} dS_* = \mathbf{0},$$

which can be written as

$$\int_{\mathfrak{D}(Z)} \hat{\mathbf{X}} \times \mathbf{T}_* \mathbf{e} dS_* = \int_{\mathfrak{D}(0)} \hat{\mathbf{X}} \times \mathbf{T}_* \mathbf{e} dS_* - Z\mathbf{e} \times \int_{\mathfrak{D}(Z)} \mathbf{T}_* \mathbf{e} dS_*.$$

Thus

$$\begin{aligned} \mathbf{m}\left(Z\right) &= \int_{\mathfrak{D}(0)} \hat{\mathbf{X}} \times \left(\mathbf{t} + \sigma \mathbf{e}\right) dS_* - Z\mathbf{e} \times \int_{\mathfrak{D}(Z)} \mathbf{t} dS_*, \\ &= \int_{\mathfrak{D}(0)} \hat{\mathbf{X}} \times \mathbf{t} dS_* - \int_{\mathfrak{D}(0)} \left(*\hat{\mathbf{X}}\right) \sigma dS_* - Z * \mathbf{f}_0^p, \quad \forall Z \in [0, L], \end{aligned}$$

that is $(3.21)_2$.

Quantities $\mathbf{f}$ and $\mathbf{m}$ equal, respectively, the resultant force and the resultant moment that are applied to the basis. From the St.-Venant principle they represent ∞^6 equivalence classes of possible distribution of loads on the basis.

Substitution for stresses in terms of displacement gradient (recall the constitutive relations (3.13)) (3.21), (3.22) and (3.23) gives the following relationships between $\mathbf{f}_0^p, f_0^a, \mathbf{m}^p(z), m^a$ and the displacement field $\mathbf{v}, w$.

$$\begin{aligned}
\mathbf{f}_0^p &= \int_{\mathfrak{D}(Z)} \mu\left(\mathbf{v}'+gradw\right) dS_* = \int_{\mathfrak{D}(0)} \mu\left(\mathbf{v}'+gradw\right) dS_*, \\
f_0^a &= \int_{\mathfrak{D}(Z)} \left[(\lambda+2\mu) w' + \lambda div\mathbf{v}\right] dS_*, \\
&= \int_{\mathfrak{D}(0)} \left[(\lambda+2\mu) w' + \lambda div\mathbf{v}\right] dS_*, \\
\mathbf{m}^p(Z) &= \int_{\mathfrak{D}(Z)} \left(-\left((\lambda+2\mu) w' + \lambda div\mathbf{v}\right)\left(*\hat{\mathbf{X}}\right)\right) dS_*, \\
&= -\int_{\mathfrak{D}(0)} \left(*\hat{\mathbf{X}}\right)\left((\lambda+2\mu) w' + \lambda div\mathbf{v}\right) dS_* \\
&\quad -Z * \int_{\mathfrak{D}(0)} \mu\left(\mathbf{v}'+gradw\right) dS_*, \\
m^a &= \int_{\mathfrak{D}(Z)} \mu\left(\left(*\hat{\mathbf{X}}\right)\cdot\left(\mathbf{v}'+gradw\right)\right) dS_*, \\
&= \int_{\mathfrak{D}(0)} \mu\left(\left(*\hat{\mathbf{X}}\right)\cdot\left(\mathbf{v}'+gradw\right)\right) dS_*.
\end{aligned} \tag{3.25}$$

For the St.-Venant's problem, we list below all equations in terms of displacements:

$$\begin{aligned}
&\mu \triangle_R w + (\lambda+\mu) div\mathbf{v}' + (\lambda+2\mu)w'' = 0, && \forall \mathbf{X} \in \mathfrak{C}_*, \\
&div\left(2\mu Symgrad\mathbf{v}+\lambda\hat{\mathbf{1}}div\mathbf{v}\right) + (\lambda+\mu)\, gradw' + \mu\mathbf{v}'' = \mathbf{0}, && \forall \mathbf{X} \in \mathfrak{C}_*; \\
&\left(2\mu Symgrad\mathbf{v}+\lambda\hat{\mathbf{1}}div\mathbf{v}\right)\mathbf{N} = -\lambda(w')\mathbf{N}, && \forall \mathbf{X} \in \partial\mathfrak{D}\times[0,L], \\
&(gradw)\cdot\mathbf{N} = -\mathbf{v}'\cdot\mathbf{N}, && \forall \mathbf{X} \in \partial\mathfrak{D}\times[0,L]; \\
&\mathbf{f}_0^p = \int_{\mathfrak{D}(Z)} \mu\left(\mathbf{v}'+gradw\right) dS_*, \\
&f_0^a = \int_{\mathfrak{D}(Z)} \left[(\lambda+2\mu) w' + \lambda div\mathbf{v}\right] dS_*,
\end{aligned} \tag{3.26}$$

$$\mathbf{m}_0^p + Z\mathbf{m}_1 = \int_{\mathfrak{D}(Z)} \left(-\left((\lambda + 2\mu)\, w' + \lambda div\mathbf{v} \right) \left(*\hat{\mathbf{X}} \right) \right) dS_*,$$

$$m_0^a = \int_{\mathfrak{D}(Z)} \mu \left(\left(*\hat{\mathbf{X}} \right) \cdot (\mathbf{v}' + gradw) \right) dS_*.$$

Solution of St.-Venant's problem by a semi-inverse method

We seek a solution of equations (3.26) of the form

$$\begin{aligned} w\left(\mathbf{X}\right) &= \sum_{i=0}^{M} \frac{1}{i!} w_i \left(\hat{\mathbf{X}} \right) Z^i, \qquad \forall \mathbf{X} \in \mathfrak{C}_*, \\ \mathbf{v}\left(\mathbf{X}\right) &= \sum_{i=0}^{M} \frac{1}{i!} \mathbf{v}_i \left(\hat{\mathbf{X}} \right) Z^i, \qquad \forall \mathbf{X} \in \mathfrak{C}_*. \end{aligned} \tag{3.27}$$

With this hypothesis equations (3.26), in terms of the unknown functions $\mathbf{v}_i$ and w_i, become

$$\begin{aligned} &\mu \triangle_R w_i + (\lambda + \mu)\, div\mathbf{v}_{i+1} + (\lambda + 2\mu) w_{i+2} = 0, && \forall \hat{\mathbf{X}} \in \mathfrak{D}, \\ &div \left(2\mu Symgrad\mathbf{v}_i + \lambda \hat{\mathbf{1}} div\mathbf{v}_i \right) + (\lambda + \mu)\, gradw_{i+1} + \mu \mathbf{v}_{i+2} = \mathbf{0}, && \forall \hat{\mathbf{X}} \in \mathfrak{D}; \\ &\left(2\mu Symgrad\mathbf{v}_i + \lambda \hat{\mathbf{1}} div\mathbf{v}_i \right) \mathbf{N} = -\lambda (w_{i+1}) \mathbf{N}, && \forall \hat{\mathbf{X}} \in \partial\mathfrak{D}, \\ &gradw_i \cdot \mathbf{N} = -\mathbf{v}_{i+1} \cdot \mathbf{N}, && \forall \hat{\mathbf{X}} \in \partial\mathfrak{D}; \\ &\int_{\mathfrak{D}(Z)} \mu \left(\mathbf{v}_{i+1} + gradw_i \right) dS_* = \mathbf{0}, && i > 0, \\ &\int_{\mathfrak{D}(Z)} \left[(\lambda + 2\mu)\, w_{i+1} + \lambda div\mathbf{v}_i \right] dS_* = 0, && i > 0, \\ &\int_{\mathfrak{D}(Z)} \left(-\left((\lambda + 2\mu)\, w_{i+1} + \lambda div\mathbf{v}_i \right) \left(*\hat{\mathbf{X}} \right) \right) dS_* = 0, && i > 1, \\ &\int_{\mathfrak{D}(Z)} \mu \left(\left(*\hat{\mathbf{X}} \right) \cdot \left(\mathbf{v}_{i+1} + gradw_i \right) \right) dS_* = \mathbf{0}, && i > 0. \end{aligned} \tag{3.28}$$

Thus the three dimensional problem has been reduced to a sequence of two dimensional problems.

In terms of the boundary-value problems A, B, C and D given in the Appendix, the boundary-value problems (3.28) can be identified as follows.

For $i = M$, the boundary-value (b-v) problem defined by equations $(3.28)_{1,4}$ is problem A and it has the solution $w_M = w_M^0$. The b-v problem defined by equations $(3.28)_{2,3}$ is problem B whose solution is $\mathbf{v}_M = \mathbf{v}_M^0 + \varpi_M \left(*\hat{\mathbf{X}} \right)$, where ϖ_M is a constant. Here and below, quantities with superscript zero signify constants.

Similarly for $i = M-1$, equations $(3.28)_{1,4}$, $(3.28)_{2,3}$, $(3.28)_{5,8}$ and $(3.28)_{6,7}$ define respectively problems A, B, C and D of the Appendix and their solutions are

$$\begin{gathered} grad w_{M-1} = -\left[\mathbf{v}_M^0 + \varpi_M \left(*\hat{\mathbf{X}} \right) \right] + 2\varpi_M \mathbf{f}_M \left(\hat{\mathbf{X}} \right); \\ div \mathbf{v}_{M-1} = \frac{-\lambda w_M^0}{(\lambda + \mu)}; \end{gathered}$$

$$\varpi_M = 0, \qquad \mathbf{v}_M = \mathbf{v}_M^0 \qquad and \qquad w_{M-1} = w_{M-1}^0 - \mathbf{v}_M^0 \cdot \hat{\mathbf{X}};$$

$$w_M = 0 \qquad and \qquad \mathbf{v}_{M-1} = \mathbf{v}_{M-1}^0 + \varpi_{M-1} \left(*\hat{\mathbf{X}} \right).$$

Following the same procedure once more, we obtain

$$\begin{aligned} grad w_{M-2} &= -\left[\mathbf{v}_{M-1}^0 + \varpi_{M-1} \left(*\hat{\mathbf{X}} \right) \right] + 2\varpi_{M-1} \mathbf{f}_{M-1} \left(\hat{\mathbf{X}} \right); \\ div \mathbf{v}_{M-2} &= \frac{-\lambda \left(w_{M-1}^0 - \mathbf{v}_M^0 \cdot \hat{\mathbf{X}} \right)}{(\lambda + \mu)}; \end{aligned}$$

$$\varpi_{M-1} = 0, \qquad \mathbf{v}_{M-1} = \mathbf{v}_{M-1}^0 \qquad and \qquad w_{M-2} = w_{M-2}^0 - \mathbf{v}_{M-1}^0 \cdot \hat{\mathbf{X}};$$

$$w_{M-1} = 0, \qquad \mathbf{v}_{M-2} = \mathbf{v}_{M-2}^0 + \varpi_{M-2} \left(*\hat{\mathbf{X}} \right) \qquad and \qquad \mathbf{v}_M = \mathbf{0}.$$

We can apply the same reasoning until $i = 2$ and thus get

$$\mathbf{v}_{4+i} = \mathbf{0}, \qquad i = 0, ..., M-4;$$

$$w_{3+i} = 0, \qquad i = 0, ..., M-3;$$

$$\mathbf{v}_3 = \mathbf{v}_3^0,$$

$$\mathbf{v}_2 = \mathbf{v}_2^0 + \varpi_2 \left(*\hat{\mathbf{X}}\right),$$

$$w_2 = w_2^0 - \mathbf{v}_3^0 \cdot \hat{\mathbf{X}}.$$

For $i = 1$, equations $(3.28)_{1,4}$, $(3.28)_{2,3}$, $(3.28)_{5,8}$ and $(3.28)_{6,7}$ define respectively problems A, B, C and D_* of the Appendix and their solutions are

$$grad w_1 = -\left[\mathbf{v}_2^0 + \varpi_2 \left(*\hat{\mathbf{X}}\right)\right] + 2\varpi_2 \mathbf{f}_2 \left(\hat{\mathbf{X}}\right),$$

$$div \mathbf{v}_1 = \frac{-\lambda \left(w_2^0 - \mathbf{v}_3^0 \cdot \hat{\mathbf{X}}\right)}{(\lambda + \mu)},$$

$$\varpi_2 = 0, \qquad \mathbf{v}_2 = \mathbf{v}_2^0 \qquad and \qquad w_1 = w_1^0 - \mathbf{v}_2^0 \cdot \hat{\mathbf{X}},$$

$$w_2^0 = \mathbf{v}_3^0 \cdot \hat{\mathbf{b}}. \tag{3.29}$$

For $i = 0$, equations $(3.28)_{2,3}$ form problem B of the Appendix, and have the solution

$$div \mathbf{v}_0 = \frac{-\lambda \left(w_1^0 - \mathbf{v}_2^0 \cdot \hat{\mathbf{X}}\right)}{(\lambda + \mu)}.$$

Using (6.38) and (3.29) we get

$$\begin{aligned} \mathbf{v}_1 &= \mathbf{v}_1^0 + \left(*\varphi_1^0\right) \hat{\mathbf{X}} - \nu \left(\mathbf{v}_3^0 \cdot \hat{\mathbf{b}}\right) \hat{\mathbf{X}} + \nu \left\{ sym \left[\hat{\mathbf{X}} \otimes \left(*\hat{\mathbf{X}}\right)\right]\right\} \left(*\mathbf{v}_3^0\right), \quad \forall \mathbf{X} \in \mathfrak{D}; \\ \mathbf{v}_0 &= \mathbf{v}_0^0 + \left(*\varphi_0^0\right) \hat{\mathbf{X}} - \nu w_1^0 \hat{\mathbf{X}} + \nu \left\{ sym \left[\hat{\mathbf{X}} \otimes \left(*\hat{\mathbf{X}}\right)\right]\right\} \left(*\mathbf{v}_2^0\right), \quad \forall \mathbf{X} \in \mathfrak{D}. \end{aligned}$$

Since the divergence of $\mathbf{v}_1$ is not zero we cannot use the solution of problem A. Equations $(3.28)_{1,4}$ for $i = 0$ give

$$
\begin{aligned}
&\triangle_R w_0 + 2\mathbf{v}_3^0 \cdot \left(\hat{\mathbf{b}} - \hat{\mathbf{X}}\right) = 0, \qquad\qquad \forall \mathbf{X} \in \mathfrak{D};\\
&grad w_0 \cdot \mathbf{N} = -\mathbf{v}_1^0 \cdot \mathbf{N} + \left[(*\varphi_1^0)\,\hat{\mathbf{X}}\right] \cdot \mathbf{N} - \nu\left(\mathbf{v}_3^0 \cdot \hat{\mathbf{b}}\right)\left(\hat{\mathbf{X}} \cdot \mathbf{N}\right)\\
&\quad +\nu\left\{sym\left[\hat{\mathbf{X}} \otimes \left(*\hat{\mathbf{X}}\right)\right]\right\}(*\mathbf{v}_3^0) \cdot \mathbf{N}, \quad \forall \mathbf{X} \in \partial\mathfrak{D}.
\end{aligned}
\tag{3.30}
$$

A solution of equations (3.26) of the form (3.27) is[4]

$$
\begin{aligned}
&\mathbf{v}_{4+i} = \mathbf{0}, \qquad i = 0, ..., M-4;\\
&\mathbf{v}_3 = \mathbf{v}_3^0,\\
&\mathbf{v}_2 = \mathbf{v}_2^0,\\
&\mathbf{v}_1 = \mathbf{v}_1^0 + (*\varphi_1^0)\,\hat{\mathbf{X}} - \nu\left(\mathbf{v}_3^0 \cdot \hat{\mathbf{b}}\right)\hat{\mathbf{X}} + \nu\left\{sym\left[\hat{\mathbf{X}} \otimes \left(*\hat{\mathbf{X}}\right)\right]\right\}(*\mathbf{v}_3^0),\\
&\mathbf{v}_0 = \mathbf{v}_0^0 + (*\varphi_0^0)\,\hat{\mathbf{X}} - \nu w_1^0 \hat{\mathbf{X}} + \nu\left\{sym\left[\hat{\mathbf{X}} \otimes \left(*\hat{\mathbf{X}}\right)\right]\right\}(*\mathbf{v}_2^0),\\
&w_{3+i} = 0, \qquad i = 0, ..., M-3;\\
&w_2 = \mathbf{v}_3^0 \cdot \left(\hat{\mathbf{b}} - \hat{\mathbf{X}}\right),\\
&w_1 = w_1^0 - \mathbf{v}_2^0 \cdot \hat{\mathbf{X}},\\
&w_0 \quad \textit{is a solution of } (3.30).
\end{aligned}
\tag{3.31}
$$

Generalized Axisymmetric Problems

The only shape for the cross-section that is possible in this case is a circle with one circular hole. In terms of the cylindrical coordinates with the origin at the centroid of the cross-section, we can write

$$
\mathfrak{D} \equiv \left\{\hat{\mathbf{X}} = r\hat{\mathbf{r}} \quad : \quad r \in [a, b] \quad , \quad \theta \in [0, 2\pi]\right\}.
$$

[4] It can be seen that $\mathbf{v}_3^0$, $\mathbf{v}_2^0$, φ_1^0 and w_1^0 are proportional to the loads on the basis.

For a generalized axisymmetric problem, the displacement field does not depend upon θ. However the circumferential or the tangential displacement need not vanish. The solution (3.31) becomes

$$v_{(2+i)r} = 0, \qquad i = 0, ..., M-4;$$

$$v_{(2+i)\theta} = 0, \qquad i = 0, ..., M-4;$$

$$v_{1r} = 0,$$

$$v_{1\theta} = \varphi_1^0 r,$$

$$v_{0r} = -\nu w_1^0 r,$$

$$v_{0\theta} = \varphi_0^0 r,$$

$$w_{2+i} = 0, \qquad i = 0, ..., M-3;$$

$$w_1 = w_1^0.$$

The function w_0 satisfies

$$\begin{cases} \triangle_R w_0 = 0, & \forall \mathbf{X} \in \mathfrak{D}; \\ grad w_0 \cdot \mathbf{N} = \left[(*\varphi_1^0)\, \hat{\mathbf{X}} \right] \cdot \mathbf{N}, & \forall \mathbf{X} \in \partial\mathfrak{D}. \end{cases} \tag{3.32}$$

Equations (3.32) have the solution[5]:

$$grad w_0 = -\mathbf{v}_1^0 + \varphi_1^0 \left(*\hat{\mathbf{X}} \right) + 2\varphi_1^0 \mathbf{f} \left(\hat{\mathbf{X}} \right),$$

where

$$div \mathbf{f} = 0, \qquad \forall \mathbf{X} \in \mathfrak{D};$$

$$rot \mathbf{f} = 1, \qquad \forall \mathbf{X} \in \mathfrak{D};$$

$$\mathbf{f} \cdot \mathbf{N} = 0, \qquad \forall \mathbf{X} \in \partial\mathfrak{D}.$$

For a circular cross-section,

[5] See the problem A in the Appendix.

$$\triangle_R w_0 = 0, \qquad \forall \mathbf{X} \in \mathfrak{D};$$

$$grad w_0 \cdot \mathbf{N} = 0, \qquad \forall \mathbf{X} \in \partial\mathfrak{D};$$

and therefore

$$w_0 = w_0^0, \qquad \forall \mathbf{X} \in \mathfrak{D}.$$

Summarizing the preceding results, the solution of the generalized axisymmetric St.-Venant problem is

$$\mathbf{v}_i = 0, \qquad i = 2, ..., M;$$

$$v_{1r} = 0,$$

$$v_{1\theta} = \varphi_1^0 r,$$

$$v_{0r} = -\nu w_1^0 r,$$

$$v_{0\theta} = \varphi_0^0 r,$$

$$w_i = 0, \qquad i = 2, ..., M;$$

$$w_1 = w_1^0,$$

$$w_0 = w_0^0,$$

where it is possible to prove that φ_1^0 and w_1^0 are proportional respectively to the axial resultant moment and the axial resultant force on the basis.

Uniform pressure in the reference configuration

Formulation of the problem

We now study the case when the state of stress in the reference configuration is that of uniform pressure,

$$\mathbf{T}_0 = -p\mathbf{1}.$$

Since the body is in equilibrium in the reference configuration, therefore

$$\begin{aligned} Div\mathbf{T}_0 &= -Grad\,(p) = \mathbf{0}, \qquad \forall \mathbf{X} \in \mathfrak{C}_*; \\ \mathbf{T}_0\mathbf{N} &= -p\mathbf{N} = \bar{\mathbf{t}}_0, \qquad \forall \mathbf{X} \in \partial\mathfrak{C}_*. \end{aligned}$$

When no additional loads are applied on the mantle of the cylindrical body, the displacement $\mathbf{u}$ is governed by

$$\begin{aligned} Div\,(\mathbf{H}\mathbf{T}_0 + \mathbb{C}\mathbf{E}) = \mathbf{0}, \qquad \forall \mathbf{X} \in \mathfrak{C}_*; \\ (\mathbf{H}\mathbf{T}_0 + \mathbb{C}\mathbf{E})\,\mathbf{N} = \mathbf{0}, \qquad \forall \mathbf{X} \in \partial\mathfrak{D} \times [0, L]\,. \end{aligned} \tag{3.33}$$

We use the decompositions

$$\begin{aligned} \mathbf{T}_* &= \begin{pmatrix} \hat{\mathbf{T}} & \mathbf{t} \\ \tilde{\mathbf{t}} & \sigma \end{pmatrix}, \qquad \mathbf{E} = \begin{pmatrix} \hat{\mathbf{E}} & \gamma \\ \gamma & \varepsilon \end{pmatrix}, \qquad \mathbf{H} = \begin{pmatrix} grad\mathbf{v} & \mathbf{v}' \\ gradw & \varepsilon \end{pmatrix}, \\ \mathbf{T}_0 &= -p\begin{pmatrix} \hat{\mathbf{1}} & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix}, \qquad \mathbf{H}\mathbf{T}_0 = -p\begin{pmatrix} grad\mathbf{v} & \mathbf{v}' \\ gradw & w' \end{pmatrix}, \qquad \mathbf{e} = \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix}, \\ \mathbf{T}_*\mathbf{e} &= \mathbf{t} + \sigma\mathbf{e}, \end{aligned}$$

and note that the first Piola-Kirchhoff stress tensor is not symmetric because of the prestress. Since

$$\begin{aligned} \mathbb{C}\mathbf{E} &= \lambda(w' + div\mathbf{v})\begin{pmatrix} \hat{\mathbf{1}} & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix} + 2\mu\begin{pmatrix} symgrad\mathbf{v} & \gamma \\ \gamma & w' \end{pmatrix}, \\ &= \begin{pmatrix} \lambda(w' + div\mathbf{v})\hat{\mathbf{1}} + 2\mu symgrad\mathbf{v} & \mu\,(\mathbf{v}' + gradw) \\ \mu\,(\mathbf{v}' + gradw) & \lambda(w' + div\mathbf{v}) + 2\mu w' \end{pmatrix}, \end{aligned}$$

the components of the first Piola-Kirchhoff stress tensor are

$$\begin{aligned}
\hat{\mathbf{T}} &= -p\hat{\mathbf{1}} - p\, grad\mathbf{v} + \lambda(w' + div\mathbf{v})\hat{\mathbf{1}} + 2\mu sym grad\mathbf{v}, \\
\mathbf{t} &= -p\mathbf{v}' + \mu\left(\mathbf{v}' + grad w\right), \\
\tilde{\mathbf{t}} &= -p\, grad w + \mu\left(\mathbf{v}' + grad w\right), \\
\sigma &= -p - pw' + \lambda(w' + div\mathbf{v}) + 2\mu w'.
\end{aligned} \tag{3.34}$$

Thus

$$\begin{aligned}
\mathbf{HT}_0 + \mathbb{C}\mathbf{E} &= \begin{pmatrix} \hat{\mathbf{T}} + p\hat{\mathbf{1}} & \mathbf{t} \\ \tilde{\mathbf{t}} & \sigma + p \end{pmatrix}, \\
(\mathbf{HT}_0 + \mathbb{C}\mathbf{E})\,\mathbf{e} &= \mathbf{t} + (\sigma + p)\,\mathbf{e}.
\end{aligned} \tag{3.35}$$

Substitution from (3.34) and (3.35) into (3.33) yields

$$div\left\{-p\, grad\mathbf{v} + \lambda(w' + div\mathbf{v})\hat{\mathbf{1}} + 2\mu sym grad\mathbf{v}\right\} + \left[-p\mathbf{v}' + \mu\left(\mathbf{v}' + grad w\right)\right]' = \mathbf{0}, \quad \forall \mathbf{X} \in \mathfrak{C}_*;$$

$$div\left\{-p\, grad w + \mu\left(\mathbf{v}' + grad w\right)\right\} + \left[-pw' + \lambda(w' + div\mathbf{v}) + 2\mu w'\right]' = 0, \quad \forall \mathbf{X} \in \mathfrak{C}_*;$$

$$\left[-p\, grad\mathbf{v} + \lambda(w' + div\mathbf{v})\hat{\mathbf{1}} + 2\mu sym grad\mathbf{v}\right]\hat{\mathbf{N}} = \mathbf{0}, \qquad \forall \mathbf{X} \in \partial\mathfrak{D} \times [0, L];$$

$$\left[-p\, grad w + \mu\left(\mathbf{v}' + grad w\right)\right] \cdot \hat{\mathbf{N}} = 0, \qquad \forall \mathbf{X} \in \partial\mathfrak{D} \times [0, L].$$

Global balance of forces and moments

Subtracting the resultant of forces and moments of forces acting in the present configuration from those in the reference configuration, we obtain

$$\begin{aligned}
\int_{\mathfrak{B}_{*Z}} (\mathbf{HT}_0 + \mathbb{C}\mathbf{E})\,\mathbf{N} dS_* &= \mathbf{0}, \\
\int_{\mathfrak{B}_{*Z}} (\mathbf{X} + \mathbf{u}) \times (\mathbf{HT}_0 + \mathbb{C}\mathbf{E})\,\mathbf{N} dS_* &= \mathbf{0}.
\end{aligned} \tag{3.36}$$

With

$$\begin{aligned}\mathbf{f}(Z) &= \int_{\mathfrak{D}(Z)} (\mathbf{H}\mathbf{T}_0 + \mathbb{C}\mathbf{E})\,\mathbf{e}dS_*,\\ \mathbf{m}(Z) &= \int_{\mathfrak{D}(Z)} \hat{\mathbf{X}} \times (\mathbf{H}\mathbf{T}_0 + \mathbb{C}\mathbf{E})\,\mathbf{e}dS_*,\end{aligned}$$

we can show that $\quad \forall Z \in [0, L]$

$$\begin{gathered}\mathbf{f}(Z) = \mathbf{f}_0 = \mathbf{f}_0^p + f_0^a\mathbf{e},\\ \mathbf{m}(Z) = \mathbf{m}_0 + Z\mathbf{m}_1,\\ = \mathbf{m}^p(Z) + m_0^a\mathbf{e},\\ = \mathbf{m}_0^p + m_0^a\mathbf{e} - Z * \mathbf{f}_0^p,\end{gathered} \tag{3.37}$$

where

$$\begin{gathered}\mathbf{f}_0^p = \int_{\mathfrak{D}(0)} \mathbf{t}dS_*,\\ f_0^a = \int_{\mathfrak{D}(0)} (\sigma + p)\, dS_*,\\ \mathbf{m}_0^p = -\int_{\mathfrak{D}(0)} \left(*\hat{\mathbf{X}}\right)(\sigma + p)\, dS_*,\\ m_0^a = \left[\int_{\mathfrak{D}(0)} \hat{\mathbf{X}} \times \mathbf{t}dS_*\right] \cdot \mathbf{e}.\end{gathered} \tag{3.38}$$

In order to prove (3.37), we recall $(3.33)_2$ and note that

$$\int\limits_{\mathfrak{B}_{*Z}} (\mathbf{H}\mathbf{T}_0 + \mathbb{C}\mathbf{E})\,\mathbf{N}dS_* = -\int_{\mathfrak{D}(0)} (\mathbf{H}\mathbf{T}_0 + \mathbb{C}\mathbf{E})\,\mathbf{e}dS_* + \int_{\mathfrak{D}(Z)} (\mathbf{H}\mathbf{T}_0 + \mathbb{C}\mathbf{E})\,\mathbf{e}dS_* = \mathbf{0}.$$

Now using (3.35),

$$\begin{aligned}\mathbf{f}(Z) &= \int_{\mathfrak{D}(Z)} \mathbf{T}_*\mathbf{e}dS_* = \int_{\mathfrak{D}(0)} (\mathbf{H}\mathbf{T}_0 + \mathbb{C}\mathbf{E})\,\mathbf{e}dS_* = \int_{\mathfrak{D}(0)} \mathbf{t}dS_* + \int_{\mathfrak{D}(0)} (\sigma + p)\,\mathbf{e}dS_*,\\ &= \mathbf{f}_0^p + f_0^a\mathbf{e} = \mathbf{f}_0, \quad \forall Z \in [0, L].\end{aligned}$$

Neglecting terms in $\mathbf{u}$ of order greater than 1 and considering $(3.33)_2$, we conclude from $(3.36)_2$ that

$$\int_{\mathfrak{B}_{*Z}} (\mathbf{X}+\mathbf{u}) \times (\mathbf{H}\mathbf{T}_0 + \mathbb{C}\mathbf{E})\,\mathbf{N} dS_*$$

$$= -\int_{\mathfrak{D}(0)} \mathbf{X} \times (\mathbf{H}\mathbf{T}_0 + \mathbb{C}\mathbf{E})\,\mathbf{e} dS_* + \int_{\mathfrak{D}(Z)} \left(\hat{\mathbf{X}} + Z\mathbf{e}\right) \times (\mathbf{H}\mathbf{T}_0 + \mathbb{C}\mathbf{E})\,\mathbf{e} dS_* = \mathbf{0}.$$

Thus

$$\mathbf{m}(Z) = \int_{\mathfrak{D}(0)} \hat{\mathbf{X}} \times (\mathbf{H}\mathbf{T}_0 + \mathbb{C}\mathbf{E})\,\mathbf{e} dS_* - Z\mathbf{e} \times \int_{\mathfrak{D}(Z)} (\mathbf{H}\mathbf{T}_0 + \mathbb{C}\mathbf{E})\,\mathbf{e} dS_*,$$

$$\begin{aligned}
\mathbf{m}(Z) &= \int_{\mathfrak{D}(0)} \hat{\mathbf{X}} \times (\mathbf{t} + (\sigma + p)\,\mathbf{e})\, dS_* - Z*\mathbf{f}_0^p, \\
&= m_0^a \mathbf{e} + \mathbf{m}_0^p - Z*\mathbf{f}_0^p, \quad \forall Z \in [0, L].
\end{aligned}$$

Substituting in (3.38) for stresses in terms of displacement field (3.34) yields

$$\begin{aligned}
&\mathbf{f}_0^p = \int_{\mathfrak{D}(Z)} -p\mathbf{v}' + \mu\,(\mathbf{v}' + grad w)\, dS_* = \int_{\mathfrak{D}(0)} -p\mathbf{v}' + \mu\,(\mathbf{v}' + grad w)\, dS_*, \\
&f_0^a = \int_{\mathfrak{D}(Z)} [(\lambda + 2\mu - p)\, w' + \lambda div \mathbf{v}]\, dS_*, \\
&\quad = \int_{\mathfrak{D}(0)} [(\lambda + 2\mu - p)\, w' + \lambda div \mathbf{v}]\, dS_*, \\
&\mathbf{m}^p(Z) = -\int_{\mathfrak{D}(Z)} \left(*\hat{\mathbf{X}}\right) [(\lambda + 2\mu - p)\, w' + \lambda div \mathbf{v}]\, dS_*, \\
&\quad = -\int_{\mathfrak{D}(0)} \left(*\hat{\mathbf{X}}\right) [(\lambda + 2\mu - p)\, w' + \lambda div \mathbf{v}]\, dS_* \\
&\quad -Z * \int_{\mathfrak{D}(0)} [-p\mathbf{v}' + \mu\,(\mathbf{v}' + grad w)]\, dS_*, \\
&m^a = \mu \int_{\mathfrak{D}(Z)} \left(*\hat{\mathbf{X}}\right) \cdot [-p\mathbf{v}' + \mu\,(\mathbf{v}' + grad w)]\, dS_*, \\
&\quad = \mu \int_{\mathfrak{D}(0)} \left(*\hat{\mathbf{X}}\right) \cdot [-p\mathbf{v}' + \mu\,(\mathbf{v}' + grad w)]\, dS_*.
\end{aligned} \tag{3.39}$$

The final equations are

$$
\begin{aligned}
&(\mu-p)\bigtriangleup_R w+(\lambda+\mu)\,div\mathbf{v}' + (\lambda+2\mu-p)w''=0, && \forall\mathbf{X}\in\mathfrak{C}_*;\\
&div\left(2\mu Symgrad\mathbf{v}-pgrad\mathbf{v}+\lambda\hat{\mathbf{1}}div\mathbf{v}\right)+(\lambda+\mu)\,gradw' + (\mu-p)\,\mathbf{v}''=\mathbf{0}, && \forall\mathbf{X}\in\mathfrak{C}_*;\\
&\left(2\mu Symgrad\mathbf{v}-pgrad\mathbf{v}+\lambda\hat{\mathbf{1}}div\mathbf{v}\right)\hat{\mathbf{N}}=-\lambda w'\hat{\mathbf{N}}, && \forall\mathbf{X}\in\partial\mathfrak{D}\times[0,L]\,;\\
&(\mu-p)\,(gradw)\cdot\hat{\mathbf{N}}=-\mathbf{v}'\cdot\hat{\mathbf{N}}, && \forall\mathbf{X}\in\partial\mathfrak{D}\times[0,L]\,;\\
&\mathbf{f}_0^p=\int_{\mathfrak{D}(Z)}\left[(\mu-p)\,\mathbf{v}'+\mu gradw\right]dS_*,\\
&f_0^a=\int_{\mathfrak{D}(Z)}\left[(\lambda+2\mu-p)\,w'+\lambda div\mathbf{v}\right]dS_*,\\
&\mathbf{m}_0^p+Z\mathbf{m}_1=-\int_{\mathfrak{D}(Z)}\left[(\lambda+2\mu-p)\,w'+\lambda div\mathbf{v}\right]\left(*\hat{\mathbf{X}}\right)dS_*,\\
&m_0^a=\int_{\mathfrak{D}(Z)}\left(*\hat{\mathbf{X}}\right)\cdot\left[(\mu-p)\,\mathbf{v}'+\mu gradw\right]dS_*.
\end{aligned}
\tag{3.40}
$$

With the hypothesis (3.27) on displacements, equations (3.40) yield

$$
\begin{aligned}
&(\mu-p)\bigtriangleup_R w_i+(\lambda+\mu)\,div\mathbf{v}_{i+1}+(\lambda+2\mu-p)w_{i+2}=0, && \forall\hat{\mathbf{X}}\in\mathfrak{D};\\
&div\left(2\mu Symgrad\mathbf{v}_i-pgrad\mathbf{v}_i+\lambda\hat{\mathbf{1}}div\mathbf{v}_i\right)+(\lambda+\mu)\,gradw_{i+1}+(\mu-p)\,\mathbf{v}_{i+2}=\mathbf{0}, && \forall\hat{\mathbf{X}}\in\mathfrak{D};\\
&\left(2\mu Symgrad\mathbf{v}_i-pgrad\mathbf{v}_i+\lambda\hat{\mathbf{1}}div\mathbf{v}_i\right)\hat{\mathbf{N}}=-\lambda(w_{i+1})\hat{\mathbf{N}}, && \forall\hat{\mathbf{X}}\in\partial\mathfrak{D};\\
&(\mu-p)\,gradw_i\cdot\hat{\mathbf{N}}=-\mu\mathbf{v}_{i+1}\cdot\hat{\mathbf{N}}, && \forall\hat{\mathbf{X}}\in\partial\mathfrak{D};\\
&\int_{\mathfrak{D}(Z)}\left[(\mu-p)\,\mathbf{v}_{i+1}+\mu gradw_i\right]dS_*=\mathbf{0}, && i>0;\\
&\int_{\mathfrak{D}(Z)}\left[(\lambda+2\mu-p)\,w_{i+1}+\lambda div\mathbf{v}_i\right]dS_*=0, && i>0;\\
&\int_{\mathfrak{D}(Z)}\left[(\lambda+2\mu-p)\,w_{i+1}+\lambda div\mathbf{v}_i\right]\left(*\hat{\mathbf{X}}\right)dS_*=0, && i>1;\\
&\int_{\mathfrak{D}(Z)}\left(*\hat{\mathbf{X}}\right)\cdot\left[(\mu-p)\,\mathbf{v}_{i+1}+\mu gradw_i\right]dS_*=\mathbf{0}, && i>0.
\end{aligned}
\tag{3.41}
$$

A solution of the system (3.41) is not yet available.

Solution of St.-Venant's Problem by Signorini's Method

We apply Signorini's method to find a solution of St-Venant's problem when the stress strain relation is quadratic in displacement gradients. However, instead of solving the problem in complete generality, we assume that the first-order deformations are pure torsional. Furthermore, we adopt hypothesis (3.27) on the second order displacement field.

As noted earlier, Signorini's method transforms a non-linear problem of n-th order, into n linear problems.

The infinitesimal twist per unit length, τ, is identified as the small parameter. That is,

$$\varepsilon \equiv \varphi_1^0 = \tau,$$

and from (2.83) we have

$$\mathbf{u}\left(\varepsilon, \mathbf{X}\right) = \sum_{n=1}^{2} \tau^n \mathbf{u}_n\left(\mathbf{X}\right) = \mathbf{u}_1\left(\mathbf{X}\right)\tau + \mathbf{u}_2\left(\mathbf{X}\right)\tau^2,$$

where

$$\mathbf{u}_1\left(\mathbf{X}\right) = Z\left(*\hat{\mathbf{X}}\right) + \phi\mathbf{e}. \tag{3.42}$$

The function ϕ is the warping function and is the solution of the problem (3.30). We have eliminated the rigid body displacement by adopting the Green and Adkins condition (2.89).

From (2.84) we have

$$\mathbf{H} = \tau\mathbf{H}_1 + \tau^2\mathbf{H}_2,$$

where

$$\mathbf{H}_1 = *Z + \left(*\hat{\mathbf{X}}\right) \otimes \mathbf{e} + \mathbf{e} \otimes grad\phi,$$

$$\mathbf{E}_1 = \frac{1}{2}\left\{\left[\left(*\hat{\mathbf{X}}\right) + grad\phi\right] \otimes \mathbf{e} + \mathbf{e} \otimes \left[\left(*\hat{\mathbf{X}}\right) + grad\phi\right]\right\}.$$

Thus

$$I_{\mathbf{E}_1} = 0.$$

We write (2.69) as

$$\mathbf{T}_* = \tau \mathbf{T}_*^{(1)} + \tau^2 \mathbf{T}_*^{(2)},$$

where

$$\mathbf{T}_*^{(1)} = 2\mu \mathbf{E}_1 + \lambda I_{\mathbf{E}_1} \mathbf{I},$$

$$\mathbf{T}_*^{(2)} = \bar{\mathbf{T}}_*^{(2)} + \mu \mathbf{I}\left[\frac{\alpha_1}{2} I_{\mathbf{H}_1 \mathbf{H}_1^T} + \frac{\alpha_4}{2} I_{(\mathbf{E}_1)^2}\right] - \left(\mathbf{H}_1^T\right)^2 + \alpha_6 \mathbf{E}_1.$$

The displacement field $\mathbf{u}_2$ is a solution of the boundary value problem (2.87) and (2.88) with $n = 2$.

In order to simplify the notation, we set

$$\mathbf{u}_2(\mathbf{X}) = \mathbf{v}(\mathbf{X}) + w(\mathbf{X})\,\mathbf{e},$$

and from (3.26) obtain the following equations for the determination of $\mathbf{v}$ and w.

$$\mu \triangle_R w + (\lambda + \mu)\, div \mathbf{v}' + (\lambda + 2\mu) w'' = -2(\lambda + \mu) Z \tau^2, \qquad \forall \mathbf{X} \in \mathfrak{C}_*,$$

$$\begin{aligned} &div\left(2\mu Symgrad\mathbf{v} + \lambda \hat{\mathbf{1}} div\mathbf{v}\right) + (\lambda + \mu)\, grad w' + \mu \mathbf{v}' \\ &\quad = -\tau^2 (\lambda + \mu)\, grad\left[\left(*\hat{\mathbf{X}}\right) \cdot (grad\phi)\right] - \left(\lambda - \mu\frac{\alpha_4}{2}\right)\left[\left(*\hat{\mathbf{X}}\right) + grad\phi\right] \\ &- \mu\frac{\alpha_6}{4} div\left\{\left[\left(*\hat{\mathbf{X}}\right) + grad\phi\right] \otimes \left[\left(*\hat{\mathbf{X}}\right) + grad\phi\right]\right\}, \qquad \forall \mathbf{X} \in \mathfrak{C}_*, \end{aligned}$$

$$\left(2\mu Symgrad\mathbf{v}+\lambda\hat{\mathbf{1}}div\mathbf{v}\right)\mathbf{N}=-\lambda(w')\mathbf{N}$$
$$+\left\{\begin{array}{c}\tau^2\mu\left[\left(*\hat{\mathbf{X}}\right)\otimes grad\phi\right]+\\ -\hat{\mathbf{1}}\left[\begin{array}{c}\frac{1}{2}\left(\lambda-\mu\frac{\alpha_4}{2}\right)\left|\left(*\hat{\mathbf{X}}\right)+grad\phi\right|^2+\\ -\tau^2\lambda\left(*\hat{\mathbf{X}}\right)\cdot(grad\phi)+\tau^2\left(\lambda+\mu\right)Z^2\end{array}\right]\end{array}\right\}\mathbf{N}\quad\forall\mathbf{X}\in\partial\mathfrak{D}\times[0,L]\,,$$
$$(gradw)\cdot\mathbf{N}=-\mathbf{v}'\cdot\mathbf{N}-\mu Z\tau^2\left(\hat{\mathbf{X}}\cdot\mathbf{N}\right),\qquad\forall\mathbf{X}\in\partial\mathfrak{D}\times[0,L]\,.\tag{3.43}$$

We solve this linear problem by following the method outlined in the next chapter for analyzing a linear Almansi-Michell problem. In particular, we assume that $\mathbf{v}$ and w are given by (3.27). Since body forces present in $(3.43)_{1,2}$ are of order τ^2, we can use the solution (4.17) by setting $m=2$ and obtain

$$\begin{aligned}
&w_i=0, && i=6,...,M;\\
&w_5=\mathbf{v}_6^0\cdot\left(\hat{\mathbf{b}}-\hat{\mathbf{X}}\right),\\
&w_4=\mathbf{v}_5^0\cdot\left(\hat{\mathbf{b}}-\hat{\mathbf{X}}\right),\\
&\mathbf{v}_i=\mathbf{0}, && i=7,...,M;\\
&\mathbf{v}_i=\mathbf{v}_i^0, && i=5,6;\\
&\mathbf{v}_4=\mathbf{v}_4^0+(*\varphi_4^0)\,\hat{\mathbf{X}}-\nu\left(\mathbf{v}_6^0\cdot\hat{\mathbf{b}}\right)\hat{\mathbf{X}}+\nu\left\{sym\left[\hat{\mathbf{X}}\otimes\left(*\hat{\mathbf{X}}\right)\right]\right\}(*\mathbf{v}_6^0)\,,\\
&\mathbf{v}_3=\mathbf{v}_3^0+(*\varphi_3^0)\,\hat{\mathbf{X}}-\nu w_5^0\hat{\mathbf{X}}+\nu\left\{sym\left[\hat{\mathbf{X}}\otimes\left(*\hat{\mathbf{X}}\right)\right]\right\}(*\mathbf{v}_5^0)\,,\quad\forall\mathbf{X}\in\mathfrak{D}.
\end{aligned}$$

The other part of the displacement field can be found by solving the following system of equations:

$$\begin{aligned}
\mu\triangle_R w_0+(\lambda+\mu)\,div\mathbf{v}_1+(\lambda+2\mu)w_2 &= 0,\\
\mu\triangle_R w_1+(\lambda+\mu)\,div\mathbf{v}_2+(\lambda+2\mu)w_3 &= -2\,(\lambda+\mu)\,\tau^2,\\
\mu\triangle_R w_2+(\lambda+\mu)\,div\mathbf{v}_3+(\lambda+2\mu)w_4 &= 0, \qquad \forall\hat{\mathbf{X}}\in\mathfrak{D};
\end{aligned}$$

$$div\left(2\mu Symgrad\mathbf{v}_0+\lambda\hat{\mathbf{1}}div\mathbf{v}_0\right)+(\lambda+\mu)\,gradw_1+\mu\mathbf{v}_2$$

$$=-\tau^2(\lambda+\mu)\,grad\left[\left(*\hat{\mathbf{X}}\right)\cdot(grad\phi)\right]-\left(\lambda-\mu\frac{\alpha_4}{2}\right)\left[\left(*\hat{\mathbf{X}}\right)+grad\phi\right]$$
$$-\mu\frac{\alpha_6}{4}div\left\{\left[\left(*\hat{\mathbf{X}}\right)+grad\phi\right]\otimes\left[\left(*\hat{\mathbf{X}}\right)+grad\phi\right]\right\},$$

$$div\left(2\mu Symgrad\mathbf{v}_1+\boldsymbol{\lambda}\hat{\mathbf{1}}div\mathbf{v}_1\right)+(\lambda+\mu)\,gradw_2+\mu\mathbf{v}_3=0,$$

$$div\left(2\mu Symgrad\mathbf{v}_2+\lambda\hat{\mathbf{1}}div\mathbf{v}_2\right)+(\lambda+\mu)\,gradw_3+\mu\mathbf{v}_4=0,\qquad\forall\hat{\mathbf{X}}\in\mathfrak{D};$$

$$\left(2\mu Symgrad\mathbf{v}_0+\lambda\hat{\mathbf{1}}div\mathbf{v}_0\right)\mathbf{N}=-\lambda(w_1)\mathbf{N}$$
$$+\tau^2\mu\left[\left(*\hat{\mathbf{X}}\right)\otimes grad\phi\right]\mathbf{N}$$
$$-\left[\tfrac{1}{2}\left(\lambda-\mu\tfrac{\alpha_4}{2}\right)\left|\left(*\hat{\mathbf{X}}\right)+grad\phi\right|^2-\tau^2\lambda\left(*\hat{\mathbf{X}}\right)\cdot(grad\phi)\right]\mathbf{N},$$
$$\left(2\mu Symgrad\mathbf{v}_1+\lambda\hat{\mathbf{1}}div\mathbf{v}_2\right)\mathbf{N}=-\lambda(w_3)\mathbf{N},$$
$$\left(2\mu Symgrad\mathbf{v}_2+\lambda\hat{\mathbf{1}}div\mathbf{v}_3\right)\mathbf{N}=-\lambda(w_4)\mathbf{N}+\tau^2(\lambda+\mu)\,\mathbf{N},\quad\forall\hat{\mathbf{X}}\in\partial\mathfrak{D};$$

$$(gradw_0)\cdot\mathbf{N}=-\mathbf{v}_1\cdot\mathbf{N},$$
$$(gradw_1)\cdot\mathbf{N}=-\mathbf{v}_2\cdot\mathbf{N}-\mu\tau^2\left(\hat{\mathbf{X}}\cdot\mathbf{N}\right),$$
$$(gradw_2)\cdot\mathbf{N}=-\mathbf{v}_3\cdot\mathbf{N},\qquad\forall\hat{\mathbf{X}}\in\partial\mathfrak{D}.$$

Some remarks should be considered in the application here of the principle of Saint-Venant [25].

A complete solution of this problem is given in [18] where the global balance equations are used to show that two of the constants vanish.

Here we note the usefulness of the Almansi-Michell's solution to solve 2^{nd} order St.-Venant's problem.

Chapter 4

The Almansi-Michell's problem

The difference between the Almansi-Michell and the St.-Venant problems is that in the former the mantle of the cylinder is subjected to surface tractions and in the latter it is traction free. Here, we study a particular case of the Almansi-Michell problem for which loads can be expressed as a polynomial in the axial co-ordinate Z. That is,

$$\begin{aligned} \mathbf{T}_*\mathbf{N} \equiv \mathbf{c}(\mathbf{X}) \equiv \mathbf{p}(\mathbf{X}) + a(\mathbf{X})\,\mathbf{e} &\equiv \sum_{j=0}^{m} \frac{Z^j}{j!}\left[\mathbf{p}_j\left(\hat{\mathbf{X}}\right) + \mathbf{e}a_j\left(\hat{\mathbf{X}}\right)\right], \\ \equiv \sum_{j=0}^{m} \frac{Z^j}{j!}\mathbf{c}_j\left(\hat{\mathbf{X}}\right), &\qquad \forall \mathbf{X} \in \partial\mathfrak{D} \times [0, L]. \end{aligned} \tag{4.1}$$

As for the St.-Venant problem, we adopt the polynomial hypothesis with respect to the axial variable Z for the displacement field:

$$\begin{cases} w(\mathbf{X}) = \displaystyle\sum_{i=0}^{M} \frac{1}{i!} w_i\left(\hat{\mathbf{X}}\right) Z^i, & \forall \mathbf{X} \in \mathfrak{C}_*, \\ \mathbf{v}(\mathbf{X}) = \displaystyle\sum_{i=0}^{M} \frac{1}{i!} \mathbf{v}_i\left(\hat{\mathbf{X}}\right) Z^i, & \forall \mathbf{X} \in \mathfrak{C}_*. \end{cases} \tag{4.2}$$

We note that m and M are arbitrary integers; however, generally $M >> m$.

In terms of the displacement field $(\mathbf{v}, w)$, the boundary conditions become

$$\begin{aligned} \left(2\mu Symgrad\mathbf{v}_i + \lambda\hat{\mathbf{1}}div\mathbf{v}_i\right)\mathbf{N} = -\lambda(w_{i+1})\mathbf{N} + \mathbf{p}_i, &\quad \forall \hat{\mathbf{X}} \in \partial\mathfrak{D}, \\ \mu grad w_i \cdot \mathbf{N} = -\mu \mathbf{v}_{i+1} \cdot \mathbf{N} + a_i, &\quad \forall \hat{\mathbf{X}} \in \partial\mathfrak{D}. \end{aligned} \tag{4.3}$$

Recalling the definitions (3.20) of $\mathbf{f}(Z)$ and $\mathbf{m}(Z)$, we first show that

$$\begin{aligned} \mathbf{f}(Z) &= \sum_{i=0}^{m+1} \mathbf{f}_i \frac{Z^i}{i!}, \\ \mathbf{m}(Z) &= \sum_{i=0}^{m+2} \mathbf{m}_i \frac{Z^i}{i!}, \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} \mathbf{f}_0 &= \mathbf{f}_0^p + f_0^a \mathbf{e} = \mathbf{q} + n\mathbf{e}, \\ \mathbf{f}_0^p &= \mathbf{q} = \int_{\mathfrak{D}(0)} \mathbf{t}\, dS_*, \\ f_0^a &= n = \int_{\mathfrak{D}(0)} \sigma\, dS_*, \\ \mathbf{f}_i &= \mathbf{f}_i^p + \mathbf{e} f_i^a = -\int_{\partial\mathfrak{D}} \mathbf{c}_{i-1}\, dl_*, \\ \mathbf{f}_i^p &= -\int_{\partial\mathfrak{D}} \mathbf{p}_{i-1}\, dl_*, \\ f_i^a &= -\int_{\partial\mathfrak{D}} a_{i-1}\, dl_*; \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} \mathbf{m}_0 &= \mathbf{m}_0^p + m_0^a \mathbf{e} = \mathbf{F}_0 + T\mathbf{e}, \\ \mathbf{m}_0^p &= -\int_{\mathfrak{D}(0)} \left(*\hat{\mathbf{X}}\right) \sigma\, dS_*, \\ m_0^a &= \left(\int_{\mathfrak{D}(0)} \hat{\mathbf{X}} \times \mathbf{t}\, dS_*\right) \cdot \mathbf{e}, \\ \mathbf{m}_i^p &= -i\left(*\mathbf{f}_{i-1}^p\right) + \int_{\partial\mathfrak{D}} \left(*\hat{\mathbf{X}}\right) a_{i-1}\, dl_*, \\ m_i^a &= -\left(\int_{\partial\mathfrak{D}} \hat{\mathbf{X}} \times \mathbf{p}_{i-1}\, dl_*\right) \cdot \mathbf{e}, \\ \mathbf{m}_{m+2}^p &= -(m+1) * \mathbf{f}_{m+1}^p, \\ m_{m+2}^a &= 0. \end{aligned} \tag{4.6}$$

Here dl_* is the element of arc length of the boundary of the cross-section in the reference configuration.

Substitution from (4.1) into the global equilibrium of force $(3.19)_1$ yields

$$-\int_{\mathfrak{D}(0)} \mathbf{T}_*\mathbf{e}dS_* + \int_{\mathfrak{D}(Z)} \mathbf{T}_*\mathbf{e}dS_* + \int_{\partial\mathfrak{D}\times[0,Z]} \mathbf{c}\left(\mathbf{X}\right) dS_* = \mathbf{0}. \tag{4.7}$$

Because of $(3.9)_2$ we have

$$\int_{\mathfrak{D}(0)} \mathbf{T}_*\mathbf{e}dS_* = \int\limits_{\mathfrak{D}(0)} \mathbf{t}dS_* + \mathbf{e}\int\limits_{\mathfrak{D}(0)} \sigma dS_*. \tag{4.8}$$

Because of $(3.20)_1$, the second integral on the left hand side of (4.7) equals $\mathbf{f}\left(Z\right)$. On the mantle $dS_* = dl_* d\eta$ where η is a parameter along the axial position in the reference configuration. From (4.1) we obtain

$$\begin{aligned}\int_{\partial\mathfrak{D}\times[0,Z]} \mathbf{c}\left(\mathbf{X}\right) dS_* &= \sum_{j=0}^{m} \int_{\partial\mathfrak{D}\times[0,Z]} \frac{\eta^j}{j!}\mathbf{c}_j\left(\hat{\mathbf{X}}\right) dS_*, \qquad (4.9)\\ &= \sum_{j=0}^{m} \int_{\partial\mathfrak{D}} \mathbf{c}_j\left(\hat{\mathbf{X}}\right) dl_* \int_0^Z \frac{\eta^j}{j!} d\eta,\\ &= \sum_{j=0}^{m} \int_{\partial\mathfrak{D}} \mathbf{c}_j\left(\hat{\mathbf{X}}\right) \frac{Z^{j+1}}{(j+1)!} dl_* = \sum_{i=1}^{m+1} \frac{Z^i}{i!} \int_{\partial\mathfrak{D}} \mathbf{c}_{i-1}\left(\hat{\mathbf{X}}\right) dl_*.\end{aligned}$$

Equations (4.7) and $(3.20)_1$ give

$$\mathbf{f}\left(Z\right) = \int\limits_{\mathfrak{D}(Z)} \mathbf{T}_*\mathbf{e}dS_* = \int_{\mathfrak{D}(0)} \mathbf{T}_*\mathbf{e}dS_* - \sum_{i=1}^{m+1} \frac{Z^i}{i!} \int_{\partial\mathfrak{D}} \mathbf{c}_{i-1}\left(\hat{\mathbf{X}}\right) dl_*. \tag{4.10}$$

Thus, because of $(3.9)_2$ and $(4.5)_{2,3,4}$, (4.10) is equivalent to $(4.4)_1$.

Using (4.1) in the global equilibrium of momentum $(3.19)_2$, we obtain

$$-\int_{\mathfrak{D}(0)} \hat{\mathbf{X}} \times \mathbf{T}_*\mathbf{e}dS_* + \int_{\mathfrak{D}(Z)} \left(\hat{\mathbf{X}} + \eta\mathbf{e}\right) \times \mathbf{T}_*\mathbf{e}dS_* + \int_{\partial\mathfrak{D}\times[0,Z]} \left(\hat{\mathbf{X}} + \eta\mathbf{e}\right) \times \mathbf{c}dS_* = \mathbf{0}. \tag{4.11}$$

Because of $(3.9)_2$,

$$\int_{\mathfrak{D}(0)} \hat{\mathbf{X}} \times \mathbf{T}_*\mathbf{e}dS_* = -\int\limits_{\mathfrak{D}(0)} \left(*\hat{\mathbf{X}}\right) \sigma dS_* + \left(\int\limits_{\mathfrak{D}(0)} \hat{\mathbf{X}} \times \mathbf{t}dS_*\right) \cdot \mathbf{e}. \tag{4.12}$$

Recalling (4.1) and $(3.20)_2$, we get

$$\begin{aligned}\int_{\mathfrak{D}(Z)} \left(\hat{\mathbf{X}}+\eta\mathbf{e}\right)\times\mathbf{T}_*\mathbf{e}dS_* &= \int_{\mathfrak{D}(Z)} \hat{\mathbf{X}}\times\mathbf{T}_*\mathbf{e}dS_* + Z\left\{\mathbf{e}\times\sum_{j=0}^{m}\frac{Z^j}{j!}\mathbf{f}_j^p\right\}, \qquad (4.13)\\ &= \int_{\mathfrak{D}(Z)} \hat{\mathbf{X}}\times\mathbf{T}_*\mathbf{e}dS_* + \sum_{i=1}^{m+1}\frac{Z^i}{(i-1)!}*\mathbf{f}_{i-1}^p,\\ &= \mathbf{m}(Z) + \left\{\sum_{i=1}^{m+1}\frac{Z^i}{i!}i*\mathbf{f}_{i-1}^p\right\},\end{aligned}$$

and

$$\begin{aligned}\int_{\partial\mathfrak{D}\times[0,Z]} \left(\hat{\mathbf{X}}+\eta\mathbf{e}\right)\times\mathbf{c}dS_* &= \sum_{j=0}^{m}\int_{\partial\mathfrak{D}\times[0,Z]}\frac{\eta^j}{j!}\left(\hat{\mathbf{X}}+\eta\mathbf{e}\right)\times\mathbf{c}_j dS_*, \qquad (4.14)\\ &= \sum_{j=0}^{m}\int_{\partial\mathfrak{D}\times[0,Z]}\frac{\eta^j}{j!}\left[\hat{\mathbf{X}}\times\mathbf{c}_j+\eta\left(*\mathbf{p}_j\right)\right]dS_*,\\ &= \sum_{j=0}^{m}\int_{\partial\mathfrak{D}}\left[\left(\hat{\mathbf{X}}\times\mathbf{c}_j\right)\frac{Z^{j+1}}{(j+1)!}+\left(*\mathbf{p}_j\right)\frac{Z^{j+2}}{j!\,(j+2)}\right]dl_*,\\ &= \sum_{i=1}^{m+1}\int_{\partial\mathfrak{D}}\left(\hat{\mathbf{X}}\times\mathbf{c}_{i-1}\right)\frac{Z^i}{i!}dl_* + \sum_{i=2}^{m+2}\int_{\partial\mathfrak{D}}\left(*\mathbf{p}_{i-2}\right)\frac{Z^i}{(i-2)!i}\frac{(i-1)}{(i-1)}dl_*,\\ &= \sum_{i=2}^{m+1}\frac{Z^i}{i!}\int_{\partial\mathfrak{D}}\left[\left(\hat{\mathbf{X}}\times\mathbf{c}_{i-1}\right)+\left(*\mathbf{p}_{i-2}\right)(i-1)\right]dl_*\\ &\quad +Z\int_{\partial\mathfrak{D}}\left(\hat{\mathbf{X}}\times\mathbf{c}_0\right)dl_* + \int_{\partial\mathfrak{D}}\left(*\mathbf{p}_m\right)\frac{Z^{m+2}}{(m+2)!}(m+1)\,dl_*.\end{aligned}$$

Substitution from (4.12), (4.13) and (4.14) into (4.11) gives

$$\begin{aligned}\mathbf{0} &= \int_{\mathfrak{D}(0)}\left(*\hat{\mathbf{X}}\right)\sigma dS_* - \left(\int_{\mathfrak{D}(0)}\hat{\mathbf{X}}\times\mathbf{t}dS_*\right)\cdot\mathbf{e}\\ &\quad +\mathbf{m}(Z)+\sum_{i=1}^{m+1}\frac{Z^i}{i!}i*\mathbf{f}_{i-1}^p + Z\int_{\partial\mathfrak{D}}\left(\hat{\mathbf{X}}\times\mathbf{c}_0\right)dl_*\end{aligned}$$

$$+\sum_{i=2}^{m+1}\frac{Z^i}{i!}\int_{\partial\mathfrak{D}}\left[\left(\hat{\mathbf{X}}\times\mathbf{c}_{i-1}\right)dl_*+(*\mathbf{p}_{i-2})(i-1)\right]dl_*$$
$$+\int_{\partial\mathfrak{D}}(*\mathbf{p}_m)\frac{Z^{m+2}}{(m+2)!}(m+1)\,dl_*.$$

Thus

$$\begin{aligned}
\mathbf{m}(Z) &= -\int\limits_{\mathfrak{D}(0)}\left(*\hat{\mathbf{X}}\right)\sigma dS_*+\left(\int\limits_{\mathfrak{D}(0)}\hat{\mathbf{X}}\times\mathbf{t}dS_*\right)\cdot\mathbf{e}\\
&-\left\{\sum_{i=1}^{m+1}\frac{Z^i}{i!}i*\mathbf{f}_{i-1}^p\right\}-Z\int_{\partial\mathfrak{D}}\left(\hat{\mathbf{X}}\times\mathbf{c}_0\right)dl_*\\
&-\sum_{i=2}^{m+1}\frac{Z^i}{i!}\int_{\partial\mathfrak{D}}\left[\left(\hat{\mathbf{X}}\times\mathbf{c}_{i-1}\right)dl_*+(*\mathbf{p}_{i-2})(i-1)\right]dl_*\\
&-\int_{\partial\mathfrak{D}}(*\mathbf{p}_m)\frac{Z^{m+2}}{(m+2)!}(m+1)\,dl_*,\\
&= -\int\limits_{\mathfrak{D}(0)}\left(*\hat{\mathbf{X}}\right)\sigma dS_*+\left(\int\limits_{\mathfrak{D}(0)}\hat{\mathbf{X}}\times\mathbf{t}dS_*\right)\cdot\mathbf{e}\\
&-Z\left\{*\mathbf{f}_0^p+\int_{\partial\mathfrak{D}}\left(\hat{\mathbf{X}}\times\mathbf{c}_0\right)dl_*\right\}\\
&-\sum_{i=2}^{m+1}\frac{Z^i}{i!}\left\{i*\mathbf{f}_{i-1}^p+\int_{\partial\mathfrak{D}}\left[\left(\hat{\mathbf{X}}\times\mathbf{c}_{i-1}\right)dl_*+(*\mathbf{p}_{i-2})(i-1)\right]dl_*\right\}\\
&-\int_{\partial\mathfrak{D}}(*\mathbf{p}_m)\frac{Z^{m+2}}{(m+2)!}(m+1)\,dl_*.
\end{aligned}$$

Using $(4.5)_5$ and the definition (4.1) of $\mathbf{c}$ in terms of $\mathbf{p}$ and a we arrive at

$$\begin{aligned}
\mathbf{m}(Z) \;=\; & -\int\limits_{\mathfrak{D}(0)}\left(*\hat{\mathbf{X}}\right)\sigma dS_*+\left(\int\limits_{\mathfrak{D}(0)}\hat{\mathbf{X}}\times\mathbf{t}dS_*\right)\cdot\mathbf{e}\\
& -Z*\mathbf{f}_0^p-Z\int_{\partial\mathfrak{D}}\left(\hat{\mathbf{X}}\times\mathbf{p}_0\right)dl_*+\mathbf{e}Z\int_{\partial\mathfrak{D}}\left(*\hat{\mathbf{X}}\right)a_0dl_*\\
& -\sum_{i=2}^{m+1}\frac{Z^i}{i!}\left\{i*\mathbf{f}_{i-1}^p+\int_{\partial\mathfrak{D}}\left[\left(\hat{\mathbf{X}}\times\mathbf{p}_{i-1}\right)-\mathbf{e}\left(*\hat{\mathbf{X}}\right)(a_{i-1})\right]dl_*-\left(*\mathbf{f}_{i-1}^p\right)(i-1)\right\}\\
& +\left(*\mathbf{f}_{m+1}^p\right)\frac{Z^{m+2}}{(m+2)!}(m+1),
\end{aligned}$$

$$
\begin{aligned}
= & -\int\limits_{\mathfrak{D}(0)} \left(*\hat{\mathbf{X}}\right) \sigma dS_* + \left(\int\limits_{\mathfrak{D}(0)} \hat{\mathbf{X}} \times \mathbf{t} dS_* \right) \cdot \mathbf{e} \\
& -Z \left\{ *\mathbf{f}_0^p + \int_{\partial \mathfrak{D}} \left(\hat{\mathbf{X}} \times \mathbf{p}_0 \right) dl_* - \mathbf{e} \int_{\partial \mathfrak{D}} \left(*\hat{\mathbf{X}} \right) a_0 dl_* \right\} \\
& -\sum_{i=2}^{m+1} \frac{Z^i}{i!} \left\{ *\mathbf{f}_{i-1}^p + \int_{\partial \mathfrak{D}} \left[\left(\hat{\mathbf{X}} \times \mathbf{p}_{i-1} \right) - \mathbf{e} \left(*\hat{\mathbf{X}} \right) (a_{i-1}) \right] dl_* \right\} \\
& + \left(*\mathbf{f}_{m+1}^p \right) \frac{Z^{m+2}}{(m+2)!} (m+1) , \\
= & -\int\limits_{\mathfrak{D}(0)} \left(*\hat{\mathbf{X}}\right) \sigma dS_* + \left(\int\limits_{\mathfrak{D}(0)} \hat{\mathbf{X}} \times \mathbf{t} dS_* \right) \cdot \mathbf{e} \\
& -\sum_{i=1}^{m+1} \frac{Z^i}{i!} \left\{ *\mathbf{f}_{i-1}^p - \int_{\partial \mathfrak{D}} \left[\left(\hat{\mathbf{X}} \times \mathbf{p}_{i-1} \right) + \mathbf{e} \left(*\hat{\mathbf{X}} \right) (a_{i-1}) \right] dl_* \right\} \\
& + \frac{Z^{m+2}}{(m+2)!} (m+1) * \mathbf{f}_{m+1}^p .
\end{aligned}
$$

Using $(4.6)_{2,3,4,5,6}$, we arrive at $(4.4)_2$.

We note that $\mathbf{f}_0$ and $\mathbf{m}_0$ equal the resultant force and the resultant moment acting on the basis and $\mathbf{f}_1$, $\mathbf{m}_1$, $\mathbf{f}_2$, $\mathbf{m}_2$ ... are kinds of resultant forces and moments acting on the mantle of the cylinder.

Substitution from the constitutive relations (3.13) into (3.20) gives

$$
\begin{aligned}
\mathbf{f}(Z) &= \int_{\mathfrak{D}(Z)} \mu \left(\mathbf{v}' + gradw \right) dS_* + \mathbf{e} \int_{\mathfrak{D}(Z)} \left[(\lambda + 2\mu) w' + \lambda div \mathbf{v} \right] dS_*, \\
\mathbf{m}(Z) &= -\int_{\mathfrak{D}(Z)} \left((\lambda + 2\mu) w' + \lambda div \mathbf{v} \right) \left(*\hat{\mathbf{X}} \right) dS_* + \mathbf{e} \int_{\mathfrak{D}(Z)} \mu \left(*\hat{\mathbf{X}} \right) \cdot \left(\mathbf{v}' + gradw \right) dS_*.
\end{aligned} \tag{4.15}
$$

We now substitute the series (3.27) and (4.4) into (4.15) and equate like powers of Z on both sides to arrive at

$$
\int\limits_{\mathfrak{D}(Z)} \mu \left(\mathbf{v}_{i+1} + gradw_i \right) dS_* = \begin{cases} \mathbf{f}_i^p, & i = 0, ..., m+1, \\ \mathbf{0}, & i > m+1; \end{cases}
$$

$$\int_{\mathfrak{D}(Z)} \left[(\lambda + 2\mu)\, w_{i+1} + \lambda div \mathbf{v}_i \right] dS_* = \begin{cases} f_i^a, \; i = 0, ..., m+1, \\ 0, \qquad\quad i > m+1; \end{cases}$$

$$-\int_{\mathfrak{D}(Z)} \left[(\lambda + 2\mu)\, w_{i+1} + \lambda div \mathbf{v}_i \right] \left(*\hat{\mathbf{X}} \right) dS_* = \begin{cases} \mathbf{m}_i^p, \; i = 0, ..., m+2, \\ \mathbf{0}, \qquad\quad i > m+2; \end{cases}$$

$$\int_{\mathfrak{D}(Z)} \mu \left(\mathbf{v}_{i+1} + grad w_i \right) \cdot \left(*\hat{\mathbf{X}} \right) dS_* = \begin{cases} m_i^a, \; i = 0, ..., m+1, \\ 0, \qquad\quad i > m+1. \end{cases} \tag{4.16}$$

Following the same procedure as that used to solve the St.-Venant problem, we conclude the following $\quad \forall \mathbf{X} \in \mathfrak{D}$.

$$\begin{aligned}
& w_i = 0, && i = m+4, ..., M, \\
& w_{m+3} = \mathbf{v}_{m+4}^0 \cdot \left(\hat{\mathbf{b}} - \hat{\mathbf{X}} \right), \\
& w_{m+2} = \mathbf{v}_{m+3}^0 \cdot \left(\hat{\mathbf{b}} - \hat{\mathbf{X}} \right), \\
& \mathbf{v}_i = \mathbf{0}, && i = m+5, ..., M; \\
& \mathbf{v}_i = \mathbf{v}_i^0, && i = m+3, m+4; \\
& \mathbf{v}_{m+2} = \mathbf{v}_{m+2}^0 + \left(*\varphi_{m+2}^0 \right) \hat{\mathbf{X}} - \nu \left(\mathbf{v}_{m+4}^0 \cdot \hat{\mathbf{b}} \right) \hat{\mathbf{X}} + \nu \left\{ sym \left[\hat{\mathbf{X}} \otimes \left(*\hat{\mathbf{X}} \right) \right] \right\} \left(*\mathbf{v}_{m+4}^0 \right), \\
& \mathbf{v}_{m+1} = \mathbf{v}_{m+1}^0 + \left(*\varphi_{m+1}^0 \right) \hat{\mathbf{X}} - \nu w_{m+2}^0 \hat{\mathbf{X}} + \nu \left\{ sym \left[\hat{\mathbf{X}} \otimes \left(*\hat{\mathbf{X}} \right) \right] \right\} \left(*\mathbf{v}_{m+3}^0 \right).
\end{aligned} \tag{4.17}$$

It is hard to evaluate the remaining terms; however, we can evaluate them for a circular cylinder.

Generalized Axisymmetric Problems

We consider the case of uniform tractions on the mantle of a cylinder; a familiar case is that of pressure loading on its inner and outer surfaces.

Using formulae given in the part of the Appendix concerning cylindrical coordinates, we conclude from (4.17) the following:

$$
\begin{aligned}
&\mathbf{v}_i = 0, \qquad i = m+3, ..., M;\\
&v_{(m+2)r} = 0,\\
&v_{(m+2)\theta} = \varphi^0_{m+2} r,\\
&v_{(m+1)r} = -\nu w^0_{m+2} r,\\
&v_{(m+1)\theta} = \varphi^0_{m+1} r,\\
&w_i = 0 \qquad i = m+3, ..., M;\\
&w_{m+2} = w^0_{m+2},\\
&w_{m+1} = w^0_{m+1}.
\end{aligned} \tag{4.18}
$$

In cylindrical coordinates equilibrium equations $(3.28)_{1,2}$ become

$$
\begin{aligned}
&\mu \tfrac{1}{r} \left(r w_{i,r}\right)_{,r} + (\lambda+\mu) \tfrac{1}{r} \left(r v_{(i+1)r}\right)_{,r} + (\lambda + 2\mu) w_{i+2} = 0,\\
&(2\mu+\lambda) \left[\tfrac{1}{r} \left(r v_{ir}\right)_{,r}\right]_{,r} + (\lambda+\mu)\, w_{(i+1),r} + \mu v_{(i+2)r} = 0,\\
&\mu \left[\tfrac{1}{r} \left(r v_{i\theta}\right)_{,r}\right]_{,r} + v_{(i+2)\theta} = 0, \qquad \forall r \in \left[R_i, R_e\right].
\end{aligned} \tag{4.19}
$$

Boundary conditions (4.3) take the form

$$
\begin{aligned}
&(2\mu+\lambda)\, v_{ir,r} + \lambda \tfrac{v_{ir}}{r} = -\lambda w_{i+1} + p^e_{ir} \qquad at \qquad r = R_e,\\
&(2\mu+\lambda)\, v_{ir,r} + \lambda \tfrac{v_{ir}}{r} = -\lambda w_{i+1} - p^i_{ir} \qquad at \qquad r = R_i,\\
&\mu \left[r \left(\tfrac{1}{r} v_{i\theta}\right)_{,r}\right] = p^e_{i\theta} \qquad at \qquad r = R_e,\\
&\mu \left[r \left(\tfrac{1}{r} v_{i\theta}\right)_{,r}\right] = -p^i_{i\theta} \qquad at \qquad r = R_i,\\
&\mu w_{i,r} = -\mu v_{(i+1)r} + a^e_i \qquad at \qquad r = R_e,\\
&\mu w_{i,r} = -\mu v_{(i+1)r} - a^i_i \qquad at \qquad r = R_i.
\end{aligned} \tag{4.20}
$$

It is possible to extract from $(4.19)_1$ and $(4.20)_{5,6}$ the following equations for the unknown field w_m.

$$
\begin{aligned}
\tfrac{1}{r}\left(r w_{m,r}\right)_{,r} + 2w^0_{m+2} &= 0, \qquad \forall r \in \left[R_i, R_e\right],\\
\mu w_{m,r} &= \mu\nu w^0_{m+2} R_e + a^e_m, \qquad r = R_e,\\
\mu w_{m,r} &= \mu\nu w^0_{m+2} R_i - a^i_m, \qquad r = R_i.
\end{aligned}
$$

A simple integration gives

$$
w_m = w^0_m - \frac{R_e R_i \left(a^i_m R_e + a^e_m R_i\right)}{\mu\left[R_e^2 - R_i^2\right]} \log(r) - \frac{1}{2} w^0_{m+2} r^2,
$$

and

$$
w^0_{m+2} = -\frac{\left(a^i_m R_i + a^e_m R_e\right)}{\left[R_e^2 - R_i^2\right]\mu\left(1+\nu\right)}.
$$

It is possible to extract from $(4.19)_2$ and $(4.20)_{1,2}$ the following equations for the unknown field v_{mr}.

$$
\begin{aligned}
\left[\tfrac{1}{r}\left(r v_{mr}\right)_{,r}\right]_{,r} &= 0 \qquad \forall r \in \left[R_i, R_e\right],\\
\left(2\mu + \lambda\right) v_{mr,r} + \lambda \tfrac{v_{mr}}{r} &= -\lambda w_{m+1} + p^e_{mr}, \qquad r = R_e,\\
\left(2\mu + \lambda\right) v_{mr,r} + \lambda \tfrac{v_{mr}}{r} &= -\lambda w_{m+1} - p^i_{mr}, \qquad r = R_i.
\end{aligned}
$$

A simple integration gives

$$
v_{mr} = \left[\frac{\left(R_i R_e\right)^2 \left(p^e_{mr} + p^i_{mr}\right)}{2\mu\left[R_e^2 - R_i^2\right]}\right]\frac{1}{r} + \left[\frac{p^e_{mr} R_e^2 + p^i_{mr} R_i^2 - \lambda w^0_{m+1}\left[R_e^2 - R_i^2\right]}{2\left[R_e^2 - R_i^2\right]\left(\lambda + \mu\right)}\right] r.
$$

It is possible to extract from $(4.19)_3$ and $(4.20)_{3,4}$ the following equations for the unknown field $v_{m\theta}$.

$$
\mu\left[\frac{1}{r}\left(r v_{m\theta}\right)_{,r}\right]_{,r} + \varphi^0_{m+2} r = 0, \qquad \forall r \in \left[R_i, R_e\right],
$$

$$\mu\left[r\left(\tfrac{1}{r}v_{m\theta}\right)_{,r}\right] = p^e_{m\theta}, \qquad r = R_e,$$

$$\mu\left[r\left(\tfrac{1}{r}v_{m\theta}\right)_{,r}\right] = -p^i_{m\theta}, \qquad r = R_i.$$

A simple integration gives

$$v_{m\theta} = \tilde{v}_{m\theta 0}r + \frac{p^i_{m\theta}R_e^4R_i^2 + p^e_{m\theta}R_e^2R_i^4}{2\mu\left[R_e^4 - R_i^4\right]}\frac{1}{r} - \frac{\varphi^0_{m+2}}{8\mu}r^3,$$

and

$$\varphi^0_{m+2} = -4\frac{p^e_{m\theta}R_e^2 + p^i_{m\theta}R_i^2}{\left[R_e^4 - R_i^4\right]}.$$

Summarizing the preceding results we have, for a constant load on the mantle ($m = 0$), the following displacement field:

$$\mathbf{v}_i = \mathbf{0}, \qquad i = 3, ..., M;$$

$$v_{2r} = 0,$$

$$v_{2\theta} = \varphi^0_2 r,$$

$$v_{1r} = -\nu w^0_2 r,$$

$$v_{1\theta} = \varphi^0_1 r,$$

$$v_{0r} = \left[\frac{(R_iR_e)^2\left(p^e_{0r}+p^i_{0r}\right)}{2\mu\left[R_e^2-R_i^2\right]}\right]\frac{1}{r} + \left[\frac{p^e_{0r}R_e^2+p^i_{0r}R_i^2-\lambda w^0_1\left[R_e^2-R_i^2\right]}{2\left[R_e^2-R_i^2\right](\lambda+\mu)}\right]r,$$

$$v_{0\theta} = \tilde{v}^0_{\theta 0}r + \frac{p^i_{0\theta}R_e^4R_i^2+p^e_{0\theta}R_e^2R_i^4}{2\mu\left[R_e^4-R_i^4\right]}\frac{1}{r} - \frac{\varphi^0_2}{8\mu}r^3,$$

$$w_i = 0 \qquad i = 3, ..., M;$$

$$w_2 = -\frac{\left(a^i_0R_i+a^e_0R_e\right)}{\left[R_e^2-R_i^2\right](1+\nu)},$$

$$w_1 = w^0_1,$$

$$w_0 = w^0_0 - \frac{R_eR_i\left(a^i_0R_e+a^e_0R_i\right)}{\mu\left[(R_e)^2-(R_i)^2\right]}\log(r) - \tfrac{1}{2}w^0_2 r^2,$$

where

$$\begin{aligned} w_2^0 &= -\frac{(a_0^i R_i + a_0^e R_e)}{[R_e^2 - R_i^2]\,\mu\,(1+\nu)}, \\ \varphi_2^0 &= -4\frac{p_{0\theta}^e R_e^2 + p_{0\theta}^i R_i^2}{[R_e^4 - R_i^4]}. \end{aligned}$$

It is possible to get a solution for other values of m since v_{mr}, $v_{m\theta}$, w_m are governed by linear ordinary differential equations.

If $m = 1$ then (4.18) and the calculation involved for the case $m = 0$ give

$$\mathbf{v}_i = 0, \qquad i = 4, ..., M;$$

$$v_{3r} = 0,$$

$$v_{3\theta} = \varphi_3^0 r,$$

$$v_{2r} = -\nu w_3^0 r,$$

$$v_{2\theta} = \varphi_2^0 r,$$

$$v_{1r} = \left[\frac{(R_i R_e)^2 \left(p_{1r}^e + p_{1r}^i\right)}{2\mu\left[R_e^2 - R_i^2\right]}\right]\frac{1}{r} + \left[\frac{p_{1r}^e R_e^2 + p_{1r}^i R_i^2 - \lambda w_2^0 \left[R_e^2 - R_i^2\right]}{2\left[R_e^2 - R_i^2\right](\lambda+\mu)}\right] r,$$

$$v_{1\theta} = \tilde{v}_{0\theta}^0 r + \frac{p_{1\theta}^i R_e^4 R_i^2 + p_{1\theta}^e R_e^2 R_i^4}{2\mu\left[R_e^4 - R_i^4\right]}\frac{1}{r} - \frac{\varphi_3^0}{8\mu} r^3,$$

$$w_i = 0 \qquad i = 4, ..., M;$$

$$w_3 = -\frac{\left(a_1^i R_i + a_1^e R_e\right)}{\left[R_e^2 - R_i^2\right](1+\nu)},$$

$$w_2 = w_2^0,$$

$$w_1 = w_1^0 - \frac{R_e R_i \left(a_1^i R_e + a_1^e R_i\right)}{\mu\left[(R_e)^2 - (R_i)^2\right]}\log(r) - \frac{1}{2} w_3^0 r^2,$$

where

$$\begin{aligned} w_3^0 &= -\frac{(a_1^i R_i + a_1^e R_e)}{[R_e^2 - R_i^2]\,\mu\,(1+\nu)}, \\ \varphi_3^0 &= -4\frac{p_{1\theta}^e R_e^2 + p_{1\theta}^i R_i^2}{[R_e^4 - R_i^4]}. \end{aligned}$$

Thus,

$$(rv_{1r})_{,r} = r\frac{p^e_{1r}R_e^2 + p^i_{1r}R_i^2 - \lambda w^0_2\left[R_e^2 - R_i^2\right]}{(\lambda+\mu)\left[R_e^2 - R_i^2\right]}.$$

Differential equation for w_0 is

$$\mu\frac{1}{r}\left(rw_{0,r}\right)_{,r} + (\lambda+\mu)\frac{1}{r}\left(rv_{1r}\right)_{,r} + (\lambda+2\mu)w_2 = 0,$$

that is

$$\mu\frac{1}{r}\left(rw_{0,r}\right)_{,r} + \frac{p^e_{1r}R_e^2 + p^i_{1r}R_i^2}{\left[R_e^2 - R_i^2\right]} + 2\mu w^0_2 = 0.$$

Boundary conditions for w_0 are

$$\mu w_{0,r} = -\mu v_{1r} + a^e_0, \qquad at \qquad r = R_e,$$

$$\mu w_{0,r} = -\mu v_{1r} - a^i_0, \qquad at \qquad r = R_i.$$

A simple integration gives

$$w_0 = w^0_0 - \frac{1}{2}w^0_2 r^2 - \frac{1}{4}\left[\frac{p^e_{1r}R_e^2 + p^i_{1r}R_i^2}{\mu\left[R_e^2 - R_i^2\right]}\right]r^2 - R_eR_i\left[\frac{2\left(a^i_0R_e + a^e_0R_i\right) + \left(p^e_{1r} + p^i_{1r}\right)R_eR_i}{2\mu\left[R_e^2 - R_i^2\right]}\right]\log(r),$$

where

$$w^0_2 = -\frac{\lambda\left(p^e_{1r}R_e^2 + p^i_{1r}R_i^2\right) + 2\left(\lambda+\mu\right)\left(a^i_0R_i + a^e_0R_e\right)}{\mu\left(3\lambda+2\mu\right)\left[R_e^2 - R_i^2\right]}.$$

It is possible to give analytical solutions even for v_{0r}, $v_{0\theta}$. However expressions are too large to include in the thesis.

A Mathematica file is attached at the end of the chapter to find the complete solution for the case of $m = 1$. In this file computations of this section are done by setting $m = 1$ at the beginning and defining everything in order to make Mathematica do the calculations.

We now give a simple example.

An example

We choose the following parameters in the ISU.

$$
\begin{aligned}
R_e &= 0.5, \quad R_i = 0.2; \\
\lambda &= 200 * 10^9, \quad \mu = 77.2 * 10^9; \\
\varphi_1^0 &= 10^{-3}, \quad w_1^0 = 10^{-1}; \\
p_{1r}^e &= 10^5, \quad p_{1\theta}^e = 10^5, \quad a_1^e = 10^5; \\
p_{1r}^i &= 10^5, \quad p_{1\theta}^i = 10^5, \quad a_1^i = 10^5; \\
p_{0r}^e &= 10^5, \quad p_{0\theta}^e = 10^5, \quad a_0^e = 10^5; \\
p_{0r}^i &= 10^5, \quad p_{0\theta}^i = 10^5, \quad a_0^i = 10^5.
\end{aligned}
$$

Thus,

$$\mathbf{v}_i = 0, \qquad i = 4, ..., M;$$

$$v_{3r} = 0,$$

$$v_{3\theta} = -1.9 * 10^6 r,$$

$$v_{2r} = 1.1 * 10^{-6} r,$$

$$v_{2\theta} = -1.9 * 10^6 r,$$

$$v_{1r} = [6.2 * 10^{-8}] \, \tfrac{1}{r} + [1.6 * 10^{-6}] \, r,$$

$$v_{1\theta} = [3.1 * 10^{-8}] \, \tfrac{1}{r} + [10^{-3}] \, r - [3.1 * 10^{-6}] \, r^3,$$

$$v_{0r} = [6.7 * 10^{-8}] \, \tfrac{1}{r} - [3.6 * 10^{-2}] \, r - [3.4 * 10^{-7}] \, r^3 + [1.7 * 10^{-7}] \, r \log(r),$$

$$v_{0\theta} = [3.1 * 10^{-8}] \, \tfrac{1}{r} + [3.1 * 10^{-6}] \, r^3,$$

$$w_i = 0 \qquad i = 4, ..., M;$$

$$w_3 = -3.2 * 10^{-6},$$

$$w_2 = -3.7 * 10^{-6},$$

$$w_1 = 10^{-1} + [1.6 * 10^{-6}]\, r^2 - 4.3 * 10^{-7} \log(r),$$

$$w_0 = 1.4 * 10^{-6} r^2 - 4.9 * 10^{-7} \log(r).$$

Chapter 5

Conclusions

We have used a semi-inverse method and the Signorini method to analyze three dimensional deformations of a prismatic body made of a homogeneous and isotropic elastic material. The solution of the problem is reduced to that of solving a set of two-dimensional problems over the cross-section of the prismatic body. For a stress free reference configuration of the body, an explicit solution of these two dimensional problems, except for the 0th order problem, is given. For a simple cross-section, the 0th order problem may be solved analytically; otherwise its approximate solution can be found numerically. An explicit solution is given for a generalized axisymmetric problem in which all three components of displacement are independent of the angular position. Equations governing three dimensional deformations of a homogeneous prismatic body made of an isotropic material and subjected to a hydrostatic pressure in the reference configuration are also derived. By assuming that the three components of displacement can be expressed as a polynomial in the axial coordinate, the solution of the three dimensional problem is reduced to that of solving a set of two dimensional problems defined over the cross-section of the body. These equations are difficult to solve analytically.

By assuming that surface tractions on the mantle of a cylindrical prismatic body and the three components of displacement can be expressed as a polynomial in the axial coordinate, three dimensional elastostatics equations are reduced to a set of two-dimensional elastostatics equations. These two dimensional problems are solved analytically for generalized axisymmetric deformations of a circular cylindrical body. Numerical results are given for a hollow cylindrical body with a uniform pressure applied on its mantle and also when the pressure is a polynomial of degree one in the axial coordinate.

Chapter 6

Appendix

[1]

Decomposition of a 2^{nd} order tensor

Let $\mathfrak{A}$ and $\mathfrak{B}$ be two vector spaces. We define the vector space $\mathfrak{A} \otimes \mathfrak{B}$ as the space of the linear combination $a \otimes b$ such that $a \in \mathfrak{A}$, $b \in \mathfrak{B}$ and

$$(a \otimes b)\, c = (b \cdot c)\, a \qquad \forall c \in \mathfrak{B};$$

Let us recall the definition given in (3.3)

$$\mathfrak{U} = \mathcal{W} \perp \mathcal{V},$$

and the linear properties of the tensor product. We have the following decomposition

$$\mathfrak{U} \otimes \mathfrak{U} = (\mathcal{W} \otimes \mathcal{W}) \perp (\mathcal{W} \otimes \mathcal{V}) \perp (\mathcal{V} \otimes \mathcal{W}) \perp (\mathcal{V} \otimes \mathcal{V}) \tag{6.1}$$

of the vector space $\mathfrak{U} \otimes \mathfrak{U}$ that is the space of the linear operators LIN of $\mathfrak{U}$ in $\mathfrak{U}$:

$$LIN = \mathfrak{U} \otimes \mathfrak{U}.$$

[1] Most of the results of the first three paragraphs can be found in [1], [20], [32] and [36].

LIN can be seen as the space of second order tensors. Every second order tensor can be decomposed into the sum of its symmetric part and its skew symmetric part. Thus

$$LIN \equiv SYM \perp SKW, \tag{6.2}$$

where SYM is the space of symmetric tensors and SKW is the space of skew symmetric tensors. That is SYM is a linear combination of

$$Sym(u \otimes u') \qquad / \qquad u \in \mathfrak{U}, u' \in \mathfrak{U},$$

and SKW a linear combination of

$$Skw(u \otimes u') \qquad / \qquad u \in \mathfrak{U}, u' \in \mathfrak{U}.$$

These spaces can also be written as

$$\begin{aligned} SYM &= \mathfrak{U} \vee \mathfrak{U}, \\ SKW &= \mathfrak{U} \wedge \mathfrak{U}. \end{aligned}$$

For the subspace $\mathcal{V} \otimes \mathcal{V}$ of $\mathfrak{U} \otimes \mathfrak{U}$, we have

$$lin \equiv \mathcal{V} \otimes \mathcal{V} = sym \perp skw,$$

such that

$$\begin{aligned} sym &= \mathcal{V} \vee \mathcal{V}, \\ skw &= \mathcal{V} \wedge \mathcal{V}. \end{aligned}$$

The space $\mathcal{W} \vee \mathcal{V}$ is the space of all symmetric tensors

$$Sym(w \otimes v) \qquad / \qquad w \in \mathcal{W}, v \in \mathcal{V}.$$

Finally we have the following decomposition for a 2^{nd} order tensor:

$$\begin{aligned} SYM &= (\mathcal{W} \otimes \mathcal{W}) \perp (\mathcal{W} \vee \mathcal{V}) \perp sym, \\ SKW &= (\mathcal{V} \wedge \mathcal{W}) \perp skw. \end{aligned}$$

The star of Hodge

It is possible to give the definition of the star of Hodge in an arbitrary vector space; however we will need it only in the two-dimensional subspace $\mathcal{V}$: for the sake of simplicity we will give its definition only in this space.

The cross product of two vectors $(\mathbf{u}, \mathbf{v})$ belonging to the three dimensional space $\mathfrak{U}$ is another vector belonging to the same space. Let us suppose that the two vectors belong to $\mathcal{V}$; then, their cross product is an element of one-dimensional vector space $\mathcal{W}$ and can be characterized as a real number. Let us give the following definition:

$$-2\mathbf{u} \times \mathbf{v} = \alpha, \qquad \forall \mathbf{u}, \mathbf{v} \in \mathcal{V}, \quad \alpha \in \mathbb{R}. \tag{6.3}$$

Besides we can consider the skew symmetric tensor $\mathbf{u} \wedge \mathbf{v}$ belonging to the subspace skw. We note that it is possible to have a bijective mapping between skw and the space $\mathbb{R}$ of the real numbers. So it is possible to give the following definition for the star of Hodge $*$ in $\mathcal{V}$:

$$*\,(\mathbf{u} \wedge \mathbf{v}) \equiv \alpha, \qquad \forall \mathbf{u}, \mathbf{v} \in \mathcal{V}, \quad \alpha \in \mathbb{R}. \tag{6.4}$$

If we are in an orthonormal system we can use the index notation, then the star of Hodge has the following matrix representation:

$$[*] = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{6.5}$$

In index notation

$$[*]_{ij} = -\varepsilon_{ij},$$

where ε_{ij} is the Levi Civita skew symmetric tensor in two dimensions. In fact we have

$$(\mathbf{u} \wedge \mathbf{v})_{ij} = (u_i v_j - u_j v_i), \qquad \forall \mathbf{u}, \mathbf{v} \in \mathcal{V}. \tag{6.6}$$

The Levi Civita tensor is useful to represent the cross product:

$$\mathbf{u} \times \mathbf{v} = \varepsilon_{ij} u_i v_j, \qquad \forall \mathbf{u}, \mathbf{v} \in \mathcal{V}. \tag{6.7}$$

The matrix representation of the scalar obtained by the application of the star of Hodge (6.5) to a generic skew symmetric tensor (for example (6.6)) is the scalar product of the two tensors:

$$[* (\mathbf{u} \wedge \mathbf{v})] = [*]_{ij} (u_i v_j - u_j v_i), \qquad \forall \mathbf{u}, \mathbf{v} \in \mathcal{V}. \tag{6.8}$$

If we add a symmetric tensor in the second factor of (6.8) we do not change the result. Because of (6.7)

$$\begin{aligned} [* (\mathbf{u} \wedge \mathbf{v})] &= [*]_{ij} [(u_i v_j - u_j v_i) + (u_i v_j + u_j v_i)], \\ &= 2 [*]_{ij} u_i v_j = -2\varepsilon_{ij} u_i v_j = -2\mathbf{u} \times \mathbf{v}, \qquad \forall \mathbf{u}, \mathbf{v} \in \mathcal{V}, \end{aligned} \tag{6.9}$$

we have verified the compatibility of (6.3) and (6.4).

It is possible to apply the star of Hodge even to a scalar and we have:

$$\begin{cases} **\,\alpha = 2\alpha, & \alpha \in \mathbb{R}, \\ **\,(\mathbf{u}\wedge\mathbf{v}) = 2\,(\mathbf{u}\wedge\mathbf{v})\,, & \forall \mathbf{u},\mathbf{v} \in \mathcal{V}. \end{cases}$$

Thus

$$**\,(\mathbf{u}\wedge\mathbf{v}) = -2 * (\mathbf{u}\times\mathbf{v}) = 2\,(\mathbf{u}\wedge\mathbf{v})\,. \tag{6.10}$$

It is possible to demonstrate these in an orthonormal system via the index notation and with the formula

$$[*]_{ij}\,[*]_{ij} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} : \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = 1 + 1 = 2.$$

We note that

$$[\varepsilon]_{ij}\,[\varepsilon]_{jh} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\delta_{ih}. \tag{6.11}$$

We can apply the star of Hodge to a vector and it is possible to demonstrate that the vector is rotated in the anticlockwise direction. In an orthonormal system,

$$*\mathbf{v} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -v_2 \\ v_1 \end{pmatrix}, \qquad \forall \mathbf{v} \in \mathcal{V}.$$

Thus

$$**\,\mathbf{v} = -\mathbf{v}, \qquad \forall \mathbf{v} \in \mathcal{V}.$$

An useful definition is the following

$$\alpha \times \mathbf{v} \equiv \alpha\,(*\mathbf{v}) = \alpha * \mathbf{v}, \qquad \forall \mathbf{v} \in \mathcal{V}, \quad \forall \alpha \in \mathbb{R}. \tag{6.12}$$

The useful formula

$$\mathbf{u} \times \mathbf{v} = (*\mathbf{u}) \cdot \mathbf{v}, \qquad \forall \mathbf{u}, \mathbf{v} \in \mathcal{V}, \tag{6.13}$$

is easy to prove in an orthonormal system.

$$\mathbf{u} \times \mathbf{v} = \varepsilon_{ij} u_i v_j = -\varepsilon_{ji} u_i v_j = (*\mathbf{u})_j v_j = (*\mathbf{u}) \cdot \mathbf{v}, \qquad \forall \mathbf{u}, \mathbf{v} \in \mathcal{V}.$$

Furthermore,

$$(*\alpha)\, \mathbf{v} = \alpha\, (*\mathbf{v}), \qquad \forall \mathbf{v} \in \mathcal{V}, \qquad \forall \alpha \in \mathbb{R}, \tag{6.14}$$

in an orthonormal system is trivial. Besides we have

$$\alpha \times \mathbf{u} \equiv \alpha\, (*\mathbf{u}) = (*\alpha)\, \mathbf{u}, \qquad \forall \mathbf{u} \in \mathcal{V}, \qquad \forall \alpha \in \mathbb{R}, \tag{6.15}$$

and because of (6.13), (6.9) and (6.10),

$$*\left[(*\mathbf{u}) \cdot \mathbf{v}\right] = *\left(\mathbf{u} \times \mathbf{v}\right) = -\frac{1}{2} *\left[*\left(\mathbf{u} \wedge \mathbf{v}\right)\right] = -\left(\mathbf{u} \wedge \mathbf{v}\right). \tag{6.16}$$

Differential operator

We first define a differential operator in an absolute way, and later we specialize the definition to Cartesian and cylindrical coordinates.

Let ϖ be a field that can be either a scalar belonging to $\mathbb{R}$, or a vector belonging to $\mathfrak{U}$, or a tensor belonging to $\mathfrak{U} \otimes \mathfrak{U}$. Let $\mathcal{Z}$ be the space of ϖ. The gradient of ϖ evaluated at a point $\mathbf{X} = \left(\hat{\mathbf{X}}, Z\right)$ of the cylinder $\mathfrak{C}_*$ is an operator

$$Grad\varpi \in \mathcal{Z} \otimes \mathfrak{U} = \mathcal{Z} \otimes \mathcal{W} \perp \mathcal{Z} \otimes \mathcal{V},$$

that can be decomposed as follows:

$$Grad\varpi = \varpi' \otimes \mathbf{e} + grad\varpi,$$

where

$$\varpi' \equiv \frac{\partial \varpi}{\partial Z},$$

and

$$grad\varpi \in \mathcal{Z} \otimes \mathcal{V}$$

is the gradient on the cross-section of the cylinder. The gradient operator in the cross-section is defined as

$$grad\,(\cdot) = (\cdot) \otimes \overleftarrow{grad} = (\cdot) \otimes \left(\overleftarrow{\partial}_{X_1} \mathbf{e}_1 + \overleftarrow{\partial}_{X_2} \mathbf{e}_2 \right).$$

The last expression means that we apply the derivative on the term appearing on the left side of the last parenthesis. This definition is correct even if we use non-Cartesian coordinates; it is illustrated by means of an example at the end of this section.

For $\varpi = \varphi$, a scalar,

$$Grad\varphi = \varphi' \otimes \mathbf{e} + grad\varphi,$$

where in Cartesian coordinates

$$grad\,(\varphi) = \frac{\partial \varphi}{\partial X_1}\mathbf{e}_1 + \frac{\partial \varphi}{\partial X_2}\mathbf{e}_2.$$

Note that Grad is the 3-dimensional gradient operator but grad is the gradient operator in a plane containing the cross-section of the cylinder.

If $\varpi = \mathbf{u} = \mathbf{v} + \mathbf{e}w$ is a vector, then

$$Grad\mathbf{u} \in \mathfrak{U} \otimes \mathfrak{U} = \mathfrak{U} \otimes \mathcal{W} \perp \mathfrak{U} \otimes \mathcal{V} = (\mathcal{W} \otimes \mathcal{W}) \perp (\mathcal{V} \otimes \mathcal{W}) \perp \mathfrak{U} \otimes \mathcal{V},$$

and

$$Grad\mathbf{u} = \mathbf{u}' \otimes \mathbf{e} + grad\mathbf{u} = w'\mathbf{e} \otimes \mathbf{e} + \mathbf{v}' \otimes \mathbf{e} + grad\mathbf{u},$$

where

$$grad\mathbf{u} \in \mathfrak{U} \otimes \mathcal{V} = \mathcal{W} \otimes \mathcal{V} \perp \mathcal{V} \otimes \mathcal{V},$$

or

$$grad\mathbf{u} = \mathbf{e} \otimes (gradw) + grad\mathbf{v},$$

and

$$w' = \frac{\partial w}{\partial Z}.$$

Thus

$$Grad\mathbf{u} = w'\mathbf{e} \otimes \mathbf{e} + \mathbf{v}' \otimes \mathbf{e} + \mathbf{e} \otimes (gradw) + grad\mathbf{v}. \tag{6.17}$$

If $\varpi = \mathbf{L} = \gamma\mathbf{e} \otimes \mathbf{e} + \mathbf{u} \otimes \mathbf{e} + \mathbf{e} \otimes \mathbf{v} + \hat{\mathbf{L}}$ is a tensor field, then

$$\begin{aligned}
Grad\mathbf{L} &\in \mathfrak{U} \otimes \mathfrak{U} \otimes \mathfrak{U}, \\
&= \mathfrak{U} \otimes \mathfrak{U} \otimes \mathcal{W} \perp \mathfrak{U} \otimes \mathfrak{U} \otimes \mathcal{V}, \\
&= \mathfrak{U} \otimes \mathcal{W} \otimes \mathcal{W} \perp \mathfrak{U} \otimes \mathcal{V} \otimes \mathcal{W} \perp \mathfrak{U} \otimes \mathcal{W} \otimes \mathcal{V} \perp \mathfrak{U} \otimes \mathcal{V} \otimes \mathcal{V}, \\
&= \mathcal{W} \otimes \mathcal{W} \otimes \mathcal{W} \perp \mathcal{V} \otimes \mathcal{W} \otimes \mathcal{W} \perp \mathcal{W} \otimes \mathcal{V} \otimes \mathcal{W} \perp \mathcal{V} \otimes \mathcal{V} \otimes \mathcal{W} \\
&\perp \mathcal{W} \otimes \mathcal{W} \otimes \mathcal{V} \perp \mathcal{V} \otimes \mathcal{W} \otimes \mathcal{V} \perp \mathcal{W} \otimes \mathcal{V} \otimes \mathcal{V} \perp \mathcal{V} \otimes \mathcal{V} \otimes \mathcal{V},
\end{aligned}$$

and

$$\begin{aligned}
Grad\mathbf{L} &= \mathbf{L}'\otimes\mathbf{e}+grad\mathbf{L}=\gamma'\mathbf{e}\otimes\mathbf{e}\otimes\mathbf{e}+\mathbf{u}'\otimes\mathbf{e}\otimes\mathbf{e}+\mathbf{e}\otimes\mathbf{v}'\otimes\mathbf{e}+\hat{\mathbf{L}}'\otimes\mathbf{e}\\
&\quad +grad\left(\gamma\mathbf{e}\otimes\mathbf{e}\right)+grad\left(\mathbf{u}\otimes\mathbf{e}+\mathbf{e}\otimes v\right)+grad\hat{\mathbf{L}},\\
&= \gamma'\mathbf{e}\otimes\mathbf{e}\otimes\mathbf{e}+\mathbf{u}'\otimes\mathbf{e}\otimes\mathbf{e}+\mathbf{e}\otimes\mathbf{v}'\otimes\mathbf{e}+\hat{\mathbf{L}}\otimes\mathbf{e}\\
&\quad +\gamma\mathbf{e}\otimes\mathbf{e}\otimes\overleftarrow{grad}+\mathbf{u}\otimes\mathbf{e}\otimes\overleftarrow{grad}+\mathbf{e}\otimes v\otimes\overleftarrow{grad}+grad\hat{\mathbf{L}}.
\end{aligned}$$

The trace of this field is given by [2]

$$tr\left[Grad\mathbf{L}\right]=\gamma'\mathbf{e}+\mathbf{u}'+tr\left[grad\mathbf{v}\right]\mathbf{e}+tr\left(grad\hat{\mathbf{L}}\right).$$

Another important operator is the divergence operator; it can be derived from the gradient operator because

$$Div\varpi=tr\left(Grad\varpi\right),$$

for any field ϖ. If $\varpi=\mathbf{u}=\mathbf{v}+\mathbf{e}w$ is a vector, then

$$Div\mathbf{u}=w'+div\mathbf{v}.$$

If $\varpi=\mathbf{L}=\gamma\mathbf{e}\otimes\mathbf{e}+\mathbf{u}\otimes\mathbf{e}+\mathbf{e}\otimes\mathbf{v}+\hat{\mathbf{L}}$ is a tensor, then

$$Div\mathbf{L}=\left(\gamma'+div\mathbf{v}\right)\mathbf{e}+\mathbf{u}'+div\hat{\mathbf{L}}. \tag{6.18}$$

Another useful operator is the *rot*. It is possible to define this operator in a general way, however, for the sake of simplicity we analyze it only for a cylindrical

[2]The trace of a third order tensor of this kind is, roughly speaking, a vector where the last tensor product of the tensor is substituted by a scalar product. In rectangular Cartesian coordinates:

$$\left[tr(\mathbf{L})\right]_i=L_{ihh}.$$

body. The *rot* operator on the cross-section is defined as

$$rot\varpi = *skwgrad\varpi.$$

If $\varpi = \mathbf{v}$ is a vector, then

$$rot\mathbf{v} = *skwgrad\mathbf{v}.$$

In Cartesian coordinates, we have

$$rot\mathbf{v} = [*]_{ij}\frac{1}{2}\left(v_{i,j} - v_{j,i}\right) = \frac{1}{2}\varepsilon_{ij}\left(-\partial_j v_i + \partial_i v_j\right) = \varepsilon_{ij}\partial_i v_j = \partial_1 v_2 - \partial_2 v_1.$$

If $\varpi = \hat{\mathbf{L}}$ is a tensor,[3] then

$$rot\hat{\mathbf{L}} = *skwgrad\hat{\mathbf{L}}.$$

In Cartesian coordinates

$$\begin{aligned}\left(rot\hat{\mathbf{L}}\right)_i &= [*]_{jh}\frac{1}{2}\left(\hat{L}_{ij,h} - \hat{L}_{ih,j}\right) = -\varepsilon_{jh}\frac{1}{2}\hat{L}_{ij,h} + \varepsilon_{jh}\frac{1}{2}\hat{L}_{ih,j}, \\ &= \frac{1}{2}\varepsilon_{jh}\left(-\partial_h\hat{L}_{ij} + \partial_j\hat{L}_{ih}\right) = \varepsilon_{jh}\partial_j\hat{L}_{ih} = \partial_1\hat{L}_{i2} - \partial_2\hat{L}_{i1}.\end{aligned} \tag{6.19}$$

Thus, in direct notation

$$\left(rot\hat{\mathbf{L}}\right)\cdot\mathbf{b} = rot\left(\hat{\mathbf{L}}^T\mathbf{b}\right), \qquad \forall\mathbf{b}\in\mathcal{V}.$$

[3] We have not defined the star of Hodge of a third order tensor. However in Cartesian coordinates

$$\left(*\mathbf{L}\right)_i = [*]_{jh}L_{ijh}.$$

Furthermore

$$\left(skw\mathbf{L}\right)_{ijh} = \frac{1}{2}\left(L_{ijh} - L_{ihj}\right).$$

Proposition 3 *The following useful formulae hold:*

$$
\begin{aligned}
rot\left(\alpha\hat{\mathbf{1}}\right) &= (*grad\alpha)\,, \qquad \alpha\in\mathbb{R};\\
rot\,(*\alpha) &= -\left[grad\,(\alpha)\right], \qquad \alpha\in\mathbb{R}.
\end{aligned}
\tag{6.20}
$$

Proof. Following the definition of the operator rot, we have

$$
\begin{aligned}
\left[rot\left(\alpha\hat{\mathbf{1}}\right)\right]\cdot\mathbf{b} &= rot\,(\alpha\mathbf{b}) = *skwgrad\,(\alpha\mathbf{b})\,,\\
&= *skw\left[\mathbf{b}\otimes grad\,(\alpha)\right] = *\left[\mathbf{b}\wedge grad\,(\alpha)\right],\\
&= -\mathbf{b}\times grad\,(\alpha) = grad\,(\alpha)\times\mathbf{b} = (*grad\alpha)\cdot\mathbf{b}, \quad \forall\mathbf{b}\in\mathcal{V}, \quad \forall\alpha\in\mathbb{R},
\end{aligned}
$$

and

$$
\begin{aligned}
\left[rot\,(*\alpha)\right]\cdot\mathbf{b} &= -rot\,((*\alpha)\,\mathbf{b}) = -*skwgrad\,((*\alpha)\,\mathbf{b})\,,\\
&= -*skwgrad\,(\alpha\,(*\mathbf{b})) = -*skw\left[\alpha\,(*\mathbf{b})\otimes\overleftarrow{grad}\right],\\
&= -*skw\left[(*\mathbf{b})\otimes grad\,(\alpha)\right] = -*\left[(*\mathbf{b})\wedge grad\,(\alpha)\right],\\
&= (*\mathbf{b})\times grad\,(\alpha) = -\mathbf{b}\cdot\left[grad\,(\alpha)\right], \qquad \forall\mathbf{b}\in\mathcal{V}, \quad \forall\alpha\in\mathbb{R}.
\end{aligned}
$$

Results (6.20) can be proved in an easier way using the Cartesian coordinates. For example,

$$
\begin{aligned}
\left[rot\left(\alpha\hat{\mathbf{1}}\right)\right]_i &= \varepsilon_{jh}\partial_j\left(\alpha\delta_{ih}\right) = \varepsilon_{ji}\partial_j\alpha = [*]_{ij}\,\partial_j\alpha = (*grad\alpha)_i\,,\\
\left[rot\,(*\alpha)\right]_i &= \varepsilon_{jh}\partial_j\left([*]_{ih}\,\alpha\right) = \varepsilon_{jh}\varepsilon_{hi}\partial_j\alpha = -\delta_{ji}\partial_j\alpha = -\partial_i\alpha = -\left[grad\alpha\right]_i.
\end{aligned}
$$

■

We now review some more useful formulae. $\forall\varphi\in\mathbb{R}$ and $\forall\mathbf{u}\in\mathfrak{U}$,

$$
\left\{
\begin{aligned}
&Grad\,(\varphi\mathbf{u}) = \varphi Grad\mathbf{u} + \mathbf{u}\otimes Grad\varphi,\\
&Div\,(grad\mathbf{u})^T = Grad\,(Div\mathbf{u})\,.
\end{aligned}
\right.
\tag{6.21}
$$

These can be proved by noting that

$$Grad\left(\varphi\mathbf{u}\right)=\left(\varphi\mathbf{u}\otimes\overleftarrow{Grad}\right)=\mathbf{u}\otimes Grad\varphi+\varphi\left(\mathbf{u}\otimes\overleftarrow{Grad}\right)=\mathbf{u}\otimes Grad\varphi+\varphi Grad\mathbf{u},$$

$$Div\left(Grad\mathbf{u}\right)^{T}=Div\left(Grad\otimes\mathbf{u}\right)=tr\left[Grad\otimes\mathbf{u}\otimes\overleftarrow{Grad}\right],$$

$$=Grad\left[tr\left(Grad\mathbf{u}\right)\right]=Grad\left[Div\mathbf{u}\right].$$

$\forall\varphi\in\mathbb{R},\forall\mathbf{a},\mathbf{b},\mathbf{c}\in\mathfrak{U},\forall\mathbf{v}\in\mathcal{V},\forall\mathbf{L}\in sym,\forall\mathbf{W}\in skw$, we have

$$\begin{aligned} Div\left(\hat{\mathbf{1}}\varphi\right) &= Grad\varphi, \quad rotgrad\mathbf{v}=0, \quad tr\left(rot\mathbf{L}\right)=0, \\ \mathbf{a}\times\left(\mathbf{b}\times\mathbf{c}\right) &= \mathbf{b}\left(\mathbf{a}\cdot\mathbf{c}\right)-\mathbf{c}\left(\mathbf{a}\cdot\mathbf{b}\right). \end{aligned} \tag{6.22}$$

In Cartesian coordinates, the validity of formulae (6.22) can be easily established.

$$\left[Div\left(\hat{\mathbf{1}}\varphi\right)\right]_{i}=\partial_{j}\left(\delta_{ij}\varphi\right)=\partial_{i}\varphi=\left[Grad\varphi\right]_{i},$$

$$rotgrad\mathbf{v}=\varepsilon_{ij}\partial_{i}\left(grad\mathbf{v}\right)_{hj}=\varepsilon_{ij}\partial_{ij}v_{h}=0,$$

$$\left[tr\left(rot\mathbf{L}\right)\right]_{ii}=\varepsilon_{ihk}\partial_{h}L_{ik},$$

$$\varepsilon_{hij}\varepsilon_{hlm}=\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl},$$

$$\left[\mathbf{a}\times\left(\mathbf{b}\times\mathbf{c}\right)\right]_{i}=\varepsilon_{ijh}a_{j}\left(\mathbf{b}\times\mathbf{c}\right)_{h}=\varepsilon_{ijh}a_{j}\varepsilon_{hlm}b_{l}c_{m}=a_{j}b_{l}c_{m}\left(\varepsilon_{hij}\varepsilon_{hlm}\right),$$

$$=a_{j}b_{l}c_{m}\left(\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl}\right)=a_{m}b_{i}c_{m}-a_{l}b_{l}c_{i}=\left[\mathbf{b}\left(\mathbf{a}\cdot\mathbf{c}\right)-\mathbf{c}\left(\mathbf{a}\cdot\mathbf{b}\right)\right]_{i}.$$

All of the formulae illustrated above for vectors in a three dimensional space $\mathfrak{U}$ are also valid in a two dimensional space $\mathcal{V}$. The following two relations are also quite useful.

$$div\left(*\hat{\mathbf{X}}\right)=\partial_{i}\left(-\varepsilon_{ij}X_{j}\right)=-\varepsilon_{ij}\delta_{ij}=0, \tag{6.23}$$

$$rot\left(*\hat{\mathbf{X}}\right)=\varepsilon_{ij}\partial_{i}\left(-\varepsilon_{jh}X_{h}\right)=-\left(-\delta_{ih}\right)\delta_{ih}=2. \tag{6.24}$$

Because of (6.11) the Laplacian operator, Δ_R, is defined as

$$\Delta_{R}\mathbf{u}\equiv Div\left(Grad\mathbf{u}\right),\quad\forall\mathbf{u}\in\mathfrak{U}. \tag{6.25}$$

An Integral formula

Let the path from the origin O to a general point $\hat{\mathbf{X}}$ of the cross-section be denoted by $\hat{\Gamma}\left(O, \hat{\mathbf{X}}\right) = \hat{\Gamma}$. The integral I of a function $\mathbf{f}\left(\hat{\mathbf{R}}\right)$ on the path $\hat{\Gamma}$ can be written as

$$I = \int_{\hat{\Gamma}} \mathbf{f}\left(\hat{\mathbf{R}}\right) \cdot \mathbf{l}_{\hat{\Gamma}} ds, \tag{6.26}$$

where

$$\begin{aligned}
\mathbf{l}_{\hat{\Gamma}} &= \frac{\hat{R}}{\hat{X}}\hat{\mathbf{X}}\left(\frac{1}{\hat{R}}\right) = \frac{1}{\hat{X}}\hat{\mathbf{X}}, \\
\hat{\mathbf{R}} &= \frac{\hat{R}}{\hat{X}}\hat{\mathbf{X}}, \\
\hat{X} &= \left|\hat{\mathbf{X}}\right|, \\
\hat{R} &= \left|\hat{\mathbf{R}}\right|.
\end{aligned}$$

The integration in (6.26) can also be performed from 0 to $\hat{X}$. That is,

$$I = \int_0^{\hat{X}} \mathbf{f}\left(\frac{\hat{R}}{\hat{X}}\hat{\mathbf{X}}\right) \cdot \hat{\mathbf{X}}\frac{1}{\hat{X}} d\hat{R}. \tag{6.27}$$

Solution of four boundary value problems

Problem A

The following Neumann problem

$$\begin{cases}
\triangle_R w = 0, & \forall \mathbf{X} \in \mathfrak{D}, \\
grad w \cdot \mathbf{N} = -\left[\mathbf{v}^0 + \varpi\left(*\hat{\mathbf{X}}\right)\right] \cdot \mathbf{N}, & \forall \mathbf{X} \in \partial\mathfrak{D},
\end{cases} \tag{6.28}$$

has a unique solution to within an additive constant. A solution for the gradient of w is

$$grad w = -\left[\mathbf{v}^0 + \varpi\left(*\hat{\mathbf{X}}\right)\right] + 2\varpi\mathbf{f}, \tag{6.29}$$

where

$$\begin{aligned} div\mathbf{f} = 0, rot\mathbf{f} = 1, \qquad &\forall\hat{\mathbf{X}} \in \mathfrak{D}, \\ \mathbf{f}\cdot\mathbf{N} = 0, \qquad &\forall\hat{\mathbf{X}} \in \partial\mathfrak{D}. \end{aligned} \tag{6.30}$$

Here $\mathbf{N}$ is an outward unit normal to $\partial\mathfrak{D}$, and $\mathbf{v}^0$ and ϖ are constants.

Because of $(6.23)_1$,

$$div\left\{-\left[\mathbf{v}^0 + \varpi\left(*\hat{\mathbf{X}}\right)\right] + 2\varpi\mathbf{f}\right\} = \varpi div\left(*\hat{\mathbf{X}}\right) + 2\varpi div\mathbf{f} = 0 + 0 = 0.$$

Using (6.24), we get $\forall\mathbf{X} \in \mathfrak{D}$

$$rot\left(grad w\right) = rot\left\{-\left[\mathbf{v}^0 + \varpi\left(*\hat{\mathbf{X}}\right)\right] + 2\varpi\mathbf{f}\right\} = \varpi\left\{-rot\left(*\hat{\mathbf{X}}\right) + 2\right\} = 0.$$

Thus (6.29) satisfies (6.28) provided that (6.30) holds.

Problem B

The following boundary value problem

$$\begin{cases} div\left(2\mu Symgrad\mathbf{v} + \lambda\hat{\mathbf{1}}\left(div\mathbf{v} - \tilde{\mathbf{v}}^0\cdot\hat{\mathbf{X}}\right)\right) = \mathbf{0}, & \forall\mathbf{X} \in \mathfrak{D}, \\ \left(2\mu Symgrad\mathbf{v} + \lambda\hat{\mathbf{1}}\left(div\mathbf{v} - \tilde{\mathbf{v}}^0\cdot\hat{\mathbf{X}}\right)\right)\mathbf{N} = -\lambda w^0\mathbf{N}, & \forall\mathbf{X} \in \partial\mathfrak{D}, \end{cases} \tag{6.31}$$

has a unique solution to within a rigid motion. Here λ, μ, $\tilde{\mathbf{v}}^0$ and w^0 are constants and $\hat{\mathbf{1}}$ is the two dimensional identity tensor. A solution of (6.31) satisfies

$$2\mu Symgrad\mathbf{v} + \lambda\hat{\mathbf{1}}\left(div\mathbf{v} - \tilde{\mathbf{v}}^0\cdot\hat{\mathbf{X}}\right) = -\lambda w^0\hat{\mathbf{1}}. \tag{6.32}$$

Taking the trace of both sides of (6.32), we obtain

$$div\mathbf{v} = \frac{-\lambda\left(w^0 - \tilde{\mathbf{v}}^0 \cdot \hat{\mathbf{X}}\right)}{(\lambda+\mu)},$$

where we have implicitly assumed that $\lambda + \mu \neq 0$.

From (6.32) we can evaluate the gradient of the vector field $\mathbf{v}$:

$$2\mu Symgrad\mathbf{v} = -\lambda w^0\hat{\mathbf{1}} - \lambda\hat{\mathbf{1}}\left[\frac{-\lambda\left(w^0 - \tilde{\mathbf{v}}^0 \cdot \hat{\mathbf{X}}\right)}{(\lambda+\mu)} - \tilde{\mathbf{v}}^0 \cdot \hat{\mathbf{X}}\right],$$

$$= \lambda\left(-w^0 + \tilde{\mathbf{v}}^0 \cdot \hat{\mathbf{X}}\right)\hat{\mathbf{1}} + \lambda\hat{\mathbf{1}}\left[\frac{\left(w^0 - \tilde{\mathbf{v}}^0 \cdot \hat{\mathbf{X}}\right)\lambda}{(\lambda+\mu)}\right],$$

$$= -\lambda\hat{\mathbf{1}}\left(w^0 - \tilde{\mathbf{v}}^0 \cdot \hat{\mathbf{X}}\right)\left(1 - \frac{\lambda}{\lambda+\mu}\right) = -\mu\left(\frac{\lambda}{\lambda+\mu}\right)\left(w^0 - \tilde{\mathbf{v}}^0 \cdot \hat{\mathbf{X}}\right)\hat{\mathbf{1}}.$$

Recalling the definition

$$\nu \equiv \frac{\lambda}{2(\lambda+\mu)}$$

of the Poisson ratio and defining

$$w \equiv w^0 - \tilde{\mathbf{v}}^0 \cdot \hat{\mathbf{X}}, \tag{6.33}$$

we get

$$Symgrad\mathbf{v} = -\nu w\hat{\mathbf{1}}.$$

From the theorem of completion we know that there exists a unique skew-symmetric part of the gradient of $\mathbf{v}$ and it is denoted by the Hodge star of a scalar function $\varphi\left(\hat{\mathbf{X}}\right)$. Thus

$$grad\mathbf{v} = *\varphi - \nu w\hat{\mathbf{1}}. \tag{6.34}$$

The compatibility condition

$$\mathbf{0} = rot\,(grad\mathbf{v}) = rot\left(*\varphi - \nu w\hat{\mathbf{1}}\right),$$

gives a form for the function φ. From (6.20) we have

$$\mathbf{0} = -grad\,(\varphi) - \nu\,(*gradw)\,,$$

and now using (6.33) we arrive at

$$grad\,(\varphi) = \nu\left(*\tilde{\mathbf{v}}^0\right).$$

Hence

$$\varphi = \varphi^0 + \nu\left(*\tilde{\mathbf{v}}^0\right)\cdot\hat{\mathbf{X}}. \tag{6.35}$$

The application of the Hodge $*$ operator to both sides of (6.35) gives

$$*\varphi = *\varphi^0 + \nu * \left[\left(*\tilde{\mathbf{v}}^0\right)\cdot\hat{\mathbf{X}}\right] = *\varphi^0 - \nu\left(\tilde{\mathbf{v}}^0 \wedge \hat{\mathbf{X}}\right), \tag{6.36}$$

where we have used (6.16). Substitution from (6.36) into (6.34) gives

$$grad\mathbf{v} = *\varphi^0 - \nu\left(\tilde{\mathbf{v}}^0 \wedge \hat{\mathbf{X}}\right) - \nu w\hat{\mathbf{1}}. \tag{6.37}$$

Substituting in (6.37) for w from (6.33), and using (6.27) to integrate (6.37), we get

$$\begin{aligned}
\mathbf{v}\left(\hat{\mathbf{X}}\right) &= \mathbf{v}^0 + \int_0^{|\hat{\mathbf{x}}|}\left\{*\varphi^0 - \nu\left(\tilde{\mathbf{v}}^0 \wedge \hat{\mathbf{X}}\right)\frac{\hat{R}}{\hat{X}} - \nu\left[w^0 - \tilde{\mathbf{v}}^0\cdot\hat{\mathbf{X}}\frac{\hat{R}}{\hat{X}}\right]\hat{1}\right\}\hat{\mathbf{X}}\frac{1}{\hat{X}}d\hat{R},\\
&= \mathbf{v}^0 + \left\{*\varphi^0 - \nu\left(\tilde{\mathbf{v}}^0 \wedge \hat{\mathbf{X}}\right)\frac{1}{2} + \nu\left[\tilde{\mathbf{v}}^0\cdot\hat{\mathbf{X}}\frac{1}{2} - w^0\right]\hat{1}\right\}\hat{\mathbf{X}},\\
&= \mathbf{v}^0 + \left(*\varphi^0\right)\hat{\mathbf{X}} + \frac{1}{2}\nu\left[-\left(\tilde{\mathbf{v}}^0 \wedge \hat{\mathbf{X}}\right)\hat{\mathbf{X}} + \left(\tilde{\mathbf{v}}^0\cdot\hat{\mathbf{X}}\right)\hat{\mathbf{X}}\right] - \nu w^0\hat{\mathbf{X}}.
\end{aligned}$$

Because of (6.16)

$$-\left(\tilde{\mathbf{v}}^0\wedge\hat{\mathbf{X}}\right)\hat{\mathbf{X}}+\left(\tilde{\mathbf{v}}^0\cdot\hat{\mathbf{X}}\right)\hat{\mathbf{X}}=\left[\left(*\tilde{\mathbf{v}}^0\right)\cdot\hat{\mathbf{X}}\right]\left(*\hat{\mathbf{X}}\right)+\left[\left(*\tilde{\mathbf{v}}^0\right)\cdot\left(*\hat{\mathbf{X}}\right)\right]\hat{\mathbf{X}},$$
$$=\left[\left(*\hat{\mathbf{X}}\right)\otimes\hat{\mathbf{X}}\right]\left(*\tilde{\mathbf{v}}^0\right)+\left[\hat{\mathbf{X}}\otimes\left(*\hat{\mathbf{X}}\right)\right]\left(*\tilde{\mathbf{v}}^0\right)=2\left\{sym\left[\left(*\hat{\mathbf{X}}\right)\otimes\hat{\mathbf{X}}\right]\right\}\left(*\tilde{\mathbf{v}}^0\right).$$

Thus

$$\mathbf{v}\left(\hat{\mathbf{X}}\right)=\mathbf{v}^0+\left(*\varphi^0\right)\hat{\mathbf{X}}+\nu\left\{sym\left[\hat{\mathbf{X}}\otimes\left(*\hat{\mathbf{X}}\right)\right]\right\}\left(*\tilde{\mathbf{v}}^0\right)-\nu w^0\hat{\mathbf{X}},\quad\forall\hat{\mathbf{X}}\in\mathfrak{D}.\quad(6.38)$$

Lemma 4 *It is important to note that if*

$$\tilde{\mathbf{v}}^0=\mathbf{0},\quad w^0=0,$$

then the solution of the boundary value problem (6.31) is

$$\mathbf{v}\left(\hat{\mathbf{X}}\right)=\mathbf{v}^0+\left(*\varphi^0\right)\hat{\mathbf{X}},$$

or equivalently

$$\mathbf{v}\left(\hat{\mathbf{X}}\right)=\mathbf{v}^0+\varpi\left(*\hat{\mathbf{X}}\right),$$

which is a rigid body motion.

Problem C

For

$$\int\limits_{\mathfrak{D}(Z)}\left(\mathbf{v}+grad w\right)dS_*=\mathbf{0},\quad(6.39)$$

$$\int\limits_{\mathfrak{D}(Z)}\left(*\hat{\mathbf{X}}\right)\cdot\left(\mathbf{v}+grad w\right)dS_*=0,\quad(6.40)$$

where $\forall \mathbf{X} \in \mathfrak{D}$,

$$\begin{aligned} \mathbf{v} &= \mathbf{v}^0 + \varpi\left(*\hat{\mathbf{X}}\right), \\ grad w &= -\left[\mathbf{v}^0 + \varpi\left(*\hat{\mathbf{X}}\right)\right] + 2\varpi\mathbf{f}, \\ div\mathbf{f} &= 0, rot\mathbf{f} = 1, \end{aligned}$$

and $\forall \mathbf{X} \in \partial\mathfrak{D}$,

$$\mathbf{f} \cdot \mathbf{N} = 0,$$

find ϖ.

Solution. Substitution for $\mathbf{v}$ and $grad w$ into (6.39) and (6.40) gives

$$\int_{\mathfrak{D}(Z)} \left(\mathbf{v}^0 + \varpi\left(*\hat{\mathbf{X}}\right) - \left[\mathbf{v}^0 + \varpi\left(*\hat{\mathbf{X}}\right)\right] + 2\varpi\mathbf{f}\right) dS_* = 2\varpi \int_{\mathfrak{D}(Z)} \mathbf{f} dS_* = \mathbf{0},$$

$$2\varpi \int_{\mathfrak{D}(Z)} \left(*\hat{\mathbf{X}}\right) \cdot \mathbf{f} dS_* = 0.$$

Since conditions on $\mathbf{f}$ do not guarantee that $\int_{\mathfrak{D}(Z)} \left(*\hat{\mathbf{X}}\right) \cdot \mathbf{f} dS_*$ always vanishes, therefore we conclude that

$$\varpi = 0.$$

Problem D

Find ϖ and $\tilde{\mathbf{v}}^0$ such that

$$
\begin{cases}
\int_{\mathfrak{D}(Z)} [(\lambda+2\mu)\,w+\lambda div\mathbf{v}]\,dS_* = 0, \\
\int_{\mathfrak{D}(Z)} [(\lambda+2\mu)\,w+\lambda div\mathbf{v}]\left(*\hat{\mathbf{X}}\right) dS_* = 0, \\
div\mathbf{v} = \dfrac{-\lambda w}{(\lambda+\mu)}, \\
w = \left(w^0 - \tilde{\mathbf{v}}^0 \cdot \hat{\mathbf{X}}\right).
\end{cases}
\tag{6.41}
$$

Solution. We substitute in the first integral the given expressions for w and $div\mathbf{v}$ and obtain

$$
\begin{aligned}
&\int_{\mathfrak{D}(Z)} \left[(\lambda+2\mu)\left(w^0 - \tilde{\mathbf{v}}^0 \cdot \hat{\mathbf{X}}\right) + \lambda\frac{-\lambda}{(\lambda+\mu)}\left(w^0 - \tilde{\mathbf{v}}^0 \cdot \hat{\mathbf{X}}\right)\right] dS_* \\
&\qquad = A_{\mathfrak{D}}\left(\lambda+2\mu-\frac{\lambda^2}{(\lambda+\mu)}\right)\left(w^0 - \tilde{\mathbf{v}}^0 \cdot \hat{\mathbf{b}}\right) = 0,
\end{aligned}
\tag{6.42}
$$

where $A_{\mathfrak{D}}$ is the area of cross-section. For the integral $(6.41)_2$ we have

$$
\begin{cases}
\int_{\mathfrak{D}(Z)} \left[(\lambda+2\mu)\left(w^0 - \tilde{\mathbf{v}}^0 \cdot \hat{\mathbf{X}}\right) + \lambda\dfrac{-\lambda}{(\lambda+\mu)}\left(w^0 - \tilde{\mathbf{v}}^0 \cdot \hat{\mathbf{X}}\right)\right]\left(*\hat{\mathbf{X}}\right) dS_* \\
= \left(\lambda+2\mu-\dfrac{\lambda^2}{(\lambda+\mu)}\right)\left(A_{\mathfrak{D}} w^0\left(*\hat{\mathbf{b}}\right) - \int_{\mathfrak{D}(Z)}\left(\tilde{\mathbf{v}}^0 \cdot \hat{\mathbf{X}}\right)\left(*\hat{\mathbf{X}}\right) dS_*\right) = 0.
\end{cases}
\tag{6.43}
$$

Thus

$$
w^0 = \tilde{\mathbf{v}}^0 \cdot \hat{\mathbf{b}}. \tag{6.44}
$$

We note that (6.44) holds even if $(6.41)_2$ does not (problem D_*). Substituting (6.44) into (6.43) we get

$$
\left\{A_{\mathfrak{D}}\left[\left(*\hat{\mathbf{b}}\right)\otimes\hat{\mathbf{b}}\right]\right\}\tilde{\mathbf{v}}^0 = \left\{\int_{\mathfrak{D}(Z)}\left[\left(*\hat{\mathbf{X}}\right)\otimes\hat{\mathbf{X}}\right] dS_*\right\}\tilde{\mathbf{v}}^0.
$$

Since the matrices in braces are not equal to each other, therefore

$$
\tilde{\mathbf{v}}^0 = \mathbf{0}, w^0 = 0.
$$

Cylindrical coordinates

With the origin of the cylindrical and the Cartesian coordinates coincident with each other, the in-plane cylindrical (r, θ) and Cartesian coordinates (X_1, X_2) of the same point are related to each other by

$$X_1 = r\cos\theta,$$

$$X_2 = r\sin\theta,$$

or

$$r = \sqrt{(X_1)^2 + (X_2)^2},$$

$$\theta = \arctan\left(\tfrac{X_2}{X_1}\right).$$

Thus

$$\frac{\partial r}{\partial X_1} = \cos\theta, \frac{\partial r}{\partial X_2} = \sin\theta;$$

$$\frac{\partial \theta}{\partial X_1} = -\frac{\frac{X_2}{(X_1)^2}}{1+\left(\frac{X_2}{X_1}\right)^2} = -\frac{X_2}{r^2} = -\frac{\sin\theta}{r},$$

$$\frac{\partial \theta}{\partial X_2} = \frac{\frac{1}{X_1}}{1+\left(\frac{X_2}{X_1}\right)^2} = \frac{X_1}{r^2} = \frac{\cos\theta}{r}.$$

The in-plane position vector of a point can be written as

$$\mathbf{X} = X_1\mathbf{e}_1 + X_2\mathbf{e}_2 = r\hat{\mathbf{r}},$$

where

$$\hat{\mathbf{r}} = \cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2,$$

and the unit vector normal to it

$$\hat{\boldsymbol{\theta}} = -\sin\theta\mathbf{e}_1 + \cos\theta\mathbf{e}_2,$$

when combined with $\mathbf{e}_3$ make a right handed orthonormal set. Conversely:

$$\mathbf{e}_1 = \cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}},$$

$$\mathbf{e}_2 = \sin\theta\hat{\mathbf{r}} + \cos\theta\hat{\boldsymbol{\theta}}.$$

Whereas a constant vector $\mathbf{v}^0$ in a Cartesian co-ordinate system has constant components, its components in a cylindrical co-ordinate system are not constants. For example,

$$\begin{aligned}
\mathbf{v}^0 &= v_x^0\mathbf{e}_1 + v_y^0\mathbf{e}_2 = v_x^0\left(\cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}}\right) + v_y^0\left(\sin\theta\hat{\mathbf{r}} + \cos\theta\hat{\boldsymbol{\theta}}\right), \\
&= \left(v_x^0\cos\theta + v_y^0\sin\theta\right)\hat{\mathbf{r}} + \left(-v_x^0\sin\theta + v_y^0\cos\theta\right)\hat{\boldsymbol{\theta}}, \\
&= v_r^0\hat{\mathbf{r}} + v_\theta^0\hat{\boldsymbol{\theta}}.
\end{aligned}$$

Thus

$$v_r^0 = v_x^0\cos\theta + v_y^0\sin\theta,$$

$$v_\theta^0 = -v_x^0\sin\theta + v_y^0\cos\theta,$$

and values of v_r^0 and v_θ^0 vary with the angle θ. We note that $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ depend upon θ and not upon r. Furthermore,

$$\begin{aligned}
\frac{\partial\hat{\mathbf{r}}}{\partial r} &= \mathbf{0}, \quad \frac{\partial\hat{\boldsymbol{\theta}}}{\partial r} = \mathbf{0}, \quad \frac{\partial\hat{\mathbf{r}}}{\partial\theta} = \hat{\boldsymbol{\theta}}, \quad \frac{\partial\hat{\boldsymbol{\theta}}}{\partial\theta} = -\hat{\mathbf{r}}; \\
\frac{\partial\left(\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}\right)}{\partial\theta} &= \hat{\boldsymbol{\theta}}\otimes\hat{\mathbf{r}} + \hat{\mathbf{r}}\otimes\hat{\boldsymbol{\theta}}, \quad \frac{\partial\left(\hat{\mathbf{r}}\otimes\hat{\boldsymbol{\theta}}\right)}{\partial\theta} = \hat{\boldsymbol{\theta}}\otimes\hat{\boldsymbol{\theta}} - \hat{\mathbf{r}}\otimes\hat{\mathbf{r}}; \\
\frac{\partial\left(\hat{\boldsymbol{\theta}}\otimes\hat{\mathbf{r}}\right)}{\partial\theta} &= -\hat{\mathbf{r}}\otimes\hat{\mathbf{r}} + \hat{\boldsymbol{\theta}}\otimes\hat{\boldsymbol{\theta}}, \quad \frac{\partial\left(\hat{\boldsymbol{\theta}}\otimes\hat{\boldsymbol{\theta}}\right)}{\partial\theta} = -\hat{\mathbf{r}}\otimes\hat{\boldsymbol{\theta}} - \hat{\boldsymbol{\theta}}\otimes\hat{\mathbf{r}}.
\end{aligned}$$

The nabla operator, $\bigtriangledown$, is defined as

$$(\cdot)\bigtriangledown = (\cdot)\left(\overleftarrow{\partial}_{X_1}\mathbf{e}_1 + \overleftarrow{\partial}_{X_2}\mathbf{e}_2\right), \tag{6.45}$$

where $\overleftarrow{\partial}_x$ means the derivative with respect to x of the quantity preceding it; in the same way ∂_x means the derivative with respect to x of the quantity following it. In polar coordinates we have

$$\begin{aligned}\partial_{X_1} &= \frac{\partial r}{\partial X_1}\partial_r + \frac{\partial \theta}{\partial X_1}\partial_\theta = (\cos\theta)\,\partial_r - \frac{\sin\theta}{r}\partial_\theta, \\ \partial_{X_2} &= \frac{\partial r}{\partial X_2}\partial_r + \frac{\partial \theta}{\partial X_2}\partial_\theta = (\sin\theta)\,\partial_r + \frac{\cos\theta}{r}\partial_\theta,\end{aligned}$$

and

$$\begin{aligned}\nabla &= \mathbf{e}_1\partial_{X_1} + \mathbf{e}_2\partial_{X_2} = \left[\cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}}\right]\left[(\cos\theta)\,\partial_r - \frac{\sin\theta}{r}\partial_\theta\right] \\ &\quad + \left[\sin\theta\hat{\mathbf{r}} + \cos\theta\hat{\boldsymbol{\theta}}\right]\left[(\sin\theta)\,\partial_r + \frac{\cos\theta}{r}\partial_\theta\right], \\ &= \hat{\mathbf{r}}\partial_r + \hat{\boldsymbol{\theta}}\frac{1}{r}\partial_\theta.\end{aligned}$$

From the definition (6.45) we have the following forms for the *grad* and the *div* operators

$$\begin{cases} grad\varpi \equiv \varpi \otimes \nabla = \varpi \otimes \left(\overleftarrow{\partial}_r\hat{\mathbf{r}} + \overleftarrow{\partial}_\theta\frac{1}{r}\hat{\boldsymbol{\theta}}\right), \\ div\varpi \equiv tr\left[\varpi \otimes \nabla\right], \end{cases}$$

where ϖ is a general scalar, vector or tensor field. More specifically if $\varpi = w$ is a scalar, then

$$grad w = w_{,r}\hat{\mathbf{r}} + \frac{1}{r}w_{,\theta}\hat{\boldsymbol{\theta}},$$

and the divergence of a scalar is not defined.

For $\varpi = \mathbf{v}$, a vector,

$$grad\mathbf{v} = \left(v_r \hat{\mathbf{r}} + v_\theta \hat{\boldsymbol{\theta}}\right) \otimes \left(\overleftarrow{\partial}_r \hat{\mathbf{r}} + \overleftarrow{\partial}_\theta \frac{1}{r} \hat{\boldsymbol{\theta}}\right) = v_{r,r} \left(\hat{\mathbf{r}} \otimes \hat{\mathbf{r}}\right) + v_{\theta,r} \left(\hat{\boldsymbol{\theta}} \otimes \hat{\mathbf{r}}\right)$$

$$+ \frac{1}{r} v_{r,\theta} \left(\hat{\mathbf{r}} \otimes \hat{\boldsymbol{\theta}}\right) + \frac{1}{r} v_r \left(\hat{\boldsymbol{\theta}} \otimes \hat{\boldsymbol{\theta}}\right) + \frac{1}{r} v_{\theta,\theta} \left(\hat{\boldsymbol{\theta}} \otimes \hat{\boldsymbol{\theta}}\right) - \frac{1}{r} v_\theta \left(\hat{\mathbf{r}} \otimes \hat{\boldsymbol{\theta}}\right),$$

$$= v_{r,r} \left(\hat{\mathbf{r}} \otimes \hat{\mathbf{r}}\right) + \left(\frac{1}{r} v_{r,\theta} - \frac{1}{r} v_\theta\right) \left(\hat{\mathbf{r}} \otimes \hat{\boldsymbol{\theta}}\right) + v_{\theta,r} \left(\hat{\boldsymbol{\theta}} \otimes \hat{\mathbf{r}}\right) + \left(\frac{1}{r} v_r + \frac{1}{r} v_{\theta,\theta}\right) \left(\hat{\boldsymbol{\theta}} \otimes \hat{\boldsymbol{\theta}}\right),$$

where

$$v_{r,r} = \frac{\partial v_r}{\partial r}, etc.$$

In matrix form

$$grad\mathbf{v} = \begin{pmatrix} v_{r,r} & \frac{1}{r}\left(v_{r,\theta} - v_\theta\right) \\ v_{\theta,r} & \frac{1}{r}\left(v_r + v_{\theta,\theta}\right) \end{pmatrix},$$

and

$$div\mathbf{v} = tr\left(grad\mathbf{v}\right) = v_{r,r} + \frac{1}{r} v_r + \frac{1}{r} v_{\theta,\theta} = \frac{1}{r}\left(r v_r\right)_{,r} + \frac{1}{r} v_{\theta,\theta}.$$

When $\varpi = \mathbf{T}$ is a tensor field, we usually do not need its gradient; however we do need its divergence.

$$div\mathbf{T} = \left[T_{rr}\left(\hat{\mathbf{r}} \otimes \hat{\mathbf{r}}\right) + T_{r\theta}\left(\hat{\mathbf{r}} \otimes \hat{\boldsymbol{\theta}}\right) + T_{\theta r}\left(\hat{\boldsymbol{\theta}} \otimes \hat{\mathbf{r}}\right) + T_{\theta\theta}\left(\hat{\boldsymbol{\theta}} \otimes \hat{\boldsymbol{\theta}}\right)\right] \cdot \left(\overleftarrow{\partial}_r \hat{\mathbf{r}} + \overleftarrow{\partial}_\theta \frac{1}{r} \hat{\boldsymbol{\theta}}\right),$$

$$= \left[T_{rr,r} \hat{\mathbf{r}} + T_{\theta r,r} \hat{\boldsymbol{\theta}}\right] + \frac{1}{r}\left[T_{r\theta,\theta} \hat{\mathbf{r}} + T_{\theta\theta,\theta} \hat{\boldsymbol{\theta}}\right] + \frac{1}{r} T_{rr} \hat{\mathbf{r}} + \frac{1}{r} T_{r\theta} \hat{\boldsymbol{\theta}} + \frac{1}{r} T_{\theta r} \hat{\boldsymbol{\theta}} - \frac{1}{r} T_{\theta\theta} \hat{\mathbf{r}},$$

$$= \hat{\mathbf{r}}\left[T_{rr,r} + \frac{1}{r}\left(T_{r\theta,\theta} + T_{rr} - T_{\theta\theta}\right)\right] + \hat{\boldsymbol{\theta}}\left[T_{\theta r,r} + \frac{1}{r}\left(T_{\theta\theta,\theta} + T_{r\theta} + T_{\theta r}\right)\right].$$

In matrix form

$$div\mathbf{T} = div \begin{pmatrix} T_{rr} & T_{r\theta} \\ T_{\theta r} & T_{\theta\theta} \end{pmatrix} = \begin{pmatrix} T_{rr,r} + \frac{1}{r}\left(T_{r\theta,\theta} + T_{rr} - T_{\theta\theta}\right) \\ T_{\theta r,r} + \frac{1}{r}\left(T_{\theta\theta,\theta} + T_{r\theta} + T_{\theta r}\right) \end{pmatrix}.$$

The matrix representation for the star of Hodge operator is

$$* = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We can develop the following representations:

$$\begin{aligned}
\hat{\mathbf{X}} &= \begin{pmatrix} r \\ 0 \end{pmatrix}, \\
*\hat{\mathbf{X}} &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} r \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ r \end{pmatrix}, \\
*\varphi &= \begin{pmatrix} 0 & -\varphi \\ \varphi & 0 \end{pmatrix}, \\
(*\varphi)\,\hat{\mathbf{X}} &= \begin{pmatrix} 0 & -\varphi \\ \varphi & 0 \end{pmatrix} \begin{pmatrix} r \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \varphi r \end{pmatrix}, \\
*\mathbf{v} &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} v_r \\ v_\theta \end{pmatrix} = \begin{pmatrix} -v_\theta \\ v_r \end{pmatrix}, \\
\hat{\mathbf{X}} \otimes \left(*\hat{\mathbf{X}}\right) &= \begin{pmatrix} r \\ 0 \end{pmatrix} \begin{pmatrix} 0 & r \end{pmatrix} = \begin{pmatrix} 0 & r^2 \\ 0 & 0 \end{pmatrix}, \\
Sym\left[\hat{\mathbf{X}} \otimes \left(*\hat{\mathbf{X}}\right)\right] &= \begin{pmatrix} 0 & \frac{1}{2}r^2 \\ \frac{1}{2}r^2 & 0 \end{pmatrix}, \\
\left\{Sym\left[\hat{\mathbf{X}} \otimes \left(*\hat{\mathbf{X}}\right)\right]\right\}(*\mathbf{v}) &= \begin{pmatrix} 0 & \frac{1}{2}r^2 \\ \frac{1}{2}r^2 & 0 \end{pmatrix} \begin{pmatrix} -v_\theta \\ v_r \end{pmatrix} = \frac{r^2}{2} \begin{pmatrix} v_r \\ -v_\theta \end{pmatrix}.
\end{aligned}$$

Generalized Axisymmetric Problems

We consider the class of problems for which the displacement field is a general function of the radial co-ordinate but a polynomial in the axial co-ordinate Z. The tangential or the circumferential component of the displacement need not vanish.

We assume that the shape of the cross-section, loads and components of the displacement field satisfy following conditions.

1. The external boundary $\partial\mathfrak{D}^0$ is a circle of radius R_e and the cross- section has at most one circular hole $(\Lambda = 1)$ of radius R_i. That is

$$\mathfrak{D} \equiv \left\{ \hat{\mathbf{X}} = r\hat{\mathbf{r}} \quad : \quad r \in [R_i, R_e] \quad , \quad \theta \in [0, 2\pi] \right\}.$$

2. The end faces of the cylindrical body are subjected to only extensional and torsional loads: bending and flexure loads are zero. Thus

$$\mathbf{f}_0^p = \mathbf{0}, \mathbf{m}_0^p = \mathbf{0}.$$

The loads on the mantle (4.1) are

$$\begin{aligned} \mathbf{p}_i\left(\hat{\mathbf{X}}\right) &= p_{ir}(r)\,\hat{\mathbf{r}} + p_{i\theta}(r)\,\hat{\boldsymbol{\theta}}, \\ a_i\left(\hat{\mathbf{X}}\right) &= a_i(r). \end{aligned}$$

Hence

$$p_{ir}(r) = \begin{cases} p_{ir}^e & r = R_e, \\ p_{ir}^i & r = R_i; \end{cases}$$

$$p_{i\theta}(r)=\begin{cases} p_{i\theta}^{ext} & r=R_e, \\ p_{i\theta}^{int} & r=R_i; \end{cases}$$

$$a_i(r)=\begin{cases} a_i^{ext} & r=R_e, \\ a_i^{int} & r=R_i. \end{cases}$$

3. The displacement field has the form

$$\mathbf{v}_i\left(\hat{\mathbf{X}}\right)=v_{ir}(r)\,\hat{\mathbf{r}}+v_{i\theta}(r)\,\hat{\boldsymbol{\theta}},$$
$$w_i\left(\hat{\mathbf{X}}\right)=w_i(r).$$

In cylindrical coordinates, and recalling that displacements and stresses are independent of the angular position θ, differential operators introduced above have the following simple forms.

$$grad w=w_{,r}\hat{\mathbf{r}},$$
$$grad\mathbf{v}=v_{r,r}\left(\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}\right)-\tfrac{1}{r}v_\theta\left(\hat{\mathbf{r}}\otimes\hat{\boldsymbol{\theta}}\right)+v_{\theta,r}\left(\hat{\boldsymbol{\theta}}\otimes\hat{\mathbf{r}}\right)+\tfrac{1}{r}v_r\left(\hat{\boldsymbol{\theta}}\otimes\hat{\boldsymbol{\theta}}\right),$$
$$div\mathbf{v}=\tfrac{1}{r}\left(rv_r\right)_{,r},$$
$$div\mathbf{T}=\hat{\mathbf{r}}\left[\tfrac{1}{r}\left(rT_{rr}\right)_{,r}-\tfrac{1}{r}T_{\theta\theta}\right]+\hat{\boldsymbol{\theta}}\left[T_{\theta r,r}+\tfrac{1}{r}T_{r\theta}+\tfrac{1}{r}T_{\theta r}\right].$$

In matrix form the gradient of a vector is

$$grad\mathbf{v}=\begin{pmatrix} v_{r,r} & -\tfrac{1}{r}v_\theta \\ v_{\theta,r} & \tfrac{1}{r}v_r \end{pmatrix},$$

and its symmetric part can be written as

$$Symgrad\mathbf{v}=\begin{pmatrix} v_{r,r} & \tfrac{1}{2}r\left(\tfrac{1}{r}v_\theta\right)_{,r} \\ \tfrac{1}{2}r\left(\tfrac{1}{r}v_\theta\right)_{,r} & \tfrac{1}{r}v_r \end{pmatrix}.$$

Equivalently,

$$Symgrad\mathbf{v} = v_{r,r}\left(\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}\right) + \frac{1}{2}\left[\left(\hat{\mathbf{r}}\otimes\hat{\boldsymbol{\theta}}\right) + \left(\hat{\boldsymbol{\theta}}\otimes\hat{\mathbf{r}}\right)\right]\left[r\left(\frac{1}{r}v_\theta\right)_{,r}\right] + \frac{1}{r}v_r\left(\hat{\boldsymbol{\theta}}\otimes\hat{\boldsymbol{\theta}}\right),$$

and

$$\left[Symgrad\mathbf{v}\right]\hat{\mathbf{r}} = \hat{\mathbf{r}}v_{r,r} + \tfrac{1}{2}\left[r\left(\tfrac{1}{r}v_\theta\right)_{,r}\right]\hat{\boldsymbol{\theta}}.$$

Furthermore

$$\begin{aligned} div\left[Symgrad\mathbf{v}\right] &= \hat{\mathbf{r}}\left[\frac{1}{r}\left(rv_{r,r}\right)_{,r} - \frac{1}{r}\frac{1}{r}v_r\right] + \hat{\boldsymbol{\theta}}\left[\frac{1}{2}\left[r\left(\frac{1}{r}v_\theta\right)_{,r}\right]_{,r} + \frac{2}{r}\frac{1}{2}r\left(\frac{1}{r}v_\theta\right)_{,r}\right], \\ &= \hat{\mathbf{r}}\left[\frac{1}{r}\left(rv_r\right)_{,r}\right]_{,r} + \hat{\boldsymbol{\theta}}\frac{1}{2}\left[\frac{1}{r}\left(rv_\theta\right)_{,r}\right]_{,r}, \end{aligned}$$

and

$$\begin{aligned} grad\left[div\mathbf{v}\right] &= \left[\frac{1}{r}\left(rv_r\right)_{,r}\right]_{,r}\hat{\mathbf{r}} = \left\{div\left[Symgrad\mathbf{v}\right]\right\}_r, \\ \triangle_R w &= div\left[gradw\right] = \frac{1}{r}\left(rw_{,r}\right)_{,r}. \end{aligned}$$

The system of differential equations

$$\mu\triangle_R w + (\lambda+\mu)\,div\mathbf{v} + (\lambda+2\mu)\tilde{w} = 0,$$

$$div\left(2\mu Symgrad\mathbf{v} + \lambda\hat{\mathbf{1}}div\mathbf{v}\right) + (\lambda+\mu)\,gradw + \mu\tilde{\mathbf{v}} = \mathbf{0}, \quad \forall\hat{\mathbf{X}}\in\mathfrak{D},$$

becomes in cylindrical coordinates

$$\mu\tfrac{1}{r}\left(rw_{,r}\right)_{,r} + (\lambda+\mu)\tfrac{1}{r}\left(rv_r\right)_{,r} + (\lambda+2\mu)\tilde{w} = 0,$$

$$(2\mu+\lambda)\left[\tfrac{1}{r}\left(rv_r\right)_{,r}\right]_{,r} + (\lambda+\mu)\,w_{,r} + \mu\tilde{v}_r = 0,$$

$$\mu\left[\tfrac{1}{r}\left(rv_\theta\right)_{,r}\right]_{,r} + \mu\tilde{v}_\theta = 0, \qquad \forall r\in\left[R_i, R_e\right],$$

and the boundary conditions

$$\left(2\mu Symgrad\mathbf{v} + \lambda\hat{\mathbf{1}}div\mathbf{v}\right)\mathbf{N} = -\lambda w\mathbf{N} + \mathbf{p},$$

$$grad w \cdot \mathbf{N} = -\mathbf{v} \cdot \mathbf{N} + a,$$

become

$$(2\mu + \lambda)\, v_{r,r} + \lambda \tfrac{v_r}{r} = -\lambda w + p_r,$$

$$\mu \left[r \left(\tfrac{1}{r} v_\theta \right)_{,r} \right] = p_\theta,$$

$$w_{,r} = -v_r + a, \quad at \quad r = R_i \quad and \quad r = R_e.$$

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