# CHAPTER 2

# TRUSS SIZING WITH MATERIAL NONLINEARITY VIA DISPLACEMENT BASED OPTIMIZATION

This chapter presents theoretical formulation and procedures for elastoplastic truss design problem in the Displacement Based Optimization (DBO) setting. The follow-up chapter 3 will present numerical results of several truss examples to verify the methodology described in this chapter. Section 2.1 formulates the problem as inner and outer level problems using the DBO approach. Section 2.2 focuses on elastoplastic laws and discusses their limitation. Section 2.3 provides procedures used to scale displacement design variables. Section 2.4 pays attention to analytical gradient of objective and constraints w.r.t. displacement design variables.

## 2.1 FORMULATION OF INNER AND OUTER PROBLEMS

McKeown (1977) first proposed a two-level scheme to formulate separately outer and inner problems in order to solve DBO problems. The outer level optimization problem searches directly in the displacement design variables space. During each outer level iteration, an inner level Linear Programming (LP) problem is efficiently solved to obtain distribution of structural design parameters (i.e., cross sectional areas of bars) from the known displacement values. Despite generality of his approach, McKeown (1977) mainly tried some composite design problems and later discussed in-depth the approach for linear elastic truss design (McKeown, 1989). Recently, Missoum et al (1998) followed McKeown's scheme to solve successfully several standard linear elastic truss design problems.

The general formulation used in McKeown (1977) or Missoum et al (1998) is adopted here for truss nonlinear problems. We show the same formulation in detail for completeness and modify it for truss sizing with material nonlinearity whenever necessary.

A set of nonlinear structural system of equations in finite element analysis context can be written as:

$$\mathbf{P}(\mathbf{u}) + \mathbf{f} = \mathbf{K}(\mathbf{x},\mathbf{u})\mathbf{u} + \mathbf{f} = 0 \qquad (2.1)$$

Where $\mathbf{P}=(p_1, p_2, \dots, p_n)$ is internal force vector contributing to nodes, $\mathbf{f}=(f_1, f_2, \dots, f_n)$ is external load vector, $\mathbf{K}$ is n×n total stiffness matrix, $\mathbf{u}$ is unknown displacement vector and $\mathbf{x}=(x_1, x_2, \dots, x_m)$ the vector of cross sectional areas for case of truss structures. The stiffness matrix $\mathbf{K}$ is a function of both the cross sectional areas and the displacements. This characteristic accounts for the nonlinearity associated with the material law.

The structural optimization problem in the DBO setting is solved for optimum displacement field in its design space with the right combination of cross sectional design parameters of structural elements. The outer level optimization problem of DBO approach is:

$$\begin{aligned} \text{Min} \quad & w(\mathbf{x}(\mathbf{u})) \\ \text{s.t.} \quad & g_j(\mathbf{u}) \le 0, \quad j = 1,2,\cdots,k \\ & \text{bounds on } \mathbf{u} \end{aligned} \tag{2.2}$$

where w is the structure weight, $g_j$ is usually constraints from bounded stresses of elements. $\mathbf{x}(\mathbf{u})$ stands for the dependence of $\mathbf{x}$ to $\mathbf{u}$. After the optimizer produces a new displacement field $\mathbf{u}$ in one particular outer level iteration stage, the structural traditional design parameters $\mathbf{x}$ are recovered by solving a set of alternative representative equations of equilibrium relation (2.1) while minimizing the weight of the structure. For truss structures, the cross sectional areas of the elements $\mathbf{x}$ can be obtained from the known displacement $\mathbf{u}$ by solving an inner level problem that is a standard Linear Program (LP) as below:

$$\begin{aligned} \text{Min} \quad & w(\mathbf{x}) \\ \text{s.t.} \quad & \mathbf{T}(\mathbf{u})\mathbf{x} + \mathbf{f} = 0 \\ & \text{bounds on } \mathbf{x} \end{aligned} \tag{2.3}$$

where $\mathbf{T}$ is the LP constraints coefficients matrix of $n \times m$ and the equality constraints $\mathbf{T}(\mathbf{u})\mathbf{x}+\mathbf{f}=0$ is an alternative representation of $\mathbf{K}(\mathbf{x},\mathbf{u})\mathbf{u}+\mathbf{f}=0$.

As compared to linear elastic case, truss sizing with nonlinear material results different routines of computing stresses from displacement and different organization of LP coefficient matrix **T** in the inner problem. This **T** matrix is, however, still a reformulation from information of total stiffness matrix **K**.

When considering nonlinear material constitutive relation, it is beneficial to return to the basic equilibrium equations of solid mechanics formulation in terms of displacement **u**:

$$\int_V \mathbf{B}^T \sigma \, \mathrm{d}V + \mathbf{f} = 0 \tag{2.4}$$

where **B** is the strain-displacement relation matrix, $\sigma$ the stress and V for volume integration. The strain $\varepsilon$ is defined in terms of discretized nodal displacement **u** as

$$\varepsilon = \mathbf{B}\mathbf{u} \tag{2.5}$$

Since the governing equations (2.4) and (2.5) are based on the principles of virtual work, they are valid for any material behavior. Assuming nonlinear elastic or elastic-plastic behavior by

$$\sigma = \sigma(\varepsilon) \tag{2.6}$$

Then, relations (2.4)–(2.6) define completely the form (2.1). Above we assume the relation (2.6) is unique since there is no difference to separate the nonlinear elasticity from elastic-plastic behavior in the DBO context. Various models for (2.6) are discussed in section 2.3 and will be used in Chapter 3.

In general, it is not an easy matter to write **P**(**u**) in the form **K**(**x**,**u**)**u** because of nonlinear relation (2.6). So matrix **T** is not readily written down from **K**. For truss sizing with nonlinear material, fortunately, **T** is guaranteed by assembling its coefficients from the truss member stresses that have already computed from (2.6) and (2.5) with known displacement **u**.

In section 2.5, we will use tangential stiffness matrix $\mathbf{K}_T$ of elasoplastic truss to compute analytical gradient information of weight w.r.t. displacement $\mathbf{u}$. The incremental $\mathbf{K}_T$ matrix (Zienkiewicz, 1977) is well defined as follows:

$$\mathbf{K}_T = \frac{d\mathbf{P}}{d\mathbf{u}} = \int_V \mathbf{B}^T \frac{d\sigma}{d\varepsilon} \frac{d\varepsilon}{d\mathbf{u}} dV = \int_V \mathbf{B}^T \mathbf{D}_T \mathbf{B}\, dV \tag{2.7}$$

where for a one-dimensional truss element

$$\mathbf{D}_T = E_T = \frac{d\sigma}{d\varepsilon} \tag{2.8}$$

where $E_T$ is the tangential Young's modulus at a given strain level for a certain element.

Two more topics related to numerical experiments will be discussed before this section is closed. First, we note that initial displacement design point is computed not from costly nonlinear finite element analysis but conveniently from linear finite element analysis. For examples of Chapter 3, such an inexact displacement design point is shown to be a good initial guess within displacement design space of nonlinear structure.

Numerical experiments showed that displacement field provided by optimizer during the outer level optimization iterative process sometimes failed to be a physical displacement field and thus led to no feasible solution for the inner problem. In order to overcome this difficulty, the equality constraints of the inner problem are relaxed by introducing additional relaxed variables. Accordingly, the modified inner problem used in the program could be written as:

$$
\begin{aligned}
&\text{Min} \quad \mathrm{w}(\mathbf{x}) + c \times \sum_{\mathrm{i}=1}^{\mathrm{n}} (e_{\mathrm{i}}^{1} + e_{\mathrm{i}}^{2}) \\
&\text{s.t.} \quad \mathbf{T}(\mathbf{u})\mathbf{x} + \begin{bmatrix} e_1^1 & & & \\ & e_2^1 & & \\ & & \cdots & \\ & & & e_{\mathrm{n}}^1 \end{bmatrix} - \begin{bmatrix} e_1^2 & & & \\ & e_2^2 & & \\ & & \cdots & \\ & & & e_{\mathrm{n}}^2 \end{bmatrix} = -\mathbf{f} \\
&\qquad\ \text{bounds on } \mathbf{x}, \\
&\qquad\ e_{\mathrm{i}}^{1} \geq 0,\ e_{\mathrm{i}}^{2} \geq 0, \qquad \mathrm{i} = 1, 2, \cdots, \mathrm{n}
\end{aligned} \tag{2.9}
$$

where $c$ is a penalty coefficient usually set to be very large. Given certain displacement **u**, the above form always create a solution for the inner problem (2.9). The objective both for the inner problem (2.9) and the outer problem (2.2) includes weight plus summation of weighted relaxed variables. Physically, $e^1{}_i$-$e^2{}_i$ means the unbalanced force at the *i*th degree of freedom. If all the relaxed variables are zero, the objective becomes the true weight of the truss structure and static equilibrium of the nonlinear finite element structure is satisfied. As a concluding remark, the form (2.9) is not the only relaxed form available. Performance of different relaxed schemes are of slight difference so that other forms being tried are not included for simplicity. Fig. 2.1 gives a flow chart to show procedures of the DBO approach.

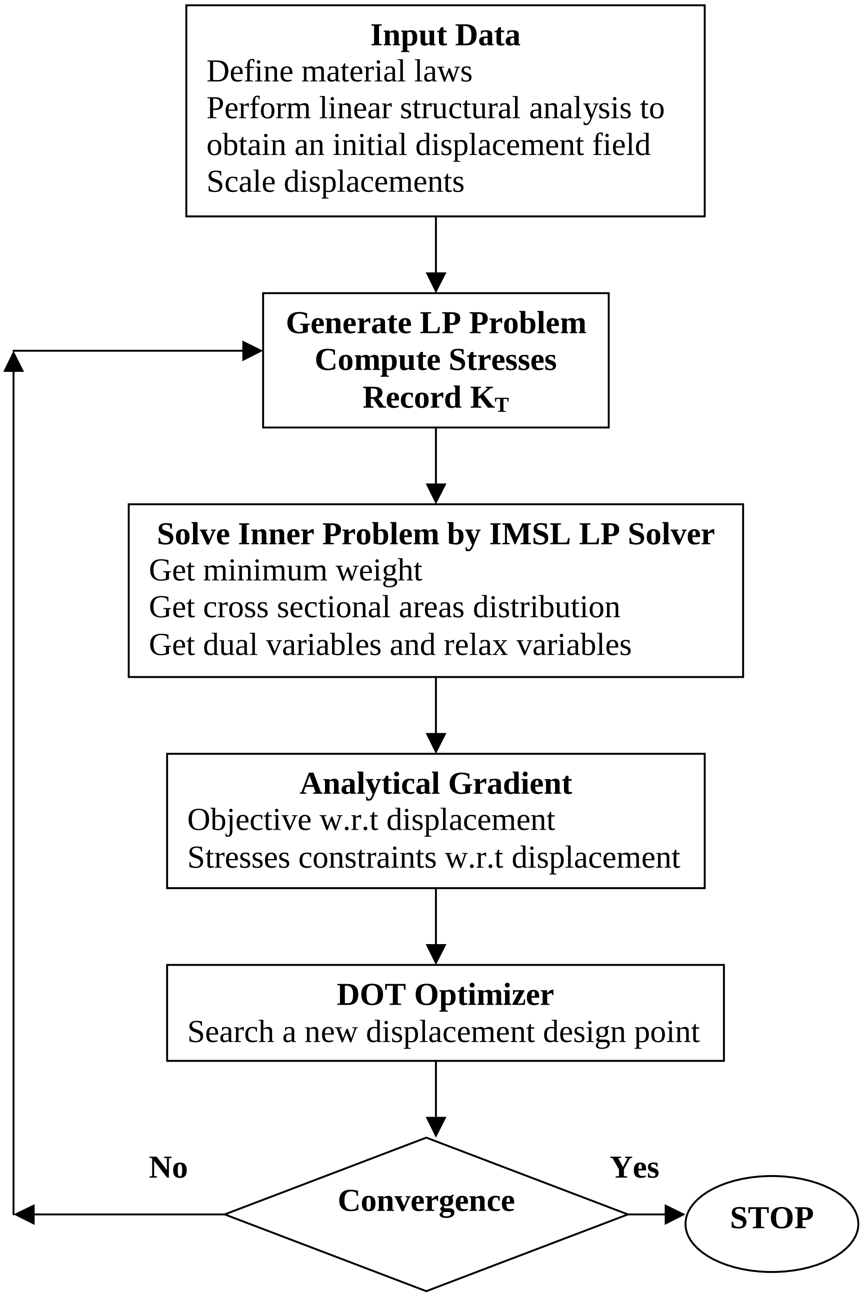


**Fig. 2.1 Flow Chart for Procedures of Displacement Based Optimization**

## 2.2 HOLONOMIC ELASTOPLASTIC LAWS

Holonomic elastoplastic behavior stands for reversible and history-independent nonlinear stress-strain relation. For truss structures, such holonomic elastoplastic laws become one dimensional stress-strain curve at element level. As noted in Tin-Loi (1999), violations of nonholonomy are likely to be insignificant for practical engineering structures. This simplified assumption is adopted by DBO study in view that current DBO approach is hard to include history-dependent behavior. Used in examples of Chapter 3, various uniaxel stress-strain curves (Mendelson 1968) are expressed as follows:

Beyond linear elasticity, a simple extension is to separate plastic part from straight-line elastic part beginning at intersection of yield point. A polynomial curve could be used for the plastic part. This realistic elastic-plastic model is of the following form

$$\sigma = \begin{cases} \mathrm{E}\varepsilon, & \varepsilon \le \varepsilon_0 \\ \mathrm{a}_0 + \mathrm{a}_1\varepsilon + \mathrm{a}_2\varepsilon^2 + \cdots + \mathrm{a}_\mathrm{k}\varepsilon^k, & \varepsilon > \varepsilon_0 \end{cases} \tag{2.10}$$

where $\varepsilon_0$ is the yield strain and E the Young's modulus. Based on experiment data, it is possible to fit the plastic part of the stress-strain curve (2.10) as accurately as desired by the polynomial of arbitrary degree k. For linear strain hardening or bi-linear law, the plastic part of (2.10) is a straight line and all the coefficients beginning with $a_2$ are zero:

$$\sigma = \begin{cases} \mathrm{E}\varepsilon, & \varepsilon \le \varepsilon_0 \\ \mathrm{a}_0 + \mathrm{a}_1\varepsilon, & \varepsilon > \varepsilon_0 \end{cases} \tag{2.11}$$

where $a_1$ is the slope of the plastic line. So let

$$a_1 = E_p = h \times E \tag{2.12}$$

where h is the hardening coefficient. $a_0$ is determined by values of the yield point connecting elastic line and plastic line. Finally, we have

$$\sigma = \begin{cases} E\varepsilon, & \varepsilon \le \varepsilon_0 \\ E\varepsilon_0 + E_p(\varepsilon - \varepsilon_0), & \varepsilon > \varepsilon_0 \end{cases} \tag{2.13}$$

As a special case of (2.13), Elastic-Perfectly Plastic (EPP) law is resulted when $E_p$ is zero.

Some popular empirical models such as power law and Ramberg-Osgood model are also studied. Applied for both tensive and compressive truss element, power law is of the form:

$$\frac{\sigma}{\sigma_0} = c \times (a + \frac{\varepsilon}{\varepsilon_0})^n, \qquad 0 \le n \le 1 \tag{2.14}$$

where normalization for both stress and strain is implemented by using yield stress $\sigma_0$ and yield strain $\epsilon_0$. n is called the strain hardening exponent. Constant a is chosen to be zero so that the stress-strain curve passes through the origin of the stress-strain diagram. The coefficient c is to be unity in order to let the curve pass the yield point. The tangential Young's modulus $E_t$ at given strain level is

$$E_t = \frac{\partial \sigma}{\partial \varepsilon} = n \times E \times (\frac{\varepsilon}{\varepsilon_0})^{n-1}, \qquad 0 \le n \le 1 \tag{2.15}$$

It is well known that power law will not usually fit at the low-strain and high-strain ends of the stress-strain curve. Moreover, (2.15) tells that value of $E_t$ is approaching infinite when $\epsilon$ approaches zero. Numerical study in Chapter 3 verified that such a case created computational difficulty. Consequently, this model is not used for numerical examples.

Ramberg-Osgood model, as shown below, can simulate a large class of material including EPP material.

$$\varepsilon = \frac{\sigma}{\mathrm{E}} + \mathrm{K}(\frac{\sigma}{\mathrm{E}})^{\mathrm{n}} \tag{2.16}$$

This three-parameter model includes parameters of K, n and elastic modulus E. For simplicity, chapter 3 will take n to be 2 by which an example is studied.

## 2.3. SCALING OF DISPLACEMENT VARIABLES

It is important to normalize design variables for design optimization problems. Although there is no general theory by which the normalization could be universally made, a good philosophy behind scaling is to let all the design variables vary near unity during optimization iterative process. This means that special scaling is needed for special design optimization problem. Considering the DBO problem, good scaling strategy is more critical for reason that DBO always searches in the displacement design space where different displacements components or degree of freedoms may be quite different in magnitude, particularly for multi-dimensional structures.

Here a two-level scaling procedure is incorporated in optimization process. First, every displacement design variable is scaled according to its upper and lower bounds during the entire optimization process. If physical bounds are not available on certain displacement, reasonably large bounds could be provided. This first-level scaling takes the form as:

$$\bar{x}_i = \frac{x_i - x_i^{\min}}{x_i^{\max} - x_i^{\min}} + \mathrm{PLT} \qquad (2.17)$$

where $x_i$, $x_i^{\min}$ and $x_i^{\max}$ are the *i*th design variable, its lower and upper bound, respectively. $\bar{x}_i$ is the *i*th normalized variable. Selecting value of positive parameter PLT is based on trial-and-error manner, usually to be 0.001. The DBO problems are always searched in normalized displacement design space, instead of un-normalized displacement space. The lower and upper bounds for all normalized displacement variables are uniformly to be PLT and 1.0+PLT, respectively. Recovery formulae of $x_i$ from $\bar{x}_i$ could be obtained from (2.17):

$$x_i = (\bar{x}_i - \mathrm{PLT})(x_i^{\max} - x_i^{\min}) + x_i^{\min} \qquad (2.18)$$

The second-level scaling is for the normalized displacement and is done automatically within commercial optimization code DOT (VR&D, 1999) used. For simplicity, detail of that scaling within DOT will not be presented here. However, we notice that second-level scaling may not be necessary and is used in trial-and-error manner. Once the second-level scaling works not very well, it could be simply switched off.

## 2.4. ANALYTICAL GRADIENT

For the truss design problems in the DBO context, analytical gradient information of the outer level weight w.r.t. the displacement components is available so that costly finite difference computation of gradient is avoided. Analytical gradient for stresses constraints can be easily computed since stresses are expressed as a linear function of displacement design variables. The analytical gradient for objective in the outer level problem (2.2) has been studied (McKeown (1977), Missoum et al (1998)). For linear elastic structures, McKeown first gave the formulae to compute sensitivity of objective, written as:

$$\frac{\partial \mathrm{w}(\mathbf{x}^*, \mathbf{u})}{\partial \mathbf{u}} = -\mathbf{K}^* \boldsymbol{\lambda}^* \tag{2.19}$$

where $\mathbf{x}^*$ is the design vector, $\mathbf{K}^*$ the linear elastic global stiffness matrix and $\lambda^*=(\lambda_1, \lambda_2, \ldots, \lambda_n)^*$ the dual variables vector at optimum for the inner problem (2.3). For the outer level problem (2.2), (2.19) gives objective gradient w.r.t. displacement variables. For inner level problem (2.3), (2.19) could be interpreted as sensitivity of optimum objective to problem parameters **u**. Missoum et al gave one way of derivation for (2.19).

For the truss sizing with material nonlinearity, (2.19) is still valid except substituting tangential stiffness matrix $\mathbf{K}_t$ for **K**. For completeness, we derive analytical gradient formulae as follows.

Since the relaxed inner problem (2.9) is used, we use F for objective instead of w. Considering theory about sensitivity of optimum solution to problem parameters (Haftka & Gurdal, 1992), we have

$$\frac{d\mathrm{F}(\mathbf{x}^*, \mathbf{u})}{d\mathbf{u}} = \frac{\partial \mathrm{F}(\mathbf{x}^*, \mathbf{e}^{1*}, \mathbf{e}^{2*})}{\partial \mathbf{u}} + \frac{\partial[\mathbf{T}(\mathbf{u})\mathbf{x}^* + \mathbf{e}^{1*} - \mathbf{e}^{2*} + \mathbf{f}]}{\partial \mathbf{u}}\boldsymbol{\mu}^* = \frac{\partial[\mathbf{T}(\mathbf{u})\mathbf{x}^*]}{\partial \mathbf{u}}\boldsymbol{\mu}^* \tag{2.20}$$

where displacement **u** is the problem parameter. $\mathbf{x}^*$ and relaxed variables $\mathbf{e}^{1*}$, $\mathbf{e}^{2*}$ at optimum and fixed load **f** do not have differentiating variable **u**. $\mathbf{T}(\mathbf{u})\mathbf{x}^*$ is again an alternative representation of $\mathbf{K}(\mathbf{x}^*,\mathbf{u})\mathbf{u}$ in structural equilibrium equations at optimum point of the relaxed inner problem (2.9), and $\mu^*=(\mu_1, \mu_2, \dots, \mu_n)^*$ the Lagrange multipliers associated with active constraints at optimum. Since all the n constraints are equality constraints, active set contains n active constraints at optimum. Noting that Lagrange multiplier and corresponding dual variable only have difference of a sign for our inner problem, and with the help of (2.7), we have from (2.20)

$$\frac{d\mathrm{F}(\mathbf{x}^*,\mathbf{u})}{d\mathbf{u}} = -\frac{\partial[\mathbf{K}(\mathbf{x}^*,\mathbf{u})\mathbf{u}]}{\partial\mathbf{u}}\boldsymbol{\lambda}^* = -\frac{\partial[\mathbf{P}(\mathbf{x}^*,\mathbf{u})]}{\partial\mathbf{u}}\boldsymbol{\lambda}^* = -\mathbf{K}_t{}^*\boldsymbol{\lambda}^* \tag{2.21}$$

The final form (2.21) gives the gradient computing formulae for outer level objective and has no relationship with the relaxed variables.