Chapter 2

Problem Formulation

2.1 Assumptions

Before beginning to formulate the mathematics of the problems to be investigated, certain assumptions will be made in order to simplify the analysis. For the analysis, the breakwater will be investigated in a planar fashion and is modeled in cross-section. Normally a breakwater would float in a body of water that would give an upward buoyant force greater than the downward gravitational force. However, in this project the breakwater is considered upside-down with the net buoyant force equal to the gravitational force, and it could also represent an object hanging by two cables (Fig. 2.1). Further assumptions are that the mooring cables are inextensible and the weight, inertia, bending resistance, and axial resistance in compression of the cables are neglected. During an "impact," when a cable becomes taut, the cable will dissipate some energy and this will be accounted for in a coefficient of restitution, e, which will be explained in more detail later. Finally, no other "damping" is considered in the system; this includes any damping that would come from the surrounding fluid, and internal dissipation of energy in the cables.

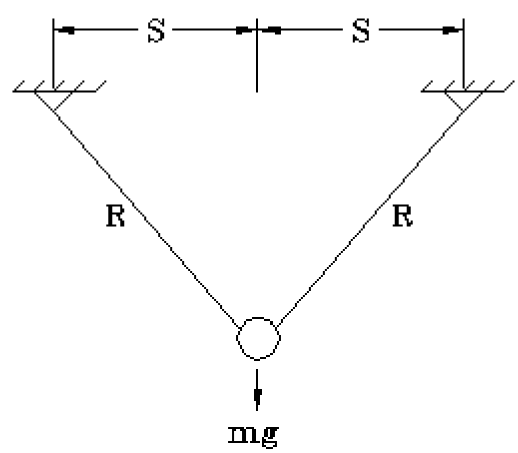


Fig. 2.1 Breakwater Configuration

2.2 Point-Mass Model

During the first part of this investigation, the breakwater will be modeled as a point mass. Free vibrations are considered first, followed by the response to harmonic forcing. The following formulation is for the point-mass configuration under free motions.

2.2.1 Geometrical Configuration

In this formulation, the breakwater has no dimension so it is seen as infinitesimally small and is treated as a point mass. The geometric configuration of the breakwater and its mooring system is arranged in such a manner that the components are symmetric. The two cables are of equal length and are suspended from the same height. The configuration of the moored breakwater, with its dimensional parameters, can be seen in Fig. 2.2. This figure also shows that the origin of the global X-Y axes is located at the center of the breakwater when it is at its lowest point, its equilibrium state with both cables taut. The coordinates x and y define the position of the point mass. Figure 2.2 shows the separation length of the supports as 2S, and the "taut" (natural) length of each cable as R. "Taut" in this sense means the cable has been stretched out and has reached its natural length. In order for the breakwater to float, the following condition must be satisfied:

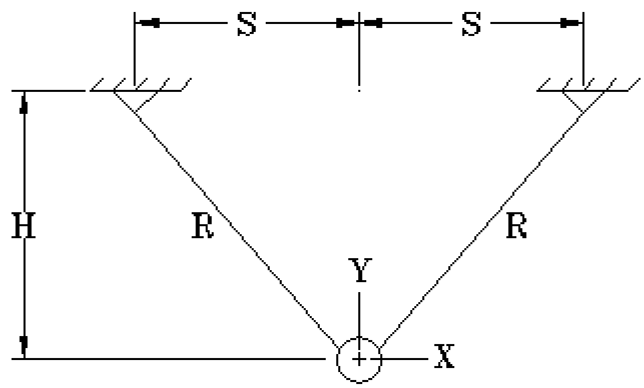


Fig. 2.2 Dimensional Parameters

$$R > S \tag{2.1}$$

With these dimensions, the distance, H, from the supports to the breakwater's lowest point in its equilibrium state may be defined by

$$S^2 + H^2 = R^2 \tag{2.2}$$

Further restrictions are placed on the breakwater of mass m. Motions of the mass must remain below the height of the supports (i.e., the breakwater not hitting the sea floor) and at or above the equilibrium position shown in Fig. 2.2. Therefore the following restriction on Y must be met:

$$0 \leq Y \leq H \tag{2.3}$$

There are two degrees of freedom in this system: X(T), Y(T), where T = time.

2.2.2 Boundaries

As stated before, the extreme vertical limits are the lowest point being Y=0, with both cables taut (Fig. 2.3), and the highest being Y=H, the sea floor. Now the horizontal limits must be defined. If the breakwater moves with one cable being taut, as shown in Fig. 2.4, one can see that a boundary is formed where that cable can be extended no further than its natural length. A left boundary, G_1=0, is formed when only the right cable is taut, and a right boundary, G_2=0, is formed when only the left cable is taut.

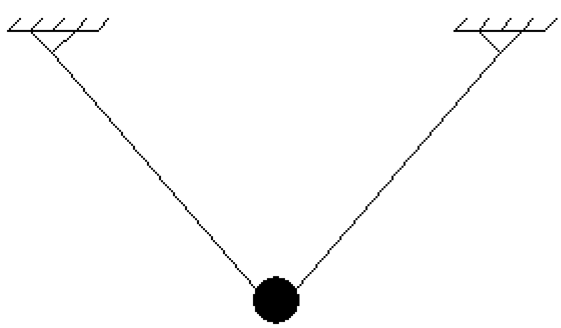


Fig. 2.3 Breakwater at Equilibrium State

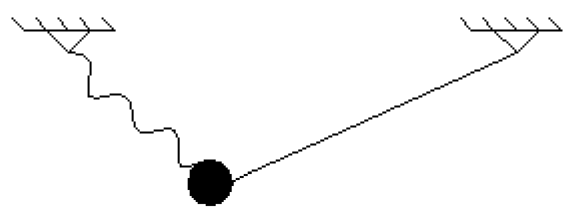


Fig. 2.4 Left Boundary, Right Cable Taut

The boundaries form circular arcs (Figs. 2.5 and 2.6 in nondimensional form) with the radius being the length of the cable and the limits being Y=0 (equilibrium state) and Y=H (sea floor). Two cable lengths were investigated, one length where the breakwater would stay inside its supports (Fig. 2.5) and one where it could go outside of its supports (Fig. 2.6). If the right cable is taut, the left circular boundary, G_1=0, may be defined by

$$(X-S)^2 + (Y-H)^2 = R^2 \tag{2.4}$$

If the left cable is taut, the circular boundary, $G_2=0$, may be defined by

$$(X+S)^2 + (Y-H)^2 = R^2 \tag{2.5}$$

Thus the motion of the breakwater mass must remain in the region U (Figs. 2.5 and 2.6) where

$$0 \le Y \le H: \quad (X-S)^2 + (Y-H)^2 - R^2 \le 0 \quad (G_1 \text{ equation}) \tag{2.6}$$

$$(X+S)^2 + (Y-H)^2 - R^2 \le 0 \quad (G_2 \text{ equation}) \tag{2.7}$$

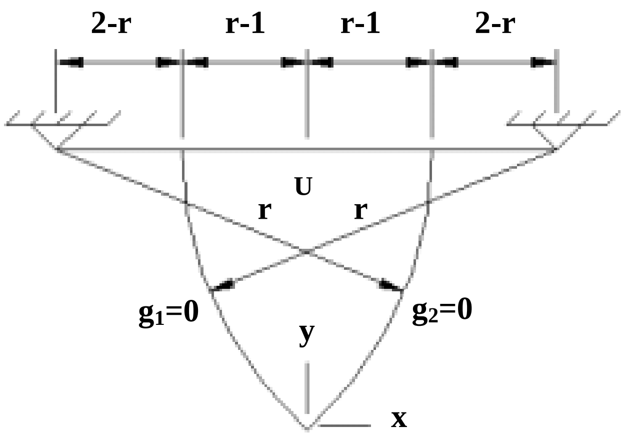


Fig. 2.5 Boundary with Radius Inside Supports

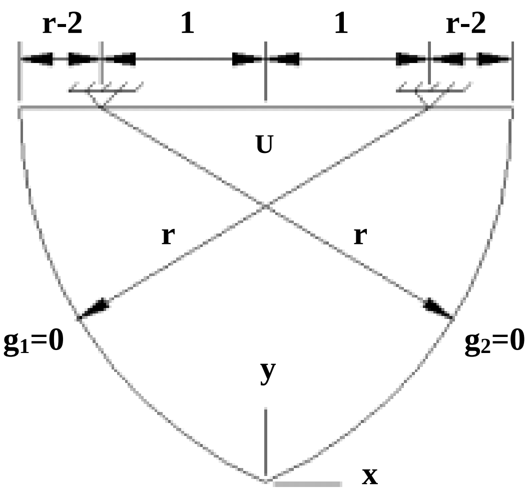


Fig. 2.6 Boundary with Radius Outside Supports

2.2.3 Analogy

The model resembles a bouncing ball in a valley or U shaped region with gravity acting on it, which causes the ball to end up at the bottom (Fig. 2.7) if no other forces are applied. The ball loses energy when it "impacts" and bounces. When it is said that the breakwater impacts, this is defined as a cable becoming taut and the breakwater reaching a boundary $G_1=0$ or $G_2=0$.

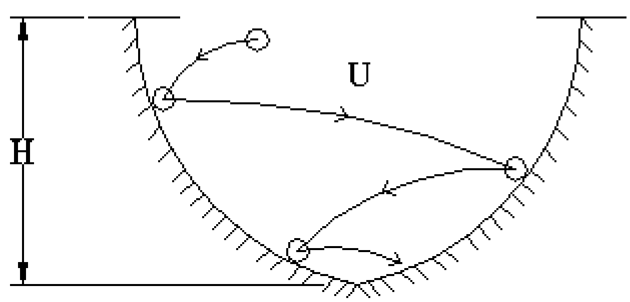


Fig. 2.7 Bouncing-Ball Analogy

2.3 Nondimensionalization

The variables used in the formulation of the problem have been nondimensionalized, so that the units will not be involved during this investigation. Length parameters were nondimensionalized by the cable spacing, S, and time by

$$\sqrt{S/g} \tag{2.8}$$

Mass is divided by itself and becomes unity. Uppercase letter symbols are used when terms have dimensions and lowercase letters are used to represent the nondimensionalized values. Thus, the parameters become

$$x=X/S \tag{2.8}$$

$$y=Y/S \tag{2.9}$$

$$r=R/S \;\; >1 \tag{2.10}$$

$$h=H/S=\sqrt{r^2-1} \tag{2.11}$$

$$t=T\sqrt{g/S} \tag{2.12}$$

The nondimensionalized geometric parameters may be seen in Fig. 2.8.

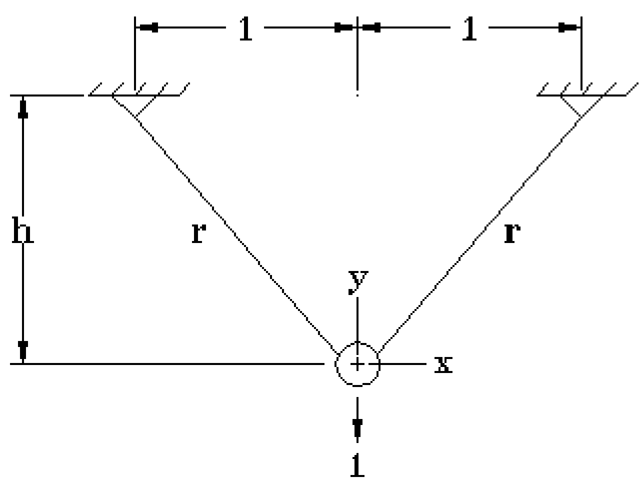


Fig. 2.8 Breakwater Configuration with Nondimensionalized Parameters

2.4 Equations of Motion

Inside the region, U, both cables are slack, and the only force acting on the breakwater during free vibration is gravity. Thus, simple Newtonian equations of motion are applicable:

$$F=ma \tag{2.13}$$

$$m\frac{d^2X}{dT^2}=0 \tag{2.14}$$

$$m\frac{d^2Y}{dT^2}=-mg \tag{2.15}$$

After the nondimensionalization, the basic equations of motion (EOM's) become

$$\ddot{x}=0 \tag{2.16}$$

$$\ddot{y}=-1 \tag{2.17}$$

where $\bullet=\frac{d}{dt}$

The solutions for this formulation are

$$x(t)=x_o+\dot{x}_o\,t \tag{2.18}$$

$$\dot{x}(t)=\dot{x}_o \tag{2.19}$$

$$y(t)=y_o+\dot{y}_o\,t-\tfrac{1}{2}t^2 \tag{2.20}$$

$$\dot{y}(t)=\dot{y}_o-t \tag{2.21}$$

where the initial conditions at t=0 are

$$x(0)=x_o,\ \dot{x}(0)=\dot{x}_o,\ y(0)=y_o,\ \dot{y}(0)=\dot{y}_o \tag{2.22}$$

2.4.1 Equations of Motion at Impact

At a time when one of the breakwater's cables becomes taut, referred to as an impact, the breakwater rebounds off a boundary and starts moving in a different direction. For this, new variables will be introduced. First a symbol will be introduced where the superscript

(-) means a variable has this value just before impact, and similarly (+) means that the variable has this value just after the impact transition time. At the time of impact, the solutions change as follows. The subscript, i, indicates that a parameter is at an impact. Thus, after an impact a new set of "initial conditions" is determined and takes the values just after impact,

$$t=t_i,\ x(t_i)=x_i^+,\ \dot{x}(t_i)=\dot{x}^+,\ y(t_i)=y_i^+,\ \dot{y}(t_i)=\dot{y}^+ \tag{2.23}$$

Then, until the next impact, the solutions are

$$x(t)=x^+ + \dot{x}^+(t-t_i) \tag{2.24}$$

$$y(t)=y^+ + \dot{y}^+(t-t_i)-\tfrac{1}{2}(t-t_i)^2 \tag{2.25}$$

For example, in the first time interval,

$$x(0)=x_o,\ \dot{x}(0)=\dot{x}_o\ \ y(0)=y_o,\ \dot{y}(0)=\dot{y}_o \tag{2.26}$$

Until the first impact, if the initial point is in the interior of U,

$$x(t)=x_o+\dot{x}_o(t-t_i) \tag{2.27}$$

$$y(t)=y_o+\dot{y}_o(t-t_i)-\tfrac{1}{2}(t-t_i)^2 \tag{2.28}$$

where t_i=0. It is necessary to determine when "impacts" occur, and subsequent "initial conditions" after impacts.

2.5 Impact

Now a detailed derivation of impact response equations will be formulated to determine the new initial conditions after impact.

2.5.1 Definition of Impact

When it is said that a breakwater impacts, this is defined as a cable becoming taut. "Taut" in this sense means that the cable reaches its natural length. When the cable becomes taut it is said that the breakwater is hitting a fictitious boundary which is defined by the mathematical equations for G_1=0 and G_2=0. Once one of the cables has become taut, an impact is felt and the breakwater rebounds in the opposite direction. This is analogous to a ball bouncing down the valley, thus hitting a side and changing direction. The taut condition is determined when the value of G_1 or G_2 or both become zero. At this instant the breakwater is on the boundary. During the solution procedure the values of G_1 and G_2 are negative when the breakwater is inside the region. After the nondimensionalization and some rearranging of terms, the boundary equations take the form of $g_1=0$ and $g_2=0$, where

$$g_1=(x-1)^2 + (y-h)^2 - r^2 \tag{2.29}$$

$$g_2=(x+1)^2 + (y-h)^2 - r^2 \tag{2.30}$$

The EOM's are valid as long as $g_1<0$ and $g_2<0$.

2.5.2 Impact Response when g_1=0

Impact parameters will now be developed for $g_1=0$. Solving Equation 2.29 for y where $g_1=0$ gives

$$y = h - \sqrt{r^2 - (x-1)^2} \tag{2.31}$$

Thus, to get the slope of the boundary at a given point, the derivative is taken:

$$\frac{dy}{dx} = \frac{x-1}{\sqrt{r^2 - (x-1)^2}} \tag{2.32}$$

2.5.2.1 Angle ϕ

Taking a small differential triangle formed by the local x and y axes (Figs. 2.9 and 2.10), the tangent to the arc at this point may be obtained by measuring the angle ϕ from the horizontal. The differential length of the hypotenuse may be obtained from the Pythagorean Theorem. Knowing the derivative of the circular arc equation, g_1, and simple trigonometry, $\sin\phi$ and $\cos\phi$ are obtained. By dividing the terms by the applicable dy or dx, the $\sin\phi$ and $\cos\phi$ terms may be simplified as seen in Equations 2.33 and 2.34:

$$\sin\phi = \frac{dy}{\sqrt{dy^2 + dx^2}} = \frac{-(dy/dx)}{\sqrt{1+(dy/dx)^2}} \tag{2.33}$$

$$\cos\phi = \frac{-dx}{\sqrt{dy^2 + dx^2}} = \frac{1}{\sqrt{1+(dy/dx)^2}} \tag{2.34}$$

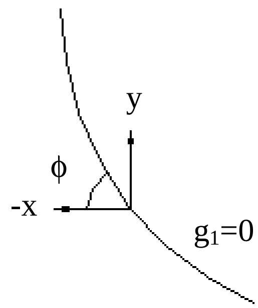


Fig. 2.9 Axes at Point on g_1=0

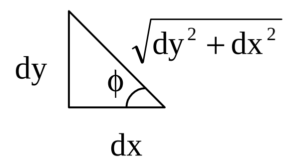


Fig. 2.10 Differential Triangle (dx<0, dy>0)

2.5.2.2 Normal and Tangential Velocities at g_1=0

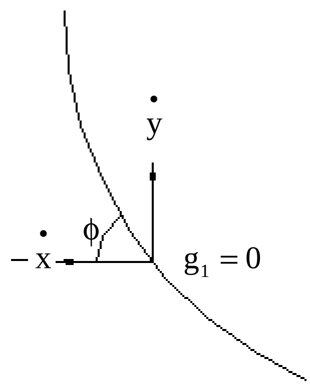
From simple geometry, the normal and tangential velocities may be obtained. By taking the vectors of the x and y velocities and then decomposing them into normal and tangential vectors, the normal and tangential velocities at a given point on the g_1=0 arc may be obtained. The normal velocity is the velocity vector occurring longitudinal to the taut cable, while the tangential velocity is the velocity that occurs tangential to a point on the boundary. Conversely, by taking the normal and tangential velocity components in the x and y directions, the x and y velocities may be obtained. This derivation is shown graphically in Figs. 2.11 and 2.12 and given in Equations 2.35–2.38.

$$v_n = -\dot{x}\sin\phi - \dot{y}\cos\phi \tag{2.35}$$

$$v_t = -\dot{x}\cos\phi + \dot{y}\sin\phi \tag{2.36}$$

$$\dot{x} = -v_t\cos\phi - v_n\sin\phi \tag{2.37}$$

$$\dot{y} = v_t\sin\phi - v_n\cos\phi \tag{2.38}$$



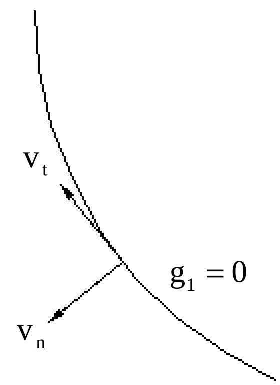


Fig. 2.11 Velocity Vectors at Point on g_1=0 **Fig. 2.12 Normal and Tangential Velocity Vectors**

It must be noted that in this derivation, the normal velocity points outward from the region in its positive direction. This is done to aid in developing graphs to interpret the data from the case solutions. For example, as the breakwater is heading towards a boundary (i.e., in a positive direction) then its normal velocity just before impact will be positive.

2.5.3 Impact Response when g_2=0

Similar to the derivation of the g_1=0 impact response, the impact parameters will now be developed for g_2=0. Solving Equation 2.30 for y where g_2=0 :

$$y = h - \sqrt{r^2 - (x+1)^2} \tag{2.39}$$

Thus, to get the slope of the boundary at a given point, the derivative is taken:

$$\frac{dy}{dx} = \frac{x+1}{\sqrt{r^2 - (x+1)^2}} \tag{2.40}$$

2.5.3.1 Angle ψ

Likewise for g_2=0, by taking a small differential triangle formed by the local x and y axes (Figs. 2.13 and 2.14), the tangent to the arc at this point may be obtained by measuring the angle ψ from the horizontal. The differential length of the hypotenuse may be obtained from the Pythagorean Theorem. Knowing the derivative of the circular arc equation, g_2, and simple trigonometry, $\sin\psi$ and $\cos\psi$ are obtained. By dividing the terms by the applicable dy or dx, the $\sin\psi$ and $\cos\psi$ terms may be simplified as given in Equations 2.40 and 2.41:

$$\sin\psi = \frac{dy}{\sqrt{dy^2 + dx^2}} = \frac{dy/dx}{\sqrt{1+(dy/dx)^2}} \tag{2.40}$$

$$\cos\psi = \frac{dx}{\sqrt{dy^2 + dx^2}} = \frac{1}{\sqrt{1+(dy/dx)^2}} \tag{2.41}$$

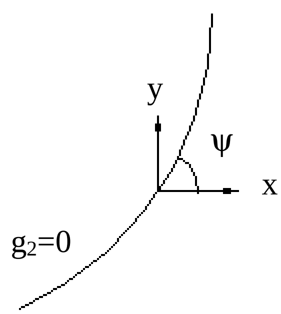


Fig. 2.13 Axes at Point on g_2=0

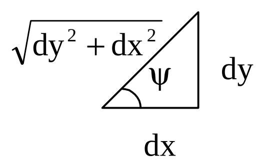


Fig. 2.14 Differential Triangle (d_x>0, d_y>0)

2.5.3.2 Normal and Tangential Velocities at g_2=0

Similar to the left boundary derivation, from simple geometry, the normal and tangential velocities may be obtained. This derivation is shown graphically in Figs. 2.15 and 2.16 and given in Equations 2.42–2.45:

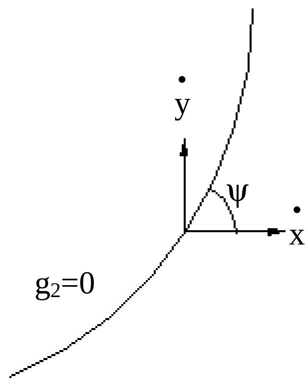
$$v_n = \dot{x}\sin\psi - \dot{y}\cos\psi \tag{2.42}$$

$$v_t = \dot{x}\cos\psi + \dot{y}\sin\psi \tag{2.43}$$

$$\dot{x} = v_t\cos\psi + v_n\sin\psi \tag{2.44}$$

$$\dot{y} = v_t\sin\psi - v_n\cos\psi \tag{2.45}$$

It must be noted that in this derivation, the normal velocity points outward from the region in its positive direction.



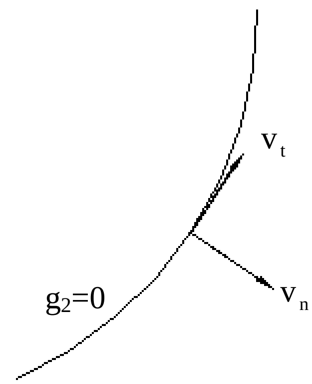


Fig. 2.15 Velocity Vectors at Point on g_2=0 **Fig. 2.16 Normal and Tangential Velocity Vectors**

2.5.4 Snap Loading Response

The breakwater feels a similar snap loading response if either the left or right boundary is hit. The only difference in the response is the equations for determining the normal and tangential velocities, which were explained previously. Snap loading in this project is referred to as an impact. This is when a cable becomes taut, and the loading in the cable at this instant is defined as the normal velocity (i.e., normal to the boundary). The mass of the breakwater may be multiplied by its change of velocity to give a resulting "impact force" or impulsive force. The normal velocity at impact is computed, just before the breakwater rebounds when the cable has become taut. When the breakwater impacts a boundary, some energy is dissipated. This dissipation of energy is absorbed by the mooring lines and is transferred to the supports. In practice the amount of energy dissipated depends on material and geometric properties of the mooring line. In this investigation, the affects of these properties are taken into account by a coefficient of restitution, e ($0<e<1$), which is common to impact problems. This reduction factor is applied to the normal velocity at the time of impact. Thus, the normal velocity is reduced during impacts, which in turn reduces the magnitudes of the horizontal and vertical velocities of the breakwater. Equation 2.46 explains this mathematically:

$$v_n^+ = -ev_n^- \tag{2.46}$$

It is assumed that the tangential velocity, v_t, remains unchanged at the time of impact and that the only parameter affected by the rebound is the normal velocity. Though the tangential velocity remains unchanged, the x and y velocities do change because they are based on both the normal and tangential velocities after impact. At an impact time, the subscript i is used to denote an impact or the next initial condition. As seen previously, new x and y values are determined after impact and these become the new initial conditions for the next range of motions until the next impact. Thus, initial conditions are seen in Equation 2.23.

2.6 Convergence to a Boundary

As seen in boundary Equations 2.29 and 2.30, the breakwater is inside of the region when g_1 and g_2 are negative. As the solution proceeds through time, the breakwater might hit a boundary, which occurs when a cable becomes taut. Mathematically, this is shown by g_1 or g_2 becoming greater than or equal to zero. The ideal case is that the breakwater's position ends up directly on a boundary (i.e., $g_{1,2}=0$). However, for practical purposes, a finite time step is used in the solution and it is more likely that the breakwater's position will end up past the boundary at an impact. In this case the exact time at which the breakwater hits the boundary is required to keep the solution accurate, and thus an impact time must be converged upon. Several convergence methods exist. Originally, the "bracketing" method known as *Regula Falsi* (the method of false position) was used to converge the breakwater to an impact time on the boundary. However, the convergence used method was later changed to Newton's Method during this investigation. This method was chosen for its simplicity and its ability to converge quickly and accurately (Johnson and Riess 1982). Newton's Method is used to find a root of a function and has the following form:

$$t_{n+1} = t_n - \frac{g_{(1,2)}(t_n)}{\dot{g}_{(1,2)}(t_n)} \tag{2.47}$$

where

$$\dot{g}_1 = 2(x-1)\dot{x} + 2(y-h)\dot{y} \qquad (2.48)$$

$$\dot{g}_2 = 2(x+1)\dot{x} + 2(y-h)\dot{y} \qquad (2.49)$$

where

$x(t)$, $\dot{x}(t)$, $y(t)$, and $\dot{y}(t)$ are functions of time as seen in Equations 2.18-2.21. With either of these methods, the value of g was converged to a tolerance of 10^{-6}, and the time at which this occurs becomes the impact time.

2.7 Solution Procedure

Once all of these parts were formulated, they were put together into a computer program which was developed to solve the problems being investigated. The program was written using the FORTRAN programming language. This program uses the equations of motion and solves for the breakwater's position and velocities at a given time. The solution procedure may be summarized as follows:

1. dimensions and other parameters are given
2. the breakwater is given initial position and velocities
3. the solution proceeds through time using the analytical solution of the equations of motion
4. when g_1 or g_2 equals zero or becomes positive (i.e., a cable has become "taut"), conditions are converged within tolerance (close to zero) using Newton's Method, and the tangential and normal velocities are determined from geometry
5. the normal velocity is reduced by a coefficient of restitution and the direction of motion is changed; this is the loss of energy and the rebound of the breakwater
6. new velocities are calculated and these conditions are made the new initial conditions, and the procedure starts over with step 3

2.8 Development of Alternate Modeling Program

The solution discussed above is an analytical solution to the problems being investigated. Before the analytical solution was applied a numerical solution was used to solve the problems. When the analytical solution was formulated, the solution process used in this research was switched to using the analytical solution because it gave realistic and accurate results. However, it was understood that more complicated forms of this same problem may be investigated in the future and an analytical solution may not exist for such problems. Therefore, a numerical solution and is explained here.

2.8.1 Numerical Modeling Program

The solutions to the differential equations discussed in the formulation section may be obtained by using a computer program written to model the motions of the point mass breakwater. A program written in FORTRAN programming language was used to obtain the solution of the ordinary differential equations (ODE's). The DIVPAG subroutine from IMSL (International Mathematical and Statistical Library) was used in solving the ODE's. DIVPAG requires the ODE's to be in first-order form, so as seen in the formulation, the ODE's to be solved are put in this form. The DIVPAG subroutine offers two numerical integration method options, the Adams-Moulton Method and Gear's Stiff Method. The Adams-Moulton Method was used in this numerical analysis.