Chapter 2

Wave Propagation in Viscous Fluid

This chapter summarizes with the derivation of the mathematical form of the acoustic wave propagation in the fluid. Before we derive the final form of the wave propagation equation in viscous fluid, we first look at two conservation (mass and momentum) of equations and state equation in the fluid. Detailed derivations can be found in the literatures [4, 6, 27, 28]. We limit our discussion only on the lossy 1-dimensional plane wave.

2.1 One-dimensional Viscous Wave Equation

2.1.1 One-dimensional Continuity Equation (Conversation of Mass)

Consider the flow of a compressible fluid through a duct of arbitrary cross section (area S) in one dimension. The control volume (CV), is the segment between x and $x + \Delta x$.

We want to know the rate at which the inside mass changes. First, we made two assumptions:

1. The CV is fixed in space
2. The flow is one-dimensional, so the mass flow only depends on *t* and *x*.

, where ρ, u are the average mass flow density and average mass flow speed, respectively. For $\Delta x \to 0$, ρ and u will become a true point function.

The time rate of increase of mass inside the CV is equal to net mass flow into the CV through the CV surfaces or in mathematical terms,

$$\frac{\partial}{\partial t}(S\rho\Delta x) = \rho u S|_x - \rho u S|_{x+\Delta x} \tag{1}$$

Since S is a constant and Δx is not a function of time, Eq. (1) may be rearranged as following:

$$\frac{\partial \rho}{\partial t} = \frac{\rho u|_x - \rho\, u|_{x+\Delta x}}{\Delta x}$$

Note that, to simplify the notation, we use the subscript (t & x) to represent the derivative of the function respective to time and/or distance. In the limit as $\Delta x \to 0$, the right-hand side becomes $-\partial(\rho u)/\partial x$, and we obtain

$$\rho_t + (\rho u)_x = 0 \tag{2}$$

This is the equation of continuity.

2.1.2 Conversation of Momentum in One Dimension

The same control volume (CV) is again used to derive the conservation of momentum relation. The momentum per unit volume is $\rho\, u$, and the momentum flux (momentum per unit area per unit time) is $\rho\, u^2 S$.

Applying Newton's second law, and taking into account the momentum inflow across the CV boundaries, the time rate of increase of momentum inside the CV equals to the sum of the net momentum inflow across boundaries and the sum of the forces acting on the CV.

Forces are generally classified into two kinds: body forces and surface forces. In this case, we make some assumptions about the forces in addition to the assumptions listed for the derivation of continuity equation.

1. We could neglect body forces. The most common body force is gravity, since we are only considering a short distance; the effect of gravity is not significant.
2. The fluid is inviscid, (we will later discuss the effect of viscosity). So the only significant surface force is that due to the pressure P at the CV end surface.

Given these four assumptions, the mathematical form of the conservation of momentum statement is

$$\frac{\partial}{\partial t}(\rho u S \Delta x) = \rho u^2 S|_x - \rho u^2 S|_{x+\Delta x} + PS|_x - PS|_{x+\Delta x}$$

Divide the above equation by $S\Delta x$, and take the limit as $\Delta x \to 0$:

$$(\rho u)_t + (\rho u^2)_x + P_x = 0 \tag{3}$$

Expanding the first two terms, we have

$$\rho_t u + \rho u_t + (\rho u)_x u + \rho u u_x + P_x = 0$$

From the continuity equation (eq (2)), the first and third term is canceled:

$$\rho (u_t + u u_x) + P_x = 0 \tag{4}$$

,which is simpler than eq (3). Thus, it is the final form of the conservation of momentum.

To include the viscous effect in momentum equation, the surface force is therefore present. From Navier-Stokes equation, we could rewrite the conservation of momentum as:

$$\rho (u_t + u u_x) + P_x = (\lambda + 2\mu) u_{zz} \tag{5}$$

, where μ is the shear viscosity coefficient and λ is the dilatational viscosity coefficient.

2.1.3 Equation of State

For fluids or gases, the equation of state relates to pressure can be expressed as a function of density and entropy. When losses are negligible, the entropy remains constant, and then the pressure is a function of density alone:

$$P = P(\rho)$$

For gases, the isentropic equation of state (so called adiabatic gas law) is often used:

$$(P/p_0) = (\rho/\rho_0)^{\gamma} \tag{6}$$

where γ is the specific heats ratio and p_0 and ρ_0 are the static values of P and ρ.

In the case of liquid, the equation of state can be expressed as the series of the general isentropic equation of state from eq. (6):

$$P = p_0 + A\frac{\rho-\rho_0}{\rho_0} + \frac{B}{2!}\left(\frac{\rho-\rho_0}{\rho_0}\right)^2 + \frac{C}{3!}\left(\frac{\rho-\rho_0}{\rho_0}\right)^3 + \cdots \tag{7}$$

The coefficient *A, B, C …* are determined from experiments or from other analyses. From some derivation coefficient *A* is equal to $\rho_0 c_0^2$. ρ_0 is the static value of the fluid's density and c_0 is the static value of the velocity of the wave, respectively.

If we take a further step, simplify eq. (7) by introducing the excess pressure

$$p \equiv P - p_0$$

and the excess density is

$$\delta\rho \equiv \rho - \rho_0$$

and incorporating these into coefficient *A*, then the isentropic state of equation for fluid (eq (7)) becomes

$$p = c_0^2 \delta\rho \, [1 + \frac{B}{2!A}\frac{\delta\rho}{\rho_0} + \frac{C}{3!A}(\frac{\delta\rho}{\rho_0})\cdots] \tag{8}$$

2.1.4 Linearization / Small-signal approximation

Eq. (2), (5), (8) are nonlinear equations, we could obtain the linear version by introducing the so-called small-signal approximation. Most sound waves disturb the status quo in the fluid only around a small region, the variables associated with sound, excess pressure, excess density, and particle velocity may be assumed to be small quantities of first order. We start by assuming

$$|\delta\rho| << \rho_0$$

Given the expression of excess density, we can expand the equation of continuity (eq (2)) as

$$\delta\rho_t + u\delta\rho_x + \rho_0 u_x + \delta\rho u_x = 0$$

The second and fourth terms are second-order terms; each contains the products of two small quantities. The first and third terms are first-order term. If the first-order terms are small, then the second-order can be neglected. Dropping them, we obtain

$$\delta\rho_t + \rho_0 u_x = 0 \tag{9}$$

The momentum equation may be linearized similarly. Expanded eq. (5) becomes

$$\rho_0 u_t + \rho_0 u u_x + \delta\rho u_t + \delta\rho u u_x + p_x = (\lambda + 2\mu) u_{xx} \tag{10}$$

Again, we can drop the higher order terms. The second, third, fourth terms are higher than first order, so we can neglect them. Then the conservation of momentum equation becomes

$$\rho_0 u_t + p_x = (\lambda + 2\mu) u_{xx} \tag{11}$$

Lastly, the isentropic equation of state (eq. (8)) can be linearized by inspection. Eq. 8 becomes

$$p = c_0^2 \, \delta\rho \tag{12}$$

2.1.5 Linear Viscous Wave Equation

Partial Discharge (PD) is a point charge emitted from the windings [19]. Since it is a sudden release of energy, it generates some acoustic wave propagating around its neighborhood. Besides, considering the point charge is released in an infinitely large amount of medium, the acoustic wave could be considered as a plane wave.

The three fundamental acoustic plane wave equations are derived in the previous section. The linearized equations are summarized as following:

Continuity: $\delta\rho_t + \rho_0 u_x = 0,$ (9)

Momentum: $\rho_0 \, u_t + p_x = (\lambda + 2\mu) u_{xx}$ (10)

State: $p = c_0^2 \, \delta\rho$ (11)

ρ is the density of the fluid, p is the excessive pressure of the fluid, u is the velocity of the fluid. The subscript (t & x) represents the derivative of the function respective to the variable. λ, ,and μ are dilatational and shear viscosity coefficients, respectively.

In this section, we combine eq. (9), (10) and (11) to find the final form of the viscous wave equation.

Since the shear viscosity coefficient μ has been measured for many fluids, the dilatational viscosity coefficient λ will not be found. By Stoke's assumption, we could approximate $\lambda = -2\mu/3$, which leads to replacement of $\lambda + 2\mu$ by $4\mu/3$. For fluids that do not follow Stokes's assumption, modern practice is to replace λ by $-2\mu/3 + \mu_B$, where μ_B is called the bulk viscosity coefficient. Therefore, $\lambda + 2\mu = \frac{4}{3}\mu + \mu_B = \mu\widetilde{V}$, where $\widetilde{V} = 4/3 + \mu_B/\mu$. In our study, we assume that the fluids follow Stoke's assumption. The momentum equation becomes

$$\rho_0 u_t + p_x = \frac{4\mu}{3} u_{xx} \qquad \textbf{(13)}$$

take the derivative respective to x on (9) and substitute u_{xx} into (12) and it becomes:

$$\rho_0 u_t + p_x = -\frac{4\mu}{3\rho_0} \delta\rho_{tx} \qquad \textbf{(14)}$$

from (11), we take the derivative respective to t and x, and we get $p_{tx} = c_0^2 \delta\rho_{tx}$, and substitute $\delta\rho_{tx}$ into (13),

$$\rho_0 u_t + p_x = -\frac{4\mu}{3\rho_0 c_0^2} p_{tx}$$

and we take the derivative respect to x of the above equation

$$\rho_0\, u_{tx} + p_{xx} = -\frac{4\mu}{3\,\rho_0 c_0^2}\, p_{txx} \tag{15}$$

From (9) and take the derivative respect to *t*, and we get $\delta\rho_{tt} = -\rho_0 u_{xt}$; also from (11) and take the 2nd derivative respective to t, and we get $\delta\rho_{tt} = 1/c_0^2\, p_{tt}$, therefore $u_{xt} = -1/\rho_0\, \delta\rho_{tt} = -1/c_0^2 \rho_0\, p_{tt}$. Finally we substitute u_{xt} into (14), and then we have

$$\frac{4\nu}{3c_0^2} p_{xxt} + p_{xx} - \frac{1}{c_0^2} p_{tt} = 0 \tag{16}$$

where $\nu = \mu/\rho_0$ is the kinematic viscosity coefficient. The three-dimensional version of this equation is shown as follows:

$$\frac{4\nu}{3c_0^2}\nabla^2 p_t + \nabla^2 p - \frac{1}{c_0^2} p_{tt} = 0 \tag{17}$$

2.2 Time-Harmonic Analytical Solutions to the Viscous Wave Equation

2.2.1 Solution

To solve eq(16), it is customary to assume a general solution of the partial differential equation and find the particular solution for the equation. Also, to take the various mechanism of dispersion in the fluid one at a time, we can determine their effect on sound propagation by following the procedure outlines in Fig 1.

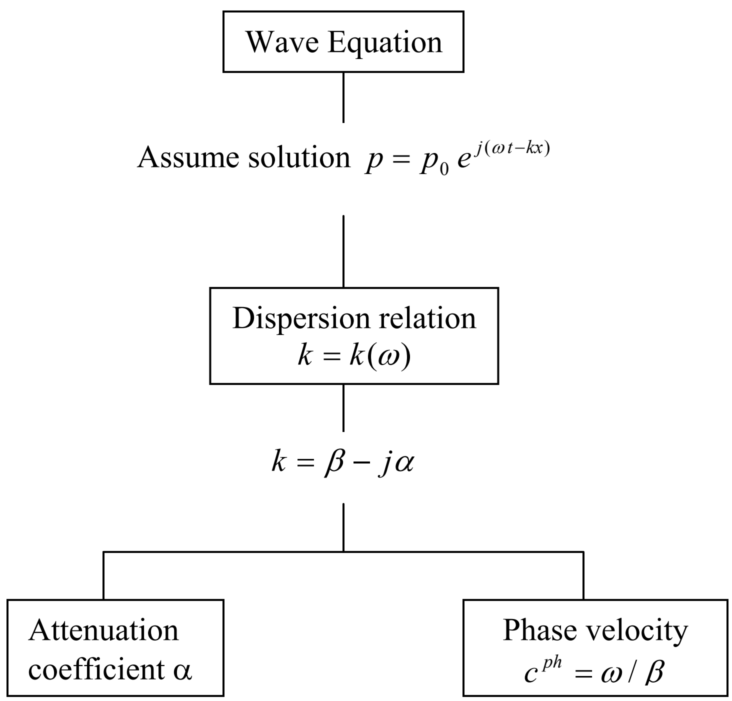

Fig 1. Dispersion relation algorithm

Since we let $p = p_0\, e^{j(\omega t - kx)}$, then $p_{xxt} = p_0(-k^2 \cdot j\omega)\, e^{j(\omega t - kx)}$, $p_{xx} = p_0(-k^2)\, e^{j(\omega t - kx)}$, and $p_{tt} = p_0(-\omega^2)\, e^{j(\omega t - kx)}$. Insert the above three derivatives into eq(15) and eliminating the constant and the common exponential terms, then we get

$$(1 + j\omega \frac{4\nu}{3c_0^2})k^2 - \frac{\omega^2}{c_0^2} = 0$$

or

$$k = \pm \frac{\omega/c_0}{\sqrt{1 + j\frac{4\delta_v}{3}}} = \pm(\beta - j\alpha)$$

, where δ_v is the dimensionless coefficient

$$\delta_v = \frac{\omega\nu}{c_0^2}$$

If we only consider the wave propagates in positive direction, which $k = \beta - j\alpha$, the time-harmonic solution of the excess pressure becomes

$$\begin{aligned} p &= p_0\, e^{j(\omega t - \beta x + j\alpha x)} \\ &= p_0\, e^{-\alpha x} e^{j(\omega t - \beta x)} \\ &= p_0\, e^{-\alpha x} e^{j\omega(t - \frac{x}{\omega/\beta})} \end{aligned} \qquad \textbf{(18)}$$

The magnitude of the attenuation coefficient α determines how fast the peak amplitude of $e^{j\omega(t - \frac{x}{\omega/\beta})}$ decays. Also, the variation of wave propagation speed $c_{ph} = \omega/\beta$ will be determined by β.

Notice that the form of above excess pressure is ideal for time-harmonic forced waves. A source emitting a signal $p = p_0\, e^{j\omega t}$ at x=0 is implied. Also, the above derivation is done in Cartesian coordinate; it can be easily extended to spherical or cylindrical coordinate.

2.2.2 Ideal 1D Acoustic Wave Propagation at Different Frequencies in Water and Oil

To study the wave propagation phenomenon in different mediums and under different frequencies, we can apply the theoretical acoustic viscous wave equation solution from section 5 with the coefficients associated with different fluid medium.

Recap assumptions on the acoustic wave theoretical derivation:

- Plane wave (1-dimension)
- Adiabatic process
- No internal molecular process
- Only viscous effect considered
- The fluid follows Stoke's assumption, for which the dilatational viscosity coefficient equals to minus two-third of viscosity coefficient.

Using caster oil as the medium, the parameters are given as following:

Density: $\rho_0 = 950\ (kg/m^3)$

Propagation Velocity: $c_0 = 1540\ (m/s)$

Viscosity coefficient: $\mu_0 = 0.96\ (N-s/m)$

Let the wave peak amplitude be 1 at the source (x=0), we simulate the wave peak amplitude, which is the term $p_0\, e^{-\alpha x}$ in eq. (17) with p_0 to be 1, vs. the distance from the source at various frequencies. The results are shown in the following two figures (in dB and linear scale):

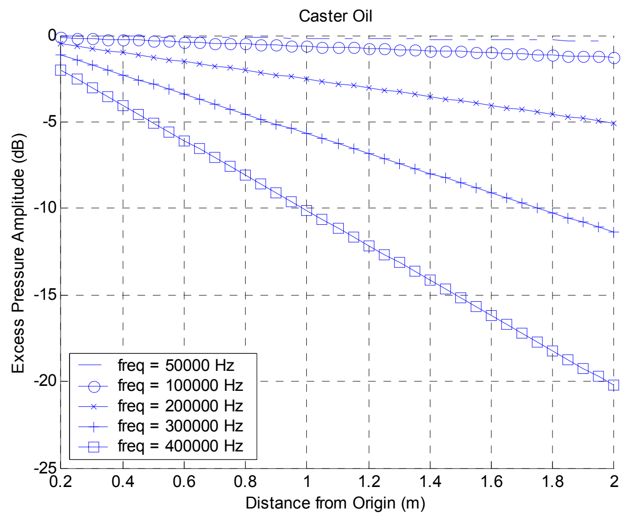

Fig 2. Excess pressure amplitude (dB) vs. the distance from the origin. The wave peak amplitude at the origin is 1.

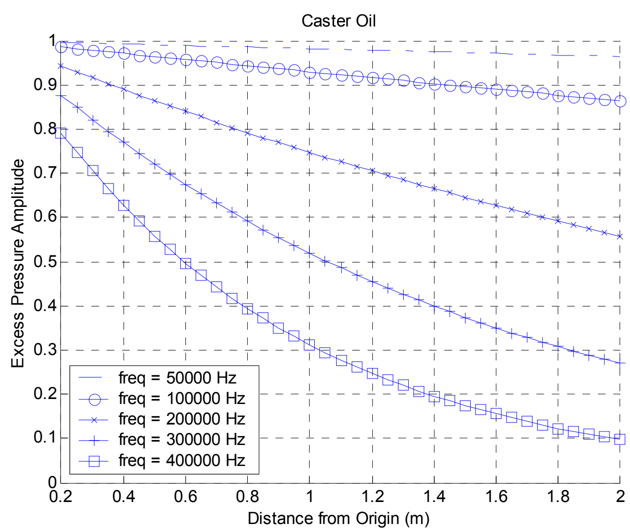

Fig 3. Excess pressure amplitude vs. the distance from the origin. The wave peak amplitude at the origin is 1.

As expected, the peak wave amplitude becomes smaller as we move further from the source. We can also notice that as the wave frequency goes higher, more attenuation can be observed at any given location. In other word, the attenuation factor depends on both the location and frequency.

Similarly, we simulate the wave peak amplitude vs. distance of water using the following parameters:

Density: $\rho_0 = 998 \ (kg/m^3)$

Velocity: $c_0 = 1481 \ (m/s)$

Viscosity coefficient: $\mu_0 = 0.001 \ (N-s/m)$

The major difference between caster oil and water, in terms of parameters, is viscosity. Water is much less viscous than oil, about a thousand times less. So we can expect that the wave peak amplitude attenuation in water as further away from the source is much less the attenuation in caster oil.

Fig. 4 and 5 shows the wave peak amplitude vs. distance from the origin in water

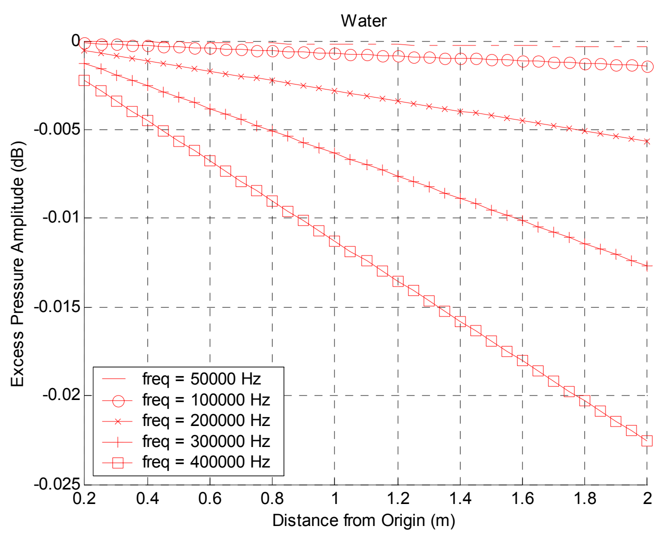

Fig 4. Excess pressure amplitude (dB) vs. the distance from the origin. The wave peak amplitude at the origin is 1.

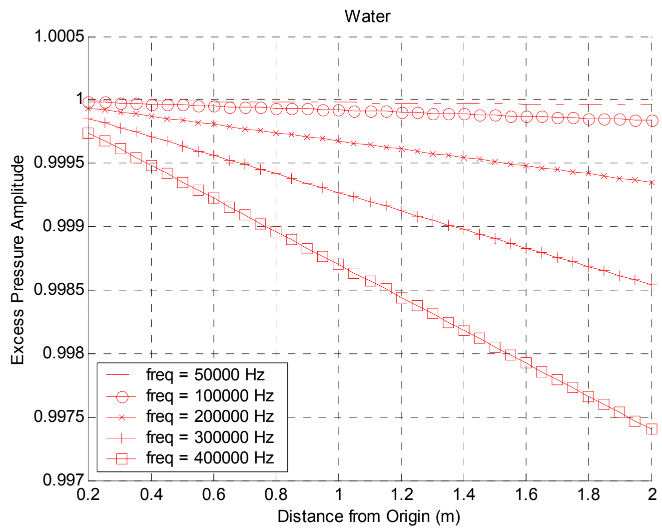

Fig 5. Excess pressure amplitude vs. the distance from the origin. The wave peak amplitude at the origin is 1.

Note that, the acoustic wave peak amplitude in water gets very minor attenuation across frequencies in this calculation, even if we are at 2 meter away from the source. In the caster oil case, we could observe large attenuation of wave peak amplitude, due to the larger coefficient of viscosity. This result shows that if we measure the acoustic wave in water at any location, assuming we are inline with the source, we will get a comparable signal density as the source, however in the caster oil case, we will need a more sensitive sensor to catch up the signal and perhaps a need for more complicated signal processing units to filter out the noises, since the signal level might be low.

2.3 Experiment

We measured the acoustic wave with a container filled with water inside. Two PZTs, one acting as the source emitter and one as the detector are then immersed inside the water. The experiment setting is shown in Fig 6.

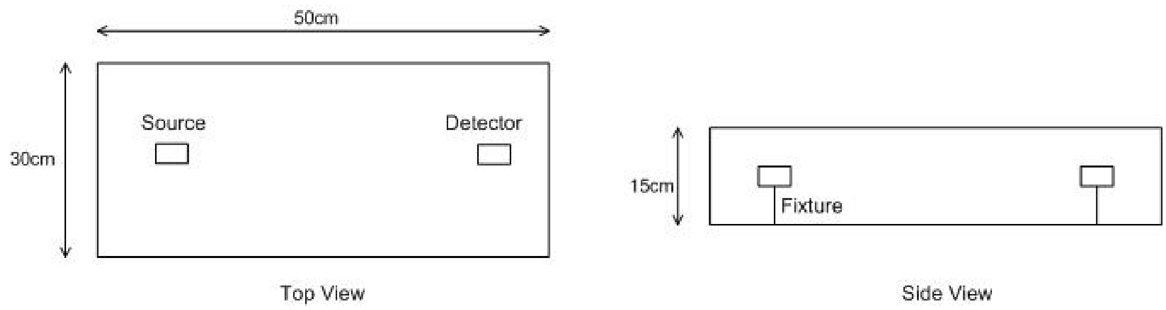

Fig 6. Experiment setup. Left: top view. Right view: side view

The source and detector are inline and both are lifted up to a same height by some steel fixture. A typical acoustic waveform is shown in fig. 7.

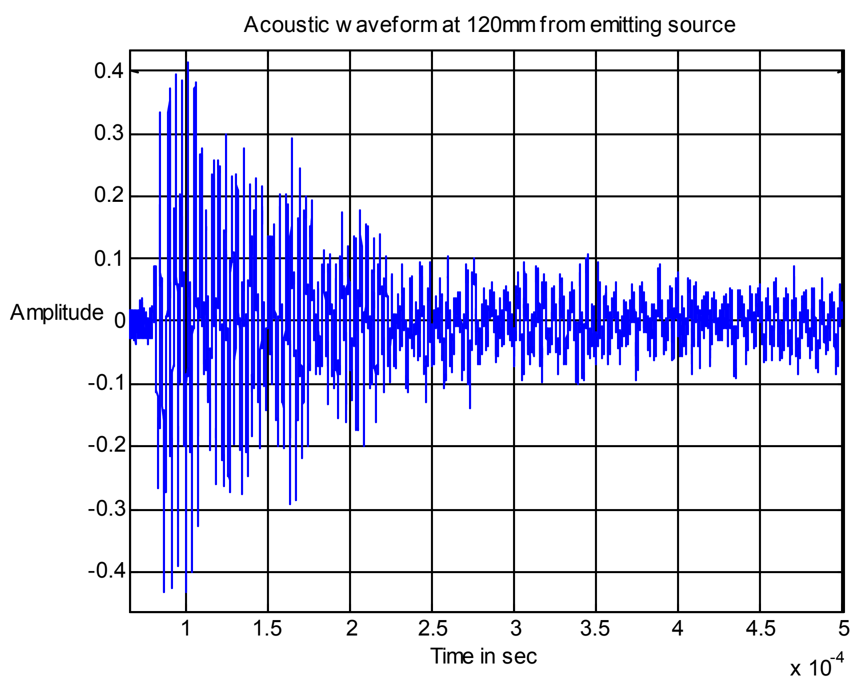

Fig 7. Typical acoustic waveform

We recorded the waveform three times at each location and did an average in time domain among the three. Then we performed Discrete Fourier Transform to get the spectrum of the waveforms. Most of the energy concentrates in the region of 300kHz to 400kHz. The following two plots show the zoom spectrum between 100kHz and 600kHz; and 280kHz and 400kHz, respectively.

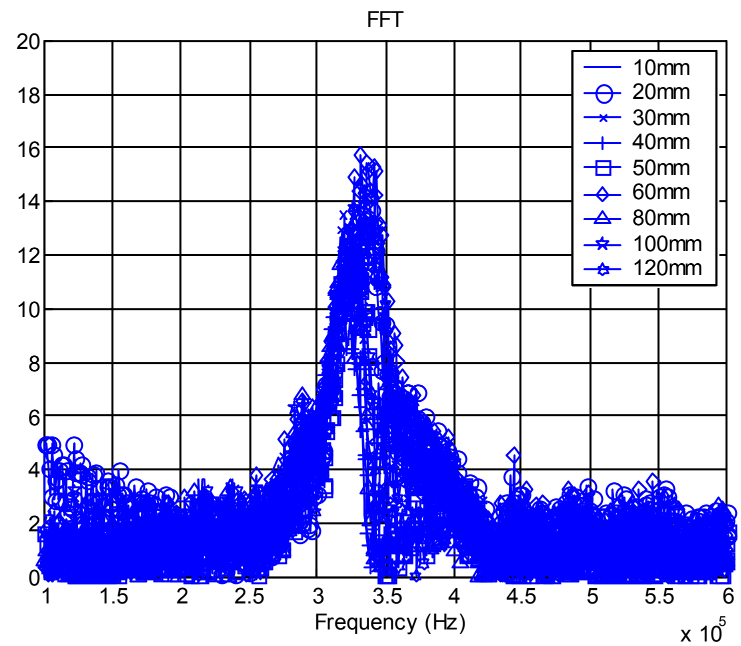

Fig 8. Zoomed FFT of the averaged acoustic wave between 100kHz and 600kHz. The legend indicates the distance between the source and measurement point

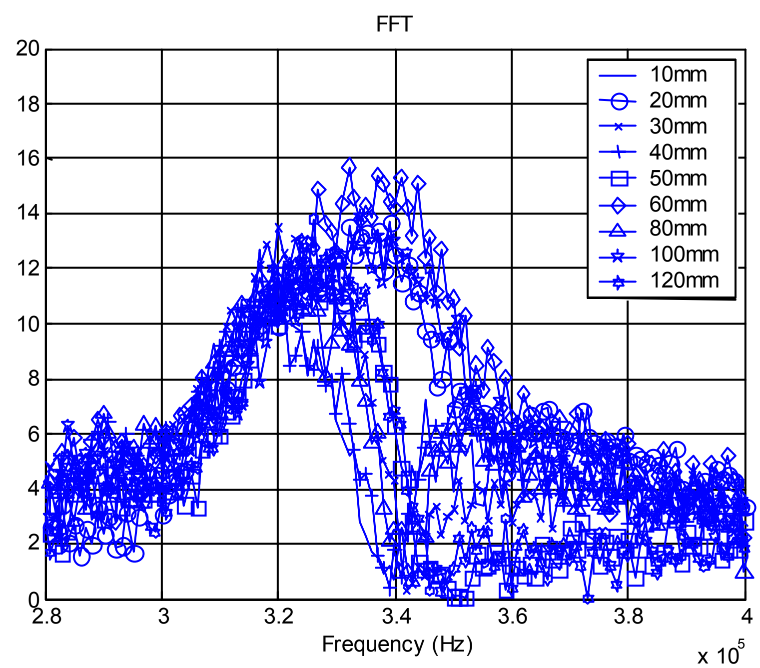

Fig 9. Zoomed FFT of the averaged acoustic wave between 280kHz and 400kHz. The legend indicates the distance between the source and measurement point

As shown in fig. 8 and fig. 9, the maximum amplitude occurs in the region of 320kHz and 340kHz across different locations. The maximum peak between 300kHz and 400kHz region (in fig. 10) shows that there is not much amplitude variation.

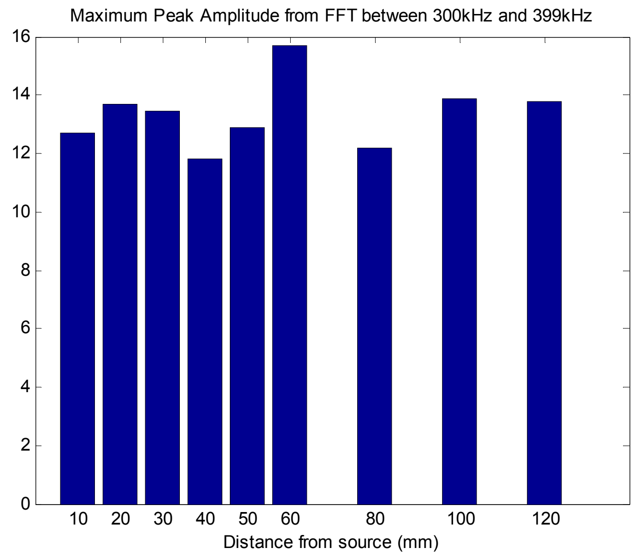

Fig 10. Maximum amplitude between 300kHz and 400kHz.

Also, we compare the peak amplitudes at 320 kHz. The peak amplitudes stay within the same level across the distances from the origin.

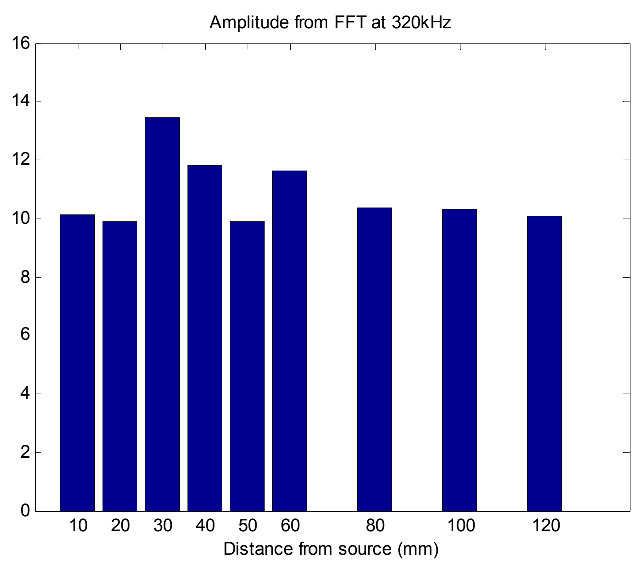

Fig 11. Amplitude at 320kHz.

The experiment results in water coincide with the findings from theoretical derivation, which the acoustic wave peak amplitude does not change with various distances from the origin.

2.4 Simulation

A computer simulation using ABAQUS is set up using the geometry of the experimental container; the acoustic wave propagation damping-rate in the experimental container can be estimated. Fig 12 shows the normalized wave amplitude at the locations with different distances from the source.

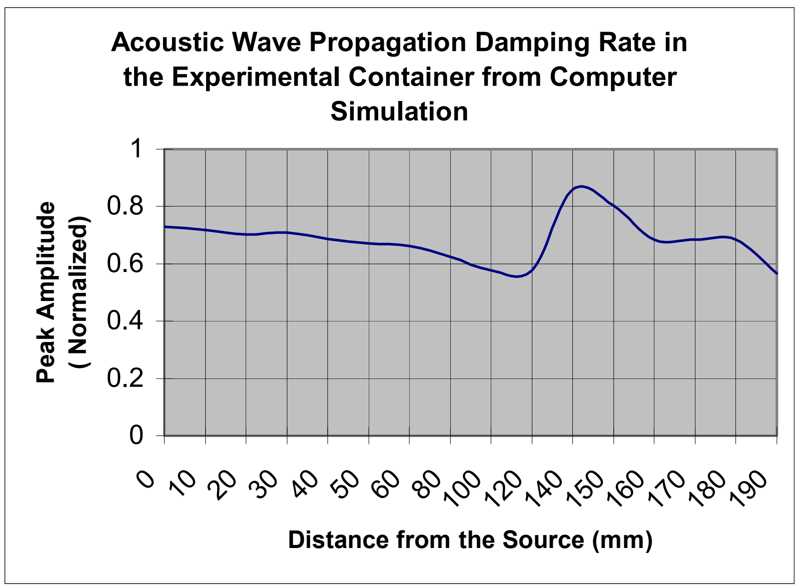

Fig 12. Propagation Damping Rate in the Experimental Container

Comparing these two figures (Fig 10 and Fig 12) obtained from the experiment and the simulation; it is found the wave propagation damping-rates are similar. The wave peak amplitude does not change much with various distances from source in water, which confirms the above conclusion when comparing the experimental result with the theoretical analysis result. But when distance to the source is larger than 120mm, the damping rate from the experiment and the computer simulation are different from that of the theoretical analysis. In the theoretical analysis, acoustic wave propagates in the infinite medium while it is not the case in the experimental container. So the discrepancy was due to the size limitation of the experimental container (same happened to the computer simulation model), and influence of its inner structures. The wave would be reflected at the rear surface of the experimental container to fortify the wave amplitude it to some extent. In this situation, the wave propagation may show different damping features from that in infinite medium in the theoretical analysis.