Optimal Control for an Impedance Boundary Value Problem

Oleksandr Bondarenko

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Terry L. Herdman
Jeff T. Borggaard
John A. Burns

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Abstract

We consider the analysis of the scattering problem. Assume that an incoming time harmonic wave is scattered by a surface of an impenetrable obstacle. The reflected wave is determined by the surface impedance of the obstacle. In this paper we will investigate the problem of choosing the surface impedance so that a desired scattering amplitude is achieved. We formulate this control problem within the framework of the minimization of a Tikhonov functional. In particular, questions of the existence of an optimal solution and the derivation of the optimality conditions will be addressed.

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Chapter 1

Introduction

1.1 Scattering Problem

We consider an acoustic wave traveling in a homogeneous medium, such as a fluid or a gas. The propagation of the wave is described by a system of nonlinear partial differential equations [5]. Since a sound wave can be considered as a small perturbation in a fluid or a gas, it is sufficient to use a linear model for wave propagation. Let $p(x,t)$ denote the pressure of the medium. From the linear model it follows that p has to satisfy the wave equation [5]

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = \Delta p, \tag{1.1}$$

where c is the speed of sound in the medium. Since the medium is homogenous, c is constant. We assume that p is represented by a time harmonic wave of the form

$$p(x,t) = \mathrm{Re}[u(x)e^{-i\omega t}], \quad x \in \mathbb{R}^3,$$

with a frequency $\omega > 0$ and a complex-valued amplitude, also called a field, u depending only on the spatial variable. Substituting p into the wave equation (1.1) it follows that the amplitude u satisfies the Helmholz equation

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^3,$$

where the wave number k is given by $k = \omega/c$. Two important examples of the amplitudes u satisfying the Helmholz equation, that we will encounter later in this paper, are the amplitude of a spherical wave

$$p(x,t) = \frac{1}{|x-x_0|}\mathrm{Re}\ e^{ik|x-x_0|-i\omega t}, \text{ i.e., } u(x) = \frac{e^{ik|x-x_0|}}{|x-x_0|},$$

with a source at x_0, and the amplitude of a plane wave

$$p(x,t) = \mathrm{Re}\ e^{ikx\cdot\hat{\theta}-i\omega t}, \text{ i.e, } u(x) = e^{ikx\cdot\theta},$$

where the vector $\theta \in S^2 = \{\hat{x} \in \mathbb{R}^3 : |\hat{x}| = 1\}$ denotes the direction of incidence of the plane wave.

Consider the case when an incident time harmonic plane wave $p^i(x,t) = \mathrm{Re}[e^{ikx\cdot\theta}e^{-i\omega t}]$ is scattered by an impenetrable obstacle. The incident field u^i is then given by $u^i = e^{ikx\cdot\theta}$, where $\theta \in S^2$ is the direction of the incidence. We assume that the obstacle can be represented by a bounded domain D with C^2 boundary. For an impenetrable object the scattered field u^s, in the most general case, has to satisfy the Robin boundary condition (also called the impedance boundary condition)

$$\frac{\partial}{\partial n}(u^i + u^s) + ik\lambda(u^i + u^s) = 0 \quad \text{on } \partial D,$$

where n is the outward unit normal to domain boundary ∂D and λ, the surface impedance, is a real valued function on ∂D. Given an impenetrable obstacle D and a surface impedance λ, one can consider the problem of finding the scattered field u^s such that

$$\Delta u^s + k^2 u^s = 0 \quad \text{in } \mathbb{R}^3 \backslash \bar{D} \tag{1.2}$$

and

$$\frac{\partial}{\partial n}(u^i + u^s) + i\lambda(u^i + u^s) = 0 \ \text{ on } \partial D. \tag{1.3}$$

However, in general, this problem does not have a unique solution. The unique and physically relevant solution also satisfies the Sommerfeld radiation condition

$$\lim_{r\to\infty} r\left[\frac{\partial}{\partial r}u^s_\lambda - iku^s_\lambda\right] = 0, \quad r = |x|, \tag{1.4}$$

uniformly in all directions $x/r \in S^2$. This condition means that at large distances, a scattered wave behaves as an outgoing spherical wave. The problem of finding u^s that satisfies the boundary value problem (1.2)-(1.4) is known as the direct scattering problem [2].

It can be shown [2] that every solution of the Helmholz equation that satisfies the Sommerfeld radiation condition has the following asymptotic behavior:

$$u^s(x) = \frac{e^{ikr}}{r}\left\{u^\infty(\hat{x}) + O\left(\frac{1}{r}\right)\right\}, \quad r = |x| \to \infty,$$

uniformly in $\hat{x} = x/r \in S^2$. The term e^{ikr}/r represents a spherical wave. The function u^∞ is called the far field pattern or the scattering amplitude.

In other scenarios, the obstacle D and the far field pattern u^∞ corresponding to an incident wave u^i, are given and the problem is to find the surface impedance λ. This is an example of an inverse scattering problem. This problem is inherently nonlinear. Moreover, it is also ill-posed [2], meaning that the solution of the inverse problem (provided it exists) does not continuously depend on the data. In particular, small perturbations of the far field can lead to large errors in the reconstruction of the impedance. In the language of the inverse problems, to obtain a stable solution (solution that depends continuously on the measured data) a regularization strategy is required [5].

In this project we consider the Tikhonov regularization method. Let $f \in L^2(S^2)$ be the given far field pattern corresponding to the incident wave u^i and let u^∞ be the far field pattern of the scattered field u^s satisfying (1.2)–(1.4). Then, the Tikhonov regularization method consists of finding λ which minimizes the functional

$$J(\lambda) = ||u^\infty - f||^2_{L^2(S^2)} + \varepsilon ||\lambda||^2_{L^2(\partial D)}, \quad \varepsilon > 0.$$

Here, ε is a regularization parameter, which is chosen based on a priori information about the solution λ.

Thus, we can view this inverse scattering problem as an optimization problem. In this project we will apply the techniques of optimal control theory to establish the necessary conditions for a minimizer of J.

1.2 Control Problem

The formulation of an optimal control problem usually contains the following parts [6]:

1. A mathematical model of the process to be controlled.
2. A statement of physical constraints.
3. Specification of a performance measure.

In our case, the process to be controlled is the scattering of the given incident wave u^i by the given impenetrable obstacle D. Thus, the mathematical model is represented by the boundary value problem (1.2) – (1.4). The physical constraints are imposed on the maximum and minimum values of the impedance λ. Mathematically, this requires λ to belong to a certain set U_{ad}. The performance measure, or the cost functional, is determined by the Tikhonov functional

$$J(\lambda) = ||u^\infty - f||^2_{L^2(S^2)} + \varepsilon ||\lambda||^2_{L^2(\partial D)}, \ \lambda \in U_{ad}.$$

In this project we consider this problem in $\mathbb{R}^2$. The boundary value problem (1.2) – (1.4) in the two-dimensional case represents the scattering of planar waves from infinitely long cylindrical obstacles. Furthermore, it serves as a model case for testing numerical approximation schemes in direct and inverse scattering [2].

Chapter 2

Statement of the Problem

2.1 Problem Formulation

Let D be a bounded, simply connected domain $D \subset \mathbb{R}^2$ with C^2 smooth boundary. Let $u = u^i + u^s_\lambda$ be the total field in $\mathbb{R}^2 \backslash \bar{D}$, where u^i is the given incident field and u^s_λ is the wave scattered by D. We assume that the incident field is given by the plane wave $u^i(x) = \exp(ikx \cdot \hat{\theta})$, where $k \in \mathbb{R}$ is the fixed positive wave number and $\hat{\theta}$ is the direction of incidence, $\hat{\theta} \in S^1 = \{\hat{x} \in \mathbb{R}^2 : |\hat{x}| = 1\}$. The scattering of the time harmonic wave by an impenetrable obstacle D can be mathematically modeled by the following set of equations:

$$\Delta u^s_\lambda + k^2 u^s_\lambda = 0 \text{ in } \mathbb{R}^2 \backslash \bar{D}, \qquad \text{(Helmholz equation)} \tag{2.1}$$

$$\frac{\partial u}{\partial n} + ik\lambda u = 0 \text{ on } \partial D, \qquad \text{(Impedance boundary condition)} \tag{2.2}$$

$$\lim_{r\to\infty} \sqrt{r}\left[\frac{\partial}{\partial r} u^s_\lambda - iku^s_\lambda\right] = 0 \text{ uniformly}, r = |x|. \qquad \text{(Radiation condition)} \tag{2.3}$$

Here, $n = n(x)$ is the unit normal vector at $x \in \partial D$ directed into the exterior of D, and $\lambda = \lambda(x)$ is the surface impedance at the point $x \in \partial D$. It can be shown [2] that every solution u^s_λ of the Helmholz equation satisfying the radiation condition has the following asymptotic behavior:

$$u^s_\lambda(x) = \frac{e^{ikr}}{\sqrt{r}}\left\{u^\infty_\lambda(\hat{x}) + O\left(\frac{1}{r}\right)\right\}, \quad r = |x| \to \infty,$$

uniformly in $\hat{x} = x/r \in S^1$. The function u^∞_λ is called the far field pattern corresponding to the scattering problem (2.1)–(2.3).

Our aim is to choose the impedance $\lambda \in U_{ad} = \{\psi \in L^\infty(\partial D) : a_- \leq \psi(x) \leq a_+ \text{ a. e. on } \partial D\}$, where $a_+ > a_- > 0$ are given constants, such that, for a given (complex valued) function $f \in L^2(S^1)$ and a regularization parameter $\varepsilon > 0$, the functional $J : L^\infty(\partial D) \supset U_{ad} \to \mathbb{R}$ given by

$$J(\lambda) = ||u^\infty_\lambda - f||^2_{L^2(S^1)} + \varepsilon||\lambda||^2_{L^2(\partial D)}, \text{ with } \lambda \in U_{ad},$$

is minimized. Here, u_λ^∞ is the far field pattern of the scattered field u_λ^s.

We now present an interpretation of the boundary condition (2.2). Here we will follow the approach based on the notion of parallel curves [1]. The impedance boundary condition (2.2) has to be understood in the following sense:

$$||n \cdot \nabla u_t + ik\lambda u_t||_{L^2(\partial D)} \to 0 \text{ as } t \to 0+, \tag{2.4}$$

where $u_t := u(x + tn(x))$ and $\nabla u_t := \nabla u(x + tn(x))$ for sufficiently small $|t|$.

2.2 Formulation of the Problem by the Boundary Integral Equation Method

In this section we use the boundary integral equation method to formulate the boundary value problem (2.1)-(2.3) as an integral equation.

The idea of the integral equation method is to express the solution in the form of a combination of a single and double layer potentials. The function

$$\Phi(x, y) := \frac{i}{4} H_0^{(1)}(k|x - y|) \quad x, y \in \mathbb{R}^2,\ x \neq y, \quad k \in \mathbb{R},\ k > 0,$$

where $H_0^{(1)}$ denotes the Hankel function of the first kind and order 0, is called a fundamental solution of the two-dimensional Helmholz equation. Given an L^2 integrable function φ on ∂D, the functions

$$U(x) := \int_{\partial D} \varphi(y)\Phi(x, y)\, ds(y), \qquad x \in \mathbb{R}^2 \backslash \partial D, \tag{2.5}$$

$$V(x) := \int_{\partial D} \varphi(y) \frac{\partial}{\partial n(y)} \Phi(x, y)\, ds(y), \qquad x \in \mathbb{R}^2 \backslash \partial D, \tag{2.6}$$

are called single and double layer potentials, respectively, with density φ, see [1]. Because of the properties of the Hankel function $H_0^{(1)}$, they are solutions of the Helmholz equation and satisfy the Sommerfeld radiation condition (2.3). Furthermore, we introduce the single and double-layer integral operators $S, D : L^2(\partial D) \to L^2(\partial D)$ by

$$(S\varphi)(x) := \int_{\partial D} \varphi(y)\Phi(x, y)\, ds(y),\ x \in \partial D \quad \text{and} \tag{2.7}$$

$$(D\varphi)(x) := \int_{\partial D} \varphi(y) \frac{\partial}{\partial n(y)} \Phi(x, y)\, ds(y),\ x \in \partial D, \tag{2.8}$$

and the boundary integral operator $D' : L^2(\partial D) \to L^2(\partial D)$ by

$$(D'\varphi)(x) := \int_{\partial D} \varphi(y) \frac{\partial}{\partial n(x)} \Phi(x, y)\, ds(y),\ x \in \partial D. \tag{2.9}$$

These integral operators are well defined and compact on $L^2(\partial D)$ [1].

It can be shown (see [1], Chapter 5) that the scattered field u_λ^s of the boundary value problem (2.1) - (2.3) can be represented as a combination of the single and double layer potentials as follows:

$$u_\lambda^s(x) = \int_{\partial D} \left[(\bar{D}D'\varphi)(y) \frac{\partial}{\partial n(y)} \Phi(x,y) - ik\varphi(y)\Phi(x,y)) \right] ds(y), \ x \in \mathbb{R}^2 \backslash \bar{D}. \tag{2.10}$$

Here, the operator $\bar{D}$ denotes the complex conjugate of D, i.e., $\bar{D} := \overline{D\bar{\varphi}}$. The density of the single layer potential is $-ik\varphi$, and for the double layer potential it is $\bar{D}D'\varphi$. By the properties of the operator D (Theorem 3 in Appendix A), the density of the double layer potential $\bar{D}D'\varphi$ is in $H^1(\partial D)$.

By the well-known jump conditions for the single and double layer potentials (Theorem 5 in Appendix A) [1], the scatted field u_λ^s satisfies the impedance boundary condition

$$\frac{\partial u_\lambda^s}{\partial n} + ik\lambda u_\lambda^s = -\frac{\partial u^i}{\partial n} - ik\lambda u^i \text{ on } \partial D$$

in the sense defined in (2.4), if and only if the density $\varphi \in L^2(\partial D)$ solves the boundary integral equation

$$T\bar{D}D'\varphi + \frac{ik}{2}\varphi - ikD'\varphi + \frac{ik\lambda}{2}\bar{D}D'\varphi + ik\lambda D\bar{D}D'\varphi + k^2\lambda S\varphi = -\frac{\partial u^i}{\partial n} - ik\lambda u^i. \tag{2.11}$$

Here, $T : H^1(\partial D) \to L^2(\partial D)$ is a well defined bounded integral operator given by

$$(T\varphi)(x) := \frac{\partial}{\partial n} \int_{\partial D} \varphi(y) \frac{\partial}{\partial n(y)} \Phi(x,y)\, ds(y), \ x \in \partial D. \tag{2.12}$$

For further analysis, it is more convenient to write equation (2.11) as

$$\varphi - A\varphi - \lambda B\varphi = g_1 + \lambda g_2, \tag{2.13}$$

where

$$A = \frac{2i}{k} T\bar{D}D' + 2D', \tag{2.14}$$

$$B = -\bar{D}D' - 2D\bar{D}D' + 2ikS, \tag{2.15}$$

$$g_1 = \frac{2i}{k}\frac{\partial u^i}{\partial n} \quad \text{and} \quad g_2 = -2u^i. \tag{2.16}$$

Since the boundary integral operators $T\bar{D}D'$, D', $\bar{D}D'$, $D\bar{D}D'$ and S are compact on $L^2(\partial D)$ and the impedance $\lambda \in L^\infty(\partial D)$ is bounded, the operators A and λB are compact.

By writing equation (2.13) in the form

$$(I - (A + \lambda B))\varphi = g_1 + \lambda g_2,$$

we recognize it as a Fredholm equation of the second kind. By Theorem 5.19 in [1], the integral equation (2.13) is uniquely solvable in $L^2(\partial D)$ for $\lambda \geq 0$ a.e. on ∂D. Thus, by the Fredholm alternative, $(I - A - \lambda B) : L^2(\partial D) \to L^2(\partial D)$ is boundedly invertible.

Furthermore, from the asymptotic behavior of the Hankels function, the far field pattern u_λ^∞ is given in terms of φ [1] by

$$u_\lambda^\infty(\hat{x}) = (F\varphi)(\hat{x}), \tag{2.17}$$

where $F : L^2(\partial D) \to L^2(S^1)$ given by

$$(F\varphi)(\hat{x}) = \gamma \int_{\partial D} \left[(\bar{D}D'\varphi)(y) \frac{\partial}{\partial n(y)} e^{-ik\hat{x}\cdot y} - ik\varphi(y)e^{-ik\hat{x}\cdot y} \right] ds(y), \hat{x} \in S^1, \tag{2.18}$$

is well-defined and compact. The constant γ is given by $\gamma = \sqrt{2/(\pi k)}e^{-i\pi/4}$.

Now we can formulate our problem as that of choosing $\lambda \in U_{ad}$ such that the functional $J : L^\infty(\partial D) \supset U_{ad} \to \mathbb{R}$

$$J(\lambda) = ||F\varphi - f||^2_{L^2(S^1)} + \varepsilon||\lambda||^2_{L^2(\partial D)}, \quad \lambda \in U_{ad}, \ \varepsilon \in \mathbb{R}, \ \varepsilon > 0$$

is minimized, where φ is related to λ through the integral equation (2.13).

Chapter 3

Existence of an Optimal Solution

3.1 Generalized Weierstrass Theorem

> With his theorem, which states that a *continuous* function of a real variable actually attains its least upper and greatest lower bounds, i.e., necessarily possesses a maximum and a minimum, Weierstrass created a tool which today is indispensable to all mathematicians for more refined analytical or arithmetical investigations.
>
> (Hilbert 1897, *Gesammelte Abh.*, vol.3, p.333)

In this chapter we turn to the question of the existence of at least one optimal solution for our optimization problem. The proof of the existence of a local minimum for J on U_{ad} is based on the generalized version of the Weierstrass Theorem. The initial theorem is called "Hauptlehrsatz"("Principal Theorem") in Weierstrass' lectures of 1861 [9]; it is stated as follows.

Theorem 3.1 If $f : [a, b] \to \mathbb{R}$ is a continuous function, then it is bounded on $[a, b]$ and admits a maximum and a minimum, i.e., there exist $m \in [a, b]$ and $M \in [a, b]$ such that

$$f(m) \leq f(x) \leq f(M)\,\text{for all}\,x \in [a, b].$$

Before we state the generalized version of the Weierstrass Theorem, we preface it with some necessary information from functional analysis.

Let H be a separable Hilbert space. A sequence $\{x_n\}$ in H is said to converge weakly (or converge in the weak topology) to x_0 if for each $y \in H$

$$(x_n, y) \to (x_0, y),$$

where $(\cdot, \cdot)$ is the given inner product on H.

For the weakly convergent sequences the following properties hold:

1) a compact operator maps a weakly convergent sequence into a strongly convergent sequence;

2) if $\{x_n\}$ is bounded in the norm, then it contains a weakly convergent subsequence.

Let X be a separable Banach space and X^* denote its dual. A sequence of elements x_n^* in X^* is said to converge weak-star (or weak*) to $x^* \in X^*$ if

$$x_n^*(x) \to x^*(x)$$

for every $x \in X$. A set $K \subset X^*$ is said to be weak* sequentially compact if every infinite sequence from K contains a weak* convergent subsequence such that its limit point belongs to K. The following properties of weak* convergent sequences hold:

1) if $\{x_n^*\}$ converges in weak* sense to x_0^*, then

$$||x_0^*|| \leq \liminf_{n\to\infty} ||x_n^*||;$$

2) if $\{x_n^*\}$ is bounded in the norm, then it contains a weak* convergent subsequence.

A functional J is called weak* sequentially lower semicontinuous on X^* if for every sequence $\{x_n^*\}$ converging in the weak* sense to an element $x^* \in X^*$ we have

$$\liminf_{n\to\infty} J(x_n^*) \geq J(x^*).$$

The Figure (3.1) illustrates a lower semicontinuous function f on the real line.

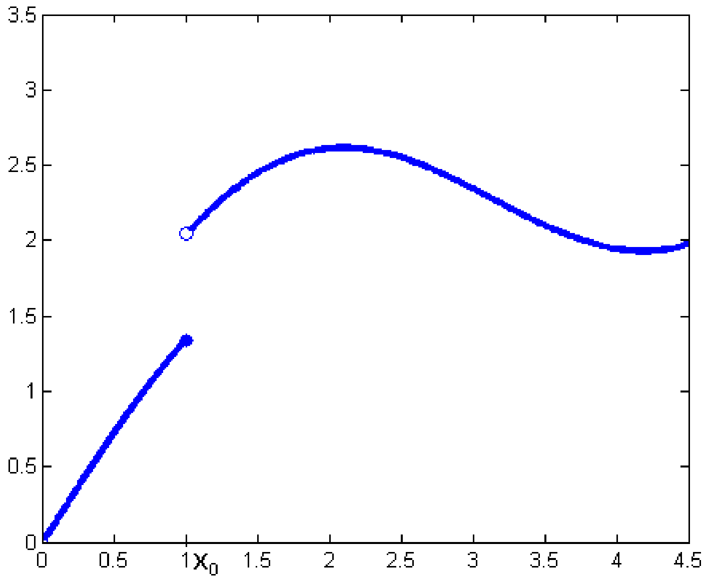

Figure 3.1: A lower semi-continuous function. The solid dot indicates $f(x_0)$.

Now we shall formulate the generalized version of the Weierstrass Theorem for lower semicontinuous functionals [8].

Theorem 3.2 (***Weierstrass***) Let X be a topological space, let $K \subset X$ be sequentially compact, and let $J : X \to [-\infty, \infty]$ be a sequentially lower semicontinuous functional. Then there exists $x_0 \in K$ such that

$$J(x_0) = \min_{x \in K} J(x).$$

For our problem we endow the space $L^\infty(\partial D)$ with weak* topology. Then the existence of the minimum point for J on $U_{ad} \subset L^\infty(\partial D)$ can be shown in four steps [8]:

1. Consider a minimizing sequence $\{\lambda_n\} \subset U_{ad}$, that is, a sequence such that

$$\lim_{n\to\infty} J(\lambda_n) = \inf_{\lambda \in U_{ad}} J.$$

2. Prove that $\{\lambda_n\}$ admits a subsequnce $\{\lambda_{n_k}\}$ that converges with respect to the weak* topology to some point $\lambda_0 \in U_{ad}$, i.e., prove that U_{ad} is weak* sequentially compact.

3. Establish the sequential lower semicontinuity of J on U_{ad} with respect to weak* topology.

4. Conclude that λ_0 is a minimum of J, because

$$\inf_{\lambda \in U_{ad}} J = \lim_{n\to\infty} J(\lambda_n) = \underbrace{\lim_{k\to\infty} J(\lambda_{n_k}) \geq J(\lambda_0)}_{\text{by lower semicontinuity of } J} \geq \inf_{\lambda \in U_{ad}} J.$$

3.2 Proof of Existence of an Optimal Solution

Proposition 3.3 The set of admissible controls U_{ad} is a weak* sequentially compact subset of $L^\infty(\partial D) = L^1(\partial D)^*$.

Proof
The space $L^\infty(\partial D)$ is the dual of $L^1(\partial D)$ which is a separable Banach space. The duality product $x^*(x)$ is given by the extension of the L^2-inner product

$$x^*(x) = \int_{\partial D} x^*(s)x(s)\,ds, \quad x \in L^1(\partial D),\ x^* \in L^\infty(\partial D).$$

Our aim is to show that every infinite sequence in U_{ad} contains a weak* convergent subsequence such that its limit point belongs to U_{ad}.

Let $\{\psi_n\}$ be a sequence in U_{ad}. Since every element of $\{\psi_n\}$ is bounded in $L^\infty(\partial D)$ norm, we can extract a weak* convergent subsequence $\{\psi_{n_k}\} \subset \{\psi_n\}$, such that $\psi_{n_k} \to \psi, \psi \in L^\infty(\partial D)$, as $k \to \infty$. Thus, we have

$$\int_{\partial D} \psi_{n_k}(s)\varphi(s)ds \to \int_{\partial D} \psi(s)\varphi(s)ds \quad \text{for any } \varphi \in L^1(\partial D).$$

In particular, for all non-negative $\varphi \in L^1(\partial D)$ the following estimates hold

$$a_- \int_{\partial D} \varphi(s)\, ds \le \int_{\partial D} \psi_{n_k}(s)\varphi(s)\, ds \le a_+ \int_{\partial D} \varphi(s)\, ds.$$

Thus, the weak* limit point ψ satisfies

$$\int\limits_{\partial D} (\psi(s) - a_+)\varphi(s)\, ds \le 0 \tag{3.1}$$

and

$$\int\limits_{\partial D} (\psi(s) - a_-)\varphi(s)\, ds \ge 0 \tag{3.2}$$

for all non-negative functions φ in $L^1(\partial D)$.

We claim

$$a_- \le \psi \le a_+ \text{ a.e. on } \partial D,$$

which implies that $\psi \in U_{ad}$. We establish the desired result by contradiction.

Assume there exists a set S with Lebesque measure $\mu(S) > 0$ such that

$$(\psi - a_+) > 0 \text{ on } S.$$

Consider the function $\hat{\varphi} := \mathbf{1}_S(\psi - a_+)$ where $\mathbf{1}_S$ is the characteristic function of S. Then

$$\int\limits_{\partial D} |\hat{\varphi}(s)| ds \le \mu(S)(||\psi||_{L^\infty(\partial D)} + a_+) < \infty,$$

i.e., $\hat{\varphi} \in L^1(\partial D)$ and since $\psi - a_+ > 0$ on S by assumption, $\hat{\varphi}$ is non-negative a.e. on ∂D. It follows that

$$\int\limits_{\partial D} (\psi(s) - a_+)\hat{\varphi}(s) ds = \int\limits_{S} |(\psi(s) - a_+)|^2 ds = ||\psi - a_+||^2_{L^2(S)} > 0. \tag{3.3}$$

The last inequality holds since $(\psi - a_+)$ is strictly greater then zero on a set of positive measure by assumption. We arrive at a contradiction since by (3.1) the integral $\int_{\partial D} (\psi(s) - a_+)\varphi(s)\, ds \le 0$ for any non-negative $\varphi \in L^1(\partial D)$.

Analogously we can show that $\psi \ge a_-$ a.e. on ∂D. We contradict the assumption that there exists a set S with Lebesque measure $\mu(S) > 0$ such that $\psi - a_- < 0$ on S. The non-negative $L^1(\partial D)$ integrable function in this case is given by $\hat{\varphi} := -\mathbf{1}_S(\psi - a_-)$.

□

To establish the weak* sequential lower continuity of J we prove the following auxiliary result, which we formulate as a lemma.

Lemma 3.4 Let $\lambda \in L^\infty(\partial D), \lambda \geq 0$ a.e. on ∂D and $\varphi \in L^2(\partial D)$ be the corresponding solution to (2.13). Assume that $\lambda_n \overset{w^*}{\to} \lambda$, then the sequence of corresponding solutions $\{\varphi_{\lambda_n}\}$ to (2.13) converges weakly in $L^2(\partial D)$ to φ, the solution of (2.13) with λ.

Proof
First we show that the mapping $\lambda \to \varphi$ is bounded [4]. Assume that this is not true. Then there exists a sequence $\{\lambda_n\} \subset L^\infty(\partial D)$, $\lambda_n \geq 0$ a.e. on ∂D, such that λ_n are bounded but $||\varphi_{\lambda_n}||_{L^2(\partial D)} \to \infty$, where $\varphi_{\lambda_n} \in L^2(\partial D)$ is a solution of (2.13) with λ set equal to λ_n.

We define $\psi_{\lambda_n} := \varphi_{\lambda_n}/||\varphi_{\lambda_n}||_{L^2(\partial D)}$. Then ψ_{λ_n} satisfies the integral equation

$$\psi_{\lambda_n} - A\psi_{\lambda_n} - \lambda_n B\psi_{\lambda_n} = \frac{1}{||\varphi_{\lambda_n}||_{L^2(\partial D)}}(g_1 + \lambda_n g_2). \tag{3.4}$$

Since the sequences $\{\psi_{\lambda_n}\}$ and $\{\lambda_n\}$ are bounded, we can extract converging subsequences, which we again denote by $\{\lambda_n\}$, $\{\psi_{\lambda_n}\}$, such that $\lambda_n \overset{w^*}{\to} \lambda$ and $\psi_{\lambda_n} \overset{w}{\to} \psi$. Every element λ_n satisfies $\lambda_n \geq 0$ a.e. on ∂D, and thus, as we have shown in the previous theorem, the weak* limit λ also satisfies $\lambda \geq 0$ a.e. on ∂D.

Let $(\cdot,\cdot)_{L^2(\partial D)}$ denote the inner product on $L^2(\partial D)$. The space $L^2(\partial D)$ is a Hilbert space. By Riesz-Fischer Theorem we can express the weak convergence $\psi_{\lambda_n} \overset{w}{\to} \psi$ in terms of the inner product:

$$(\psi_{\lambda_n}, \phi)_{L^2(\partial D)} \to (\psi, \phi)_{L^2(\partial D)}, \ \forall \phi \in L^2(\partial D).$$

Next we show that the weak limit points λ and ψ satisfy

$$(\psi_{\lambda_n} - A\psi_{\lambda_n} - \lambda_n B\psi_{\lambda_n}, \phi)_{L^2(\partial D)} \to (\psi - A\psi - \lambda B\psi, \phi)_{L^2(\partial D)}, \ \forall \phi \in L^2(\partial D).$$

Indeed,

$$(\psi_{\lambda_n}, \phi)_{L^2(\partial D)} \to (\psi, \phi)_{L^2(\partial D)}, \ \forall \phi \in L^2(\partial D)$$

is given and

$$\begin{aligned}|(A\psi_{\lambda_n} - A\psi, \phi)_{L^2(\partial D)}| &= |(A(\psi_{\lambda_n} - \psi), \phi)_{L^2(\partial D)}| \\ &\leq ||A(\psi_{\lambda_n} - \psi)||_{L^2(\partial D)} ||\phi||_{L^2(\partial D)}, \quad \forall \phi \in L^2(\partial D).\end{aligned}$$

By the compactness of A, $||A(\psi_{\lambda_n} - \psi)||_{L^2(\partial D)} \to 0$, as $n \to \infty$.

We write $\lambda_n B\psi_{\lambda_n} - \lambda B\psi$ in the form $\lambda_n B(\psi_{\lambda_n} - \psi) + (\lambda_n - \lambda)B\psi$. Then for every $\phi \in L^2(\partial D)$

$$\begin{aligned}|(\lambda_n B\psi_{\lambda_n} - \lambda B\psi, \phi)_{L^2(\partial D)}| &= \left|(\lambda_n B(\psi_{\lambda_n} - \psi) + (\lambda_n - \lambda)B\psi, \phi)_{L^2(\partial D)}\right| \\ &\leq ||\lambda_n||_{L^2(\partial D)} ||B(\psi_{\lambda_n} - \psi)||_{L^2(\partial D)} + \left|((\lambda_n - \lambda)B\psi, \phi)_{L^2(\partial D)}\right|.\end{aligned}$$

$B(\psi_{\lambda_n} - \psi)$ converges to zero strongly, since B is compact.

The functions $B\psi$ and ϕ are in $L^2(\partial D)$. By Cauchy-Schwarz inequality their product $B\psi\overline{\phi}$ is an $L^1(\partial D)$ integrable function:

$$\int_{\partial D}(B\psi)(x)\overline{\phi(x)}ds(x) \leq ||B\psi||_{L^2(\partial D)}||\phi||_{L^2(\partial D)}.$$

Since λ_n converges weak* to λ

$$|((\lambda_n-\lambda)B\psi,\phi)_{L^2(\partial D)}| = \left|\int_{\partial D}(\lambda_n-\lambda)(x)\underbrace{(B\psi)(x)\overline{\phi(x)}}_{\in L^1(\partial D)}ds(x)\right| \to 0, \text{as} \quad n\to\infty.$$

Taking limits on both sides of (3.4) we obtain

$$(\psi - A\psi - \lambda B\psi, \phi) = 0, \quad \forall \phi \in L^2(\partial D).$$

Let $\phi := \psi - A\psi - \lambda B\psi$, we have $||\psi - A\psi - \lambda B\psi||^2_{L^2(\partial D)} = 0$. Since (2.13) is uniquely solvable for $\lambda \geq 0$ a.e. on ∂D we conclude that $\psi = 0$.

Furthermore, $\lambda_n B\psi_{\lambda_n}$ converges strongly to zero:

$$||\lambda_n B\psi_{\lambda_n}||_{L^2(\partial D)} \leq ||\lambda_n||_{L^\infty(\partial D)}||B\psi_{\lambda_n}||_{L^2(\partial D)} \to 0, \text{ as } n\to\infty,$$

since $||\lambda_n||_{L^\infty(\partial D)}$ is bounded and $||B\psi_{\lambda_n}||_{L^2(\partial D)} \to ||B\psi||_{L^2(\partial D)}(=0)$ by compactness of B.

By continuity of norms, we conclude from (3.4) that ψ_{λ_n} converges strongly to 0. This is a contradiction, since $||\psi_{\lambda_n}||_{L^2(\partial D)} = 1$. Hence the mapping $\lambda \to \varphi$ is bounded.

Now let the sequence $\{\lambda_n\} \subset L^\infty(\partial D)$, $\lambda_n \geq 0$ a.e. on ∂D, be weak* convergent to $\lambda \in L^\infty(\partial D)$ and let φ_{λ_n} be the solution of (2.13) corresponding to each λ_n. We have shown that the sequence $\{\varphi_{\lambda_n}\}$ is bounded and thus it contains a weakly convergent subsequence $\{\varphi_{\lambda_{n_k}}\}$. Following the same steps as above we conclude that the limit point φ, $\varphi_{\lambda_{n_k}} \xrightarrow{w} \varphi$, satisfies

$$(\varphi - A\varphi - \lambda B\varphi, \phi) = (g_1 + \lambda g_2, \phi), \quad \forall \phi \in L^2(\partial D).$$

Again, we set $\phi := \varphi - A\varphi - \lambda B\varphi - g_1 - \lambda g_2$ then for φ, $||\varphi - A\varphi - \lambda B\varphi - g_1 - \lambda g_2||^2_{L^2(\partial D)} = 0$ holds. Thus, φ is a solution of (2.13). Since the integral equation (2.13) is uniquely solvable, there are no other limits of the sequence $\{\varphi_{\lambda_n}\}$, which implies that the whole sequence $\{\varphi_{\lambda_n}\}$ converges weakly in $L^2(\partial D)$ to φ. This completes the proof.

□

Proposition 3.5 The functional $J : L^\infty(\partial D) \supset U_{ad} \to \mathbb{R}$

$$J(\lambda) = ||F\varphi - f||^2_{L^2(S^1)} + \varepsilon||\lambda||^2_{L^2(\partial D)}, \quad \lambda \in U_{ad}, \ \varepsilon \in \mathbb{R}, \ \varepsilon > 0$$

is weak* sequentially lower semicontinuous on U_{ad}.

Proof
We show that for every sequence $\{\lambda_n\} \subset U_{ad}$ converging in the weak* sense to $\lambda \in U_{ad}$ it follows that

$$\liminf_{n\to\infty} J(\lambda_n) \geq J(\lambda).$$

Let $\lambda_n \overset{w^*}{\to} \lambda$. By Lemma (3.4) the corresponding sequence $\{\varphi_{\lambda_n}\}$ converges weakly to φ, the solution of (2.13). Since F is compact $F\varphi_{\lambda_n}$ converges to $F\varphi$ in the norm, i.e., $||F\varphi_{\lambda_n} - F\varphi||_{L^2(S^1)} \to 0$.
Therefore, by the reverse triangle inequality

$$\Big| ||F\varphi_{\lambda_n} - f||_{L^2(S^1)} - ||F\varphi - f||_{L^2(S^1)} \Big| \leq ||(F\varphi_{\lambda_n} - f) - (F\varphi - f)||_{L^2(S^1)} \to 0.$$

Thus, $||F\varphi_{\lambda_n} - f||^2_{L^2(S^1)} \to ||F\varphi - f||^2_{L^2(S^1)}$.

We may write $||\lambda_n||^2_{L^2(\partial D)} - ||\lambda||^2_{L^2(\partial D)}$ as

$$||\lambda_n||^2_{L^2(\partial D)} - ||\lambda||^2_{L^2(\partial D)} = 2(\lambda_n - \lambda, \lambda)_{L^2(\partial D)} + ||\lambda_n - \lambda||^2_{L^2(\partial D)}.$$

Then we have

$$||\lambda_n||^2_{L^2(\partial D)} - ||\lambda||^2_{L^2(\partial D)} = 2(\lambda_n - \lambda, \lambda)_{L^2(\partial D)} + ||\lambda_n - \lambda||^2_{L^2(\partial D)} \geq 2(\lambda_n - \lambda, \lambda)_{L^2(\partial D)}.$$

Since $\lambda \in U_{ad}$ is fixed and

$$\int_{\partial D} |\lambda(x)| ds(x) \leq a_+ \int_{\partial D} 1 \, ds(x) < \infty,$$

i.e., $\lambda \in L^1(\partial D)$, the fact that λ is the weak* limit point of λ_n implies

$$(\lambda_n - \lambda, \lambda)_{L^2(\partial D)} = \int_{\partial D} (\lambda_n - \lambda)(x)\lambda(x) \, ds(x) \to 0, \text{ as } n \to \infty.$$

Now we can estimate $\liminf_{n\to\infty}(||\lambda_n||^2_{L^2(\partial D)} - ||\lambda||^2_{L^2(\partial D)})$:

$$\liminf_{n\to\infty}\big(||\lambda_n||^2_{L^2(\partial D)} - ||\lambda||^2_{L^2(\partial D)}\big) \geq \liminf_{n\to\infty} 2(\lambda_n - \lambda, \lambda)_{L^2(\partial D)} = 0.$$

Hence, $\liminf_{n\to\infty} ||\lambda_n||^2_{L^2(\partial D)} \geq ||\lambda||^2_{L^2(\partial D)}$.

We have shown that

$$\begin{aligned}\liminf_{n\to\infty} J(\lambda_n) &= \liminf_{n\to\infty}(||F\varphi_{\lambda_n} - f||^2_{L^2(S^1)} + \varepsilon||\lambda_n||^2_{L^2(\partial D)}) \\ &\geq ||F\varphi - f||^2_{L^2(S^1)} + \varepsilon||\lambda||^2_{L^2(\partial D)} = J(\lambda).\end{aligned}$$

$\square$

We summarize the main result of this chapter as a theorem.

Theorem 3.6 The functional J attains its minimum on U_{ad}, i.e., there exists $\lambda_0 \in U_{ad}$ such that

$$J(\lambda_0) = \inf_{\lambda \in U_{ad}} J(\lambda).$$

Chapter 4

The Derivative of J

4.1 Introduction

A decisive role in finding the extremal points of a real valued function on a subset of $\mathbb{R}^n$ is played by its partial, or, more generally, by its directional derivatives (provided these derivatives exist). Similarly, in arbitrary vector spaces, we will use the concept of derivative to obtain necessary conditions for local extrema.

The aim of this chapter is to show that the real-valued functional J is differentiable on U_{ad}, the subset of the infinite dimensional vector space $L^\infty(\partial D)$, and to compute its derivative. In the following we introduce the Gateaux and Frechet differentials, the generalizations of directional derivatives and Jacobians on arbitrary vector spaces, and discuss their elementary properties.

Definition [7]. Let X be a vector space, Y a normed space, and J a mapping defined on a domain $U \subset X$ and having range $R \subset Y$. Let $x \in U \subset X$ and h be arbitrary in X. If the limit

$$\delta J(x;h) = \lim_{\alpha \to 0} \frac{1}{\alpha}\big[J(x+\alpha h) - J(x)\big] \tag{4.1}$$

exists, it is called the *Gateaux differential of J at x with increment h.* If the limit (4.1) exists for each $h \in X$, the mapping J is said to be *Gateaux differentiable at x.*

For the case when Y is the real line, the mapping J represents a real-valued functional on X. Thus, the Gateaux differential of J, if it exists, is

$$\delta J(x;h) = \frac{d}{d\alpha} J(x+\alpha h)\bigg|_{\alpha=0},$$

and for each fixed $x \in X$, we have to understand $\delta J(x;h)$ as a functional with respect to the variable $h \in X$.

Next we present the concept of the Frechet derivative which generalizes the notion of the total differential in finite dimensional spaces.

Definition [7]. Let J be a mapping defined on an open domain U in a normed space X and having range in a normed space Y. If for fixed $x \in U$ and each $h \in X$ there exists $J'(x)h \in Y$ which is linear and continuous with respect to h such that

$$\lim_{||h|| \to 0} \frac{||J(x+h) - J(x) - J'(x)h||}{||h||} = 0,$$

then J is said to be *Frechet differentiable at* x and $J'(x)h$ is said to be the *Frechet differential of* J *at* x *with increment* h. In particular, $J'(x) \in \mathcal{L}(X,Y)$.

When J is Frechet differentiable at every point of U, J is said to be *Frechet differentiable in* U and the mapping $J' : U \to \mathcal{L}(X,Y)$ is called *the Frechet derivative* of J on U. If the correspondence $x \to J'(x)$ is continuous at the point x_0, i.e., if given $\varepsilon > 0$ there is $\delta > 0$ such that $||x - x_0|| < \delta$ implies $||J'(x) - J'(x_0)|| < \varepsilon$, we say that J is *continuously Frechet differentiable* at x_0.

In the finite-dimensional case, when $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$, if J is Frechet differentiable in $x \in X$, the linear bounded mapping $J'(x) \in \mathcal{L}(X,Y)$ is given by the Jacobian. In particular, if Y is the real line and $X = \mathbb{R}^n$, the Frechet differential $J'(x)$ is the gradient $\nabla J(x)$.

Gateaux and Frechet differentiable mappings have similar properties as directional derivatives and Jacobians:

(1) The sum- and product rules hold both for Gateaux and Frechet derivatives.

(2) A function, which is Frechet differentiable at a point, is continuous there. As in the finite dimensional case, it does not hold in general for Gateaux differentiable functions.

(3) The chain rule holds for the Frechet derivative.

(4) Every Frechet differentiable mapping $J : X \supset U \to Y$ is also Gateaux differentiable, and the two derivatives coincide, i.e., $\delta J(u;h) = J'(u)h$ for all $h \in X$.

For the computation of the derivative of J we use the integral formulation of the initial problem, i.e., we consider the functional J, given as

$$J(\lambda) = ||F\varphi - f||^2_{L^2(S^1)} + ||\lambda||_{L^2(\partial D)}, \ \lambda \in U_{ad},$$

where F is the far field operator defined by (2.18) and φ is the solution of the boundary integral equation (2.13). Since the method of integral equations is only one of possible approaches to formulate the problem, we expect that the terms occurring in the derivative of J to not only satisfy particular integral equations, but also have their own physical interpretation, that is, they uniquely correspond to certain boundary value problems. Later in this chapter we will see that this is indeed the case.

4.2 A Method to Compute J'

Below we outline the strategy followed in this chapter to compute the derivative of J.

Consider the problem:

$$\begin{aligned}&\text{minimize } \hat{J}(\varphi,\lambda)\\ &\text{subject to } e(\varphi,\lambda)=0,\ \ (\varphi,\lambda)\in L^2(\partial D)\times U_{ad},\end{aligned} \tag{4.2}$$

with $\hat{J}: L^2(\partial D)\times L^\infty(\partial D)\to\mathbb{R}$ and $e: L^2(\partial D)\times L^\infty(\partial D)\to L^2(\partial D)$ given by

$$\begin{aligned}&\hat{J}(\varphi,\lambda)=||F\varphi-f||^2_{L^2(S^1)}+\varepsilon||\lambda||^2_{L^2(\partial D)} \text{ and}\\ &e(\varphi,\lambda)=\varphi-A\varphi-\lambda B\varphi-g_1-\lambda g_2.\end{aligned}$$

As we noted in Chater II, for $\lambda\in L^\infty(\partial D),\ \lambda\geq 0$ a.e. on ∂D, the equation

$$\varphi-A\varphi-\lambda B\varphi=g$$

is uniquely solvable for every $g\in L^2(\partial D)$. Since every element $\lambda\in U_{ad}$ is essentially bounded from below by $a_->0$ the state equation $e(\varphi,\lambda)=0$ possesses a unique solution $\varphi(\lambda)\in L^2(\partial D)$, giving rise to the solution operator $\lambda\to\varphi(\lambda)$. Inserting $\varphi(\lambda)$ in (4.2) we obtain the reduced problem

$$\begin{aligned}&\text{minimize } \hat{J}(\varphi(\lambda),\lambda)\\ &\text{subject to } e(\varphi(\lambda),\lambda)=0,\quad \lambda\in U_{ad},\end{aligned}$$

which is equivalent to our boundary value problem, since $J(\lambda)=\hat{J}(\varphi(\lambda),\lambda)$.

Let $X=(L^\infty(\partial D),||\cdot||_\infty)$ and $U\subset X$ be an open domain in X. Then the Frechet derivative of J on U can be computed as follows:

$$J'(\lambda)h=\hat{J}'(\varphi(\lambda),\lambda)h=\hat{J}_\varphi(\varphi(\lambda),\lambda)\varphi'(\lambda)h+\hat{J}_\lambda(\varphi(\lambda),\lambda)h,\ \forall h\in L^\infty(\partial D).$$

Here we used the chain rule to compute $\hat{J}'$, which requires that both $\hat{J}$ and $\varphi=\varphi(\lambda)$ are Frechet differentiable. Since $\varphi(\lambda)$ satisfies equation $e(\varphi(\lambda),\lambda)=0$ and is not given explicitly, we use the Implicit Function Theorem to show that $\varphi(\lambda)$ is Frechet differentiable on U. The equation for the derivative $\varphi'(\lambda)$ for each λ is then obtained by differentiating $e(\varphi(\lambda),\lambda)$ with respect to λ:

$$e_\varphi(\varphi(\lambda),\lambda)\varphi'(\lambda)+e_\lambda(\varphi(\lambda),\lambda)=0,$$

or

$$\varphi'(\lambda)=-e_\varphi(\varphi(\lambda),\lambda)^{-1}e_\lambda(\varphi(\lambda),\lambda).$$

The derivative $\varphi'(\lambda)$ represents a bounded linear operator which maps $L^\infty(\partial D)$ into $L^2(\partial D)$.

However, we cannot use this strategy immediately to compute the Frechet derivative of J on U_{ad}, because the set U_{ad} is not open. Indeed, let $\lambda_{a_-}:=a_-$ be the constant function on ∂D. Then no open neighborhood defined by an open ball $U(\lambda_{a_-}, \delta) := \{\psi \in L^\infty(\partial D) \mid ||\lambda_{a_-} - \psi||_{L^\infty(\partial D)} < \delta\}$ is entirely contained in U_{ad} for any positive δ (consider $\psi := a_- - \delta/2$, then $\psi \in U(\lambda_{a_-}, \delta)$ but $\psi \notin U_{ad}$). Therefore, $\lambda_{a_-} \in U_{ad}$ is not an interior point of U_{ad} and thus, U_{ad} is not open.

The strategy still works for open domains in $L^\infty(\partial D)$. Therefore, we first compute the Frechet derivative of J on an open set $\tilde{U}_{ad} \supset U_{ad}$, where $\tilde{U}_{ad} := \{\psi \in L^\infty(\partial D) : a_- - \tilde{\varepsilon} < \psi(x) < a_+ + \tilde{\varepsilon}\}$ with $\tilde{\varepsilon} \in \mathbb{R}$ sufficiently small such that $(a_- - \tilde{\varepsilon}) > 0$. The condition on $\tilde{\varepsilon}$ is chosen in this way, since it guarantees that for any element $\lambda \in \tilde{U}_{ad}$, $\lambda \geq 0$ holds a.e. on ∂D, and $I - A - \lambda B$ remains boundedly invertible. By properties of Frechet differentiable mappings, Gateaux and Frechet derivatives coincide on open sets. Finally, we restrict the Gateaux derivative of J to the set U_{ad} and specify the set of admissible increment directions $h \in L^\infty(\partial D)$ for each point $\lambda \in U_{ad}$.

4.3 Gateaux Derivative of J

Proposition 4.1 $e(\varphi^*, \lambda^*)$ is continuously Frechet differentiable for all $(\varphi^*, \lambda^*) \in L^2(\partial D) \times L^\infty(\partial D)$. The Frechet derivative of e is given by

$$e'(\varphi^*, \lambda^*)(\varphi, \lambda) = \varphi - A\varphi - \lambda^* B\varphi - (B\varphi^* + g_2)\lambda \text{ for all } (\varphi, \lambda) \in L^2(\partial D) \times L^\infty(\partial D).$$

Proof
To show that the Frechet derivative of e at $(\varphi^*, \lambda^*) \in L^2(\partial D) \times L^\infty(\partial D)$ is represented by the linear operator $e'(\varphi^*, \lambda^*)$ we need to verify that $e'(\varphi^*, \lambda^*)$ gives the best approximation to e near (φ^*, λ^*), i.e., that

$$||e(\varphi^* + \varphi, \lambda^* + \lambda) - e(\varphi^*, \lambda^*) - e'(\varphi^*, \lambda^*)(\varphi, \lambda)||_{L^2(\partial D)} \text{ is in } o(||(\varphi, \lambda)||).$$

Here, the norm of the vector (φ, λ) belonging to the Cartesian product $L^2(\partial D) \times L^\infty(\partial D)$ is given by $||(\varphi, \lambda)|| = ||\varphi||_{L^2(\partial D)} + ||\lambda||_{L^\infty(\partial D)}$.
Then

$$\begin{aligned}
&||e(\varphi^* + \varphi, \lambda^* + \lambda) - e(\varphi^*, \lambda^*) - e'(\varphi^*, \lambda^*)(\varphi, \lambda)||_{L^2(\partial D)} \\
&= ||(\varphi^* + \varphi) - A(\varphi^* + \varphi) - (\lambda^* + \lambda)B(\varphi^* + \varphi) - g_1 - (\lambda^* + \lambda)g_2 \\
&- \varphi^* + A\varphi^* + \lambda^* B\varphi^* + g_1 + \lambda^* g_2 - \varphi + A\varphi + \lambda^* B\varphi + \lambda(B\varphi^* + g_2)||_{L^2(\partial D)} \\
&= || - \lambda B\varphi||_{L^2(\partial D)} \leq ||\lambda||_{L^\infty(\partial D)} ||B|| ||\varphi||_{L^2(\partial D)} \leq ||\lambda||_{L^\infty(\partial D)} ||B|| (||\varphi||_{L^2(\partial D)} + ||\lambda||_{L^\infty(\partial D)}).
\end{aligned}$$

The linear operator B is compact and therefore bounded, i.e., $||B|| < \infty$. The estimate on the right hand side is $o(||\varphi, \lambda||)$ since

$$\frac{||\lambda||_{L^\infty(\partial D)} ||B|| (||\varphi||_{L^2(\partial D)} + ||\lambda||_{L^\infty(\partial D)})}{||\varphi||_{L^2(\partial D)} + ||\lambda||_{L^\infty(\partial D)}} = ||\lambda||_{L^\infty(\partial D)} ||B|| \to 0 \text{ as } ||(\varphi, \lambda)|| \to 0.$$

We establish the continuous Frechet differentiability of $e(\varphi^*, \lambda^*)$ by showing that the correspondence $(\varphi^*, \lambda^*) \to e'(\varphi^*, \lambda^*)$ is continuous for every point $(\varphi^*, \lambda^*) \in L^2(\partial D) \times L^\infty(\partial D)$, i.e., that for a given $\varepsilon > 0$ there is $\delta > 0$ such that $||(\varphi^*, \lambda^*) - (\hat{\varphi}^*, \hat{\lambda}^*)|| < \delta$

implies $||e'(\varphi^*, \lambda^*) - e'(\hat{\varphi}^*, \hat{\lambda}^*)|| < \varepsilon$. Here the norm $||e'(\varphi^*, \lambda^*)||$ is the operator norm of the linear operator $e'(\varphi^*, \lambda^*)$.

Let $\varepsilon > 0$ be given, then for any $(\varphi^*, \lambda^*), (\hat{\varphi}^*, \hat{\lambda}^*) \in L^2(\partial D) \times L^\infty(\partial D)$ such that

$$||(\varphi^*, \lambda^*) - (\hat{\varphi}^*, \hat{\lambda}^*)|| < \frac{\varepsilon}{||B|| + \varepsilon}$$

and for any $(\varphi, \lambda) \in L^2(\partial D) \times L^\infty(\partial D), (\varphi, \lambda) \neq 0$ we can estimate the ratio

$$\begin{aligned}
&\frac{||g'(\varphi^*, \lambda^*)(\varphi, \lambda) - g'(\hat{\varphi}^*, \hat{\lambda}^*)(\varphi, \lambda)||_{L^2(\partial D)}}{||(\varphi, \lambda)||} \\
&= \frac{||\varphi - A\varphi - \lambda^* B\varphi - (B\varphi^* + g_2)\lambda - \varphi + A\varphi + \hat{\lambda}^* B\varphi + (B\hat{\varphi}^* + g_2)\lambda||_{L^2(\partial D)}}{||(\varphi, \lambda)||} \\
&= \frac{||(\hat{\lambda^*} - \lambda^*)B\varphi + B(\hat{\varphi}^* - \varphi^*)\lambda||}{||(\varphi, \lambda)||_{L^2(\partial D)}} \\
&\leq \frac{||\lambda^* - \hat{\lambda}^*||_{L^\infty}\, ||B||\, ||(\varphi, \lambda)|| + ||B||\, ||\varphi^* - \hat{\varphi}^*||_{L^2(\partial D)} ||(\varphi, \lambda)||}{||(\varphi, \lambda)||} \\
&= \frac{||B||\, ||(\varphi^*, \lambda^*) - (\hat{\varphi}^*, \hat{\lambda}^*)||\, ||(\varphi, \lambda)||}{||(\varphi, \lambda)||} < \frac{\varepsilon}{1 + \varepsilon/||B||}.
\end{aligned}$$

Thus,

$$\sup\left\{ \frac{||g'(\varphi^*, \lambda^*)(\varphi, \lambda) - g'(\hat{\varphi}^*, \hat{\lambda}^*)(\varphi, \lambda)||}{||(\varphi, \lambda)||} : (\varphi, \lambda) \in L^2(\partial D) \times L^\infty(\partial D) \text{ with } (\varphi, \lambda) \neq 0 \right\}$$
$$\leq \frac{\varepsilon}{1 + \varepsilon/||B||}.$$

By definition of an operator norm $||g'(\varphi^*, \lambda^*) - g'(\hat{\varphi}^*, \hat{\lambda}^*)|| < \varepsilon$. This completes the proof. □

Proposition 4.2 $\varphi(\lambda^*)$ is continuously Frechet differentiable on $\tilde{U}_{ad}$. The derivative $\varphi'(\lambda^*) : L^\infty(\partial D) \to L^2(\partial D)$ satisfies

$$\varphi'(\lambda^*) - A\varphi'(\lambda^*) - \lambda^* B\varphi'(\lambda^*) = (g_2 + B\varphi(\lambda^*))I \text{ for all } \lambda^* \in \tilde{U}_{ad}, \tag{4.3}$$

where $\varphi(\lambda^*) \in L^2(\partial D)$ is a solution of (2.13) with λ set equal to λ^*, and $I : L^\infty(\partial D) \to L^\infty(\partial D)$ is the identity operator.

Proof
The key to the proof is the Implicit Function Theorem. We first verify that the assumptions of the Implicit Function Theorem are fulfilled.

1. For each $\lambda^* \in \tilde{U}_{ad}$ the equation $e(\varphi(\lambda^*), \lambda^*) = 0$ possesses a unique solution $\varphi(\lambda^*) \in L^2(\partial D)$. By the choice of $\tilde{\varepsilon}$ we have that $\lambda^* > (a_- - \tilde{\varepsilon}) > 0$ a.e. on ∂D. Therefore, for each $\lambda^* \in \tilde{U}_{ad}$ the operator

$$I - A - \lambda^* B$$

is boundedly invertible on $L^2(\partial D)$ and thus, there is a unique function $\varphi(\lambda^*) \in L^2(\partial D)$ satisfying

$$\varphi(\lambda^*) - A\varphi(\lambda^*) - \lambda^* B\varphi(\lambda^*) = g_1 + \lambda^* g_2,$$

or $e(\varphi(\lambda^*), \lambda^*) = 0$.

2. The function e is continuously Frechet differentiable on $L^2(\partial D) \times L^\infty(\partial D)$ and the operator $e_\varphi(\varphi(\lambda^*), \lambda^*) : L^2(\partial D) \to L^2(\partial D)$ is boundedly invertible for each $\lambda^* \in \tilde{U}_{ad}$.

The continuous differentiability of e on $L^2(\partial D) \times L^\infty(\partial D)$ was established in the previous proposition. The partial derivatives of $e(\varphi^*, \lambda^*)$ at $(\varphi^*, \lambda^*) \in L^2(\partial D) \times L^\infty(\partial D)$ with respect to φ and λ are given by

$$e_\varphi(\varphi^*, \lambda^*) = I - A - \lambda^* B, \qquad e_\lambda(\varphi^*, \lambda^*) = -(B\varphi^* + g_2)I.$$

The partial derivative $e_\varphi(\varphi^*, \lambda^*)$ at (φ^*, λ^*) represents an operator mapping $L^2(\partial D)$ to $L^2(\partial D)$; $e_\lambda(\varphi^*, \lambda^*)$ is an operator which maps $L^\infty(\partial D)$ to $L^2(\partial D)$. As noted above, $e_\varphi(\varphi^*, \lambda^*) = I - A - \lambda^* B$ is boundedly invertible for all $\lambda^* \in \tilde{U}_{ad}$.

The assumptions of the Implicit Function Theorem are satisfied. Therefore, $\varphi(\lambda^*)$ is continuously differentiable for all $\lambda^* \in \tilde{U}_{ad}$. The derivative $\varphi'(\lambda^*) : L^\infty(\partial D) \to L^2(\partial D)$ satisfies

$$\varphi'(\lambda^*) = -e_\varphi(\varphi(\lambda^*), \lambda)^{-1} e_\lambda(\varphi(\lambda^*), \lambda^*)$$

or

$$\varphi'(\lambda^*) - A\varphi'(\lambda^*) - \lambda^* B\varphi'(\lambda^*) = g_2 + B\varphi(\lambda^*).$$

In particular, for a vector $\lambda \in L^\infty(\partial D)$ the function $\varphi'(\lambda^*)\lambda \in L^2(\partial D)$ satisfies the equation

$$\varphi'(\lambda^*)\lambda - A\varphi'(\lambda^*)\lambda - \lambda^* B\varphi'(\lambda^*)\lambda = g_2\lambda + B\varphi(\lambda^*)\lambda.$$

$\square$

Proposition 4.3 $\hat{J}$ is Frechet differentiable on $L^2(\partial D) \times L^\infty(\partial D)$. The Frechet derivative of $\hat{J}$ is given by

$$\hat{J}'(\varphi^*, \lambda^*)(\varphi, \lambda) = 2\mathrm{Re} \int_{S^1} \overline{(F\varphi^* - f)} F\varphi \, d\hat{\theta} + 2\varepsilon \int_{\partial D} \lambda\lambda^* \, ds,$$

for all $(\varphi, \lambda) \in L^2(\partial D) \times L^\infty(\partial D)$.

Proof
Again, as in the proof of Proposition 4.1, we verify that

$$|\hat{J}(\varphi^* + \varphi, \lambda^* + \lambda) - \hat{J}(\varphi^*, \lambda^*) - \hat{J}'(\varphi^*, \lambda^*)(\varphi, \lambda)| \text{ is in } o(||(\varphi, \lambda)||):$$

$$
\begin{aligned}
&|\hat{J}(\varphi^* + \varphi, \lambda^* + \lambda) - \hat{J}(\varphi^*, \lambda^*) - \hat{J}'(\varphi^*, \lambda^*)(\varphi, \lambda)| \\
&= \Big| ||F(\varphi^* + \varphi) - f||^2_{L^2(S^1)} + \varepsilon||\lambda^* + \lambda||^2_{L^2(\partial D)} - ||F\varphi^* - f||^2_{L^2(S^1)} - \varepsilon||\lambda^*||^2_{L^2(\partial D)} \\
&- 2\mathrm{Re}\int_{S^1} \overline{(F\varphi^* - f)} F\varphi \, dx - 2\varepsilon \int_{\partial D} \lambda\lambda^* \, ds \Big| \\
&= \Big| ||F\varphi^* - f||^2_{L^2(S^1)} + 2\mathrm{Re}\int_{S^1} \overline{(F\varphi^* - f)} F\varphi \, dx + ||F\varphi||^2_{L^2(S^1)} + \varepsilon||\lambda^*||^2_{L^2(\partial D)} + 2\varepsilon \int_{\partial D} \lambda^*\lambda \, ds \\
&+ \varepsilon||\lambda||^2_{L^2(\partial D)} - ||F\varphi^* - f||^2_{L^2(S^1)} - \varepsilon||\lambda^*||^2_{L^2(\partial D)} - 2\mathrm{Re}\int_{S^1} \overline{(F\varphi^* - f)} F\varphi \, dx - 2\varepsilon \int_{\partial D} \lambda\lambda^* \, ds \Big| \\
&= ||F\varphi||^2_{L^2(S^1)} + \varepsilon||\lambda||^2_{L^2(\partial D)} \leq \max\{||F||^2, \varepsilon\}(||\varphi||^2_{L^2(\partial D)} + ||\lambda||^2_{L^\infty(\partial D)}) \\
&\leq \max\{||F||^2, \varepsilon\}(||\varphi||_{L^2(\partial D)} + ||\lambda||_{L^\infty(\partial D)})^2 = \max\{||F||^2, \varepsilon\}||(\varphi, \lambda)||^2.
\end{aligned}
$$

The estimate is $o(||(\varphi, \lambda)||)$ since

$$
\frac{\max\{||F||^2, \varepsilon\}||(\varphi, \lambda)||^2}{||(\varphi, \lambda)||} = \max\{||F||^2, \varepsilon\}||(\varphi, \lambda)|| \to 0 \text{ as } ||(\varphi, \lambda)|| \to 0.
$$

□

Theorem 4.4 The functional J is Frechet differentiable on $\tilde{U}_{ad}$. The Frechet derivative of J is given by

$$
J'(\lambda^*)h = 2\mathrm{Re}\int_{S^1} \overline{(F\varphi(\lambda^*) - f)(\hat{\theta})} F(\varphi'(\lambda^*)h)(\hat{\theta}) d\hat{\theta} + 2\varepsilon \int_{\partial D} \lambda^*(x)h(x) ds(x),
$$

for all $h \in L^\infty(\partial D)$.

Proof

The proof is obtained by applying the chain rule and Propositions 4.1-4.3. □

We formulate the main result of this section as a corollary of the above theorem.

Corollary 4.5 The functional $J(\lambda^*)$ is Gateaux differentiable for all $\lambda^* \in U_{ad}$ in the direction $h = \lambda - \lambda^*, \lambda \in U_{ad}$. The Gateaux derivative of J on U_{ad} is given by

$$
\delta J(\lambda^*; h) = 2\mathrm{Re}\int_{S^1} \overline{\big(F\varphi(\lambda^*) - f\big)(\hat{\theta})} F(\varphi'(\lambda^*)h)(\hat{\theta}) d\hat{\theta} + 2\varepsilon \int_{\partial D} h\lambda^* \, ds.
$$

Proof

We show that for any $\lambda^* \in U_{ad}$ vector $h \in L^\infty(\partial D)$ is an admissible direction at λ^* if and only if there is a vector $\lambda \in U_{ad}$ such that $h = \lambda - \lambda^*$.

If $h \in L^\infty(\partial D)$ is an admissible direction at $\lambda^* \in U_{ad}$, then by definition, there exists $\alpha_1 > 0$ such that $\lambda^* + \alpha h \in U_{ad}$ for $0 < \alpha < \alpha_1$. Thus, for each fixed α there is $\lambda \in U_{ad}$ such that $\lambda^* + \alpha h = \lambda$. Hence, $\alpha h = \lambda - \lambda^*$ and so h is co-directed with $\lambda - \lambda^*$.

The other implication, that for any $\lambda \in U_{ad}$ the vector $\lambda - \lambda^*$ is an admissible direction at λ^*, follows from the observation U_{ad} is convex.

Indeed, we can show that for two elements x and y of U_{ad}, the sum $\alpha x + (1-\alpha)y$ is in U_{ad} for $\alpha \in [0,1]$. For any $\alpha \in [0,1]$ the function $\alpha x + (1-\alpha)y \in L^\infty(\partial D)$ is bounded by a_+ a.e. on ∂D:

$$\alpha x + (1-\alpha)y \leq \alpha a_+ + (1-\alpha)a_+ = a_+ \text{ a. e. on } \partial D.$$

It is also bounded from below by a_- a.e. on ∂D, since

$$\alpha x + (1-\varepsilon)y \geq \alpha a_- + (1-\alpha)a_- = a_- \text{ a. e. on } \partial D.$$

Thus, $\alpha x + (1-\alpha)y \in U_{ad}$ and we conclude that U_{ad} is convex.

For any elements $\lambda^*, \lambda \in U_{ad}$, the sum $\alpha\lambda + (1-\alpha)\lambda^*$, or equivalently $\lambda^* + \alpha(\lambda - \lambda^*)$, belongs to U_{ad} for $0 \leq \alpha \leq 1$. By definition, the vector $h = \lambda - \lambda^*$ is an admissible direction at λ^*.

□

4.4 Interpretation of the Derivative

We now show that the terms $F\varphi(\lambda^*)$ and $F\varphi'(\lambda^*)h$ occuring in the derivative of J have their own physical meaning: they correspond to two particular scattering problems.

The function $\varphi(\lambda^*) \in L^2(\partial D)$ is the solution of the boundary integral equation

$$\varphi(\lambda^*) - A\varphi(\lambda^*) - \lambda B\varphi(\lambda^*) = g_1 + \lambda^* g_2,$$

or expanding the operators A and B:

$$\begin{aligned} T\bar{D}D'\varphi(\lambda^*) + \frac{ik}{2}\varphi(\lambda^*) - ikD'\varphi(\lambda^*) + \frac{ik\lambda^*}{2}\bar{D}D'\varphi(\lambda^*) \\ + ik\lambda^* D\bar{D}D'\varphi(\lambda^*) + k^2\lambda^* S\varphi(\lambda^*) = -\frac{\partial u^i}{\partial n} - ik\lambda^* u^i. \end{aligned} \tag{4.4}$$

We recall that the equation (4.4) uniquely corresponds to the scattering problem

$$\begin{aligned} &\Delta u^s_{\lambda^*} + k^2 u^s_{\lambda^*} = 0 \text{ in } \mathbb{R}^2 \backslash \bar{D} \\ &\frac{\partial u^s_{\lambda^*}}{\partial n} + ik\lambda^* u^s_{\lambda^*} = -\frac{\partial u^i}{\partial n} - ik\lambda^* u^i \text{ on } \partial D \\ &\lim_{r\to\infty} \sqrt{r}\left[\frac{\partial}{\partial r}u^s_{\lambda^*} - iku^s_{\lambda^*}\right] = 0 \text{ uniformly, } r = |x|. \end{aligned} \tag{4.5}$$

The far field operator F maps the density $\varphi(\lambda^*)$ to the far field of $u^s_{\lambda^*}$. Thus, $F\varphi(\lambda^*)$ is the far field $u^\infty_{\lambda^*}$ of the scattered field $u^s_{\lambda^*}$.

By Lemma 4.2 $\varphi'(\lambda^*)h \in L^2(\partial D)$ satisfies

$$\varphi'(\lambda^*)h - A\varphi'(\lambda^*)h - \lambda^* B\varphi'(\lambda^*)h = \big(g_2 + B\varphi(\lambda^*)\big)h,$$

or

$$T\bar{D}D'\varphi'(\lambda^*)h + \frac{ik}{2}\varphi'(\lambda^*)h - ikD'\varphi'(\lambda^*)h + \frac{ik\lambda^*}{2}\bar{D}D'\varphi'(\lambda^*)h + ik\lambda^* D\bar{D}D'\varphi'(\lambda^*)h$$
$$+ k^2\lambda^* S\varphi'(\lambda^*)h = -ikh\Big(u^i + \Big[\frac{1}{2}\bar{D}D'\varphi(\lambda^*) + D\bar{D}D'\varphi(\lambda^*) - ikS\varphi(\lambda^*)\Big]\Big). \tag{4.6}$$

The right hand side of the equation (4.6) can be simplified. We recall that the scattered field $u^s_{\lambda^*}$ can be represented as the combination of the single and double layer potentials

$$u^s_{\lambda^*}(x) = \int_{\partial D}\Big[(\bar{D}D'\varphi(\lambda^*))(y)\frac{\partial}{\partial n(y)}\Phi(x,y) - ik\varphi(\lambda^*)(y)\Phi(x,y)\Big]\,ds(y), x \in \bar{D}^c.$$

By the jump conditions, the extension of $u^s_{\lambda^*}$ to the boundary ∂D is given by

$$u^s_{\lambda^*}|_{\partial D} = \frac{1}{2}\bar{D}D'\varphi(\lambda^*) + D\bar{D}D'\varphi(\lambda^*) - ikS\varphi(\lambda^*). \tag{4.7}$$

The term $(1/2)\bar{D}D'\varphi(\lambda^*) + D\bar{D}D'\varphi(\lambda^*) - ikS\varphi(\lambda^*)$ appears on the right hand side of (4.6). Therefore, we may write the boundary integral equation (4.6) as

$$T\bar{D}D'\varphi'(\lambda^*)h + \frac{ik}{2}\varphi'(\lambda^*)h - ikD'\varphi'(\lambda^*)h + \frac{ik\lambda^*}{2}\bar{D}D'\varphi'(\lambda^*)h + ik\lambda^* D\bar{D}D'\varphi'(\lambda^*)h$$
$$+ k^2\lambda^* S\varphi'(\lambda^*)h = -ikh\Big(u^i + u^s_{\lambda^*}\Big). \tag{4.8}$$

Looking at the equation (4.8), we realize that it is of the same form as (2.11), and thus, uniquely corresponds [1] to the scattering problem

$$\begin{aligned}
&\Delta u^s_{\lambda^*,h} + k^2 u^s_{\lambda^*,h} = 0 \text{ in } \mathbb{R}^2\backslash\bar{D}\\
&\frac{\partial u^s_{\lambda^*,h}}{\partial n} + ik\lambda^* u^s_{\lambda^*,h} = -ikh(u^i + u^s_{\lambda^*}) \text{ on } \partial D\\
&\lim_{r\to\infty}\sqrt{r}\Big[\frac{\partial}{\partial r}u^s_{\lambda^*,h} - iku^s_{\lambda^*,h}\Big] = 0 \text{ uniformly, } r = |x|.
\end{aligned}$$

The scattered field $u^s_{\lambda^*,h}$ can be written in terms of the single and double layer potentials

$$u^s_{\lambda^*,h}(x) = \int_{\partial D}\Big[\big(\bar{D}D'\varphi'(\lambda^*)h\big)(y)\frac{\partial}{\partial n(y)}\Phi(x,y) - ik\big(\varphi'(\lambda^*)h\big)(y)\Phi(x,y)\big)\Big]\,ds(y).$$

Similarly, $F(\varphi'(\lambda^*)h) \in L^2(S^1)$ is the far field pattern $u^\infty_{\lambda^*,h}$ of $u^s_{\lambda^*,h}$.

We close this chapter with the following theorem.

Theorem 4.6 The functional $J(\lambda^*)$ is Gateaux differentiable for all $\lambda^* \in U_{ad}$ in the direction $h = \lambda - \lambda^*, \lambda \in U_{ad}$. The Gateaux derivative of J on U_{ad} is given by

$$\delta J(\lambda^*; h) = 2\text{Re}\int_{S^1} \overline{\big(u^\infty_{\lambda^*} - f\big)(\hat{\theta})}\, u^\infty_{\lambda^*,h}(\hat{\theta})d\hat{\theta} + 2\varepsilon\int_{\partial D} h\lambda^*\, ds$$

with $u^\infty_{\lambda^*}$ and $u^s_{\lambda^*,h}$ are the far fields of the scattered waves $u^s_{\lambda^*} \in C^2(\mathbb{R}^2\backslash\bar{D})$ and $u^s_{\lambda^*} \in C^2(\mathbb{R}^2\backslash\bar{D})$ satisfying

$$\begin{aligned}
&\Delta u^s_{\lambda^*} + k^2 u^s_{\lambda^*} = 0 \text{ in } \mathbb{R}^2\backslash\bar{D}, && \Delta u^s_{\lambda^*,h} + k^2 u^s_{\lambda^*,h} = 0 \text{ in } \mathbb{R}^2\backslash\bar{D},\\
&\frac{\partial u^s_{\lambda^*}}{\partial n} + ik\lambda^* u^s_{\lambda^*} = -\frac{\partial u^i}{\partial n} - ik\lambda^* u^i \text{ on } \partial D, \quad \text{and} && \frac{\partial u^s_{\lambda^*,h}}{\partial n} + ik\lambda^* u^s_{\lambda^*,h} = -ikh(u^i + u^s_{\lambda^*}) \text{ on } \partial D,\\
&\lim_{r\to\infty} \sqrt{r}\left[\frac{\partial}{\partial r} u^s_{\lambda^*} - iku^s_{\lambda^*}\right] = 0, && \lim_{r\to\infty} \sqrt{r}\left[\frac{\partial}{\partial r} u^s_{\lambda^*,h} - iku^s_{\lambda^*,h}\right] = 0,\\
&\text{uniformly, } |x| = r. && \text{uniformly, } |x| = r,
\end{aligned}$$

respectively.

Chapter 5

Necessary Optimality Conditions

5.1 Introduction

In the finite dimensional case we used the differential calculus to examine the local behavior of a function and to find its local extrema. So, for a real valued continuous function f of one variable that is differentiable in the interior of the interval $[a, b] \subset \mathbb{R}$, the necessary condition for a local extremum x_0 in the interior of $[a, b]$ is that $f'(x_0) = 0$. For the endpoints a and b one-sided derivative may be sufficient to determine whether there is an extremum: if $f'(a) > 0$, i.e., the right-hand derivative is positive, then there is a local minimum at a, if $f'(a) < 0$ there is a local maximum. For the boundary point b, if the left-hand derivative $f'(b)$ is positive, then $f(b)$ is a local maximum, if $f'(b) < 0$ then $f(b)$ is a local minimum.

The technique developed in the one dimensional case for real valued functions can be extended in a natural way to obtain necessary conditions for local extrema of real-valued functionals on infinite dimensional linear spaces.

Theorem 5.1 [7] Let the real-valued functional f have a Gateaux differential on a vector space X. A necessary condition for f to have an extremum at $x_0 \in X$ is that $\delta f(x_0; h) = 0$ for all $h \in X$.

Proof
For every $h \in X$, the function $f(x_0+\alpha h)$ of the real variable α must achieve an extremum at $\alpha = 0$. Thus, the results for real valued function of one variable apply and

$$\frac{d}{d\alpha} f(x_0 + \alpha h)\bigg|_{\alpha=0} = 0.$$

$\square$

To state the necessary conditions for our problem we will use the following generalization of the Theorem 5.1.

Theorem 5.2 [7] Let f be a real-valued functional defined on a vector space X. Suppose that x_0 minimizes f on the convex set $\Omega \subset X$ and that f is Gateaux differentiable at x_0.

Then

$$\delta f(x_0; x - x_0) \geq 0$$

for all $x \in \Omega$.

Proof
Since Ω is convex, $(1-\alpha)x_0 + \alpha x \in \Omega$ or, after rearranging the terms, $x_0 + \alpha(x - x_0) \in \Omega$ for $0 \leq \alpha \leq 1$. Hence, the necessary condition for the minimum at x_0 is that

$$\frac{d}{d\alpha} f(x_0 + \alpha(x - x_0))\Big|_{\alpha=0} \geq 0.$$

□

5.2 The Optimality System

In Proposition 4.5 we have shown that the set U_{ad} is convex. Therefore, for our problem we can apply Theorem 5.1. A necessary optimality condition for J on U_{ad} has the form

$$\delta J(\lambda^*; \lambda - \lambda^*) \geq 0 \text{ for all } \lambda \in U_{ad}.$$

By writing it directly we have

$$\delta J(\lambda^*; \lambda - \lambda^*) = 2\mathrm{Re} \int_{S^1} \overline{(F\varphi(\lambda^*) - f)(\hat{\theta})} F(\varphi(\lambda^*)(\lambda - \lambda^*))(\hat{\theta})\, d\hat{\theta}$$
$$+2\varepsilon \int_{\partial D} \lambda^*(x)(\lambda - \lambda^*)(x)\, ds(x) = 2\mathrm{Re} \int\limits_{\partial D} \overline{\big(\varphi'(\lambda^*)^* F^*(F\varphi(\lambda^*) - f\big)(x)}(\lambda - \lambda^*)(x) ds(x)$$
$$+ 2\varepsilon \int\limits_{\partial D} (\lambda - \lambda^*)(x)\lambda^*(x)\, ds(x) = 2\mathrm{Re} \int\limits_{\partial D} \Big[\overline{\big(\varphi'(\lambda^*)^* F^*(F\varphi(\lambda^*) - f\big)} + \varepsilon\lambda^*\Big](\lambda - \lambda^*)\, ds \geq 0.$$

The necessary optimality condition stated in this form is of limited use, since it requires the use of a particular method, the method of integral equations, and particular operators F and $\varphi'(\lambda^*)$ to verify, whether a point λ^* is a local minimizer for J.

The work of A.Kirsch [3] motivates us to write the Gateaux differential $\delta J(\lambda^*; h)$

$$\delta J(\lambda^*; h) = 2\mathrm{Re} \int\limits_{S^1} \overline{\big(u^\infty_{\lambda^*} - f\big)(\hat{\theta})} u^\infty_{\lambda^*,h}(\hat{\theta}) d\hat{\theta} + 2\varepsilon \int\limits_{\partial D} h\lambda^*\, ds$$

in the form

$$\delta J(\lambda^*; h) = 2\mathrm{Re} \int\limits_{\partial D} \big[w(x)u(x) + \varepsilon\lambda^*(x)\big] h(x)\, ds(x),$$

where w is the solution of the adjoint scattering problem

$$\begin{aligned}
&w = w^i + w^s,\\
&\Delta w + k^2 w = 0 \text{ in } \mathbb{R}^2 \backslash \bar{D},\\
&\frac{\partial w}{\partial n} + ik\lambda^* w = 0 \text{ on } \partial D,\\
&\lim_{r\to\infty} \sqrt{r}\left[\frac{\partial}{\partial r} w^s_{\lambda^*} - ikw^s_{\lambda^*}\right] = 0 \text{ uniformly, } r = |x|.
\end{aligned} \tag{5.1}$$

The incident field w^i has the form

$$w^i(x) = \gamma_1 \int\limits_{S^1} \overline{\big(u^\infty_{\lambda^*} - f\big)(\hat{x})} e^{-ik\hat{x}\cdot x} ds(\hat{x}) \text{ for } x \in \mathbb{R}^2, \tag{5.2}$$

with constant $\gamma_1 \in \mathbb{C}$ to be determined.

Since the adjoint problem has been formulated, our goal is to verify that the adjoint state w satisfies the equality

$$\int\limits_{S^1} \overline{(u^\infty_{\lambda^*} - f)(\hat{\theta})} u^\infty_{\lambda^*,h}(\hat{\theta})\, d\hat{\theta} = \int\limits_{\partial D} w(x)u(x)h(x)ds(x),$$

and to find the constant γ_1.

Lemma 5.3

$$\int\limits_{S^1} \overline{(u^\infty_{\lambda^*} - f)} u^\infty_{\lambda^*,h}\, d\hat{\theta} = \int\limits_{\partial D} wuh\, dx, \tag{5.3}$$

with $u = u^i + u^s_{\lambda^*}$ and $w = w^i + w^s$, where

$$w^i(x) = 2ik\gamma \int\limits_{S^1} \overline{\big(u^\infty_{\lambda^*} - f\big)(\hat{x})} e^{-ik\hat{x}\cdot x} ds(\hat{x}) \text{ for } x \in \mathbb{R}^2,\ \gamma = \sqrt{\frac{2}{\pi k}} e^{-i\pi/4}.$$

Proof
Throughout the proof we will refer to the far field patterns $u^\infty_{\lambda^*}$ and $u^\infty_{\lambda^*,h}$ written in the form $F\varphi(\lambda^*)$ and $F\varphi'(\lambda^*)h$, where $\varphi(\lambda^*) \in L^2(\partial D)$ and $\varphi'(\lambda^*)h \in L^2(\partial D)$ are the solutions of the boundary integral equations (4.4) and (4.6), respectively. Therefore, we first write (5.3) as

$$\int\limits_{S^1} \overline{(F\varphi(\lambda^*) - f)} F\varphi'(\lambda^*)h\, d\hat{\theta} = \int\limits_{\partial D} wuh\, dx.$$

We represent the scattered field w^s as the combination of the single and double layer potentials [1]

$$w^s(x) = \int_{\partial D} \left[(\bar{D}D'\varphi_w)(y) \frac{\partial}{\partial n(y)} \Phi(x,y) - ik\varphi_w(y)\Phi(x,y))\right] ds(y),\ x \in \mathbb{R}^2 \backslash \bar{D}.$$

with density $\varphi_w \in L^2(\partial D)$.

The densities $\varphi'(\lambda^*)h \in L^2(\partial D)$ and $\varphi_w \in L^2(\partial D)$ satisfy

$$T\bar{D}D'\varphi'(\lambda^*)h + \frac{ik}{2}\varphi'(\lambda^*)h - ikD'\varphi'(\lambda^*)h + \frac{ik\lambda^*}{2}\bar{D}D'\varphi'(\lambda^*)h + ik\lambda^* D\bar{D}D'\varphi'(\lambda^*)h$$
$$+ k^2\lambda^* S\varphi'(\lambda^*)h = -ikh\Big(u^i + u^s_{\lambda^*}\Big), \tag{5.4}$$

and

$$T\bar{D}D'\varphi_w + \frac{ik}{2}\varphi_w - ikD'\varphi_w + \frac{ik\lambda^*}{2}\bar{D}D'\varphi_w + ik\lambda^* D\bar{D}D'\varphi_w + k^2\lambda^* S\varphi_w = -\frac{\partial w^i}{\partial n} - ik\lambda^* w^i \tag{5.5}$$

respectively.

By jump relations (Theorem 5 in Appendix A), extending the fields w^s and $u^s_{\lambda^*,h}$ to the boundary we obtain that

$$w^s\big|_{\partial D} = \frac{1}{2}\bar{D}D'\varphi_w + D\bar{D}D'\varphi_w - ikS\varphi_w$$

and

$$u^s\big|_{\partial D} = \frac{1}{2}\bar{D}D'\varphi'(\lambda^*)h + D\bar{D}D'\varphi'(\lambda^*)h - ikS\varphi'(\lambda^*)h.$$

The idea of the proof is simple. We

(1) multiply (5.4) by $(w^i + w^s)$ and integrate over ∂D,

(2) multiply (5.5) by $u^s_{\lambda^*,h}$ and integrate over ∂D,

(3) subtract the result obtained in (2) from the result of (1).

The last step yields the desired equality (5.3).

(1) To make the computations less cumbersome we write (5.4) in the form

$$T\bar{D}D'\varphi'(\lambda^*)h + \frac{ik}{2}\varphi'(\lambda^*)h - ikD'\varphi'(\lambda^*)h + ik\lambda^* u^s_{\lambda^*,h} = -ikh\Big(u^i + u^s_{\lambda^*}\Big).$$

Multiplying (5.4) by $(w^i + w^s)$ and integrating over ∂D yields:

$$\int_{\partial D} \frac{\partial}{\partial n(x)}\left(\int_{\partial D} \bar{D}D'\varphi'(\lambda^*)h(y)\frac{\partial}{\partial n(y)}\Phi(x,y)\,ds(y)\right)(w^i + w^s)(x)ds(x)$$
$$+ \int_{\partial D} \frac{ik}{2}\varphi'(\lambda^*)h(x)\,(w^i + w^s)(x)\,ds(x) - ik\int_{\partial D} D'\varphi'(\lambda^*)h(x)\,(w^i + w^s)(x)\,ds(x)$$
$$+ ik\int_{\partial D} \lambda^*(x)u^s_{\lambda^*,h}\,(w^i + w^s)(x)\,ds(x) = -ik\int_{\partial D} h(x)u(x)w(x)\,ds(x).$$

(2) Multiplying (5.5) by $u^s_{\lambda^*,h}$ and integrating over ∂D we obtain:

$$
\begin{aligned}
&\int_{\partial D} \frac{\partial}{\partial n(x)} \left(\int_{\partial D} (\bar{D}D'\varphi_w)(y) \frac{\partial}{\partial n(y)} \Phi(x,y)\, ds(y) \right) u^s_{\lambda^*,h}(x) ds(x) \\
&+ \int_{\partial D} \frac{ik}{2} \varphi_w(x)\, u^s_{\lambda^*,h}(x)\, ds(x) - ik \int_{\partial D} D'\varphi_w(x)\, u^s_{\lambda^*,h}(x)\, ds(x) \\
&+ ik \int_{\partial D} \lambda^*(x)(w^i + w^s)(x)\, u^s_{\lambda^*,h}\, ds(x) + \int_{\partial D} \frac{\partial}{\partial n(x)} w^i(x)\, u^s_{\lambda^*,h}(x)\, ds(x) = 0.
\end{aligned}
$$

(3) Now we subtract the equality obtained in (2) from that obtained in (1):

$$
\begin{aligned}
&\int_{\partial D} \frac{\partial}{\partial n(x)} \left(\int_{\partial D} \bar{D}D'\varphi'(\lambda^*)h(y) \frac{\partial}{\partial n(y)} \Phi(x,y)\, ds(y) \right) (w^i + w^s)(x) ds(x) \\
&+ \int_{\partial D} \frac{ik}{2} \varphi'(\lambda^*)h(x)\, (w^i + w^s)(x)\, ds(x) - ik \int_{\partial D} D'\varphi'(\lambda^*)h(x)\, (w^i + w^s)(x)\, ds(x) \\
&+ ik \cancel{\int_{\partial D} \lambda^*(x) u^s_{\lambda^*,h}\, (w^i + w^s)(x)\, ds(x)} \\
&- \int_{\partial D} \frac{\partial}{\partial n(x)} \left(\int_{\partial D} (\bar{D}D'\varphi_w)(y) \frac{\partial}{\partial n(y)} \Phi(x,y)\, ds(y) \right) u^s_{\lambda^*,h}(x) ds(x) \\
&- \int_{\partial D} \frac{ik}{2} \varphi_w(x)\, u^s_{\lambda^*,h}(x)\, ds(x) + ik \int_{\partial D} D'\varphi_w(x)\, u^s_{\lambda^*,h}(x)\, ds(x) \\
&- ik \cancel{\int_{\partial D} \lambda^*(x) u^s_{\lambda^*,h}\, (w^i + w^s)(x)\, ds(x)} - \int_{\partial D} \frac{\partial}{\partial n(x)} w^i(x)\, u^s_{\lambda^*,h}(x)\, ds(x) \\
&= -ik \int_{\partial D} h(x) u(x) w(x)\, ds(x).
\end{aligned}
\tag{5.6}
$$

It is hard to see it at the first sight, but there are more terms which cancel out. In order to show it we introduce the following rules:

1. For $\varphi_1, \varphi_2 \in H^1(\partial D)$

$$
\int\limits_{\partial D} \frac{\partial}{\partial n(x)} D\varphi_1(x)\varphi_2(x)\, ds(x) = \int\limits_{\partial D} \frac{\partial}{\partial n(x)} D\varphi_2(x)\varphi_1(x)\, ds(x)
$$

2. For two functions φ and ψ satisfying the Helmholz equation on D holds

$$
\int\limits_{\partial D} \frac{\partial}{\partial n(x)} \varphi(x)\psi(x) ds(x) = \int\limits_{\partial D} \varphi(x) \frac{\partial}{\partial n(x)} \psi(x) ds(x).
$$

The proof follows immediately by Green's second identity.

3.

$$
\int\limits_{\partial D} \bar{D}D'\varphi_1(x)\varphi_2(x)\, dx = \int\limits_{\partial D} \varphi_1(x) \bar{D}D'\varphi_2(x)\, dx.
$$

4.

$$\int\limits_{\partial D} D\varphi_1(x)\varphi_2(x)\,dx = \int\limits_{\partial D} \varphi_1(x)D'\varphi_2(x)\,dx$$

5.

$$\int\limits_{\partial D} S\varphi_1(x)\varphi_2(x)\,dx = \int\limits_{\partial D} \varphi_1(x)S\varphi_2(x)\,dx$$

The justification for each rule can be found in Appendix.

We claim that

$$\begin{aligned}
&\int_{\partial D} \frac{\partial}{\partial n(x)}\left(\int_{\partial D} \bar{D}D'\varphi'(\lambda^*)h(y)\frac{\partial}{\partial n(y)}\Phi(x,y)\,ds(y)\right) w^s(x)ds(x)\\
&+\int_{\partial D} \frac{ik}{2}\varphi'(\lambda^*)h(x)\,w^s(x)\,ds(x) - ik\int_{\partial D} D'\varphi'(\lambda^*)h(x)\,w^s(x)\,ds(x)\\
&-\int_{\partial D} \frac{\partial}{\partial n(x)}\left(\int_{\partial D} (\bar{D}D'\varphi_w)(y)\frac{\partial}{\partial n(y)}\Phi(x,y)\,ds(y)\right) u^s_{\lambda^*,h}(x)ds(x)\\
&-\int_{\partial D} \frac{ik}{2}\varphi_w(x)\,u^s_{\lambda^*,h}(x)\,ds(x) + ik\int_{\partial D} D'\varphi_w(x)\,u^s_{\lambda^*,h}(x)\,ds(x) = 0,
\end{aligned}$$

or, expanding w^s and $u^s_{\lambda^*,h}$:

$$\begin{aligned}
&\int_{\partial D} \frac{\partial}{\partial n(x)}\left(\int_{\partial D} \bar{D}D'\varphi'(\lambda^*)h(y)\frac{\partial}{\partial n(y)}\Phi(x,y)\,ds(y)\right)\left(\frac{1}{2}\bar{D}D'\varphi_w + D\bar{D}D'\varphi_w - ikS\varphi_w\right)(x)ds(x)\\
&+\int_{\partial D} \frac{ik}{2}\varphi'(\lambda^*)h(x)\left(\frac{1}{2}\bar{D}D'\varphi_w + D\bar{D}D'\varphi_w - ikS\varphi_w\right)(x)\,ds(x)\\
&-ik\int_{\partial D} D'\varphi'(\lambda^*)h(x)\left(\frac{1}{2}\bar{D}D'\varphi_w + D\bar{D}D'\varphi_w - ikS\varphi_w\right)(x)\,ds(x)\\
&-\int_{\partial D} \frac{\partial}{\partial n(x)}\left(\int_{\partial D} (\bar{D}D'\varphi_w)(y)\frac{\partial}{\partial n(y)}\Phi(x,y)\,ds(y)\right)\Bigg(\frac{1}{2}\bar{D}D'\varphi'(\lambda^*)h\\
&\qquad\qquad + D\bar{D}D'\varphi'(\lambda^*)h - ikS\varphi'(\lambda^*)h\Bigg)(x)ds(x)\\
&-\int_{\partial D} \frac{ik}{2}\varphi_w(x)\left(\frac{1}{2}\bar{D}D'\varphi'(\lambda^*)h + D\bar{D}D'\varphi'(\lambda^*)h - ikS\varphi'(\lambda^*)h\right)ds(x)\\
&+ik\int_{\partial D} D'\varphi_w(x)\left(\frac{1}{2}\bar{D}D'\varphi'(\lambda^*)h + D\bar{D}D'\varphi'(\lambda^*)h - ikS\varphi'(\lambda^*)h\right)(x)\,ds(x) = 0.
\end{aligned}$$

On the next page we indicate how each term gets canceled by the other by giving one term a number A and its counterpart the number A′.

$$\underbrace{\int_{\partial D} \frac{\partial}{\partial n(x)} \left(\int_{\partial D} \bar{D} D' \varphi'(\lambda^*) h(y) \frac{\partial}{\partial n(y)} \Phi(x,y)\, ds(y) \right) \frac{1}{2} \bar{D} D' \varphi_w(x) ds(x)}_{①}$$

$$+ \underbrace{\int_{\partial D} \frac{\partial}{\partial n(x)} \left(\int_{\partial D} \bar{D} D' \varphi'(\lambda^*) h(y) \frac{\partial}{\partial n(y)} \Phi(x,y)\, ds(y) \right) D \bar{D} D' \varphi_w(x) ds(x)}_{②}$$

$$- \underbrace{\int_{\partial D} \frac{\partial}{\partial n(x)} \left(\int_{\partial D} \bar{D} D' \varphi'(\lambda^*) h(y) \frac{\partial}{\partial n(y)} \Phi(x,y)\, ds(y) \right) ikS\varphi_w(x) ds(x)}_{③}$$

$$+ \underbrace{\int_{\partial D} \frac{ik}{2} \varphi'(\lambda^*) h(x)\, \frac{1}{2} \bar{D} D' \varphi_w\, ds(x)}_{④} + \underbrace{\int_{\partial D} \frac{ik}{2} \varphi'(\lambda^*) h(x)\, D \bar{D} D' \varphi_w\, ds(x)}_{⑤}$$

$$- \underbrace{ik \int_{\partial D} \frac{ik}{2} \varphi'(\lambda^*) h(x)\, S\varphi_w(x)\, ds(x)}_{⑥} - \underbrace{ik \int_{\partial D} D' \varphi'(\lambda^*) h(x)\, \frac{1}{2} \bar{D} D' \varphi_w(x)\, ds(x)}_{⑤'}$$

$$- \underbrace{ik \int_{\partial D} D' \varphi'(\lambda^*) h(x)\, D \bar{D} D' \varphi_w(x)\, ds(x)}_{⑦} + \underbrace{(ik)^2 \int_{\partial D} D' \varphi'(\lambda^*) h(x)\, ikS\varphi_w(x)\, ds(x)}_{⑧}$$

$$- \underbrace{\int_{\partial D} \frac{\partial}{\partial n(x)} \left(\int_{\partial D} (\bar{D} D' \varphi_w)(y) \frac{\partial}{\partial n(y)} \Phi(x,y)\, ds(y) \right) \frac{1}{2} \bar{D} D' \varphi'(\lambda^*) h(x) ds(x)}_{①'}$$

$$- \underbrace{\int_{\partial D} \frac{\partial}{\partial n(x)} \left(\int_{\partial D} (\bar{D} D' \varphi_w)(y) \frac{\partial}{\partial n(y)} \Phi(x,y)\, ds(y) \right) D \bar{D} D' \varphi'(\lambda^*) h(x) ds(x)}_{②'}$$

$$+ \underbrace{\int_{\partial D} \frac{\partial}{\partial n(x)} \left(\int_{\partial D} (\bar{D} D' \varphi_w)(y) \frac{\partial}{\partial n(y)} \Phi(x,y)\, ds(y) \right) ikS\varphi'(\lambda^*) h(x) ds(x)}_{⑦'}$$

$$- \underbrace{\int_{\partial D} \frac{ik}{2} \varphi_w(x)\, \frac{1}{2} \bar{D} D' \varphi'(\lambda^*) h(x)\, ds(x)}_{④'} - \underbrace{\int_{\partial D} \frac{ik}{2} \varphi_w(x)\, D \bar{D} D' \varphi'(\lambda^*) h(x)\, ds(x)}_{⑨}$$

$$+ \underbrace{ik \int_{\partial D} \frac{ik}{2} \varphi_w(x)\, S\varphi'(\lambda^*) h\, ds(x)}_{⑥'} + \underbrace{ik \int_{\partial D} D' \varphi_w(x)\, \frac{1}{2} \bar{D} D' \varphi'(\lambda^*) h(x)\, ds(x)}_{⑨'}$$

$$+ \underbrace{ik \int_{\partial D} D' \varphi_w(x)\, D \bar{D} D' \varphi'(\lambda^*) h(x)\, ds(x)}_{③'} - \underbrace{(ik)^2 \int_{\partial D} D' \varphi_w(x)\, S\varphi'(\lambda^*) h(x)\, ds(x)}_{⑧'} = 0.$$

Terms ① and ①$'$ cancel by Rule **1**,
② and ②$'$ by Rule **2**,
③ and ③$'$ by Rule **3**, note that $\frac{\partial}{\partial n(x)} S\varphi(x) = D'\varphi(x)$.
④ and ④$'$ by Rule **3**,
⑤ and ⑤$'$ by Rule **4**,
⑥ and ⑥$'$ by Rule **5**,
⑦ and ⑦$'$ by Rule **3**,
⑧ and ⑧$'$ by Rule **2**,
⑨ and ⑨$'$ by Rule **4**.

After the cancellation of terms, equation (5.6) results in

$$
\begin{aligned}
&\int_{\partial D} \frac{\partial}{\partial n(x)} \left(\int_{\partial D} \bar{D} D' \varphi'(\lambda^*) h(y) \frac{\partial}{\partial n(y)} \Phi(x,y)\, ds(y) \right) w^i(x) ds(x) \\
&+ \int_{\partial D} \frac{ik}{2} \varphi'(\lambda^*) h(x)\, w^i(x)\, ds(x) - ik \int_{\partial D} D' \varphi'(\lambda^*) h(x)\, w^i(x)\, ds(x) \\
&- \int_{\partial D} \frac{\partial}{\partial n(x)} w^i(x)\, u^s_{\lambda^*,h}(x)\, ds(x) \\
&= -ik \int_{\partial D} h(x) u(x) w(x)\, ds(x),
\end{aligned}
$$

or, after expanding $u^s_{\lambda^*,h}$, in

$$
\begin{aligned}
&\underbrace{\int_{\partial D} \frac{\partial}{\partial n(x)} \left(\int_{\partial D} \bar{D} D' \varphi'(\lambda^*) h(y) \frac{\partial}{\partial n(y)} \Phi(x,y)\, ds(y) \right) w^i(x) ds(x)}_{①} \\
&+ \int_{\partial D} \frac{ik}{2} \varphi'(\lambda^*) h(x)\, w^i(x)\, ds(x) - ik \underbrace{\int_{\partial D} D' \varphi'(\lambda^*) h(x)\, w^i(x)\, ds(x)}_{②} \\
&- \int_{\partial D} \frac{\partial}{\partial n(x)} w^i(x)\, \frac{1}{2} \bar{D} D' \varphi'(\lambda^*) h(x)\, ds(x) - \underbrace{\int_{\partial D} \frac{\partial}{\partial n(x)} w^i(x)\, D \bar{D} D' \varphi'(\lambda^*) h(x)\, ds(x)}_{①'} \\
&+ \underbrace{\int_{\partial D} \frac{\partial}{\partial n(x)} w^i(x)\, ik S \varphi'(\lambda^*) h(x)\, ds(x)}_{②'} = -ik \int_{\partial D} h(x) u(x) w(x)\, ds(x).
\end{aligned}
$$

Terms ① and ①$'$, ② and ②$'$ cancel by Rule **2**.

So far we have shown that (5.6) reduces to

$$
\begin{aligned}
&-\frac{2\gamma\, ik}{2}\Bigg[\int\limits_{\partial D} -ik\varphi'(\lambda^*)h(x)\int\limits_{S^1}\overline{(F\varphi(\lambda^*)-f)(\hat{x})}e^{-ik\hat{x}\cdot x}\,ds(\hat{x})\,ds(x)\\
&\qquad\qquad + \int\limits_{\partial D}\bar{D}D'\varphi'(\lambda^*)h(x)\frac{\partial}{\partial n(x)}\int\limits_{S^1}\overline{(F\varphi(\lambda^*)-f)(\hat{x})}e^{-ik\hat{x}\cdot x}\,ds(\hat{x})\,ds(x)\Bigg]\\
&\quad = -ik\int\limits_{\partial D} h(x)u(x)w(x)\,ds(x). \qquad (5.7)
\end{aligned}
$$

Since $\overline{(F\varphi(\lambda^*)-f)(\hat{x})e^{-ik\hat{x}\cdot x}}$ is differentiable with respect to x and

$$\overline{(F\varphi(\lambda^*)-f)(\hat{x})}\frac{\partial}{\partial n(x)}e^{-ik\hat{x}\cdot x}$$

is integrable on ∂D, differentiation and integration in the second term can be interchanged. The densities $\varphi'(\lambda^*)h$ and $\bar{D}D'\varphi'(\lambda^*)h$ and the incident field w^i and its derivative $\frac{\partial}{\partial n}w^i$ are $L^2(\partial D)$ integrable. By Fubini's Theorem the order of integration in the first and second term can be interchanged, since the integrals satisfy

$$\int\limits_{\partial D}\Big|-ik\varphi'(\lambda^*)h(x)\int\limits_{S^1}\overline{(F\varphi(\lambda^*)-f)(\hat{x})}e^{-ik\hat{x}\cdot x}\,ds(\hat{x})\Big|\,ds(x) \leq k||\varphi'(\lambda^*)h||_{L^2(\partial D)}||w^i||_{L^2(\partial D)} < \infty.$$

and

$$
\begin{aligned}
&\int\limits_{\partial D}\Big|\bar{D}D'\varphi'(\lambda^*)h(x)\frac{\partial}{\partial n(x)}\int\limits_{S^1}\overline{(F\varphi(\lambda^*)-f)(\hat{x})}e^{-ik\hat{x}\cdot x}\,ds(\hat{x})\Big|\,ds(x)\\
&\qquad\qquad \leq ||\bar{D}D'\varphi'(\lambda^*)h||_{L^2(\partial D)}\left|\left|\frac{\partial}{\partial n}w^i\right|\right|_{L^2(\partial D)} < \infty.
\end{aligned}
$$

Therefore, finally, we can write

$$
\begin{aligned}
&\int\limits_{S^1}\overline{(F\varphi(\lambda^*)-f)(\hat{x})}\;\underbrace{\gamma\int\limits_{\partial D}\bar{D}D'\varphi'(\lambda^*)h(x)\frac{\partial}{\partial n(x)}e^{-ik\hat{x}\cdot x} - ik\varphi'(\lambda^*)h(x)e^{-ik\hat{x}\cdot x}\,ds(\hat{x})}_{F\varphi(\lambda^*)h(\hat{x})}\,ds(x)\\
&= \int\limits_{\partial D} h(x)u(x)w(x)\,ds(x),
\end{aligned}
$$

The proof is complete. □

The necessary optimality condition can now be stated as follows:

$$\delta J(\lambda^*;\lambda-\lambda^*) = 2\mathrm{Re}\int\limits_{\partial D}[w(x)u(x)+\varepsilon\lambda^*(x)](\lambda-\lambda^*)(x)\,ds(x) \geq 0 \;\text{ for all }\; \lambda\in U_{ad}.$$

Lemma 5.4
If

$$\text{Re} \int_{\partial D} [w(x)u(x) + \varepsilon\lambda^*(x)](\lambda - \lambda^*)(x)\, ds(x) \geq 0 \ \text{ for all } \ \lambda \in U_{ad},$$

then for any constant $M \in \mathbb{R}$, $a_- \leq M \leq a_+$,

$$\text{Re}[wu + \varepsilon\lambda^*](M - \lambda^*) \geq 0 \quad \text{a.e. on } \partial D. \tag{5.8}$$

Proof
The proof follows by contradiction [3]. Assume there exists a subset $S_1 \subset \partial D$ with the Lebesque measure $\mu(S_1) > 0$ such that

$$\text{Re}[w(x)u(x) + \varepsilon\lambda^*(x)](M - \lambda^*(x)) < 0 \ \text{ for every } x \in S_1,$$

where M is some constant, $a_- \leq M \leq a_+$. Since the function $\text{Re}[wu + \varepsilon\lambda^*](M - \lambda^*)$ is measurable, by Lusin's theorem there exists a subset $S_2 \subset \partial D$, with $\mu(S_2) < \mu(S_1)$ such that $[wu + \varepsilon\lambda^*](M - \lambda^*)$ is continuous on $\partial D \backslash S_2$.

Since $\mu(S_1) > \mu(S_2)$ there exists a point $x_0 \in S_1 \backslash S_2$. The function $\text{Re}[wu+\varepsilon\lambda^*](M-\lambda^*)$ is continuous at x_0 and is negative at x_0 by assumption. Therefore, there exists a subset $S_3 \subset \partial D, \mu(S_3) > 0$ such that

$$\text{Re}[w(x)u(x) + \varepsilon\lambda^*(x)](M - \lambda^*(x)) < 0 \ \text{ for all}\, x \in S_3,$$

and $\text{Re}[wu + \varepsilon\lambda^*](M - \lambda^*)$ is continuous on S_3.

Consider the function

$$\hat{\lambda}(x) = \begin{cases} M & \text{if} \quad x \in S_3 \\ \lambda^*(x) & \text{if} \quad x \in \partial D \backslash S_3. \end{cases}$$

Then $\hat{\lambda} \in U_{ad}$, and since $\text{Re}[wu + \varepsilon\lambda^*](M - \lambda^*)$ is continuous on S_3

$$\text{Re} \int_{\partial D} [w(x)u(x)+\varepsilon\lambda^*(x)](\hat{\lambda}-\lambda^*)(x)\, ds(x) = \text{Re} \int_{S_3} [w(x)u(x)+\varepsilon\lambda^*(x)](M-\lambda^*(x))\, ds(x) < 0,$$

which contradicts (5.8). $\square$

Corollary 5.5 Let S be a set such that $\lambda^*(x) \in (a_-, a_+)$ for every $x \in S$. Then

$$\text{Re}\ [w(x)u(x) + \varepsilon\lambda^*(x)] = 0, \quad \text{for every } x \in S. \tag{5.9}$$

Let N be a set such that $\text{Re}[w(x)u(x) + \varepsilon\lambda^*(x)] = 0$ for all $x \in N$. Then the set S introduced in the corollary above is a subset of N. It depends on ε, whether the equation $\text{Re}[w(x)u(x) + \varepsilon\lambda^*(x)] = 0$ has a solution. Consequently ε controls the measure of N and thus, the measure of S. For example, if ε is chosen "too big", then, since $|wu|$ is bounded and $\lambda^* \geq a_- > 0$, $\text{Re}[wu + \varepsilon\lambda^*] > 0$ a.e. on ∂D. Thus, N and S are empty sets. In this

case the minimizer of J is unique, and it is given by $\lambda^* := a_-$ a.e. on ∂D.
For $\varepsilon = 0$ and $J(\lambda) > 0$ on U_{ad}, except for the cases when $w(x) = 0$ or $u(x) = 0$, an optimal solution λ^* attains only the the values $\{a_-, a_+\}$.

We conclude this paper by stating the necessary conditions for a local minimum of $J : L^\infty(\partial D) \supset U_{ad} \to \mathbb{R}$

$$J(\lambda) = ||u^s_\lambda - f||^2_{L^2(S^1)} + \varepsilon||\lambda||^2_{L^2(\partial D)}$$

on U_{ad}.

The optimality system consists of the inequality

$$\mathrm{Re}[wu + \varepsilon\lambda^*](M - \lambda^*) \geq 0 \quad \text{for any } \; a_- \leq M \leq a_+ \; \text{ a.e. on } \; \partial D,$$

the scattering problem

$$\begin{aligned}
&\Delta u^s_{\lambda^*} + k^2 u^s_{\lambda^*} = 0 \text{ in } \mathbb{R}^2\backslash\bar{D} \\
&\frac{\partial u^s_{\lambda^*}}{\partial n} + ik\lambda^* u^s_{\lambda^*} = -\frac{\partial u^i}{\partial n} - ik\lambda^* u^i \text{ on } \partial D \\
&\lim_{r\to\infty} \sqrt{r}\left[\frac{\partial}{\partial r}u^s_{\lambda^*} - iku^s_{\lambda^*}\right] = 0 \text{ uniformly, } r = |x|,
\end{aligned}$$

and the adjoint problem

$$\begin{aligned}
&w = w^i + w^s, \\
&\Delta w + k^2 w = 0 \text{ in } \mathbb{R}^2\backslash\bar{D}, \\
&\frac{\partial w}{\partial n} + ik\lambda^* w = 0 \text{ on } \partial D, \\
&\lim_{r\to\infty} \sqrt{r}\left[\frac{\partial}{\partial r}w^s_{\lambda^*} - ikw^s_{\lambda^*}\right] = 0 \text{ uniformly, } r = |x|,
\end{aligned}$$

where

$$w^i(x) = \gamma_1 \int\limits_{S^1} \overline{(u^\infty_{\lambda^*} - f)(\hat{x})} e^{-ik\hat{x}\cdot x} ds(\hat{x}) \text{ for } x \in \mathbb{R}^2, \text{ and } \gamma_1 = 2ik\sqrt{\frac{2}{\pi k}}e^{-i\pi/4}.$$

Appendix A

Single and Double Layer Potentials

Definition A.1 *The function*

$$\Phi(x,y) := \frac{i}{4} H_0^{(1)}(k|x-y|) \text{ for } x \neq y, \quad k \in \mathbb{R},\ k > 0, \tag{A.1}$$

is called the two dimensional fundamental solution of the Helmholz equation $\Delta u + k^2 u = 0$*, where* $H_0^{(1)}$ *denotes the Hankel function of the first type and order 0. Given an* $L^2(\partial D)$ *integrable function* φ *the functions*

$$u(x) := \int_{\partial D} \varphi(y)\Phi(x,y)\, ds(y), \qquad x \in \mathbb{R}^2 \backslash \partial D, \tag{A.2}$$

$$v(x) := \int_{\partial D} \varphi(y) \frac{\partial}{\partial n(y)} \Phi(x,y)\, ds(y), \qquad x \in \mathbb{R}^2 \backslash \partial D, \tag{A.3}$$

are called single and double layer potentials, respectively, with density φ*.*

The single and double layer potentials satisfy the Helmholz equation and the radiation condition.

Theorem A.2 *The boundary integral operators* S*,* D*,* D'*:* $L^2(\partial D) \to L^2(\partial D)$

$$(S\varphi)(x) := \int_{\partial D} \varphi(y)\Phi(x,y)\, ds(y),\ x \in \partial D \tag{A.4}$$

$$(D\varphi)(x) := \int_{\partial D} \varphi(y) \frac{\partial}{\partial n(y)} \Phi(x,y)\, ds(y),\ x \in \partial D \tag{A.5}$$

$$(D'\varphi)(x) := \int_{\partial D} \varphi(y) \frac{\partial}{\partial n(x)} \Phi(x,y)\, ds(y),\ x \in \partial D. \tag{A.6}$$

are well-defined and compact.

Theorem A.3 *Let $\varphi \in H^1(\partial D)$. Then the operator $T : H^1(\partial D) \to L^2(\partial D)$*

$$(T\varphi)(x) := \frac{\partial}{\partial n} \int_{\partial D} \varphi(y) \frac{\partial}{\partial n(y)} \Phi(x,y)\, ds(y),\ x \in \partial D \tag{A.7}$$

is well defined and bounded.
The operator D from (A.5) is bounded from $L^2(\partial D)$ into $H^1(\partial D)$.

Theorem A.4 *For an operator K in some function space we denote by $\bar{K}$ the complex conjugate of K, i.e., $\bar{K} := \overline{K\bar{\varphi}}$. Then we have*

(a) $< \bar{D}\varphi, \psi >_{L^2(\partial D)} \; = \; < \varphi, D'\psi >_{L^2(\partial D)}$ for all $\varphi, \psi \in L^2(\partial D)$, i.e., $\bar{D}$ and D' are L^2-adjoint of each other.

(b) $< \bar{S}\varphi, \psi >_{L^2(\partial D)} \; = \; < \varphi, S\psi >_{L^2(\partial D)}$ for all $\varphi, \psi \in L^2(\partial D)$, i.e., $\bar{S}$ and S are L^2-adjoint of each other.

(c) $< \bar{T}\varphi, \psi >_{L^2(\partial D)} \; = \; < \varphi, T\psi >_{L^2(\partial D)}$ for all $\varphi, \psi \in L^2(\partial D)$, i.e., $\bar{T}$ and T are L^2-adjoint of each other.

Extending the single and double layer potentials and their derivatives to the boundary ∂D we arrive at following relations, known as "jump" conditions.

Theorem A.5 *For the single layer potential u and double layer potential v hold the following jump conditions:*

$$||u_{\pm t} - S\varphi||_{L^2(\partial D)} \to 0,\ t \to 0+, \tag{A.8}$$

$$||n \cdot \nabla u_{\pm t} \pm \varphi - D'\varphi||_{L^2(\partial D)} \to 0,\ t \to 0+, \tag{A.9}$$

$$||v_{\pm t} \mp \varphi - D\varphi||_{L^2(\partial D)} \to 0,\ t \to 0+, \tag{A.10}$$

$$||n \cdot \nabla v_{+t} - n \cdot \nabla v_{-t}||_{L^2(\partial D)} \to 0,\ t \to 0+\,. \tag{A.11}$$

Here, $n(x)$ is a unit normal directed into the exterior D, and

$$u_{\pm t}(x) := u(x \pm t n(x)) \quad \text{and} \quad \nabla u_{\pm t}(x) := \nabla u(x \pm t n(x)).$$

Remark. The normal derivative of the double layer potential v does not exist for L^2 densities. It does exists for densities in the Sobolev space $H^1(\partial D)$.

Theorem A.6 *Let φ and ψ be $L^2(\partial D)$ integrable. Then for operators D, D' and S defined in Theorem A.2 the following holds:*

(a)

$$\int\limits_{\partial D} \bar{D}D'\varphi_1(x)\varphi_2(x)\, dx = \int\limits_{\partial D} \varphi_1(x)\bar{D}D'\varphi_2(x)\, dx.$$

(b)

$$\int\limits_{\partial D} D\varphi_1(x)\varphi_2(x)\, dx = \int\limits_{\partial D} \varphi_1(x)D'\varphi_2(x)\, dx.$$

(c)

$$\int_{\partial D} S\varphi_1(x)\varphi_2(x)\,dx = \int_{\partial D} \varphi_1(x)S\varphi_2(x)\,dx.$$

(d) *For* $\varphi_1, \varphi_2 \in H^1(\partial D)$

$$\int_{\partial D} \frac{\partial}{\partial n(x)} D\varphi_1(x)\varphi_2(x)\,ds(x) = \int_{\partial D} \frac{\partial}{\partial n(x)} D\varphi_2(x)\varphi_1(x)\,ds(x).$$

Proof

(a)

$$\int_{\partial D} \bar{D}D'\varphi_1(x)\varphi_2(x)\,ds(x) = \int_{\partial D}\int_{\partial D}\int_{\partial D} \varphi_1(y_1)\frac{\partial}{\partial n(y)}\Phi(y,y_1)\,ds(y_1)\frac{\partial}{\partial n(y)}\overline{\Phi(x,y)}\,ds(y)\,\varphi_2(x)\,ds(x)$$
$$= \int_{\partial D}\int_{\partial D} \varphi_1(y_1)\varphi_2(x)\int_{\partial D} \frac{\partial}{\partial n(y)}\Phi(y,y_1)\,\frac{\partial}{\partial n(y)}\overline{\Phi(x,y)}\,ds(y)\,ds(x)ds(y_1)$$
$$= \int_{\partial D}\int_{\partial D} \varphi_1(y_1)\varphi_2(x)\int_{\partial D} \frac{\partial}{\partial n(y)}\overline{\Phi(y,y_1)}\,\frac{\partial}{\partial n(y)}\Phi(x,y)\,ds(y)\,ds(x)ds(y_1)$$
$$= \int_{\partial D}\int_{\partial D}\int_{\partial D} \varphi_2(x)\frac{\partial}{\partial n(y)}\Phi(x,y)\,ds(x)\,\frac{\partial}{\partial n(y)}\overline{\Phi(y,y_1)}\,ds(y)\varphi_1(y_1)ds(y_1)$$
$$= \int_{\partial D} \bar{D}D'\varphi_2(y_1)\varphi_1(y_1)\,ds(y_1)$$

The order of integration in the second and the forth line can be interchanged by applying the Fubini's Theorem.

(b)

$$\int_{\partial D} D\varphi_1(x)\varphi_2(x)\,ds(x) = \int_{\partial D}\int_{\partial D} \varphi_1(y)\frac{\partial}{\partial n(y)}\Phi(x,y)\,ds(y)\varphi_2(x)\,ds(x)$$
$$= \int_{\partial D}\int_{\partial D} \varphi_2(x)\frac{\partial}{\partial n(y)}\Phi(x,y)\,ds(x)\varphi_1(y)\,ds(y) = \int_{\partial D} D'\varphi_2(y)\varphi_1(y)\,ds(y)$$

(c)

$$\int_{\partial D} S\varphi_1(x)\varphi_2(x)\,ds(x) = \int_{\partial D}\int_{\partial D} \Phi(x,y)\varphi_1(y)\,ds(y)\varphi_2(x)\,ds(x)$$
$$= \int_{\partial D}\int_{\partial D} \varphi_2(x)\Phi(x,y)\,ds(x)\varphi_1(y)\,ds(y) = \int_{\partial D} S\varphi_2(y)\varphi_1(y)\,ds(y)$$

(d)

$$
\begin{aligned}
&\int\limits_{\partial D} \frac{\partial}{\partial n(x)} D\varphi_1(x)\varphi_2(x)\,ds(x) = \int\limits_{\partial D} \frac{\partial}{\partial n(x)} \int\limits_{\partial D} \varphi_1(y)\frac{\partial}{\partial n(y)}\Phi(x,y)\,ds(y)\varphi_2(x)\,ds(x)\\
&= \int\limits_{\partial D}\int\limits_{\partial D} \varphi_1(y)\varphi_2(x)\frac{\partial}{\partial n(x)}\frac{\partial}{\partial n(y)}\Phi(x,y)\,ds(y)\,ds(x) \text{ [by Schwarz's Theorem]}\\
&= \int\limits_{\partial D}\int\limits_{\partial D} \varphi_1(y)\varphi_2(x)\frac{\partial}{\partial n(y)}\frac{\partial}{\partial n(x)}\Phi(x,y)\,ds(y)\,ds(x)\\
&= \int\limits_{\partial D} \frac{\partial}{\partial n(y)} \int\limits_{\partial D} \varphi_2(x)\frac{\partial}{\partial n(x)}\Phi(x,y)\,ds(x)\varphi_1(y)\,ds(y)\\
&= \int\limits_{\partial D} \frac{\partial}{\partial n(y)} D\varphi_2(y)\varphi_1(y)\,ds(y)
\end{aligned}
$$

$\square$

Appendix B

Weak convergence

Definition B.1 Let X be a normed space and X^* its dual.

(a) A sequence $\{x_n\}_{n=1}^{\infty} \subset X$ is said to converge strongly, or with respect to the norm, provided $||x - x_n|| \to 0$ as $n \to \infty$.

(b) A sequence $\{x_n\}_{n=1}^{\infty} \subset X$ is said to converge weakly to $x \in X$, provided $x^*(x_n) \to x^*(x)$, for all $x^* \in X^*$. Weak convergence is often denoted by $x_n \overset{w}{\to} x$.

(c) A sequence $\{x_n^*\}_{n=1}^{\infty} \subset X^*$ is said to converge in weak sence to $x^* \in X^*$, provided $x_n^*(x) \to x^*(x)$, for every $x \in X$. Weak* convergence is often denoted by $x_n^* \overset{w^*}{\to} x^*$.*

Definition B.2 *Let X be a normed space and X^* its dual.*

(a) A set $U \subset X^*$ is called weak sequentially closed, if the limit point of every weak* convergent sequence $\{x_n^*\}_{n=1}^{\infty} \subset U$ belongs to U.*

(b) A set $U \subset X^*$ is called weak sequentially compact, if every sequence $\{x_n^*\}_{n=1}^{\infty} \subset U$ contains a weak* convergent subsequence such that its limit point belongs to U.*

(c) A functional $J : X^* \to \mathbb{R}$ is called weak sequentially lower semi-continuous provided for every sequence $\{\varphi_k\}_{k=1}^{\infty} \subset X^*$ converging in weak* sense to an element $\varphi \in X^*$ we have*

$$\liminf_{k\to\infty} J(\varphi_k) \geq J(\varphi).$$

Theorem B.3 *Let X be a separable Banach space. Then every bounded sequence $\{x_k^*\} \subset X^*$ in X^* contains a weak*-convergent subsequence.*

Theorem B.4 *Let H be a separable Hilbert space. Then every bounded sequence $\{x_k\} \subset H$ in H contains a weakly convergent subsequence.*

Theorem B.5 *Any weak* sequentially lower semi-continuous functional $J : X^* \to \mathbb{R}$ attains its minima on any weak* sequentially compact subset $U \in X^*$, i.e., there exists $\varphi^0 \in X^*$ such that*

$$J(\varphi^0) = \inf_{\varphi \in U} J(\varphi).$$

Appendix C

The Frechet Derivative

Definition C.1 *Let X and Y be normed spaces over the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, $U \subset X$ an open subset, $\hat{x} \in U$, and $T : X \supset U \to Y$ be a (possibly nonlinear) mapping.*

(a) *T is called Frechet differentiable at $\hat{x}$ if there exists a linear bounded operator $A : X \to Y$ (depending on $\hat{x}$) such that*

$$\lim_{h \to 0} \frac{1}{||h||} ||T(\hat{x} + h) - T(\hat{x}) - Ah|| = 0.$$

We write $T'(\hat{x}) := A$. In particular, $T'(\hat{x}) \in \mathcal{L}(X, Y)$.

(b) *The mapping T is called continuously Frechet differentiable for $\hat{x} \in U$ if T is Frechet differentiable in a neighborhood V of $\hat{x}$ and the mapping $T' : V \to \mathcal{L}(X, Y)$ is continuous in $\hat{x}$.*

Definition C.2 *Let the mapping*

$$T : U \subset X \times Y \to Z$$

be given on an open subset U, where X, Y and Z are normed spaces over $\mathbb{F}$. For fixed $\hat{y} \in Y$, $T(x, \hat{y})$ is a function of x whose Frechet derivative at $\hat{y}$, if it exists, is called the partial Frechet derivative of T with respect to x, and is denoted by $T_x(\hat{x}, \hat{y})$. The partial Frechet derivative $T_y(\hat{x}, \hat{y})$ is defined similarly.

Theorem C.3 *If the mapping T is Frechet differentiable at $(\hat{x}, \hat{y})$, then the partial Frechet derivatives $T_x(\hat{x}, \hat{y})$ and $T_y(\hat{x}, \hat{y})$ exist, and*

$$T'(\hat{x}, \hat{y})(x, y) = T_x(\hat{x}, \hat{y})x + T_y(\hat{x}, \hat{y})y, \quad x \in X,\ y \in Y. \tag{C.1}$$

Conversely, if the partial Frechet derivatives $T_x(\hat{x}, \hat{y})$ and $T_y(\hat{x}, \hat{y})$ exist in a neighborhood of $(\hat{x}, \hat{y})$ and are continuous at $(\hat{x}, \hat{y})$, then T is Frechet differentiable at $(\hat{x}, \hat{y})$ and (C.1) holds.

Theorem C.4

(a) *Let $T, S : X \supset U \to Y$ be Frechet differentiable for $x \in U$. Then $T + S$ and αT are also Frechet differentiable for all $\alpha \in \mathbb{F}$ and*

$$(T+S)'(x) = T'(x) + S'(x), \quad (\alpha T)'(x) = \alpha T'(x).$$

(b) *Chain rule: Let $T : X \supset U \to V \subset Y$ and $S : Y \supset V \to Z$ be Frechet differentiable for $x \in U$ and $T(x) \in V$, respectively. Then ST is also Frechet differentiable in x and*

$$(ST)'(x) = \underbrace{S'(T(x))}_{\in \mathcal{L}(Y,Z)} \underbrace{T'(x)}_{\mathcal{L}(X,Y)} \in \mathcal{L}(X, Z).$$

Theorem C.5 *(Implicit Function Theorem) Let X, Y and Z be normed spaces and let $T : X \times Y \supset U \to Z$ be a continuously Frechet differentiable mapping, where $U \subset X \times Y$ is an open set. Let $(\hat{x}, \hat{y}) \in U$ be such that $T(\hat{x}, \hat{y}) = 0$ and that $T_y(\hat{x}, \hat{y}) \in \mathcal{L}(Y, Z)$ has a bounded inverse.*
Then there exists an open neighborhood $U_X(\hat{x}) \times U_Y(\hat{y}) \subset U$ of $(\hat{x}, \hat{y})$ and a unique continuous function $e : U_X(\hat{x}) \to Y$ such that

(a) $e(\hat{x}) = \hat{y}$;

(b) *for all $x \in U_X(\hat{x})$ there exists exactly one $y \in U_Y(\hat{y})$ with $T(x, y) = 0$, namely $y = e(x)$. Moreover, the mapping $e : U_X(\hat{x}) \to Y$ is continuously Frechet differentiable with derivative*

$$e'(x) = T_y(x, e(x))^{-1} T_x(x, e(x)).$$

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