ELECTRICAL ANALOGS FOR PLATE EQUATIONS AND THEIR APPLICATIONS IN MECHANICAL VIBRATION SUPPRESSION BY P.Z.T. ACTUATORS

BY

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(ABSTRACT)

Before the beginning of digital-computers era, a lot of research was carried out in order to find electric circuits the governing equations of which were analogous to the ones of mechanical systems. The mentioned circuits were called ectro-mechanical analogs. They were used as analogical computers for the simulation and the design of mechanical systems. The actual technological development of piezoelectric actuators, which are devices able to efficiently transduce energy between the electrical and mechanical form, induced us to consider again those electro-mechanical analogs in order to create coupled piezo-electro-mechanical systems. Our idea is that the coupling between electro-mechanical phenomena is maximum when the propagation of both electrical and mechanical waves are governed by similar equations. Let us remark that because of the propagating mechanical wave-speed is much lower than the light-speed for every material, it is

not possible to search for an efficient electro-mechanical coupling inside a piezoelectric continuum. Consequently circuits able to support the propagation of electric-potential waves have been considered. In this work, the equations for the elastica and for the plate are considered and their circuital analogs are derived using their finite-difference approximation. Afterwards, the coupling between the two structures is modelled considering piezoelectric actuators uniformly distributed on the mechanical system and connected to the nodes of the electric circuit. Then the electro-mechanical coupled equations are derived, and an analytical solution is found for a particular case. Finally some numerical simulations showing the efficiently energy exchange is presented.

To my Teacher Francesco dell'Isola.

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Chapter 1

Mechanical models

In this chapter the mechanical equations for the elastica and for the plate are derived considering the hypotheses of linear elasticity.

Elastica model for the beam

The elastica equation is derived considering the balance laws of the forces and the bending-moments upon the Bernoulli-Navier beam.

Kinematics

Let us assume the following Bernoulli-Navier hypotheses for the beam:

- The reference configuration is a cylinder (i.e. its shape is the Cartesian product of an axis and a cross section).
- The maximum diameter of the section is much smaller than the length of the body.
- The cross sections are non-deformable, which means that they can only undergo translations and rotations along the axis.

We will consider only plane beams with constant cross section, hence their reference configuration is given by $\mathcal{S} \times \mathcal{I}$, where $\mathcal{I} = [0, l] \in \mathbb{R}$ is the beam axis, and $\mathcal{S}$ is an interval of $\mathbb{R}^2$ representing its cross section.

Let us consider an observer, which is characterized by the origin o and the basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ on the space of traslations. Three scalar fields are chosen as the kinematical descriptors for the beam: $w(s,t)$, $u(s,t)$, $\vartheta(s,t)$, where $s \in \mathcal{I}$ is the abscissa defined on the axis and $t \in [0, \infty]$ is the time variable. Moreover, these fields are assumed to be continuous with all the derivatives on the entire domain: $C^{\infty}(\mathcal{I} \times [0, \infty])$. The field $w(s,t)$ represents the longitudinal displacement of the axis, while $u(s,t)$ is the transverse displacement and $\vartheta(s,t)$ represents the variation of attitude.

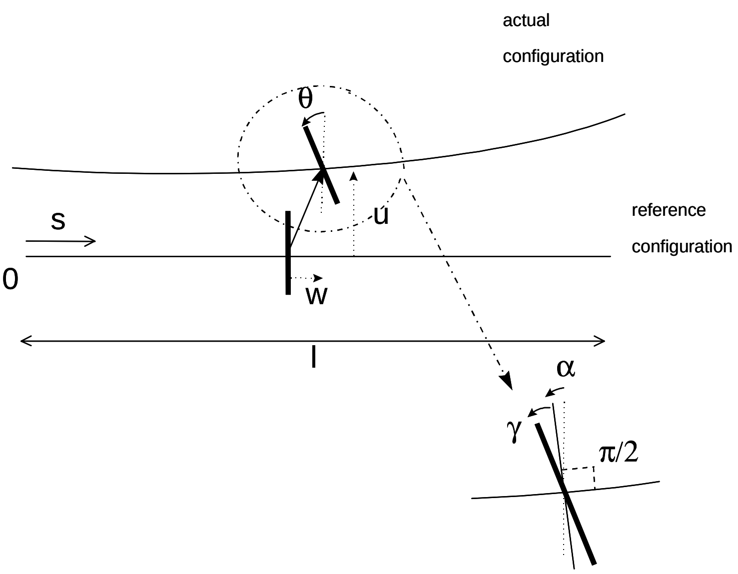


Figure 1.1: kinematical descriptors of the beam.

The evolution of the beam is described by the **motion** π defined as:

$$\pi : t \longrightarrow \left[w\left(s,t\right), u\left(s,t\right), \vartheta\left(s,t\right) \right], \qquad \forall s \in \mathcal{I}.$$

Small deformations

Defining α as the angle between the position of a section in the reference configuration and in the actual one (figure (1.1)), it is immediate to see that equals the angle between the reference configuration axis and its tangent in the actual configuration. Hence the following relation holds ([1]):

$$\tan\alpha = u'.$$

The hypotheses of small deformation require that:

$$\tan\alpha \simeq \alpha \simeq u'.$$

In these hypotheses, let us derive the small deformation fields (the cross sections of the beam, in the reference configuration, are assumed to be orthogonal to the axis):

- Shear deformation represents the lack of orthogonality of the sections in the actual configuration:

$$\chi\left(s,t\right) = u'\left(s,t\right) - \vartheta\left(s,t\right),$$

[1] The subscript ($'$) represent the derivative with respect to the space variable s.

- Axial deformation is due to the different axial displacement that two sections s and $(s + ds)$ undergo passing from the actual to the reference configuration:

$$\varepsilon_w(s,t) = \frac{w(s+ds,t) - w(s,t)}{ds} = w'(s,t).$$

- Bending deformation: is the change in attitude which two sections s and $(s + ds)$ have passing from the reference to the actual configuration:

$$\varepsilon_\vartheta(s,t) = \frac{\vartheta(s+ds,t) - \vartheta(s,t)}{ds} = \vartheta'(s,t).$$

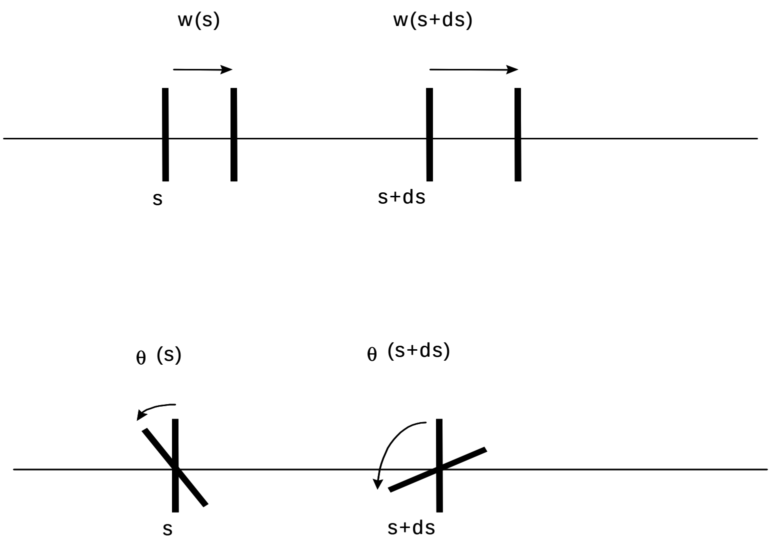


Figure 1.2: Beam deformation

Dynamics

Considering a Cauchy cut along the axis s, the beam is divided in two parts

$I^- = [0, s^-)$ and $I^+ = (s^+, L]$.

According to the Cauchy hypothesis, the interaction between these parts is represented by two vectorial fields:

- $\mathbf{t}(s)$: represent the contact force of s^+ upon s^- and it expend power on the axial deformation.

- $\mathbf{M}(s)$: represent the banding moment of s^+ upon s^- and it expends power on the bending deformation.

Let us now derive the equation of motion considering the balance laws for the applied forces and the moments on a generic body $\mathcal{B}$ ([2]):

$$\mathbf{R} = \int_{\mathcal{B}} d\mathbf{f} = 0\,; \qquad \mathbf{M} = \int_{\mathcal{B}} (P - O) \times d\mathbf{f} = 0.$$

Force balance-law

Let us consider a part of the beam included between the sections s_1 and s_2:

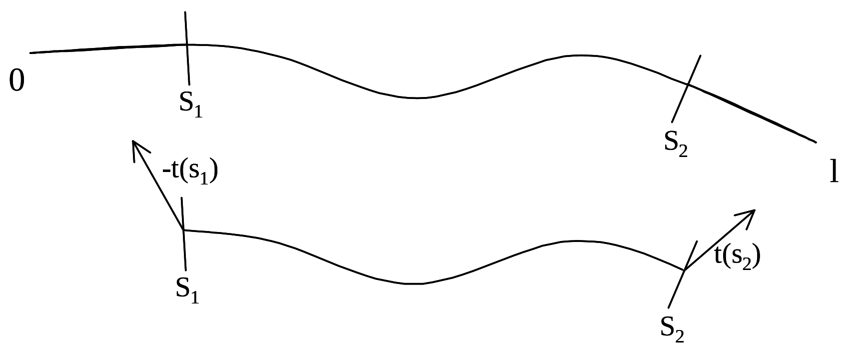


Figure 1.3: Balance of forces

[2] O is the pole with respect to which the bending moment is considered.

the balance of forces on that part assumes the following form:

$$\int_{s_1}^{s_2} \mathbf{b}(s)\, ds + \mathbf{t}(s_2) - \mathbf{t}(s_1) = 0,$$

where $\mathbf{t}(s_1)$ and $\mathbf{t}(s_2)$ represent the action of the rest of the beam on the considered part while $\mathbf{b}(s)$ is the external actions distributed through the part. Using the Torricelli-Barrow theorem we get:

$$\int_{s_1}^{s_2} (\mathbf{b}(s) + \mathbf{t}'(s))\ ds = 0 \qquad \forall s_1\,,\ s_2, \tag{1.1}$$

Since it must hold for every s_1, s_2 we obtain the **first local balance law** for the beam:

$$\mathbf{b}(s) + \mathbf{t}'(s) = 0.$$

Moment balance-law

The balance of moments on a generic part of the beam included in s_1, s_2 is:

$$\int_{s_1}^{s_2} [\mathbf{r}(s) \times \mathbf{b}(s) + \boldsymbol{\mu}(s)]\, ds + \mathbf{M}(s_2) - \mathbf{M}(s_1) + \mathbf{r}(s_2) \times \mathbf{t}(s_2) - \mathbf{r}(s_1) \times \mathbf{t}(s_1) = 0,$$

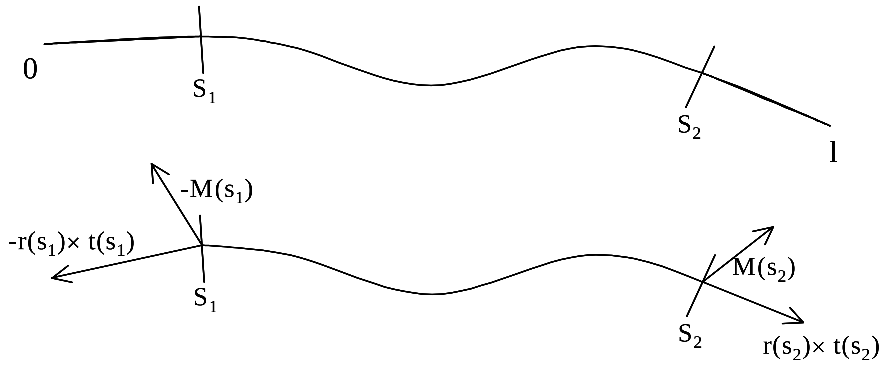


Figure 1.4: Balance of moments

where $\mathbf{M}(s_1)$ and $\mathbf{M}(s_2)$ are the contact moments on the beam given by the rest of the body, $[\mathbf{r}(s) \times \mathbf{b}(s) + \boldsymbol{\mu}(s)]$ is the moment source term and the other terms are the contribution of the contact forces $\mathbf{t}(s_1)$, $\mathbf{t}(s_2)$. Using the Torricelli-Barrow theorem we get:

$$\int_{s_1}^{s_2} [\mathbf{M}'(s) + [\mathbf{r}(s) \times \mathbf{t}(s)]' + \mathbf{r}(s) \times \mathbf{b}(s) + \boldsymbol{\mu}(s)]ds = 0 \qquad \Rightarrow$$

$$\Rightarrow \int_{s_1}^{s_2} [\mathbf{M}'(s) + \mathbf{r}'(s) \times \mathbf{t}(s) + \mathbf{r}(s) \times [\mathbf{t}'(s) + \mathbf{b}(s)] + \boldsymbol{\mu}(s)]\, ds = 0,$$

from the equilibrium of the forces we have $\mathbf{t}'(s) + \mathbf{b}(s) = 0$, and hence

$$\int_{s_1}^{s_2} [\mathbf{M}'(s) + \mathbf{r}'(s) \times \mathbf{t}(s) + \boldsymbol{\mu}(s)]\, ds = 0 \qquad \forall s_1, s_2.$$

Because this must hold for every s_1, s_2 we obtain the **local moment balance law**:

$$\mathbf{M}'(s) + \mathbf{r}'(s) \times \mathbf{t}(s) + \boldsymbol{\mu}(s) = 0.$$

Straight-line axis beam

Let us consider a beam of straight-line axis, and a reference orthogonal frame as in figure

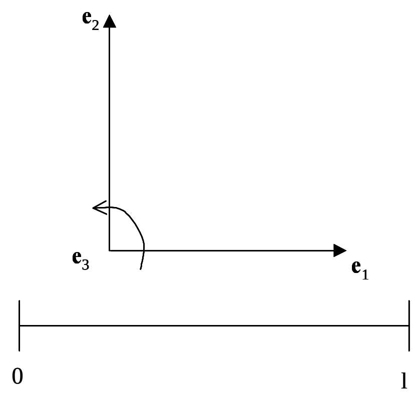


Figure 1.5: Reference frame.

The balance equations are:

$$\left[\begin{array}{l} \mathbf{t}' + \mathbf{b} = 0 \\ \\ \mathbf{M}' + \mathbf{r}' \times \mathbf{t} + \boldsymbol{\mu} = 0 \end{array}\right. . \tag{1.2}$$

If all the forces belong to the plane $(\mathbf{e}_1, \mathbf{e}_2)$, the quantities in the balance equations assume the following form:

$$\mathbf{t} = \begin{pmatrix} N \\ T \\ 0 \end{pmatrix}, \quad \mathbf{M} = \begin{pmatrix} 0 \\ 0 \\ M \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} b_N \\ b_T \\ 0 \end{pmatrix}, \quad \boldsymbol{\mu} = \begin{pmatrix} 0 \\ 0 \\ \mu \end{pmatrix}$$

and the projection of (1.2) gives ([3]):

$$\left[\begin{array}{l} N' + b_N = 0, \\ T' + b_T = 0, \\ M' + T + \mu = 0. \end{array} \right. \tag{1.3}$$

Constitutive relations

The constitutive relations represent the properties of a given medium, they relate the kinematical descriptors to the dynamic actions. Considering a linear material, we assume:

a) **Constitutive relation for the contact shear-stress**

$$T = k_T\,(u' - \vartheta). \tag{1.4}$$

We will consider the beam unshearable, i.e. the sections of the beam remain

[3] We assumed $\mathbf{r}' \equiv \mathbf{e}_1$,i.e.:

$$\mathbf{r}' \times \mathbf{t} = \mathbf{e}_1 \times (N\mathbf{e}_1 + T\mathbf{e}_2) = T\mathbf{e}_3$$

always orthogonal to the axis:

$$k_T = \infty \; ; \; u' - \vartheta = 0, \tag{1.5}$$

this implies that the constitutive relation (1.4) loses its meaning for our hypotheses.

b) **Constitutive relation for the contact normal-stress**

$$N = k_N \, w'.$$

c) **Constitutive relation for the contact bending-stress**

$$M = k_M \, \vartheta'.$$

d) **Absence of external actions and inertial constitutive relations ([4])**

$$\begin{aligned} \mu &= 0, \\ b_T &= \rho \ddot{u}, \\ b_N &= \rho \ddot{w}. \end{aligned}$$

Elastica equations

Let us consider the derivative of the third equilibrium equation $(1.3)_3$ ([5])

$$M'' + T' + \mu' = 0$$

[4] The superscript $(\dot{\cdot})$ represent the time derivative.
[5] With the subscript on the formula number $()_i$ we intend the i-th equation of the formula.

from the $(1.3)_2$ we get $T' = -b_T$, hence:

$$M'' - b_T + \mu' = 0$$

From the hypothesis (1.5) we get

$$u'' = \vartheta'.$$

Hence the constitutive relation for the moment becomes

$$M = k_M \ddot{u}, \tag{1.6}$$

which when substituted into $(1.3)_3$, together with the relation for the external actions and the inertial term, gives the first elastica equation:

$$(k_M u'')'' - \rho\ddot{u} = 0. \tag{1.7}$$

Substituting the constitutive relation for N and for the inertial term b_N into the equilibrium equation $(1.3)_1$ we get the second elastica equation:

$$(k_N w')' + \rho w'' = 0.$$

We are interested in studying bending vibrations of the beam, hence the elastica evolution will be described only by the equation (1.7).

Plate model

The derivation of the plate model will be done using a reduction procedure from the 3D continuum Cauchy model. Hence some fundamental concepts of continuum mechanics will be here recalled.

Kinematics

Continuum mechanics, which includes the study of deforming bodies, is based on the concept of **material point** $\mathfrak{p}$, which is considered as a physical object, having infinitesimal dimensions and occupying a **position p** in an Euclidean space $\mathcal{E}$. A **body** $\mathcal{C}$ is simply defined as a set of material points.

Considering the Euclidean space $\mathcal{E}$ and the time axis $\mathcal{T}$, we define the **motion** as the application $\pi(\mathfrak{p},t) : \mathcal{C} \times \mathcal{T} \longrightarrow \mathcal{E} \times \mathcal{T}$, which maps, for each instant $t \in \mathcal{T}$, the material point $\mathfrak{p} \in \mathcal{C}$ into a the position $\mathbf{p} \in \mathcal{E}$. We call **placement** the map $\pi(\mathfrak{p},\bar{t})$ at a fixed $\bar{t}$. Only regular placements of the body $\mathcal{C}$ will be considered, i.e. the material-points positions of the body $\mathcal{C}$ occupies always regions with regular and bounded bounder. Moreover the body cannot penetrate itself. Mathematically, this means that

$$\det\left[\boldsymbol{\nabla}_{\mathfrak{p}}\pi(\mathfrak{p},\bar{t})\right] > 0,$$

where we have assumed $\pi(\mathfrak{p},\bar{t})$ to be continuous and differentiable.

Remark 1 *In the following let us omit the time dependence in the formulas.*

Using a Lagrangian approach, we define the **transport** mapping $f(\mathbf{p})$ as a function that displaces a generic material point $\mathbf{p}$ from a referential configuration to the actual one (see fig 1.6).

Hence we can define the **displacement** as the vectorial field $\mathbf{u}(\mathbf{p})$:

$$\mathbf{u}(\mathbf{p}) = f(\mathbf{p}) - \mathbf{p},$$

which represents the displacement of a point $\mathbf{p}$ from the reference configuration to the actual one. Let us define also the tensorial fields $\mathbf{F}(\mathbf{p})$ and $\mathbf{H}(\mathbf{p})$:

$$\begin{aligned} \mathbf{F} &= \boldsymbol{\nabla} f(\mathbf{p}), \\ \mathbf{H} &= \boldsymbol{\nabla} \mathbf{u}(\mathbf{p}), \end{aligned}$$

which are the gradient of f and $\mathbf{u}$ defined in the Euclidean space.

The displacement $\mathbf{u}$ and its gradient $\mathbf{H}$ are chosen as the kinematic descriptors of the body.

Remark 2 *Let us observe that whereas* $f(\mathbf{p})$ *maps the material point from the referential configuration to the actual one,* $\mathbf{F}(\mathbf{p})$ *(in the limits of a local analysis), maps a vector of material points between the two configurations (see fig 1.6). Hence* $\mathbf{F}(\mathbf{p})$ *describes the local deformation of the body. In fact, let us consider a vector of material points* $\mathbf{d} = \mathbf{p} - \mathbf{p}_0$*, if* $|\mathbf{d}|$ *is small enough, we can approximate its image in the actual configuration with the vector* $f(\mathbf{p}) - f(\mathbf{p}_0) = \mathbf{q} - \mathbf{q}_0$ *(see fig 1.6). Moreover:*

$$f(\mathbf{p}) = f(\mathbf{p}_0 + \mathbf{d}) \cong f(\mathbf{p}_0) + \mathbf{F}|_{\mathbf{p}_0} \mathbf{d} = \mathbf{q}_0 + \mathbf{F}|_{\mathbf{p}_0} \mathbf{d},$$

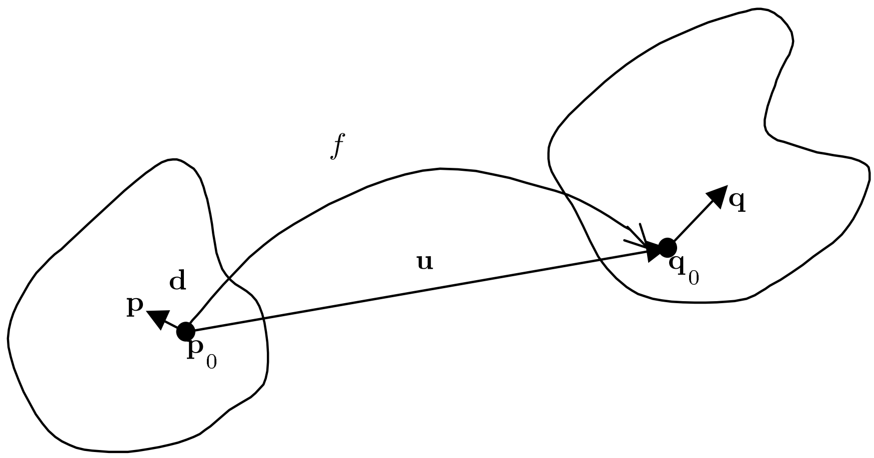


Figure 1.6: Transport f and displacement $\mathbf{u}$ maps.

hence it is immediate to see how $\mathbf{F}$ *maps local vectors of material points in the actual configuration:*

$$\mathbf{q} - \mathbf{q}_0 \cong \mathbf{F}|_{\mathbf{p}_0} (\mathbf{p} - \mathbf{p}_0).$$

In order to mathematically describe the deformation, let us first define the class of rigid transports:

Definition 3 *Given a body* $\mathcal{C}$ *and given two vectors of material points* $\mathbf{d}_1, \mathbf{d}_2,$ *a transport* f *is* ***rigid*** *if it does not change their lengths and does not change the angles between them, i.e. the scalar product is preserved (*[6]*) :*

$$\mathbf{d}_1 \cdot \mathbf{d}_2 = \mathbf{F}\mathbf{d_1} \cdot \mathbf{F}\mathbf{d_2} \qquad \forall \mathbf{d}_1, \mathbf{d}_2,$$

[6] The simbol $\cdot$ represent the dot product between vectors in the Euclidean space.

By the previous definition we get ([7])

$$\mathbf{d_1}\cdot\mathbf{d_2} = \mathbf{d_1}\cdot\left(\mathbf{F}^T\mathbf{F}\right)\mathbf{d_2} \quad \Longrightarrow \quad \mathbf{F}^T\mathbf{F} = \hat{\mathbf{1}}.$$

Therefore, the definition of deformation tensor can be given as:

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \hat{\mathbf{1}}), \qquad \mathbf{C} = \mathbf{F^T F}. \tag{1.8}$$

Roughly speaking, $\mathbf{E}$ gives information on how far a transport differs a the rigid one.

Remark 4 *The tensor* $\mathbf{C}$ *is called the right Cauchy-Green deformation tensor.*

In order to represent $\mathbf{E}$ in terms of the state variable $\mathbf{u}$ in a simple way, let us consider small displacements $\mathbf{u}$ of a position $\mathbf{p}$ from the reference configuration ([8]).

$$f(\mathbf{p}) = \mathbf{q} = \mathbf{p} + \varepsilon\mathbf{u}.$$

Taking the gradient we obtain

$$\mathbf{F} = \hat{\mathbf{1}} + \varepsilon\nabla\mathbf{u}.$$

Hence the deformation tensor becomes

$$\mathbf{E} = \frac{1}{2}(\mathbf{F^T F} - \hat{\mathbf{1}}) = \frac{1}{2}(\mathbf{H}^T + \mathbf{H}) + \varepsilon^2\mathbf{H}^T\mathbf{H},$$

[7] $\hat{\mathbf{1}}$ is the identity tensor.

[8] ε represent a smallness parameter.

since the second term on the right hand side is an infinitesimal of higher order we have ([9]):

$$\mathbf{E} = \frac{1}{2}(\mathbf{H}^T + \mathbf{H}) = Sym\boldsymbol{\nabla}(\mathbf{u}).$$

Finally, let us define the velocity field of the kinematical descriptors ([10]):

$$\mathbf{v} = \dot{\mathbf{u}},$$

$$\boldsymbol{\nabla}\mathbf{v} = \dot{(\boldsymbol{\nabla}\mathbf{u})} \simeq \boldsymbol{\nabla}\dot{\mathbf{u}}.$$

Dealing with a rigid displacement, both $\boldsymbol{\nabla}\mathbf{u}$, $\boldsymbol{\nabla}\mathbf{v}$ are antisymmetric: $\boldsymbol{\nabla}\mathbf{u}, \boldsymbol{\nabla}\mathbf{v} \epsilon Skw$ ([11]).

Remark 5 *For more detail about continuum mechanics see [4]*

Plate model

We will use the Kirchhoff-Love model for a plate.

In order to derive the form of the state field $\mathbf{u}$, let us give a mathematical definition of a plate.

---

[9] Sym represent the symmetric part of the tensor.

[10] The superscript $\cdot$ represents the time derivative.

[11] $\boldsymbol{\nabla}\mathbf{u}\epsilon Skw$ is immediately from the definition of the deformation $\mathbf{E}$. While considering the time derivative of the Cauchy tensor for a rigid displacement ($\mathbf{F}^T\mathbf{F} = \hat{\mathbf{1}}$) we obtain:

$$\dot{\mathbf{F}}^T\mathbf{F} + \mathbf{F}^T\dot{\mathbf{F}} = 0 \quad \Rightarrow \quad \left(\dot{\mathbf{F}}\mathbf{F}^T\right)^T = -\left(\dot{\mathbf{F}}\mathbf{F}^T\right);$$

hence $\dot{\mathbf{F}}\mathbf{F}^T \epsilon Skw$.So

$$\boldsymbol{\nabla}\mathbf{v} = \dot{\mathbf{F}}\mathbf{F}^T \quad \Rightarrow \quad \boldsymbol{\nabla}\mathbf{v}\epsilon Skw.$$

Definition 6 *A plate is a body $\mathcal{C}$ occupying a region $\mathcal{D} = \mathcal{S} \times \mathcal{I}$ (where $\mathcal{S}$ is a plane surface and $\mathcal{I}$ is the interval $[-h_t, h_t] \in \mathbb{R}$), in which the thickness $2h_t$ is small when compared to the diameter of the surface $\mathcal{S}$.*

Hence, the following decomposition for the position vector can be assumed (see fig. 1.7):

$$\mathbf{x} = \mathbf{r} + \zeta \mathbf{e},$$

where $\mathbf{r}$ is the position vector in $\mathcal{S}$, $\mathbf{e}$ is the unit vector perpendicular to $\mathcal{S}$ and $\zeta \in \mathcal{I}$ is the component of $\mathbf{x}$ in the direction of $\mathbf{e}$. Thus the following decomposition for the displacement $\mathbf{u}$ is considered:

$$\mathbf{u}(\mathbf{r},\xi) = \mathbf{v}(\mathbf{r},\zeta) + w(\mathbf{r},\zeta)\,\mathbf{e}.$$

The hypothesis of small thickness allows us to assume that $\mathbf{u}(\mathbf{r},\xi)$ has a linear dependence on ζ:

$$\mathbf{v}(\mathbf{r},\zeta) = \mathbf{v}_0(\mathbf{r}) + \zeta \mathbf{v}_1(\mathbf{r}),$$

$$w(\mathbf{r},\zeta) = w_0(\mathbf{r}) + \zeta w_1(\mathbf{r}).$$

This implies that a vector of material points orthogonal to $\mathcal{S}$ in the reference configuration is not be bent.

Let us give a physical meaning to each component of the displacement. The field $\mathbf{v}_0(\mathbf{r})$ takes into account the stretching of the surface $\mathcal{S}$ (see fig (1.8)). $\mathbf{v}_1(\mathbf{r})$

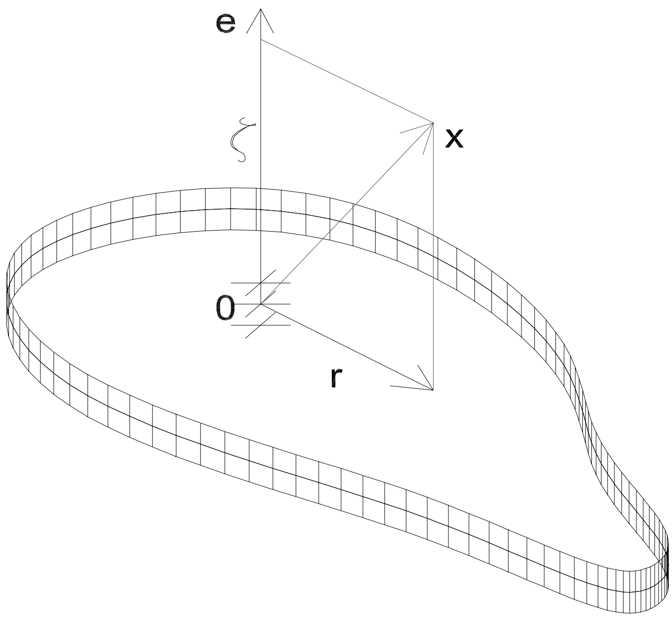


Figure 1.7: Decomposition of displacement $\mathbf{u}$ on the plate reference configuration.

impresses a rotation of the straight lines of material-points perpendicular to the surface (see fig (1.9)). The field $w_0(\mathbf{r})$ is the vertical deflection of $\mathcal{S}$ (see fig (1.10)). $w_1(\mathbf{r})$ gives a vertical stretch of the straight lines of material-points perpendicular to $\mathcal{S}$ (see fig (1.11)).

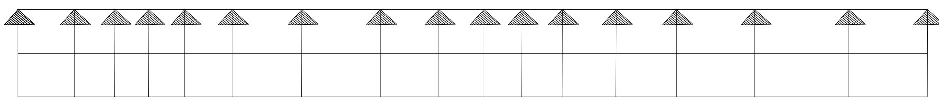


Figure 1.8: Horizontal streching of $\mathcal{S}$ due to $\mathbf{v}_0(\mathbf{r})$.

We will use a simplified model in which:

1) The stretch of the middle surface $\mathcal{S}$ is negligible, i.e. $\mathbf{v}_0(\mathbf{r}) = 0$.

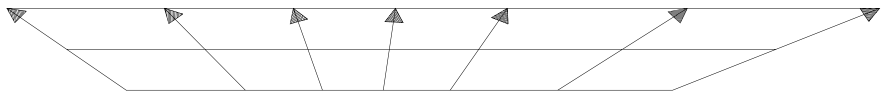


Figure 1.9: Rotation of the straight lines perpendicular to $\mathcal{S}$ due to $\zeta\mathbf{v}_1(\mathbf{r})$.

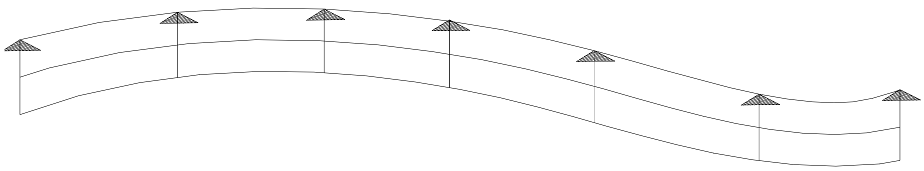


Figure 1.10: Vertical deflection of $\mathcal{S}$ due to $\mathbf{w}_0(\mathbf{r})$.

2) There is no vertical deformation of material-points vectors orthogonal to $\mathcal{S}$: $w_1(\mathbf{r}) = 0$.

3) **Kirchhoff- Love** hypothesis holds ([12]):

$$\mathbf{v}_1(\mathbf{r}) = -\nabla w_0(\mathbf{r}) .$$

This implies that straight lines orthogonal to $\mathcal{S}$ before deformation, remain perpendicular to $\mathcal{S}$ after deformation.

Consequently the displacement for a plate assumes the following form:

$$\mathbf{u}(\mathbf{r},\zeta) = w_0(\mathbf{r})\,\mathbf{e} - \zeta\nabla w_0(\mathbf{r}) . \tag{1.9}$$

[12] With the symbol ∇ we intend the gradient defined on $\mathcal{S}$.

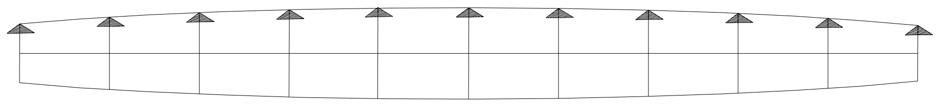


Figure 1.11: Vertical deformation of the stright lines perpendicular to $\mathcal{S}$ due to $w_1(\mathbf{r})$.

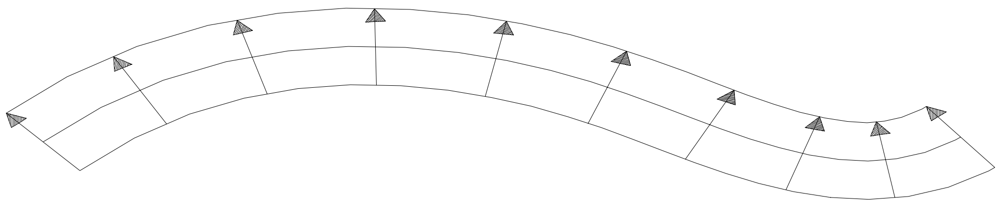


Figure 1.12: Allowed deformation.

The deformation $\mathbf{E}$ becomes:

$$\mathbf{E} = Sym\left[\boldsymbol{\nabla}\mathbf{u}\left(\mathbf{r},\zeta\right)\right] = -\zeta Sym\left[\nabla\nabla w_o\left(\mathbf{r}\right)\right]. \tag{1.10}$$

Dynamics, 'principle of virtual power'

In order to derive the equation of motion for a continuum (hence for the plate), the energy will be considered as a primitive concept and a balance of the internal and external power in the body (instead of balance of forces), will be assumed.

Let us consider the following definitions:

Definition 7 *The energy time-variation inside the body, i.e. the* ***internal power***

is:

$$\mathfrak{W}_{\mathfrak{i}} = \int_{\mathcal{D}} (\boldsymbol{\tau} \cdot \mathbf{v} + \mathbf{T} \cdot \boldsymbol{\nabla} \mathbf{v}) \, .$$

Where $\mathcal{D}$ represents the domain $\pi(\mathcal{C}, \bar{t})$, which the body $\mathcal{C}$ occupies at each instant $\bar{t}$.

Definition 8 *The energy time-variation outside the body, i.e. the* ***external power*** *is:*

$$\mathfrak{W}_{\mathfrak{e}} = \int_{\mathcal{D}} (\mathbf{b} \cdot \mathbf{v}) + \int_{\partial\mathcal{D}} (\mathbf{t} \cdot \mathbf{v}) \, ,$$

The coefficients $\mathbf{b}, \mathbf{t}$ are the body and surface external forces respectively, while $\mathbf{T}$ is the stress tensor and $\boldsymbol{\tau}$ is an internal body force which, in our case, will vanish. In fact it is assumed that the internal power equals zero in a rigid motion, i.e. $\mathfrak{W}_{\mathfrak{i}} = 0$ for each couple $(\mathbf{v}, \nabla \mathbf{v})$ belonging to a rigid motion, thus:

$$\begin{aligned} \boldsymbol{\tau} &= 0, \\ \mathbf{T} &\in Sym. \end{aligned}$$

Let us enunciate the principle of virtual power as follows:

Claim 9 *For every test couple $(\mathbf{v}, \nabla \mathbf{v})$, The internal power equals the external one*

$$\mathfrak{W}_{\mathfrak{i}} = \mathfrak{W}_{\mathfrak{e}},$$

This implies that

$$\int_{\mathcal{D}} \mathbf{T} \cdot \boldsymbol{\nabla} \mathbf{v} - \int_{\mathcal{D}} \mathbf{b} \cdot \mathbf{v} = \int_{\partial \mathcal{D}} (\mathbf{t} \cdot \mathbf{v}) \, . \tag{1.11}$$

Remark 10 *Let us observe that the first term on the left hand side can be written as:*

$$\int_{\mathcal{D}} \mathbf{T} \cdot \boldsymbol{\nabla} \mathbf{v} = \int_{\mathcal{D}} \boldsymbol{\nabla} \cdot (\mathbf{T}\mathbf{v}) - \int_{\mathcal{D}} (\boldsymbol{\nabla} \cdot \mathbf{T}) \cdot \mathbf{v}, \tag{1.12}$$

using the divergence theorem on the first term of the right hand side of (1.12) we obtain ([13]):

$$\int_{\mathcal{D}} \boldsymbol{\nabla} \cdot (\mathbf{T}\mathbf{v}) = \int_{\partial \mathcal{D}} (\mathbf{T}\mathbf{v}) \cdot \mathbf{n} = \int_{\partial \mathcal{D}} \mathbf{T}\mathbf{n} \cdot \mathbf{v},$$

which substituted into the relation (1.12) gives the following power balance for the (1.11) :

$$\int_{\mathcal{D}} (\boldsymbol{\nabla} \cdot \mathbf{T} + \mathbf{b}) \cdot \mathbf{v} = \int_{\partial \mathcal{D}} (\mathbf{T}\mathbf{n} - \mathbf{t}) \cdot \mathbf{v}.$$

Because this must hold for every test couple $(\mathbf{v}, \nabla \mathbf{v})$*, the Cauchy-Navier equation and the Cauchy theorem are found:*

$$\boldsymbol{\nabla} \cdot \mathbf{T} + \mathbf{b} = 0 \qquad on \qquad D, \qquad \mathbf{T} \epsilon Sym,$$

$$\mathbf{T}\mathbf{n} = \mathbf{t} \qquad on \qquad \partial D.$$

Because of the symmetry of the stress tensor the balance relation (1.11) can be also written in the following form:

$$\int_{\mathcal{D}} \mathbf{T} \cdot Sym(\boldsymbol{\nabla} \mathbf{v}) - \int_{\mathcal{D}} \mathbf{b} \cdot \mathbf{v} = \int_{\partial \mathcal{D}} (\mathbf{t} \cdot \mathbf{v}) \, ; \tag{1.13}$$

[13] $\mathbf{n}$ is the outward normal to the boundary.

Kirchhoff-Love Plate model equations

Let us now consider the power balance (1.13) for the plate.

From the formulas of the displacement (1.9) and the deformation (1.10) of a plate we obtain ([14]):

$$\mathbf{v} = \dot{u}\,(\mathbf{r})\,\mathbf{e} - \zeta\nabla\dot{u}\,(\mathbf{r})\,,$$

$$\dot{\mathbf{E}} = Sym\left[\boldsymbol{\nabla}\mathbf{v}\,(\mathbf{r},\zeta)\right] = -\zeta Sym\left[\nabla\nabla\dot{u}\,(\mathbf{r})\right].$$

Moreover we will assume the external forces on the boundary of the plate vanish, i.e. $\mathbf{t} = 0$. Hence the power balance (1.13) for the plate becomes:

$$\int_{\mathcal{S}}\int_{\mathcal{I}} \zeta\mathbf{T}\cdot Sym\left[\nabla\nabla\dot{u}\,(\mathbf{r})\right] + \int_{\mathcal{S}}\int_{\mathcal{I}} \mathbf{b}\cdot\dot{u}\,(\mathbf{r})\,\mathbf{e} - \int_{\mathcal{S}}\int_{\mathcal{I}} \zeta\mathbf{b}\cdot\nabla\dot{u}\,(\mathbf{r}) = 0. \qquad (1.14)$$

Let us define the following quantities ([15])

$$\mathbf{M} = \int_{\mathcal{I}} \zeta\ \mathbf{T}|_{\mathcal{S}}\,, \qquad \beta = -\int_{\mathcal{I}} \mathbf{b}\cdot\mathbf{e}, \qquad \mathbf{B} = \int_{\mathcal{I}} \zeta\ \mathbf{b}|_{\mathcal{S}}\,,$$

where $\mathbf{M}$ will represent the bending moment tensor for the plate, while β and $\mathbf{B}$ will take into account the inertial forces. The balance relation (1.14) becomes:

$$\int_{\mathcal{S}} \left[\mathbf{M}\cdot\ Sym\left[\nabla\nabla\dot{u}\right] + \beta\,\dot{u} + \mathbf{B}\cdot\nabla\dot{u}\right] = 0.$$

Using integration by parts and the divergence theorem on the first term on the right hand side (as it was done in the 3D case) we get ([16]):

$$\int_{\mathcal{S}} \left[\beta - \nabla\cdot\mathbf{B} + \nabla\cdot(\nabla\cdot\mathbf{M})\right]\,\dot{u} + \int_{\partial\mathcal{S}} \left\{\left[\mathbf{B}\cdot\mathbf{n} - (\nabla\cdot\mathbf{M})\cdot\mathbf{n}\right]\dot{u} + (\mathbf{M}\mathbf{n})\cdot\nabla\dot{u}\right\} = 0,$$

[14] In what follows we will use u instead of w_o for the deflection.
[15] The simbols $|_{\mathcal{S}}$ means that the inside quantity is evaluated on $\mathcal{S}$.
[16] $\mathbf{n}$ is now the outworld normal to the boundary line of $\mathcal{S}$.

Since this must hold for each test couple $(\dot{u}, \nabla\dot{u})$, from the first integral we get the dynamic equation for the plate:

$$\nabla \cdot (\nabla \cdot \mathbf{M}) - \nabla \cdot \mathbf{B} + \beta = 0, \qquad \text{on } \mathcal{S}, \tag{1.15}$$

while the second one gives us the allowable boundary conditions:

$$\forall\,(\dot{u}, \nabla\dot{u}) \quad \text{admissible} \quad \left[\begin{array}{l} [\mathbf{B}\cdot\mathbf{n} - (\nabla\cdot\mathbf{M})\cdot\mathbf{n}]\;\; \dot{u} = 0 \\ \\ (\mathbf{M}\mathbf{n})\cdot\nabla\dot{u} = 0 \end{array}\right. \qquad \text{on } \partial\mathcal{S}. \tag{1.16}$$

Remark 11 *Let us observe that the relations (1.16) are satisfied considering the completely-clamped boundary conditions:*

$$\left[\begin{array}{l} u|_{\partial\mathcal{S}} = 0, \\ \dfrac{\partial u}{\partial \mathbf{n}} = \nabla u|_{\partial\mathcal{S}} = 0, \end{array}\right.$$

or the ***simply-supported*** *ones ([17]):*

$$\left[\begin{array}{l} u = 0 \\ \\ \mathbf{M}\mathbf{n} = 0 \end{array}\right. \qquad \text{on } \partial\mathcal{S}.$$

Constitutive relations

The constitutive relations relate the dynamic actions (stress tensor $\mathbf{T}$ and external body force $\mathbf{b}$) to the state variables and permit us to derive the wave equation in the considered medium.

If we consider a linear isotropic, homogenous material, the constitutive relations for $\mathbf{T}$ can be expressed in the following form:

$$\mathbf{T} = 2\,\mu_L\,\mathbf{E} + \lambda_L\,tr\,(\mathbf{E})\;\,\hat{\mathbf{1}}, \tag{1.17}$$

[17] Which means that the banding moment vanishes on the boundary.

where the coefficients μ_L and λ_L represent the Lame' constants.

Considering the deformation tensor of a plate: $\mathbf{E} = -\xi Sym\left[\nabla\nabla u\left(\mathbf{r}\right)\right]$, we get

$$\mathbf{T} = -2\;\mu_L\;\xi\;Sym\left[\nabla\nabla u\left(\mathbf{r}\right)\right] - \lambda_L\;\xi\;tr\left\{Sym\left[\nabla\nabla u\left(\mathbf{r}\right)\right]\right\}\hat{\mathbf{1}}.$$

Consequently, the banding moment tensor $\mathbf{M} = \int_{\mathcal{I}} \zeta\;\mathbf{T}|_{\mathcal{S}}$ becomes:

$$\mathbf{M} = J_{\mathcal{I}}\left[2\,\mu_L\,Sym\left[\nabla\nabla u\left(\mathbf{r}\right)\right] + \lambda_L\;\left[\nabla^2 u\left(\mathbf{r}\right)\right]\;\hat{\mathbf{1}}\right], \qquad J_{\mathcal{I}} = \int_{\mathcal{I}} \zeta^2 = \frac{2}{3}h_t^3 \quad (1.18)$$

The constitutive relation for the inertial force $\mathbf{b}$ is simply:

$$\mathbf{b} = -\rho\ddot{\mathbf{u}}.$$

hence considering the Kirchhoff-Love displacement in the plate we get:

$$\mathbf{b} = -\rho\left[\ddot{u}\left(\mathbf{r}\right)\mathbf{e} - \boldsymbol{\xi}\nabla\ddot{u}\left(\mathbf{r}\right)\right], \qquad (1.19)$$

and the constitutive relations for the inertial terms $\mathbf{B} = \int_{\mathcal{I}} \zeta\;\mathbf{b}|_{\mathcal{S}}$ and $\beta = -\int_{\mathcal{I}} \mathbf{b}\cdot\mathbf{e}$ become:

$$\begin{aligned} \mathbf{B} &= J\rho\nabla\ddot{u}, \qquad (1.20) \\ \beta &= -2h_t\rho\ddot{u}. \end{aligned}$$

Finally, substituting the constitutive relations (1.19, 1.20) into the equilibrium for the plate (1.15), the plate wave equation is found:

$$S_p\nabla^2\nabla^2 u - J_{\mathcal{I}}\,\rho\,\nabla^2\ddot{u} + 2\,h_t\,\rho\,\ddot{u} = 0,$$

where:

$$\begin{aligned} J_{\mathcal{I}} &= \frac{2}{3}h_t^3, \\ S_p &= J_{\mathcal{I}}\left(2\mu_L + \lambda_L\right). \end{aligned}$$

Remark 12 *If we consider the constitutive relation, the simply-connected boundary conditions ($u|_{\partial\mathcal{S}} = 0$,$\mathbf{Mn}|_{\partial\mathcal{S}} = 0$) become:*

$$\left[\begin{array}{l} u|_{\partial\mathcal{S}} = 0, \\ \dfrac{\partial^2 u}{\partial \mathbf{n}^2} = 0. \end{array}\right.$$

Conclusion 13 *The wave equation for the plate and the boundary conditions which have been considered are here summarized:*

- *Wave equation ([18]):*

$$\frac{h_t^2 Y}{3\rho}\nabla^2\nabla^2 u - \frac{h_t^2}{3}\nabla^2\ddot{u} + \ddot{u} = 0.$$

- *completely-clamped plate-BC:*

$$u|_{\partial\mathcal{S}} = \frac{\partial u}{\partial \mathbf{n}} = 0.$$

- *Simply-connected plate-BC:*

$$u|_{\partial\mathcal{S}} = \frac{\partial^2 u}{\partial \mathbf{n}^2} = 0.$$

[18] Here $Y = 2\mu_L + \lambda_L$ represents the Young modulus for the considered matherial.

Chapter 2

Some useful mathematical tools

Before proceeding into the derivation of the electric analogs for the presented mechanical structures, let us recall some mathematical tools which will be used.

Laplace transforms

The Laplace transform allows us to find a simpler representation of the input /output relation for linear systems (in particular it makes possible the representation of the response function of a linear system as the product of transformed excitation and the transformed response function of the system).

In order to define the Laplace transform, let us consider the class of functions $\mathcal{DR}$ as the functions $f : \mathbb{R} \longrightarrow \mathbb{C}$ which satisfy the following properties:

- The domain of $f(t)$ is $(-\infty, +\infty)$,
- $f(t)$ is integrable on all its domain,
- $f(t)$ is piecewise continuously differentiable on each piecewise interval,
- $f(t) = \frac{1}{2}\left[f(t^+) + f(t^-)\right]$ for each t, where it is discontinuous.

Moreover define the class of the Laplace-transformable functions $\mathcal{T}$ as the set of real valued functions defined in $\mathbb{R}$, the generic element of which satisfies the following properties:

- $y(t) = 0$ for $t < 0$,
- There is a real number α such that $y(t)e^{-\alpha t}$ belongs to $\mathcal{DR}$.

Definition 14 *The Laplace transform of a function $y(t) \in \mathcal{T}$ is the complex valued function:*

$$\mathfrak{L}[y(t)]\,(s) = Y\,(s) = \int_{-\infty}^{+\infty} y(t)e^{-st}dt, \qquad s \in \mathbb{C}$$

defined when possible. It can be proven (see e.g. [1]) that there exist a real number α such that the function $Y\,(s)$ is well defined on the open half-plane $\mathrm{Re}(s) > \alpha$ *($s = \sigma + \iota\omega$ being the complex variable). For our purpose we will consider functions such that $\alpha \leq 0$, and restrict the domain of $Y\,(s)$ be $(0, +\infty)$.*

It can be seen that Laplace-mapping creates an isomorphism between the transformable functions $y(t)$ and their transformed function $Y\,(s)$. The possibility of facing the analysis of problems limiting our attention to the s-domain (called also frequency domain) instead of considering the t-domain is due to the inverse theorem that permit us to return from the Laplace-domain to the time-domain:

Theorem 15 *Let us consider $y(t) \in \mathcal{T}$, with Laplace-transform $Y\,(s)$. Suppose σ*

to be a real number in $Y(s)$ domain. Then for all real numbers t:

$$\mathfrak{L}^{-1}[y(t)](s) = y(t) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} Y(s)\, e^{st} ds.$$

Remark 16 *The previous theorem shows that $y(t)$ is uniquely determined by its Laplace transform $Y(s)$*

Some useful properties of Laplace-transform

In order to apply the analysis of a linear system in the frequency-domain, some important properties of the Laplace-transform are to be recalled here.

Let us consider two transformable function: $y_1(t)$, $y_2(t)$ and their transformed functions $Y_1(s)$, $Y_2(s)$ then:

a)

$$\mathfrak{L}[y_1(t) + \gamma y_2(t)](s) = F_1(s) + \gamma F_2(s), \qquad \gamma \in \mathbb{C}$$

which represents the **linearity of the Laplace mapping**.

b)

$$\mathfrak{L}[y_1(t-\tau)](s) = F(s-\gamma)\, e^{-s\tau}, \qquad \tau \in \mathbb{R}^+,$$

$$\mathfrak{L}[e^{\gamma t} y_1(t)](s) = F(s-\gamma), \qquad \gamma \in \mathbb{C},$$

these show how a translation of the function is transformed from one domain to the other.

c)

$$\mathfrak{L}[y_1(\gamma t)]\,(s) = \frac{1}{\gamma} F\left(\frac{s}{\gamma}\right),$$

which gives the **relation for the change of variable**.

d) The following is one of the most important properties for Laplace-transforms and concerns the **rule of transformation of the derivatives of a function from one domain to the other**:

$$\mathfrak{L}[y_1^{(n)}(t)]\,(s) = s^n F\,(s) - s^{n-1} f\left(0^+\right) - s^{n-2} \dot{f}\left(0^+\right) - \ldots - f^{(n-1)}\left(0^+\right) \qquad n \in \mathbb{N},$$

$$\mathfrak{L}[t^n y_1(t)]\,(s) = (-1)^n F^{(n)}\,(s) \qquad n \in \mathbb{N}.$$

e) Finally recall the **rule of transformation of integrals**. Suppose that $y\,(t)$ and its primitive:

$$p(t) = \int_0^t y\,(\xi)\,d\xi$$

belong to the transformable functions $\mathcal{T}$, and suppose that there exists a real number α such that $p(t) \to 0$ as $t \to 0$, then:

$$\mathfrak{L}[p(t)]\,(s) = \frac{1}{s} Y\,(s)$$

Convolution theorem

The Convolution theorem is a fundamental result of the Laplace transform theory; it allows for the representation of an input/output transfer function of a linear system in the frequency domain:

Theorem 17 *The Laplace transform of the convolution of two function* $y_1(t)$*,* $y_2(t)$ *belonging to* $\mathcal{T}$ *belongs itself to* $\mathcal{T}$ *and it is equal to the product of the transformed function* $Y_1(s)$*,* $Y_2(s)$*:*

$$\mathfrak{L}[y_1(t) * y_2(t)]\,(s) = Y_1\,(s)\,Y_2\,(s)\,. \tag{2.1}$$

For a proof see ([1]).

Finite differences

The finite differences (FD) method is used to reduce a differential equation (DE) to an algebraic one. Derivatives are transformed into finite differences and an approximation of the solution of DFs is found by simply solving linear systems. In our case the finite-difference equations obtained from the elastica and the plate equations, transformed into the Laplace domain will be used to synthesize an electrical network, the governing equations of which will be identical to the FD-equations of their mechanical counterpart. In particular the electrical network used will be circuits of two terminal networks. This procedure is similar to that one used in designing analog calculators, prior to the development of digital computers.

In order to present the FD- method, let us consider a linear n-order differential equation $g : \mathbb{R}^{n+1} \to \mathbb{R}$:

$$g\left(f\,(x)\,, f'\,(x)\,, f''\,(x)\,, ..., f^{(n)}\,(x)\right) = 0, \qquad x \in \mathbb{R}^m,$$

where the function f is the solution which we want to approximate, defined on a compact of $\mathbb{R}^m$ ([1]):

$$f : \mathcal{D} \subset \mathbb{R}^m \to \mathbb{R}.$$

It is known that when $n = 1$, the solution is determined once exactly n boundary conditions are assigned:

$$b_k\left(f(x), f'(x), f'(x), ..., f^{(n-1)}(x)\right) = 0, \qquad k = 1, 2, ..., n,$$

where x is suitably chosen in $\partial\mathcal{D}$ (which represents the boundary of $\mathcal{D}$). When $m = 2$ the boundary conditions are relations valid $\forall x \in \partial\mathcal{D}$.

The first step of the finite-differences method consists in a discretization of the domain of the solution f using a grid (we will choose a grid of uniform intervals for the elastica and a grid of rectangular elements of constant edge h for the plate)

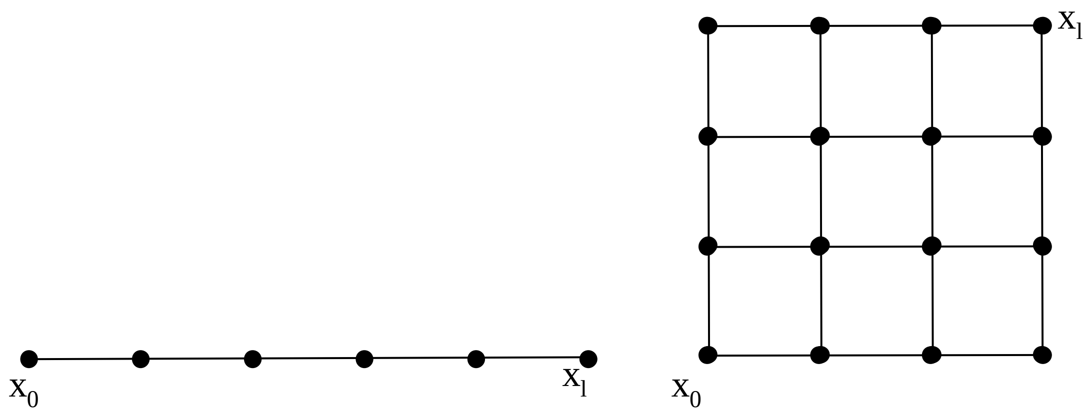


Figure 2.1: Exumples of FD one dimensional and two dimensional grids.

[1] m, in our case, equals 1 for the elastica and equals 2 for the plate equation.

so that a set of nodes can be associated to the domain ([2]):

$$x \to \{x_i\}, \qquad i = 1, .., l.$$

Hence the values which f assumes on the nodes of the grid are chosen as the approximating set for f:

$$f \to \{f(x_i)\} = \{f_i\}, \qquad i \in \mathbb{N}.$$

The approximations for the derivatives of the function f are realized using the relative differences upon the set $\{f_i\}$:

$$f^{(n)}(x_i) \to \Delta_f^{(n)}(x_i) \equiv \Delta_{f_i}^{(n)},$$

which are defined in the following subsection.

Finite differences for constant grids

Let consider a one dimensional domain: $x \in \mathcal{I}$ and a discretization $\{x_i\}$ of it. The first derivative of the function f can be defined in three different ways:

- Backward first finite-differences:

$$\overleftarrow{\Delta}'_{f_i} = \frac{f(x_i) - f(x_{i-1})}{(x_i - x_{i-1})} = \frac{f_i - f_{i-1}}{h}.$$

  From a geometrical point of view, we are approximating the first derivative with the slope of the straight line passing through (x_{i-1}, f_{i-1}) and (x_i, f_i) (see fig 2.2).

[2] Here l is the total number of the nodes of the grid.

- <u>Forward first finite-different</u>:

$$\overrightarrow{\Delta}'_f(x_i) = \frac{f(x_{i+1}) - f(x_i)}{(x_{i+1} - x_i)} = \frac{f_{i+1} - f_i}{h}.$$

The geometrical interpretation is the approximation of the first derivative by the slope of the straight line passing through (x_i, f_i) and (x_{i+1}, f_{i+1}) (see fig 2.2).

- <u>Centered central finite-different</u>:

$$\Delta'_f(x_i) = \frac{\overleftarrow{\Delta}'_{f_i} + \overrightarrow{\Delta}'_f(x_i)}{2} = \frac{f(x_{i+1}) - f(x_{i-1})}{2(x_{i+1} - x_{i-1})} = \frac{f_{i+1} - f_{i-1}}{2h}.$$

From a geometrical point of view, we are approximating the first derivative with the slope of the straight line passing through (x_{i-1}, f_{i-1}) and (x_{i+1}, f_{i+1}) (see fig 2.2). It is the average of the backward and forward finite differences. It can be seen to be more accurate than the others.The higher order finite

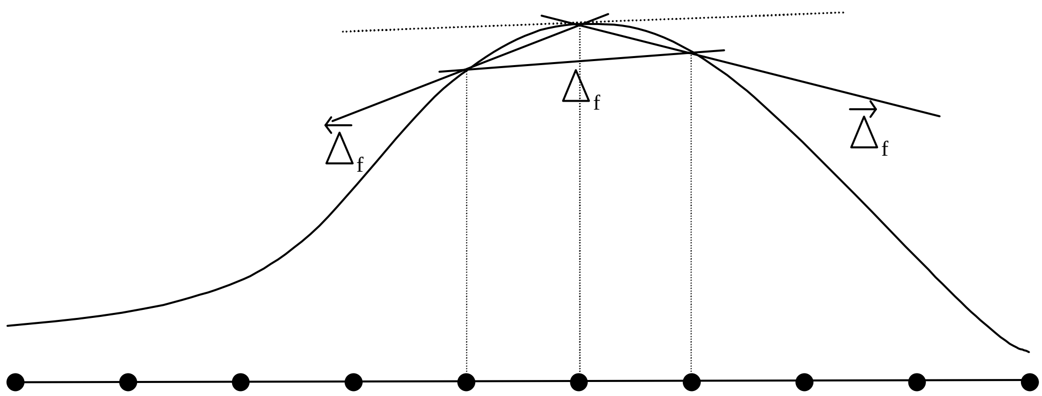


Figure 2.2: First finite differences.

differences can be easily obtained by a recursive procedure, i.e. the n-order

FDs is the first difference of the $(n-1)$-order one:

$$\Delta_{f_i}^{(n)} = \Delta'_f \left(\Delta_f^{(n-1)} (x_i) \right).$$

Remark 18 ***Example 19*** *As we will need it later, let us consider the calculation of the second centered difference:*

$$\begin{aligned} \Delta''_f (x_i) &= \overleftarrow{\Delta}'_f \left(\overrightarrow{\Delta}'_f (x_i) \right) = \\ &= \frac{\overrightarrow{\Delta}'_{f_i} - \overrightarrow{\Delta}'_{f_{i-1}}}{h} = \frac{f_{i+1} - 2f_i + f_{i-1}}{h^2}. \end{aligned}$$

Remark 20 *Let us consider a two dimensional domain $(x, y) \in \mathcal{I}$ and the rectangular, h-step grid on $\mathcal{I}$, which identifies the sampling nodes $\{x_i, y_j\}$. The mixed finite-differences, which approximate its relative derivatives, can be easily derived with the following recursive formula:*

$$\frac{\partial^{(n_x+n_y)} f(x,y)}{\partial x^{n_x} \partial y^{n_y}} \simeq \Delta_{f_x}^{(n_x,n_y)} (x_i, y_j) = \Delta_{f_x}^{(n_x)} \left(\Delta_{f_y}^{(n_y)} (x_i, y_j) \right).$$

Example 21 *Let us for instance consider the forward approximation of the first mixed derivative:*

$$\begin{aligned} \frac{\partial^2 f(x,y)}{\partial x \partial y} &\simeq \overrightarrow{\Delta}'_{f_x} \left(\overrightarrow{\Delta}'_{f_y} (x_i, y_j) \right) = \frac{\overrightarrow{\Delta}'_{f_y} (x_{i+1}, y_j) - \overrightarrow{\Delta}'_{f_y} (x_i, y_j)}{h} = \\ &= \frac{f(x_{i+1}, y_{i+1}) - f(x_{i+1}, y_j) - f(x_i, y_{j+1}) + f(x_i, y_j)}{h^2}. \end{aligned}$$

Approximate solution

The last step of the FD-method is the replacement of the differential equation by the algebraic finite-difference one obtained by substituting the approximations of its derivatives:

$$g\left(f\left(x_i\right),\Delta_f\left(x_i\right),\Delta_f^2\left(x_i\right),...,\Delta_f^{(n)}\left(x_i\right)\right)=0,\qquad i=1,..,l.$$

The approximation of the boundary-conditions are obtained by replacing the values of f and its derivatives on the boundary nodes of the grid.

$$b_k\left(f\left(\bar{x}_i\right),\Delta_f\left(\bar{x}_i\right),\Delta_f^2\left(\bar{x}_i\right),...,\Delta_f^{(n)}\left(\bar{x}_i\right)\right)=0,\qquad k=1,2,...,N.$$

where $\bar{x}_i$ are the boundary nodes, and N depends on the type of DE considered. Let us remark that the boundary nodes are the sets of the nearest $(n-1)$-nodes to the boundary $\partial\mathcal{D}$.

The solution of the algebraic system, obtained from the previous two sets of equations gives an approximation of the solution f on the nodes of the grid. Even if a theorem for the convergence of the approximate FD-solution to a generic DE cannot be proven in general, we expect to obtain an accuracy inversely proportional to the step h of the grid. In particular, the sampling, operated by the finite-difference method upon the solution, does not permit us to reproduce the wave lengths of f smaller than the step of the grid.

Remark 22 *For more detail about FD see [3].*

Chapter 3

Electrical circuits

We are interested in coupling electrical and mechanical waves, and we know that a deformation field on a piezoelectric material generates and electric field in it an vice versa. However, it is not possible to obtain an efficient energy exchange between the electrical and mechanical waves propagating in a macroscopic structure of piezoelectric material, because the propagating speed of an electric field, in such a material, equals light-speed and it is greater then the propagating speed of every deformation field, i.e. greater than sound-speed. This means that the electric field, generated by a deformation field on a PZT-system, is not supported and efficiently transmitted by the system itself, so that the energy cannot be efficiently transformed from one to the other form. This suggested to search an electric structure able to support a propagating wave, the speed of which can be controlled by electric parameters and then couple it to the mechanical device using the piezoelectric effect. Since some electric circuits are able to support the propagation of the electric potential associated with its nodes, the idea is to look for a circuit, the equation of which, is formally identical to the one for the mechanical device and to couple these two structures by piezoelectric actuators uniformly distributed. In order to understand how this is possible, we will derive a circuit analog for the equations of the elastica and the plate.

A circuit is constituted by lumped elements. Therefore we start to study the form of the general electromagnetic equations under the lumped assumption and subsequently we will present its limits. Then the physically realizable two terminal networks will be derived. Furthermore, a typical way of performing a circuital analysis will be shown in order to present the ideas that produced the searched circuital analog. Finally, the electronic way of performing the non physically realizable two-terminal-networks, which we need, will be presented.

Lumped hypothesis

When the time t, which the electromagnetic field needs in its propagation between two points of the region of interest, is much smaller than the time $t_{\min}$, in which the electric quantities we are considering can have a meaningful variation, then it is reasonable to assume that the light-speed is infinity. In this case, the physical dimensions of the region that we are considering are unnecessary in the model of the physical system, so that the lumped hypothesis can be assumed. Let us observe that if we are dealing with a region, the linear-dimensions of which are at most L, then the time t which the electromagnetic field spends to go beyond the entire region is:

$$t = \frac{L}{c},$$

(c being the light-speed); moreover let us remark that the characteristic time $t_{\min}$ of an application is simply related to the inverse of the highest mechanical frequency $f_{\max}$ which we wish to couple to electromagnetic fields:

$$t_{\min} = \frac{1}{2f_{\max}};$$

thus it is reasonable to consider the lumped approximation when:

$$t \ll t_{\min} \qquad \text{or} \qquad 2\frac{L}{c}f_{\max} \ll 1.$$

Example 23 *Let us consider a structure, which has linear dimension of ten meters at most:*

$$L = 10m$$

The light-speed in every material is very close to the light-speed in the vacuum:

$$c = 3 \cdot 10^8 \frac{m}{s}.$$

Hence the highest frequencies that we can consider in the lumped hypothesis are:

$$f_{\max} \ll \frac{c}{2L} = 15 \cdot 10^6 Hz.$$

Let us remark that the result found is about one thousand times higher than the maximum audible frequency.

From Maxwell equations to circuit models

When the Lumped hypotheses are assumed, the Maxwell equation can be written in its simplest form. Also different kinds of media can be characterized in a very elementary way, so that they can be used to realize the physical components of a circuit. In order to develop the lumped model, let us recall the distributed general model of the electromagnetic interaction in a region of the space. The following electromagnetic quantities must be defined:

- Electric field $\mathbf{E}$,
- Magnetic field $\mathbf{H}$,
- Electric induction $\mathbf{D}$: this quantity represents the effect that the electric field has upon polarizing the considered medium.
- Magnetic induction $\mathbf{B}$: this quantity represents the effect that the magnetic field has upon medium magnetization.
- conduction-current density $\mathbf{J}$.
- Electric external charge density ρ.

The relation among the quantities introduced are given by the Maxwell Equations:

$$\begin{aligned} &\nabla \times \mathbf{E} = -\tfrac{\partial \mathbf{B}}{\partial t}, \\ &\nabla \times \mathbf{H} = \mathbf{J} + \tfrac{\partial \mathbf{D}}{\partial t}, \\ &\nabla \cdot \mathbf{B} = 0, \\ &\nabla \cdot \mathbf{D} = \rho. \end{aligned} \tag{3.1}$$

Let us assume a linear, permanent, homogenized and isotropic medium; this implies that the response to an applied electromagnetic field is independent of the field orientation, the instant of application and the material point. Moreover the effect induced on the medium is proportional to the excitation, and the superposition principle holds. Under these hypotheses, the behavior of the medium can be characterized by the following scalar constants:

- Conductivity γ,
- permittivity constant ε,
- permeability constant μ,
- Electric-charge density ρ:

and its constitutive relations can be written in the following way:

$$\begin{aligned} \mathbf{D} &= \varepsilon \mathbf{E}, \\ \mathbf{B} &= \mu \mathbf{H}, \\ \mathbf{J} &= \gamma \left(\mathbf{E} - \mathbf{E}_o \right) + \mathbf{J}_{o.} \end{aligned} \tag{3.2}$$

where $\mathbf{E}_o$ and $\mathbf{J}_o$ are impressed quantities.

The wave equation for the electric field (or for the magnetic one) can be easily derived: applying a curl operator on the Maxwell equation $(3.1)_1$, We get:

$$\nabla \times \nabla \times \mathbf{E} = -\frac{\partial}{\partial t} \left(\nabla \times \mathbf{B} \right),$$

where under suitable regularity conditions, the spatial and time derivatives can be interchanged. Moreover knowing that:

$$\nabla \times \nabla \times \mathbf{E} = \nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$$

and using the constitutive equations (3.2) the damped wave equation for $\mathbf{E}$ is obtained:

$$\nabla^2 \mathbf{E} - \mu\gamma \frac{\partial \mathbf{E}}{\partial \mathbf{t}} - \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = \mathbf{f}(\rho, \mathbf{E}_o, \mathbf{J}_o),$$

from which it can be seen that the electromagnetic (or light) speed into the media is:

$$c = \frac{1}{\sqrt{\mu\varepsilon}}.$$

The Lumped hypothesis implies that the light-speed is infinite, i.e. requires that

$$\mu\varepsilon = 0.$$

Let us underline that the previous condition can be verified in three different ways, which define different kinds of propagation regions, that can be used for obtaining lumped elements.

Region 1

In this kind of region both μ and ε equals zero, this implies that the electric and magnetic induction $\mathbf{D}$ and $\mathbf{B}$ vanish. Since the electric and magnetic energydensities in such a region are defined as:

$$\text{Electric energy: } \mathcal{E}_e = \tfrac{1}{2}\mathbf{D}\cdot\mathbf{E} = \tfrac{\varepsilon}{2}\left|\mathbf{E}\right|^2,$$

$$\text{Magnetic energy: } \mathcal{E}_m = \tfrac{1}{2}\mathbf{B}\cdot\mathbf{H} = \tfrac{\mu}{2}\left|\mathbf{H}\right|^2,$$

there is no stored electric or magnetic energy.

Moreover, since $\mathbf{D}$ equals zero, the Maxwell equation $(3.1)_4$ implies that electric charge-density vanishes, hence the electric current that goes in that region equals the one that goes out. With current we mean the flux of electric charge per unit time that passes through a closed surface that bounds the region.

Furthermore, since $\mathbf{B}$ equals zero, the Maxwell equation $(3.1)_1$ becomes:

$$\nabla\times\mathbf{E} = 0,$$

which means that $\mathbf{E}$ is conservative and can be represented by a scalar, potential-function (electrical tension), so that there is no ambiguity in defining the difference of potential between two points in the region.

A further specification of the properties of the region 1 can be obtained considering different values for γ.

Region 1A:

This subregion is characterized by the absence of impressed quantities $(\mathbf{E}_o, \mathbf{J}_o)$, and the vanishing of the conductivity:

$$\gamma = 0,$$

so that the third of the constitutive relations (3.2) becomes:

$$\mathbf{J} = 0.$$

As can be seen, the phenomena of transformation of energy from electrical form into another are taken into account by the term

$$\mathbf{J} \cdot \mathbf{E},$$

since this term equals zero, in a 1A region there is no electric energy loss.

Remark 24 *In the lumped model this region represents a vacuum, i.e. the environment in which all the other regions (elements of the circuit) are immersed.*

Region 1B:

This subregion also is characterized by the absence of impressed quantities $(\mathbf{E}_o, \mathbf{J}_o)$, but the conductivity tends to infinity:

$$\gamma = \infty,$$

from the equation of $(3.2)_3$, it is evident that the value of $\mathbf{E}$ in this region must vanish,

$$\mathbf{E} = \mathbf{0},$$

since the electric potential is defined as the scalar function v such that:

$$\nabla v = \mathbf{E}$$

we have that the electric potential in this region is constant

Remark 25 *It is evident that the region 1B models the short circuit, that is, the connections among the other different regions (or elements).*

Region 1C:

Also in this subregion $(\mathbf{E}_o, \mathbf{J}_o)$ vanishes, moreover γ has finite values

$$\gamma \in (0, \infty),$$

which implies an irreversible energy transformation from the electrical form to another one. The quantity of energy that is dissipated here is

$$\mathcal{P}_e = \gamma \left|\mathbf{E}\right|^2.$$

Remark 26 *This kind of region is used to realize one of the most important lumped component: the resistor.*

Region 1D:

It is characterized by:

$$\gamma = 0,$$

$$\mathbf{E}_o = 0,$$

$$\mathbf{J}_o \neq 0.$$

This region is able to impose a reversible energy-exchange from the environment into an electric form. It impresses a current distribution $\mathbf{J}_o$ and the transformed power per unit volume is:

$$\mathcal{P}_e = \mathbf{J}_o \cdot \mathbf{E}_o.$$

Remark 27 *In the lumped model, this region will furnish the independent current generator.*

Region 1E:

The properties of this region are:

$$\gamma = \infty,$$

$$\mathbf{E}_o \neq 0,$$

$$\mathbf{J}_o = 0.$$

As in the previous case it is possible to impose a reversible energy transformation. The power density supplied in this case is:

$$\mathcal{P}_e = \frac{\varepsilon}{2} |\mathbf{E}|^2 = \frac{\varepsilon}{2} |\nabla v|^2 .$$

Remark 28 *This is used to synthesize the independent voltage generator.*

Region 2

In this kind of region only ε equals zero, while μ has a finite non zero value. This implies, from the formulas (3.2), that only $\mathbf{D}$ vanishes. Consequently (see equations (3.1)) there is no charge density stored and the current entering the region equals the current exiting. In addition, since $\mathbf{B}$ is non zero, it is present a non zero density of magnetic energy is present

$$\mathcal{E}_m = \frac{\mu}{2} |\mathbf{H}|^2 .$$

Remark 29 *In this region it is not possible to define an electric potential function. In the case in which γ equals zero a simple relation links the electric field to the density current*

$$\nabla^2 \mathbf{E} = \mu \frac{\partial \mathbf{J}}{\partial t}. \tag{3.3}$$

Remark 30 *The lumped element inductor is physically constructed inducing electromagnetic fields in media which can be regarded as being of region-B type.*

Region 3

This is the dual of the region 2 and is characterized by a finite non-zero value of ε and a vanishing value of μ. Thus the magnetic induction $\mathbf{B}$ and the stored magnetic energy vanishes (see formula 3.2). Conversely there is a finite value for the electric energy:

$$\mathcal{E}_e = \frac{\varepsilon}{2} |\mathbf{E}|^2 ,$$

moreover, since $\mathbf{B}$ equals zero, it is possible to define an electric potential function v. Let us remark that for region C, in which γ also equals zero, the relation between electric field and current density is:

$$\nabla \cdot \mathbf{J} = -\varepsilon \frac{\partial}{\partial t} (\nabla \cdot \mathbf{E}) = -\varepsilon \frac{\partial}{\partial t} \left(\nabla^2 v \right). \tag{3.4}$$

Remark 31 *This region is used to obtain lumped capacitors.*

Lumped system

As a result, the hypothesis that the light-speed can be assumed to be infinite implies that the domain in which we are considering the electromagnetic phenomena can be regarded as an opportune union of the different regions presented. In particular, the 1A-region, that models the vacuum, is the medium in which all the other regions are immersed. Let us remark that in this region it is possible to define, without ambiguity, the electric-tension between two points and moreover it does not store electric charge. Since the 1B-region has a constant potential and does not store charge (perfect conductor) it can be considered as the "connector" between the other regions. In particular, a lumped element can be regarded as a generic region immersed in the 1A-region and connected to the rest of the domain by two terminals of an 1B-region.

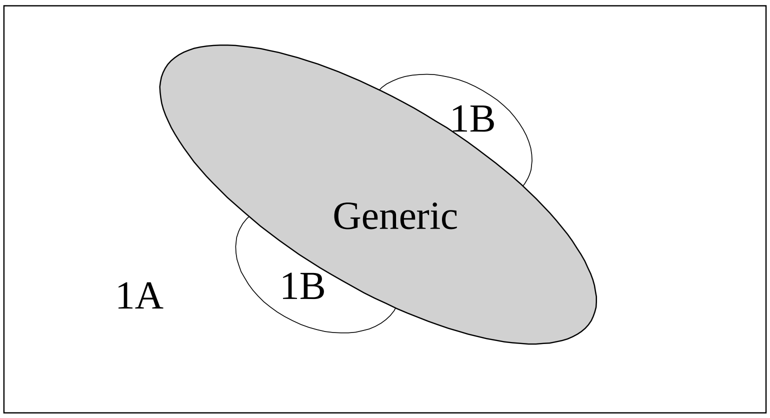


Figure 3.1: Lumped element.

This means that it is possible to speak about the electric-tension on a lumped element as the potential difference between its terminals. Moreover it is possible to associate a current passing through the element. In fact let us consider a surface surrounding the element and intersecting only its terminals; the electric charge stored by the element is given by

$$Q = \int_{Vol} \rho dV.$$

From the Maxwell equation $(3.1)_4$ and using the divergence theorem we obtain

$$Q = \int_{Vol} \nabla \cdot \mathbf{D} dV = \int_{Sur} \mathbf{D} \cdot \mathbf{n} dV.$$

Because in a 1B region, the electric induction $\mathbf{D}$ equals zero, the electric charge stored in the element vanishes. This implies that the net current entering the elements equals zero:

$$I = \int_{Vol} \mathbf{J} \cdot \mathbf{n} dV = \frac{\partial}{\partial t} \int_{Vol} \rho dV = 0. \tag{3.5}$$

Therefore, the current entering in a terminal is equal to the current exiting, so that it is possible to associate a current to each element.

Kirchhoff laws The Kirchhoff laws can be derived considering a closed surface $\mathcal{S}$ and a closed line $\mathcal{L}$ which intersect the elements of the domain only in the 1A or 1B regions. Using a reasoning similar to the one used before, and considering the surface $\mathcal{S}$ bounding several elements, we can derive the first Kirchhoff equation.

The algebraic sum of the currents entering $\mathcal{S}$ equals zero.

Moreover, since the electric field is irrotational along the closed line $\mathcal{L}$, its integral on $\mathcal{L}$ equals zero. Since this integral equals the algebraic sum of the tensions of the terminals 1B, which are intersected by $\mathcal{L}$, we obtain the second Kirchhoff equation:

The algebraic sum of the terminal-tensions intersected by a closed path equals zero.

Constitutive relation Let us finally derive a formulation for the constitutive relations of lumped elements from the properties of their relative regions:

Resistor: The microscopic constitutive relation for a lumped element, the medium of which belongs to the region 1C is:

$$\mathbf{J}(\mathbf{P},t) = \gamma \mathbf{E}(\mathbf{P},t), \tag{3.6}$$

where $\mathbf{P}$ is the generic material-point of the region and t the time variable. Since the medium is linear, the electric field $\mathbf{E}$, for each given material-point, is proportional to the potential difference v between the edges of the element. Thus the electric field can be represented as the product of a function $\mathbf{f}$ depending only upon the material point $\mathbf{P}$ and the electric potential v depending only upon time t:

$$\mathbf{E}(\mathbf{P},t) = \mathbf{f}(\mathbf{P}) v(t).$$

Similarly the current density $\mathbf{J}$ is proportional to the current i through its edges by mean a function $\mathbf{g}(\mathbf{P})$:

$$\mathbf{J}(\mathbf{P}, t) = \mathbf{g}(\mathbf{P}) i(t).$$

Thus, from the micro constitutive-equation (3.6), the following tension-current relation can be derived:

$$v(t) = \frac{1}{\gamma} \left| \frac{\mathbf{g}(\mathbf{P})}{\mathbf{f}(\mathbf{P})} \right| i(t).$$

The coefficient of the right hand side depends only upon the geometry of the component, so that it must be the constant value R that characterizes the resistor:

$$R = \frac{1}{\gamma} \left| \frac{\mathbf{g}(\mathbf{P})}{\mathbf{f}(\mathbf{P})} \right|.$$

Consequently, the macroscopical constitutive-relation for the element has been obtained:

$$v(t) = Ri(t).$$

Inductor: Using the relation (3.3) obtained for the region B, the considerations presented in the previous case lead to the following relation:

$$v(t) = \mu \left| \frac{\mathbf{g}(\mathbf{P})}{\nabla^2 \mathbf{f}(\mathbf{P})} \right| \frac{\partial i(t)}{\partial t},$$

and the characteristic constant of the inductor becomes:

$$L = \mu \left| \frac{\mathbf{g}(\mathbf{P})}{\nabla^2 \mathbf{f}(\mathbf{P})} \right|;$$

the macroscopic constitutive relation gives:

$$v(t) = L\frac{\partial i(t)}{\partial t}.$$

Capacitor: The relation between the electric field and the current density in a C-region is given by the equation (3.4):

$$\nabla \cdot \mathbf{J} = -\varepsilon \frac{\partial}{\partial t}(\nabla \cdot \mathbf{E}),$$

hence the following relation between the tension and the current on the element can be derived:

$$i(t) = -\varepsilon \frac{|\nabla \cdot \mathbf{f}(\mathbf{P})|}{|\nabla \cdot \mathbf{g}(\mathbf{P})|}\frac{\partial}{\partial t}(v(t)).$$

The macroscopic constitutive-relation for the capacitor becomes:

$$i(t) = C\frac{\partial v(t)}{\partial t}.$$

Remark 32 *In this subsection we have deduced the constitutive relations for circuit elements postulating, at the micro-level, the Maxwell equations and accepting the lumped elements hypothesis. This procedure finds its mechanical counterpart in the reasoning, based on Saint-Venant results, which allow us to determine the constitutive equations for elastica starting from Cauchy 3-D theory.*

Circuit model

From a topological point of view a circuit can be described by its associated graph, the algebraic structure of which is given by the Kirchhoff laws. Thus let us introduce same useful elements of the graph theory.

Remark 33 *Although the theory of graphs has been extensively used by engineers working in circuit analysis, it is evident that its methods and ideas are of great relevance also in the theory of mechanical structures. In particular, truss beams, truss plates and every multicomponent structure has a mechanical behavior which depends on:*

I) the properties of all structural members

II) the interconnection geometry.

For instance, the same structural members and constraints can form statically undetermined or determined structures depending on the chosen interconnection modalities. We refrain from discussing rigorously all parallels between the theory of circuits and the theory of structures. However we will underline those similarities which will seem to us more meaningful and striking.

Elements of graph theory for an electric circuit

A **graph** is a set of **nodes** connected by several oriented **edges** or **branches**. The **degree of a node** is the number of edges that reach this node. For every circuit the associated graph will be considered.

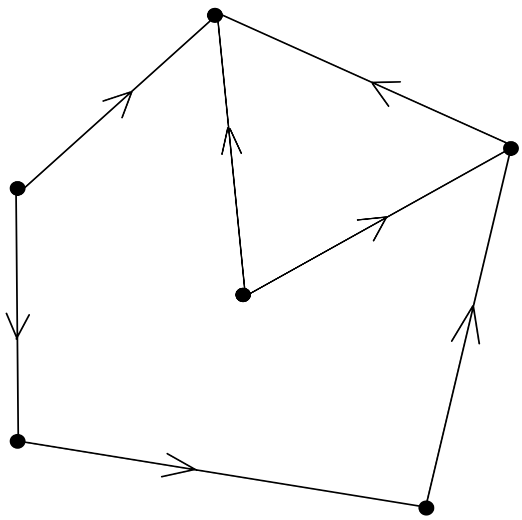


Figure 3.2: Topological representation of a circuit.

The state of a circuit is known once the tensions between the nodes of each edge and the currents through each edge are known. Hence a circuit can be characterized by a vector of currents and a vector of electric-tensions.

Remark 34 *Since the meaningful tensions in a circuit are just the differences of electric potential and not absolute quantities, it is often useful to chose a reference node (ground) and define the tensions at each node as the voltage differences between the nodes and the ground, so that it is possible to associate unequivocally a tension to each node of the circuit.*

Remark 35 *A circuit is a finite dimensional Lagrangian system, exactly as truss structures.*

In order to approach the analysis of a circuit, let us introduce the following topological concepts:

Definition 36 *A **loop** is a closed line that intersects a set of edges, each one two times.*

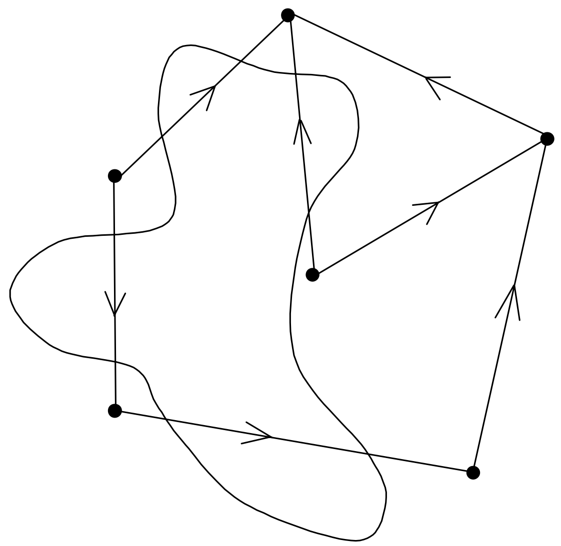


Figure 3.3: Loop.

Remark 37 *A loop is often improperly confused with the closed path that can run along the circuital edges which it intersects; this is not correct, in fact the edges of a circuit are physically constituted by electric elements, and we saw that it is not always possible to assign a potential on the entire element but only on its edges. However, in what follows, it is convenient sometime to use the second definition that is more intuitive.*

Definition 38 *A **cut** is a closed line which cuts each of the inclosed branches at most one time. Let us remark that this line divides the set of the branches in a circuit into two disconnected subsets.*

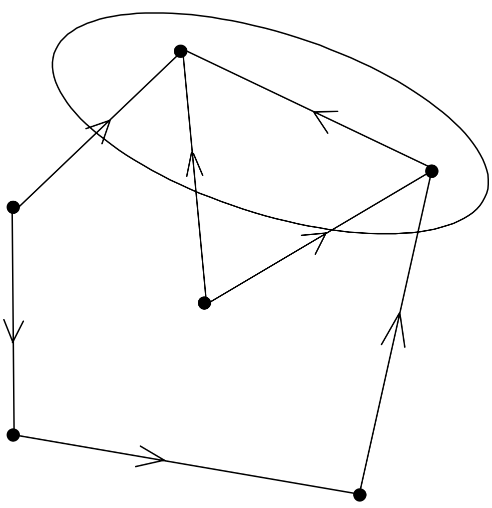


Figure 3.4: Cut.

Definition 39 *A **tree** is a subset of connected edges of the circuit that reaches all the nodes of the circuit, but does not create any closed loop (see fig. 3.5).*

Definition 40 *A **co-tree** is the complement set of the tree, i.e. all the branches that do not belong to a tree belongs to a co-tree (see fig. 3.5).*

Definition 41 ***Fundamental loop** is obtained by adding a co-tree branch to a tree.*

Definition 42 *A **Fundamental cut** is a cut which contains only one branch of the tree. Let us remark that the absence of a closed loop in a tree implies that the elimination of a branch in a tree divides the circuit - edges set into two subsets, so that it is always possible to create a cut intersecting only one tree-branch and other edges of the co-tree (see fig. 3.5).*

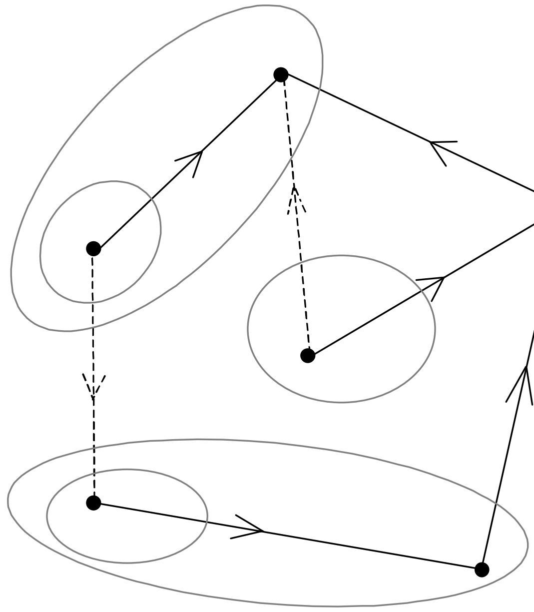


Figure 3.5: Tree (black lines), co-tree (dashed lines), fundamental cuts.

It is now easy to define the Kirchhoff laws, which give algebraic structure to the graph.

I Kirchhoff law: The sum of the currents entering a cut equals zero.

II Kirchhoff law: The sum of the tensions of the nodes intersected by a loop equals zero.

Remark 43 *These laws are also called "the equilibrium equation for the currents" and "the equilibrium equation of the tensions", respectively.*

Remark 44 *The first and second Kirchhoff laws find their analogs in the theory of* *truss structures. Indeed the first one corresponds to the balance of forces at a given joint, while the second to the Kinematic compatibility condition on admissible displacements.*

Because of the Kirchhoff equations, not all the currents and the tensions in a circuit are independent. In order to solve the circuit several sets of independent quantities can be found. We choose to use the **method of cuts**, which defines a set of independent tensions as unknowns of the problem and to use a set of independent current-equilibrium equations to obtain the solution of the circuit.

Solution method

As a result of the absence of loops in a tree, the second Kirchhoff law implies that it is impossible to express a tension of one of its nodes by means of tensions on the others, so that the set of the tensions in a tree is independent. Moreover, adding each branch of the co-tree to the tree, the fundamental loops are obtained and the tension equilibrium-law permits us to express the co-tree tensions as a function of the tensions in the tree. In conclusion: **the tensions of the nodes in a tree is a complete set of independent tensions of the circuit**.

It can be seen that as consequence of the fact that one and only one edge for each fundamental cut belongs to the tree of the circuit, the currents in the co-tree are independent; this implies that **the current-equilibrium equations for the fundamental cuts are independent**.

Remark 45 *Let us observe that the definition of fundamental cut implies that the number of the edges in the tree equals the number of fundamental cuts.*

In conclusion, if we are dealing with a circuit of n nodes, a set of $n-1$ independent equations for the currents (one for each fundamental cut), and a set of $n-1$ independent unknowns (node tensions referred to the ground) is determined once a tree is chosen. In order to solve the circuit we have to find the relation between the tensions and the currents in the circuit, which is given by the constitutive equations of the circuit elements.

The fundamental hypothesis, that is assumed for the constitutive relations of the elements in the circuit, is that **all the edges are constituted by two-terminal-networks**. This implies that the voltage difference between two nodes depends only on the current between the nodes; or vice versa: the current through each edge depends only upon the voltage difference between the two adjacent nodes. Hence let us define what a two terminal network is, which are the properties it can have, and the fundamental two-terminal networks used.

Two terminal network model

A two terminal network (TTN) is, in general, an element the state of which can be characterized by two quantities: (v, i). The first one, v, is called across quantity and, in our case, it is represented by the tension between the terminals. The second one, i, is called through quantity, and, in the electric case, is represented by the current which passes through the two terminal network.

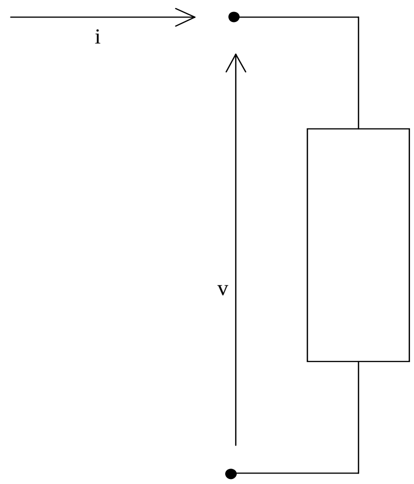


Figure 3.6: Two terminal network.

Moreover in a TTN, the quantities (v, i) are related by a function :

$$f : \mathcal{S} \to \mathcal{S}, \qquad f \in \mathcal{D}$$

Where $\mathcal{S}$ is the signal space to which i and v belong, defined as:

$$\mathcal{S} : \{x \in \mathcal{C}^\infty, x(t) = 0 \qquad \text{when } t \leq 0\},$$

and $\mathcal{D} \supset \mathcal{S}$ is the space of the distributions with support included in $(0, +\infty)$. For more details see [2].

Since the relation f is characteristic of the two terminal network, the TTN can be described in terms of the properties of the function f.

Definition 46 *The instantaneous power expended by the TTN is the function* $p : \mathcal{S} \to \mathcal{S}$*:*

$$p(t) = i(t)\, v(t).$$

Definition 47 *The energy delivered to the TTN at time* t *is the function* $\mathcal{E} : \mathcal{S} \to \mathcal{S}$*:*

$$\mathcal{E}(t) = \int_{-\infty}^{t} p(\xi)\, d\xi.$$

Typical two terminal networks

The input\output relation for some fundamental electric TTN, which will be used in our synthesis are presented here:

- Resistor (see fig (**??**)):

$$v(t) = \pm R i(t), \qquad R \in \mathbb{R}^{+}.$$

- Inductor (see fig (3.8)):

$$v(t) = \pm L \frac{di(t)}{dt}, \qquad L \in \mathbb{R}^{+}.$$

- Super-inductor (see fig (3.9)):

$$v(t) = \pm D \frac{d^2 i(t)}{dt^2}, \qquad D \in \mathbb{R}^{+}.$$

- Capacitor (see fig (3.10)):

$$i(t) = \pm C \frac{dv(t)}{dt}, \qquad C \in \mathbb{R}^{+}.$$

- Super capacitor (see fig (3.11)):

$$i(t) = \pm F \frac{dv(t)}{dt}, \qquad F \in \mathbb{R}^{+}.$$

- Current generator (see fig (3.12)):

$$i(t) = \bar{\imath}(t), \qquad \bar{\imath}(t) \in \mathcal{S}.$$

- Voltage generator (see fig (3.13)):

$$v(t) = \bar{v}(t), \qquad \bar{v}(t) \in \mathcal{S}.$$

- Norator (see fig (3.14)):

$$\begin{aligned} v(t) &\in \mathcal{S}, \\ i(t) &\in \mathcal{S}. \end{aligned}$$

- Nullator (see fig (3.15)):

$$\begin{aligned} v(t) &= 0, \\ i(t) &= 0. \end{aligned}$$

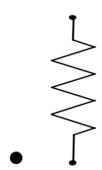


Figure 3.7: Resistor

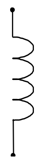


Figure 3.8: Inductor

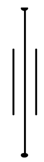


Figure 3.9: Supe-inductor

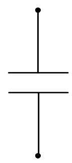


Figure 3.10: Capacitor

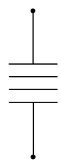


Figure 3.11: Super-capacitor

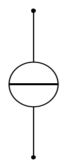


Figure 3.12: Current generator

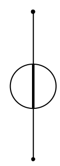


Figure 3.13: Voltage generator

Figure 3.14: Norator

Figure 3.15: Nullator

Some properties

In order to classify the TTN let consider the following properties:

- **Linearity**: This is one of the two fundamental properties that is required so as to find a representation for a TTN. It allows the application of the principle of superposition, which requires:

  - The response r of several-excitations sum $(\sum e_i)$ must be equal to the sum of the responses r_i of the single excitations.
  - The response to an α-scaled excitation αe_i equals the response to e_i times the scale α In formulas:

$$f(e_1 + \alpha e_2) = f(e_1) + \alpha f(e_2), \qquad \forall \alpha \in \mathbb{R}, \quad \forall e_1, e_2.$$

Remark 48 *The resistor, the inductor and super-inductor, the capacitor and super-capacitor, the nullator and the norator are all linear, while the voltage and current generator are not.*

- **Time invariance**: this is the second important requirement for the TTN representation. It means that the behavior of the TTN does not depend on time. In other words, considering a fixed instant τ and the response

$$r_\tau = f(e(\tau)),$$

  Then:

$$r_{t+\tau} = f(e(t+\tau)), \qquad \forall t, \tau \in \mathbb{R}.$$

Remark 49 *A voltage or current generator is not time-invariant, since it can generate a variable signal, while the other presented TTN are.*

- **Passivity**: this implies that the TTN cannot deliver energy to the external world. In other words:

$$\mathcal{E}(t) = \int_{-\infty}^{t} i(\xi)\, v(\xi)\, d\xi \geq 0 \qquad \forall i(\xi)\, v(\xi) \in \mathcal{S}, \quad \forall t \in \mathbb{R}.$$

Example 50 *Let consider the energy associated to the TTN considered:*

-
  - Since the energy delivered to a resistor is:

    $$\mathcal{E}_R(t) = \int_{-\infty}^{t} i(\xi) v(\xi)\, d\xi = \pm R \int_{-\infty}^{t} i^2(\xi) d\xi,$$

    only the positive resistors are passive TTN.

  - The energy for an inductor is:

    $$\mathcal{E}_L(t) = \int_{-\infty}^{t} i(\xi) v(\xi)\, d\xi = \pm L \int_{-\infty}^{t} \frac{di(\xi)}{d\xi} i(\xi) d\xi = \pm \frac{L}{2} i^2(t),$$

    and hence only the positive inductors are passive.

  - The energy for a super-inductor is:

    $$\mathcal{E}_L(t) = \int_{-\infty}^{t} i(\xi) v(\xi)\, d\xi = \pm D \int_{-\infty}^{t} \frac{d^2 i(\xi)}{d\xi^2} i(\xi) d\xi \gtreqless 0.$$

    Hence a super-inductor is not passive.

  - The energy delivered to a capacitor is:

    $$\mathcal{E}_C(t) = \int_{-\infty}^{t} i(\xi) v(\xi)\, d\xi = \pm C \int_{-\infty}^{t} \frac{dv(\xi)}{d\xi} v(\xi) d\xi = \pm \frac{C}{2} v^2(t).$$

    Again only the positive capacitors are passive.

– The energy delivered to a super-capacitor is

$$\mathcal{E}_C(t) = \int_{-\infty}^{t} i(\xi)v(\xi)\,d\xi = \pm F \int_{-\infty}^{t} \frac{d^2 v(\xi)}{d\xi^2} v(\xi) d\xi \gtreqless 0.$$

Consequently super-capacitor not passive.

– The nullator is obviously passive, while norator, current and voltage generator trivially are not.

Remark 51 *Current and voltage generators are the only active elements realizable using the regions introduced in the previous subsection.*

In order to introduce the losslessness property, we need to define the concept of an augmented two terminal network.

Definition 52 *If we consider a TTN f, its augmented TTN f_a is a two terminal network such that for each pair (i, v) satisfying f, the pair $(v + i, v)$ satisfies f_a. An augmented TTN can be regarded as a two terminal network constituted by the series connection of the TTN itself and a resistor of unit value.*

Solvability: a TTN network is said to be solvable if and only if for every excitation function $e \in \mathcal{S}$, there exist a unique pair (i, v), satisfying the characteristic function f such that

$$i + v = e$$

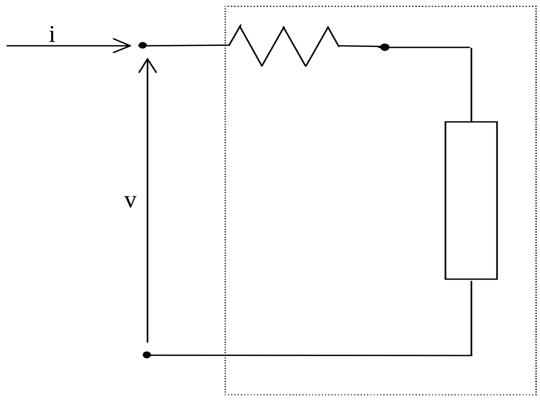


Figure 3.16: Augmented TTN

This means that there is a unique pair (i, v) satisfying f, which is the unique solution for the simple circuit realized connecting the TTN to a generic voltage generator of value e.

Remark 53 *The nullator and the norator are not solvable, in fact the first one satisfies the relation only when* $e = 0$*, while the second one has an infinite number of pairs of* (i, v)*, which satisfies it.*

- **Losslessness**: this means that the net energy delivered by the TTN equals zero:

  - TTN is solvable,
  - TTN is passive,
  - $\forall i(t), v(t) \in \mathcal{S} \Rightarrow \mathcal{E}(\infty) = 0.$

Remark 54 *The positive inductor and capacitor are the unique lossless elements.*

- **Reciprocity**: This implies that the system is independent of the exchange between the excitation and the response:

$$\forall (v_1, i_1), (v_2, i_2) \in \mathcal{S} \Rightarrow v_1 * i_1 = v_2 * i_1$$

Where the symbol'$*$' represents the convolution defined as:

$$g_1 * g_2 = \int_{-\infty}^{+\infty} g_1 (t - \xi) g_2 (\xi) C.$$

Remark 55 *The resistor, the inductor and the super-inductor and the capacitor and super-capacitor and the nullator are examples of reciprocal elements, while the norator is not.*

Time domain representation of a TTN

Linearity and time-invariance permit us to consider the time-domain representation of the relation f using the Schwartz theorem:

Theorem 56 *Let us consider a linear and time invariant TTN represented by a mapping $f : \mathcal{S} \to \mathcal{S}$. The TTN-response $r(t)$ which belongs to the space $\mathcal{D}$ can be obtained as the convolution of the excitation $e(t) \in \mathcal{S}$ and a kernel function $y(t)$:*

$$r(t) = y(t) * e(t) = \int_{-\infty}^{\infty} y(t - \xi) e(\xi) \, d\xi.$$

Remark 57 *Not all of the considered TTN have a time domain representation. In fact the voltage and the current generator, which do not satisfy the linear and time-invariance properties, cannot be represented as a convolution of an excitation* *and a transfer function. We expected this since they represent the actions of the external world on a circuit.*

Circuit analysis

Once the typical TTN has been introduced, a further development of the circuit analysis already introduced is possible. Let us first deal with a generic circuit of n nodes connected using two-terminal-networks constituted only by resistors, inductors, capacitances and current generators. Once a tree has been chosen on the graph of the circuit, $n-1$ equilibrium-equations for the currents can be written (one for each fundamental cut). Let us underline that the current generators will represent the non homogenous terms of the equilibrium equations. Afterwards, as all the TTN are solvable, substituting the constitutive relations of the other components into the equilibrium equations, a linear integro-differential system of $n-1$ equations in $n-1$ unknowns is obtained, where the unknown are the tensions of the tree-edges (or equivalently the potential of the nodes, once a ground is chosen). The solution of the system is given once the initial conditions for the inductors and the capacitors relations are fixed, although it is not easy to calculate the solution of the system because of the presence of integral and differential operators.

Dealing with Laplace-transformable quantities, it is useful to consider the frequency-domain analysis of this circuit in the frequency domain. In fact the L-transform of the integral and the differential operators become simply a product or a division by the L-variable s.

Since both the Laplace operator and the Kirchhoff-equations are linear, the L-transform of the equilibrium-equation for each fundamental cut is equal to the equilibrium-equation of the L-transformed currents in the circuit:

$$\mathfrak{L}\left[\sum_{k=1}^{n-1} i_k(t) = \sum_{l=1}^{m} \bar{\imath}_l(t)\right](s) \Rightarrow \sum_{k=1}^{n} \mathfrak{L}\left[i_k(t)\right](s) = \sum_{l=1}^{m} \mathfrak{L}\left[\bar{\imath}_l(t)\right](s)$$
$$\Rightarrow \sum_{k=1}^{n} I_k(s) = \sum_{l=1}^{m} \bar{I}_l(s)$$

Let us observe that the L-transform of the m-current-generator terms will simply become the non-homogenous terms of the equilibrium-equation for the transformed currents. Moreover, also the TTN constitutive relations are linear, so that it is possible to analyze the entire problem considering the L-transforms of the tensions and the currents and their constitutive relations instead of their related time-dependent quantities. The advantage of this approach is that the constitutive relations of the TTN, in the frequencies domain, are simpler and the integro-differential system becomes an algebraic system. Once the complete solution for a circuit is found, the Laplace inverse-theorem permits us to find the time domain solution.

Remark 58 *If a voltage generator is present in the circuit, its current is unknown. Its applied tension will supply a non-homogenous term in the law of equilibrium of tensions. Since the Laplace transform method can be applied also to* *non-homogenous linear systems, the transformed equations in the frequency domain will be correspondingly non homogenous.*

Remark 59 *This circumstance finds a parallel in the theory of structures. Indeed a pin joint connecting a beam to the "ground" is a constraint which imposes a displacement and leaves unknown the needed exerted constraint reaction.*

Remark 60 *Let us observe that only if the number of norators in a circuit equals the number of nullators is the system solvable; in fact while the norators do not introduce any constitutive equation, the nullator is characterized by two of them. Hence, if there are the same numbers of the two components the system is still determined and it can be subject to the previous analysis, Let us underline explicitly that since both of them are linear, the analysis in the Laplace domain is still valid.*

Let us finally consider the Laplace transform relations for the TTN considered:

- Resistor:

$$V(s) = \pm R I(s), \qquad R \in \mathbb{R}^+.$$

- Inductor

$$V(s) = \pm L\left[sI(s) - i\left(0^+\right)\right], \qquad L \in \mathbb{R}^+.$$

- Super-inductor

$$V(s) = \pm D\left[s^2 I(s) - s\dot{i}\left(0^+\right) - i\left(0^+\right)\right], \qquad D \in \mathbb{R}^+.$$

- Capacitor

$$I(s) = \pm C\left[sV(s) - V\left(0^+\right)\right], \qquad C \in \mathbb{R}^+.$$

- Super-capacitor

$$I(s) = \pm F\left[s^2V(s) - s\dot{V}\left(0^+\right) - V\left(0^+\right)\right], \qquad F \in \mathbb{R}^+.$$

- Current generator

$$I(s) = \bar{I}(s), \qquad \bar{I}(s) \in \mathcal{S}.$$

- Voltage generator

$$V(s) = \bar{V}(s), \qquad \bar{V}(s) \in \mathcal{S}.$$

- Norator

$$\begin{aligned} V(s) &\in \mathcal{S}, \\ I(s) &\in \mathcal{S}. \end{aligned}$$

- Nullator

$$\begin{aligned} V(s) &= 0, \\ I(s) &= 0. \end{aligned}$$

Remark 61 *For more details about circuit analysis see [5].*

Electronic synthesis of TTN

In the section in which the lumped model of the Maxwell equations has been derived , it was seen how, from the constitutive relations of the regions A, B and C, only a few TTN can be obtained; in particular, resistors, inductors and capacitors of positive values and independent generators. However it is possible to simulate the behavior of other TTN using electronic circuits. In order to do that we need to introduce the concept of **port** and of **driving immittance** for a n-**port network**:

Definition 62 *Let us consider a cut which divides a circuit in two parts (A: inside the cut, and B: outside the cut) intersecting of only two of its edges in the points* T_1 *and* T_2*. If the current entering in the part A from* T_1 *is equal to the current exiting A from* T_2*, then the two points* (T_1, T_2) *represent a port of the circuit.*

Definition 63 *A* n*- port network is a part of a circuit which is accessible from the rest of the circuit from n-ports.*

Remark 64 *This implies that an n- port network is unequivocally characterizable (with respect to the rest of the circuit) by the vector of the tensions and the currents on its ports. Hence a TTN can be regard as a one port network. In fact it is unequivocally characterized by the current and the tension between its terminals.*

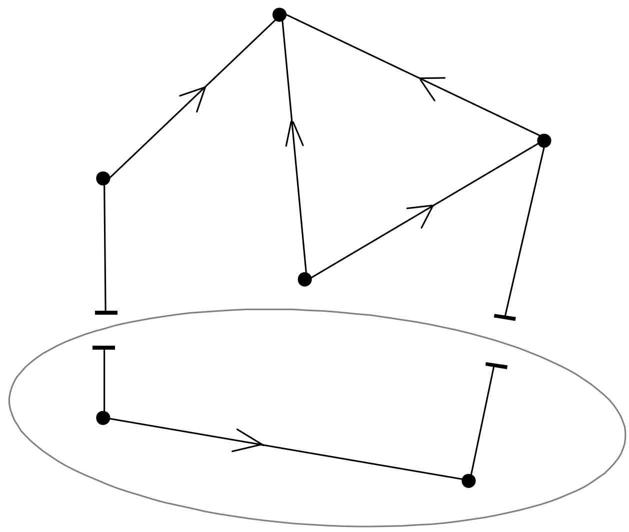


Figure 3.17: Port of a circuit

Definition 65 *The ratio between the current I_A entering in a one port network and the tension V_A between the terminals T_1 and T_2 is the driving immittances Y_{dr} of the part A relative to the port (T_1, T_2):*

$$Y_{dr} = \frac{I_A}{V_A}.$$

Remark 66 *The immittance for a TTN can be regarded as the immittance of a one port network.*

Hence our goal is to find a one port electronic-network the immittance of which equals the immittance of the TTN already introduced.

The basic building blocks of the electronic circuits which realize this are **operational amplifiers**. Hence let us discuss what they physically are and how they can be modelled in a circuit analysis.

Operational amplifier

The operational amplifier (OP AMP) is a high-gain electronic amplifier realized using transistors. Hence, in order to operate, it needs a power supply, which (together with the transistors used) determines a range of tension and current in which it can operate linearly way. Beside the power-supply terminals, the OP AMP presents a differential signal-input and a single-ended signal-output referred to the ground:

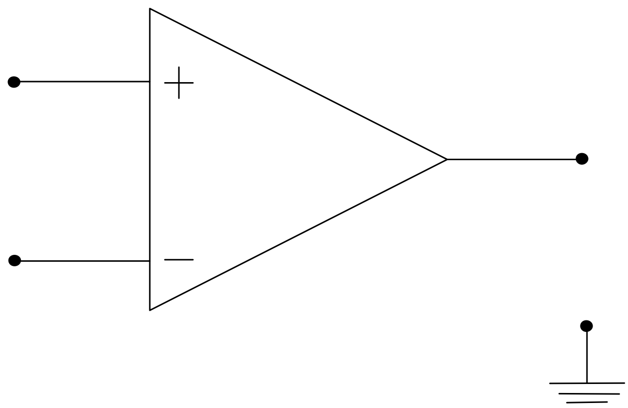


Figure 3.18: Operational amplifier

Consequently, for the electric signals, it is regarded as a two port network with one terminal of the output-port connected to the ground. The current entering in the input port of an OP AMP is assumed to be always zero[1]. In order to let the amplifier work in its region of linearity, the great value of the gain implies that

[1] Using the MOSFET technology, it is possible to reach an excellent approximation of this assumption.

its input must be almost zero. Hence also the tension of the input, in the ideal analysis, is set to zero. This means that the input port behaves like a nullator.

In the ideal model, the output tension depends only upon the input, consequently the current will be determined only by the driving immittance seen by the output-port. Hence neither the tension nor the current depend upon the output port which behaves like a norator:

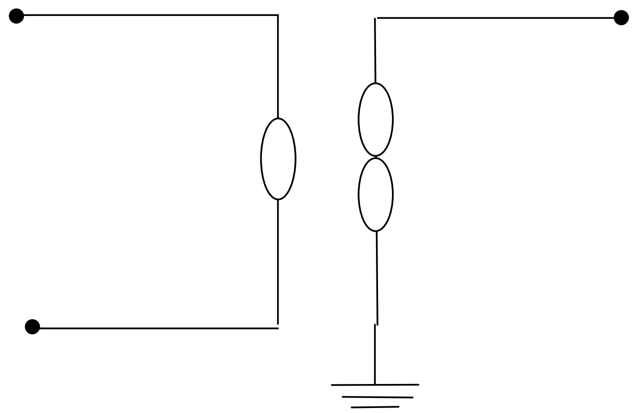


Figure 3.19: Ideal operational-amplifier model

Consequently, it is possible to realize circuits (in an electronic way) which, beside the physically realizable elements, presents couples of nullator-norator, where each norator is connected to the ground.

When nullators and norators are present in a circuit, it is easy to derive a circuit analysis from the previously presented one, in the following way:

- Each norator is substituted by a independent current generator of unknown value.

- Each nullator is substituted by an open circuit.
- When the set of solving equations is derived, the second equation for the nullator is taken into account (i.e. the potential is the same at its nodes).
- Since the current generator of the norators are additional unknowns, the additional equations of the nullator lead to a system of an equal number of equations and unknowns.

When the circuits are simple, it is easier to consider the equations of the nullator directly on the graph of the circuit and proceed with a direct analysis.

Let us derive, in this way, two kinds of one port networks able to simulate the behavior of all the TTN introduced in the previous section.

One port network simulating TTN

There are many one port networks proposed in the literature for the simulation of a generic TTN. Here we will present two classes of them, the first one is used to realize active components such negative resistors, inductors and super-inductors, capacitors and super-capacitors. The other one can simulate the respective positive versions of each of these.

First configuration

The first proposed configuration is based upon the following one port network, which is able to simulate only TTNs which have one edge connected to the ground.

It is realized using only one operational amplifier:

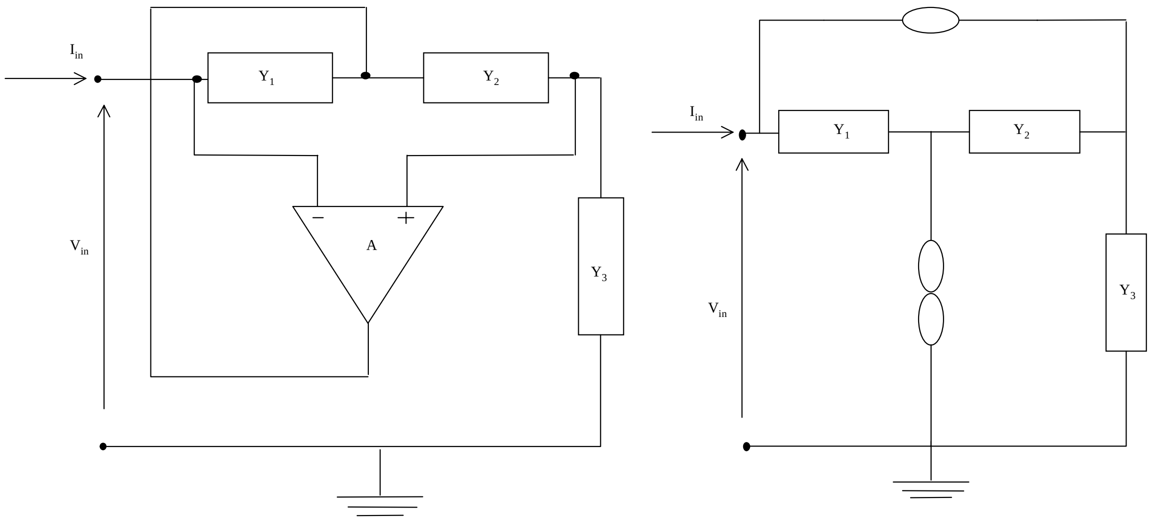
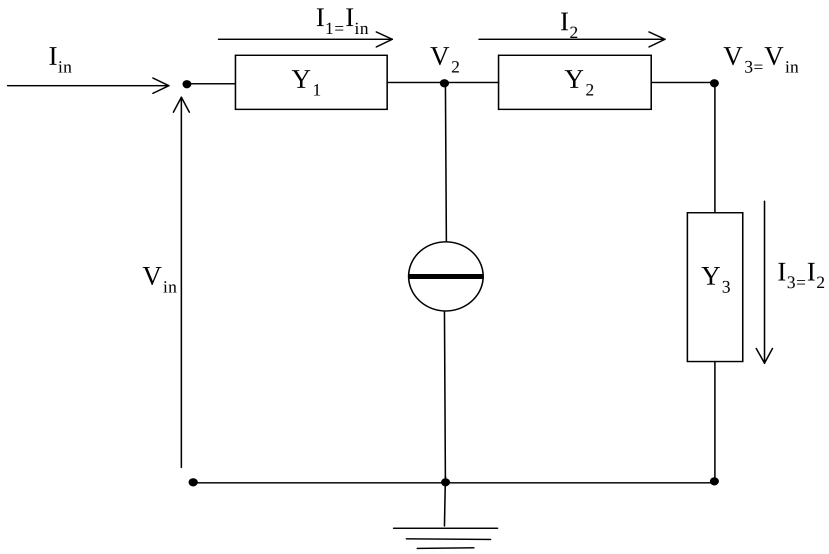


Figure 3.20: Basic element for the first configuration and ideal circuital model.

where Y_1, Y_2, Y_3 are some generic, physically-realizable immittances (resistors, inductors, capacitors of positive values) ([2]).

Our goal is to find the driving immittance $Y_{in} = I_{in}/V_{in}$ of this one port network. This can be done considering a direct analysis of the circuit model of the following figure, where the norator is replaced with an unknown current generator, while both the current an the tension on the nullator edge are already assumed to vanish.

[2] In the same figure, the ideal circuit model of this configuration is also presented.



Circuital analisys model

Let us consider the constitutive relations for all the immittances in the circuit:

$$(V_i - V_{i+1})\,Y_i = I_i,$$

which in our case become (see previous figure):

$$\left[\begin{array}{c} I_{in} = (V_{in} - V_2)\,Y_1 \\ \\ I_2 = (V_2 - V_{in})\,Y_2 \\ \\ I_2 = V_{in} Y_3 \end{array}\right.$$

From the second and the third relation we get:

$$V_2 = V_{in}\frac{Y_2 + Y_3}{Y_3},$$

which substituted into the first one gives the value of the driving immittance:

$$Y_{in} = \frac{I_{in}}{V_{in}} = -\frac{Y_1 Y_3}{Y_2}.$$

This shows how from positive values for Y_1, Y_2, Y_3 we get negative values of the grounded driving-immittance.

The circuit able to simulate floating TTNs is constituted by two of the previous circuits connected as in figure

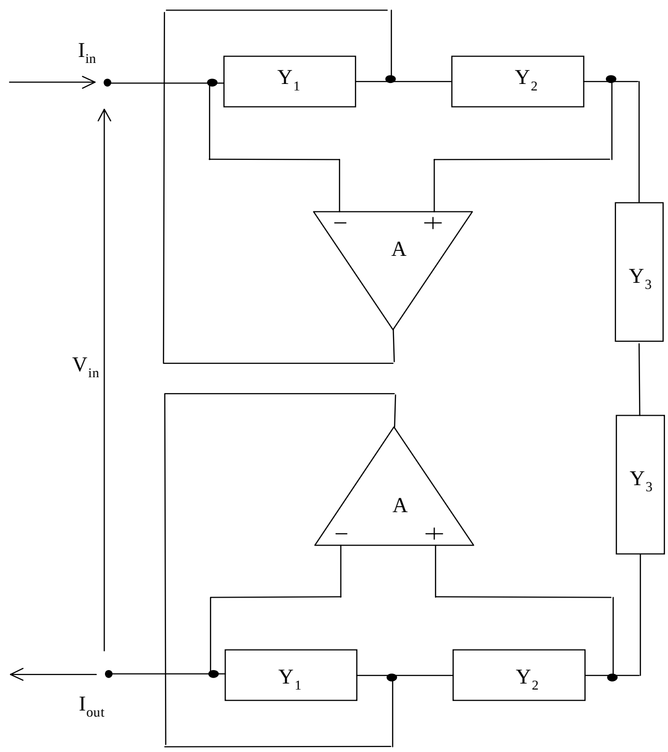


Figure 3.21: Floating one port network simulating negative TTN values.

First of all we must show that this really behaves as a one port circuit, i.e. $I_{in} = I_{out}$. Let us consider the equivalent circuit model:

If we consider the constitutive relations for the two immittances Y_2:

$$
\begin{aligned}
I_2 &= (V_2 - V_1)\, Y_2, \\
I_2 &= (V_3 - V_4)\, Y_2,
\end{aligned}
$$

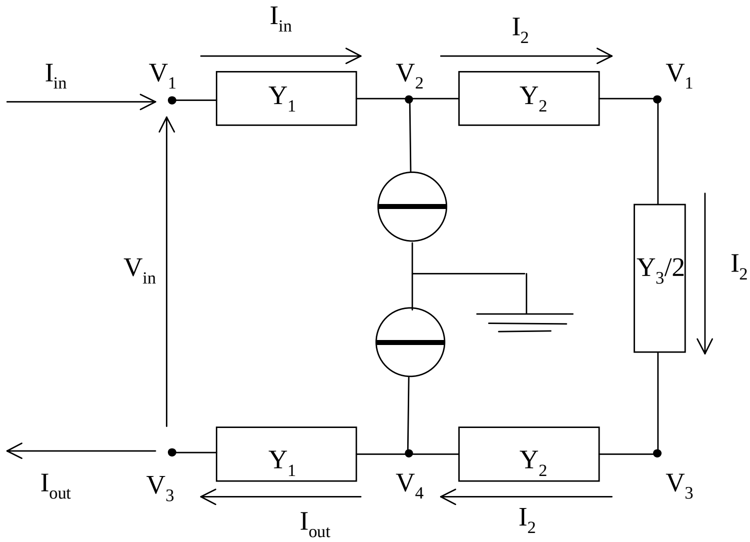


Figure 3.22: Ideal equivalent circuit of the floating-TTN.

we get

$$(V_2 - V_1) = -(V_4 - V_3),$$

Hence looking at the relations for the immittances Y_1 we have that this circuit behaves as a one port network:

$$\left[\begin{array}{l} I_{in} = (V_1 - V_2)\,Y_1 \\ \\ I_{out} = (V_4 - V_3)\,Y_1 \end{array}\right. \Rightarrow \left[\begin{array}{l} I_{in} = -\dfrac{Y_1}{Y_2} I_2 \\ I_{out} = -\dfrac{Y_1}{Y_2} I_2, \end{array}\right. \Rightarrow I_{in} = I_{out}.$$

Observing that $V_{in} = V_1 - V_3$, the constitutive relation of Y_3:

$$I_2 = (V_1 - V_3)\,Y_3,$$

which implies that:

$$I_{in} = -\frac{Y_1 Y_3}{Y_2} V'_{in}$$

gives the driving immittance:

$$Y_{in} = -\frac{Y_1 Y_3}{Y_2}.$$

Example 67 *If we consider* $Y_1 = Y_2$ *then:*

$$Y_{in} = -Y_3,$$

hence the circuit is able to chance the sign of the immittance Y_3

Example 68 *A super-inductor* $\left(Y_{si} = D\frac{1}{s^2}\right)$ *can be easily obtained, for instance considering a resistor* $Y_2 = \frac{1}{R_2}$ *and the inductors* $Y_1 = \frac{1}{sL_1}, Y_3 = \frac{1}{sL_3}$*:*

$$Y_{in} = \frac{-1}{R_2 L_1 L_3}\frac{1}{s^2}.$$

Or it is possible to use a resistor Y_1*, an inductor* Y_3 *and a capacitor* $Y_2 = sC_2$*:*

$$Y_{in} = \frac{-R_1}{C_2 L_3}\frac{1}{s^2}.$$

Exercise 69 *In the same way it is possible to obtain a super-capacitor* $(Y_{sc} = Fs^2)$*, considering both* Y_1, Y_3 *as capacitors and a resistor* Y_2*:*

$$Y_{in} = -\frac{C_1 C_3}{R_2}s^2.$$

Otherwise we can consider Y_1 *a resistor,* Y_2 *an inductor and* Y_3 *a capacitor:*

$$Y_{in} = -R_1 L_2 C_3 s^2.$$

Second configuration

The second configuration permits us to obtain TTN with positive values of immittances, it is realized using two OP AMP. The basic version permit us to realize TTNs referred to ground:

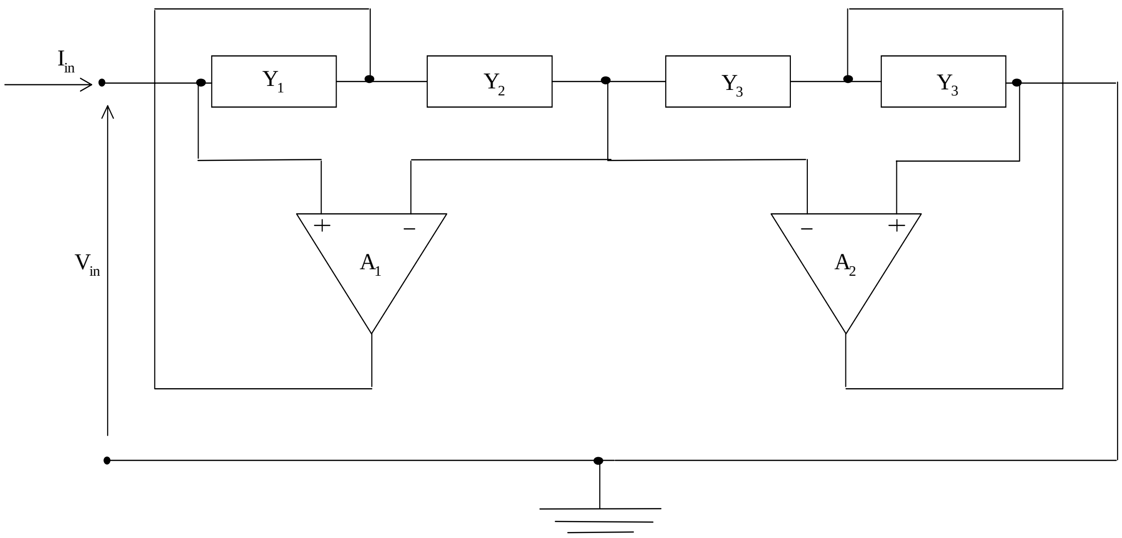


Figure 3.23: Basic element of the second configuration.

The driving immittance can be found considering the constitutive relations of the immittances Y_1, Y_2, Y_3, Y_4, Y_5 in the previous figure

$$I_{in} = (V_{in} - V_2)\, Y_1,$$

$$I_2 = (V_2 - V_{in})\, Y_2,$$

$$I_2 = (V_{in} - V_3)\, Y_3,$$

$$I_3 = (V_3 - V_{in})\, Y_4,$$

$$I_3 = V_{in} Y_5,$$

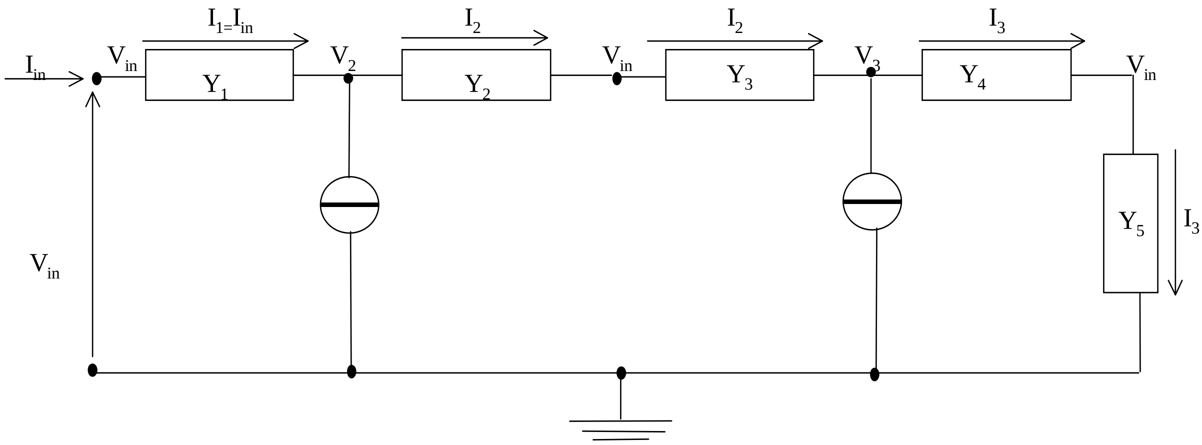


Figure 3.24: Ideal circuit of the second electronic TTN

From the third and the fourth relations we get:

$$V_3 = \frac{Y_5 + Y_4}{Y_4} V_{in}.$$

Hence from the second one:

$$V_2 = \left(1 - \frac{Y_5 Y_3}{Y_2 Y_4}\right) V_{in},$$

which substituted into the first relation gives the values of the driving immittance:

$$Y_{in} = \frac{I_{in}}{V_{in}} = \frac{Y_1 Y_3 Y_5}{Y_2 Y_4}.$$

The floating version of the TTN realized using the second configuration is obtained (as in the previous case) by using two of the ground referred circuits connected as in figure

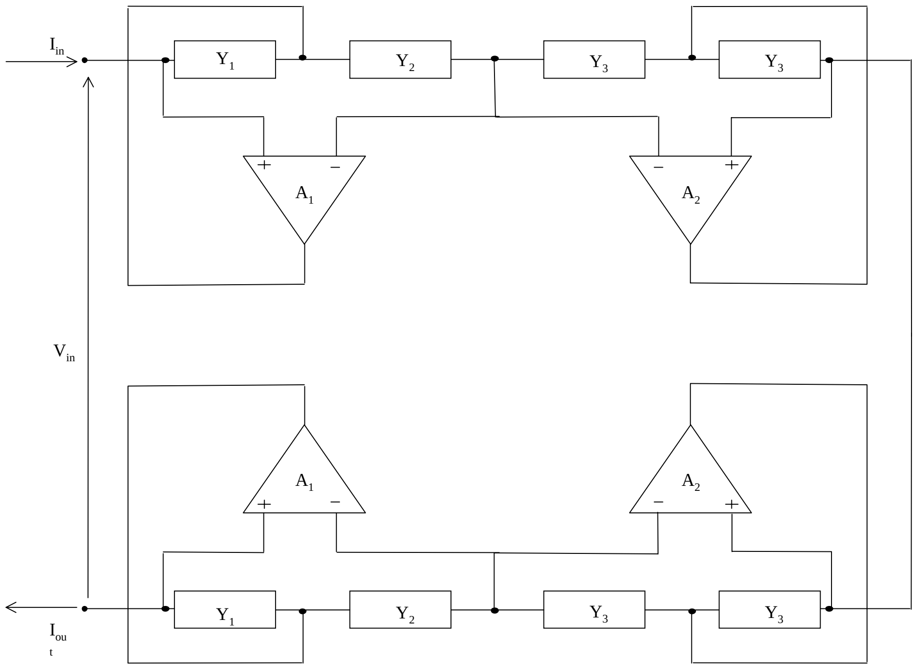


Figure 3.25: Floating operational circuital-model of the second configuration.

This behaves as a one port network. In fact from the constitutive relations for the immittances Y_4 we get:

$$\left[\begin{array}{l} I_3 = (V_3 - V_1)\, Y_4 \\ I_3 = (V_4 - V_5)\, Y_4 \end{array}\right. \Rightarrow (V_3 - V_1) = (V_4 - V_5)\,,$$

which, together with the relations for the immittances Y_3 implies that $I_2 = I_4$:

$$\left[\begin{array}{l} I_2 = -\,(V_3 - V_1)\, Y_3 \\ I_4 = -\,(V_4 - V_5)\, Y_3 \end{array}\right. \Rightarrow I_4 = I_2.$$

From the relations for the immittances Y_2 we get:

$$\left[\begin{array}{l} I_2 = (V_2 - V_1)\, Y_2 \\ I_4 = I_2 = (V_4 - V_6)\, Y_2 \end{array}\right. \Rightarrow (V_4 - V_6) = (V_2 - V_1)\,,$$

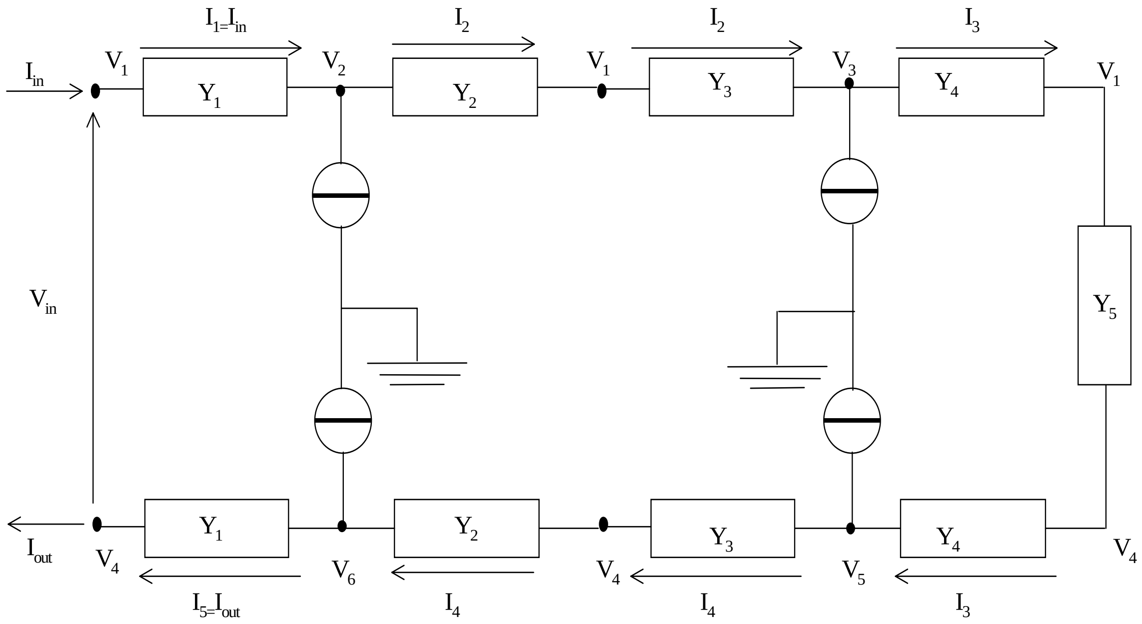


Figure 3.26: Floating ideal circuital model of the second configuration.

which together with the relations for Y_1 gives:

$$\left[\begin{array}{l} I_{in} = (V_1 - V_2)\, Y_1 \\ \\ I_{out} = (V_6 - V_4)\, Y_1 \end{array}\right. \Rightarrow I_{in} = I_{out},$$

i.e. the circuit is a one port network.

In order to derive the driving immittances V_{in}/I_{in}, let us remark that $V_{in} = V_1 - V_4$ which is also the tension upon the immittance Y_5, thus:

$$V_{in} = \frac{I_3}{Y_5}.$$

From the relations for Y_3 and Y_4 we get:

$$\left[\begin{array}{l} I_3 = (V_3 - V_1)\, Y_4 \\ \\ I_2 = -\,(V_3 - V_1)\, Y_3 \end{array}\right. \Rightarrow I_3 = -\frac{Y_4}{Y_3} I_2.$$

While observing the relations for Y_1 and Y_2 we have:

$$\left[\begin{array}{l} I_2 = (V_2 - V_1)\, Y_2 \\ \\ I_{in} = (V_1 - V_2)\, Y_1 \end{array}\right. \Rightarrow I_2 = -\frac{Y_2}{Y_1} I_{in}.$$

Hence the following value for the current I_3, which flows through the immittance Y_5, is found

$$I_3 = \frac{Y_4 Y_2}{Y_3 Y_1} I_{in},$$

which permits us to express the driving immittance of the circuit:

$$Y_{in} = \frac{I_{in}}{V_{in}} = \frac{I_{in}}{I_3 Y_5} = \frac{Y_4 Y_2 Y_5}{Y_3 Y_1}.$$

Remark 70 *As it was seen for the first configuration, many different choices for the immittances Y_1, Y_2, Y_3, Y_4, Y_5 are possible for the simulation of each of the TTNs presented in the previous section.*

Chapter 4

Electro-mechanical circuit analog

In this chapter the electric circuit analog for the mechanical systems, presented in the previous chapter, are derived. From the homogenized model of the mechanical devices, a lumped model will be considered using the finite-difference method. Then, working with dimensionless quantities, an electric circuit, the equations of which are the same of the FD-mechanical ones, will be derived. Finally also the homogenized version of the electrical equations will be considered.

Circuit analog of a beam elastica

Let us consider the elastica equation obtained in the first chapter:

$$k_M \frac{\partial^4 u(x,t)}{\partial x^4} + \rho \frac{\partial^2 u(x,t)}{\partial t^2} = 0,$$

and the boundary conditions for a beam, the length of which is l:

- Completely-clamped boundary conditions

$$u(x,t)|_{x=0} = u(x,t)|_{x=l} = 0,$$

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0} = \left.\frac{\partial u(x,t)}{\partial x}\right|_{x=l} = 0.$$

- Simply supported boundary conditions:

$$u(x,t)|_{x=0} = u(x,t)|_{x=l} = 0,$$

$$\left.\frac{\partial^2 u(x,t)}{\partial^2 x}\right|_{x=0} = \left.\frac{\partial^2 u(x,t)}{\partial^2 x}\right|_{x=l} = 0.$$

Frequency domain analysis

In order to find the electric analog of the Elastica, let us transform the previous equation into the Laplace domain ([1]):

$$k_M \frac{\partial^4 U(s)}{\partial x^4} + \rho \left[s^2 U(s) - s\dot{u}\left(0^+\right) - u\left(0^+\right)\right] = 0. \tag{4.1}$$

- Transformed completely-clamped boundary conditions

$$U(x,s)|_{x=0} = U(x,s)|_{x=l} = 0,$$

$$\left.\frac{\partial U(x,s)}{\partial x}\right|_{x=0} = \left.\frac{\partial U(x,s)}{\partial x}\right|_{x=l} = 0.$$

- Transformed simply-supported boundary conditions:

$$U(x,s)|_{x=0} = U(x,s)|_{x=l} = 0,$$

$$\left.\frac{\partial^2 U(x,s)}{\partial^2 x}\right|_{x=0} = \left.\frac{\partial^2 U(x,s)}{\partial^2 x}\right|_{x=l} = 0.$$

We are interested in developing a circuit structure, the equations of which show the same structure as the elastica one. This structure must be inde-

[1] Here $U(s) = \mathcal{L}[u(t)](s)$, in what follows the Capital letters will represent the L-transformed quantities of the small ones.

pendent of the initial conditions of the solution. Hence let us set:

$$u(t)|_{t=0^+} = 0,$$

$$\dot{u}(t)|_{t=0^+} = 0,$$

This means that the displacement and the velocity for each material point equals zero at t equal zero, so that the equation (4.1) becomes:

$$k_M \frac{\partial^4 U(s)}{\partial x^4} + \rho s^2 U(s) = 0. \tag{4.2}$$

Dimensionless analysis

Furthermore let us consider the dimensionless form of the equation. The characteristic quantities for the function and the variables involved have to be chosen:

$$\begin{aligned} x &= l_o \tilde{x}, \\ s &= s_o \tilde{s}, \\ U &= u_o \tilde{U}. \end{aligned}$$

Hence the relation (4.2) becomes:

$$k_M \frac{\partial^4 u_o \tilde{U}(s_o \tilde{s})}{\partial (l_o \tilde{x})^4} + \rho s_o^2 \tilde{s}^2 u_o \tilde{U}(s_o \tilde{s}) = 0,$$

and its dimensionless form can be written:

$$\frac{\partial^4 \tilde{U}}{\partial \tilde{x}^4} + \alpha \tilde{s}^2 \tilde{U} = 0, \qquad \alpha = \frac{\rho l_o^4 s_o^2}{k_M},$$

while the dimensionless boundary conditions are:

- Dimensionless transformed completely-clamped-boundary-conditions

$$\tilde{U}\left(l_o\tilde{x}, s_o\tilde{s}\right)\Big|_{\tilde{x}=0} = \tilde{U}\left(l_o\tilde{x}, s_o\tilde{s}\right)\Big|_{\tilde{x}=\frac{l}{l_o}} = 0,$$

$$\left.\frac{\partial\tilde{U}\left(l_o\tilde{x}, s_o\tilde{s}\right)}{\partial\tilde{x}}\right|_{\tilde{x}=0} = \left.\frac{\partial\tilde{U}\left(l_o\tilde{x}, s_o\tilde{s}\right)}{\partial\tilde{x}}\right|_{\tilde{x}=\frac{l}{l_o}} = 0.$$

- Dimensionless transformed simply-supported boundary-conditions:

$$\tilde{U}\left(l_o\tilde{x}, s_o\tilde{s}\right)\Big|_{\tilde{x}=0} = \tilde{U}\left(l_o\tilde{x}, s_o\tilde{s}\right)\Big|_{\tilde{x}=\frac{l}{l_o}} = 0,$$

$$\left.\frac{\partial^2\tilde{U}\left(l_o\tilde{x}, s_o\tilde{s}\right)}{\partial\tilde{x}^2}\right|_{\tilde{x}=0} = \left.\frac{\partial^2\tilde{U}\left(l_o\tilde{x}, s_o\tilde{s}\right)}{\partial\tilde{x}^2}\right|_{\tilde{x}=\frac{l}{l_o}} = 0.$$

We now proceed with a finite element approximation of the equation in order to find a lumped model for the elastica, which will permit us to derive the circuit analog.

Finite difference approximation

First of all let us select an ε-step grid of uniformly distributed nodes on the $\tilde{x}$-domain; on those nodes we will consider the values of the function we are going to approximate.

Since we are dealing with a fourth order differential equation, we expect to consider, for each node, at least the nearest four ones, so that the fourth finite difference on it can be defined.

Remark 71 *Let us remark that if n is the number of the nodes on the domain l, it will be divided into $N = n - 1$ parts of length h:*

$$h = \frac{l}{N},$$

and the dimensionless step ε becomes:

$$\varepsilon = \frac{h}{l_o} = \frac{1}{N}\frac{l}{l_o}.$$

Let us also recall the first forward finite difference on the i-node:

$$\overrightarrow{\Delta}_i = \frac{\tilde{U}_{i+1} - \tilde{U}_i}{\varepsilon}.$$

The second finite-difference is obtained as the backward FD on $\overrightarrow{\Delta}_i$:

$$\Delta_i^2 = \overleftarrow{\Delta}_i\left(\overrightarrow{\Delta}_i\right) = \frac{\overrightarrow{\Delta}_i - \overrightarrow{\Delta}_{i-1}}{\varepsilon} = \frac{\tilde{U}_{i+1} - 2\tilde{U}_i + \tilde{U}_{i-1}}{\varepsilon^2}.$$

The forward finite difference operating on the second FD gives the following result for the third one:

$$\Delta_i^3 = \overrightarrow{\Delta}\left(\Delta_i^2\right) = \frac{\Delta_{i+1}^2 - \Delta_i^2}{\varepsilon} = \frac{\tilde{U}_{i+2} - 3\tilde{U}_{i+1} + 3\tilde{U}_i - \tilde{U}_{i-1}}{\varepsilon^3}.$$

Then the fourth finite difference chosen is simply the backward difference of the third one:

$$\Delta_i^4 = \overleftarrow{\Delta}_i\left(\Delta_i^3\right) = \frac{\Delta_i^2 - \Delta_{i-1}^2}{\varepsilon} = \frac{\tilde{U}_{i+2} - 4\tilde{U}_{i+1} + 6\tilde{U}_i - 4\tilde{U}_{i-1} + \tilde{U}_{i-2}}{\varepsilon^4}.$$

Finally the FD-approximation of the elastica equation is derived:

$$\frac{\tilde{U}_{i+2} - 4\tilde{U}_{i+1} + 6\tilde{U}_i - 4\tilde{U}_{i-1} + \tilde{U}_{i-2}}{\varepsilon^4} + \alpha\tilde{s}^2\tilde{U}_i = 0, \qquad i = 1, 2.., n - 4. \tag{4.3}$$

The FD-boundary conditions are given as follows:

- Dimensionless completely-clamped FD boundary-conditions

$$\left[\begin{array}{l} \tilde{U}_1 = \tilde{U}_n = 0, \\ \\ \overrightarrow{\Delta}_1 = \dfrac{\tilde{U}_2 - \tilde{U}_1}{\varepsilon} = 0, \\ \overleftarrow{\Delta}_{ni} = \dfrac{\tilde{U}_n - \tilde{U}_{n-1}}{\varepsilon} = 0. \end{array}\right. \Rightarrow \tilde{U}_1 = \tilde{U}_2 = \tilde{U}_{n-1} = \tilde{U}_n = 0, \tag{4.4}$$

- Dimensionless simply-supported FD boundary-conditions:

$$\left[\begin{array}{l} \tilde{U}_1 = \tilde{U}_n = 0, \\ \\ \overrightarrow{\Delta}_1^2 = \overrightarrow{\Delta}_1\left(\overrightarrow{\Delta}_1\right) = \dfrac{\tilde{U}_3 - 2\tilde{U}_2 + \tilde{U}_1}{\varepsilon^2} = 0, \\ \overleftarrow{\Delta}_n^2 = \overleftarrow{\Delta}_1\left(\overleftarrow{\Delta}_1\right) = \dfrac{\tilde{U}_n - 2\tilde{U}_{n-1} + \tilde{U}_{n-2}}{\varepsilon^2} = 0. \end{array}\right. \Rightarrow \left[\begin{array}{l} \tilde{U}_1 = \tilde{U}_n = 0, \\ \\ \tilde{U}_3 = 2\tilde{U}_2, \\ \tilde{U}_{n-2} = 2\tilde{U}_{n-1}. \end{array}\right. \tag{4.5}$$

It will be useful to consider the following relations obtained from the equation (4.3):

$$\begin{array}{ll} \frac{1}{\tilde{s}}\tilde{U}_{i+2} - \frac{4}{\tilde{s}}\tilde{U}_{i+1} + \left(\frac{6}{\tilde{s}} + \alpha\varepsilon^4\tilde{s}\right)\tilde{U}_i - \frac{4}{\tilde{s}}\tilde{U}_{i-1} + \frac{1}{\tilde{s}}\tilde{U}_{i-2} = 0 & \\ \\ \tilde{U}_{i+2} - 4\tilde{U}_{i+1} + \left(6 + \alpha\varepsilon^4\tilde{s}^2\right)\tilde{U}_i - 4\tilde{U}_{i-1} + \tilde{U}_{i-2} = 0 & i = 1, 2.., n-4. \\ \\ \frac{1}{\tilde{s}^2}\tilde{U}_{i+2} - \frac{4}{\tilde{s}^2}\tilde{U}_{i+1} + \left(\frac{6}{\tilde{s}^2} + \alpha\varepsilon^4\right)\tilde{U}_i - \frac{4}{\tilde{s}^2}\tilde{U}_{i-1} + \frac{1}{\tilde{s}^2}\tilde{U}_{i-2} = 0 & \end{array} \tag{4.6}$$

Let us observe that the first and the third ones are obtained simply by dividing by the frequency $\tilde{s}$ and $\tilde{s}^2$. In the time domain this means that we are considering the first and the second time integral of the previous equations, hence the electric analog for the deflection will become the first and second time integral of the potential.

Remark 72 *The FD approximation transforms the elastica equation into a set of* $n-4$ *algebraic equations on the central nodes of the grid. Moreover the boundary conditions are transformed into a set of* 4 *algebraic equations (which, besides the two boundary nodes involve at most the two nodes nearest to them). In conclusion, the problem of finding the displacement for the elastica has been transformed in the solution of a linear algebraic system of* n*-equations with* n*-unknowns (which are the displacements on the nodes).*

Electro-mechanical analog

Our goal is to find lumped electrical networks, the governing equations of which are the same as the ones derived for the FD-elastica problem. Let us recall that, in the electric case, the potentials on the nodes of the circuit can be chosen as a set of independent unknowns. Consequently, at least a circuit of n nodes must be taken into account, (one electric-node for each grid-node). The potential on each electric-node represents the electric analog of the displacement on each sampling node of the mechanical domain. Once a tree of the circuit is chosen, the equilibrium of the currents at each fundamental cut gives the electric analog of the mechanical-equations system.

In order to obtain the same structure for all the equations on the $n-4$ central nodes (as in the mechanical case), we can consider a tree in which all the edges are connected to the ground G. This implies that each fundamental cut simply becomes a closed line around each node, thus the current terms for the governing equation will be exactly the currents of the edges belonging to the node itself.

The equation at the i-th node shows terms of the four adjacent ones, this implies that, if only TTN are used, each node must be connected to the four adjacent ones. Hence the simplest topology of the circuit is the following:

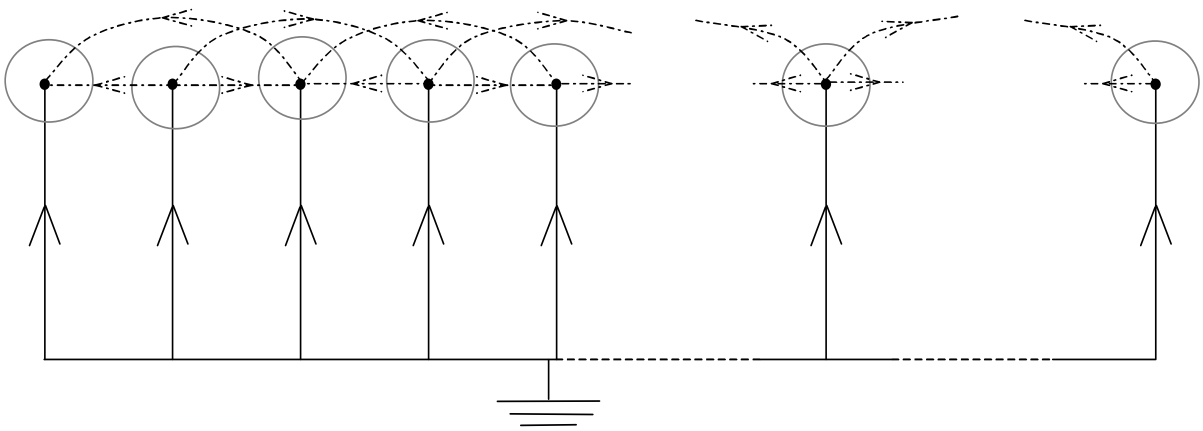


Figure 4.1: Elastica circuit: Tree (black-lines), co-tree (dashed-lines), fund.cuts (gray-lines)

Finally the symmetry of the equation around the i-node implies that the TTN which connects the i-node to $i+1$ one is equal to the TTN between the i-th and the $i-1$-th node, and the TTN between the i-th and the $i+2$-th node has to be equal to the TTN between the i-th and the $i-2$ node. Finally the following element for the circuit is derived:

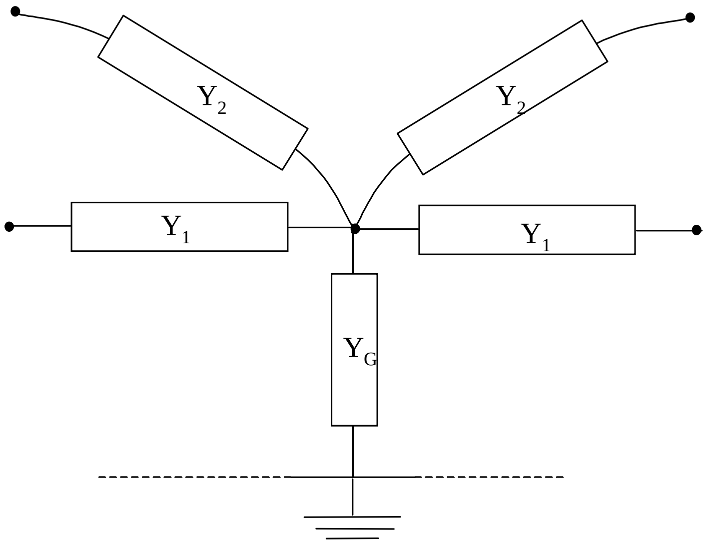


Figure 4.2: Element of the circuit analog for the elastica.

The equation for each fundamental cut of the generic central node is:

$$I_{i+2} + I_{i+1} + I_{i-1} + I_{i-2} + I_{G_i} = 0. \tag{4.7}$$

Let us synthesize the circuit with only linear TTN, the constitutive relation of which (into the L-domain) can be expressed as:

$$I(s) = Y(s)\, V(s),$$

where $V(s)$ is the potential difference upon the TTN.

Remark 73 *This implies that, among the presented linear TTN, we are not considering norators or nullators.*

As a consequence, the equilibrium equation (4.7) becomes:

$$(V_{i+2} - V_i)\,Y_2 + (V_{i+1} - V_i)\,Y_1 + (V_{i-1} - V_i)\,Y_1 + (V_{i-2} - V_i)\,Y_2 + V_i Y_G = 0$$

that is a set of $n - 4$ equations each one having, beside the central term i, its adjacent four ones:

$$Y_2 V_{i+2} + Y_1 V_{i+1} - (2Y_1 + Y_G)\,V_i + Y_1 V_{i-1} + Y_2 V_{i-2} = 0 \tag{4.8}$$

In order to derive the electro-mechanical analog, the coefficients of the deflection in the FD-elastica equation must be compared to the ones of the tensions in the equilibrium equation (4.8). This can be done once the dimensionless form of eqn (4.8) is derived. Hence, besides the characteristic frequency s_o, let us define a characteristic tension V_o and resistance R_o such that:

$$\begin{aligned} &s = s_o\tilde{s}, \\ &V_i = V_o\tilde{V}_i, \\ &Y_k = \tfrac{1}{R_o}\tilde{Y}, \qquad k \in \{1, 2, G\}. \end{aligned}$$

The dimensionless equilibrium relation (4.8) becomes:

$$\begin{aligned} \tilde{Y}_2\,(s_o\tilde{s})\,\tilde{V}_{i+2} + \tilde{Y}_1\,(s_o\tilde{s})\,\tilde{V}_{i+1} - \left[2\tilde{Y}_1\,(s_o\tilde{s}) + 2\tilde{Y}_2\,(s_o\tilde{s}) + \tilde{Y}_G\,(s_o\tilde{s})\right]\tilde{V}_i \\ + \tilde{Y}_1\,(s_o\tilde{s})\,\tilde{V}_{i-1} + \tilde{Y}_2\,(s_o\tilde{s})\,\tilde{V}_{i-2} = 0. \end{aligned}$$

Let us equate the coefficients of the previous dimensionless equation to each of the three expressions (4.6) found for the elastica, this will produce three different expressions for the immittances of the TTN of our circuit:

1) Case1

$$\begin{bmatrix} \tilde{Y}_1(s_o\tilde{s}) = \dfrac{4}{\tilde{s}} \\ \tilde{Y}_2(s_o\tilde{s}) = -\dfrac{1}{\tilde{s}} \\ \tilde{Y}_G(s_o\tilde{s}) = \alpha\varepsilon^4\tilde{s} \end{bmatrix} \Rightarrow \begin{bmatrix} Y_1(s_o\tilde{s}) = \dfrac{4}{R_o}\dfrac{1}{\tilde{s}} \\ Y_2(s_o\tilde{s}) = -\dfrac{1}{R_o}\dfrac{1}{\tilde{s}} \\ Y_G(s_o\tilde{s}) = \dfrac{\alpha\varepsilon^4}{R_o}\tilde{s} \end{bmatrix} \tag{4.9}$$

2) Case 2

$$\begin{bmatrix} \tilde{Y}_1(s_o\tilde{s}) = 4 \\ \tilde{Y}_2(s_o\tilde{s}) = -1 \\ \tilde{Y}_G(s_o\tilde{s}) = \alpha\varepsilon^4\tilde{s}^2 \end{bmatrix} \Rightarrow \begin{bmatrix} Y_1(s_o\tilde{s}) = \dfrac{4}{R_o} \\ Y_2(s_o\tilde{s}) = -\dfrac{1}{R_o} \\ Y_G(s_o\tilde{s}) = \dfrac{\alpha\varepsilon^4}{R_o}\tilde{s}^2 \end{bmatrix} \tag{4.10}$$

3) Case3

$$\begin{bmatrix} \tilde{Y}_1(s_o\tilde{s}) = \dfrac{4}{\tilde{s}^2} \\ \tilde{Y}_2(s_o\tilde{s}) = -\dfrac{1}{\tilde{s}^2} \\ \tilde{Y}_G(s_o\tilde{s}) = \alpha\varepsilon^4 \end{bmatrix} \Rightarrow \begin{bmatrix} Y_1(s_o\tilde{s}) = \dfrac{4}{R_o}\dfrac{1}{\tilde{s}^2} \\ Y_2(s_o\tilde{s}) = -\dfrac{1}{R_o}\dfrac{1}{\tilde{s}^2} \\ Y_G(s_o\tilde{s}) = \dfrac{\alpha\varepsilon^4}{R_o} \end{bmatrix} \tag{4.11}$$

Case 1

If we consider the general form for inductive and capacitive immittances in the Laplace domain:

$$\begin{bmatrix} Y_L = \dfrac{1}{sL} = \dfrac{1}{s_oL}\dfrac{1}{\tilde{s}}, \\ Y_C = sC = s_oC\tilde{s}, \end{bmatrix}$$

We can immediately interpret the case-one formulas (4.9): the immittance Y_G of the TTN between the ground and each node is capacitive:

$$Y_G = \frac{\alpha\varepsilon^4}{R_o}\tilde{s} = s_oC\tilde{s},$$

the values C of the capacitance are:

$$C = \frac{\alpha\varepsilon^4}{R_o s_o},$$

while Y_1 is the immittance of an inductor:

$$Y_1 = \frac{4}{R_o}\frac{1}{\tilde{s}} = \frac{1}{s_o L_1}\frac{1}{\tilde{s}},$$

its relative value of inductance L_1 is:

$$L_1 = \frac{R_o}{4s_o},$$

and so it is for the immittance Y_2:

$$Y_2 = -\frac{1}{R_o}\frac{1}{\tilde{s}} = \frac{1}{s_o L_2}\frac{1}{\tilde{s}},$$

where the values of the inductance L_2 is:

$$L_2 = -\frac{R_o}{s_o}.$$

In conclusion, choosing a characteristic-positive inductance of value L:

$$L = \frac{R_o}{s_o},$$

the TTN-values of the sought circuit are:

$$\left[\begin{array}{l} C = \dfrac{\alpha\varepsilon^4}{R_o s_o} \\ L_1 = \dfrac{L}{4} \\ L_2 = -L \end{array}\right. , \qquad \text{where } L = \frac{R_o}{s_o}, \tag{4.12}$$

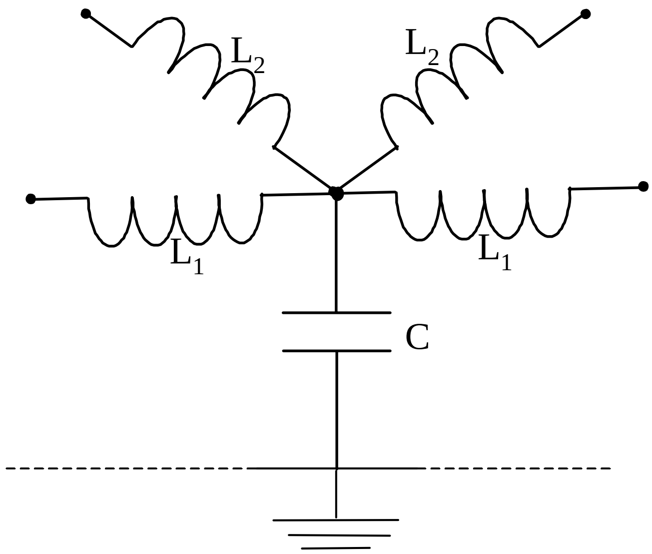


Figure 4.3: Element of the elastica circuit analog.

Remark 74 *Let us underline that the negative value of L_2 means that its relative TTN is not passive, so that active one port networks, of the first configuration, presented in the previous section, have to be used.*

Let us consider now the electrical analog of the boundary-conditions and what they imply in the electrical circuit (consider formulas (4.4, 4.5):

- Electrical completely-clamped-BC:

$$V_1 = V_2 = V_{n-1} = V_n = 0,$$

Remark 75 *This condition is simply realized by connecting the first two nodes and the last two ones directly to the ground.*

- Electrical simply supported-BC:

$$V_1 = V_n = 0,$$

$$V_3 = 2V_2,$$

$$V_{n-2} = 2V_{n-1},$$

Remark 76 *The first relation can be realized connecting the external nodes directly to the ground, while the second and the third one are obtained connecting the node* 3 *(and* $n-2$*) to the second (and* $n-1$*-th) one using the electronic circuit in figure (4.4):*

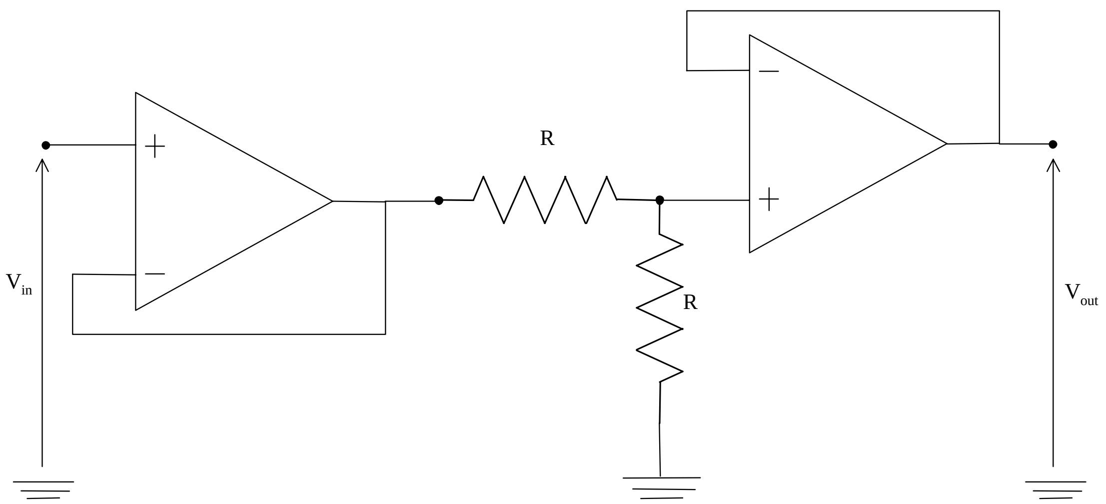


Figure 4.4: BC-circuit.

This circuit is based upon the voltage follower amplifier configuration: which is a buffer able to:

1) Measure the input tension of a node in a circuit without perturbing it.

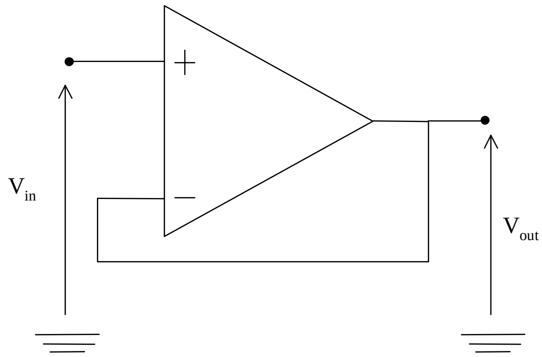


Figure 4.5: Voltage follower

2) Supply an output voltage equal to the input.

Consequently the immediate interpretation of the BC-circuit is: the first voltage follower measures the tension on one node without perturbing the system. The resistors realize the given ratio between the input and output voltage. Finally the second buffer imposes the output tension on the boundary node uncoupling the partitioning resistors with the external world. The equivalent ideal circuit is presented in figure (4.6):

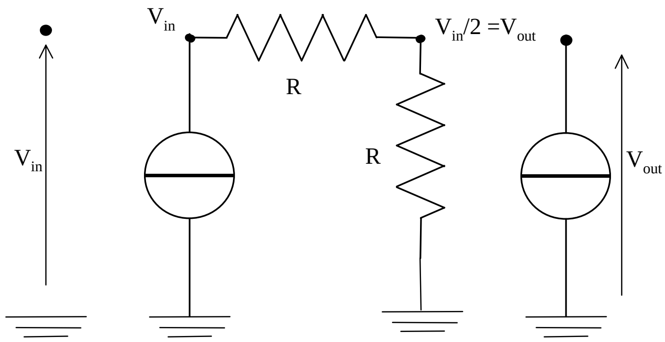


Figure 4.6: Ideal equivalent circuit for the BC.

Remark 77 *More details about electronic circuits can be found in [8].*

Case 2

If we consider the immittance value of a resistance R in L-domain:

$$Y_R = \frac{1}{R},$$

We can immediately interpreted the immittances Y_1 and Y_2 from the case-two relations (4.10):

$$\begin{aligned} Y_1 &= Y_{R_1} = \frac{4}{R_o}, \\ Y_2 &= Y_{R_2} = -\frac{1}{R_o} : \end{aligned}$$

Y_1 and Y_2 are immittances of resistors having values:

$$\begin{aligned} R_1 &= 4R_o, \\ R_2 &= -R_o. \end{aligned}$$

The ground immittance $Y_G = \frac{\alpha\varepsilon^4}{R_o}\tilde{s}^2$ represents a super-capacitor $F = \frac{\alpha\varepsilon^4}{R_o}$ realizable with one of the circuits shown in the previous section.

Case 3

The third case presents resistive-immittances for Y_G:

$$R_G = \frac{R_o}{\alpha\varepsilon^4},$$

while the values for Y_1 and Y_2 represent a superconductor $Y_L = \frac{4}{R_o}\frac{1}{\tilde{s}^2}$ of value $D = \frac{R_o}{4}$.

Conclusions

In order to observe some properties of the electrical waves propagating through the circuit, it is useful to consider the homogenized equation related to the electrical solution system. Its dimensionless form has obviously the same expression as the elastica one:

$$\frac{\partial^4 \tilde{v}}{\partial \tilde{x}^4} + \alpha \frac{\partial^2 \tilde{v}}{\partial \tilde{t}^2} = 0, \tag{4.13}$$

([2]) in which the independent variable is now the dimensionless electric-potential $\tilde{v}$. The coefficient α can be derived from relations (4.12):

$$\left[\begin{array}{l} C = \dfrac{\alpha \varepsilon^4}{R_o s_o} \\ L = \dfrac{R_o}{s_o} \end{array} \right. \quad \Rightarrow \qquad \alpha = \frac{s_o^2 LC}{\varepsilon^4}. \tag{4.14}$$

Since $\varepsilon = \frac{h}{l_o}$ it is useful to consider the following expression for α

$$\alpha = \frac{LC}{h^4} s_o^2 l_o^4.$$

Phase speed

Let us observe how the coefficient α is related to the phase speed for the waves in this medium.

If we consider a dimensionless elementary-wave

$$\tilde{v}(\tilde{x}, \tilde{t}) = A e^{j(\frac{2\pi}{\lambda}\tilde{x} - \tilde{\omega}\tilde{t})},$$

[2] $\tilde{t}$ is the dimensionless time, $\tilde{t} = s_o t$, where s_o is the chosen characteristic frequency.

of dimensionless wavelength $\tilde{\lambda}$ and frequency $\tilde{\omega}$:

$$\left[\begin{array}{l} \tilde{\lambda} = \dfrac{\lambda}{l_o}, \\ \tilde{\omega} = \dfrac{\omega}{s_o}, \end{array}\right.$$

and its propagation into a medium (the wave equation of which is the relation (4.13)) the spectral equation for the medium is obtained:

$$\left(\frac{2\pi}{\tilde{\lambda}}\right)^4 - \alpha\tilde{\omega}^2 = 0.$$

Then considering the dimensionless phase speed of a wave, defined as:

$$\tilde{v} = \frac{\tilde{\omega}\tilde{\lambda}}{2\pi},$$

the dispersive relation of the medium $\tilde{v}\left(\tilde{\lambda}\right)$ is derived:

$$\tilde{v}\left(\tilde{\lambda}\right) = \pm\frac{2\pi}{\tilde{\lambda}}\sqrt{\frac{1}{\alpha}}.$$

Remark 78 *The phase speed is inversely proportional to the square root of the parameters α, moreover the dispersion relation $\tilde{v}\left(\tilde{\lambda}\right)$ is not constant, this implies that the media is dispersive. The shorter wave-lengths of a signal propagating in it travel faster then the longer ones.*

It is useful to derive the dimensional form of the phase speed, considering the mechanical and electrical expressions for the dimensionless coefficient α, :

$$v = l_o s_o \tilde{v},$$

- Mechanical phase speed:

$$\alpha_m = \frac{\rho}{k_M} s_o^2 l_o^4 \qquad \Rightarrow \qquad |\upsilon_m| = \frac{2\pi}{\lambda}\sqrt{\frac{k_M}{\rho}},$$

- Electrical phase speed:

$$\alpha_e = \frac{LC}{h^4} s_o^2 l_o^4 \qquad \Rightarrow \qquad |\upsilon_e| = \frac{2\pi}{\lambda}\frac{h^2}{\sqrt{LC}},$$

Remark 79 *The dimensional electric-wave equation becomes:*

$$\frac{\partial^4 \upsilon}{\partial x^4} + \frac{LC}{h^4}\ddot{\upsilon} = 0.$$

Remark 80 *In order to obtain the same phase speed in the mechanical and electrical cases, the following relation must hold:*

$$LC = h^4 \frac{\rho}{k_M}$$

Remark 81 *Since the quantities L and C are defined upon a module of length $h = \frac{l}{N}$, let us define the total amount of inductance L_T and capacitance C_T along the domain as:*

$$L_T = NL,$$
$$C_T = NC.$$

The amount of the quantity $L_T C_T$, which realizes the electrical analog, must satisfy *the following relation:*

$$|v_e| = |v_m| \qquad \Rightarrow \qquad \frac{Nh^2}{\sqrt{L_T C_T}} = \sqrt{\frac{k_M}{\rho}} \qquad \Rightarrow$$

$$\Rightarrow L_T C_T = \frac{\rho}{k_M} l^2 h^2.$$

It is important to underline that this quantity depends not only upon the length of the bar, but also upon the step used.

Electric analog for the plate

In order to develop the electric analog for the plate, let us recall the plate-equation:

$$S_p \nabla^2 \nabla^2 u(x,y,t) - J_{\mathcal{I}} \rho \nabla^2 \frac{\partial^2 u(x,y,t)}{\partial t^2} + 2 h_t \rho \frac{\partial^2 u(x,y,t)}{\partial t^2} = 0$$

where:

$$\begin{aligned} J_{\mathcal{I}} &= \tfrac{2}{3} h_t^3 \\ S_p &= J_{\mathcal{I}} \left(2\mu_L + \lambda_L \right) = J_{\mathcal{I}} Y \end{aligned} \tag{4.15}$$

Let us consider a square plate of edge l, on which it is easy to define a set of boundary conditions:

- Completely-clamped boundary conditions:

$$u|_{\partial\mathcal{D}} = 0 \Rightarrow u|_{x=0} = u|_{x=l} = u|_{y=0} = u|_{y=l} = 0,$$

$$\left.\frac{\partial u}{\partial \mathbf{n}}\right|_{\partial\mathcal{D}} = 0 \Rightarrow \left.\frac{\partial u}{\partial x}\right|_{x=0} \left.\frac{\partial u}{\partial x}\right|_{x=l} \left.\frac{\partial u}{\partial y}\right|_{y=0} \left.\frac{\partial u}{\partial y}\right|_{y=l} = 0.$$

- Simply supported boundary conditions:

$$u|_{\partial\mathcal{D}} = 0 \Rightarrow u|_{x=0} = u|_{x=l} = u|_{y=0} = u|_{y=l} = 0,$$

$$\left.\frac{\partial^2 u}{\partial^2 \mathbf{n}}\right|_{\partial\mathcal{D}} = 0 \Rightarrow \left.\frac{\partial^2 u}{\partial^2 x}\right|_{x=0} \left.\frac{\partial^2 u}{\partial^2 x}\right|_{x=l} \left.\frac{\partial^2 u}{\partial^2 y}\right|_{y=0} \left.\frac{\partial^2 u}{\partial^2 y}\right|_{y=l} = 0.$$

Frequency domain analysis

Since the circuit analysis is simpler using the Laplace variable, it is useful to transform the mechanical equation into the frequency domain:

$$S_p\nabla^2\nabla^2U(s) - J_{\mathcal{I}}\rho s^2\nabla^2U\left(s\right) + 2h_t\rho s^2U\left(s\right) + \left(J_{\mathcal{I}}\rho\nabla^2 - 2h_t\rho\right)\left[s\dot{u}\left(0^+\right) + u\left(0^+\right)\right] = 0$$

Again, since the structure of the equation, which we are going to synthesize in an electrical way, must be independent of the initial conditions, we set them equal to zero:

$$u\left(x,y,t\right)|_{t=0^+} = 0,$$

$$\dot{u}\left(x,y,t\right)|_{t=0^+} = 0,$$

Hence the set of the plate equation and boundary conditions can be summarized in the following form:

$$S_p\nabla^2\nabla^2U - J_{\mathcal{I}}\rho s^2\nabla^2U + 2h_t\rho s^2U = 0$$

- Completely-clamped boundary conditions

$$U|_{x=0} = U|_{x=l} = U|_{y=0} = U|_{y=l} = 0$$

$$\left.\frac{\partial U}{\partial x}\right|_{x=0} \left.\frac{\partial U}{\partial x}\right|_{x=l} \left.\frac{\partial U}{\partial y}\right|_{y=0} \left.\frac{\partial U}{\partial y}\right|_{y=l} = 0$$

- Simply-supported boundary-conditions:

$$U|_{x=0} = U|_{x=l} = U|_{y=0} = U|_{y=l} = 0$$

$$\left.\frac{\partial^2 U}{\partial^2 x}\right|_{x=0} \left.\frac{\partial^2 U}{\partial^2 x}\right|_{x=l} \left.\frac{\partial^2 U}{\partial^2 y}\right|_{y=0} \left.\frac{\partial^2 U}{\partial^2 y}\right|_{y=l} = 0$$

Dimensional analysis

As was done in the elastica case, the dimensionless form of the equation is derived. The aim of the characteristic quantities for the function and the variable involved gives:

$$\begin{aligned} x &= l_o\tilde{x}, \qquad y = l_o\tilde{y}, \\ s &= s_o\tilde{s}, \\ U &= u_o\tilde{U}. \end{aligned}$$

The dimensionless plate equation becomes ([3]):

$$\frac{S_p}{l_o^4}\widetilde{\nabla^2\nabla^2}\tilde{U} - \frac{J_{\mathcal{I}}\rho}{l_o^2}s_o^2\tilde{s}^2\widetilde{\nabla^2}\tilde{U} + 2h_t\rho s_o^2\tilde{s}^2\tilde{U} = 0,$$

Considering the given expressions for $J_{\mathcal{I}}$ and S_p (see formulas (4.15)) we can write:

$$\frac{h_t^2\left(2\mu_L+\lambda_L\right)}{3\rho}\frac{1}{s_o^2 l_o^4}\widetilde{\nabla^2\nabla^2}\tilde{U} - \frac{h_t^2}{3l_o^2}\tilde{s}^2\widetilde{\nabla^2}\tilde{U} + \tilde{s}^2\tilde{U} = 0.$$

When the wave-lengths involved are much bigger than the thickness h_t of the plate, the following inequality holds:

$$\frac{h_t^2}{3l_o^2}\left\|\widetilde{\nabla^2}\tilde{U}\right\| \ll \left\|\tilde{U}\right\|, \tag{4.16}$$

and the plate equation becomes:

$$\widetilde{\nabla^2\nabla^2}\tilde{U} + \alpha\tilde{s}^2\tilde{U} = 0, \qquad \alpha = \frac{3\rho}{h_t^2\left(2\mu_L+\lambda_L\right)}s_o^2 l_o^4. \tag{4.17}$$

[3] The simbol ~ on the Laplacian and double laplacian operator represent its dimensionless form

Remark 82 *Accepting the inequality (4.16) means that equation (4.17) does not describe the component of a signal propagating in a plate when the wave-lengths are comparable to the thickness of the plate. Since the plate model holds when its thickness is much smaller than the dimension of the plate, this will not limit seriously the application range of the considered model. Moreover the finite difference approximation is sampling the deflection field in a finite set of locations, so that if the sampling step is bigger than the thickness of the plate, the error given by the previous position becomes negligible.*

Finite difference approximation

Let us now proceed with the finite element approximation of the equation and its relative boundary conditions. In order to do that, an uniform square grid of $n \times n$ nodes on the domain is chosen, each edge of which is of dimensionless with step $\varepsilon = h/l_o$. The values $\tilde{U}_{ij}$ of the dimensionless deflection $\tilde{U}$ on the ij-th node will be sampled.

First of all let us find the finite difference form of the double laplacian operator, the explicit form of which is:

$$\widetilde{\nabla^2 \nabla^2} = \frac{\partial^4}{\partial \tilde{x}^4} + \frac{\partial^4}{\partial \tilde{y}^4} + 2\frac{\partial^4}{\partial \tilde{x}^2 \partial \tilde{y}^2},$$

this will be approximated by the following FD-operator:

$$\left(\widetilde{\nabla^2\nabla^2} U\right)_{ij} = \Delta_i^4(\tilde{U}_{ij}) + \Delta_j^4(\tilde{U}_{ij}) + 2\Delta_j^2(\Delta_i^2\left(\tilde{U}_{ij}\right)).$$

The expression of the first two terms on the right hand side are simply the fourth order differences on the x and y domains:

$$\Delta_i^4(\tilde{U}_{ij}) = \frac{\tilde{U}_{i+2j} - 4\tilde{U}_{i+1j} + 6\tilde{U}_{ij} - 4\tilde{U}_{i-1j} + \tilde{U}_{i-2j}}{\varepsilon^4},$$

$$\Delta_j^4(\tilde{U}_{ij}) = \frac{\tilde{U}_{ij+2} - 4\tilde{U}_{ij+1} + 6\tilde{U}_{ij} - 4\tilde{U}_{ij-1} + \tilde{U}_{ij-2}}{\varepsilon^4},$$

while the third term can be calculated considering the expressions of the centered FD of second order, which gives:

$$\begin{aligned}\Delta_j^2(\Delta_i^2\tilde{U}_{ij}) = \frac{4\tilde{U}_{ij}}{\varepsilon^4} - 2\frac{\tilde{U}_{i+1j} + \tilde{U}_{i-1j}}{\varepsilon^4} - \\ - 2\frac{\tilde{U}_{ij+1} + \tilde{U}_{ij-1}}{\varepsilon^4} + \frac{\tilde{U}_{i+1j+1} + \tilde{U}_{i-1j+1} + \tilde{U}_{i+1j-1} + \tilde{U}_{i-1j-1}}{\varepsilon^4},\end{aligned}$$

The FD-double-laplacian operator becomes:

$$\begin{aligned}\left(\nabla^2\nabla^2\tilde{U}\right)_{ij} &= \frac{16\tilde{U}_{ij}}{\varepsilon^4} + \frac{\tilde{U}_{i+2j} - 6\tilde{U}_{i+1j} - 6\tilde{U}_{i-1j} + \tilde{U}_{i-2j}}{\varepsilon^4} + \\ &\quad \frac{\tilde{U}_{ij+2} - 6\tilde{U}_{ij+1} - 6\tilde{U}_{ij-1} + \tilde{U}_{ij-2}}{\varepsilon^4} + \\ &\quad \frac{\tilde{U}_{i+1j+1} + \tilde{U}_{i-1j+1} + \tilde{U}_{i+1j-1} + \tilde{U}_{i-1j-1}}{\varepsilon^4},\end{aligned}$$

and the expression of the FD-system for the plate:

$$\left(\widetilde{\nabla^2\nabla^2}\tilde{U}\right)_{ij} + \alpha\tilde{s}^2\tilde{U}_{ij} = 0,$$

assumes the following expression:

$$\left(\frac{16}{\varepsilon^4}+\alpha\tilde{s}^2\right)\tilde{U}_{ij}+\frac{\tilde{U}_{i+2j}-6\tilde{U}_{i+1j}-6\tilde{U}_{i-1j}+\tilde{U}_{i-2j}}{\varepsilon^4}+$$
$$+\frac{\tilde{U}_{ij+2}-6\tilde{U}_{ij+1}-6\tilde{U}_{ij-1}+\tilde{U}_{ij-2}}{\varepsilon^4}+$$
$$+\frac{\tilde{U}_{i+1j+1}+\tilde{U}_{i-1j+1}+\tilde{U}_{i+1j-1}+\tilde{U}_{i-1j-1}}{\varepsilon^4}=0.$$

Remark 83 *The first term on the left hand side is relative to the central node, the second and the third ones involve respectively the nodes along the x-direction and the y-one. Up to now we simply have the projection of the elastica FD-equation on the axial directions. However, The fourth term is found only in the 2D case and represents a relation between the central node and nodes on the diagonal directions. In the electric case this implies additional connections in the circuit relative to each central node.*

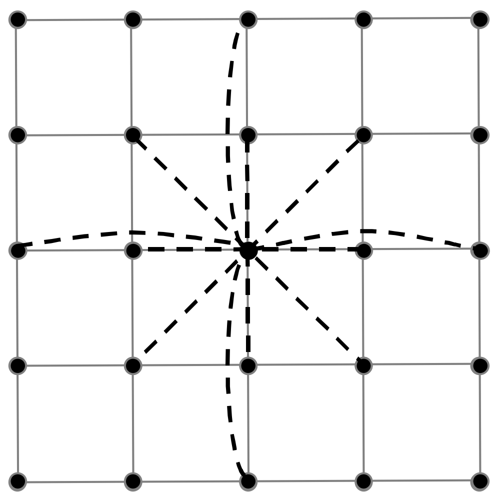


Figure 4.7: Nodes involved for each FD-equation.

As was done for the elastica case, it is useful to consider three different forms of the plate FD-equation, where the first and the third of which are obtained simply by dividing once for the dimensionless Laplace variable $\tilde{s}$ and then for $\tilde{s}^2$:

- First form:

$$\left(\frac{16}{\tilde{s}} + \tilde{s}\varepsilon^4\alpha\right)\tilde{U}_{ij} + \frac{1}{\tilde{s}}(\tilde{U}_{i+2j} - 6\tilde{U}_{i+j} - 6\tilde{U}_{i-j} + \tilde{U}_{i-2j} +$$
$$+\tilde{U}_{ij+2} - 6\tilde{U}_{ij+1} - 6\tilde{U}_{ij-1} + \tilde{U}_{ij-2} + 2\tilde{U}_{i+j+1} + 2\tilde{U}_{i-j+1} + 2\tilde{U}_{i+j-1} + 2\tilde{U}_{i-j-1}) = 0.$$

- Second form:

$$\left(16 + \tilde{s}^2\varepsilon^4\alpha\right)\tilde{U}_{ij} + (\tilde{U}_{i+2j} - 6\tilde{U}_{i+j} - 6\tilde{U}_{i-j} + \tilde{U}_{i-2j} +$$
$$+\tilde{U}_{ij+2} - 6\tilde{U}_{ij+1} - 6\tilde{U}_{ij-1} + \tilde{U}_{ij-2} + 2\tilde{U}_{i+j+1} + 2\tilde{U}_{i-j+1} + 2\tilde{U}_{i+j-1} + 2\tilde{U}_{i-j-1}) = 0.$$

- Third form:

$$\left(\frac{16}{\tilde{s}^2} + \tilde{s}\varepsilon^4\alpha\right)\tilde{U}_{ij} + \frac{1}{\tilde{s}^2}(\tilde{U}_{i+2j} - 6\tilde{U}_{i+j} - 6\tilde{U}_{i-j} + \tilde{U}_{i-2j} +$$
$$+\tilde{U}_{ij+2} - 6\tilde{U}_{ij+1} - 6\tilde{U}_{ij-1} + \tilde{U}_{ij-2} + 2\tilde{U}_{i+j+1} + 2\tilde{U}_{i-j+1} + 2\tilde{U}_{i+j-1} + 2\tilde{U}_{i-j-1}) = 0.$$

It is easy to see that the boundary conditions become the following expressions for the displacement on the boundary nodes:

- Dimensionless transformed completely-clamped boundary-conditions:

$$\left[\begin{array}{l} \tilde{U}\Big|_{\partial\mathcal{D}} = 0 \Rightarrow \tilde{U}_{1j} = \tilde{U}_{nj} = \tilde{U}_{i1} = \tilde{U}_{in} = 0, \qquad i,j = 1,..,n \\ \\ \dfrac{\partial U}{\partial \mathbf{n}}\Big|_{\partial\mathcal{D}} = 0 \Rightarrow \overrightarrow{\Delta}_{1j} = \overleftarrow{\Delta}_{nj} = \overrightarrow{\Delta}_{i1} = \overleftarrow{\Delta}_{in} = 0, \qquad i,j = 1,..,n \end{array}\right. \Rightarrow$$

$$U_{2j} = U_{n-1j} = U_{i2} = U_{in-1} = 0, \qquad i,j = 1,..,n$$

- Dimensionless transformed simply-supported boundary-conditions:

$$\left[\begin{array}{l} \tilde{U}\Big|_{\partial\mathcal{D}} = 0 \Rightarrow \tilde{U}_{1j} = \tilde{U}_{nj} = \tilde{U}_{i1} = \tilde{U}_{in} = 0, \qquad i,j = 1,..,n \\ \\ \dfrac{\partial U}{\partial \mathbf{n}}\Big|_{\partial\mathcal{D}} = 0 \Rightarrow \overrightarrow{\Delta}^2_{1j} = \overleftarrow{\Delta}^2_{nj} = \overrightarrow{\Delta}^2_{i1} = \overleftarrow{\Delta}^2_{in} = 0, \qquad i,j = 1,..,n \end{array}\right. \Rightarrow$$

$$\tilde{U}_{3j} = 2\tilde{U}_{2j}, \qquad \tilde{U}_{n-2j} = 2\tilde{U}_{n-1j}, \qquad \tilde{U}_{i3} = 2\tilde{U}_{i2}, \qquad \tilde{U}_{in-2} = 2\tilde{U}_{in-1}, \qquad i,j = 1,..,n.$$

Remark 84 *The plate equation becomes a set of* $(n-4)\times(n-4)$ *algebraic equations on the central nodes of the grid. While the boundary conditions are transformed into a set of equations on the boundary nodes. Not all the boundary conditions we considered are independent, in particular, for* i,j *equal* 1 *or* n*, the vertex nodes are taken into account two times. Hence the independent subset boundary conditions has a number of equations given by:*

$$(4n-4) + [4(n-2) - 4] = 8(n-2),$$

([4]) since the central equations are:

$$(n-4)^2 = n^2 - 8\,(n-2)\,,$$

a set of $n \times n$ equations in $n \times n$ unknowns is found.

Electro-mechanical analog

In order to develop the electric analog of the plate, a circuit of $n \times n$-nodes is considered: one electrical node for each node of the FD-grid. Then their tensions $\{V_{ij}\}$ are considered as the set of electrical unknowns, to be associated to the deflections $\tilde{U}_{ij}$ of the FD-nodes.

Each node is connected to the ground by an immittance of value Y_G. Hence the chosen tree of the circuit can be simply constituted by the set of all the edges between these nodes and the ground. This implies that an independent set of equations is given by the equilibrium of the currents getting in each node.

Recall that, the current flowing in a TTN depends only upon the difference of potential between its terminals. Hence, since only TTN are used, the current entering in a node ij, from each edge, is proportional only to the tension on the edge itself. As a result, each term of the equilibrium equation (once the TTN constitutive-relations are considered) will only depend upon the immittance of the relative edge.

Each FD-equation, besides the ij term, involves four terms for each axial

[4] They are the sum of the nodes of the external and internal boundary perimeter.

direction and four for the diagonal one, so that the following sets of nodes are involved:

$$\begin{aligned} \mathcal{N}_x &: \{i+1j, i-1j, i+2j, i-2j\} \\ \mathcal{N}_y &: \{ij+1, ij-1, ij+2, ij-2\} \\ \mathcal{N}_{xy} &: \{i+1j+1, i-1j+1, i+1j-1, i-1j-1\} . \end{aligned}$$

Consequently we must complete the tree of the circuit with co-tree edges which connect the ij node to all the nodes of the sets $(\mathcal{N}_x, \mathcal{N}_y, \mathcal{N}_{xy})$. From the symmetry of the FD-equations, it follows that:

$$\begin{aligned} Y_{i+1j} &= Y_{i-1j} = Y_{ij+1} = Y_{ij-1} = Y_1, \\ Y_{i+2j} &= Y_{i-2j} = Y_{ij+2} = Y_{ij-2} = Y_2, \\ Y_{i+1j+1} &= Y_{i-1j+1} = Y_{i+1j-1} = Y_{i-1j-1} = Y_3. \end{aligned}$$

The equilibrium of the currents on the ij-th node gives:

$$\begin{aligned} &(I_{i+2j} + I_{i+1j} + I_{i-1j} + I_{i-2j}) + (I_{ij+2} + I_{ij+1} + I_{ij-1} + I_{ij-2}) + \\ &\qquad + (I_{i+1j+1} + I_{i-1j+1} + I_{i+1j-1} + I_{i-1j-1}) + I_{G_i} = 0. \end{aligned}$$

Once the constitutive relations of the TTNs connected between the i-th and the j-th node are considered:

$$I_{i+p,j+q} = Y_{i+p,j+q} \left(V_{i+p,j+q} - V_{i,j}\right), \quad p, q \in \mathbb{N}.$$

the equilibrium equations become:

$$[Y_G + 4(Y_1 + Y_2 + Y_3)]\, V_{ij} - [(Y_2 V_{i+2j} + Y_1 V_{i+1j} + Y_1 V_{i-1j} + Y_2 V_{i-2j}) +$$

$$+ (Y_2 V_{ij+2} + Y_1 V_{ij+1} + Y_1 V_{ij-1} + Y_2 V_{ij-2}) + Y_3 (V_{i+1j+1} + V_{i-1j+1} + V_{i+1j-1} + V_{i-1j-1}) = 0$$

In order to compare the electrical to the mechanical set of equations, let us first consider the dimensionless form for the electrical ones. Hence the characteristic frequency, potential and immittance are introduced:

$$\left[\begin{array}{ll} s = s_o \tilde{s}, & \\ V_i = V_o \tilde{V}_i, & \\ Y_k = \frac{1}{R_o} \tilde{Y}_k, & \quad k \in \{1, 2, 3, G\}, \end{array}\right.$$

and the following dimensionless form of the electrical equilibrium equation is obtained:

$$\left[\tilde{Y}_M + 4\left(\tilde{Y}_1 + \tilde{Y}_2 + \tilde{Y}_3\right)\right] \tilde{V}_{ij} - (\tilde{Y}_2 \tilde{V}_{i+2j} + \tilde{Y}_1 \tilde{V}_{i+j} + \tilde{Y}_1 \tilde{V}_{i-j} + \tilde{Y}_2 \tilde{V}_{i-2j} +$$

$$+ \tilde{Y}_2 \tilde{V}_{ij+2} + \tilde{Y}_1 \tilde{V}_{ij+1} + \tilde{Y}_1 \tilde{V}_{ij-1} + \tilde{Y}_2 \tilde{V}_{ij-2} + \tilde{Y}_3 \tilde{V}_{i+j+1} + \tilde{Y}_3 \tilde{V}_{i-j+1} + \tilde{Y}_3 \tilde{V}_{i+j-1} + \tilde{Y}_3 \tilde{V}_{i-j-1}) = 0$$

The electric analog of the plate is found comparing the coefficients of the previous relation to the coefficients of each of the three different forms of the mechanical equations which has been found. Consequently, three different sets of TTN-immittances able to synthesize the circuit are derived:

- Case 1: matching the coefficient of the electric analog to the ones of the first mechanical equation we get:

$$\begin{cases} Y_1\left(s_o\tilde{s}\right) = \dfrac{6}{\alpha\varepsilon^4 R_o}\dfrac{1}{\tilde{s}}, \\ Y_2\left(s_o\tilde{s}\right) = -\dfrac{1}{\alpha\varepsilon^4 R_o}\dfrac{1}{\tilde{s}}, \\ Y_3\left(s_o\tilde{s}\right) = -\dfrac{1}{\alpha\varepsilon^4 R_o}\dfrac{1}{\tilde{s}}, \\ Y_M\left(s_o\tilde{s}\right) = \dfrac{1}{R_o}\tilde{s}, \end{cases}$$

Since Y_1, Y_2, Y_3 are proportional to $\frac{1}{s}$ they are immittances characteristic of inductances L_1, L_2, L_3, while Y_M, which is proportional to s, is characteristic of a capacitance C_M. If we consider the following characteristic value for an inductance L and a capacitance C:

$$\begin{aligned} L &= \alpha\varepsilon^4 \frac{R_o}{s_o}, \\ C &= \frac{1}{R_o s_o}, \end{aligned} \tag{4.18}$$

the values of the inductors and capacitors that must be used in the circuit are:

$$\begin{aligned} L_1 &= \frac{L}{6}, \\ L_2 &= L_3 = -L, \\ C_M &= C = \frac{1}{R_o s_o}. \end{aligned}$$

- Case 2: matching the coefficient of the electric analog to the ones of the second mechanical equation we get:

$$\left[\begin{array}{l} Y_1\left(s_o\tilde{s}\right) = \dfrac{6}{\alpha\varepsilon^4 R_o}, \\ Y_2\left(s_o\tilde{s}\right) = Y_3\left(s_o\tilde{s}\right) = -\dfrac{1}{\alpha\varepsilon^4 R_o}, \\ Y_M\left(s_o\tilde{s}\right) = \dfrac{1}{R_o}\tilde{s}^2. \end{array}\right.$$

Let us remark that Y_M is an immittance of a super-capacitor of value

$$F = \frac{1}{s_o^2 R_o}$$

while Y_1 and Y_2 are the immittance of resistors of values R_1 and R_2:

$$\begin{aligned} R_1 &= \frac{\alpha\varepsilon^4 R_o}{6}, \\ R_1 &= -\alpha\varepsilon^4 R_o. \end{aligned}$$

- Case 3: matching the coefficient of the electric analog to the ones of the third mechanical equation we get:

$$\left[\begin{array}{l} Y_1\left(s_o\tilde{s}\right) = \dfrac{6}{\alpha\varepsilon^4 R_o}\dfrac{1}{\tilde{s}^2}, \\ Y_2\left(s_o\tilde{s}\right) = Y_3\left(s_o\tilde{s}\right) = -\dfrac{1}{\alpha\varepsilon^4 R_o}\dfrac{1}{\tilde{s}^2}, \\ Y_M\left(s_o\tilde{s}\right) = \dfrac{1}{R_o}, \end{array}\right.$$

Here Y_M is a resistor of value

$$R_M = R_o$$

and the immittances Y_1,Y_2 are super-inductors of value

$$\begin{aligned} D_1 &= \frac{\alpha\varepsilon^4 R_o}{6}, \\ D_2 &= -\alpha\varepsilon^4 R_o \end{aligned}$$

Boundary conditions

The first boundary conditions requirements for both the completely-clamped and the simply-connected case, implies electric potential equal to zero on the boundary nodes. This simply means that all the external nodes of the plate have a short circuit to the ground.

The second relation for the completely-clamped case implies that also the potential on the internal boundary nodes must vanish, so that, again, all of the nodes are directly connected to the ground.

The second requirement for the simply-connected boundary conditions requests the potential on the internal boundary nodes to be equal to the potential on its adjacent external central-nodes. This can be obtained using the proper circuit presented in the elastica case.

Conclusions

Let us consider the homogenized equation related to the electrical solution system, the dimensionless form of which is:

$$\widetilde{\nabla^2\nabla^2}\tilde{v} + \alpha\tilde{s}^2\tilde{v} = 0,$$

in which the independent variable is now the dimensionless electric-potential $\tilde{v}$. The coefficient α can be derived from the relations (4.18):

$$\begin{bmatrix} L = \alpha\varepsilon^4 \dfrac{R_o}{s_o} \\ C = \dfrac{1}{R_o s_o} \end{bmatrix} \quad \Rightarrow \quad \alpha = \frac{s_o^2 LC}{\varepsilon^4} \quad \Rightarrow \quad \alpha = \frac{LC}{h^4} s_o^2 l_o^4.$$

Remark 85 *Since the phase speed is strictly related to the dimensionless coefficient α ([5]):*

$$\tilde{v}\left(\tilde{\lambda}\right) = \pm \frac{2\pi}{\tilde{\lambda}} \sqrt{\frac{1}{\alpha}}$$

than the phase speed of a plane-wave propagating in the plate equals the phase speed of the wave in the plate circuit having the same dimensionless coefficient α, i.e.:

$$\alpha_m = \frac{3\rho}{h_t^2 \left(2\mu_L + \lambda_L\right)} s_o^2 l_o^4 \qquad \equiv \qquad \alpha_e = \frac{LC}{h^4} s_o^2 l_o^4.$$

Hence the analog is obtained when:

$$LC = \frac{3\rho h^4}{h_t^2 \left(2\mu_L + \lambda_L\right)}.$$

[5] Where now with $\tilde{\lambda}$ we intend the modulus of the dimensionless wave-length vector.

Chapter 5

Coupled electro-mechanical systems

In this chapter we analyze the properties of the electro-mechanical systems which are realized by coupling, through piezoelectric effect, the mechanical devices and their electric analogs.

Let us remark how both lumped and homogenized models have been considered for both the mechanical and electrical systems introduced. In fact, in the first chapter, we developed the wave equations for the mechanical systems using homogenized models. Then, using the finite elements method, we found their relative lumped models, thus deriving their electric lumped analog. Moreover we synthesized those lumped models using electrical circuits; finally, in order to present some wave-propagation properties of the electric systems, the homogenized models for the circuits have been introduced.

The coupling between the mechanical system and its electric circuit-analog will be physically realized by connecting piezoelectric actuators uniformly distributed on the mechanical device to the electric circuit. From an electrical point of view, it is easy to take into account the piezoelectric effect using the lumped model of the circuit, while from a mechanical viewpoint it will be easier to consider a distributed piezoelectric effect upon the entire domain. Finally an homogenized model of the coupled system will be developed.

In order to do this, let us introduce what a piezoelectric material is, how it can be used to realize an electro-mechanical actuator, how a piezoelectric actuator can be modelled and how it is possible to take into account the piezoelectric effect which it generates into the electro-mechanical system.

Piezoelectric materials

A piezo-electric (PZT) material is a media in which the effect of a mechanical deformation field $\mathbf{E}$ produces a stress field $\mathbf{T}$ and an electric-induction field $\mathbf{D}$. Vice versa, the medium response to an electric field $\mathbf{e}$ is given by an electrical induction and a mechanical force field. This implies that the state variables of a piezoelectric material are given by a couple of a mechanical and an electrical variable, for instance, the deformation and the electric field: $(\mathbf{E}, \mathbf{e})$. In general, the constitutive relations of a linear, time invariant, homogenous PZT medium can be expressed in the following form:

$$\begin{bmatrix} \mathbf{T} \\ \mathbf{D} \end{bmatrix} = \begin{bmatrix} \mathbf{K}_{mm} & \mathbf{K}_{me} \\ \mathbf{K}_{em} & \mathbf{K}_{ee} \end{bmatrix} \begin{bmatrix} \mathbf{E} \\ \mathbf{e} \end{bmatrix},$$

where $\mathbf{K}_{ee}$ is a second order constant tensor which represents the permittivity, $\mathbf{K}_{mm}$ is a fourth-order constant stress-strain tensor. $\mathbf{K}_{em}$ and $\mathbf{K}_{me}$ are third order piezoelectric tensors which are responsible for the coupling between electrical and mechanical phenomena. In order to satisfy the first principle of thermodynamics the following relations between $\mathbf{K}_{em}$ and $\mathbf{K}_{me}^{\top}$ must hold:

$$\mathbf{K}_{em} = -\mathbf{K}_{me}^{\top};$$

Remark 86 *This implies that the energy flowing away from the mechanical form is transmitted to the electrical one and vice versa.*

The previous relation shows that a piezoelectric medium is an anisotropic material. **Transversely isotropic** dielectrics are the simplest piezoelectric materials, to which we will limit our attention. They have an axis of transverse isotropy, i.e. every rotation about it belongs to its material symmetry group.

Piezoelectric materials are used to realize both electro-mechanical sensors and actuators. The main difference between those two kinds of devices is: sensors are used to measure mechanical deformations, so that their interaction with the mechanical system must be minimized. Actuators are used to drive mechanical systems, so that the energy exchange between the mechanical and electrical form must be maximized.

Piezo-electric actuators

Piezo-electric actuators are realized using elements of transversely isotropic PZT-materials, which show the highest coupling effect. These elements are connected to various ways to electrodes that make them electrically accessible, from the external world.

Different kinds of actuators have been conceived depending on the particular kind of deformation we want to couple with, for instance extension or bending deformations. Moreover they can be described just using average quantities representative of their internal behavior. Consequently, the potential between the electrodes (instead of the electric field on it) will represent its electric state-variable; and a stress or bending-moment on its axial directions (instead of the stress field) will represent the mechanical state variables. (for more details about PZT actuators see [6])

Extensional actuators

If we are interested in coupling with an extension (or compression) of a mechanical device, the extensional actuators can be used. The variation of lengths Δ_x, Δ_y along the axial directions can be considered as mechanical state variables, the tension between its terminals, v, as the electrical one. In this case considering the forces F_x, F_y and the stored electric charge, Q, the constitutive relations of the actuators can be expressed in the following form ([1]):

$$\begin{bmatrix} F_x \\ F_y \\ Q \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} & -k_{em} \\ k_{12} & k_{22} & -k_{em} \\ k_{em} & k_{em} & k_{ee} \end{bmatrix} \begin{bmatrix} \Delta_x \\ \Delta_y \\ v \end{bmatrix}.$$

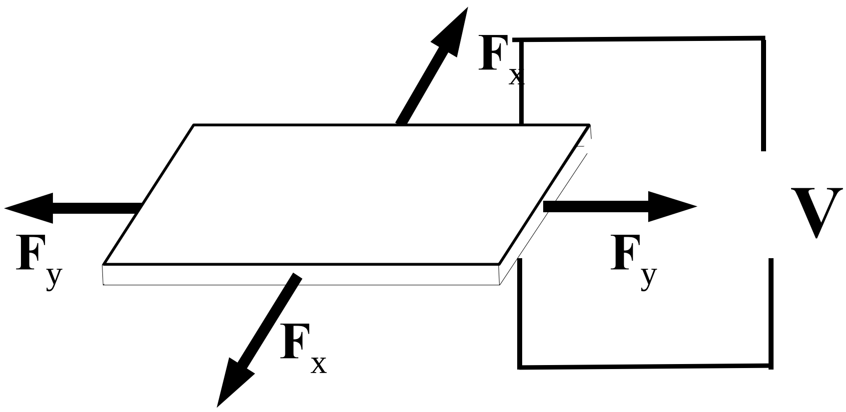


Figure 5.1: Extensional actuator

`Bending actuators`

If we are interested in coupling with a deflection, it is convenient to consider bending actuators. Here the integral constitutive relations (which relate the variation of attitude ϑ_x, ϑ_y on the axial directions and the tension v to the bending moments M_x, M_y and the charge Q) can be expressed in the following form ([2]):

$$\begin{bmatrix} M_x \\ M_y \\ Q \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} & -g_{em} \\ g_{12} & g_{22} & -g_{em} \\ g_{em} & g_{em} & g_{ee} \end{bmatrix} \begin{bmatrix} \vartheta_x \\ \vartheta_y \\ v \end{bmatrix}.$$

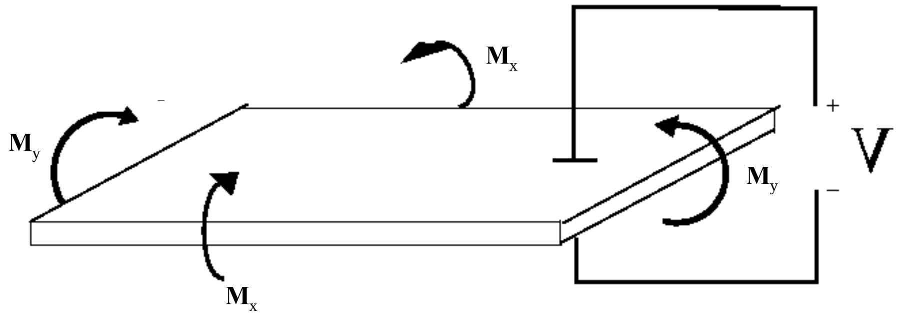


Figure 5.2: Bending PZT actuator

Remark 87 *In the electro-mechanical coupling which will be considered, only bending actuators will be used.*

Model of the actuators effect in a piezo-electric system

With piezoelectric system we intend one of the mechanical structures presented, a beam or a plate, connected to its relative circuit analog by PZT actuators uniformly distributed on its surface. In particular, the actuators will be glued with their directions of action parallel to the same preferred axis characteristic of the mechanical system considered.

[1] The variation in length $\Delta_{x/y}$ is simply related to the deformation $\varepsilon_{x/y}$: $\Delta_{x/y} = l_p \varepsilon_{x/y}$, where l_p is the length of the patch.

[2] The variation of attitude $\vartheta_{x/y}$ is simply related to the curvature $\zeta_{x/y}$: $\vartheta_{x/y} = l_p \zeta_{x/y}$.

PZT interaction model

Let us recall that all the considerations we did in developing the electrical analog hold when the wavelengths involved are sufficiently greater than the step used in sampling the spatial domain (which must be greater or at least equal to the dimensions of the piezoelectric actuators). Consequently it is reasonable to consider the actuators as elements concentrated on the nodes of the domain. This permits us to consider a lumped model of the PZT interaction.

Let us consider the mechanical homogenized model and the electrical lumped one, for a given piezo-electro-mechanical (PEM) system. Hence, besides the continuum mechanical domain, we consider a set of discrete nodes which sample the domain corresponding to each piezoelectric actuator. Each actuator supplies an electro-mechanical effect concentrated on its relative node, which also corresponds to one node of the electric circuital analog. Thus the variation of attitude ϑ_x, ϑ_y induced by the system on each actuator will be represented by the curvatures ζ_x, ζ_y which are related to the second derivatives of the deflection u on the ij-node:

$$\begin{aligned} \vartheta_x &\simeq l_p \zeta_x = l_p \left. \frac{\partial^2 u(x,y)}{\partial x^2} \right|_{ij} = l_p \left. u_{,xx} \right|_{ij}, \\ \vartheta_y &\simeq l_p \zeta_y = l_p \left. \frac{\partial^2 u(x,y)}{\partial x^2} \right|_{ij} = l_p \left. u_{,yy} \right|_{ij}. \end{aligned}$$

The resultant bending moments M_x, M_y, on the actuators, can be replaced by the components of a PZT bending-moments tensor $\mathbf{M}_a$ applied in the nodes ([3]):

$$M_x \simeq l_p \, M_{a_{xx}}|_{ij} \, ,$$

$$M_y \simeq l_p \, M_{a_{yy}}|_{ij} \, .$$

The electrical tension v on each actuator will simply represent the tension between two nodes of the electrical circuit. In order to consider the tension of each actuator as the potential of its representative node, one of the two terminals of each actuator will be connected to the ground.

The electric charge Q of the actuator will be the charge stored between the electrical node and the ground.

Assuming that the behavior of the PZT actuator is the same in the two axial directions, i.e. $g_{11} = g_{22} = g_{mm}$, the constitutive relations for the PZT at each node are:

$$\begin{bmatrix} M_{a_{xx}} \\ M_{a_{yy}} \\ Q \end{bmatrix}_{ij} = \begin{bmatrix} g_{mm} & 0 & -\frac{g_{em}}{l_p} \\ 0 & g_{mm} & -\frac{g_{em}}{l_p} \\ l_p g_{em} & l_p g_{em} & g_{ee} \end{bmatrix} \begin{bmatrix} u_{,xx} \\ u_{,yy} \\ v \end{bmatrix}_{ij} . \tag{5.1}$$

Remark 88 *If we are dealing with a 1D problem in 1D x-domain, the mechanical state variable can be considered as the integral of the 2D deflection u along the*

[3] It will be assumed that the components of $\mathbf{M}_a$ expending power on the curvature components $u_{,xy}$, $u_{,yx}$ vanish, i.e.$g_{12} = g_{21} = 0$. The PZT bending moment tensor associated with the concentrated effect of the actuator on the mechanical structure can be represented as:

$$\mathbf{M}_a = M_{a_{xx}} \left(\hat{\mathbf{x}} \otimes \hat{\mathbf{x}}\right) + M_{a_{yy}} \left(\hat{\mathbf{y}} \otimes \hat{\mathbf{y}}\right) - g_{me} v \hat{\mathbf{1}}.$$

where $(\hat{\mathbf{x}} \otimes \hat{\mathbf{x}})$ and $(\hat{\mathbf{y}} \otimes \hat{\mathbf{y}})$ are the tensor projections in the axial directions.

y-domain, and the same holds for the bending moment tensor. With an abuse of notation we will continue to call the 1D quantities with the same symbol as the 2D ones, so that the constitutive relations can be expressed in the following way ([4])*:*

$$\begin{bmatrix} M_a \\ Q \end{bmatrix}_i = \begin{bmatrix} l_p g_{mm} & -g_{em} \\ l_p g_{em} & g_{ee} \end{bmatrix} \begin{bmatrix} u'' \\ v \end{bmatrix}_i . \tag{5.2}$$

Electric point of view In order to find a circuit interpretation of the piezo-electric effect, let us observe that, from an electrical point of view, the actuator can be regarded as an element connecting one node to the ground. The relations between the current and the tension on it can be derived considering the last constitutive relation of (5.1), (or 5.2 for 1D case), which is:

$$Q = g_{ee}v + l_p g_{em} \nabla^2 u, \qquad (\text{or } Q = g_{ee}v + l_p g_{em} u'').$$

since it depend upon the charge Q stored in the actuator, the explicit relation between the current and the tension is obtained simply differentiating it with respect to time:

$$i = g_{ee}\dot{v} + l_p g_{em} \nabla^2 \dot{u}, \qquad (\text{or } i = g_{ee}\dot{v} + l_p g_{em} \dot{u}_{,xx}).$$

The previous relation shows that the current i is a composition of two terms

$$i = i_1 + i_2,$$

[4] Let us recall the simbol ($\prime$) represents the derivative in the space domain.

$$i_1(t) = g_{ee}\frac{\partial v(t)}{\partial t} \qquad \Rightarrow \qquad I_1(s) = g_{ee}sV(s)$$

$$i_2(t) = l_p g_{em}\nabla^2 \dot{u}(t) \qquad \Rightarrow \qquad I_2(s) = s l_p g_{em}\nabla^2 U(s),$$

$$(\text{or } i(t) = l_p g_{em}\dot{u}''(t) \Rightarrow I_2(s) = s l_p g_{em} U''(s)\,)$$

which can be represented by two different TTN is between the node of the actuator and the ground. In the first term the current depends directly upon the time derivative of the tension, hence it represents a capacitor of capacitance g_{ee}, the immittance of which (in the frequency domain) is $Y_1 = sg_{ee}$.

The second term does not depend upon the tension on the actuator, it simply impresses a value of current due to the deformation of the mechanical system. Hence it is the electro-mechanical coupling term and will be represented by a current generator of impressed value $\bar{\imath}_2$.

Remark 89 *Let us recall that, for each mechanical device, we have derived one circuit electric-analog in which each node was connected to the ground by capacitances. This suggests that we can realize the electric circuit using the piezoelectric actuators as the capacitances between each node and the ground. The current generators will transform the circuit equations simply adding to them a non-homogenous term due to the coupling.*

Mechanical point of view From a mechanical point of view it is useful to consider the piezoelectric effect uniformly distributed upon the domain. This implies that we will consider fields of the mechanical quantities defined on the entire domain, instead of their values on each node. Consequently, since the electric quantities have been defined only on the nodes, we must consider them uniformly distributed on their area (or length in 1D case) of influence, which is given by area (length) of the domain divided by the number of the actuators. If we consider the influence region as the square of an edge (step of length) l_T/N, where N is the total number of actuators, l_T is the length of the plate (or beam), then the distributed piezoelectric effect can be obtained applying the scaling factor $\eta = Nl_p^2/l_T^2$ (or $\eta = Nl_p/l_T$) to the dynamic-actions vector $\begin{bmatrix} M_{a_{xx}} & M_{a_{yy}} \end{bmatrix}$. Hence, defining the new dynamic vector:

$$\begin{bmatrix} M_{pz_{xx}} \\ M_{pz_{yy}} \\ Q \end{bmatrix} = \begin{bmatrix} \eta M_{a_{xx}} \\ \eta M_{a_{yy}} \\ Q \end{bmatrix},$$

the homogenized constitutive relation becomes the following:

- 2D case

$$\begin{bmatrix} M_{pz_{xx}} \\ M_{pz_{yy}} \\ Q \end{bmatrix} = \eta \begin{bmatrix} \eta g_{mm} & 0 & -\eta \frac{g_{em}}{l_p} \\ 0 & \eta g_{mm} & -\eta \frac{g_{em}}{l_p} \\ l_p g_{em} & l_p g_{em} & g_{ee} \end{bmatrix} \begin{bmatrix} u_{,xx} \\ u_{,yy} \\ v \end{bmatrix} \tag{5.3}$$

- 1D case

$$\begin{bmatrix} M_{pz} \\ Q \end{bmatrix} = \eta \begin{bmatrix} \eta g_{mm} l_p & -\eta g_{em} \\ g_{em} l_p & g_{ee} \end{bmatrix} \begin{bmatrix} u'' \\ v \end{bmatrix} \tag{5.4}$$

Piezo-electro-mechanical elastica

The piezo-electro-mechanic (PEM) beam is a device realized by gluing n bending piezoelectric-actuators along a beam of length l. Hence each actuator will have an influence length h:

$$h = \frac{l}{n}.$$

One electrical terminal of each actuator will be connected to the ground node of the electric analog circuit, the other one will represent one node of the circuit, which is realized using the first scheme developed in the previous section ([5]). In this way the piezoelectric actuator will play the role of a capacitor in the electric analog. Then each node i is connected to its adjacent ones $i+1$ and $i-1$ by an inductance of value $L/4$ ([6]), and to the nodes $i+1$and $i-1$ by an inductance $-L$.

In order to see how the PZT can play the role of the given capacitance C ([7]) in the circuit, let us recall how, in coupling the mechanical and electrical system, we expect to obtain the maximum of energy exchange when the two isolated systems support the propagation of an elementary wave with the same phase speed (i.e.

[5] It was realized connecting capacitances C between each node and the ground and different inductances between each node and its four adjacent ones.

[6] The value of the characteristic inductance L defined in the previous chapter was:

$$L = R_o/s_o$$

[7] The value of the characteristic capacitance C defined in the previous chapter was:

$$C = \frac{\alpha}{R_o s_o}\left(\frac{h}{l_o}\right)^4$$

they have the same dispersive equation). In deriving the electric analog for the elastica, it was seen that the dependence of the phase speed is related to the electric parameters by the term $1/\sqrt{LC}$. Hence the product between L and C, has to match a given value imposed by the mechanical system. This degree of freedom in choosing the electric parameters allows us to set the characteristic capacitance C equal to the piezoelectric one:

$$C = g_{ee}.$$

As a consequence the inductance of the electric circuit will be determined.

Electrical coupled equation

From the electrical point of view we have exactly the circuit elastica analog which has been analyzed in the previous chapter, excited by the coupling term of the PZT actuators. In order to find the source term arising from the PZT coupling, let us consider the PZT constitutive relation in the frequency domain:

$$I_{T_i} = sCV_i + sl_p g_{em} U_i'',$$

where I_{T_i} (the total current flowing from the i node to the ground) is divided in two components:

$$\begin{aligned} I_{G_i} &= sCV_i, \\ \bar{I}_{G_i} &= sl_p g_{em} U_i'', \end{aligned}$$

The first component is due to the capacitance of the actuator while the second is the impressed coupling term, which simply adds a non-homogenous term to the equilibrium equation of the elastica circuit (4.7):

$$I_{i+2} + I_{i+1} + I_{i-1} + I_{i-2} + I_{G_i} = -\bar{I}_{G_i}.$$

considering the value of the immittances of the edges connected to the i-th node:

$$\begin{cases} Y_{i+1} = Y_{i-1} = \dfrac{4}{sL}, \\ Y_{i+2} = Y_{i-2} = \dfrac{-1}{sL}, \\ Y_i = sC, \end{cases}$$

the following form of the electrical equations can be derived:

$$(V_{i+2} - 4V_{i+1} + 6V_i - 4V_{i-1} + V_{i-2}) + s^2 LCV_i = -s^2 Ll_p g_{em} U_i''.$$

In order to analyze the behavior of the electromechanical system it is useful to consider the homogenized form of the previous equation. It can be derived going backwards from a finite difference form to the differential one. The fourth FD-difference in the equation is rebuilt just dividing all of the expression by the fourth power of the actuator influence-length h:

$$\frac{V_{i+2} - 4V_{i+1} + 6V_i - 4V_{i-1} + V_{i-2}}{h^4} + s^2 \frac{LC}{h^4} V_i = -s^2 \frac{Ll_p g_{em}}{h^4} U_i''$$

Then considering the potential uniformly distributed on all of the domain l of the beam, the homogenized form of the electrical equation for the piezo-electro-mechanical elastica is found:

$$V^{IV} + \frac{LC}{h^4} s^2 V = -s^2 \frac{Ll_p g_{em}}{h^4} U''.$$

In the time domain the equation becomes:

$$v^{IV} + \frac{LC}{h^4}\ddot{v} + \frac{Ll_p g_{em}}{h^4}\ddot{u}'' = 0, \qquad \text{or} \qquad \frac{\partial^4 v}{\partial x^4} + \frac{LC}{h^4}\frac{\partial^2 v}{\partial t^2} + \frac{Ll_p g_{em}}{h^4}\frac{\partial^4 u}{\partial x^2 \partial t^2} = 0. \tag{5.5}$$

Let us recall that the scheme we are using was developed equating the coefficients of the mechanical equation divided by the frequency variable s, it was underlined how this implies that we were matching, not just the elastica equation, but its time integral. Hence the electrical analog of the deflection we must consider is not exactly the potential v but its time integral ψ:

$$v = \frac{d\psi}{dt} = \dot{\psi},$$

The piezo-electro-mechanical equation in terms of ψ is obtained as the time integral of the equation (5.5):

$$\frac{\partial^4 \psi}{\partial x^4} + \frac{LC}{h^4}\frac{\partial^2 \psi}{\partial t^2} + \frac{Ll_p g_{em}}{h^4}\frac{\partial^3 u}{\partial x^2 \partial t} = 0.$$

Let us consider its dimensionless form:

$$\frac{v_o}{s_o l_o^4}\frac{\partial^4 \tilde{\psi}}{\partial \tilde{x}^4} + s_o v_o \frac{LC}{h^4}\frac{\partial^2 \tilde{\psi}}{\partial \tilde{t}^2} + \frac{Ll_p g_{em}}{h^4}\frac{s_o u_o}{l_o^2}\frac{\partial^3 \tilde{u}}{\partial \tilde{x}^2 \partial \tilde{t}} = 0.$$

Finally the following electrical coupled equation is obtained:

$$\frac{\partial^4 \tilde{\psi}}{\partial \tilde{x}^4} + \alpha_e \frac{\partial^2 \tilde{\psi}}{\partial \tilde{t}^2} + \beta_e \frac{\partial^3 \tilde{u}}{\partial \tilde{x}^2 \partial \tilde{t}} = 0, \qquad \alpha_e = s_o^2 l_o^4 \frac{LC}{h^4}, \qquad \beta_e = \frac{s_o^2 l_o^2 u_o}{v_o}\frac{Ll_p g_{em}}{h^4}.$$

Mechanical coupled equation

The piezoelectric effect, from a mechanical point of view, can be regarded as an additional contribution M_{pz} to the purely mechanical moment M_m of the beam.

Hence the total moment M_T which has to be considered in the beam equation is:

$$M_T = M_m + M_{pz},$$

where the expressions for M_m and M_{pz} are given by the constitutive relations of the beam and actuator:

$$\begin{aligned} M_m &= K_M u'', \\ M_{pz} &= \eta l_p g_{mm} u'' - \eta g_{em} v. \end{aligned}$$

The total bending moment for the piezo-electro-mechanical beam is:

$$M_T = (K_M + \eta l_p g_{mm})\, u'' - \eta g_{em} v$$

It is reasonable to assume that the bending-stiffness K_M of the beam is much greater that the purely mechanical one of the actuator:

$$K_M \gg \eta l_p g_{mm},$$

thus the following constitutive relation for the beam is assumed:

$$M = K_M u'' - \eta g_{em} v. \tag{5.6}$$

In order to develop the coupled mechanical equation for our system let us recall the equilibrium equation for the bending moment in a beam:

$$M_T'' - \rho \frac{\partial^2 u}{\partial t^2} = 0.$$

Substituting the constitutive relation (5.6) into this, the second piezo-electro-mechanical equation is obtained:

$$u^{IV} + \frac{\rho}{K_M}\ddot{u} - \frac{g_{em}}{K_M}v'' = 0, \qquad \text{or} \qquad \frac{\partial^4 u}{\partial x^4} + \frac{\rho}{K_M}\frac{\partial^2 u}{\partial t^2} - \frac{\eta g_{em}}{K_M}\frac{\partial^2 v}{\partial x^2} = 0.$$

In order to compare this to the first one, let us consider the time integral of the potential ψ instead of the electric potential v:

$$\frac{\partial^4 u}{\partial x^4} + \frac{\rho}{K_M}\frac{\partial^2 u}{\partial t^2} - \frac{\eta g_{em}}{K_M}\frac{\partial^3 \psi}{\partial x^2 \partial t} = 0.$$

Let us consider its dimensionless form:

$$\frac{u_o}{l_o^4}\frac{\partial^4 u}{\partial x^4} + s_o^2 u_o \frac{\rho}{K_M}\frac{\partial^2 u}{\partial t^2} - \frac{v_o}{l_o^2}\frac{\eta g_{em}}{K_M}\frac{\partial^3 \psi}{\partial x^2 \partial t} = 0.$$

Finally the mechanical coupled equation is obtained:

$$\frac{\partial^4 \tilde{u}}{\partial \tilde{x}^4} + \alpha_m \frac{\partial^2 \tilde{u}}{\partial \tilde{t}^2} - \beta_m \frac{\partial^3 \tilde{\psi}}{\partial \tilde{t} \partial \tilde{x}^2} = 0 \qquad \alpha_m = s_o^2 l_o^4 \frac{\rho}{K_M} \qquad \beta_m = l_o^2 \frac{v_o}{u_o}\frac{\eta g_{em}}{K_M}.$$

PzEM System

The evolution of the piezo-electro-mechanical system is described by the following system of coupled equations:

$$\frac{\partial^4 \tilde{u}}{\partial \tilde{x}^4} + \alpha_m \frac{\partial^2 \tilde{u}}{\partial \tilde{t}^2} - \beta_m \frac{\partial^3 \tilde{\psi}}{\partial \tilde{t} \partial \tilde{x}^2} = 0 \qquad \alpha_m = s_o^2 l_o^4 \frac{\rho}{K_M} \qquad \beta_m = l_o^2 \frac{v_o}{u_o}\frac{\eta g_{me}}{K_M}.$$

$$\frac{\partial^4 \tilde{\psi}}{\partial \tilde{x}^4} + \alpha_e \frac{\partial^2 \tilde{\psi}}{\partial \tilde{t}^2} + \beta_e \frac{\partial^3 \tilde{u}}{\partial \tilde{x}^2 \partial \tilde{t}} = 0, \qquad \alpha_e = s_o^2 l_o^4 \frac{LC}{h^4}, \qquad \beta_e = \frac{s_o^2 l_o^2 u_o}{v_o}\frac{L l_p g_{em}}{h^4}.$$

In order to obtain a perfect analog we must set the electrical dimensional coefficients equal to the mechanical ones:

$$\alpha_m = \alpha_e \Rightarrow LC = \frac{\rho h^4}{K_M}$$

Since the capacitance C is given by the actuator value g_{ee}, we get a value of the inductance L:

$$L = \frac{\rho h^4}{CK_M}.$$

We can obtain the equality of the coupling coefficients $\beta_e = \beta_m$ by setting the characteristic tension of the problem:

$$v_o = s_o u_o \sqrt{\frac{l_p \rho}{\eta C}}.$$

Remark 90 *Further comments to this result will be developed for the plate case.*

Piezo-electro-mechanical plate

The piezo-electro-mechanical plate is realized by gluing $n \times n$ actuators uniformly on the plate. Hence the region of influence of each patch will be an area h^2, where h is the step between two actuators along an axial direction. As done in the elastica case, one terminal of each actuator will be connected to the ground, and the other will represent one node of the circuit plate analog. Hence, the actuator itself will represent the capacitance of the circuit, which is completed connecting each node to its adjacent ones in both the axial and diagonal directions by the inductance $-L$ ([8]) and connecting the two step length nodes in the axial direction with an inductance of value $\frac{L}{6}$. The degrees of freedom we have in tuning the phase speed of the waves into the circuit allow us to set the capacitance of the circuit equal to the piezoelectric one:

$$C = g_{ee}.$$

Electrical coupling equation

From the electrical point of view we have exactly the circuit analog of the plate excited by the coupling term of the actuators. Let us recall the electrical

[8] In the case of the plate the characteristic values for L and C were:

$$\begin{aligned} L &= \alpha \varepsilon^4 \frac{R_o}{s_o} \\ C &= \frac{1}{R_o s_o} \end{aligned}$$

constitutive relation of the actuators in the 2D case:

$$i = g_{ee}\dot{v} + l_p g_{em} \nabla^2 \dot{u},$$

The current is divided into its capacitive term i_{G_i} and its impressed one $\bar{\imath}_{G_i}$.In the Laplace domain, we get:

$$I_{T_i} = I_{G_i} + \bar{I}_{G_i}$$

where:

$$\left[\begin{array}{l} I_{G_i} = g_{ee} sV, \\ \bar{I}_{G_i} = s l_p g_{em} \nabla^2 U. \end{array}\right.$$

As in the elastica case, the impressed component will only represent the non-homogenous term in the current equilibrium equations of the plate analog circuit:

$$(I_{i+2j} + I_{i+1j} + I_{i-1j} + I_{i-2j}) + (I_{ij+2} + I_{ij+1} + I_{ij-1} + I_{ij-2}) +$$
$$+ (I_{i+1j+1} + I_{i-1j+1} + I_{i+1j-1} + I_{i-1j-1}) + I_{G_i} = -\bar{I}_{G_i}.$$

considering the value of the immittances of the edges connected to the i-th node:

$$\left[\begin{array}{l} Y_{i+1j} = Y_{i-1j} = Y_{ij+1} = Y_{ij-1} = Y_{i-1j+1} = Y_{i+1j-1} = Y_{i+1j+1} = Y_{i-1j-1} = -\dfrac{1}{sL}, \\ Y_{i+2j} = Y_{i-2j} = Y_{ij+2} = Y_{ij-2} = \dfrac{6}{sL}, \\ Y_i = sC, \end{array}\right.$$

the following form of the electrical equations can be derived:

$$\left(16 + s^2 LC\right) V_{ij} + (V_{i+2j} - 6V_{i+j} - 6V_{i-j} + V_{i-2j} +$$

$$+V_{ij+2} - 6V_{ij+1} - 6V_{ij-1} + V_{ij-2} + 2V_{i+j+1} + 2V_{i-j+1} + 2V_{i+j-1} + 2V_{i-j-1}) = -s^2 Ll_p g_{em} \nabla^2 U.$$

In order to derive the homogenized form of the previous equation, let us derive the finite difference approximation of the double laplacian operator. Hence let us divide all of the expression by h^4 and assemble the terms in order to isolate the FD double laplacian operator:

$$\left(\frac{16V_{ij}}{h^4} + \frac{V_{i+2j} - 6V_{i+1j} - 6V_{i-1j} + V_{i-2j}}{h^4} + \frac{V_{ij+2} - 6V_{ij+1} - 6V_{ij-1} + V_{ij-2}}{h^4} \right.$$
$$\left. + \frac{V_{i+1j+1} + V_{i-1j+1} + V_{i+1j-1} + V_{i-1j-1}}{h^4} \right) + s^2 \frac{LC}{h^4} V = s^2 \frac{Ll_p g_{em}}{h^4} \nabla^2 U,$$

Then considering the potential uniformly distributed throughout the domain of the plate, the homogenized form of the electrical coupled equation becomes:

$$\nabla^2 \nabla^2 V + s^2 \frac{LC}{h^4} V + s^2 \frac{Ll_p g_{em}}{h^4} \nabla^2 U = 0.$$

Its form in the time domain is:

$$\nabla^2 \nabla^2 v + \frac{LC}{h^4} \frac{\partial^2 v}{\partial t^2} + \frac{Ll_p g_{em}}{h^4} \nabla^2 \left(\frac{\partial^2 u}{\partial t^2} \right) = 0.$$

As in the elastica case, the analog we are using was developed from the time integral of the plate equation, i.e. the electrical analog of the deflection is the time integral ψ of the potential v. Hence let us consider the equation expressed in the variable ψ, obtained as the time integral of the previous one:

$$\nabla^2 \nabla^2 \psi + \frac{LC}{h^4} \frac{\partial^2 \psi}{\partial t^2} + \frac{Ll_p g_{em}}{h^4} \nabla^2 \left(\frac{\partial u}{\partial t} \right) = 0.$$

Let us consider its dimensionless form:

$$\frac{v_o}{s_o l_o^4}\widetilde{\nabla^2\nabla^2}\tilde{\psi} + s_o v_o \frac{LC}{h^4}\frac{\partial^2\tilde{\psi}}{\partial t^2} + \frac{s_o u_o}{l_o^2}\frac{Ll_p g_{em}}{h^4}\widetilde{\nabla^2}\left(\frac{\partial\tilde{u}}{\partial t}\right) = 0.$$

Finally the electrical coupled equation for the plate is:

$$\widetilde{\nabla^2\nabla^2}\tilde{\psi} + \alpha_e\frac{\partial^2\tilde{\psi}}{\partial t^2} + \beta_e\widetilde{\nabla^2}\left(\frac{\partial\tilde{u}}{\partial t}\right) = 0, \qquad \alpha_e = l_o^4 s_o^2\frac{LC}{h^4}, \qquad \beta_e = \frac{s_o^2 l_o^2 u_o}{v_o}\frac{Ll_p g_{em}}{h^4}.$$

Mechanical coupled equation

As introduced in the previous subsection, the piezoelectric effect will generate a change in the bending moment response of the material, which can be taken into account by simply adding to the bending moment tensor of the plate $\mathbf{M}_m$ the bending-moment PZT-tensor $\mathbf{M}_{pz}$:

$$\mathbf{M}_T = \mathbf{M}_m + \mathbf{M}_{pz},$$

where the expression of $\mathbf{M}_m$ and $\mathbf{M}_{pz}$ are:

$$\begin{aligned}\mathbf{M}_m &= J_{\mathcal{I}}\left[2\mu_L Sym\left(\nabla\nabla u\right) + \lambda_L\nabla^2 u\hat{\mathbf{1}}\right],\\ \mathbf{M}_{pz} &= \eta g_{mm}\left[\left(\hat{\mathbf{x}}\otimes\hat{\mathbf{x}}\right)\frac{\partial^4 u}{\partial x^4} + \left(\hat{\mathbf{y}}\otimes\hat{\mathbf{y}}\right)\frac{\partial^4 u}{\partial y^4}\right] - \eta\frac{g_{em}}{l_p}v\hat{\mathbf{1}}.\end{aligned}$$

([9]) Plugging the previous expression into the bending moment equilibrium equation (1.15), the coupling term will explicitly appear:

$$S_p\nabla^2\nabla^2 u + \eta g_{mm}\left[\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4}\right] - J_{\mathcal{I}}\rho\nabla^2\frac{\partial^2 u}{\partial t^2} + 2h_t\rho\frac{\partial^2 u}{\partial t^2} - \eta\frac{g_{em}}{l_p}\nabla^2 v = 0,$$

[9]Recall $J_{\mathcal{I}} = \frac{2}{3}h_t^3$.

([10]) Since it is reasonable to assume the bending stiffness of the plate is much greater then the purely mechanical one of the PZT effect, the term depending on ηg_{mm} is negligible. Moreover (as it was already seen in the previous chapter) the third term on the right hand side can be neglected because of the smallness of the plate thickness compared to its surface diameter. Finally the following form for the mechanical coupled equation is obtained:

$$\nabla^2\nabla^2 u + \frac{2h_t\rho}{S_p}\frac{\partial^2 u}{\partial t^2} - \frac{\eta g_{em}}{l_p S_p}\nabla^2 v = 0, \qquad S_p = \frac{2}{3}h_t^3\left(2\mu_L + \lambda_L\right).$$

Let us consider it in terms of the time integral of the potential:

$$\nabla^2\nabla^2 u + \frac{2h_t\rho}{S_p}\frac{\partial^2 u}{\partial t^2} - \frac{\eta g_{em}}{l_p S_p}\nabla^2\left(\frac{\partial\psi}{\partial t}\right) = 0, \qquad S_p = \frac{2}{3}h_t^3\left(2\mu_L + \lambda_L\right).$$

Moreover let us consider its dimensionless form:

$$\frac{u_o}{l_o^4}\nabla^2\nabla^2\tilde{u} + s_o^2 u_o\frac{2h_t\rho}{S_p}\frac{\partial^2\tilde{u}}{\partial\tilde{t}^2} - \frac{v_o}{l_o^2}\frac{\eta g_{em}}{l_p S_p}\nabla^2\left(\frac{\partial\tilde{\psi}}{\partial\tilde{t}}\right) = 0.$$

Finally the mechanical coupled equation for the plate is:

$$\widetilde{\nabla^2\nabla^2}\tilde{u} + \alpha_m\frac{\partial^2\tilde{u}}{\partial\tilde{t}^2} - \beta_m\widetilde{\nabla^2}\left(\frac{\partial\tilde{\psi}}{\partial\tilde{t}}\right) = 0, \qquad \alpha_m = s_o^2 l_o^4\frac{2h_t\rho}{S_p} \qquad \beta_m = l_o^2\frac{v_o}{u_o}\frac{\eta g_{em}}{l_p S_p}.$$

[10] Recall that $S_p = J_{\mathcal{I}}\left(2\mu_L + \lambda_L\right)$.

PzEM System

The evolution of the piezo-electro-mechanical plate is described by the following system of coupled equations:

$$\widetilde{\nabla^2\nabla^2}\tilde{u} + \alpha_m \frac{\partial^2 \tilde{u}}{\partial \tilde{t}^2} - \beta_m \widetilde{\nabla^2}\left(\frac{\partial \tilde{\psi}}{\partial \tilde{t}}\right) = 0, \qquad \alpha_m = s_o^2 l_o^4 \frac{2h_t\rho}{S_p} \qquad \beta_m = l_o^2 \frac{v_o}{u_o}\frac{\eta g_{me}}{l_p S_p}. \tag{5.7}$$

$$\widetilde{\nabla^2\nabla^2}\tilde{\psi} + \alpha_e \frac{\partial^2 \tilde{\psi}}{\partial t^2} + \beta_e \widetilde{\nabla^2}\left(\frac{\partial \tilde{u}}{\partial t}\right) = 0, \qquad \alpha_e = s_o^2 l_o^4 \frac{LC}{h^4}, \qquad \beta_e = \frac{s_o^2 l_o^2 u_o}{v_o}\frac{L l_p g_{em}}{h^4}.$$

In order to match the phase speed of the two equations (i.e. $\alpha_m = \alpha_e$), we must set a value for the inductance given by:

$$L = h^4 \frac{2h_t\rho}{CS_p}$$

The coupling terms β_e, β_m are equal once the following value of the characteristic tension is chosen:

$$v_o = u_o s_o l_P \sqrt{\frac{2h_t\rho}{\eta C}}.$$

Finally the coupled system can be written in the following form:

$$\nabla^2\nabla^2 u + \alpha\ddot{u} - \beta\nabla^2\,\dot{\psi} = 0, \tag{5.8}$$

$$\nabla^2\nabla^2\psi + \alpha\ddot{\psi} + \beta\nabla^2\,\dot{u} = 0.$$

In the following section some considerations of the results we have found will be derived. In particular, our attention will be focused upon the plate case. Let us remark that, since the elastica coupled equations are formally analogous to the plate ones, the considerations which follow will have value for the elastica too.

Final considerations

First of all the dispersive relation of the piezo-electro-mechanical plate will be derived and compared to the uncoupled electro-mechanical system. Afterwards, a spectral analysis of the problem will be considered so as to find an approximate solution and to realize some numerical simulations of the PEM-plate behavior. Also an analytical solution for particular cases will be derived.

Dispersive relation of the coupled system

In order to consider the dispersive relation of the coupled electromechanical plate, let us assume the following wave-form for the deflection u and the potential primitive ψ which propagate into the plate:

$$\begin{aligned} \tilde{u}(\mathbf{r},t) &= Ae^{j(\mathbf{k}_m\cdot\mathbf{r}-\omega_m t)} \\ \tilde{\psi}(\mathbf{r},t) &= Be^{j(\mathbf{k}_e\cdot\mathbf{r}-\omega_e t)} \end{aligned}$$

where:

$\mathbf{r}$ is the position vector in the plate surface $\mathcal{S}$,

$\mathbf{k}_m, \mathbf{k}_e$ are the mechanical and electric wave vectors,

ω_m, ω_e are the mechanical and electric wave frequencies.

From the previous expressions, the following form for the equations of motion (5.8) is obtained:

$$
\begin{aligned}
(\mathbf{k}_m^4 - \alpha\omega_m^2)\tilde{u} + i\beta\omega_e \mathbf{k}_e^2 \tilde{\psi} &= 0, \\
(\mathbf{k}_e^4 - \alpha\omega_e^2)\tilde{\psi} - i\beta\omega_m \mathbf{k}_m^2 \tilde{u} &= 0.
\end{aligned}
$$

Since these must be satisfied for every frequency and wave-vector we must assume:

$$
\begin{aligned}
\mathbf{k}_e &= \mathbf{k}_m = \mathbf{k} = (k_x\mathbf{i} + k_y\mathbf{j}) = k(\mathbf{i}\cos\varphi + \mathbf{j}\sin\varphi), \\
\omega_e &= \omega_m = \omega,
\end{aligned}
$$

so that the equations of motion lead to an homogeneous system of two equations in the two unknowns A, B:

$$
\begin{aligned}
(k^4 - \alpha\omega^2)A + i\beta\omega k^2 B &= 0, \\
-i\beta\omega k^2 A + (k^4 - \alpha\omega^2)B &= 0,
\end{aligned}
\tag{5.9}
$$

A non-trivial solution of the problem can be found by setting the determinant of the system to zero, which permits us to determine the dispersive equation for the PEM plate:

$$
(k^4 - \alpha\omega^2)^2 - \left(\beta\omega k^2\right)^2 = 0 \tag{5.10}
$$

The previous expression can be seen to be an algebraic second order equation in ω^2. This means that for every wave vector k there are four different waves, the frequencies of which are $\pm\omega_1$, $\pm\omega_2$. Hence it is enough to consider only the pulsations ω_1, ω_2 (the others mean only an opposite direction of wave-propagation).

Solving the equation (5.10), the values for the positive characteristic frequencies ω_1, ω_2 are derived:

$$\begin{aligned} \omega_1 &= \frac{\beta}{2\alpha}\left(\sqrt{1+\frac{4\alpha}{\beta^2}}+1\right)k^2, \\ \omega_2 &= \frac{\beta}{2\alpha}\left(\sqrt{1+\frac{4\alpha}{\beta^2}}-1\right)k^2. \end{aligned}$$

The relation between the electric and mechanical wave-amplitudes can be found by substituting the expressions of ω_1 and ω_2 in one of the two equation (5.9):

$$\frac{A}{B} = -\frac{i\beta\omega_n k^2}{(k^4-\alpha\omega_n^2)} = i \qquad n = 1, 2,$$

which means that the electrical and mechanical modes have always the same amplitude and show a phase difference of $\frac{\pi}{2}$. Hence every electromechanical propagating-wave shows an equal amount of electrical and mechanical component, i.e. the electro-mechanical system is perfectly coupled for every kind of evolution.

Let now consider the dispersive relation of the coupled system (phase speeds of these waves versus k):

$$v_n(k) = \frac{\omega_n}{k} \qquad n = 1, 2,$$

an let us compare it with the phase speed of the uncoupled systems (which can be obtained from the phase speed of the coupled one when the coupling term β

tends to zero):

$$\upsilon_m(k) = \upsilon_e(k) = \sqrt{\frac{1}{\alpha}}k.$$

The coupling phenomena generate a splitting of the uncoupled dispersion-relation, which is constituted by two coincident straight lines of slope $\sqrt{\frac{1}{\alpha}}$. The plot (5.3) shows the uncoupled dispersion-relation in grey and the coupled one in black, for values of α and β used for the simulations presented in the next section.

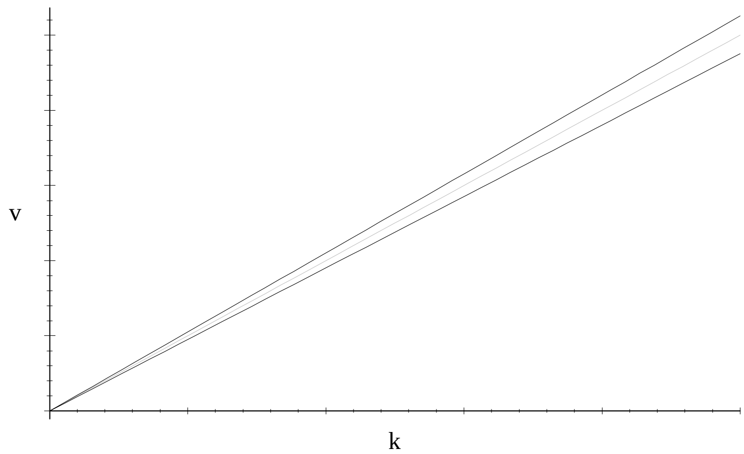


Figure 5.3: Dispersive relation for the uncoupled (grey) and coupled (black) cases.

Spectral analysis

Our goal here is to transform the PDE of the coupled system into a set of ODE for the coefficients of the solution in a given base. In order to do that, let us consider the following vectorial operators:

$$\mathbf{L} \equiv \begin{bmatrix} \frac{1}{\alpha}\widetilde{\nabla^2\nabla^2} & 0 \\ 0 & \frac{1}{\alpha}\widetilde{\nabla^2\nabla^2} \end{bmatrix}, \qquad \mathbf{G} \equiv \begin{bmatrix} 0 & -\frac{\beta}{\alpha}\widetilde{\nabla^2} \\ \frac{\beta}{\alpha}\widetilde{\nabla^2} & 0 \end{bmatrix},$$

and the following form of the PEM-plate equations:

$$\frac{1}{\alpha}\begin{bmatrix} \widetilde{\nabla^2\nabla^2} & 0 \\ 0 & \widetilde{\nabla^2\nabla^2} \end{bmatrix}\begin{bmatrix} \tilde{u} \\ \tilde{\psi} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\ddot{\begin{bmatrix} \tilde{u} \\ \tilde{\psi} \end{bmatrix}} + \frac{\beta}{\alpha}\begin{bmatrix} 0 & -\widetilde{\nabla^2} \\ \widetilde{\nabla^2} & 0 \end{bmatrix}\dot{\begin{bmatrix} \tilde{u} \\ \tilde{\psi} \end{bmatrix}} = 0.$$

Hence defining the vectorial variable:

$$\mathbf{u} \equiv \begin{bmatrix} \tilde{u} \\ \tilde{\psi} \end{bmatrix},$$

the problem assumes the form of a vectorial equation:

$$\mathbf{L}(\tilde{\mathbf{u}}) + \alpha\mathbf{u}'' + \beta\mathbf{G}(\tilde{\mathbf{u}}') = \mathbf{0},$$

Because the operators $\widetilde{\nabla^2\nabla^2}$ is self-adjoint in the space $L^2(\mathcal{D})$ ([11]) and the block decomposition of the coupling operator $\mathbf{G}$ is antisymmetric, when simply-supported or completely-clamped boundary conditions are considered, it can be seen that the compositions of the bases (one of the purely mechanical problem, the other of the purely electric one) is a complete basis of the solution of the coupled problem.

[11] Where $L^2(\mathcal{D})$ is the Hilbert space of the functions L^2 defined upon a compact $\mathcal{D}$ of $\mathbb{R}^2$.

Let us consider the eigenvalue problem of the uncoupled operator (which compose $\mathbf{L}$)

$$\frac{1}{\alpha}\widetilde{\nabla^2\nabla^2}m_\lambda = \lambda m_\lambda.$$

Since the set of the eigenvectors $\{m_\lambda\}$ is a base for the double laplacian operator, the generic solution of the coupled problem can be decomposed using a composition of two eigenbases $\{m_\lambda\}$ one for the electric, the other for the mechanical problem:

$$\begin{aligned}\tilde{u} &= \sum_\lambda p_\lambda(t)\, m_\lambda, \\ \tilde{\psi} &= \sum_\lambda q_\lambda(t)\, m_\lambda.\end{aligned} \tag{5.11}$$

Substituting the previous expression into the couple equations for the plate (5.7), we get:

$$\begin{aligned}&\frac{1}{\alpha}\widetilde{\nabla^2\nabla^2}\sum_\lambda p_\lambda m_\lambda + \sum_\lambda \ddot{p}_\lambda m_\lambda - \frac{\beta}{\alpha}\widetilde{\nabla^2}\sum_\lambda \dot{q}_\lambda m_\lambda = 0, \\ &\frac{1}{\alpha}\widetilde{\nabla^2\nabla^2}\sum_\lambda q_\lambda m_\lambda + \sum_\lambda \ddot{q}_\lambda m_\lambda + \frac{\beta}{\alpha}\widetilde{\nabla^2}\sum_\lambda \dot{p}_\lambda m_\lambda = 0,\end{aligned}$$

Since the coefficients $p_\lambda(t)$, $q_\lambda(t)$ do not depend upon the spatial variable, we get:

$$\begin{aligned}&\sum_\lambda\left(p_\lambda\frac{1}{\alpha}\widetilde{\nabla^2\nabla^2}m_\lambda + \ddot{p}_\lambda m_\lambda - \frac{\beta}{\alpha}\dot{q}_\lambda\widetilde{\nabla^2}m_\lambda\right) = 0, \\ &\sum_\lambda\left(q_\lambda\frac{1}{\alpha}\widetilde{\nabla^2\nabla^2}m_\lambda + \ddot{q}_\lambda m_\lambda + \frac{\beta}{\alpha}\dot{p}_\lambda\widetilde{\nabla^2}m_\lambda\right) = 0.\end{aligned}$$

Hence considering that $\{m_\lambda\}$ are eigenvectors of the double laplacian operator:

$$\sum_\lambda \left[(\lambda p_\lambda + \ddot{p}_\lambda)\, m_\lambda - \frac{\beta}{\alpha} \dot{q}_\lambda \widetilde{\nabla}^2 m_\lambda \right] = 0,$$
$$\sum_\lambda \left[(\lambda q_\lambda + \ddot{q}_\lambda)\, m_\lambda + \frac{\beta}{\alpha} \dot{p}_\lambda \widetilde{\nabla}^2 m_\lambda \right] = 0.$$

Finally, if we consider the inner product of the previous formulas with the generic vector m_k in the chosen basis:

$$\sum_\lambda \left[(\lambda p_\lambda + \ddot{p}_\lambda) < m_\lambda, m_k > - \frac{\beta}{\alpha} \dot{q}_\lambda < \widetilde{\nabla}^2 m_\lambda, m_k > \right] = 0,$$
$$\sum_\lambda \left[(\lambda q_\lambda + \ddot{q}_\lambda) < m_\lambda, m_k > + \frac{\beta}{\alpha} \dot{p}_\lambda < \widetilde{\nabla}^2 m_\lambda, m_k > \right] = 0,$$

we find the ODEs for the coefficients p_λ, q_λ. If we chose $\{m_k\}$ to be orthonormal base (which is always possible because of the Gram-Smith theorem [9]), we get:

$$\lambda p_\lambda + \ddot{p}_\lambda - \frac{\beta}{\alpha} \sum_\lambda A_{k\lambda} \dot{q}_\lambda = 0,$$
$$\lambda q_\lambda + \ddot{q}_\lambda + \frac{\beta}{\alpha} \sum_\lambda A_{k\lambda} \dot{p}_\lambda = 0,$$

where:

$$A_{k\lambda} = \frac{\beta}{\alpha} < \widetilde{\nabla}^2 m_\lambda, m_k > .$$

Let us underline that, once a set of M modes for the approximating solution is considered, the solution is derived by solving a set of M ordinary-differential second-order equations, which implies the setting of 2 initial conditions for each of the chosen modes.

The vectorial form for the system of the coefficients becomes:

$$\begin{bmatrix} \ddot{p} \\ \ddot{q} \end{bmatrix} + \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} + \frac{\beta}{\alpha} \begin{bmatrix} 0 & -A \\ A & 0 \end{bmatrix} \begin{bmatrix} \dot{p} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{5.12}$$

where, considering the first M modes of the system, the previous quantities assume the following expressions:

$$p \equiv \begin{bmatrix} p_1 \\ p_2 \\ . \\ p_M \end{bmatrix}, \quad q \equiv \begin{bmatrix} q_1 \\ q_2 \\ . \\ q_M \end{bmatrix}, \quad \lambda \equiv \begin{bmatrix} \lambda_1 & 0 & . & 0 \\ 0 & \lambda_2 & . & 0 \\ . & . & . & . \\ 0 & 0 & . & \lambda_M \end{bmatrix}, \quad A \equiv \begin{bmatrix} A_{\lambda_1\lambda_1} & A_{\lambda_1\lambda_2} & . & A_{\lambda_1\lambda_M} \\ A_{\lambda_2\lambda_1} & A_{\lambda_2\lambda_2} & . & A_{\lambda_2\lambda_M} \\ . & . & . & . \\ A_{\lambda_M\lambda_1} & A_{\lambda_M\lambda_2} & . & A_{\lambda_M\lambda_M} \end{bmatrix}.$$

Hence, the solution is given once the following initial conditions are known:

$$p_0 = p(0), \qquad q_0 = q(0), \qquad \dot{p}_0 = \left.\frac{dp(t)}{dt}\right|_{t=0}, \qquad \dot{q}_0 = \left.\frac{dq(t)}{dt}\right|_{t=0}.$$

Let us recall the form of the eigenvectors and eigenvalues of the uncoupled plate operator in the case of the completely-clamped and the simply-supported boundary-conditions in a square plate of edge l.

- Simply-supported BCs: In this case it is possible to solve the eigenvalue problem by finding an analytical solution of the eigenvectors which are simply the Cartesian product of the elastica-operator eigenvectors:

$$m_{ij}(x,y) = A \sin\left(\frac{i\pi}{l}x\right) \sin\left(\frac{j\pi}{l}y\right), \qquad n, m \in \mathbb{N}. \tag{5.13}$$

The relative eigenvectors are given by:

$$\lambda_{ij} = \left[\lambda_i^2 + \lambda_j^2\right]^2 = \frac{1}{\alpha}\left[\left(\frac{i\pi}{l}\right)^2 + \left(\frac{j\pi}{l}\right)^2\right]^2.$$

- completely-clamped BCs: In this case there does not exist a closed analytical expression for the eigenvectors, hence it is better to use another basis similar to the previous one, which is obtained as the Cartesian product of the eigenbasis of the completely-clamped elastica in both the axial directions:

  $$m_{ij}(x, y) = f_i(\frac{x}{l})\ f_j(\frac{y}{l}),$$

  where:

  $$f_n(\sigma) = \cosh(\lambda_n\sigma) - \cos(\lambda_n\sigma) + \frac{\cosh(\lambda_n\sigma) - \cos(\lambda_n\sigma)}{\sinh(\lambda_n\sigma) - \sin(\lambda_n\sigma)}\left[\sin(\lambda_n\sigma) - \sinh(\lambda_n\sigma)\right].$$

  It is possible to show that this basis is complete In fact it is the eigenbasis of the self-adjoint operator obtained by dropping the mixed term in the double laplacian. If this basis is a good approximation of the true eigenbasis, then the eigenvalues are well approximated by the Rayleigh ratio defined as:

  $$R_{mn} = \frac{\int_0^l \int_0^l \frac{1}{\alpha}\nabla^2\nabla^2 m_{ij}\ m_{ij}dxdy}{\int_0^l \int_0^l m_{ij}m_{ij}dxdy}.$$

Remark 91 *The solution of the system (5.12), once the boundary and the initial conditions are assumed, permits us to derive the numerical solution of the coupling problem shown in the next section.*

Explicit solution form

In order to develop an explicit solution form of the coupling problem, let us write it in terms of a first order differential vectorial equation. This can be easily done considering also the time derivatives of the deflection-coefficient $\dot{p}$ and the electric potential-coefficient $\dot{q}$ as unknowns of the problem. Consequently the problem assumes the following form:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{S}\mathbf{x}(t), \\ \mathbf{x}(0) &= \mathbf{x}_0, \end{aligned} \tag{5.14}$$

where:

$$\mathbf{x} \equiv \begin{bmatrix} p \\ q \\ \dot{p} \\ \dot{q} \end{bmatrix} \qquad \mathbf{S} \equiv \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\lambda & 0 & 0 & -A \\ 0 & -\lambda & A & 0 \end{bmatrix}, \qquad \mathbf{x}_0 \equiv \begin{bmatrix} p_0 \\ q_0 \\ \dot{p}_0 \\ \dot{q}_0 \end{bmatrix}.$$

The solution of this problem is known:

$$\mathbf{x}(t) = e^{\mathbf{S}t}\mathbf{x}_0.$$

In order to express it in an easy way, let us consider the eigenvalues problem for the exponential matrix $\mathbf{S}$:

$$\mathbf{S}\mathbf{V} = \mathbf{V}\Omega,$$

where Ω is the diagonal matrix of the eigenvectors ξ_i of $\mathbf{S}$:

$$\Omega = \begin{bmatrix} \xi_1 & 0 & . & 0 \\ 0 & \xi_2 & . & 0 \\ . & . & . & . \\ 0 & 0 & . & \xi_{4M} \end{bmatrix}$$

and $\mathbf{V}$ is the matrix whose columns are the eigenvectors $\mathbf{v}_n$:

$$\mathbf{V} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & . & \mathbf{v}_{4M} \end{bmatrix}.$$

If all the eigenvalues ξ_n are distinct, the eigenvectors are orthogonal Hence the matrix $\mathbf{V}$ is non singular and can be used as a base for the solution for the coupled problem. Hence considering $\mathbf{y}(t)$ the solution vector in the new basis we get:

$$\dot{\mathbf{y}}(t) = \Omega\mathbf{y}(t),$$

$$\Omega = \mathbf{V}^{-1}\mathbf{S}\mathbf{V};$$

and the explicit form of the solution becomes:

$$\mathbf{y}(t) = e^{\Omega t}\mathbf{y}_0,$$

Since Ω is diagonal, the exponential matrix e^{Ω} has a simple representation:

$$e^{\Omega} = \Lambda = \begin{bmatrix} e^{\xi_1} & 0 & . & 0 \\ 0 & e^{\xi_2} & . & 0 \\ . & . & . & . \\ 0 & 0 & 0 & e^{\xi_{4M}} \end{bmatrix}.$$

Hence the solution $\mathbf{x}(t)$ in the original basis can be written as:

$$\mathbf{x}(t) = \left(\mathbf{V}\Lambda\mathbf{V^{-1}}\right)\mathbf{x_0} = \mathbf{V}\Lambda\left(\mathbf{V}^{-1}\mathbf{x}_0\right).$$

Considering $\{w_{ij}\}$ the elements of the inverse of $\mathbf{V}$ of the solution can be written in the following form:

$$x_m(t) = \sum_{l,k} e^{\xi_l t} v_{ml} w_{kl} x_{0k}. \tag{5.15}$$

Analytical solution for the electro-mechanical coefficients

Let us consider the simply-supported boundary condition case. Considering the eigenbasis m_{ij} (see 5.13) the coupling term

$$A_{\lambda_i k_j} = \frac{\beta}{\alpha} < \widetilde{\nabla}^2 m_{ij}, m_{ij} >= \frac{\beta}{\alpha}\int_0^l\int_0^l \widetilde{\nabla}^2 m_{ij}\ m_{ij} dxdy$$

becomes

$$A_{\lambda_i k_j} = -\frac{1}{4}\frac{\beta}{\alpha}\left[\left(\frac{i\pi}{l}\right)^2 + \left(\frac{j\pi}{l}\right)^2\right]\boldsymbol{\delta}_{ij},$$

where $\boldsymbol{\delta}_{ij}$ is the Kronecker symbol. This means that, in the simply-supported problem, each mode of the coupled structure does not communicate to the others. For each mode we get two scalar equations for the electrical and mechanical coefficients:

$$\lambda p(t) + \ddot{p}(t) - A\dot{q}(t) = 0,$$

$$\lambda q(t) + \ddot{q}(t) + A\dot{p}(t) = 0,$$

and the explicit solution (see 5.15) becomes:

$$p(t) = \sum_{l,k} e^{\xi_l t} v_{1l} w_{kl} x_{0k},$$

$$q(t) = \sum_{l,k} e^{\xi_l t} v_{2l} w_{kl} x_{0k}.$$

In this case **S** is no longer a block-matrix, because of the modal uncoupling, it is a scalar matrix:

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\lambda & 0 & 0 & -A \\ 0 & -\lambda & A & 0 \end{bmatrix}.$$

The eigenvalues of this can be expressed in the following form:

$$\left[\begin{array}{l} \xi_{1,2} = \pm i\kappa_1, \\ \xi_{3,4} = \pm i\kappa_2, \end{array}\right.$$

where the coefficients $\kappa_{1/2}$ of the eigenvalues are:

$$\kappa_{1/2} = \frac{1}{\sqrt{2}} \sqrt{A^2 + 2\lambda \mp A\sqrt{(A^2 + 4\lambda)}}.$$

Considering the following equality:

$$\sqrt{a \mp \sqrt{a^2 - b^2}} = \frac{1}{\sqrt{2}} \left(\sqrt{a+b} \mp \sqrt{a-b} \right),$$

immediately the alternative form for $\kappa_{1/2}$ is

$$\left[\begin{array}{l} \kappa_1 = \frac{1}{2}\left(\sqrt{A^2+4\lambda}-A\right), \\ \kappa_2 = \frac{1}{2}\left(\sqrt{A^2+4\lambda}+A\right). \end{array}\right.$$

The matrix of eigenvectors $\mathbf{V}$ can be written in the form:

$$\mathbf{V} = \frac{1}{A\lambda}\begin{bmatrix} -\frac{i}{\lambda}A\kappa_1\kappa_2^2 & \frac{i}{\lambda}A\kappa_1\kappa_2^2 & -\frac{i}{\lambda}A\kappa_2\kappa_1^2 & \frac{i}{\lambda}A\kappa_2\kappa_1^2 \\ \kappa_2^2-\lambda & \kappa_2^2-\lambda & \kappa_1^2-\lambda & \kappa_1^2-\lambda \\ \lambda A & \lambda A & \lambda A & \lambda A \\ -i\kappa_1\left(-\kappa_2^2+\lambda\right) & i\kappa_1\left(-\kappa_2^2+\lambda\right) & -i\kappa_2\left(-\kappa_1^2+\lambda\right) & i\kappa_2\left(-\kappa_1^2+\lambda\right) \end{bmatrix}$$

and its inverse is:

$$\mathbf{V^{-1}} = \frac{1}{2\left(\kappa_2^2-\kappa_1^2\right)}\begin{bmatrix} i\left(-\kappa_1^2+\lambda\right)\frac{\lambda}{\kappa_1} & \lambda A & -\kappa_1^2+\lambda & -i\kappa_1 A \\ -i\left(-\kappa_1^2+\lambda\right)\frac{\lambda}{\kappa_1} & \lambda A & -\kappa_1^2+\lambda & i\kappa_1 A \\ -i\left(-\kappa_2^2+\lambda\right)\frac{\lambda}{\kappa_2} & -\lambda A & \kappa_2^2-\lambda & i\kappa_2 A \\ i\left(-\kappa_2^2+\lambda\right)\frac{\lambda}{\kappa_2} & -\lambda A & \kappa_2^2-\lambda & -i\kappa_2 A \end{bmatrix}$$

In order to further develop the calculations, let us assume the following initial conditions:

$$\mathbf{x}_0 = \begin{bmatrix} p_o \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

which means that there is only a deflection of one mode non vanishing at $t = 0$ for the given mode. Consequently, the solution of the mechanical coefficients $p(t)$ and $q(t)$ become:

$$\left[\begin{array}{l} p(t) = \sum_{l=1}^{4} e^{\kappa_l t} v_{1l} w_{1l} x_{01} = \frac{1}{2}p\left(1 + \frac{A}{\sqrt{(A^2+4\lambda)}}\right)\cos\kappa_1 t + \frac{1}{2}p\left(1 - \frac{A}{\sqrt{(A^2+4\lambda)}}\right)\cos\kappa_2 t. \\ \\ q(t) = \sum_{l=1}^{4} e^{\kappa_l t} v_{2l} w_{1l} x_{01} = \frac{1}{2}p\left(1 - \frac{A}{\sqrt{(A^2+4\lambda)}}\right)\sin\kappa_2 t - \frac{1}{2}p\left(\frac{A}{\sqrt{(A^2+4\lambda)}} + 1\right)\sin\kappa_1 t. \end{array}\right.$$

Using well-known trigonometric formulas:

$$\left[\begin{array}{l} A_1\cos\omega_1 t + A_2\cos\omega_2 t = \\ \qquad = (A_1 + A_2)\cos[\frac{\omega_1-\omega_2}{2}t]\cos[\frac{\omega_1+\omega_2}{2}t] - (A_1 - A_2)\sin[\frac{\omega_1-\omega_2}{2}t]\sin[\frac{\omega_1+\omega_2}{2}t], \\ \\ A_1\sin\omega_1 t + A_2\sin\omega_2 t = \\ \qquad = (A_1 + A_2)\cos[\frac{\omega_1-\omega_2}{2}t]\sin[\frac{\omega_1+\omega_2}{2}t] + (A_1 - A_2)\sin[\frac{\omega_1-\omega_2}{2}t]\cos[\frac{\omega_1+\omega_2}{2}t], \end{array}\right.$$

we get the following useful expression:

$$\left[\begin{array}{l} p(t) = p\cos[\frac{\kappa_2-\kappa_1}{2}t]\cos[\frac{\kappa_2+\kappa_1}{2}t] + p\frac{A}{\sqrt{(A^2+4\lambda)}}\sin[\frac{\kappa_2-\kappa_1}{2}t]\sin[\frac{\kappa_2+\kappa_1}{2}t], \\ \\ q(t) = -\frac{p}{\sqrt{(A^2+4\lambda)}}A\cos[\frac{\kappa_2-\kappa_1}{2}t]\sin[\frac{\kappa_2+\kappa_1}{2}t] + p\sin[\frac{\kappa_2-\kappa_1}{2}t]\cos[\frac{\kappa_2+\kappa_1}{2}t], \end{array}\right.$$

where:

$$\left[\begin{array}{l}\dfrac{\kappa_2-\kappa_1}{2}=\dfrac{1}{8}\beta\pi^2\left(m^2+n^2\right),\\ \dfrac{\kappa_2+\kappa_1}{2}=\dfrac{1}{8}\sqrt{\left(\beta^2+64\alpha\right)}\pi^2\left(m^2+n^2\right).\end{array}\right.$$

Let us underline that if the coupling term tending to zero, the uncoupled solution trivially is:

$$\left[\begin{array}{l}p(t)|_{A=0}=p\cos\sqrt{\lambda}t,\\ q(t)|_{A=0}=0.\end{array}\right.$$

Comparing the coupled and uncoupled case, it is evident that the coupling phenomena gives a modulation of the uncoupled solution for $p(t)$, which is composed of one term in phase and the second in opposition of phase proportional to the coupling coefficient A. Moreover, a non vanishing electrical signal in opposite phase with respect to the mechanical one is found (the following plot are realized using values of the coefficients presented in the next chapter):

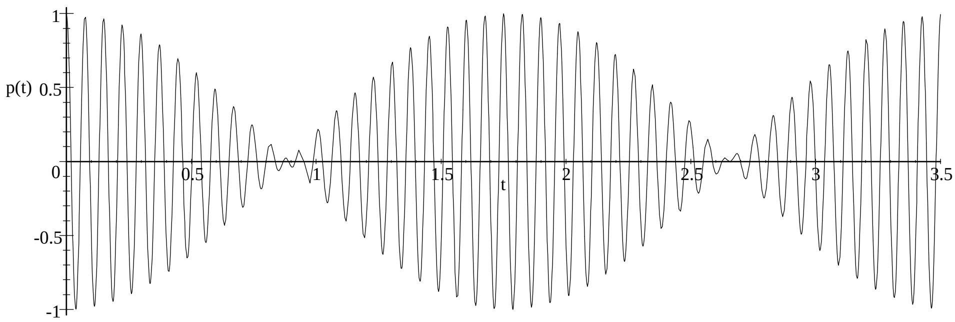


Figure 5.4: Mechanical coefficient $p(t)$

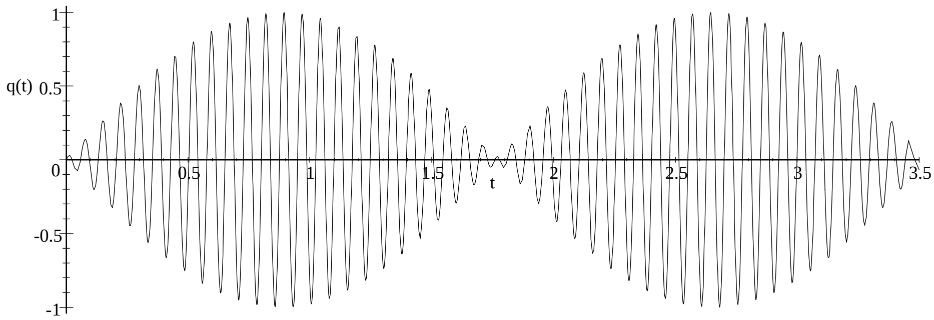


Figure 5.5: Electrical coefficient $q(t)$

Chapter 6

Numerical simulations and conclusions

The numerical simulations of the evolution for a completely-clamped square-plate are presented here. We considered the first nine modes in the evolution for a square plate, i.e. the following approximation for the deflection $\tilde{u}(x, y, t)$ and the potential time-integral $\tilde{\psi}(x, y, t)$:

$$\begin{aligned} \tilde{u}(x,y,t) &= \sum_{\lambda=1}^{9} p_\lambda(t)\, m_\lambda(x,y), \\ \tilde{\psi}(x,y,t) &= \sum_{\lambda=1}^{9} q_\lambda(t)\, m_\lambda(x,y), \end{aligned} \tag{6.1}$$

let us recall that λ are the eigenvalues and m_λ the eigenvectors of the uncoupled problem:

$$\frac{1}{\alpha}\widetilde{\nabla^2\nabla^2} m_\lambda(x,y) = \lambda m_\lambda(x,y),$$

which, for the completely-clamped BCs assumes the form presented in the previous section. We have solved the following system for the electro-mechanical coefficients:

$$\lambda p_\lambda(t) + \ddot{p}_\lambda(t) - \frac{\beta}{\alpha}\sum_{k=1}^{9} A_{k\lambda}\dot{q}_\lambda(t) = 0,$$
$$\lambda q_\lambda(t) + \ddot{q}_\lambda(t) + \frac{\beta}{\alpha}\sum_{k=1}^{9} A_{k\lambda}\dot{p}_\lambda(t) = 0,$$

where:

$$A_{k\lambda} = \frac{\beta}{\alpha} < \widetilde{\nabla}^2 m_\lambda, m_k >,$$

$$\alpha = s_o^2 l_o^4 \frac{2h_t\rho}{S_p},$$

$$\beta = l_o^2 \frac{v_o}{u_o}\frac{\eta g_{me}}{l_p S_p}.$$

The characteristics of the system are presented in the table (6.2). The actuator considered are the commercial bending actuators produced by ACX:

$$\begin{array}{ll}
\multicolumn{2}{l}{{*}\textit{Characteristic quantities:}} \\
\text{Characteristic length:} & l_o = 1\,[m]\,, \\
\text{Characteristic deflection:} & u_o = 1\,[mm]\,, \\
\text{Characteristic frequency:} & s_o = 4\pi, \\
\text{Characteristic tension:} & v_o = u_o s_o l_P\sqrt{\frac{2h_t\rho}{\eta C}} = 5.89\,[V]\,. \\
\multicolumn{2}{l}{{*}\textit{Plate dimensions}} \\
\text{Edge length:} & l = 1\,[m]\,, \\
\text{Thickness:} & h_t = 2\,[mm]\,.
\end{array} \tag{6.2}$$

**Plate mechanical characteristics*

Mass density: $\rho = 2700 \left[\frac{kg}{m^2}\right]$,

Young modulus: $Y = 70 \times 10^9 \, [Pa]$,

Rigidity coefficient $S_p = \frac{2}{3} h_p^3 Y = 373 \, [Nm]$.

PZT actuators characteristics

PZT capacitance: $g_{ee} = 0.6 \times 10^{-6} \, [F]$,

PZT coupling coefficient: $k_{em} = 2 \quad 10^{-3} \left[\frac{N}{V} m = C\right]$,

PZT edge $l_p = 4 \quad 10^{-2} \, [m]$,

PZT Scaling factor $\eta = .2$.

The characteristic tension v_o corresponds to the potential on the actuators associated with a characteristic displacement u_o. Let us remark that its value is much lower than the maximum value permitted on an actuator which is about one hundred volts.

Completely clamped simulation

The initial condition considered is a unit deflection u_o for the mechanical component of the first mode, i.e. $p_1(t)|_{t=0} = 1$. The plot for the first five electromechanical coefficients are the following:

The modulation of the first mechanical coefficient has opposite phase with respect to the electrical one, this shows how the energy oscillates back and forth

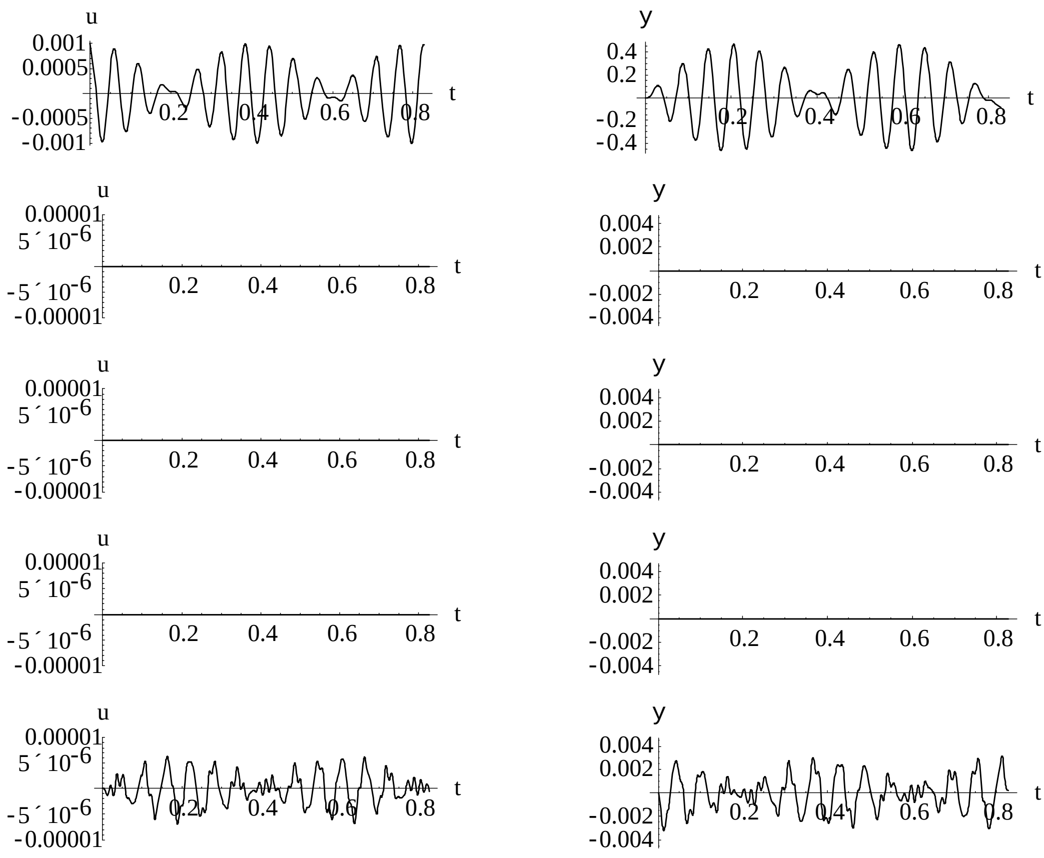


Figure 6.1: Evolution of the elecytro-mechanical coefficients.

from the mechanical to the electrical form in about one half second. The fifth mode shows a small coupling with the first one, which however gives a small contribution to the total displacement.

The evolution of the PEM plate is derived considering the formulas (6.1) and represents the first half period of the modulation in which almost all the mechanical energy is transduced into the electrical form (see fig 6.2, 6.3, 6.4, 6.5).

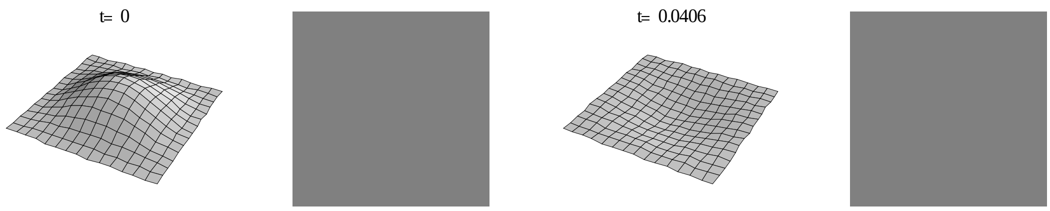


Figure 6.2: Evolution 1

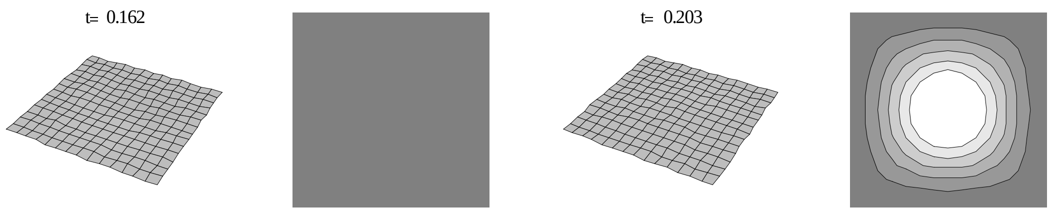


Figure 6.3: Evolution 2

Conclusions

The solutions of the PEM-plate-equations for the simply-supported and the completely-clamped boundary conditions shows that the electro-mechanical system conceived is able to efficiently transduce the energy from the mechanical to the electrical form. Because we coupled the mechanical system to an electrical circuit having the same governing equations, the energy transduction is independent of the particular modal excitation on the plate. Moreover the system can be realized using commercial actuators. As a consequence of the low values of the tensions shown in the simulations it is possible to realize the electric components using low-cost commercial electronic-devices.

A further development of this work would be the design of a dissipative electric system. For instance, the insertion of resistive impedances in the circuit would allow the dissipation of the electric energy transduced from the mechanical form, hence damping the vibrations on the mechanical systems. Moreover, considering devices able to store the energy instead of dissipating it, the transduced energy could be recycled, which is a critical problem in spatial applications.

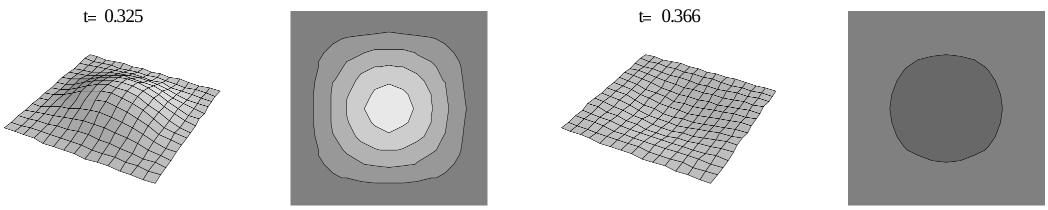


Figure 6.4: Evolution 3

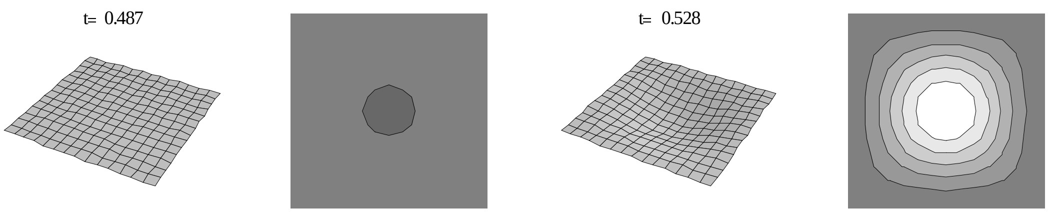


Figure 6.5: Evolution 4

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VITA

Silvio Alessandroni was born to Gino Alessandroni and Franca Battisti on July 5, 1973 in Rome, Italy. He graduated from high school in 1992 at Liceo Scientifico Archimede, Rome.

Upon graduation from high school, Mr. Alessandroni enrolled at Universita' di Roma La Sapienza, Rome. He graduated with a Master of science in Electrical Engineering in 1999. In the same year he enrolled at Virginia Polytechnic Institute and State University for master work in Engineering Science and Mechanics. In November,1999 Mr. Alessandroni began his Ph.D. studies in Theoretical and Applied Mechanics.

Mr Alessandroni thinks that life is just a wood stage to trumple on every day!