CHAPTER 3

SCALING LAWS FOR A PR MANIPULATOR

3.1 Introduction

The problem of scaling a manipulator divides itself into three categories. They are: scaling a trajectory reference planner; scaling the control law; and scaling the dynamical system. As mentioned in Chapter 1, and developed in Appendix A, scaling the system dynamics for this application only involves scaling the dynamics of a PM d.c. motor. This is because low gearing in both the prototype and model systems uncouples the link dynamics. Thus in reality, this chapter sets out to scale: motors, a trajectory planner, and a model based control law.

The chapter begins with the development of the PR manipulator's kinematic equations which are then used with a cubic polynomial trajectory path to form a nondimensional trajectory planner. Dimensional analysis is next applied to the problem of sizing motors. Similitude methods are used to nondimensionalize motor equations that include friction and load terms. The Buckingham Pi theorem is used next to develop scaling laws for sizing motors. The chapter concludes by introducing a strategy for control of the prototype's kinematics and dynamics.

3.2 The Trajectory Planner

As defined by Fu, Gonzalez, and Lee (1987), robot *kinematics* is "the analytical study of the geometry science of motion of a robot arm with respect to a fixed reference coordinate system without regard to the forces/moments that cause the motion." Rephrased, it is the time based study of the geometrical properties of a manipulator. Developing the kinematic equations for the PR manipulator begins with a review of the coordinate systems as stated by Mabie and Reinholtz (1987).

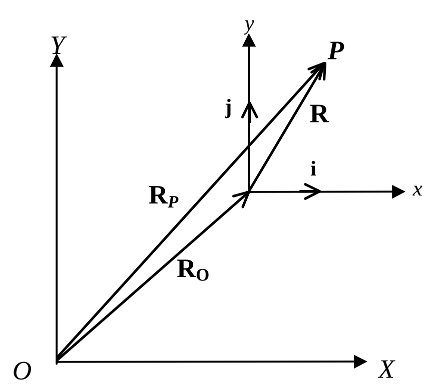

Figure 3.1: Local and global coordinate system example

Position P is first established in the (local) xy-coordinate system. (This is the manipulator's workspace introduced in Chapter 1.) P is moving relative to the fixed (global) XY-coordinate system, and is defined in terms of vectors as:

$$R_P = R_O + R \tag{3.1}$$

If unit vectors (**i**,**j**) are fixed to the local coordinate axes, then:

$$R = x\boldsymbol{i} + y\boldsymbol{j} \tag{3.2}$$

Using standard vector calculus techniques, equations (3.1) and (3.2) can be differentiated to obtain the higher order terms of velocity and acceleration. The inverse of these derivative equations is known as "inverse kinematics." From these equations, link angles are generated and used as control variables in a setpoint programmer.

Derivation of the kinematic and inverse kinematic equations for the PR manipulator's application under study is developed by way of example. It is graphically illustrated in Figure 3.2.

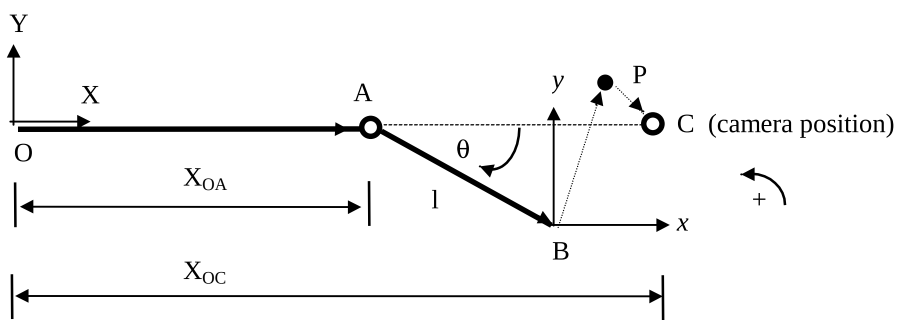

Figure 3.2: Kinematic illustration

It is desired to move point **P**, designated in the local x,y-coordinate system, to point **C**, the position directly beneath the camera. The initial vector that defines **P** in the global X,Y-coordinate system is:

$$V_{OP} = V_{OA} + V_{AB} + V_{BP} \tag{3.3}$$

or

$$V_{OP} = (X_{OA} + l\cos\theta + x)i + (0 + l\sin\theta + y)j \tag{3.4}$$

The new position stated in terms of the global XY-coordinate system is:

$$X_{OP} = X_{OA} + l\cos\theta + x \tag{3.5}$$

and

$$Y_{OP} = lsin\theta + y \tag{3.6}$$

The global distance from point P to the camera position, **C** is:

$$V_{PC} = V_{OC} - V_{OP} \tag{3.7}$$

where

$$V_{OC} = X_{OC}i + 0j \tag{3.8}$$

then

$$V_{PC}\, i = X_{OC} - (X_{OA} + lcos\theta + x) = \Delta X_{sp} \tag{3.9}$$

$$V_{PC}\, j = 0 - (lsin\theta + y) = \Delta Y_{sp} \tag{3.10}$$

Equations (3.9) and (3.10) form the basis of a trajectory planner. As described by Craig (1989), trajectory planning refers to a "time history of position, velocity, and acceleration for each degree of freedom." In essence, it is the problem of moving an object from an initial starting position along a desired path to a final position in a predetermined amount of time.

The trajectory planner developed in this study was designed to meet two special purposes. In order to test for adequate friction compensation, the reference planner had to produce a smooth, S-curve reference signal. A prototype system with uncompensated friction will stick when forced to follow this position reference at low velocities. The second requirement was that it had to be scaleable.

A cubic polynomial incorporated in a trajectory path produces the desired S-curve motion path. Because it is smooth and continuous between two points, many controls applications in the process industries use the cubic reference planner to avoid tearing or jerking product. Examples include the paper, textile, and plastics industries.

The development of the desired path planner begins by imposing the following four constraints on any smooth motion which starts and ends at rest:

$$\begin{aligned} \theta(0) &= \theta_O \\ \theta(t_f) &= \theta_f \\ \dot{\theta}(0) &= 0 \\ \dot{\theta}(t_f) &= 0 \end{aligned} \tag{3.11}$$

where θ_O is the initial position, θ_f is the final positions and t_f is the final time. These four constraints are satisfied by the cubic polynomial (Craig, 1989):

$$\theta(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 \tag{3.12}$$

The joint velocity and acceleration along this path are:

$$\dot{\theta}(t) = a_1 + 2a_2 t + 3a_3 t^2 \tag{3.13}$$

$$\ddot{\theta}(t) = 2a_2 + 6a_3 t \tag{3.14}$$

Combining (3.12-3.14) with the constraints in (3.4) yields four equations in four unknowns:

$$\begin{aligned} \theta_0 &= a_0 \\ \theta_f &= a_0 + a_1 t_f + a_2 t_f^2 + a_3 t_f^3 \\ 0 &= a_1 \\ 0 &= a_1 + 2a_2 t_f + 3a_3 t_f^2 \end{aligned} \tag{3.15}$$

Solving for the a_i coefficients produces:

$$\begin{aligned} a_0 &= \theta_0 \\ a_1 &= 0 \\ a_2 &= \frac{3}{t_f^2}\left(\theta_f - \theta_0\right) \\ a_3 &= -\frac{2}{t_f^3}\left(\theta_f - \theta_0\right) \end{aligned} \tag{3.16}$$

Substituting these values into (3.12) yields:

$$\theta(t) = \theta_0 + \frac{3}{t_f^2}\left(\theta_f - \theta_0\right)t^2 - \frac{2}{t_f^3}\left(\theta_f - \theta_0\right)t^3 \tag{3.17}$$

Equation (3.17) is the dimensional form of a cubic trajectory path planner with respect to position. Craig (1989) illustrates how the planner works with the following example:

> The initial angle of a rotational link is θ_0 = 15 degrees while the final desired angle is θ_f = 75 degrees which the system must move to in 3 seconds. Substituting these values into (3.17) and taking the first and second derivatives produces:

$$\theta(t) = 15.0 + 20t^2 - 4.44t^3$$

$$\dot{\theta}(t) = 40t - 13.33t^2 \tag{3.18}$$

$$\ddot{\theta}(t) = 40 - 26.66t$$

Figure 3.3 shows the simulated position, velocity, and acceleration trajectories for this example.

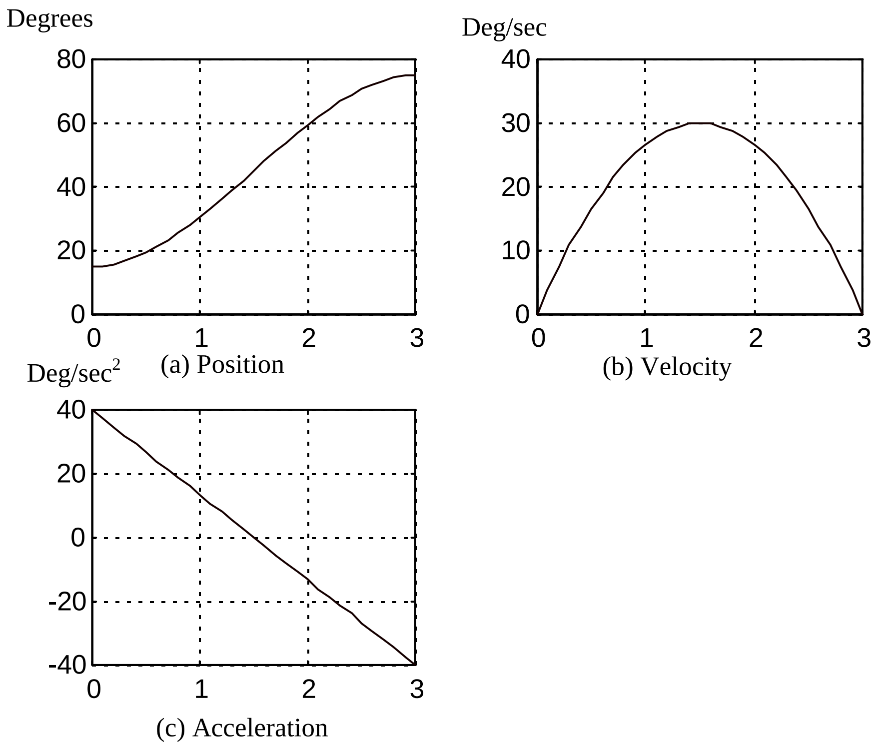

Figure 3.3: (a) Position, (b) Velocity, and (c) Acceleration profiles for a single cubic segment which starts and ends at rest

Now consider using the results stated in equation (3.17) together with the PR manipulator's kinematic equations. The process begins by substituting the results of equation (3.5) and (3.6) in for θ_0 in equation (3.17). These terms represent the initial conditions for the cubic trajectory path planner. Equation (3.17) is rewritten as:

$$X_{sp}(t) = X_{OP} + \frac{3}{t_f^2}\left(X_f - X_{OP}\right)t^2 - \frac{2}{t_f^3}\left(X_f - X_{OP}\right)t^3$$

and

$$Y_{sp}(t) = Y_{OP} + \frac{3}{t_f^2}\left(Y_f - Y_{OP}\right)t^2 - \frac{2}{t_f^3}\left(Y_f - Y_{OP}\right)t^3$$

Next, substitute ΔX_{sp} and ΔY_{sp}, which are defined by equations (3.9) and (3.10), in for $X_f - X_{OP}$ and $Y_f - Y_{OP}$ respectively. ΔX_{sp} and ΔY_{sp} represent the differential setpoints in the global coordinate system for the translational and rotational links. The equations become:

$$\begin{aligned} X_{sp}(t) &= X_{OP} + \frac{3}{t_f^2}\left(\Delta X_{sp}\right)t^2 - \frac{2}{t_f^3}\left(\Delta X_{sp}\right)t^3 \\ Y_{sp}(t) &= Y_{OP} + \frac{3}{t_f^2}\left(\Delta Y_{sp}\right)t^2 - \frac{2}{t_f^3}\left(\Delta Y_{sp}\right)t^3 \end{aligned} \tag{3.19}$$

Following the dimensionless product and similitude examples presented in Chapter 2, these equations are nondimensionalized by dividing each with the delta (Δ) terms. Recall that any equation can be nondimensionalized if it is dimensionally homogenous and is divided by a term of like-dimension. Equation (3.19) is thus rewritten as:

$$\begin{aligned} \frac{X_{sp}(t)}{\Delta X_{sp}} &= \frac{X_{OP}}{\Delta X_{sp}} + 3\left(\frac{t}{t_f}\right)^2 - 2\left(\frac{t}{t_f}\right)^3 \\ \frac{Y_{sp}(t)}{\Delta Y_{sp}} &= \frac{Y_{OP}}{\Delta Y_{sp}} + 3\left(\frac{t}{t_f}\right)^2 - 2\left(\frac{t}{t_f}\right)^3 \end{aligned} \tag{3.20}$$

or simplified, it is:

$$\begin{aligned} \bar{X}_{sp} &= \bar{X}_0 + 3\bar{t}^2 - 2\bar{t}^3 \\ \bar{Y}_{sp} &= \bar{Y}_0 + 3\bar{t}^2 - 2\bar{t}^3 \end{aligned} \tag{3.21}$$

Thus, the PR manipulator kinematic equations are dimensionless. However, this form is inconvenient for programming and for deriving the inverse kinematics. The equations are returned to a form that is dimensional in position and nondimensional in time. This form, however, is still amenable to any dimensioned PR manipulator as will be demonstrated in Chapter 4.

$$\begin{aligned} X_{sp}(\bar{t}) &= X_{OP} + \left(3\bar{t}^2 - 2\bar{t}^3\right)\Delta X_{sp} \\ Y_{sp}(\bar{t}) &= Y_{OP} + \left(3\bar{t}^2 - 2\bar{t}^3\right)\Delta Y_{sp} \end{aligned} \tag{3.22}$$

Equation (3.22) represents a cubic trajectory position planner for the PR manipulator's application under study. It generates new setpoints in the global *XY*-coordinate system as time goes from t_0 to t_f. The next step is to reformulate this set of equations so that it is acceptable to a controller. Inverse kinematics performs this task, and the procedure is developed by way of example.

Consider a new global (X_{sp} ,Y_{sp}) setpoint produced by (3.22) for some time, *t* less than t_f. Figure (3.4) depicts this new setpoint.

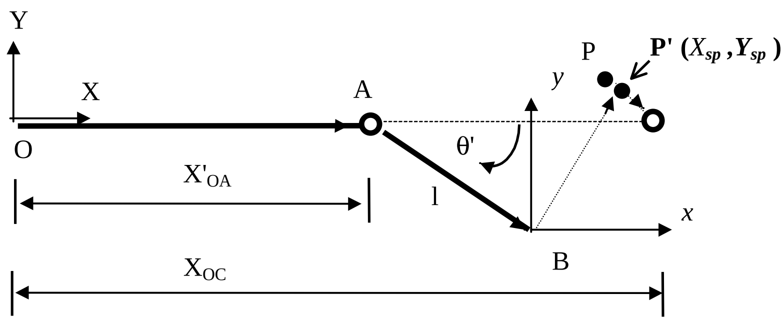

Figure 3.4: Example for inverse kinematics

The XY components of the new position vector OP' are:

$$X_{OP'} = X_{sp} = X'_{OA} + l\cos\theta' + x \qquad (3.23)$$

$$Y_{OP'} = Y_{sp} = l\sin\theta' + y \qquad (3.24)$$

Recall that the *x* term in (3.23) and the *y* term in (3.24) are established in the local coordinate system and are considered fixed as the manipulator moves. Thus, the new control variables are found by taking the inverse of (3.23) and (3.24), or:

$$\begin{aligned} \theta' &= \sin^{-1}\left(\frac{Y_{sp} - y}{l}\right) \\ X'_{OA} &= X_{sp} - l\cos(\theta') - x \end{aligned} \qquad (3.25)$$

If computer resources are available, a digital controller computes a new setpoint and performs a new inverse calculation every time it cycles through a control loop. If computer time is a premium, then the trajectory references and inverses are precomputed off-line and stored in a reference array. The latter method was chosen for this study. Thus, equation (3.25) represents the complete setpoint programmer that is dimensionless in time and scaleable for any PR manipulator.

3.3 Applying Similitude Methods to PM d.c. Motors.

This section begins by developing the speed versus torque curve for a PM d.c. motor. Many motor manufacturers provide this curve to application engineers to facilitate the motor selection process. The following discussion extends the results produced by Ipsen (1960) which produced a component of the speed vs. torque curve as part of his analysis. Note, unlike Ipsen, this derivation neglects the inductance term and the second pole in the motor equation. This is possible because the PM motors under test are all assumed to have poles that result in $\tau_m > 10(\tau_e)$. The test data in Chapter 5 proves that this assumption is valid.

The speed versus torque curve is important for two reason. First, it graphically establishes the idea of internal motor losses due to friction, and secondly, its (xy) intercept lays the ground work for similitude analysis. Note that for this derivation, the equations are stated in terms of angular velocity rather than position.

The curve is derived from:

$$T_g = K_t I = J_1 \frac{d\omega}{dt} + D_1\omega + T_f + T_L \tag{3.26}$$

$$V = IR + K_b\omega \tag{3.27}$$

$$I = \frac{T_g}{K_t} \tag{3.28}$$

and

$$V = \frac{T_g R}{K_t} + K_b\omega \tag{3.29}$$

Solving for ω produces:

$$\omega = \frac{V}{K_b} - \frac{T_g R}{K_t K_b} = \frac{V}{K_b} - \frac{T_g}{K_D} \tag{3.30}$$

Equation (3.30) defines the speed versus torque curve. The term $K_D = \frac{K_t K_b}{R}$ in equation (3.30) is known as the motor's damping constant (Saner, 1993) and desribes the slope of the curve.

Setting the motor's internally generated torque term, T_g to zero in equation (3.30) produces the Y intercept for the curve. This point is commonly known as the theoretical no load speed, or:

if

$$T_g = 0$$

then

$$\omega_{NL} = \frac{V}{K_b} \qquad \text{[No Load Speed, Theoretical case]} \tag{3.31}$$

If internal motor torque losses are considered, only T_L is set to zero in equation (3.26). The no load speed is thus:

$$\omega_{NL_{True}} = \frac{V}{K_b} - \frac{\left(J_1\dot{\omega} + V_1\omega + T_c\,\mathrm{sgn}(\omega)\right)}{K_D} \qquad \text{[True case]} \tag{3.32}$$

Furthermore, the inertia term in (3.32) goes to zero if the motor is being evaluated after the acceleration transient.

To determine the motor's stall torque, the speed term is set to zero in equation (3.30) and the curve's X intercept is solved for, or:

if

$$\omega = 0$$

then

$$T_{gs} = \left(\frac{V}{K_b}\right)\left(\frac{K_t K_b}{R}\right) = \frac{VK_t}{R} \qquad \text{[Stall Torque]} \tag{3.33}$$

Figure 3.5 shows the speed vs. torque curve generated from equation (3.30). The X intercept is defined by equation (3.33) and the theoretical intercept, Y', is defined by (3.31). The true operating speed is defined by (3.32) and is less than the idealized case due to static and Coulomb friction.

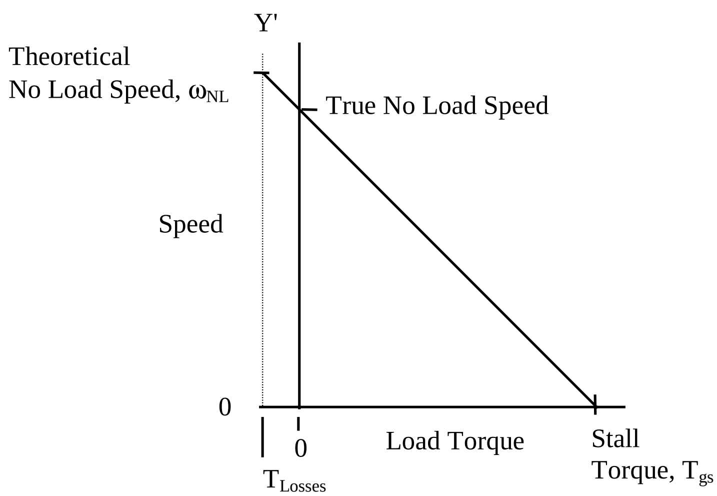

Figure 3.5: Speed vs. Torque curve for PM d.c. motor

Note that both speed and torque are functions of the applied voltage *V*, and the speed versus torque curve shifts as applied voltage changes. The torque term T_{losses} in Figure 3.5 is attributed to static (breakaway) and kinetic (dynamic) friction torques. From a motor manufacturer's perspective, the breakaway torques are "a function of cogging (changes in magnetic circuit reluctance), brush friction, and bearing friction. These are affected by bearing type and preload, brush material and force, air gap flux density, and the magnetic circuit configuration. Dynamic torque losses are caused by magnetic hysteresis, eddy currents, windage, brush friction and bearing losses" (Saner, 1993).

Nondimensionalizing a PM d.c motor using similitude methods begins by combining equations (3.26) through (3.29) to form a dimensionally homogenous equation in units of volts, [V]. This first derivation is in terms of angular velocity and includes loading, viscous damping, and Coulomb friction terms. The internally generated torque equation is:

$$T_g = K_t I = J_1 \frac{d\omega}{dt} + D_1\omega + T_C \,\mathrm{sgn}(\omega) + T_L \tag{3.34}$$

Substituting equation (3.34) into (3.29) yields:

$$V = \frac{J_1 R}{K_t}\frac{d\omega}{dt} + \frac{R}{K_t}D_1\omega + \frac{R}{K_t}T_C \,\mathrm{sgn}(\omega) + \frac{R}{K_t}T_L + K_b\omega \tag{3.35}$$

Nondimensionalizing (3.35) begins with forming the dimensionless product:

$$V = \overline{V} V_{ref} \tag{3.36}$$

Using (3.36) in (3.35) produces:

$$\overline{V} = \frac{J_1 R}{K_t V_{ref}}\frac{d\omega}{dt} + \frac{R}{K_t V_{ref}}D_1\omega + \frac{R}{K_t V_{ref}}T_C \,\mathrm{sgn}(\omega) + \frac{R}{K_t V_{ref}}T_L + \frac{K_b\omega}{V_{ref}} \tag{3.37}$$

Two observations can be made about (3.37). First,

$$\frac{K_b\omega}{V_{ref}} \text{ is really } \frac{\omega}{\frac{V_{ref}}{K_b}} = \frac{\omega}{\omega_{NL}} = \overline{\omega} \tag{3.38}$$

which is the motor's speed divided by the theoretical no load speed as stated in equation (3.31) and presented by Ipsen (1960). Equation (3.38) restated as a dimensionless product is:

$$\overline{\omega} = \frac{\omega}{\omega_{NL}} = \frac{\omega}{\omega_{ref}} \text{ or, } \omega = \omega_{ref}\overline{\omega} \tag{3.39}$$

The second observation is:

$$\frac{R}{K_t V_{ref}} = \frac{1}{\frac{K_t V_{ref}}{R}} = \frac{1}{T_{gs}} \tag{3.40}$$

which was presented in equation (3.33) as the motor's stall torque. Again, this term is a function of the voltage reference, V_{ref}. Substituting (3.39) into (3.37) produces:

$$\overline{V} = \frac{J_1 R}{K_t V_{ref}}\frac{d(\omega_{ref}\overline{\omega})}{dt} + \frac{R}{K_t V_{ref}}D_1(\omega_{ref}\overline{\omega}) + \frac{R}{K_t V_{ref}}T_C \,\mathrm{sgn}(\omega_{ref}\overline{\omega}) + \frac{R}{K_t V_{ref}}T_L + \overline{\omega} \tag{3.41}$$

Note that:

$$\frac{d\left(\omega_{ref}\overline{\omega}\right)}{dt} = \frac{d}{dt}\left(\frac{V_{ref}}{K_b}\right)\overline{\omega} = \frac{V_{ref}}{K_b}\frac{d}{dt}\overline{\omega}\text{, and } \operatorname{sgn}\left(\omega_{ref}\overline{\omega}\right) = \operatorname{sgn}\left(\overline{\omega}\right) \tag{3.42}$$

Substituting (3.40) and (3.42) into (3.41) yields:

$$\overline{V} = \frac{J_1 R}{K_t V_{ref}}\frac{V_{ref}}{K_b}\frac{d}{dt}\overline{\omega} + \frac{R}{K_t V_{ref}} D_1\left(\frac{V_{ref}}{K_b}\overline{\omega}\right) + \frac{T_C}{T_{gs}}\operatorname{sgn}\left(\overline{\omega}\right) + \frac{T_L}{T_{gs}} + \overline{\omega} \tag{3.43}$$

or

$$\overline{V} = \frac{J_1 R}{K_t K_b}\frac{d}{dt}\overline{\omega} + \frac{D_1}{K_D}\overline{\omega} + \frac{T_C}{T_{gs}}\operatorname{sgn}\left(\overline{\omega}\right) + \frac{T_L}{T_{gs}} + \overline{\omega} \tag{3.44}$$

where K_D is the damping constant as previously defined. Notice that these substitutions produced $\frac{J_1 R}{K_t K_b}$, which is the motor's mechanical time constant. If this term is taken as a time reference variable, t_{ref}, then a dimensionless product is formed by:

$$\tau_m = \frac{J_1 R}{K_t K_b} = t_{ref}\text{ , } \bar{t} = \frac{t}{t_{ref}}\text{, or } t = t_{ref}\bar{t} \tag{3.45}$$

Taking the time derivative of (3.45) produces:

$$dt = d\left(t_{ref}\bar{t}\right) = t_{ref}\,d\bar{t} \tag{3.46}$$

Substituting (3.45) and (3.46) into (3.44) yields:

$$\overline{V} = \frac{d}{d\bar{t}}\overline{\omega} + \frac{D_1}{K_D}\overline{\omega} + \frac{T_C}{T_{gs}}\operatorname{sgn}\left(\overline{\omega}\right) + \frac{T_L}{T_{gs}} + \overline{\omega} \tag{3.47}$$

or

$$\overline{V} = \dot{\overline{\omega}} + \overline{D}_1\overline{\omega} + \overline{T}_C\operatorname{sgn}\left(\overline{\omega}\right) + \overline{T}_L + \overline{\omega} \tag{3.48}$$

Equation (3.48) is completely dimensionless and can be used to compare the responses of a family of PM d.c. motors as discussed in Chapter 2. For purposes of preliminary analysis, assume the angular acceleration is zero, then equation (3.48) can be written to represent a nondimensional Speed versus Torque curve, or:

$$\overline{\omega} = \left(\frac{1}{\overline{D}_1 + 1} \right) \left(\overline{V} - \overline{T}_C \operatorname{sgn}(\overline{\omega}) - \overline{T}_L \right) \tag{3.49}$$

As advocated in Chapter 2, equation (3.49) was nondimensionalized in a manner that produced dimensionless variables that have physical meaning and are intuitively satisfying. Simply stated, equation (3.49) is composed of dimensionless speed, viscous and Coulomb friction, voltage, and load terms.

If equation (3.48) is linearized, then a closed form solution can be derived to further assist with the analysis. Assume velocity is for one direction only, and steps are applied to the voltage, Coulomb, and load terms. Then taking the Laplace transform of (3.48) produces:

$$\frac{\overline{V}(s)}{s} = s\overline{\omega}(s) + \overline{D}_1\overline{\omega}(s) + \frac{\overline{T}_C(s)}{s} + \frac{\overline{T}_L(s)}{s} + \overline{\omega}(s)$$

$$\frac{\overline{V}(s) - \overline{T}_C(s) - \overline{T}_L(s)}{s} = \left(s + \left(\overline{D}_1 + 1\right)\right)\overline{\omega}(s)$$

$$\overline{\omega}(s) = \frac{\overline{V}(s) - \overline{T}_C(s) - \overline{T}_L(s)}{s\left(s + \left(\overline{D}_1 + 1\right)\right)} \tag{3.50}$$

[It should be noted that Ghanekar *et al.* (1995) nondimensionalized the Laplace variable *s* which the author found unnecessary for this analysis.]

Applying the inverse Laplace transform of

$$\mathrm{L}^{-1}\left[\frac{1}{s(s+a)} \right] = \frac{1}{a}\left(1 - e^{-at}\right)$$

to equation (3.50) produces:

$$\overline{\omega}(\bar{t}) = \left(\overline{V} - \overline{T}_C - \overline{T}_L\right)\left(1 - e^{-\left(\overline{D}_1 + 1\right)\bar{t}}\right) \tag{3.51}$$

In many applications, this closed form solution would be sufficient for rapidly assessing candidate PM d.c. motors to be used in a prototype system. Expanding this equation to include static friction, initial condition terms, and writing it in terms of position, rather than velocity makes it even more useful for the application under study. The procedure for including these terms is presented next.

To simplify the derivation, the static friction term is treated as a square pulse, and again, only one velocity direction is considered. Figure 3.6 (A) graphically represents the full friction model, and Figure 3.6 (B) represents the part considered for purposes of deriving the closed form solution.

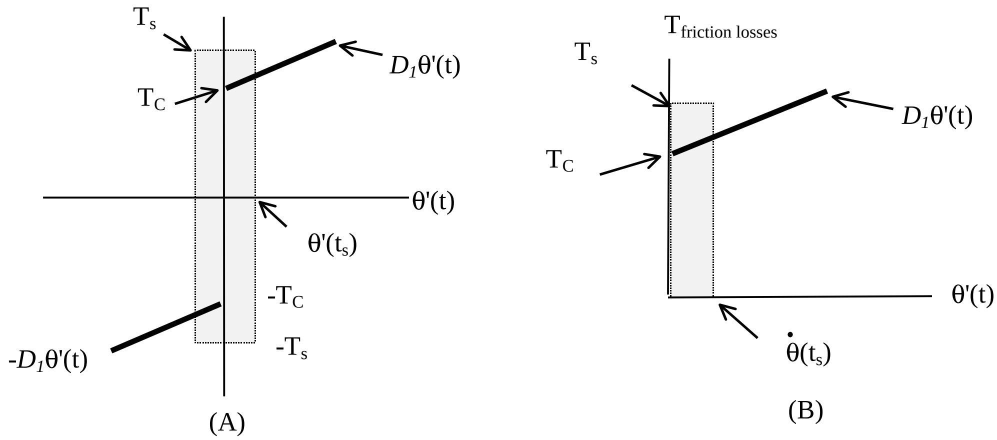

Figure 3.6: Friction model used for closed form solution

The shaded region represents the area where the system is considered to be under the influence of stiction, T_s. When the angular velocity reaches $\dot{\theta}\ (t_s)$ (where t_s is the time of transition) static friction is turned off and only Coulomb T_c and viscous $D_1\dot{\theta}\ (t)$ frictions are affecting the system.

The static friction pulse is described mathematically as a cut-off function and is presented here for review (Thomson, 1960). Figure 3.7 illustrates the concept.

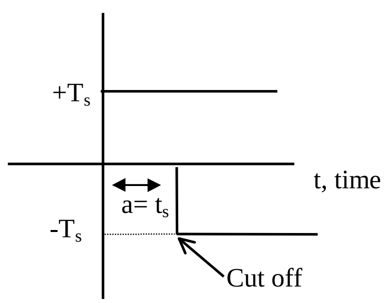

Figure 3.7: Cut-off function

The cut-off function represented in the time domain is:

$$f(t) = u(t) - u(t-a) \tag{3.52}$$

Recall that the Laplace transform of (3.52) is:

$$T_s(\mathrm{L}[u(t)-u(t-a)]) = \frac{T_s}{s} - T_s\int_0^\infty e^{-st}u(t-a)dt$$
$$= \frac{T_s}{s} - T_s\left[\int_0^a e^{-st}(o) + \int_a^\infty e^{-st}(1)dt\right]$$
$$= \frac{T_s}{s} - T_s\left[-\frac{1}{s}e^{-st}\right]_a^\infty = \frac{T_s}{s} - T_s\left[-\frac{1}{s}e^{-s\infty} - \left(-\frac{1}{s}e^{-sa}\right)\right]$$

and

$$f(s) = \frac{T_s}{s} - \frac{T_s}{s}e^{-sa} \tag{3.53}$$

where T_S is the magnitude of static friction.

The derivation of the dimensionless closed form solution for position begins by adding equation (3.53) to equation (3.35) written in terms of θ and assumed for one direction only. If $a = t_s$, then the dimensioned equation is:

$$\begin{aligned} V &= \frac{J_1 R}{K_t}\frac{d^2\theta}{dt^2} + \frac{R}{K_t}D_1\frac{d\theta}{dt} + \frac{R}{K_t}T_C + \frac{R}{K_t}T_L + K_b\frac{d\theta}{dt} \\ &+ \frac{R}{K_t}(T_s - T_C)\big(u(t) - u(t-t_s)\big) \end{aligned} \tag{3.54}$$

The following dimensionless products and derivative terms are used to nondimensionalize (3.54).

$$\begin{aligned} V &= \bar{V}V_{ref} \\ \theta &= \bar{\theta}\theta_{ref} \\ t &= \bar{t}t_{ref} \\ \theta_{ref} &= \omega_{ref}t_{ref} \\ \frac{d\theta}{dt} &= \frac{d\left(\bar{\theta}\theta_{ref}\right)}{d\left(\bar{t}t_{ref}\right)} = \frac{\theta_{ref}d\bar{\theta}}{t_{ref}d\bar{t}} = \omega_{ref}\frac{d\bar{\theta}}{d\bar{t}} = \frac{V_{ref}}{K_b}\frac{d\bar{\theta}}{d\bar{t}} \\ \frac{d^2\theta}{dt^2} &= \frac{d}{dt}\left(\omega_{ref}\frac{d\bar{\theta}}{d\bar{t}}\right) = \frac{\omega_{ref}}{d(\bar{t}t_{ref})}\left(\frac{d^2\bar{\theta}}{d\bar{t}}\right) = \frac{\omega_{ref}}{t_{ref}}\left(\frac{d^2\bar{\theta}}{d\bar{t}^2}\right) = \frac{V_{ref}}{t_{ref}K_b}\left(\frac{d^2\bar{\theta}}{d\bar{t}^2}\right) \end{aligned} \tag{3.55}$$

Substituting the terms in (3.55) into (3.54) produces:

$$\begin{aligned} \bar{V} &= \frac{J_1 R V_{ref}}{K_t V_{ref} K_b t_{ref}}\frac{d^2\bar{\theta}}{d\bar{t}^2} + \frac{RV_{ref}}{K_t K_b V_{ref}}D_1\frac{d\bar{\theta}}{d\bar{t}} + \frac{R}{K_t V_{ref}}T_C + \frac{R}{K_t V_{ref}}T_L + \frac{K_b V_{ref}}{V_{ref}K_b}\frac{d\bar{\theta}}{d\bar{t}} \\ &+ \frac{R}{K_t V_{ref}}(T_s - T_C)\big(u(\bar{t}) - u(\bar{t} - \bar{t}_s)\big) \end{aligned}$$

or

$$\bar{V} = \frac{\tau_m}{t_{ref}} \frac{d^2\bar{\theta}}{d\bar{t}^2} + \frac{D_1}{K_D} \frac{d\bar{\theta}}{d\bar{t}} + \frac{T_C}{T_{gs}} + \frac{T_L}{T_{gs}} + \frac{d\bar{\theta}}{d\bar{t}} + \frac{(T_s - T_C)}{T_{gs}} \left(u(\bar{t}) - u(\bar{t} - \bar{t}_s) \right) \qquad (3.56)$$

If the mechanical time constant is set equal to t_{ref}, and the coefficients are written in dimensionless variables, then:

$$\bar{V} = \frac{d^2\bar{\theta}}{d\bar{t}^2} + \left(\bar{D}_1 + 1 \right) \frac{d\bar{\theta}}{d\bar{t}} + \bar{T}_C + (\bar{T}_s - \bar{T}_C)\left(u(\bar{t}) - u(\bar{t} - \bar{t}_s) \right) + \bar{T}_L \qquad (3.57)$$

Equation (3.55) is the dimensionless form of the motor equation stated in terms of position. It is suitable for numerical simulation, and accounts for viscous, Coulomb, and "pulse type" static friction, and torque loading.

The closed form solution to (3.57) is generated next with initial conditions. Recall the functional operators:

$$f'(t) = s\bar{f}(s) - f(0)$$
$$f''(t) = s^2 \bar{f}(s) - sf(0) - f'(0)$$

Applying these functions to the dimensionless angular position derivative terms of (3.55) yields:

$$\begin{aligned} \frac{d^2\bar{\theta}}{d\bar{t}^2} &= s^2\bar{\theta} - s\bar{\theta}_0 - \frac{d\bar{\theta}_0}{dt} \\ \frac{d\bar{\theta}}{d\bar{t}} &= s\bar{\theta} - \bar{\theta}_0 \end{aligned} \qquad (3.58)$$

The Laplace transform of (3.57) using the expressions in (3.58) and assuming steps on all constants is:

$$\frac{\bar{V}}{s} = s^2\bar{\theta} - s\bar{\theta}_0 - \frac{d\bar{\theta}_0}{dt} + \left(\bar{D}_1 + 1\right)\left(s\bar{\theta} - \bar{\theta}_0\right) + \frac{\bar{T}_C}{s} + \frac{(\bar{T}_s - \bar{T}_C)}{s} - \frac{(\bar{T}_s - \bar{T}_C)}{s} e^{-s\bar{t}_s} + \frac{\bar{T}_L}{s} \qquad (3.59)$$

Using (3.59), $\bar{\theta}(s)$ is solved in the Laplace domain by:

$$s^2\bar{\theta} - s\bar{\theta}_0 - \frac{d\bar{\theta}_0}{dt} + \left(\bar{D}_1 + 1\right)\left(s\bar{\theta} - \bar{\theta}_0\right) = \frac{\bar{V}}{s} - \frac{\bar{T}_C}{s} - \frac{(\bar{T}_s - \bar{T}_C)}{s} + \frac{(\bar{T}_s - \bar{T}_C)}{s}e^{-s\bar{t}_s} - \frac{\bar{T}_L}{s}$$

$$\left(s^2 + (\bar{D}_1 + 1)s\right)\bar{\theta} - \left(s+1\right)\bar{\theta}_0 - \frac{d\bar{\theta}_0}{dt} = \frac{\bar{V}}{s} - \frac{\bar{T}_C}{s} - \frac{(\bar{T}_s - \bar{T}_C)}{s} + \frac{(\bar{T}_s - \bar{T}_C)}{s}e^{-s\bar{t}_s} - \frac{\bar{T}_L}{s}$$

$$\left(s^2 + (\bar{D}_1 + 1)s\right)\bar{\theta} = \frac{\bar{V} - \bar{T}_s - \bar{T}_L}{s} + \frac{(\bar{T}_s - \bar{T}_C)}{s}e^{-s\bar{t}_s} + \left(s+1\right)\bar{\theta}_0 + \frac{d\bar{\theta}_0}{dt}$$

$$\bar{\theta}(s) = \frac{\bar{V} - \bar{T}_s - \bar{T}_L}{s^2\left(s + (\bar{D}_1 + 1)\right)} + \frac{(\bar{T}_s - \bar{T}_C)}{s^2\left(s + (\bar{D}_1 + 1)\right)}e^{-s\bar{t}_s} + \frac{\left(s+1\right)\bar{\theta}_0}{s\left(s + (\bar{D}_1 + 1)\right)} + \frac{1}{s\left(s + (\bar{D}_1 + 1)\right)}\frac{d\bar{\theta}_0}{dt}$$

Recall:

$$\mathrm{L}^{-1}\left[\frac{1}{s^2\left(s+a\right)}\right] = \frac{1}{a^2}\left(e^{-at} + at - 1\right)$$

then

$$\bar{\theta}(t) = \frac{1}{\left(\bar{D}_1 + 1\right)^2}(\bar{V} - \bar{T}_s - \bar{T}_L)\left(e^{-(\bar{D}_1+1)\bar{t}} + (\bar{D}_1 + 1)\bar{t} - 1\right) + \frac{1}{\left(\bar{D}_1 + 1\right)^2}(\bar{T}_s - \bar{T}_C)\left(e^{-(\bar{D}_1+1)(\bar{t}-\bar{t}_s)} + (\bar{D}_1 + 1)\left(\bar{t} - \bar{t}_s\right) - 1\right)\left(u\left(\bar{t} - \bar{t}_s\right)\right)$$
$$+\bar{\theta}_0 e^{-(\bar{D}_1+1)\bar{t}} + \bar{\theta}_0\frac{\left(1 - e^{-(\bar{D}_1+1)\bar{t}}\right)}{\left(\bar{D}_1 + 1\right)} + \left(\frac{d\bar{\theta}_0}{d\bar{t}}\right)\frac{\left(1 - e^{-(\bar{D}_1+1)\bar{t}}\right)}{\left(\bar{D}_1 + 1\right)}$$

If $\bar{D}_1$ is very close to zero and the initial velocity is always zero per the requirements of the trajectory planner, then the final version of the closed form solution can be approximated as:

$$\bar{\theta}(t) = (\bar{V} - \bar{T}_s - \bar{T}_L)\left(e^{-(\bar{D}_1+1)\bar{t}} + (\bar{D}_1 + 1)\bar{t} - 1\right) + (\bar{T}_s - \bar{T}_C)\left(e^{-(\bar{D}_1+1)(\bar{t}-\bar{t}_s)} + (\bar{D}_1 + 1)\left(\bar{t} - \bar{t}_s\right) - 1\right)\left(u\left(\bar{t} - \bar{t}_s\right)\right) + \bar{\theta}_0$$

Equation (3.60)

where

$$u\left(\bar{t} - \bar{t}_s\right) = 0 \quad \text{for } \bar{t} < \bar{t}_s$$
$$u\left(\bar{t} - \bar{t}_s\right) = 1 \quad \text{for } \bar{t} \geq \bar{t}_s \qquad (3.61)$$

and $\bar{\theta}_0$ is the initial angular shaft position. For purposes of testing and comparing motors, the load term, T_L is set to zero. If the motor under test is geared and connected to a pulley system that causes linear displacement, then the initial position is:

$$\bar{\theta}_0 = \frac{\theta_0}{\theta_{ref}}$$
$$\theta_0 = \frac{x_0}{N_T r} \qquad (3.62)$$
$$\bar{\theta}_0 = \frac{x_0}{N_T r \theta_{ref}} = \frac{x_0}{N_T r \omega_{ref} t_{ref}}$$

and the position variable is:

$$\bar{\theta} = \frac{x}{N_T r \theta_{ref}} = \frac{x}{N_T r \omega_{ref} t_{ref}} = \frac{x K_b^2 K_t}{N_T r V_{ref} JR} \qquad (3.63)$$

Equation (3.63) will be shown to be a key scaling term, and necessary for redimensionalizing the systems' simulation results.

Thus, equation (3.60) is a dimensionless closed form solution suitable for simulating and scaling a family of PM d.c. gearmotors. The following restates the definitions of the terms associated with this equation.

$x = \bar{\theta} N_T r \omega_{ref} t_{ref}$	[dimensionless position]
$x_0 = \bar{\theta}_0 N_T r \omega_{ref} t_{ref}$	[dimensionless initial position]
$\bar{V} = \frac{V}{V_{ref}}$	[dimensionless voltage]
$\bar{T}_s = \frac{T_s}{T_{gs}}$	[dimensionless static friction]
$\bar{T}_C = \frac{T_C}{T_{gs}}$	[dimensionless Coulomb friction]
$\bar{T}_L = \frac{T_L}{T_{gs}}$	[dimensionless torque load]
$\bar{D}_1 = \frac{D_1}{K_D}$	[dimensionless viscous friction]
$\bar{t} = \frac{t}{\tau_m}$, and $\bar{t}_s = \frac{t_s}{\tau_m}$	[dimensionless time, and cut-off time]

Buckingham's Pi theorem is used next for purposes of grouping these dimensionless variables. The desired result is a reduced set of scaling laws.

3.4 Development of Buckingham Pi Terms for PR Manipulator Motors

Forming dimensionless Pi terms for the purposes of scaling PM d.c. motors follows the steps outlined in Chapter 2. The process begins by revisiting the dimensional equation (3.54):

$$V = \left[J \frac{d^2\theta}{dt^2} + D_1 \frac{d\theta}{dt} + T_C \operatorname{sgn}\left(\frac{d\theta}{dt}\right) + (T_s - T_C)\big(u(t) - u(t - t_s)\big) + T_L \right] \frac{R}{K_t} + K_b \frac{d\theta}{dt}$$

Recall that the Buckingham Pi theorem does not require explicit knowledge of the equation that describes the process under study. Thus, only its variables and a general function statement are required of (3.54). The procedure is as follows:

Step 1: *State the postulated functional relationship. List and count variables*

$$V = f\left(R, J, K_t, K_b, T_C, T_s, D_1, T_L, \theta, t\right) \text{ } (n\text{=11 variables})$$

Step 2: *List all variables and state their dimensions*

Variable	V	R	J	K_t	K_b	T_C	T_s	D_1	T_L	θ	t
Dimensions	$[V]$	$\left[\frac{V}{I}\right]$	$\left[\frac{FLT^2}{A}\right]$	$\left[\frac{FL}{I}\right]$	$\left[\frac{VT}{A}\right]$	$[FL]$	$[FL]$	$\left[\frac{FLT}{A}\right]$	$[FL]$	$[A]$	$[T]$

where the fundamental units are defined as:

$$\begin{bmatrix} F = \text{Force} \\ L = \text{Length} \\ T = \text{Time} \\ I = \text{Current} \\ V = \text{Voltage} \\ A = \text{Angle} \end{bmatrix}$$

(Ipsen's (1960) example was used as a guide for selecting these fundamental units.)

Step 3: *Determine j and k*

There are 6 fundamental units. Therefore, j= 6, and k=n-j=11-6=5. Thus, the analysis anticipates producing 5 independent dimensionless Pi terms.

Step 4: *Select j variables that do not form a Pi group*

After trial and error, and guided by experience gained from work with similitude analysis, the base terms from which all Pi terms will be formed are:

$$\left[T_C, J, R, K_t, K_b, V\right]$$

Step 5: *Form 5 Pi terms with the remaining k=n-j variables*

$$\Pi_1 = \left[T_C^a J^b R^c K_t^d K_b^e V^f D_1^1\right] = \left[(FL)^a \left(\frac{FLT^2}{A}\right)^b \left(\frac{V}{I}\right)^c \left(\frac{FL}{I}\right)^d \left(\frac{VT}{A}\right)^e (V)^f \left(\frac{FLT}{A}\right)^1\right] = F^0 L^0 T^0 V^0 I^0 A^0$$

$$\Pi_2 = \left[T_C^a J^b R^c K_t^d K_b^e V^f T_L^1\right] = \left[(FL)^a \left(\frac{FLT^2}{A}\right)^b \left(\frac{V}{I}\right)^c \left(\frac{FL}{I}\right)^d \left(\frac{VT}{A}\right)^e (V)^f (FL)^1\right] = F^0 L^0 T^0 V^0 I^0 A^0$$

$$\Pi_3 = \left[T_C^a J^b R^c K_t^d K_b^e V^f \theta^1\right] = \left[(FL)^a \left(\frac{FLT^2}{A}\right)^b \left(\frac{V}{I}\right)^c \left(\frac{FL}{I}\right)^d \left(\frac{VT}{A}\right)^e (V)^f (A)^1\right] = F^0 L^0 T^0 V^0 I^0 A^0$$

$$\Pi_4 = \left[T_C^a J^b R^c K_t^d K_b^e V^f t^1\right] = \left[(FL)^a \left(\frac{FLT^2}{A}\right)^b \left(\frac{V}{I}\right)^c \left(\frac{FL}{I}\right)^d \left(\frac{VT}{A}\right)^e (V)^f (T)^1\right] = F^0 L^0 T^0 V^0 I^0 A^0$$

$$\Pi_5 = \left[T_C^a J^b R^c K_t^d K_b^e V^f T_s^1\right] = \left[(FL)^a \left(\frac{FLT^2}{A}\right)^b \left(\frac{V}{I}\right)^c \left(\frac{FL}{I}\right)^d \left(\frac{VT}{A}\right)^e (V)^f (FL)^1\right] = F^0 L^0 T^0 V^0 I^0 A^0$$

For each Pi term, equate exponents and solve the system of Ax=0 simultaneous equations. Equation (3.64) demonstrates this for the first Pi term, and it is written so the damping term $[D_1]^1$ is brought to the right side of the equation.

$$\begin{bmatrix} A_{matrix} & T_C & J & R & K_t & K_b & V \\ F & 1 & 1 & 0 & 1 & 0 & 0 \\ L & 1 & 1 & 0 & 1 & 0 & 0 \\ T & 0 & 2 & 0 & 0 & 1 & 0 \\ V & 0 & 2 & 0 & 0 & 1 & 1 \\ I & 0 & 0 & -1 & -1 & 0 & 0 \\ A & 0 & -1 & 0 & 0 & -1 & 0 \end{bmatrix} * \begin{bmatrix} X_{vector} \\ a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = \begin{bmatrix} 0_{vector} * [D_1]^{-1} \\ -1 \\ -1 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad (3.64)$$

The first two rows in the A matrix are identical. Thus this matrix is rank deficient and not invertible. Full rank can be achieved by defining one of the exponents and using this expression to replace one of the duplicate rows. Experience and work with similitude analysis established that the exponent *a* should be set to 1. This expression replaces the first row in (3.64). The equation is rewritten as:

$$\begin{bmatrix} A_{matrix} & T_C & J & R & K_t & K_b & V \\ F & 1 & 0 & 0 & 0 & 0 & 0 \\ L & 1 & 1 & 0 & 1 & 0 & 0 \\ T & 0 & 2 & 0 & 0 & 1 & 0 \\ V & 0 & 2 & 0 & 0 & 1 & 1 \\ I & 0 & 0 & -1 & -1 & 0 & 0 \\ A & 0 & -1 & 0 & 0 & -1 & 0 \end{bmatrix} * \begin{bmatrix} X_{vector} \\ a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = \begin{bmatrix} 0_{vector} * [D_1]^{-1} \\ 1 \\ -1 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{3.65}$$

Solving the system of equations $x = A^{-1} \cdot 0 \cdot [D_1]^{-1}$ yields:

$$\begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ -2 \\ -1 \\ -1 \end{bmatrix}$$

and produces a Pi term of:

$$\Pi_1 = \frac{T_C D_1 R^2}{K_t^2 K_b V} = \left[\left(\frac{T_C}{T_{gs}} \right) \left(\frac{D_1}{K_D} \right) \right] \tag{3.66}$$

Observe that the first Pi term is formed by combining the dimensionless Coulomb and viscous friction variables that were generated using similitude methods. The remaining Pi terms are generated in a similar manner. The only difference being that exponent *a* is redefined as 0 instead of 1. The terms are:

$$\begin{aligned} \Pi_2 &= \frac{RT_L}{K_t V} = \left[\frac{T_L}{T_{gs}} \right] \\ \Pi_3 &= \frac{K_b}{V} \frac{K_t K_b}{JR} \theta = \left[\frac{\theta}{\omega_{NL} \tau_m} \right] \\ \Pi_4 &= \frac{K_b K_t}{JR} t = \left[\frac{t}{\tau_m} \right] \\ \Pi_5 &= \frac{RT_s}{K_t V} = \left[\frac{T_s}{T_{gs}} \right] \end{aligned} \tag{3.67}$$

The terms in (3.67) are exactly the same as those generated from similitude analysis. If the motors are coupled to a gearbox and position is linear rather than angular displacement, then the following substitutions are made:

$$\begin{aligned} T_L &= N_T T_{L_{pre-geared}} \\ \theta &= \frac{x}{N_T r} \end{aligned} \tag{3.68}$$

(Note, the results in equations (3.68) were developed in Appendix A.)

Thus, to successfully scale PM d.c. brush commutated motors that have fast electrical time constants, the following relationships must hold true.

$$\begin{aligned}
\Pi_1 &= \left[\left(\frac{T_C}{T_{gs}}\right)\left(\frac{D_1}{K_D}\right)\right]_{\text{model}} = \left[\left(\frac{T_C}{T_{gs}}\right)\left(\frac{D_1}{K_D}\right)\right]_{\text{prototype}} \\
\Pi_2 &= \left[\frac{N_T T_{L_{pre-geared}}}{T_{gs}}\right]_{\text{model}} = \left[\frac{N_T T_{L_{pre-geared}}}{T_{gs}}\right]_{\text{prototype}} \\
\Pi_3 &= \left[\frac{x}{N_T r \omega_{NL} \tau_m}\right]_{\text{model}} = \left[\frac{x}{N_T r \omega_{NL} \tau_m}\right]_{\text{prototype}} \\
\Pi_4 &= \left[\frac{t}{\tau_m}\right]_{\text{model}} = \left[\frac{t}{\tau_m}\right]_{\text{prototype}} \\
\Pi_5 &= \left[\frac{T_s}{T_{gs}}\right]_{\text{model}} = \left[\frac{T_s}{T_{gs}}\right]_{\text{prototype}}
\end{aligned} \tag{3.69}$$

(It is assumed that boundary and initial conditions are met between the model and the prototype systems.)

The results of the dimensional analysis presented in (3.69) are easily summarized. In order to have similar position responses, the model and prototype motors must have equal ratios of:

- Coulomb to stall torque and static friction to stall torque (This is a function of the reference voltage for both systems.)
- viscous friction to the damping constant
- torque load to stall torque
- position reference to the product of gearing, pulley radius, no load speed, and mechanical time constant.
- time of run to mechanical time constant.

As Saunders *et .al* (1996) points out, achieving exact similarity between systems is often difficult because of resource and economic limitations. This is especially true for scaling motors that belong to the more inexpensive ceramic PM class. At face value, locating off-the-shelve motors that are suitable for use in a scaled prototype system can be an insurmountable challenge. However, there are simple control and design techniques that can assist with achieving similarity among dissimilar motors. This methodology is presented next.

3.5 Control Strategy for Scaled PR Manipulators

The controller presented here pre-computes the torque necessary for the actuators to move the system to a position setpoint while following a desired trajectory path in a prespecified amount of time. Consider the translational axis of the PR manipulator. A simple block diagram illustrates the forward loop of this mechanism's control algorithm.

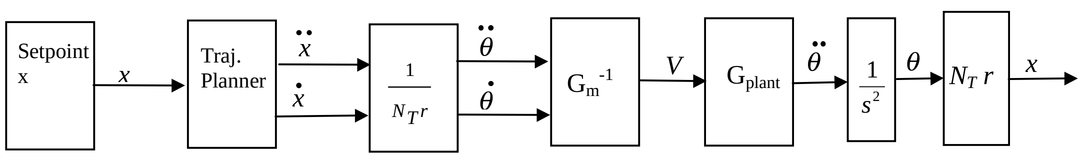

Figure 3.8: Computed torque control algorithm (translation axis)

Essentially, the controller is comprised of two parts. The first part is the scaleable trajectory planner which was developed in section 3.2. The second part is based on the computed torque control method introduced in section 2.3. The inverse motor model G_m^{-1} in Figure 3.8 forms the basis of this control technique. Derivative terms generated by the trajectory planner are fed to G_m^{-1} which computes a voltage that is subsequently turned into a torque by the motor.

The following set of equations form the basis of the controller under study. The trajectory planner's inverse kinematic conversion produces the following control variables. They are:

$$\begin{aligned} \theta &= \sin^{-1}\left(\frac{Y_{sp} - y}{l}\right) \\ X_{OA} &= X_{sp} - l\cos(\theta) - x \end{aligned} \tag{3.70}$$

The rotational and linear displacements in (3.70) are twice differentiated and post multiplied by $\frac{1}{N_T r}$ which essentially acts like a gain on the system. These acceleration and velocity terms are passed to the inverse model G_m^{-1} which is described by equation (3.71). Note that this model uses the pulse function for state friction.

$$V = \left[J\frac{d^2\theta}{dt^2} + D_1\frac{d\theta}{dt} + T_C \operatorname{sgn}\left(\frac{d\theta}{dt}\right) + (T_s - T_C)\big(u(t) - u(t - t_s)\big)\operatorname{sgn}\left(\frac{d\theta}{dt}\right) + T_L \right]\frac{R}{K_t} + K_b\frac{d\theta}{dt} \tag{3.71}$$

For the simulations presented in Chapter 4, the "plant" motors were modeled using the so-called Stribeck curve to describe the transition from static to kinetic friction. It is presented in equation (3.72).

$$\frac{d^2\theta}{dt^2} = \frac{VK_t}{JR} - \left[D_1\frac{d\theta}{dt} + T_C \operatorname{sgn}\left(\frac{d\theta}{dt}\right) + \left((T_S - T_C)e^{-(\dot{\theta}/\dot{\theta}_S)^2}\right)\operatorname{sgn}\left(\frac{d\theta}{dt}\right) + T_L \right]\frac{1}{J} - \frac{K_b K_t}{JR}\frac{d\theta}{dt} \tag{3.72}$$

Scaling the model and prototype controllers is accomplished by: 1) entering the system dimensions and setpoints into the trajectory planner formed by (3.70); and 2) substituting into (3.71) the motor parameters that satisfied the Pi terms of (3.69). If the motor parameters were properly identified, then the controllers are automatically scaled and tuned.

To satisfy the first and fifth Pi terms, motors must be selected that have similar friction characteristics. However, this is rarely achieved. In fact, Saner (1993) states that a motor manufacturer's specified friction terms may vary as much as 50% for different windings and voltage ratings for within one motor frame size. However, if the friction terms are modeled correctly in (3.71), then the motor friction terms in (3.72) cancel and the friction related Pi groupings effectively go to zero. The idea is illustrated below.

$$\Pi_1 = \left[\left(\frac{T_C}{T_{gs}}\right)\left(\frac{D_1}{K_D}\right)\right]_{\text{model}} = \left[\left(\frac{T_C}{T_{gs}}\right)\left(\frac{D_1}{K_D}\right)\right]_{\text{prototype}} \to 0$$

as (T_C and D_1) $\to 0$ in both systems

and

$$\Pi_5 = \left[\frac{T_s}{T_{gs}}\right]_{\text{model}} = \left[\frac{T_s}{T_{gs}}\right]_{\text{prototype}} \to 0$$

as $T_s \to 0$ in both systems

However, the degree to which this can be successfully achieved is dependent on the magnitudes of the friction terms. This will be further explained in the next two chapters.

3.6 Summary

This chapter developed a nondimensional trajectory planner for a PR manipulator. In addition, dimensionless closed form solutions were developed for velocity and position step responses. Through similitude methods, dimensionless variables were produced to scale and evaluate PM d.c. motors. From the knowledge and insights gained in this exercise, a reduced set of Buckingham Pi terms were formulated. Both the dimensionless variables and the Buckingham Pi terms were in agreement with each other and provided direct insight to the scaling problem. A control law was proposed that used the scaleable trajectory planner with the computed torque method. It was proposed that scaling and tuning this controller could be accomplished by using the motor parameters that satisfied the Pi terms.