3.0 Mathematical Model of the Wireless Alarm System

The wireless alarm system has been modeled mathematically to obtain an analytical expression for the probability that an alarm unit in a given system will fail to receive an alarm message because of message collisions.

In a three-unit system, assuming that all three units are in range of at least one other unit, there is no chance that message collisions will prevent any unit from receiving an alarm message.

In a four-unit system, message collisions may become a threatening factor depending upon the configuration. If all four units are within operating range of one another, there is no chance that message collisions will prevent any unit from receiving an alarm message. Furthermore, message collisions do not affect a serial four-unit configuration.

There are two four-unit configurations in which message collisions can degrade system performance. The first configuration, shown in Figure 3.1, is a worst-case four-unit configuration. In this case, both units two and three are in range of units one and four. However, units two and three are *not* in range of each other. Since the system relies upon a Carrier Sense Multiple Access (CSMA) technique, collisions in this configuration can be caused not only by synchronization of the holdoff timers in units two and three, but also by the inability of units two and three to detect one another's transmissions.

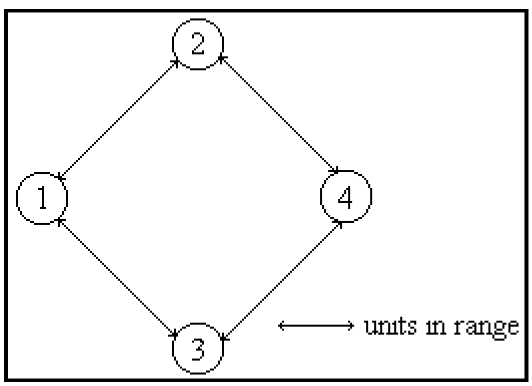

Figure 3.1: Worst Case Four-Unit Configuration

Figure 3.2 demonstrates a better or intermediate situation, in which units two and three are in range of units one and four, and are also in range of each other. In this configuration, units two and three can sense one another's transmissions. As a result, message collisions are caused only by synchronization of the holdoff timers in units two and three.

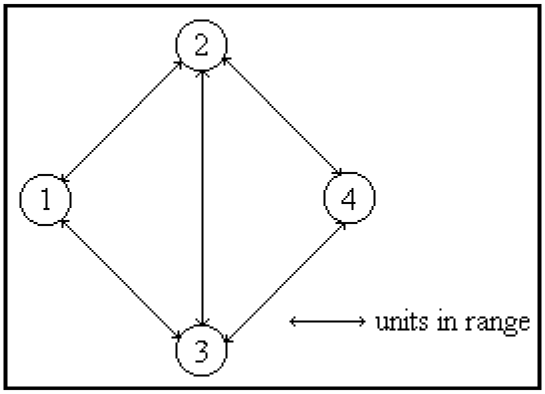

Figure 3.2: Intermediate Four-Unit Configuration

In order to analyze the probability of message collisions in each of these configurations, it is necessary first to determine the circumstances under which all of the messages generated by units two and three collide with one another.

3.1 Definitions and Assumptions

Table 3.1 summarizes some parameters that will be used in the analysis. Note that constant times are denoted by *T*, while random time variables are given by τ.

Table 3.1: System Parameters

Symbol	Description	Value (sec)
T_m	Message duration	0.015625
T_{tx}	Continuous transmission duration	24
T_r	Receive window duration	0.09375
T_{rw}	Time interval between receive windows	18.84375
T_{dmax}	Maximum value a unit can have in its holdoff timer	6
τ_{tx2}	Time at which unit two begins transmitting	
τ_{tx3}	Time at which unit three begins transmitting	
i	Index of unit four's receive window being examined	

The following four assumptions are made for the analysis:

(1) There is no capture effect. If any part of two messages overlap, both messages are destroyed.
(2) There is a zero propagation delay in the system. The time required for a message to propagate from one unit to another is zero.
(3) The time at which unit one generates the original alarm message is a random variable with a uniform distribution.
(4) The value in any unit's holdoff timer is a random variable with a uniform distribution.

3.2 Determination of Transmission Intervals

The first step of the analysis is to determine all of the possible scenarios in which unit four does not receive a message originating from unit one. More specifically, we focus on the instances when all messages generated by units two and three collide with one another during unit four's receive windows. Using this approach, it can be determined that there are two significant cases which must be examined.

Recall that when transmitting, the alarm units continuously resend the message for a time period of 24 seconds. This time interval is denoted T_{tx}. Recall also that alarm units are capable of receiving messages during an interval of 93.75 ms every 18.84375 seconds. The interval during which units can receive messages is called T_r, while the time period between these receive intervals is T_{rw}.

The first significant case which must be examined is shown in Figure 3.3. This case occurs when the transmissions of units two and three overlap exactly one of unit four's receive windows. Since $T_{tx} > T_{rw}+2T_r$, it is also possible that the transmissions of units two and three will overlap two of unit four's receive intervals. This is the second significant case, and is depicted in Figure 3.4.

It is possible that units two and three may transmit such that their messages overlap one of unit four's receive windows and only part of a second receive window. This scenario is negligible because it occurs with only a very small probability relative to the other two cases. The overall probability of the collision of all messages generated by units two and three is simply the sum of the probabilities of the occurrence of each of the two significant cases.

3.2.1 Case I: Unit Two Transmits Over One of Unit Four's Receive Windows

Case I, shown in Figure 3.3, demonstrates that there is a range of times over which unit two's transmissions can begin such that they overlap exactly one of unit four's receive windows. Note that the dotted rectangles in the figure represent T_{tx} seconds, the amount of time required for all of unit two's transmissions. Assuming that unit two initiates its transmissions within the interval marked with an arrow in Figure 3.3, it is then possible to determine under what circumstances unit three's transmissions overlap all those of unit two.

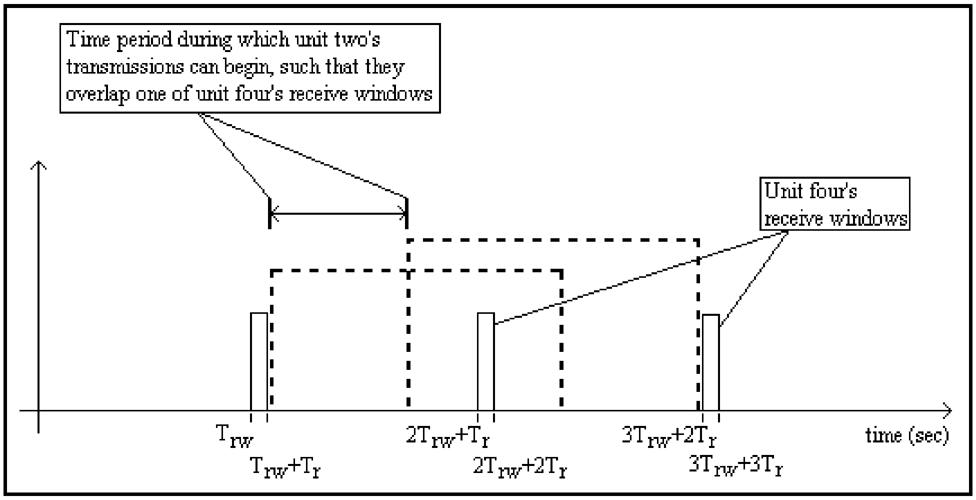

Figure 3.3: Case I – Unit 2 Transmits Over One Receive Window

From Figure 3.3, it can be seen that unit two's transmissions overlap exactly one of unit four's receive windows if unit two begins transmitting within the time interval specified in equation (3.1).

$$(T_{rw} + T_r)(i - 1) < \tau_{tx2} < T_{rw}(i + 1) + i\,T_r - T_{tx} \qquad (3.1)$$

Recall that receivers in the wireless alarm system consider a message valid only if two identical messages are received consecutively. Therefore, all messages generated by unit three will collide with those generated by unit two if unit three initiates its transmissions within $2T_m$ before or after the interval defined for the origin of unit two's transmissions. The interval over which unit three's transmissions can begin such that they collide with messages generated by unit two is given by equation (3.2).

$$(T_{rw} + T_r)(i - 1) - 2T_m < \tau_{tx3} < T_{rw}(i + 1) + i\,T_r - T_{tx} + 2T_m \qquad (3.2)$$

3.2.2 Case II: Unit Two Transmits Over Two of Unit Four's Receive Windows

Case II is the situation in which all of unit two's transmissions occur over exactly two of unit four's receive windows. Figure 3.4 demonstrates that there is a range of times over which unit two's transmissions can begin, such that they overlap exactly two of unit four's receive windows. Again, the dotted rectangles in the figure represent an interval of T_{tx} seconds, the amount of time required for all of unit two's continuous retransmissions. Assuming that unit two initiates its transmissions within the time interval marked with an arrow in Figure 3.4, it is possible to determine under what circumstances unit three's transmissions overlap all those of unit two.

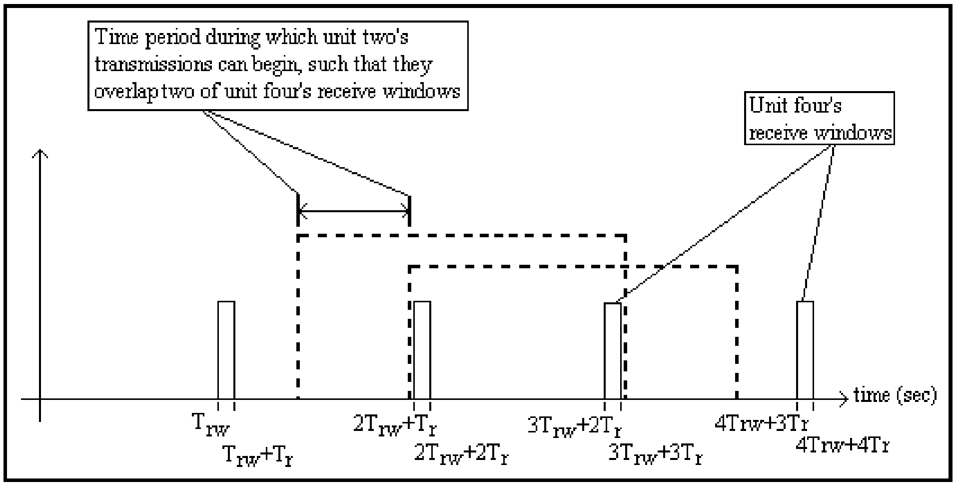

Figure 3.4: Case II – Unit 2 Transmits Over Two Receive Windows

From Figure 3.4, it can be seen that unit two's transmissions overlap exactly two of unit four's receive windows if unit two's transmissions begin within the time interval specified by equation (3.3).

$$(i + 1)(T_{rw} + T_r) - T_{tx} \leq \tau_{tx2} < iT_{rw} + (i\text{-}1)T_r \qquad (3.3)$$

Recall from the Case I analysis that all of unit three's messages will collide with those generated by unit two if unit three initiates its transmissions within $2T_m$ before or after the interval defined for the origin of unit two's transmissions. The interval over which unit three's transmissions can begin such that they collide with messages generated by unit two is given by equation (3.4).

$$(i + 1)(T_{rw} + T_r) - T_{tx} - 2\,T_m < \tau_{tx3} < iT_{rw} + (i\text{-}1)T_r + 2\,T_m \qquad (3.4)$$

3.3 Collision Analysis

Recall that there are two four-unit configurations which merit collision analysis. The first of these is shown in Figure 3.1 and represents a worst-case four-unit configuration. An improved four-unit configuration which still merits analysis is shown in Figure 3.2. Using the transmission time intervals just derived, it is possible to determine the probability that collisions prevent unit four from receiving a message in both of these configurations.

3.3.1 Four-Unit Worst-Case Configuration Collision Analysis

The probability that unit four does not receive an alarm message in the worst-case four-unit configuration is simply the sum of the probabilities of the Case I and Case II situations just discussed. Recall that both cases require that unit two initiates its transmissions within some defined interval, *and* that unit three initiates its transmissions within a slightly larger interval. The probability that unit four does not receive a message in the four-unit worst-case configuration is thus given by equation (3.5).

Pr {Unit four does not receive message}
= Pr {Case I} + Pr {Case II}
$$=\{Pr\ [(T_{rw} + T_r)(i-1) < \tau_{tx2} < T_{rw}(i+1) + i\,T_r - T_{tx}]\} \times$$
$$\{Pr\ [(T_{rw} + T_r)(i-1) - 2T_m < \tau_{tx3} < T_{rw}(i+1) + i\,T_r + 2T_m - T_{tx}]\} +$$
$$\{Pr[(i+1)(T_{rw} + T_r) - T_{tx} \le \tau_{tx2} < iT_{rw} + (i-1)T_r]\} \times$$
$$\{Pr[(i+1)(T_{rw} + T_r) - T_{tx} - 2\,T_m < \tau_{tx3} < iT_{rw} + (i-1)T_r + 2\,T_m]\} \qquad (3.5)$$

We will assume that the time at which unit two initiates its transmissions is a random variable uniformly distributed over one period of unit four's receive intervals. Therefore, for our analysis, we assume that τ_{tx2} is uniform over
$[(i-1)(T_{rw}+T_r), i(T_{rw}+T_r)]$.

Recall that in this worst case scenario, unit three is out of range of unit two, and cannot detect unit two's transmissions. The time at which unit three begins its transmissions is independent of the time at which unit two starts transmitting. This allows us to model the start of unit three's transmissions as a uniform random variable.

The interval over which unit three's transmissions must occur is almost identical to that over which unit two's transmissions must occur. In fact, the only difference is that unit three's transmission interval is $4T_m$, or 62.5 ms larger. This increase in time interval is very small relative to the time interval itself. Therefore, we can approximate that the probability that unit three begins transmitting within its interval is identical to the probability that unit two begins transmitting within the slightly smaller interval we have defined.

The probability that unit four does not receive a message in this four-unit worst-case configuration results in equation (3.6).

Pr {Unit four does not receive message}
$$=\{Pr\ [(T_{rw} + T_r)(i-1) < \tau_{tx2} < T_{rw}(i+1) + i\,T_r - T_{tx}]\} \times$$
$$\{Pr\ [(T_{rw} + T_r)(i-1) - 2T_m < \tau_{tx3} < T_{rw}(i+1) + i\,T_r + 2T_m - T_{tx}]\} +$$
$$\{Pr[(i+1)(T_{rw} + T_r) - T_{tx} \le \tau_{tx2} < iT_{rw} + (i-1)T_r]\} \times$$
$$\{Pr[(i+1)(T_{rw} + T_r) - T_{tx} - 2\,T_m < \tau_{tx3} < iT_{rw} + (i-1)T_r + 2\,T_m]\}$$
$$= \left[\int_{(T_{rw}+T_r)(i-1)}^{T_{rw}(i+1)+iT_r-T_{tx}} \frac{d\tau_{tx2}}{T_{rw}}\right]^2 + \left[\int_{(i+1)(T_{rw}+T_r)-T_{tx}}^{iT_{rw}+(i-1)T_r} \frac{d\tau_{tx2}}{T_{rw}}\right]^2$$
$$= (0.7313)^2 + (0.2637)^2$$
$$= 0.6043 \qquad (3.6)$$

3.3.2 Four-Unit Intermediate-Case Configuration Collision Analysis

The collision analysis for the four-unit intermediate-case configuration is different than that for the four-unit worst-case configuration. Recall that in the intermediate configuration, units two and three can hear one another's transmissions. If unit two begins transmitting before unit three, unit three will sense the channel busy until unit two finishes its retransmissions and vice versa. Under normal circumstances, message collisions are only a threat in this intermediate configuration if units two and three sense the channel idle simultaneously. This occurs only if the holdoff timers in units two and three are synchronized with one another.

Each unit's holdoff timer is essentially a six-bit counter which increments every 93.75 ms and resets every 6 sec. Assuming that the value in any unit's holdoff timer is a random variable uniformly distributed over a time interval of 6 seconds, the probability that the holdoff timers in any two units are synchronized is simply 1/64, or about 0.016.

There are hardware delays in the wireless alarm system which were unexpected by the designer. These delays have made the system more vulnerable to collisions. Ideally, once an alarm unit has sensed that the channel is idle, it should begin its transmissions immediately. However, an unanticipated hardware delay has caused a 187.5 ms delay from the time at which a unit senses the channel idle to the time at which the unit actually begins its transmissions.

This means that even though one unit has sensed the channel idle and has scheduled its transmissions, the channel will still appear idle to other units 187.5 ms later. A hypothetical example of the effect this has on the system is shown in Figure 3.5.

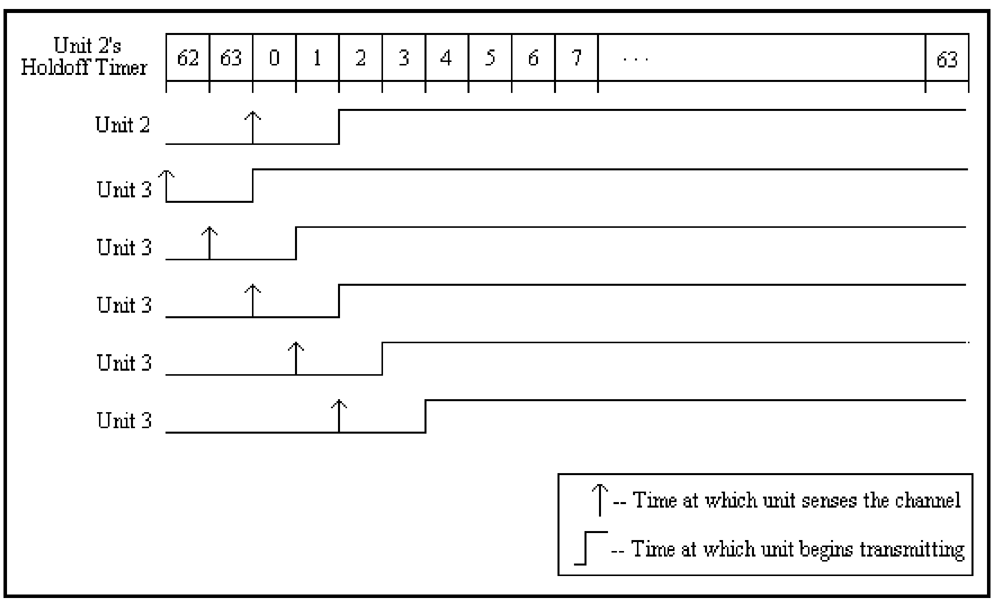

Figure 3.5: Effect of Hardware Delays on Transmission Timing

In the example shown in Figure 3.5, the value in unit two's holdoff timer is shown. Note that when this value becomes zero, unit two senses the channel as is depicted by the vertical arrow on the line representing the activity of unit two. Since no other units are transmitting at that particular time, unit two initiates its transmissions. Note, however, that because of hardware delays, unit two's transmissions do not begin until two holdoff timer increments later.

The remaining portion of Figure 3.5 represents the possible times at which unit three may sense the channel idle, yet transmit such that its messages collide with those generated by unit two.

If we let τ_{s2} represent the time at which unit two senses the channel, we see that unit three's transmissions will collide with those generated by unit two if unit three senses the channel within the interval *[τ_{s2} - 187.5 ms, τ_{s2} + 187.5 ms]*. Again, assuming that the value in a unit's holdoff timer is a uniform random variable, we can determine the probability that the value of unit three's holdoff timer is within 187.5 ms of that in unit two's holdoff timer as shown in equation (3.7).

Pr {Unit three's holdoff timer is within 187.5 ms of unit two's holdoff timer}

$$= \int_{\tau_{s2}-0.1875}^{\tau_{s2}+0.1875} \frac{d\tau}{6}$$

$$= 0.0625 \qquad (3.7)$$

This result represents the probability that message collisions will prevent unit four from receiving an alarm message in the four-unit intermediate-case configuration. Note that the intermediate configuration is a significant improvement over the worst-case configuration, which yielded a probability of 0.6043 that unit four would not receive an alarm message.

3.4 Four-Unit Installation Recommendations

Four-unit configurations in which message collisions are not a threat are acceptable installations of the wireless alarm system. This includes a topology in which all four units are within range of one another and a serial arrangement of four units.

In this analysis, two configurations in which message collisions are a threat were analyzed. Of these, one yields a probability of link failure of 0.6043, while the other yields a probability of 0.0625.

At this time, the project sponsor is planning to include a disclaimer with the wireless alarm system, indicating that the system may exhibit up to a 0.0625 probability of unit-to-unit link failure. It is assumed that an installation of the wireless alarm system is acceptable if the probability of link failure is less than or equal to 0.0625. Besides the configurations in which message collisions are not a threat, the only acceptable four-unit topology is the intermediate configuration shown in Figure 3.2.