# A POWER STUDY OF MULTIPLE RANGE
## AND MULTIPLE F TESTS

by

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## I.  INTRODUCTION

In recent years, many multiple comparisons tests have been presented.  Among the authors of these tests are Newman (1939), Duncan(1947, 1951, and 1955), Tukey (1949, 1951, and 1953), Scheffé (1953), and Hartley (1954).  These procedures are used for testing differences between single means or comparisons among more than two means from among several.  The properties and uses of each tests are discussed by the authors.  In this paper a method is developed and expressions are formulated for comparing certain of these tests.  Also, some interesting developments are made in the problem of enumerating the total number of decisions.

Some common multiple range and multiple F tests are the

1.  Multiple t test,

2.  Least significant difference test,

3.  Newman-Keuls multiple range test,

4.  Duncan's multiple comparisons test,

5.  Tukey's test based on allowances,

6.  Scheffé's test based on contrasts,

7.  Duncan's new multiple range test, and

8.  Tukey's X procedure.

Some of these tests are no more than mentioned in this paper. Others are discussed in great detail. At least one, the multiple t test, should never be used to test differences among more than three means.

The procedures as we use and compare them involve testing differences between pairs of means. Thus, they are defined as follows:

Assume that n sample means are arranged according to increasing size. A _multiple_ _range_ _test_ is one in which the difference between any two means is declared significant provided the range of every subset of p means, $p \leq n$, which contains the two given means, is significant according to a range test on p means. A _multiple_ _F_ _test_ is one in which the difference between any two means is declared significant provided the variance of every subset of p means, $p \leq n$, which contains the two given means, is significant according to an F test on p means. This is not the most general definition of an F test, but it is sufficient for our purposes as is explained in IX.

Multiple comparisons tests may be classified according to their system of significance levels. Corresponding to each multiple range test there is a multiple F test which has the same system of significance levels. For example, 5 and 6 as well as 7 and 4 are companion

multiple range and multiple F tests, respectively. The
choice as to which of the multiple range tests (multiple
F tests) is best depends mainly on one's choice of a
system of significance levels. On the other hand, the
relative merits of a multiple range test versus the
corresponding multiple F test are very obscure. This
is the type comparison considered in this paper.

In order to compare the companion multiple range and
multiple F tests, it is necessary to derive and evaluate
sets of power functions. Each of these functions is one
whose use has been indicated by Lehmann (1950), Duncan
(1953 and 1955), Scheffé (1953), Hartley (1954), and
Harter (1954) and is concerned with measuring the correct
decision probabilities and error probabilities for de-
cisions involving differences between two means. Thus,
in testing the differences in a set of n means, n(n-1)
power functions of this type are formulated. These
power functions are thought of as components of a vector,
called a power vector. Half of the components in each
power vector represent probabilities of correct decisions,
and the other half represent probabilities of errors.
Averages of the correct decision probability components
and error probability components of a power vector,
called correct decision vector and error vector, respec-
tively, are most useful for test comparisons.

In considering experiments involving n means, it is very important that we do not confuse the multiple comparison problem in which we are interested with the apparently similar but very different problem of testing a single homogeneity hypothesis of n means. Tang's (1938) power function is for an F test of the homogeneity of means. In our case, interest is primarily centered in tests of hypotheses for the means considered two at a time, making the power vector approach much more appropriate.

Expressions of power for the multiple range tests are given for n=2,3,4, and the method for writing the power function for any $n > 4$ is discussed. Approximation methods are formulated for the power of the multiple range tests and are used to compute the power for Duncan's multiple range test of four means. Numerical integration is used to find power for Duncan's multiple comparisons test, making a comparison of his two tests possible.

The notation could not be kept consistent throughout this paper. However, the most important things do carry the same notation from topic to topic. It should be noted that for any two sets of points A and B, A+B denotes all points in either A or B, A·B denotes all points common to both A and B, and A-B denotes all points of A which are not in B.

## II. DECISIONS AND REGIONS

### 1. Assumptions.

Let

$$m_1, \ m_2, \ \ldots, \ m_n$$

denote a random sample of n observed means from n normal populations with true means

$$\mu_1, \ \mu_2, \ \ldots, \ \mu_n,$$

respectively, and a common standard error, $\sigma_m$. If this standard error is unknown, assume the usual estimate, $s_m$, is available, is independent of the observed means, and is based on a number of degrees of freedom, denoted by $n_2$.

In this paper, it is assumed that $n_2 = \infty$, in which case $s_m = \sigma_m$. It is also assumed that $\sigma_m = 1$ in most of the paper. These assumptions simplify the work involved by lessening the calculations and the writing considerably, but they do not restrict us when comparing two tests. It should be noted that, even though the t-test becomes the normal deviate test and the F-test becomes the $\chi^2$-test, when $n_2 = \infty$, we do not change the names, multiple t-test and multiple F-test, since the general procedure happens to be illustrated only for the case when $n_2 = \infty$.

### 2. Decisions and Regions for Range Tests.

A review of the various decisions and their corresponding regions of acceptance for the type tests under

consideration is given at this time. In the simplest
case, with any pair of means, $m_i$ and $m_j$, there are three
possible decisions. They are

    1. $m_i$ is significantly less than $m_j$,

    2. $m_i$ and $m_j$ are not significantly different, and

    3. $m_j$ is significantly less than $m_i$,

where $i, j = 1, 2, \ldots, n$, and $i \neq j$. Following Duncan's
notation, these decisions are denoted by $D(i,j)$, $D(\underline{i,j})$
and $D(j,i)$, respectively. The order of the numbers in
$D(i,j)$ and $D(j,i)$ indicates the ranking of the means, and
$D(\underline{i,j})$ indicates the means are not ranked. The decisions
$D(i,j)$, $D(\underline{i,j})$, and $D(j,i)$ may also be interpreted to
mean $\mu_i$ is less than $\mu_j$, $\mu_i$ is unranked relative to $\mu_j$, and
$\mu_j$ is less than $\mu_i$, respectively, and are denoted by
$\mu_i < \mu_j$, $\mu_i \overset{\cdot}{=} \mu_j$, and $\mu_j < \mu_i$. It should be noted that
"$\mu_i$ is unranked relative to $\mu_j$" means that $\mu_i$ may be
greater than, equal to, or less than $\mu_j$.

The ultimate objective in testing $n$ means is to rank
each of the $_nC_2$ pairs of true means. This is done by ex-
tending the ideas and notation for two means, as discussed
above, to three or more means. For three means, $m_1$, $m_2$,
$m_3$, we may make 19 consistent-pairwise (see III for dis-
cussion of term) decisions denoted as follows:

D(1,2,3)  D(1,<u>2,3</u>)  D(<u>1,2</u>,3)  D(<u>1,2,3</u>)  D(<u>1,2,3</u>)

D(1,3,2)  D(2,<u>1,</u>3)  D(<u>1,</u>3,2)  D(<u>1,3,2</u>)

D(2,1,3)  D(3,<u>1,</u>2)  D(<u>2,</u>3,1)  D(<u>2,1,</u>3)

D(2,3,1)                                 D(<u>2,3,</u>1)

D(3,1,2)                                 D(<u>3,1,</u>2)

D(3,2,1)                                 D(<u>3,2,</u>1)

Clearly, only five distinctly different patterns occur among these 19 different possible consistent-pairwise decisions on three means. It is sufficient to define the leading decision in each column. $D(1,2,3)$ may be read as $m_1$ is significantly less than $m_2$, $m_2$ is significantly less than $m_3$, and $m_1$ is significantly less than $m_3$. $D(1,\underline{2,3})$ may be read as $m_1$ is significantly less than $m_2$, $m_2$ and $m_3$ are not significantly different, and $m_1$ is significantly less than $m_3$. $D(\underline{1,2},3)$ may be read as $m_1$ and $m_2$ are not significantly different, $m_2$ is significantly less than $m_3$, and $m_1$ is significantly less than $m_3$. $D(\underline{1,2,3})$ may be read as $m_1$ and $m_2$ are not significantly different, $m_2$ and $m_3$ are not significantly different, and $m_1$ is significantly less than $m_3$. $D(\underline{1,2,3})$ may be read as $m_1$ and $m_2$, $m_2$ and $m_3$, and $m_1$ and $m_3$ are not significantly different. In general, for n means, $m_1$, $m_2$, ..., $m_n$, any decision is denoted in a similar manner. The means are ordered in some fashion with lines under some of them. Any pair of

means not grouped by a common line are significantly different, the first listed being the smaller. Any pair of means grouped by at least one line are not significantly different.

For further discussion on the total number of patterns for n means and the total number of decisions for 4, 5, and 6 means, see III.

Let D( ) denote any decision. That collection of all possible values of the set of sample differences between pairs of means for which we made this decision is called its decision region and is denoted by deleting D from the corresponding decision notation. Thus, (i,j) denotes the decision region for making decision D(i,j).

## 3. Geometry of Decision Regions.

In later discussions, it is convenient to think of decision regions geometrically. To illustrate how the regions look for the various multiple range tests, the simple but non-practical (for $n > 2$) multiple t test regions are used. The decision regions for other tests, which are discussed in more detail later, are easily defined assuming an understanding of the multiple t regions.

For the geometrical discussion more notation is introduced. Let

(1) $$d_{ij} = m_i - m_j,$$

where $i = 1, \ldots, n-1$, $j = 2, \ldots, n$, and $i < j$, denote the
ordered difference for sample means $m_i$ and $m_j$. Let

$$(2) \qquad x_i = (\sum_{k=1}^{i} m_k - i m_{i+1}) / \left( \frac{i(i+1)}{2} \right)^{\frac{1}{2}} \,,$$

where $i = 1, \ldots, n-1$, denote the axes in the space of the
differences $d_{ij}$ which is called the reduced-sample-space by
Duncan. Sometimes we wish to think of $x_i$ in terms of dif-
ferences. Thus we write, using (2),

$$x_i = \left( (m_1 - m_{i+1}) + \ldots + (m_i - m_{i+1}) \right) / \left( \frac{i(i+1)}{2} \right)^{\frac{1}{2}}$$

$$= (d_{1,i+1} + \ldots + d_{i,i+1}) / \left( \frac{i(i+1)}{2} \right)^{\frac{1}{2}}, \text{ or}$$

$$(3) \qquad x_i = \sum_{j=1}^{i} d_{j,i+1} / \left( \frac{i(i+1)}{2} \right)^{\frac{1}{2}} .$$

Now the geometric regions of the t test for two means
are defined. The best test for choosing from among three
possible decisions $D(1,2)$, $D(\underline{1,2})$, and $D(2,1)$ is given by
the following rule: make the decision $D(1,2)$ if
$m_1 - m_2 = d_{12} < -\sqrt{2} \, t_\alpha s_m$, the decision $D(\underline{1,2})$ if
$|m_1 - m_2| = |d_{12}| \leq \sqrt{2} \, t_\alpha s_m$, or the decision $D(2,1)$ if
$m_1 - m_2 > \sqrt{2} \, t_\alpha s_m$, where $t_\alpha$ is the two-tail $\alpha$-level signi-
ficant value of t based on $n_2$ degrees of freedom and $s_m$
is the standard error of the means. Under our restrictive
assumptions $\sqrt{2} \, t_\alpha s_m = \sqrt{2} \, (1.96) \cdot 1 = 2.77$ when $\alpha = 5\%$. Geo-
metrically, the sample-space is represented by a straight
line, $x_1 = d_{12} = m_1 - m_2$, as shown in Figure 1. A

particular value on this line, representing an observed difference $m_1 - m_2 = d_{12}$, is called a sample point. The test rule divides the line into three intervals, or regions, $d_{12} < -2.77$, $-2.77 \leq d_{12} \leq 2.77$, and $d_{12} > 2.77$ which are denoted by $(1,2)$, $(\underline{1},2)$, and $(2,1)$, respectively. Decision $D(1,2)$, $D(\underline{1},2)$, or $D(2,1)$ is made depending upon whether the sample point falls in decision region $(1,2)$, $(\underline{1},2)$, or $(2,1)$.
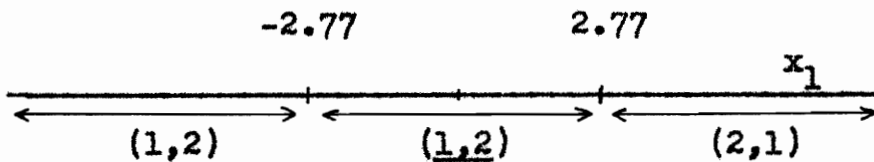


Figure 1

Regions of 5% Level Multiple Normal Deviate Test

$(n_2 = \infty,\ \sigma_m = 1,\ n = 2)$

To discuss the decision regions of the multiple t test involving three means, $m_1$, $m_2$, and $m_3$, a reduced two-dimensional sample-space is required. Let the horizontal axis be $x_1 = m_1 - m_2 = d_{12}$ and the vertical axis be the comparison $x_2 = (m_1 + m_2 - 2m_3) / \sqrt{3} = (d_{13} + d_{23}) / \sqrt{3}$, since $x_1$ and $x_2$ are clearly orthogonal. Because we are interested in how much each of the $d_{ij}$'s differ from zero, we wish to solve for the $d_{ij}$'s as a function of the $x_i$'s. This is always possible as is shown in IV. Using theorems 3 and 4 of IV gives

(4)     $d_{12} = x_1$, $d_{13} = (x_1 + \sqrt{3} \, x_2)/2$, $d_{23} = (-x_1 + \sqrt{3} \, x_2)/2$.

Consider what happens when $d_{12} = 0$, $d_{13} = 0$, $d_{23} = 0$. Geometrically, we have three lines through the origin, intersecting so that the angle cut out by any pair of lines is $60^o$. Remembering in the two mean case that the interval $-\sqrt{2} \, t_\alpha s_m \leq d_{12} \leq \sqrt{2} \, t_\alpha s_m$ about $d_{12} = 0$ was selected as the region for the decision $D(\underline{1,2})$, we take similar intervals about $d_{12} = 0$, $d_{13} = 0$, $d_{23} = 0$ to make the decisions $D(\underline{1,2})$, $D(\underline{1,3})$, $D(\underline{2,3})$, respectively. However, in two dimensions the intervals are plane bands, $-\sqrt{2} \, t_\alpha s_m \leq d_{ij} \leq \sqrt{2} \, t_\alpha s_m$, where $ij = 12, 13, 23$. These bands intersect in a symmetrical fashion about the origin cutting the plane into 19 parts as shown in Figure 2. These are the decision regions for the 19 decisions for three means given in Section 2. Letting $\sqrt{2} \, t_\alpha s_m = q_2$, we may also think of the 19 regions as being cut out by the symmetrical intersection about the origin of the following three pairs of parallel lines:

$$x_1 = \pm \, q_2,$$
(5)     $$x_1 + \sqrt{3} \, x_2 = \pm \, 2q_2, \text{ and}$$
$$-x_1 + \sqrt{3} \, x_2 = \pm \, 2q_2.$$

In the multiple t test involving four means, $m_1$, $m_2$, $m_3$, and $m_4$, a reduced three-dimensional sample-space is required. The orthogonal axes are, from (3),
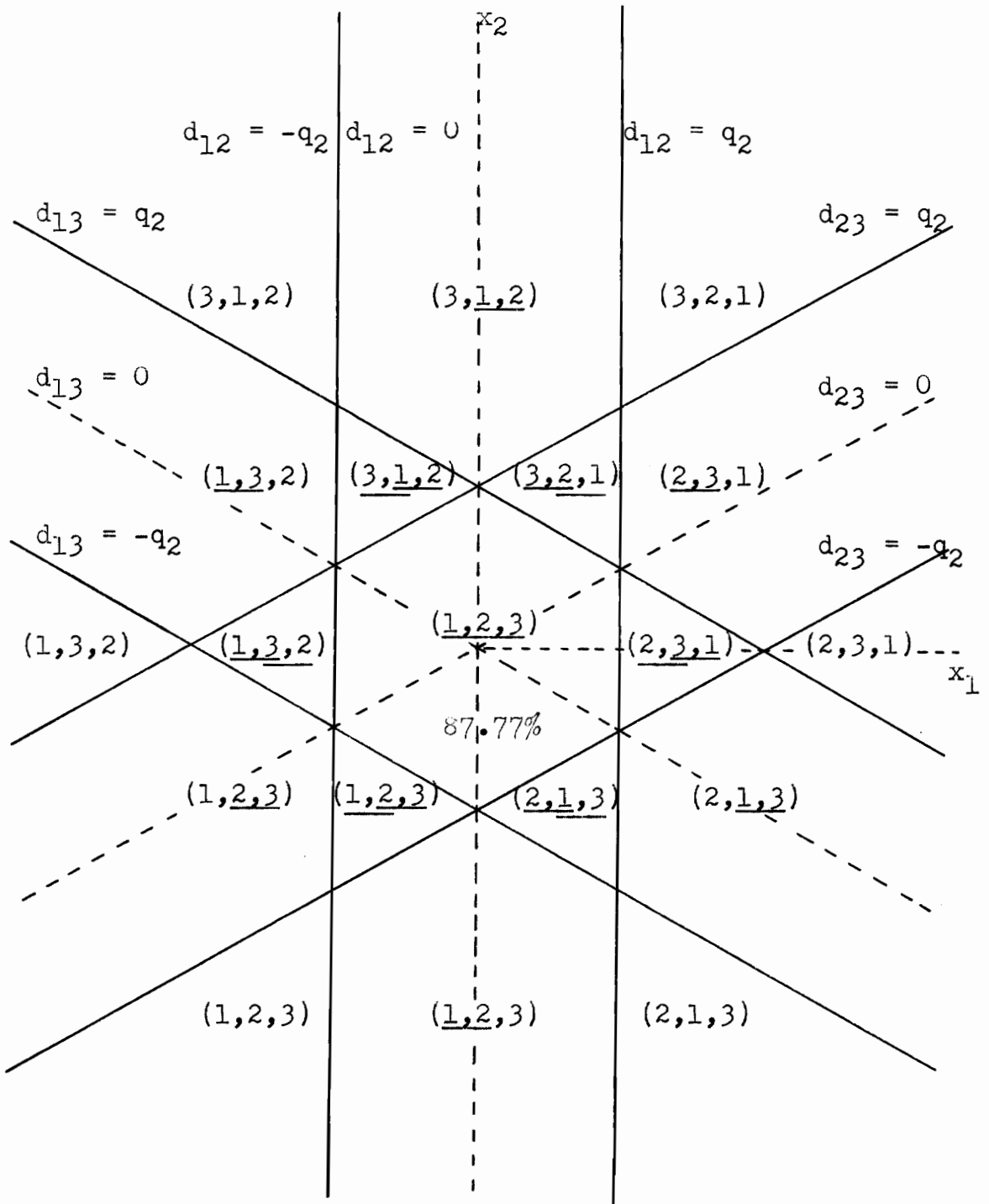
Figure 2

Regions of 5% Level Multiple Normal-Deviate Test.

$(n_2 = \infty, \; \sigma_m = 1, \; n = 3)$

$$x_1 = d_{12},$$

(6) $\qquad x_2 = (d_{13}+d_{23})/\sqrt{3},$ and

$$x_3 = (d_{14}+d_{24}+d_{34})/\sqrt{6}.$$

From IV, we find $d_{ij}$, $ij = 12, 13, 23, 14, 24, 34$, in terms of $x_i$, $i = 1, 2, 3$. Setting $d_{ij} = \pm q_2$ and simplifying, gives (5) plus the following equations:

$$3x_1 + \sqrt{3}\, x_2 + 2\sqrt{6}\, x_3 = \pm 6q_2$$

(7) $\qquad -3x_1 + \sqrt{3}\, x_2 + 2\sqrt{6}\, x_3 = \pm 6q_2,$ and

$$- 2\sqrt{3}\, x_2 + 2\sqrt{6}\, x_3 = \pm 6q_2.$$

The 6 pairs of parallel planes in (5) and (7) intersect symmetrically about the origin in a three dimensional space to form 183 decision regions (see III) for the four mean multiple t test. There are 183 consistent-pairwise decisions, each of which corresponds to one and only one of the 183 decision regions.

In the multiple t test involving n means, $m_1$, $m_2$, ..., $m_n$, a reduced (n-1)-dimensional sample-space is required. The orthogonal axes are

$$x_1 = d_{12},$$

$$x_2 = (d_{13}+d_{23})/\sqrt{3}$$

(8) $\qquad \ldots\ldots\ldots\ldots\ldots$

$$x_{n-1} = \left(\sum_{j=1}^{n-1} d_{j,n}\right)\Big/\sqrt{\frac{(n-1)n}{2}}\ .$$

From IV, we find $d_{ij}$, where $i=1, \ldots, n-1$, $j=2, \ldots, n$, and $i < j$, in terms of $x_i$, $i=1, 2, \ldots, n-1$. On setting

$d_{ij} = \pm q_2$ and simplifying, we get

$$x_1 = \pm q_2,$$
$$x_1 + \sqrt{3}\, x_2 = \pm 2q_2,$$
$$-x_1 + \sqrt{3}\, x_2 = \pm 2q_2,$$
$$- - - - - - -$$

(9)
$$d_{1,n} = \pm q_2,$$
$$d_{1,n} - x_1 = \pm q_2,$$
$$- - - - - - -$$
$$d_{1,n} + \cdots + d_{n-2,n} - \sqrt{\frac{(n-2)(n-1)}{2}}\, x_{n-2} = \pm (n-2)q_2,$$

where

$$d_{1,n} = \left[ (3 \cdot 4 \cdots (n-1))x_1 + (4 \cdot 5 \cdots (n-1))\sqrt{3}\, x_2 \right.$$
$$+ \cdots + (n-3)! \sqrt{\frac{(n-2)(n-1)}{2}}\, x_{n-2} + (n-2)!$$
$$\left. \sqrt{\frac{(n-1)n}{2}}\, x_{n-1} \right] \Big/ (n-1)!$$

and the other d's are given by (11) in IV. These $(n-1)n/2$
pairs of parallel hyperplanes intersect symmetrically about
the origin in n-1 space. The decision regions for an n
mean test are determined by these intersections. There is
a one-to-one correspondence between these regions and the
consistent-pairwise decisions on n means.

## 4. Comparison of Regions for Different Tests.

The three mean multiple t test regions may be used to
define corresponding regions for other symmetric multiple
range tests. The multiple t region (1,2,3) is symmetric

about the origin (center at the origin) and the two decision
region $(\underline{1},2) = (\underline{1},2,3) + (\underline{1},2,3) + (3,\underline{1},2) + (\underline{1},2,3) + (3,\underline{1},2)$
$+ (2,\underline{1},3) + (3,\underline{2},1)$ is symmetric about $x_1 = d_{12} = 0$ and the origin.
The region $(\underline{1},2,3)$ for any symmetric multiple range test may
be obtained from the multiple t test region $(\underline{1},2,3)$ by uni-
formly moving all its boundaries to a new position, keeping
them parallel to their original position.  The boundaries of
$(\underline{1},2)$, with the exception of those common to $(\underline{1},2,3)$, are
termed the two decision boundaries of $(\underline{1},2)$.  The region
$(\underline{1},2)$ for any symmetric multiple range test is obtained,
after the multiple range region $(\underline{1},2,3)$ is defined, by uni-
formly moving the two decision boundaries of the multiple
t test region $(\underline{1},2)$ to a new position, keeping them parallel
to their original position.  The multiple range test regions,
$(\underline{1},3)$ and $(\underline{2},3)$ in a symmetric test, are fixed once $(\underline{1},2)$
is defined.  Thus, the 19 decision regions for any symmetric
multiple range test are defined when the boundaries of $(\underline{1},2,3)$
and $(\underline{1},2)$ for that test are fixed.  In a similar manner, four
decision, three decision, and two decision boundaries for a
four mean multiple t test may be defined.  The decision re-
gions for a multiple range test of four means may then be
determined by moving the four, three, and two decision bound-
aries of the multiple t test to a new position, keeping them
parallel to their original position.

The symmetric central region in all the tests discussed
in this paper is expanded as well as most of the two decision,

three decision, ..., n-1 decision regions. These regions are never expanded enough to change the number of n mean decision regions in an n mean test. (Of course, the size of any region may change.)

Tukey's test based on allowances is the same as the multiple t test, except that the parallel hyperplanes (lines) are moved out until the central region is much larger. The Newman-Keuls' test and Duncan's new multiple range test are obtained by moving the three decision, ..., n decision boundaries. Tukey's 1953 multiple range test is a compromise between his test based on allowances and the Newman-Keuls' test. (There are many possible multiple range tests. One difficulty in selecting the best one for a given experiment comes in choosing a system of significance levels.)

The multiple F tests may be described using the corresponding range tests as guides. In these, the symmetric center is replaced by a sphere, the three decision, ..., n-1 decision region boundaries are replaced by cylinders, and the two decision region boundaries remain the same (planes). This introduces more regions for testing whether a given contrast is significantly different from zero.

## III. THE ENUMERATION OF DECISION PATTERNS

### 1. Introduction.

This chapter contains a mathematical treatment of the enumeration of consistent decision patterns obtained in the pairwise comparison of n means. In the comparison of n means, there are all together $_nC_2$ pairwise comparisons, and each individual comparison between two means, say $m_1$ and $m_2$, must be a preference for $m_1$, a preference for $m_2$ or an indifference between the two. Symbolically, these three alternatives are written as $m_1 < m_2$, $m_2 < m_1$, and $m_1 \doteq m_2$, respectively.

There are, thus, all together $3^{nC_2}$ possible decision sets in the comparison of n means, a decision set consisting of the $_nC_2$ pairwise comparisons. There are many inconsistent or circular decision sets. For example, for n = 3, the decision set $m_1 < m_2$, $m_2 < m_3$, $m_3 < m_1$ is circular and does not permit a linear ordering of the preference expressed for these means.

If a decision set is non-circular, it can be represented symbolically with the use of the following scheme:

If $m_1 < m_2$, the letter $m_1$ is written to the left of $m_2$; if $m_1 \doteq m_2$, the letters $m_1$ and $m_2$ are underlined with what we shall call an indifference line (if $m_1 \doteq m_2$, it does not matter whether we write $\underline{m_1m_2}$ or $\underline{m_2m_1}$). We shall also write $\underline{m_1m_2m_3}$ to express the fact that $m_1 \doteq m_2$,

$m_2 \doteq m_3$, and $m_1 \doteq m_3$.  In general, the fact that $m_1 \doteq m_2$ will be expressed by an indifference line common to $m_1$ and $m_2$ as well as all other means in between.

The following are two simple examples illustrating this representation of decision sets:

(i)  The decision set $m_1 < m_2$, $m_1 < m_3$, $m_1 < m_4$, $m_2 \doteq m_3$, $m_2 < m_4$, and $m_3 \doteq m_4$ is written as

$$m_1 \quad \underline{m_2 \quad m_3} \quad m_4,$$

and

(ii)  The decision set $m_1 \doteq m_2$, $m_1 \doteq m_3$, $m_1 < m_4$, $m_2 \doteq m_3$, $m_2 \doteq m_4$, and $m_3 \doteq m_4$ is written as

$$\underline{m_1 \quad m_2 \quad m_3} \quad m_4 \quad .$$

If the only difference between the schematic presentation of two decision sets is a permutation of the means $m_1$, $m_2$, $m_3$, ..., $m_n$, they are said to belong to the same decision pattern.  A decision pattern is, therefore, characterized by the number of means and the number and the arrangement of the indifference lines.  The decision pattern corresponding to a given decision set will be indicated by replacing the means with dots.  The decision pattern corresponding to the first example above is

$$\bullet \quad \underline{\bullet \quad \bullet} \quad \bullet \quad ,$$

and that of the second example is

$$\underline{\bullet \quad \bullet \quad \bullet} \quad \bullet \quad \bullet$$

Two important points which must be observed in the construction of decision patterns are

(i)   No indifference line is completely covered by another indifference line.  For example, it is not permissible to write . $\underline{\ldots\ldots}$ . . The shorter line is evidently redundant.

(ii)  Two indifference lines extending in the same direction can not terminate at the same dot. For example, it is permissible to have $\underline{\ldots}$ . , but it is not permissible to have . $\underline{\ldots}$ . This is an immediate consequence of (i).

Having defined decision patterns and decision sets, one may now ask

(i)   What is the total number of distinct and consistent, that is, non-circular, decision sets in the pairwise comparison of n means?

(ii)  What is the total number of distinct and consistent, that is, non-circular, decision patterns in the pairwise comparison of n means?

In this discussion, an answer will be provided to question (ii), and formulas will be developed for the enumeration of the total number of decision patterns.  Although question (i) can, of course, be answered by direct enumeration for small values of n, the general treatment of this problem is as yet unsolved.

## 2. Basic Relations.

In order to derive formulas giving the total number of decision patterns, consider the last k dots on the right in a pattern which has n dots, $n \geq k$. Beneath each pair of dots, $a_i$ and $a_{i+1}$, where $i = 1, 2, \ldots, k-1$, j line segments, $j = 0, 1, 2, \ldots$, may be drawn, each line being part of or a whole indifference line. The step from $a_i$ to $a_{i+1}$, called the _ith step_, may be made in many ways as indicated by the number of line segments underlining the pair of dots. Let $s_j$ denote a step with j line segments. It should be noted that each line segment under dots $a_i$ and $a_{i+1}$ may or may not be part of an indifference line including several other dots. A dot $a_i$ is called a _right terminal dot_ (_left terminal dot_) of an indifference line whenever the indifference line does not extend to $a_{i+1}$ $(a_{i-1})$.

Let $f_j(x)$, $j = 0, 1, 2, \ldots$, denote the total number of decision patterns possible when the first step of k dots is $s_j$.

It can be seen easily that

(1) $$f_0(k) = f_0(k-1) + f_1(k-1), \ k \geq 3,$$

since the number of decision patterns for k dots with first step $s_0$ is the same as the sum of the number of decision patterns for k-1 dots with the first steps $s_0$ and $s_1$ (k cannot be $\leq 2$, since $f_1(k-1)$ would be undefined).

If the first step is $s_1$, then the right terminal dot of $s_1$ is either $a_2$ or some dot to its right. If it is $a_2$, the second step must be either $s_0$ or $s_1$. If it were $s_j$, with $j \geq 2$, there would have to be more than one indifference line with $a_2$ as a left terminal dot. As was pointed out before, this is not permissible. For similar reasons, the second step must be either $s_1$ or $s_2$. The recursion formula can therefore be written as

$$f_1(k) = \left[ f_0(k-1) + f_1(k-1) \right] + \left[ f_1(k-1) + f_2(k-1) \right] ,$$

or

(2) $\quad f_1(k) = f_0(k-1) + 2f_1(k-1) + f_2(k-1), \ k \geq 3$ .

Next, suppose the first step is $s_2$, in which case it is necessary that $n \geq k+1$ so that two indifference lines will not have a common left terminal dot at $a_1$. The right terminal dots may be (i) one at $a_2$ and the other to the right of $a_2$, or (ii) both to the right of $a_2$. For (i), the second step can be $s_1$ or $s_2$, and for (ii), the second step can be $s_2$ or $s_3$. Thus the total number of decision patterns with k dots where the first step is $s_2$ is given by the recursion form

$$f_2(k) = \left[ f_1(k-1) + f_2(k-1) \right] + \left[ f_2(k-1) + f_3(k-1) \right] ,$$

or

(3) $\quad f_2(k) = f_1(k-1) + 2f_2(k-1) + f_3(k-1), \ k \geq 3.$

Similarly,

(4) $\quad f_3(k) = f_2(k-1) + 2f_3(k-1) + f_4(k-1), \ k \geq 3.$

The general recursion formula for $e = 1, 2, 3, \ldots$
may be written as

(5) $\quad f_e(k) = f_{e-1}(k-1) + 2f_e(k-1) + f_{e+1}(k-1), \ k \geq 3$ .

To prove (5), assume that $s_e$ is the first step, in which
case it is necessary that $n \geq k+e-1$. It must be large
enough so that no two of the $e$ indifference lines of the
first step have $a_1$ or any dot to the left of $a_1$ as a com-
mon left terminal dot. At least $e-1$ of the indifference
lines in step one must be continued beyond $a_2$ since two
indifference lines can not have a common right terminal
dot at $a_2$. The second step, thus, has at least $e-1$ in-
difference lines. On the other hand, not more than $e+1$
indifference lines are possible in the second step since
two indifference lines would otherwise have $a_2$ as a common
left terminal dot. Thus, if the first step is $s_e$ and only
one of its indifference lines terminates at $a_2$, the second
step is $s_{e-1}$ or $s_e$; if the first step is such that no
indifference lines terminate at $a_2$, the second step is $s_e$
or $s_{e+1}$.

For certain values of $k$, $s_e$ is an impossible first
step and $f_e(k)$ is set equal to zero. (It will be assumed
here that $n$ is sufficiently large so that not more than
one indifference line has a left terminal dot at $a_1$.) The
problem is to see how large $k$ must be so that there is at
most one indifference line with a right terminal dot at $a_2$.

For k=2, $s_1$ is a possible first step, but $s_j$, where j = 2, 3,....., is not. Thus, $f_e(2) = 0$, when e = 2, 3, ... .

For k = 3, $s_2$ is a possible first step since one indifference line may terminate at $a_2$ and the other at $a_3$. If the first step were $s_3$, either $a_2$ or $a_3$ would have to be a common right terminal dot for two indifference lines. Thus, $f_e(3) = 0$, when e = 3, 4, ... .

If k is an arbitrary positive integer, say r, then $s_{r-1}$ is a possible first step since each $a_i$, where i = 2, 3, ..., r, may be a right terminal dot for exactly one indifference line. If the first step were $s_r$, then some point $a_i$ would have to be a common right terminal dot for two indifference lines. Thus $f_e(r) = 0$, when e = r, r+1, ..., and, in general,

(6)                         $f_e(k) = 0$,

where e = k, k+1, ..., and k = 2, 3, ... .

Let f(n) denote the total number of decision patterns for n dots. Clearly,

(7)                     $f(n) = f_0(n) + f_1(n)$,

since $s_0$ and $s_1$ are the only possible first steps.

Since f(k) depends only on $f_0(k)$ and $f_1(k)$, equations (1) and (5) are sufficient to determine f(k) for any k $\geq$ 3. Evidently for k = 2, $f_0(2) = 1$ and $f_1(2) = 1$. Combining this result with (1) and (5) it follows that

$f_0(3) = f_0(2) + f_1(2) = 1 + 1 = 2,$

$f_1(3) = f_0(2) + 2f_1(2) + f_2(2) = 1 + 2 \cdot 1 + 0 = 3,$

$f_2(3) = f_1(2) + 2f_2(2) + f_3(2) = 1 + 2 \cdot 0 + 0 = 1,$

$f_e(3) = 0, \ e \geq 3,$

and, therefore, $f(3) = 2 + 3 = 5.$ Also,

$f_0(4) = f_0(3) + f_1(3) = 2 + 3 = 5,$

$f_1(4) = f_0(3) + 2f_1(3) + f_2(3) = 2 + 2 \cdot 3 + 1 = 9,$

$f_2(4) = f_1(3) + 2f_2(3) + f_3(3) = 2 + 2 \cdot 1 + 0 = 5,$

and so on. This procedure becomes rather laborious, since for a given value of k the quantity $f_1(2k-1)$ cannot be found without first calculating all of the $f_e(k)$.

## 3. Derivation of Formulas and Results.

It would be desirable to express $f_0(k)$ as well as $f_1(k)$ in terms of the functions $f_0(i)$ and $f_1(i)$, where $i < k$, since $f(k)$ depends only on $f_0(k)$ and $f_1(k)$. In the case of $f_0(k)$, it follows immediately that $f_0(k) = f(k-1)$. The recursion formula for $f_1(k)$ is derived in two steps.

First $f_{e+1}(k)$ is expressed as a function of $f_0(k)$, $f_1(k)$, $f_1(k+1)$, ..., $f_1(k+e)$. This is straightforward. From (1) and (5), it follows that

(8)             $f_1(k) = f_0(k+1) - f_0(k),$ and

(9)             $f_{e+1}(k) = f_e(k+1) - 2f_e(k) - f_{e-1}(k).$

The repeated application of these formulas gives the following formulas:

(10)   $f_2(k) = f_1(k+1) - 2f_1(k) - f_0(k) ,$

(11) $\quad f_3(k) = f_1(k+2) - 4f_1(k+1) + 2f_1(k) + f_0(k),$

(12) $\quad f_4(k) = f_1(k+3) - 6f_1(k+2) + 9f_1(k+1) - f_1(k),$

(13) $\quad f_5(k) = f_1(k+4) - 8f_1(k+3) + 20f_1(k+2) - 15f_1(k+1)$
$$- f_0(k),$$

(14) $\quad f_6(k) = f_1(k+5) - 10f_1(k+4) + 35f_1(k+3) - 49f_1(k+2)$
$$+ 21f_1(k+1) + f_0(k),$$

(15) $\quad f_7(k) = f_1(k+6) - 12f_1(k+5) + 54f_1(k+4) - 111f_1(k+3)$
$$+ 99f_1(k+2) - 27f_1(k+1) + f_1(k),$$

(16) $\quad f_8(k) = f_1(k+7) - 14f_1(k+6) + 77f_1(k+5) - 209f_1(k+4)$
$$+ 286f_1(k+3) - 176f_1(k+2) + 34f_1(k+1)$$
$$- 2f_1(k) - f_0(k),$$

and so on.

To obtain a general rule for writing $f_{e+1}(k)$ in terms of $f_1(k+e)$, ..., $f_1(k)$ and $f_0(k)$, let it be assumed that

(17) $\quad f_e(k) = f_1(k+e-1) + a_{1,e}f_1(k+e-2) + a_{2,e}f_1(k+e-3)$
$$+ a_{3,e}f_1(k+e-4) + \ldots + a_{e-2,e}f_1(k+1)$$
$$+ a_{e-1,e}f_1(k) + a_{0,e}f_0(k),$$

where $a_{j,e}$, $j = 0, 1, \ldots, e-1$, are integers. Using (9) and (17), it follows that

$$f_{e+1}(k) = f_1(k+e) + a_{1,e}f_1(k+e-1) + a_{2,e}f_1(k+e-2) + \ldots$$
$$+ a_{e-1,e}f_1(k+1) + a_{0,e}f_0(k+1) - 2f_1(k+e-1)$$
$$- 2a_{1,e}f_1(k+e-2) - \ldots - 2a_{e-2,e}f_1(k+1)$$
$$- 2a_{e-1,e}f_1(k) - 2a_{0,e}f_0(k) - f_1(k+e-2) - \ldots$$
$$- a_{e-3,e-1}f_1(k+1) - a_{e-2,e-1}f_1(k) - a_{0,e-1}f_0(k),$$

or

$$(18) \quad f_{e+1}(k) = f_1(k+e) + (a_{1,e} - 2) f_1(k+e-1)$$
$$+ (a_{2,e} - 2a_{1,e} - 1) f_1(k+e-2)$$
$$+ (a_{3,e} - 2a_{2,e} - a_{1,e-1}) f_1(k+e-3)$$
$$+ \ldots + (a_{e-1,e} - 2a_{e-2,e} - a_{e-3,e-1})$$
$$f_1(k+1) + (a_{0,e} - 2a_{e-1,e} - a_{e-2,e-1})$$
$$f_1(k) + (-a_{0,e} - a_{0,e-1}) f_0(k),$$

since

$$(-2a_{e-1,e} - a_{e-2,e-1})f_1(k) + a_{0,e}f_0(k+1) + (-2a_{0,e} - a_{0,e-1}) \cdot$$
$$f_0(k) = (-2a_{e-1,e} - a_{e-2,e-1}) f_1(k) + a_{0,e}f_1(k)$$
$$+ (-a_{0,e} - a_{0,e-1}) f_0(k)$$
$$= (a_{0,e} - 2a_{e-1,e} - a_{e-2,e-1})f_1(k) + (-a_{0,e} - a_{0,e-1})$$
$$f_0(k).$$

The second step is that of using formulas (10) to (16) to express $f_1(k)$ in terms of $f_0(k-e-1)$, $f_1(k-1)$, ..., $f_1(k-e-1)$, $e = 0, 1, 2, \ldots$ . Solving (10) to (16) for $f_1(k+e-1)$, replacing $k+e-1$ by $k$, we get

$$(19) \quad f_1(k) = 2f_1(k-1) + f_0(k-1) + f_2(k-1), \quad k \geq 3,$$

$$(20) \quad f_1(k) = 4f_1(k-1) - 2f_1(k-2) - f_0(k-2) + f_3(k-2), \quad k \geq 4,$$

$$(21) \quad f_1(k) = 6f_1(k-1) - 9f_1(k-2) + f_1(k-3) + f_4(k-3), \quad k \geq 5,$$

$$(22) \quad f_1(k) = 8f_1(k-1) - 20f_1(k-2) + 15f_1(k-3) + f_0(k-4)$$
$$+ f_5(k-4), \quad k \geq 6,$$

$$(23) \quad f_1(k) = 10f_1(k-1) - 35f_1(k-2) + 49f_1(k-3) - 21f_1(k-4)$$
$$- f_0(k-5) + f_6(k-5), \quad k \geq 7,$$

(24) $f_1(k) = 12f_1(k-1) - 54f_1(k-2) + 111f_1(k-3) - 99f_1(k-4)$
$\qquad + 27f_1(k-5) - f_1(k-6) + f_7(k-6),\ k \geq 8,$ and

(25) $f_1(k) = 14f_1(k-1) - 77f_1(k-2) + 209f_1(k-3) - 286f_1(k-4)$
$\qquad + 176f_1(k-5) - 34f_1(k-6) + 2f_1(k-7) + f_o(k-7)$
$\qquad + f_8(k-7),\ k \geq 9.$

Since $f_e(k) = 0$ when $k \leq e$, we write the following formulas which are used in the actual calculations of $f_1(k)$:

(26) $f_1(k) = 2f_1(k-1) + f_o(k-1),\ k=3,$

(27) $f_1(k) = 4f_1(k-1) - 2f_1(k-2) - f_o(k-2),\ k=4,5,$

(28) $f_1(k) = 6f_1(k-1) - 9f_1(k-2) + f_1(k-3),\ k=5,6,7,$

(29) $f_1(k) = 8f_1(k-1) - 20f_1(k-2) + 15f_1(k-3) + f_o(k-4),$
$\qquad k = 6,7,8,9,$

(30) $f_1(k) = 10f_1(k-1) - 35f_1(k-2) + 49f_1(k-3) - 21f_1(k-4)$
$\qquad - f_o(k-5),\ k = 7$ to $11,$

(31) $f_1(k) = 12f_1(k-1) - 54f_1(k-2) + 111f_1(k-3) - 99f_1(k-4)$
$\qquad + 27f_1(k-5) - f_1(k-6),\ k=8$ to $13,$ and

(32) $f_1(k) = 14f_1(k-1) - 77f_1(k-2) + 209f_1(k-3) - 286f_1(k-4)$
$\qquad + 176f_1(k-5) - 34f_1(k-6) + 2f_1(k-7) + f_o(k-7),$
$\qquad k=9$ to $15.$

Similarly,

(33) $f_1(k) = 16f_1(k-1) - 104f_1(k-2) + 351f_1(k-3) - 650f_1(k-4)$
$\qquad + 637f_1(k-5) - 287f_1(k-6) + 43f_1(k-7) - 2f_1(k-8)$
$\qquad - f_o(k-8),\ k=10$ to $17,$ and

$$(34) \quad f_1(k) = 18f_1(k-1) - 135f_1(k-2) + 545f_1(k-3)$$
$$- 1275f_1(k-4) + 1728f_1(k-5) - 1275f_1(k-6)$$
$$+ 441f_1(k-7) - 54f_1(k-8) + f_1(k-9), \quad k=11 \text{ to } 19.$$

With the aid of formulas (26) to (34) the total number of decision patterns is easily tabulated. Using $f_0(2)=1$, $f_1(2) = 1$, and $f_0(k) = f(k-1)$ together with (7), (26), (27), (28), (30), and (34), table 1 is computed.

### Table 1

#### Number of Consistent Decision Patterns

| k | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|
| $f_0(k)$ | 1 | 2 | 5 | 14 | 42 | 132 | 429 | 1,430 |
| $f_1(k)$ | 1 | 3 | 9 | 28 | 90 | 297 | 1,001 | 3,432 |
| $f(k)$ | 2 | 5 | 14 | 42 | 132 | 429 | 1,430 | 4,862 |

| k | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|
| $f_0(k)$ | 4,862 | 16,796 | 58,786 | 208,012 | 742,900 |
| $f_1(k)$ | 11,934 | 41,990 | 149,226 | 534,888 | 1,931,540 |
| $f(k)$ | 16,796 | 58,786 | 208,012 | 742,900 | 2,674,440 |

| k | 15 | 16 | 17 | 18 |
|---|---|---|---|---|
| $f_0(k)$ | 2,674,440 | 9,694,845 | 35,357,670 | 129,644,790 |
| $f_1(k)$ | 7,020,405 | 25,662,825 | 94,287,120 | 347,993,910 |
| $f(k)$ | 9,694,845 | 35,357,670 | 129,644,790 | 477,638,700 |

It will be noticed that some of the formulas (26) to (34) do not depend on $f_0$. Actually, every third formula, starting with (28), does not have a term in $f_0$. As k becomes larger, it is more desirable to use only those formulas which have $f_1(k)$ expressed as a function of $f_1(k-1)$, $f_1(k-2)$, ..., $f_1(k-e)$. It is suggested that (26), (27), (28), (31), and (34) form the most suitable combination of formulas for the computation of the number of decision patterns for $k = 3, \ldots, 19$.

To show that in every third formula there is no function $f_0$, we observe from (18) that the coefficient of $f_0(k)$ is

(35)
$$a_{o,e+1} = - a_{o,e} - a_{o,e-1} \cdot$$

Setting e equal to t and t+1, (35) is written as

$$a_{o,t+1} = - a_{o,t} - a_{o,t-1} \text{ and}$$

$$a_{o,t+2} = - a_{o,t+1} - a_{o,t}, \text{ respectively.}$$

Solving these equations for $a_{o,t+2}$ gives

(36)
$$a_{o,t+2} = a_{o,t-1} \cdot$$

It follows that for any two values of e which are equal (mod 3) the values of $a_{o,e}$ must be the same. Hence, the $a_{o,e}$ assume only 3 distinct values depending on whether $e \equiv 0, 1,$ or 2 (mod 3). Since $a_{o,2} = -1$, $a_{o,3} = 1$, and $a_{o,4} = 0$ in (10), (11), and (12), respectively, it follows that the $a_{o,e}$ assume no values other than 0, -1, and 1.

4. <u>Counting Decision Patterns and Sets.</u>

The formulas which have been derived above give the total number of decision patterns for n means. Now we shall enumerate the total number of decision sets for the cases when n = 2, 3, 4, 5 in a manner which will also give a check on the total number of decision patterns determined by formulas. The method is straightforward, so little explanation is necessary.

For the sake of convenience, suppose the letters a, b, d, ..., which represent means, are placed in alphabetical order in each decision pattern. Thus, to enumerate the total number of decisions, all we do is list the decision patterns, find the number of permutations possible in each pattern, and total the results. In order to save space, a whole family of decision patterns is listed together. For example, (i) (<u>ab</u>cd) represents the patterns (<u>ab</u>cd), (ab<u>cd</u>), and (ab<u>cd</u>) and (ii) (<u>ab</u>cde) represents the patterns (<u>ab</u>cde), (ab<u>cd</u>e), (abc<u>de</u>), and (ab<u>cd</u>e). The total number of patterns is indicated by summing the first factor in the expression for the number of decisions, or decision sets. The enumerations are as follows:

| Patterns for two means | Number of decisions | | |
|---|---|---|---|
| (ab) | $1 \cdot 2!$ | = | 2 |
| (a̲b̲) | $1 \cdot 1$ | = | 1 |
| Total number of patterns | 2 | | |
| Total number of decisions | | | 3 |

| Patterns for three means | Number of decisions | | |
|---|---|---|---|
| (abc) | $1 \cdot 3!$ | = | 6 |
| (ab̲c̲) | $2(3!/2!)$ | = | 6 |
| (a̲b̲c̲) | $1 \cdot 3!$ | = | 6 |
| (a̲b̲c̲) | $1 \cdot 1$ | = | 1 |
| Total number of patterns | 5 | | |
| Total number of decisions | | | 19 |

| Patterns for four means | Number of decisions | | |
|---|---|---|---|
| (abcd) | $1 \cdot 4!$ | = | 24 |
| (a̲b̲cd) | $3(4!/2!)$ | = | 36 |
| (a̲b̲c̲d) | $2(4!/3!)$ | = | 8 |
| (a̲b̲cd) | $2 \cdot 4!$ | = | 48 |
| (a̲b̲c̲d̲) | $1(4!/(2!2!))$ | = | 6 |
| (a̲b̲c̲d) | $2(4!/2!)$ | = | 24 |
| (a̲b̲c̲d̲) | $1(4!/2!)$ | = | 12 |
| (a̲b̲c̲d) | $1 \cdot 4!$ | = | 24 |
| (a̲b̲c̲d̲) | $1 \cdot 1$ | = | 1 |
| Total number of patterns | 14 | | |
| Total number of decisions | | | 183 |

| Patterns for five means | Number of decisions | | |
|---|---|---|---|
| (abcde) | 1·5! | = | 120 |
| (abcde) | 4(5!/2!) | = | 240 |
| (abcde) | 3(5!/3!) | = | 60 |
| (abcde) | 2(5!/4!) | = | 10 |
| (abcde) | 1·1 | = | 1 |
| (abcde) | 3·5! | = | 360 |
| (abcde) | 3(5!/(2!2!) ) | = | 90 |
| (abcde) | 4(5!/2!) | = | 240 |
| (abcde) | 2(5!/(2!3!) ) | = | 20 |
| (abcde) | 2(5!/2!) | = | 120 |
| (abcde) | 1(5!/(2!2!) ) | = | 30 |
| (abcde) | 2(5!/3!) | = | 40 |
| (abcde) | 2(5!/(2!2!) ) | = | 60 |
| (abcde) | 1(5!/3!) | = | 20 |
| (abcde) | 2·5! | = | 240 |
| (abcde) | 2(5!/2!) | = | 120 |
| (abcde) | 2(5!/2!) | = | 120 |
| (abcde) | 1·5! | = | 120 |
| (abcde) | 2(5!/2!) | = | 120 |
| (abcde) | 1·5! | = | 120 |
| (abcde) | 1·5! | = | 120 |

Total number of patterns    42

Total number of decisions       2371

These results and the totals in the case for six means are tabulated in Table 2.

Table 2

Number of Decisions

| Number of Means | Number of Indifference Lines | | | | | | Totals |
|---|---|---|---|---|---|---|---|
| | 0 | 1 | 2 | 3 | 4 | 5 | |
| 2 | 2 | 1 | | | | | 3 |
| 3 | 6 | 7 | 6 | | | | 19 |
| 4 | 24 | 45 | 90 | 24 | | | 183 |
| 5 | 120 | 311 | 980 | 840 | 120 | | 2,371 |
| 6 | 720 | 2,383 | 9,950 | 16,650 | 8,280 | 720 | 38,703 |

Theorem. The product of the number of decision patterns and $n!$ is an upper bound for the number of decisions involving n means.

Proof. Clearly, the second factor for the number of decisions for each pattern cannot exceed $n!$.

Table 3 shows how the number of decisions compares with the upper bound given by the theorem.

Table 3

Upper Bound for the Number of Decisions

| Number of Means | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| Number of Decisions | 3 | 19 | 183 | 2,371 | 38,703 |
| Bound Given by Theorem | 4 | 30 | 336 | 5,040 | 97,200 |

## IV. FORMULAS RELATING $x_i$ AND $d_{ij}$

Let $d_{ij}$ and $x_i$ be defined by equations (1) and (3) of II, respectively. Clearly,

$$(1) \qquad\qquad d_{ii} = 0,$$

and

$$(2) \qquad\qquad d_{ij} = -d_{ji}.$$

Using (1) of II and (2), we write

$$(3) \qquad d_{ij} = m_i - m_j = (m_k - m_j) - (m_k - m_i) = d_{kj} - d_{ki} \text{ and}$$

$$(4) \qquad d_{ij} = d_{ik} + d_{kj},$$

where $k = 1, 2, \ldots, n$.

From 3 of II, we see that $x_i$ is expressed in terms of $d_{ij}$. In some of our work, it is necessary that each $d_{ij}$ be expressed in terms of $x_i$. For this purpose we need the theorems below which follow from (1) and (3) of II together with (1), (2), (3), and (4).

Theorem 1.

$$(5) \quad \sum_{j=1}^{i} d_{j,i+1} = i d_{\alpha,i+1} - \sum_{j=1}^{i} d_{\alpha j},$$

where $\alpha = 1, 2, \ldots, i$, and $i = 1, 2, \ldots, n-1$.

Proof. By (3),

$$\sum_{j=1}^{i} d_{j,i+1} = \sum_{j=1}^{i} (d_{\alpha,i+1} - d_{\alpha j}) = i d_{\alpha,i+1} - \sum_{j=1}^{i} d_{\alpha j}.$$

From (5), we get

$$(6) \quad d_{\alpha,i+1} = \frac{1}{i}\left[ \sum_{j=1}^{i} d_{\alpha j} + \sum_{j=1}^{i} d_{j,i+1} \right],$$

where $\alpha = 1, 2, \ldots, i$ and $i = 1, 2, \ldots, n-1$.

Theorem 2.

$$(7) \quad d_{i,i+1} = \frac{1}{i}\left[ \sqrt{\frac{i(i+1)}{2}}\, x_i - \sqrt{\frac{i(i-1)}{2}}\, x_{i-1} \right],$$

where $i = 1, 2, \ldots, n-1$.

Proof. From (3) of II together with (3) above,

$$\sqrt{\frac{i(i+1)}{2}}\, x_i - \sqrt{\frac{(i-1)i}{2}}\, x_{i-1} = \sum_{j=1}^{i} d_{j,i+1} - \sum_{j=1}^{i-1} d_{j,i}$$

$$= \sum_{j=1}^{i-1}(d_{j,i+1}-d_{j,i}) + d_{i,i+1}$$

$$= \sum_{j=1}^{i-1} d_{i,i+1} + d_{i,i+1}$$

$$= i\, d_{i,i+1} .$$

Theorem 3.

$$(8) \quad d_{1,i+1} = \sum_{j=1}^{i-1} \frac{(j-1)!}{(j+1)!} \sqrt{\frac{j(j+1)}{2}}\, x_j + \frac{1}{i}\sqrt{\frac{i(i+1)}{2}}\, x_i .$$

Proof. Prove (8) by induction. When $i=1$ in (8), $d_{12} = x_1$, which checks with (3) of II when $i=1$. When $i=2$ in (8), $d_{13} = x_1/2! + (\sqrt{3}x_2)/2 = [x_1 + \sqrt{3}x_2]/2$. But from (3) of II and (6), $d_{13} = \frac{1}{2}[d_{12} + \sqrt{3}x_2] = \frac{1}{2}[x_1 + \sqrt{3}x_2]$. Thus (8) is true for $i=1$ and 2. Assume (8) true for $i=k$, and show that (8) holds when $i=k+1$. Thus we assume

$$(9) \quad d_{1,k+1} = \left[ \frac{x_1}{2!} + \frac{\sqrt{3}x_2}{3!} + \ldots + \frac{(k-2)!}{k!}\sqrt{\frac{(k-1)k}{2}}\, x_{k-1} \right]$$

$$+ \frac{1}{k}\sqrt{\frac{k(k+1)}{2}}\, x_k .$$

Add

$$\frac{(k-1)!}{(k+1)!} \sqrt{\frac{k(k+1)}{2}}\, x_k + \frac{1}{k+1} \sqrt{\frac{(k+1)(k+2)}{2}}\, x_{k+1} - \frac{1}{k} \sqrt{\frac{k(k+1)}{2}}\, x_k$$

$$= \frac{1-(k+1)}{k(k+1)} \sqrt{\frac{k(k+1)}{2}}\, x_k + \frac{1}{k+1} \sqrt{\frac{(k+1)(k+2)}{2}}\, x_{k+1}$$

$$= d_{k+1,k+2}$$

to both sides of (9) to obtain

$$d_{1,k+1} + d_{k+1,k+2} = \left[ \frac{x_1}{2!} + \frac{\sqrt{3}x_2}{3!} + \cdots + \frac{(k-2)!}{k!} \sqrt{\frac{(k-1)k}{2}}\, x_{k-1} \right.$$

$$+ \frac{(k-1)!}{(k+1)!} \sqrt{\frac{k(k+1)}{2}}\, x_k \right]$$

$$+ \frac{1}{k+1} \sqrt{\frac{(k+1)(k+2)}{2}}\, x_{k+1} \ .$$

Using (4) gives

$$(10)\quad d_{1,k+2} = \sum_{j=1}^{k} \frac{(j-1)!}{(j+1)!} \sqrt{\frac{j(j+1)}{2}}\, x_j + \frac{1}{k+1} \sqrt{\frac{(k+1)(k+2)}{2}}\, x_{k+1},$$

which is (8) with i = k+1.

Theorem 4.

$$(11)\quad d_{\alpha,i+1} = \frac{1}{\alpha-1} \left[ \sum_{j=1}^{\alpha-1} d_{j,i+1} - \sqrt{\frac{(\alpha-1)\alpha}{2}}\, x_{\alpha-1} \right],$$

where $\alpha = 2, 3, \ldots, i$.

Proof. Let j range from 1 to $\alpha$-1 in (6). Thus

$$d_{\alpha,i+1} = \frac{1}{\alpha-1} \left[ \sum_{j=1}^{\alpha-1} d_{j,i+1} + \sum_{j=1}^{\alpha-1} d_{\alpha j} \right]$$

$$= \frac{1}{\alpha-1} \left[ \sum_{j=1}^{\alpha-1} d_{j,i+1} - \sum_{j=1}^{\alpha-1} d_{j\alpha} \right], \text{ by (2)}$$

$$= \frac{1}{\alpha-1} \left[ \sum_{j=1}^{\alpha-1} d_{j,i+1} - \sqrt{\frac{(\alpha-1)\alpha}{2}} \, x_{\alpha-1} \right], \text{ by (1) of II.}$$

In order to list the expressions for $d_{12}$, $d_{13}$, ..., $d_{67}$, (8) and (11) are arrayed in the following more useful form:

$$d_{1,i+1} = \left[ (3 \ldots i)x_1 + (4 \ldots i)\sqrt{3}x_2 + \ldots + (i-2)! \sqrt{\frac{(i-1)i}{2}} \, x_{i-1} \right.$$

$$\left. + (i-1)! \sqrt{\frac{i(i+1)}{2}} \, x_i \right] / \, i!$$

$$d_{2,i+1} = d_{1,i+1} - x_1$$
$$d_{3,i+1} = \left[ d_{1,i+1} + d_{2,i+1} - \sqrt{3} \, x_2 \right]/2$$
$$d_{4,i+1} = \left[ d_{1,i+1} + d_{2,i+1} + \sqrt{6} \, x_3 \right]/3$$

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$$d_{j,i+1} = \left[ (d_{1,i+1} + \ldots + d_{j-1,i+1}) - \sqrt{\frac{(j-1)j}{2}} \, x_{j-1} \right] / \, (j-1),$$

where $j = 2, \ldots, i$.

Letting $i = 1, 2, \ldots, 6$ in these formulas gives

$$d_{12} = x_1$$

$$d_{13} = (x_1 + \sqrt{3}x_2)/2$$
$$d_{23} = (-x_1 + \sqrt{3}x_2)/2$$

$$d_{14} = (3x_1 + \sqrt{3}x_2 + 2\sqrt{6}x_3)/3!$$
$$d_{24} = (-3x_1 + \sqrt{3}x_2 + 2\sqrt{6}x_3)/3!$$
$$d_{34} = ( \qquad -2\sqrt{3}x_2 + 2\sqrt{6}x_3)/3!$$

$$d_{15} = (12x_1 + 4\sqrt{3}x_2 + 2\sqrt{6}x_3 + 6\sqrt{10}x_4)/4!$$

$$d_{25} = (-12x_1 + 4\sqrt{3}x_2 + 2\sqrt{6}x_3 + 6\sqrt{10}x_4)/4!$$

$$d_{35} = (\qquad -8\sqrt{3}x_2 + 2\sqrt{6}x_3 + 6\sqrt{10}x_4)/4!$$

$$d_{45} = (\qquad\qquad -6\sqrt{6}x_3 + 6\sqrt{10}x_4)/4!$$

$$d_{16} = (60x_1 + 20\sqrt{3}x_2 + 10\sqrt{6}x_3 + 6\sqrt{10}x_4 + 24\sqrt{15}x_5)/5!$$

$$d_{26} = (-60x_1 + 20\sqrt{3}x_2 + 10\sqrt{6}x_3 + 6\sqrt{10}x_4 + 24\sqrt{15}x_5)/5!$$

$$d_{36} = (\qquad -40\sqrt{3}x_2 + 10\sqrt{6}x_3 + 6\sqrt{10}x_4 + 24\sqrt{15}x_5)/5!$$

$$d_{46} = (\qquad\qquad -30\sqrt{6}x_3 + 6\sqrt{10}x_4 + 24\sqrt{15}x_5)/5!$$

$$d_{56} = (\qquad\qquad\qquad -24\sqrt{10}x_4 + 24\sqrt{15}x_5)/5!$$

$$d_{17} = (360x_1 + 120\sqrt{3}x_2 + 60\sqrt{6}x_3 + 36\sqrt{10}x_4 + 24\sqrt{15}x_5 + 120\sqrt{21}x_6)/6!$$

$$d_{27} = (-360x_1 + 120\sqrt{3}x_2 + 60\sqrt{6}x_3 + 36\sqrt{10}x_4 + 24\sqrt{15}x_5 + 120\sqrt{21}x_6)/6!$$

$$d_{37} = (\qquad -240\sqrt{3}x_2 + 60\sqrt{6}x_3 + 36\sqrt{10}x_4 + 24\sqrt{15}x_5 + 120\sqrt{21}x_6)/6!$$

$$d_{47} = (\qquad\qquad -180\sqrt{6}x_3 + 36\sqrt{10}x_4 + 24\sqrt{15}x_5 + 120\sqrt{21}x_6)/6!$$

$$d_{57} = (\qquad\qquad\qquad -144\sqrt{10}x_4 + 24\sqrt{15}x_5 + 120\sqrt{21}x_6)/6!$$

$$d_{67} = (\qquad\qquad\qquad\qquad -120\sqrt{15}x_5 + 120\sqrt{21}x_6)/6!$$

It should be noted that the leading term of $d_{\alpha,i+1}$ is minus the sum of like terms in $d_{1,i+1}, \ldots, d_{\alpha-1,i+1}$. All other terms are the same as in $d_{1,i+1}, \ldots, d_{\alpha-1,i+1}$. Thus, once $d_{1,i+1}$ is written, $d_{\alpha,i+1}$, where $\alpha = 2, \ldots, i$, follows immediately.

## V.  POWER FUNCTIONS

### 1.  Introduction.

In this paper, the tests under discussion are multiple decision tests and as such are typical candidates for the type of analysis proposed by Wald (1939 and 1941). Thus any two tests would be judged by the relative magnitudes of their average weighted risks, the test having the lower average weighted risk being considered better. Such an analysis of any test would involve a careful consideration of all its decisions and related errors.

The two main steps in this analysis would be the selection of an appropriate error weight function and an anticipated frequency function of the parameters. That is, expressions would have to be chosen which give the relative seriousness of various errors which may be made and the relative expected frequencies of differences in parameter means, respectively.

Selecting these functions is non-mathematical and rests on experimental considerations of errors and possible parameter locations. So often the research worker, who must decide which weight functions and anticipated frequency functions to use, has little interest in the mathematics involved or lacks time enough to carry through such an analysis. On the other hand, a worker not trained to know which errors are most important for

the various decisions and which parameter locations are
most likely (or the worker who is not willing to put a
lot of time on making the selection of the error func-
tions and anticipated frequency function) can expect
little from the analysis after it is finished.  It is
most important that appropriate functions be chosen
since one test may have a lower average weighted risk
than another for one class of these functions, but the
reverse might be true for another class of functions.
This method of analysis is also difficult since the num-
ber of decisions would be so great and the weighting of
errors so intricate that the work involved becomes too
cumbersome for more than three means.

## 2. Discussion of Power.

For these reasons, as well as for a natural desire
to make the work as straightforward and simple as pos-
sible, we compare two tests with the use of sets of func-
tions similar to the Neyman-Pearson power function for a
two-decision test.  This approach is an extension of
methods used by Lehmann (1950), Hartley (1954), Harter
(1954), and more fully by Duncan (1953 and 1955).  The
relation of this type of analysis to a full average
weighted risk analysis is summarized in the following
quotation from Duncan (1953):  "This way of analyzing
these multiple decision procedures is put forward as a

simple step toward a complete average risk analysis, and
not as a substitute for the latter.  It bears the same
relationship to the latter as does a Neyman-Pearson analy-
sis in a simple, familiar, two-decision case.  That is,
it fails to consider details of relative error weights
and Bayes [anticipated frequency] functions, but it is
simpler in conception and arrives at similar results."

In obtaining the power functions for these tests,
our first difficulty is that none of them, including the
case involving only two means, is a two-decision proce-
dure.  However, a power function as defined by Neyman
and Pearson (1933 and 1950) is strictly a two-decision
test concept.  A typical definition of power function is
the following given by Hoel (1954):

"The power function $P(\delta)$ of a test is the

function of the parameter that gives the

(1)   probability that the sample point will fall

in the critical region of the test when $\delta$ is

the true value of the parameter."

For example, consider a 5% level test of the hypothesis
$H_o$: $\delta = 0$, where $\delta = \mu - \mu_o$, against the alternative hypothesis
$H_a$: $\delta > 0$ based on a single observation d, where d is
normally distributed with mean $\mu$ and variance two.  (A
variance of two is used because it is a consequence of
the restriction, $\sigma_m = 1$, made in II.)  The critical region

is d > 2.77 and the power function is the familiar curve
shown in Figure 1.  The merits of a power function of this
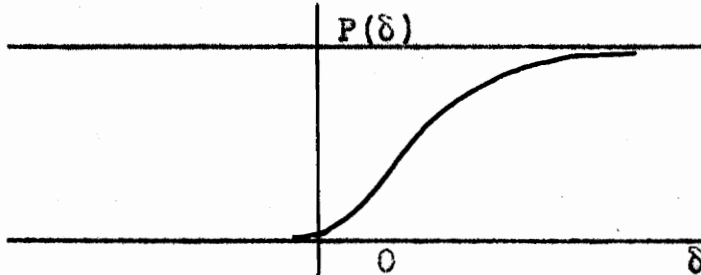


Figure 1

Power Function for a One Sided Normal Deviate Test

type are well known.  Over the parameter points $\delta > 0$, com-
prising the alternative hypothesis, the function gives prob-
abilities of correct decisions, that is, to reject $H_0$.  Over
the points $\delta \leq 0$, the function gives probabilities of the
type I error.  The two sections of the curve completely
describe the operating characteristics of the tests.

In passing, it should be noted that although it is
customary to refer to this procedure as a test of $H_0 : \delta = 0$
against the alternative $H_a : \delta > 0$, its main merits in
many applications rest upon the fact that it provides a
test of $H_0 : \delta \leq 0$ against the same alternative $H_a : \delta > 0$.
$H_0 : \delta \leq 0$ may be termed the complete null hypothesis to
distinguish it from $H_0 : \delta = 0$, which may be termed the point
null hypothesis.  In many situations, the value of mak-
ing an $\alpha$-level test of the point null hypothesis lies in
the fact that it provides an $\alpha'$-level test of any single

hypothesis from the complete composite null hypothesis where $\alpha' \leq \alpha$.

In a two mean three-decision test one way (which proves to be unsatisfactory) to adjust to the two-decision test concept would be to group the two decisions $D(1,2)$ and $D(2,1)$ together as the decision that $m_1$ and $m_2$ are significantly different, that is, accept the hypothesis $\mu_1 \neq \mu_2$. The decision that $m_1$ and $m_2$ are significantly different would be denoted by $D(1\neq2)$. Thus, the three-decision test, with decisions $D(1,2)$, $D(\underline{1,2})$, and $D(2,1)$, would be reduced to a two-decision test with decisions $D(1\neq2)$ and $D(\underline{1,2})$. The test would then be analyzed as an $\alpha$-level test of $\mu_1 = \mu_2$ against the two-sided alternative $\mu_1 \neq \mu_2$, $\alpha$ being the significance level. According to definition (1), the power function would be denoted by

(2) $\qquad p(1\neq2) = P[D(1\neq2)|\, \xi \, , \, \sigma_m \,]$ ,

where $\xi = \mu_1 - \mu_2$ is the true difference. The power function would be the probability of making the decision $D(1\neq2)$, which is equivalent to rejecting the hypothesis $\mu_1 = \mu_2$, given any $\xi$. The test, considered in this way, would be a two-decision two-sided test of $H_0 : \mu_1 = \mu_2$.

The trouble with this approach, however, is that by pooling the probabilities of the decisions $D(1,2)$ and $D(2,1)$ for any $\xi \neq 0$, we would combine the probability of

the correct decision with the probability of the most
incorrect decision. That is, taking any point, $\xi < 0$
for example, we would be adding the probability of the
correct decision D(1,2) to the probability of the wrong
decision D(2,1). Certainly a function which combines
correct decision probabilities with such serious error
probabilities would not be desirable. Thus, this method
of attack would not be appropriate for the tests under
considerations.

A more useful analysis for the three-decision test
of two means is one in which it is regarded as a combina-
tion of two distinct two-decision tests, where one is a
test of the hypothesis $\mu_1 \leq \mu_2$ against the alternative
$\mu_2 < \mu_1$ and the other is a test of the hypothesis $\mu_2 \leq \mu_1$
against the alternative $\mu_1 < \mu_2$. Thus for two means, a
three-decision test has the two power functions

(3) $$P_{21}(\xi) = P[D(2,1)|\ \xi\ ,\ \sigma_m\ ]\ ,$$

and

(4) $$P_{12}(\xi) = P[D(1,2)|\ \xi\ ,\ \sigma_m\ ]\ ,$$

which are the Neyman-Pearson power functions of the tests
of $\mu_1 \leq \mu_2$ and $\mu_2 \leq \mu_1$, respectively. For any value of
$\xi$, these functions give the probability of a correct or
an incorrect decision. In particular, $P_{21}(\xi)$ is the prob-
ability of deciding that $\mu_1 > \mu_2$. Thus, $P_{21}(\xi)$ is a
correct decision probability function over the interval

$\xi = \mu_1 - \mu_2 > 0$ and an _error probability function_ over the interval $\xi \leq 0$. Similarly, $P_{12}(\xi)$ is a correct decision probability function over the interval $\xi < 0$ and an error probability function over the interval $\xi \geq 0$. Each power function should be as high as possible over the region in which it is a correct decision probability function and as low as possible over the region in which it is an error probability function. Formalizing this idea, we may express each power function of a three-decision two-sided test as two functions. For $P_{12}(\xi)$, the correct decision probability function is given by

(5)     $C_{12}(\xi) = P[D(1,2)|\ \xi\ ,\ \sigma_m\ ]$ when $\mu_1 < \mu_2$,

and the error probability function is given by

(6)     $E_{12}(\xi) = P[D(1,2)|\ \xi\ ,\ \sigma_m\ ]$ when $\mu_1 \geq \mu_2$.

Likewise, the correct decision probability function and the error probability function for $P_{21}(\xi)$ are given by

(7)     $C_{21}(\xi) = P[D(2,1)|\ \xi\ ,\ \sigma_m\ ]$ when $\mu_2 < \mu_1$ and

(8)     $E_{21}(\xi) = P[D(2,1)|\ \xi\ ,\ \sigma_m\ ]$ when $\mu_2 \geq \mu_1$,

respectively. These relations may be written more compactly as

(9)     $P_{12}(\xi) = \begin{cases} C_{12}(\xi) & \text{when } \mu_1 < \mu_2 \\ E_{12}(\xi) & \text{otherwise} \end{cases}$, and

(10)    $P_{21}(\xi) = \begin{cases} C_{21}(\xi) & \text{when } \mu_2 < \mu_1 \\ E_{21}(\xi) & \text{otherwise} \end{cases}$ .

This approach avoids the difficulty of adding the serious error probability to the correct decision probability which is inherent in the power function $p(1 \neq 2)$. These functions, whose uses have been indicated in part or wholly by Lehmann, Duncan, Hartley, Harter, and Scheffé, are the ones which are concerned with measuring the probability of correct decisions for individual differences. These are the functions to be discussed and compared in this paper. Their use is readily extended to more than two means as will be shown later.

The graphs of equations (2), (3), and (4) may shed some light on the relations among equations (2) to (10). (Assume $\sigma_m = 1$ for each graph.) Figure 2 shows the graphs of $P_{12}(\xi)$ and $P_{21}(\xi)$ for the same significance level. Each of these curves is divided into two parts by the vertical axis, denoted by $P(\xi)$. The upper part of $P_{21}(\xi)$ to the right of $P(\xi)$ is $C_{21}(\xi)$, and the lower part to the left and including the $P(\xi)$ axis, is $E_{21}(\xi)$. In passing, it should be observed that corresponding to each $\xi \neq 0$ there is one correct decision probability and one error probability. For $\xi = 0$, there are two error probabilities which are equal in this case. Figure 3 shows the graph of (2). It is the composite graph of the two curves in Figure 2.
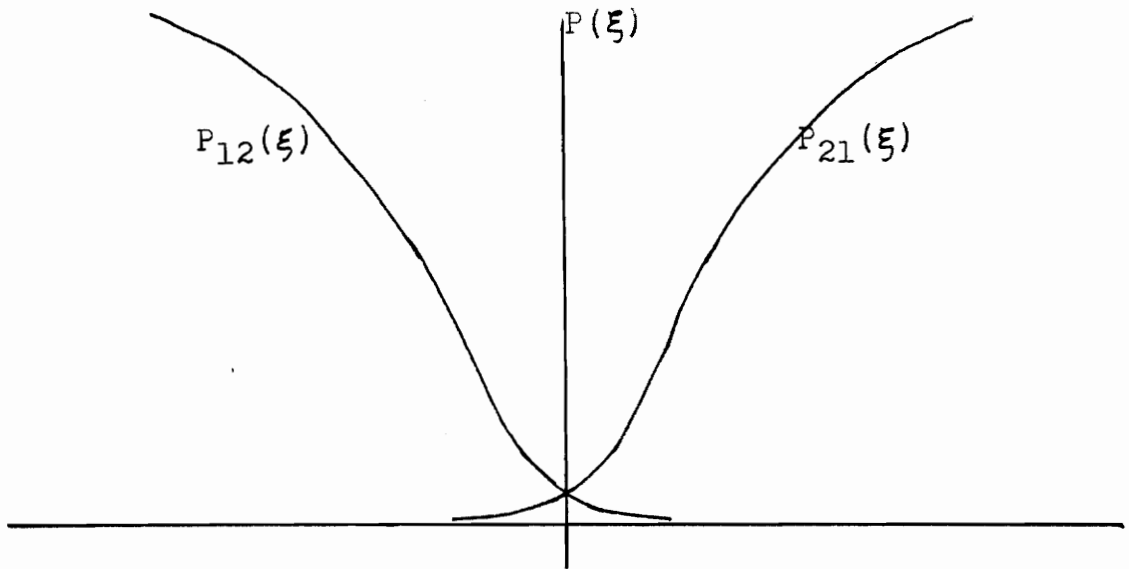
Figure 2

Graphs of $P_{12}(\xi)$ and $P_{21}(\xi)$ at the Same
Significance Level $\alpha_0$.
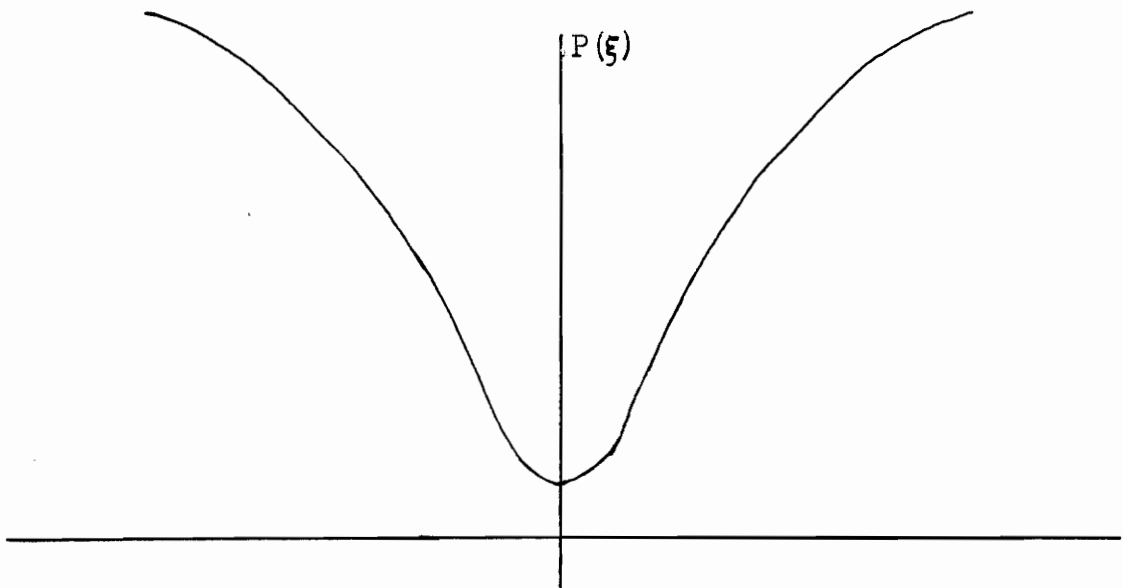


Figure 3

Graph of $p(1 \neq 2)$ for the $\alpha_0$-level Two
Sided Test.

## 3. Power Vectors.

For the general treatment of power, more notation is necessary. Let

(11) $$\delta_{ij} = E(d_{ij}) = \mu_i - \mu_j,$$

where $i=1,\ldots,n-1$, $j=2,\ldots,n$, and $i < j$, denote the ordered difference for true means $\mu_i$ and $\mu_j$. Let

(12) $$\xi_i = E(x_i),$$

where $i=1,\ldots,n-1$, denotes the axes in the spaces of true differences $\delta_{ij}$, which is called the reduced-parameter-space. Let

(13) $$\xi = (\xi_1,\ldots,\xi_{n-1}) \quad ,$$

a vector of n-1 components, denote a point in the parameter-space.

From IV, it is clear that corresponding to any set of values for the $n(n-1)/2$ differences $(d_{12},d_{13},\ldots,d_{n-1,n})$ there is a unique point $x=(x_1,\ldots,x_{n-1})$ in the sample-space, and conversely. Likewise, for any set of values for the $n(n-1)/2$ components of the vector of true differences,

(14) $$\delta = (\delta_{12},\delta_{13},\ldots,\delta_{n-1,n}),$$

there is a unique point $\xi$ in the parameter-space, and conversely.

The approach to power considerations discussed in Section 2 above for the two mean case is readily extended to three or more means. For each pair of means, $\mu_1$ and $\mu_2$, $\mu_1$ and $\mu_3$, and $\mu_2$ and $\mu_3$, two power functions are defined

which correspond to (3) and (4). These six power functions, defined for the three mean case, are given by

(15) $\qquad P_{ij}(\xi) = P[D(i,j)| \xi ,\sigma_m ]$, and

(16) $\qquad P_{ji}(\xi) = P[D(j,i)| \xi ,\sigma_m ]$,

where ij denotes 12, 13, or 23 and $\xi =(\xi_1,\xi_2)$. Each of these functions may be broken into two functions, a correct decision probability function, $C_{ij}(\xi)$ and $C_{ji}(\xi)$, and an error probability function, $E_{ij}(\xi)$ and $E_{ji}(\xi)$, following the method used above. For a fixed $\xi \neq 0$, any pair of power function $P_{ij}(\xi)$ and $P_{ji}(\xi)$ has two components, a correct decision probability component, $C_{\overline{ij}}(\xi) = C_{\overline{ij}}$, and an error probability component, $E_{\overline{ij}}(\xi) = E_{\overline{ij}}$, such that exactly one comes from each power function. For example, if $C_{\overline{ij}}(\xi)$ comes from $P_{ij}(\xi)$, $E_{\overline{ij}}(\xi)$ must come from $P_{ji}(\xi)$. Thus, for a fixed $\xi \neq 0$, (15) and (16) furnish a set of three correct decision probabilities and a set of three error probabilities.

When considering power, we are mainly interested in how $P_{ij}(\xi)$ changes as $\delta_{ij} = \mu_i - \mu_j$ varies even though the other differences make small changes in $P_{ij}(\xi)$ for $\delta_{ij}$ fixed. That is, we wish to think of $P_{ij}(\xi)$ as a function of $\delta_{ij}$. Since

$$P[D(i,j)| \xi ,\sigma_m ] = P[D(i,j)| \delta ,\sigma_m ]$$
$$= P[D(i,j)| \delta_{ij},\delta',\sigma_m ] ,$$

where $\delta'$ is the set of all differences except $\delta_{ij}$, we write

(17) $\qquad P_{ij}(\delta_{ij}) = P[D(i,j)| \delta_{ij},\delta',\sigma_m ]$,

expressing $P_{ij}(\xi)$ as a function of $\delta_{ij}$.

Corresponding to each difference, $\delta_{ij}$, two power functions, $P_{ij}(\xi)$ and $P_{ji}(\xi)$ ($P_{ij}(\delta_{ij})$ and $P_{ji}(\delta_{ij})$), are defined. Thus, corresponding to each parameter point $\xi = (\xi_1, \xi_2)$ (set of differences $\delta_{12}, \delta_{13}, \delta_{23}$), we define a power vector, $P_3$, to be that vector which has as its components the six power functions given in (15) and (16). We write

(18)     $P_3 = (P_{12}, P_{21}, P_{13}, P_{31}, P_{23}, P_{32})$ ,

where $P_{ij}$ is an abbreviation of $P_{ij}(\ )$ and the subscript on $P_3$ denotes the number of means compared. The power vector, $P_3$, may also be written as

(19)     $P_3 = (C_{\overline{12}}, E_{\overline{12}}, C_{\overline{13}}, E_{\overline{13}}, C_{\overline{23}}, E_{\overline{23}})$, or

(20)     $P_3 = (C_3, E_3)$ ,

where

(21)     $C_3 = (C_{\overline{12}}, C_{\overline{13}}, C_{\overline{23}})$

and

(22)     $E_3 = (E_{\overline{12}}, E_{\overline{13}}, E_{\overline{23}})$ .

In the general case of $n$ means, there are $n(n-1)/2$ differences $\delta_{ij}$. For each difference $\delta_{ij}$, there are two power functions given by (15) and (16), where $i = 1, \ldots, n-1$, $j = 2, \ldots, n$, $i < j$, and $\xi = (\xi_1, \ldots, \xi_{n-1})$. Each $P_{ij}(\delta_{ij})$ is the Neyman-Pearson power function of a test of the hypothesis $\mu_j \leq \mu_i (\delta_{ij} \geq 0)$ against the alternative hypothesis $\mu_i < \mu_j (\delta_{ij} < 0)$. That is, $P_{ij}(\delta_{ij})$ gives the probability

of a correct decision for all values of $\mu_i$ and $\mu_j$ for which $\mu_i < \mu_j$ and the probability of a wrong decision for all values of the means for which $\mu_j \leq \mu_i$. Similar statements may be made about $P_{ji}(\delta_{ij})$. For a fixed $\xi \neq 0$, the correct decision probability component and error probability component of the pair of power functions $P_{ij}(\delta_{ij})$ and $P_{ji}(\delta_{ij})$ are denoted by $C_{\overline{ij}}(\xi) = C_{\overline{ij}}$ and $E_{\overline{ij}}(\xi) = E_{\overline{ij}}$, respectively.

The power vector for n means at any point $\xi$ is given by

(23) $\qquad P_n = (P_{12}, P_{21}, \ldots, P_{ij}, P_{ji}, \ldots, P_{n-1,n}, P_{n,n-1})$ ,

where the $n(n-1)$ components are the power functions, $P_{ij}(\delta_{ij}) = P_{ij}$ and $P_{ji}(\delta_{ij}) = P_{ji}$. The power vector, $P_n$, may also be written as

(24) $\quad P_n = (C_{\overline{12}}, E_{\overline{12}}, \ldots, C_{\overline{ij}}, E_{\overline{ij}}, \ldots, C_{\overline{n-1,n}}, E_{\overline{n-1,n}})$, or

(25) $\quad P_n = (C_n, E_n)$,

where

(26) $\quad C_n = (C_{\overline{12}}, \ldots, C_{\overline{ij}}, \ldots, C_{\overline{n-1,n}})$

and

(27) $\quad E_n = (E_{\overline{12}}, \ldots, E_{\overline{ij}}, \ldots, E_{\overline{n-1,n}})$ .

## 4. Significance Levels.

Many other correct decision probability and error probability functions could be defined for decisions ranking p means at one time, where $p = 3, \ldots, n$. For example, in a test of four means, the function

(28) $\qquad C_{4,2,3,1}(\xi) = P[D(4,2,3,1)| \xi, \sigma_m ]$,

which is a correct decision probability function over the
parameter-space region defined by $\mu_4 < \mu_2 < \mu_3 < \mu_1$, might
be considered. Likewise, the function

(29)        $E_{4,2,3,1}(\xi) = P[DC(4,2,3,1)| \xi ,\sigma_m ]$ ,

where $C(4,2,3,1)$ denotes the region complementary to the
decision region $(4,2,3,1)$ in the sample-space of four means,
might be considered a relevant error probability function
over the parameter-space region $\mu_4 \leq \mu_2 \leq \mu_3 \leq \mu_1$. Similar
functions could be defined for the $4! - 1$ remaining four
mean decisions of the type $D(i_1,i_2,i_3,i_4)$, where $i_1,i_2,i_3$,
and $i_4$ assume the values 1, 2, 3, and 4 in some order. Also,
$4 \cdot 3!$ three mean correct decision probability and error prob-
ability functions of the types

(30)        $C_{2,1,4}(\xi) = P[D(2,1,4)| \xi ,\sigma_m ]$, and

(31)        $E_{2,1,4}(\xi) = P[DC(2,1,4)| \xi ,\sigma_m ]$ ,

respectively, may be defined. $C_{2,1,4}(\xi)$ is defined over the
parameter-space region $\mu_2 < \mu_1 < \mu_4$, and $E_{2,1,4}$ is defined
over the region $\mu_2 \leq \mu_1 \leq \mu_4$. In general, in a test of n
means, $n!/(n-p)!$ p-mean correct decision probability and
error probability functions of the above type could be de-
fined.

These p-mean functions, however, appear to be of secon-
dary importance and convey little additional information.
Provided upper bounds are placed on the maximum values of
the error probability functions for p mean decisions, it

appears that the merits of a test can be judged in terms of its two-decision power functions alone. Suitable upper bounds of this nature are provided by a set of probabilities defined as follows:

$$P[DC(\underline{i_1, i_2}) \mid \mu_{i_1} = \mu_{i_2}] ,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$(32) \qquad P[DC(\underline{i_1, \ldots, i_p}) \mid \mu_{i_1} = \ldots = \mu_{i_p}] ,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$P[DC(\underline{1, \ldots, n}) \mid \mu_1 = \ldots = \mu_n] ,$$

where $i_p = 1, \ldots, n$,

$i_1, \ldots, i_p$ is some arrangement of p of n means, and $C(\underline{i_1, \ldots, i_p})$ denotes the region complementary to the decision region $(\underline{i_1, \ldots, i_p})$ in the sample-space of n means.

Each of these probabilities serves a function similar to that of a significance level in a two-decision test, and, following Duncan (1955), may be termed a p-mean significance level, $\alpha(i_1, \ldots, i_p)$. Thus, $\alpha(i_1, \ldots, i_p)$ denotes the maximum probability of finding at least one wrong significant difference among p observed means.

To illustrate the fact that the probabilities in (32) are upper bounds of the corresponding error probability functions, we show that in the three mean case

$$(33) \qquad P[DC(3,2,1) \mid \xi \text{ belongs to } \mu_1 = \mu_2 = \mu_3]$$

is an upper bound of

(34) $E_{(3,2,1)_1}(\xi) = P[DC(3,2,1)_1 \mid \xi \in \text{region } \mu_3 \leq \mu_2 \leq \mu_1]$,

where $(3,2,1)_1$ denotes the region of acceptance for the
hypothesis $\mu_3 \leq \mu_2 \leq \mu_1$, and $\in$ is to be read "belongs to."
To prove this for the multiple t test, we appeal to geometry.
Referring to Figure 2 of II, we let

$$R_1 = (\underline{1,2,3}) \,,$$

(35) $\quad \begin{aligned} R_2 &= (\underline{1,2,3}) + (\underline{3,1},2) + (3,\underline{2},1) + (3,\underline{2},1) \,, \\ R_3 &= (\underline{1,2,3}) + (2,\underline{3},1) + (3,\underline{2},1) + (\underline{3},2,1) \,, \text{ and} \\ R_4 &= R_2 + R_3 + (3,2,1) \,, \end{aligned}$

where the corresponding sets of parameter points are given by

$$S_1 : \mu_3 = \mu_2 = \mu_1 \,,$$

(36) $\quad \begin{aligned} S_2 &: \mu_3 \leq \mu_2 = \mu_1 \,, \\ S_3 &: \mu_3 = \mu_2 \leq \mu_1 \,, \text{ and} \\ S_4 &: \mu_3 \leq \mu_2 \leq \mu_1 \,, \end{aligned}$

respectively. Let the axes of the parameter-space be $\xi_1 = x_1$
and $\xi_2 = x_2$. Hence $S_1$ is the origin, $S_2$ the positive $\xi_2$ axis,
$S_3$ the positive part of the line $\mu_2 = \mu_3$, and $S_4$ is the region
bounded by $S_2$ and $S_3$ (one sixth of the parameter-space).
Let $C(R_q)$, where $q=1,2,3,4$, denote the complement of region
$R_q$ in the sample-space. In this notation, the problem be-
comes one of showing that

(37) $\quad P[DC(R_q) \mid \xi \in S_q] \leq P[DC(R_1) \mid \xi \in S_1]$.

If $q=1$, the left and right members are identically equal.
If $q=2$, we have the situation shown in Figure 4. Let $p_0$
be any point in $S_2$, except $\mu_3 = \mu_2 = \mu_1$. Construct a regular
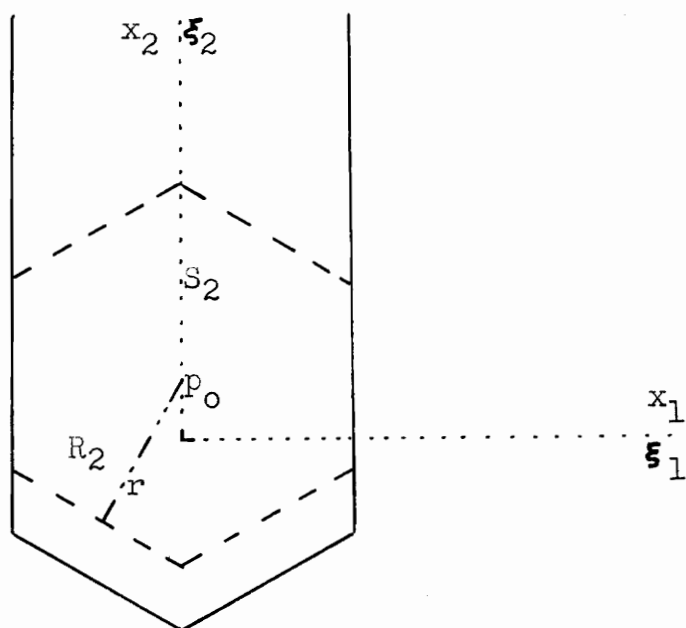
Figure 4

Graph of Region $R_2$

hexagon, H, with radius r, center at $p_o$, and two sides fall-ing on the vertical boundary lines of $R_2$. Clearly, H is contained in $R_2$. Thus, $CR_2$ is contained in CH. Hence,

$$P[D(CR_2)|\xi \in S_2] \leq P[D(CH)|\xi \in S_2] = P[D(CR_1)|\xi \in S_1] .$$

The proof is similar when q=3 and 4. Therefore, (37) is established. There are five more error functions,

$$E_{(i_1,i_2,i_3)_1}(\xi) = P[DC(i_1,i_2,i_3)_1|\ \xi \in \text{region } \mu_{i_1} \leq \mu_{i_2} \leq \mu_{i_3}]$$

where $i_p(p=1,2,3)$ is one of the numbers 1, 2, and 3, which have (33), that is $\alpha(1,2,3)$, as their upper bound. Thus, $\alpha(1,2,3)$ is the maximum probability of finding at least one wrong significant difference among three observed means.

Having indicated the method which shows that the

significance levels defined by (32) are upper bounds of the
p-mean error probabilities, we may now state that

(38)    two tests, $T_1$ and $T_2$, (for example, multiple range
versus a multiple F) may be compared at any para-
meter point $\xi$ using their correct decision vectors
$C_{n1}$ and $C_{n2}$, respectively, provided their corres-
ponding p-mean significance levels, $\alpha(i_1,\ldots,i_p)$,
are equal.

## 5.   Comparison of Tests by Use of Averages.

Let $c_{kj}$ denote the kth component of test j in the cor-
rect decision vector, $C_{nj}$, where $k=1,\ldots,n(n-1)/2$ and $j=1,2$.
If, at a point $\xi$, $c_{k1} > c_{k2}$ for each k, then test $T_1$ is
clearly better than $T_2$ at $\xi$ (provided, of course, that the
significance levels are equal).  However, this situation
does not always occur.  Suppose $c_{k1} > c_{k2}$ for some k, and
$c_{k1} \leq c_{k2}$ for all other k, then what can we say about the
two tests?  Obviously, further considerations are necessary.

A natural desire is to determine some kind of average
values of $C_{n1}$ and $C_{n2}$ to compare the two tests, $T_1$ and $T_2$.
A useful average correct decision probability may be defined
by

(39)                    $C_w = \Sigma\ w_{ij}C_{ij},$

where

(40)                    $w_{ij} = |\delta_{ij}/\Sigma\delta_{ij}|$

This seems a useful choice of weights for obtaining such an

average since the relative importance of one correct deci-
sion probability to another is roughly proportional to the
relative magnitudes of the two differences, $|\delta_{ij}|$ and $|\delta'_{ij}|$, in-
volved. For example, this assumes that the importance of
the probability of a correct decision for a difference of
two units is twice that of the importance of a probability
of a correct decision for a difference of one unit. The
divisor, $\Sigma|\delta_{ij}|$ is used so that $C_W$ will never exceed one.

Likewise, an average error probability, $E_W$, may be de-
fined by

(41) $$E_W = \Sigma w_{ij} E_{ij} \, ,$$

where $w_{ij}$ is defined by (40). Here again making the weights
proportional to the difference $|\delta_{ij}|$ will usually provide
an acceptable average error probability. We say that the
test with the lower $E_W$ has a smaller type I error probabil-
ity.

Other weights could be used in defining $C_w$ and $E_w$, but
(39) and (41) seem the most desirable in the absence of
information concerning the probabilities of $|\delta_{ij}|$. In a
given experiment, if something is known about the distribu-
tion of $|\delta_{ij}|$ it is possible that $w_{ij}$ could be improved and
thus $C_W$ and $E_W$ would become more sensitive to any contrasts
between two tests.

As an illustration of the use of these averages Table
4 in Appendix A may be used to compare Duncan's multiple
range and multiple F tests at a given parameter point when

three means are involved. For example, at $\xi = (4,3)$, or $|\delta_{12}|=4.000000$, $|\delta_{13}|=4.598076$, and $|\delta_{23}|=.598076$, we find for the multiple range and multiple F tests

$C_{w1} = .8029$ and $E_{w1} = .000538$, and

$C_{w2} = .8002$ and $E_{w2} = .000537$

respectively. Thus, we conclude that the correct decision probability of the multiple range test is higher than for the multiple F test, but the type I error probability of the multiple F test is lower at the point $\xi = (4,3)$. It is interesting to observe from Table 4 that the probability of correct decisions is uniformly better for the multiple range test in the case of three means.

## VI.  DERIVATION OF POWER EXPRESSIONS

### 1.  Background Summary.

In this paper, we are fundamentally interested in the power of a test, its definition, its expression, its evaluation, and its use in the comparison of two tests.  This being the case, we shall omit any general discussion of significance levels, protection levels, and comparison of the tests relative to these concepts, except as they come up in the discussion of power or in the sections on computation.  For a careful treatment of these definitions and comparisons, see Duncan (1955).  However, as a guide to the comparison of certain multiple range tests, we give the following table due to Duncan (1955):

Table 1

Comparison of Significant Ranges for 5%
Level Tests of 20 Means

| Test | Subset Size | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | 2 | 3 | 4 | 5 | 6 | 8 | 10 | 14 | 20 |
| Tukey's Test Based on Allowances | 5.01 | 5.01 | 5.01 | 5.01 | 5.01 | 5.01 | 5.01 | 5.01 | 5.01 |
| Tukey's 1953 Test | 3.89 | 4.16 | 4.32 | 4.44 | 4.52 | 4.65 | 4.74 | 4.88 | 5.01 |
| Newman-Keuls' Test | 2.77 | 3.32 | 3.63 | 3.86 | 4.03 | 4.29 | 4.47 | 4.74 | 5.01 |
| New Multiple Range Test | 2.77 | 2.92 | 3.02 | 3.09 | 3.15 | 3.23 | 3.29 | 3.38 | 3.47 |

For our purposes, the work is restricted to the standard normal deviate case, which simplifies the calculations considerably. That is, we assume the degrees of freedom for error to be infinite and the standard error of a mean to be unity. The significance level of a test is taken as 5%. In other words, that system of significance levels is selected which yields a test which the author of the test would call a 5% level test. Since all the tests under consideration are symmetric, we can further restrict our problem to that of deriving an expression for and computing $P_{21}(\xi)$. Each test considered has the property that for every set of values of the means under consideration any pair may be selected as $m_1$ and $m_2$.

The multiple range tests of Newman-Keuls, Duncan, and Tukey (1953) are considered together at this time since the corresponding decision regions of these tests are similar and have parallel boundaries. Expressing the power, $P_{21}(\xi)$, for any range test similar to Tukey's test based on allowances, that is, any test with a single plane being the boundary of (2,1), is trivial and will not be considered here. Before the exact formulas giving power of $P_{21}(\xi)$ are derived, the decision regions for these multiple range tests are described. For three means only a review is given, but for four or more means a more detailed discussion of the regions is presented.

## 2. Decision Regions for Three Multiple Range Tests.

Consider the case of three means. First, we describe the critical region (2,1). The decision region for the two mean decision D(2,1) is the sum of the following six three mean decision regions:

(2,1,3), (2,3,1), (2,_1,3_), (_2,3,1_), (3,1,2), (_3,1_,2). The boundary equations are given by

$$x_1 = q_2 \text{ when } |x_2| \geq (2q_3 - q_2)/\sqrt{3}$$
$$x_1 = q_3 \text{ when } -q_3/\sqrt{3} \leq x_2 \leq q_3/\sqrt{3}$$
$$x_1 + \sqrt{3}x_2 = 2q_3 \text{ when } q_2 \leq x_1 \leq q_3, \text{ and}$$
$$-x_1 + \sqrt{3}x_2 = -2q_3 \text{ when } q_2 \leq x_1 \leq q_3,$$

where $q_2$ and $q_3$ are the least significant ranges for sample size of two and three, respectively, and $q_2$ and $q_3$ depend on the test used and the significance level. In each test, $q_3 > q_2$. The graph for Duncan's range test is shown in Figure 1. With the aid of Table 1, the two mean and three mean boundaries of the Newman-Keuls' and Tukey's 1955 tests may be located by drawing lines parallel to the boundary lines of Figure 1.

Before describing further the decision regions for the multiple range tests, the multiple t test regions for four means, whose boundaries are defined by (5) and (7) of II, are discussed in some detail. The two mean and three mean decision regions, as well as the four mean decision regions, are of interest and they are formed by adding certain of the
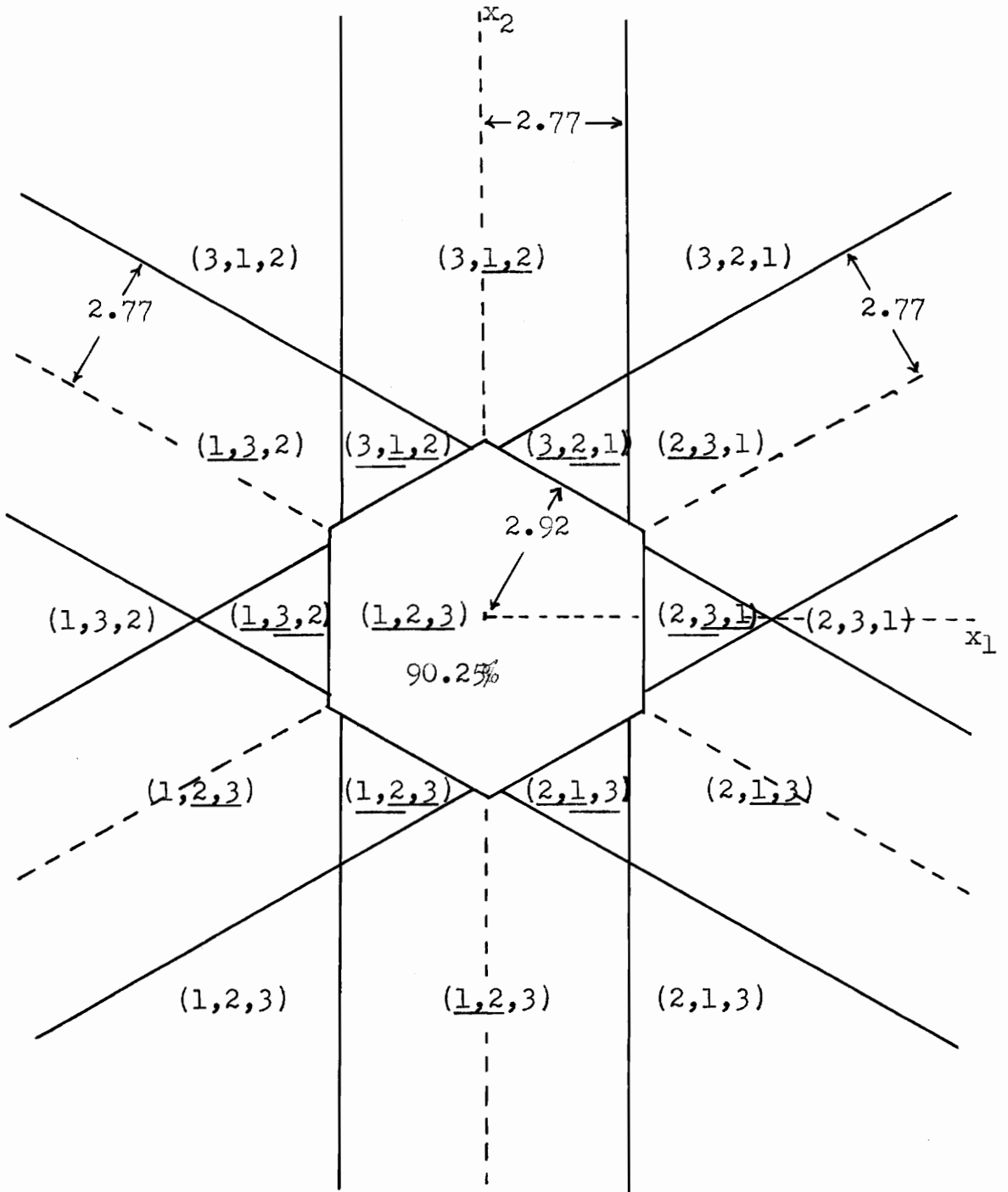
Figure 1

Duncan's 5% Level Multiple Range Test

$(n_2 = \infty, \; \sigma_m = 1)$

four mean decision regions. For example, the decision region (1,2,3) is the sum of the following 15 four mean decision regions:

(1,2,3,4)  (4,1,2,3)  (1,2,3,4)  (1,2,3,4)  (4,1,2,3)

(1,3,2,4)  (2,3,1,4)  (4,2,1,3)  (4,3,1,2)  (4,1,2,3)

(1,2,3,4)  (2,1,3,4)  (3,1,2,4)  (4,1,3,2)  (4,2,3,1) .

The region for the two mean decision D(2,1) is the sum of the following 57 four mean decision regions:

(2,1,3,4)  (2,1,4,3)  (2,3,1,4)  (2,4,1,3)  (2,3,4,1)

(2,4,3,1)  (3,2,1,4)  (4,2,1,3)  (3,2,4,1)  (4,2,3,1)

(3,4,2,1)  (4,3,2,1)  (2,3,1,4)  (2,4,1,3)  (2,3,4,1)

(2,4,3,1)  (3,4,2,1)  (2,1,3,4)  (2,1,4,3)  (2,3,4,1)

(3,2,4,1)  (4,2,3,1)  (2,1,3,4)  (2,3,1,4)  (2,4,1,3)

(3,2,1,4)  (4,2,1,3)  (2,3,4,1)  (2,3,1,4)  (2,4,1,3)

(2,3,4,1)  (2,4,3,1)  (3,2,4,1)  (4,2,3,1)  (4,3,2,1)

(3,4,2,1)  (2,3,1,4)  (2,4,1,3)  (2,3,4,1)  (2,4,3,1)

(3,2,4,1)  (4,2,3,1)  (2,1,3,4)  (2,1,4,3)  (2,3,1,4)

(2,4,1,3)  (2,3,4,1)  (2,4,3,1)  (2,3,4,1)  (2,4,3,1)

(2,3,4,1)  (2,3,1,4)  (2,4,1,3)  (2,3,4,1)  (2,4,3,1)

(3,2,4,1)  (4,2,3,1) .

The region for the decision D(1,2) is the sum of 57 regions obtained by interchanging 1 and 2 in the 57 regions above. Thus, out of 183 four mean decision regions, the region (1,2) is the sum of the remaining 69 regions.

We now describe the shape and position of the decision
regions (1,2,3,4), (1,2,3), (1,2,4), (1,2), and (2,1).
This is done with the aid of Figures 2 and 3 which show
plane sections when $x_1=0$ and $x_1=q_2$, respectively. The lines
on these figures are the intersections of the planes in (5)
and (7) of II, excepting $x_1=\pm q_2$, with the planes $x_1=0$ and
$x_1=q_2$. Note that only planes $-2\sqrt{3}x_2 + 2\sqrt{6}x_3 = \pm 6q_2$ cut $x_1=0$
and $x_1=q_2$ at right angles. The other eight planes make the
same size solid angle, $60^o$, with the planes $x_1=0$ and $x_1=q_2$.

Using these figures, it is easy to visualize the re-
quired decision regions. The region (1,2,3,4) is the
rhombic-dodecahedron with center at the origin formed by
the symmetric intersection about the origin of the six pairs
of parallel planes in (5) and (7) of II. The rhombus
$B_1B_2B_3B_4$ in Figure 3 is one of the twelve faces, and the
hexagon $A_1A_2A_3A_4A_5A_6$ in Figure 2 is formed by the plane,
$d_{12}=x_1=0$, which cuts the rhombic-dodecahedron into halves.
The vertices $A_2$, $A_3$, $B_1$, and $B_2$ belong to one rhombus which
is in the plane $d_{14}=q_2$. Likewise, each of the following
three sets of vertices belong to one rhombus: $A_3$, $A_4$, $B_2$,
$B_3$; $A_1$, $A_6$, $B_1$, $B_4$; and $A_6$, $A_5$, $B_4$, $B_3$. The line $A_1A_2$ bi-
sects the rhombus with vertices $B_1$, $A_1$, $A_2$, and $B_1'$, where $B_1'$
is the symmetric point to $B_1$ about line $A_1A_2$ in the plane
$d_{34}=q_2$. Similarly, the line $A_4A_5$ bisects the rhombus with
three of its vertices $B_3$, $A_4$ and $A_5$.
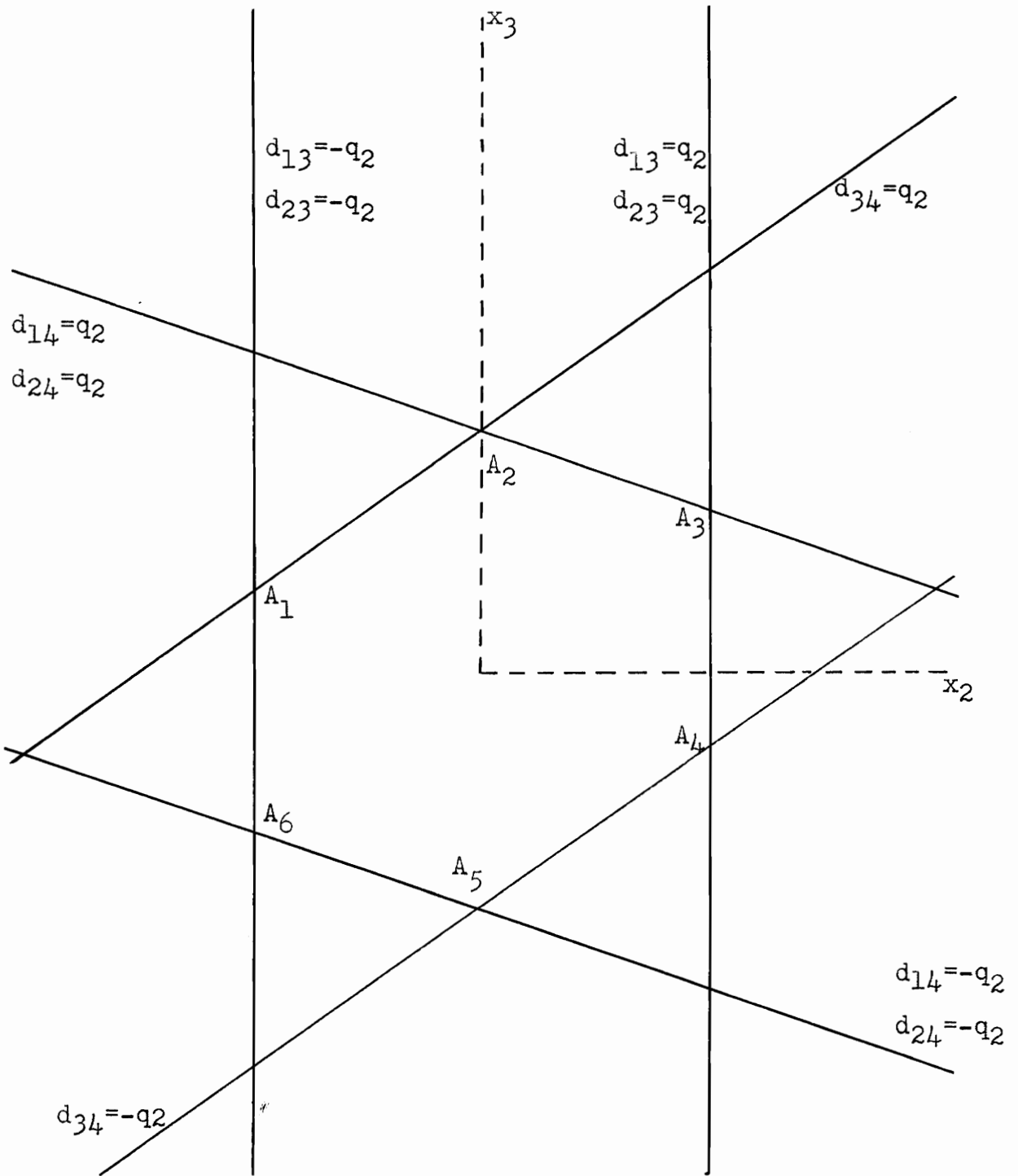
Figure 2

Intersections of the Multiple t Test Boundary Planes on

$d_{12} = x_1 = 0$ $(d_{ij} = q_2 = 2.77, \ n_2 = \infty, \ \sigma_m = 1)$.
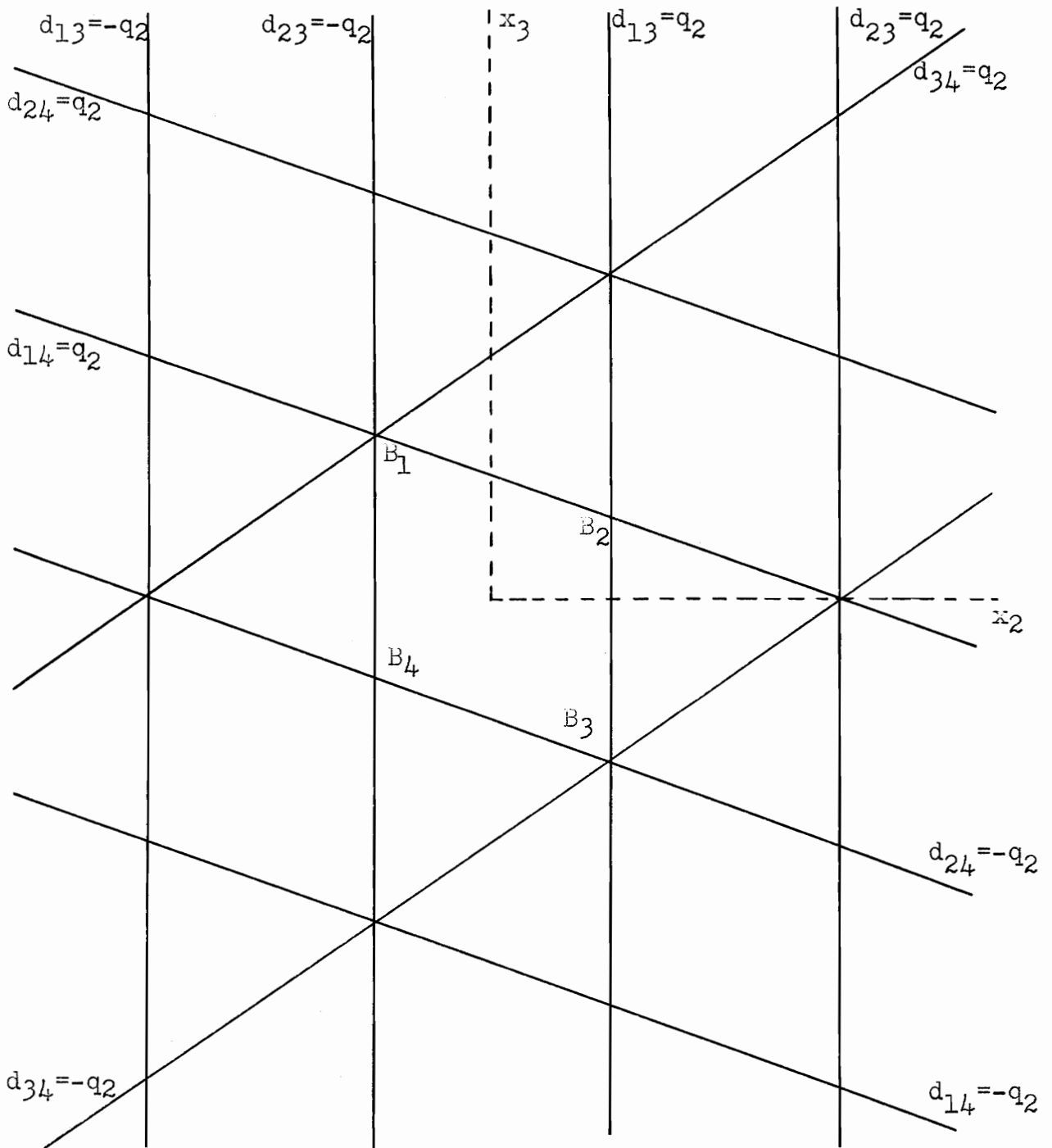
Figure 3

Intersection of the Multiple t Test Boundary Planes on

$d_{12} = x_1 = q_2$ ($d_{ij} = q_2 = 2.77$, $n_2 = \infty$, $\sigma_m = 1$).

The region $(\underline{1},\underline{2},\underline{3})$ is a right regular hexagonal cylinder
with its center line the $x_3$ axis. That part of the plane
$d_{12}=q_2$ between the intersection of $d_{12}=q_2$ with $d_{13}=q_2$ and
$d_{23}=-q_2$ in Figure 3 is one of the six faces of this cylinder.
That part of the plane $d_{13}=q_2$ cut off by the planes $x_1=0$ and
$x_1=q_2$ is an adjacent face to the one in Figure 3. Similarly,
that part of the plane $d_{23}=-q_2$ cut off by the planes $x_1=0$
and $x_1=q_2$ is an adjacent face to the one in Figure 3. The
other three faces of the regular cylinder are symmetric to
these about the plane $x_1=0$. The region $(\underline{1},\underline{2},\underline{4})$ is a right
regular hexagonal cylinder with its center line passing
through the points $A_1$ and $A_4$. The faces of this cylinder are
easily located following the method used above.

The region $(\underline{1},\underline{2})$ is the solid region between the two
planes $x_1=-q_2$ and $x_1=q_2$. The region $(2,1)$ is the solid re-
gion above the $x_1=q_2$ plane, that is, it is the region for
which $x_1 > q_2$.

Other regions of less importance in this paper are also
located. The region $(\underline{2},\underline{3},\underline{4},1)$ is the pyramidal shaped solid
bounded by the planes $d_{13}=q_2$, $d_{23}=-q_2$, $d_{14}=q_2$, $d_{24}=-q_2$, and
$x_1=q_2$. The base of this pyramid is the rhombus $B_1B_2B_3B_4$.
Eleven more regions of this type may similarly be located.
The region $(\underline{2},\underline{4},\underline{3},1)$ is the solid figure bounded by the planes
$d_{13}=q_2$, $d_{34}=q_2$, $d_{14}=q_2$, and $x_1=q_2$. There are a total of 24
such regions. The region $(2,\underline{1},\underline{3},\underline{4})$ is the solid figure

bounded by the planes $d_{12}=q_2$, $d_{13}=-q_2$, $d_{14}=-q_2$, and $d_{34}=-q_2$.
There are 48 such regions. Thus, we have described how to
locate 85 of the 183 decision regions enumerated in IV.

We are now in a better position to discuss the multiple
range regions. The boundary equations for the decision re-
gions are given by the following 36 formulas:

(2) $$d_{ij} = \pm q_k,$$

where $i < j$, $i=1,2,3$, $j=2,3,4$, $k=2,3,4$, and $q_2$, $q_3$, and $q_4$
are the least significant ranges for samples of size two,
three, and four, respectively. The size of $q_k$ depends on the
test used and the significance level. The significance level
used in this work is generally the one which corresponds to
the usual 5% level. In each test $q_2 < q_3 < q_4$.

The multiple range regions (1,2,3,4), (1,2,3), (1,2,4),
(1,2), and (2,1) are now defined. The region (1,2,3,4) is
bounded by the 12 planes obtained by letting k=4 in (2).
This region has the same shape as the four mean multiple t
region (1,2,3,4) described above, and it is in the same posi-
tion relative to the axes. However, it is larger than the
multiple t region since the bounding planes are now a dis-
tance of $q_4$ (see Table 1) from the origin. The region
(1,2,3,4) is symmetric about both the origin and the plane
$x_1=0$. This region is shown in Figure 4 for Duncan's new
multiple range test.

The multiple range region (1,2,3) has a different shape
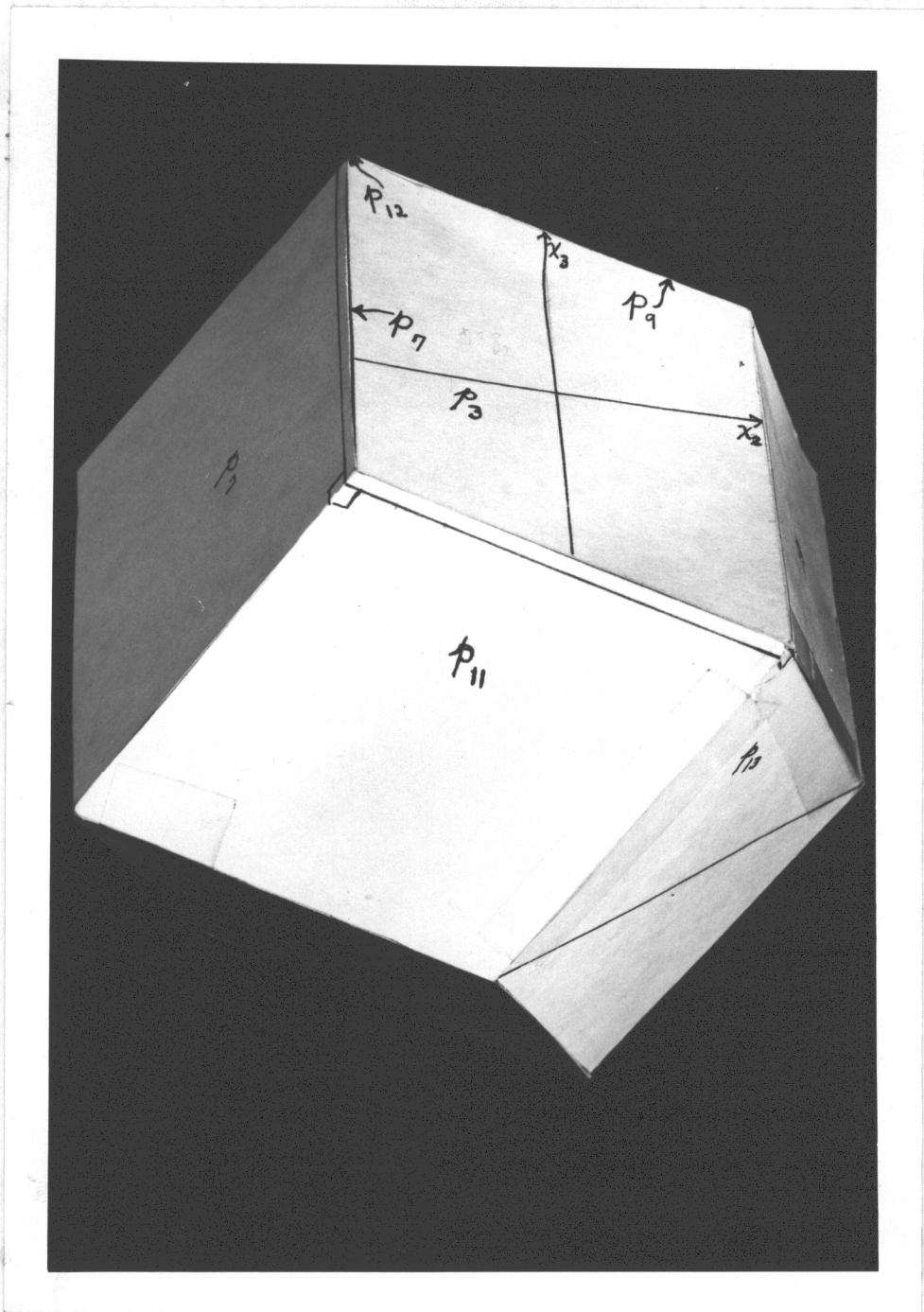
Figure 4

The Solid Region (<u>1,2,3,4</u>). (Rhombic-dodecahedron).

from the multiple t region $(\underline{1,2,3})$. It is the sum of two
solid regions given by the formula

(3) $\qquad (\underline{1,2,3}) = (\underline{1,2,3,4}) + (\overline{1,2,3},4)$ ,

where $(\overline{1,2,3},4)$ is the region bounded by

(4) $\qquad d_{12} = \pm q_3, \; d_{13} = \pm q_3, \text{ and } d_{23} = \pm q_3.$

This region is symmetric about the plane $d_{12}=x_1=0$, since it
is the sum of two parts each of which is symmetric about the
plane $x_1=0$. The region $(\underline{1,2,4})$ is similarly described and
is symmetric about the plane $x_1=0$.

The multiple range region $(\underline{1,2})$ is the sum of four over-
lapping multiple range regions, namely

(5) $(\underline{1,2}) = (\underline{1,2,3,4}) + (\underline{1,2,3}) + (\underline{1,2,4}) + (\overline{1,2},3,4)$

where $(\overline{1,2},3,4)$ is the region bounded by

(6) $\qquad d_{12} = -q_2 \text{ and } d_{12} = q_2$ .

This region is symmetric about the plane $d_{12}=0$, since it is
the sum of four regions each of which is symmetric about the
plane $d_{12}=x_1=0$. The complement of $(\underline{1,2})$ is the sum of two
regions, $(1,2)$ and $(2,1)$. In the region $(2,1)$, $x_1$ is always
positive, and in $(1,2)$, $x_1$ is always negative.

In a similar manner the other five regions $(\underline{1,3})$, $(\underline{1,4})$,
$(\underline{2,3})$, $(\underline{2,4})$, and $(\underline{3,4})$ may be defined. These regions are
the same size and shape as $(\underline{1,2})$, but $(\underline{i,j})$ is symmetric
about the plane $d_{ij}=0$, where i and j satisfy the conditions
of (2).

Figure 5 shows the boundaries of the critical region

for the decision that $\mu_1 \leq \mu_2$, that is, the region $(2,1)$. The boundary equations are given by

$$p_1: d_{12} = q_2, \qquad x_1 \qquad\qquad\qquad = q_2,$$
$$p_2: d_{12} = q_3, \qquad x_1 \qquad\qquad\qquad = q_3,$$
$$p_3: d_{12} = q_4, \qquad x_1 \qquad\qquad\qquad = q_4,$$
$$p_4: d_{13} = q_3, \qquad x_1 + \sqrt{3}x_2 \qquad = 2q_3,$$
$$p_5: d_{13} = q_4 \qquad x_1 + \sqrt{3}x_2 \qquad = 2q_4,$$
$$p_6: d_{23} = -q_3, \qquad -x_1 + \sqrt{3}x_2 \qquad = -2q_3,$$
$$(7) \qquad p_7: d_{23} = -q_4, \text{or } -x_1 + \sqrt{3}x_2 \qquad = -2q_4,$$
$$p_8: d_{14} = q_3, \qquad 3x_1 + \sqrt{3}x_2 + 2\sqrt{6}x_3 = 6q_3,$$
$$p_9: d_{14} = q_4, \qquad 3x_1 + \sqrt{3}x_2 + 2\sqrt{6}x_3 = 6q_4,$$
$$p_{10}: d_{24} = -q_3, \quad -3x_1 + \sqrt{3}x_2 + 2\sqrt{6}x_3 = -6q_3,$$
$$p_{11}: d_{24} = -q_4, \quad -3x_1 + \sqrt{3}x_2 + 2\sqrt{6}x_3 = -6q_4,$$
$$p_{12}: d_{34} = q_4, \qquad -2\sqrt{3}x_2 + 2\sqrt{6}x_3 = 6q_4,$$
$$p_{13}: d_{34} = -q_4, \qquad -2\sqrt{3}x_2 + 2\sqrt{6}x_3 = -6q_4,$$

where $p_r$ denotes the rth plane reading from top to bottom of (7), and r=1,...,13. These bounding planes are indicated in Figure 5.

From this discussion, it is clear how the regions $(1,2,3,4,5)$, $(1,2,3,4)$, $(1,2,3,5)$, $(1,2,4,5)$, $(1,2,3)$, $(1,2,4)$, $(1,2,5)$, $(1,2)$, and $(2,1)$ are defined for the multiple range tests of five means. The region $(1,2,3,4,5)$ is a solid figure bounded (enclosed) by the 20 hyperplanes

$$(8) \qquad\qquad\qquad d_{ij} = \pm q_5$$

where $i < j$, i=1,2,3,4, j=2,3,4,5, and $q_5$ is the least
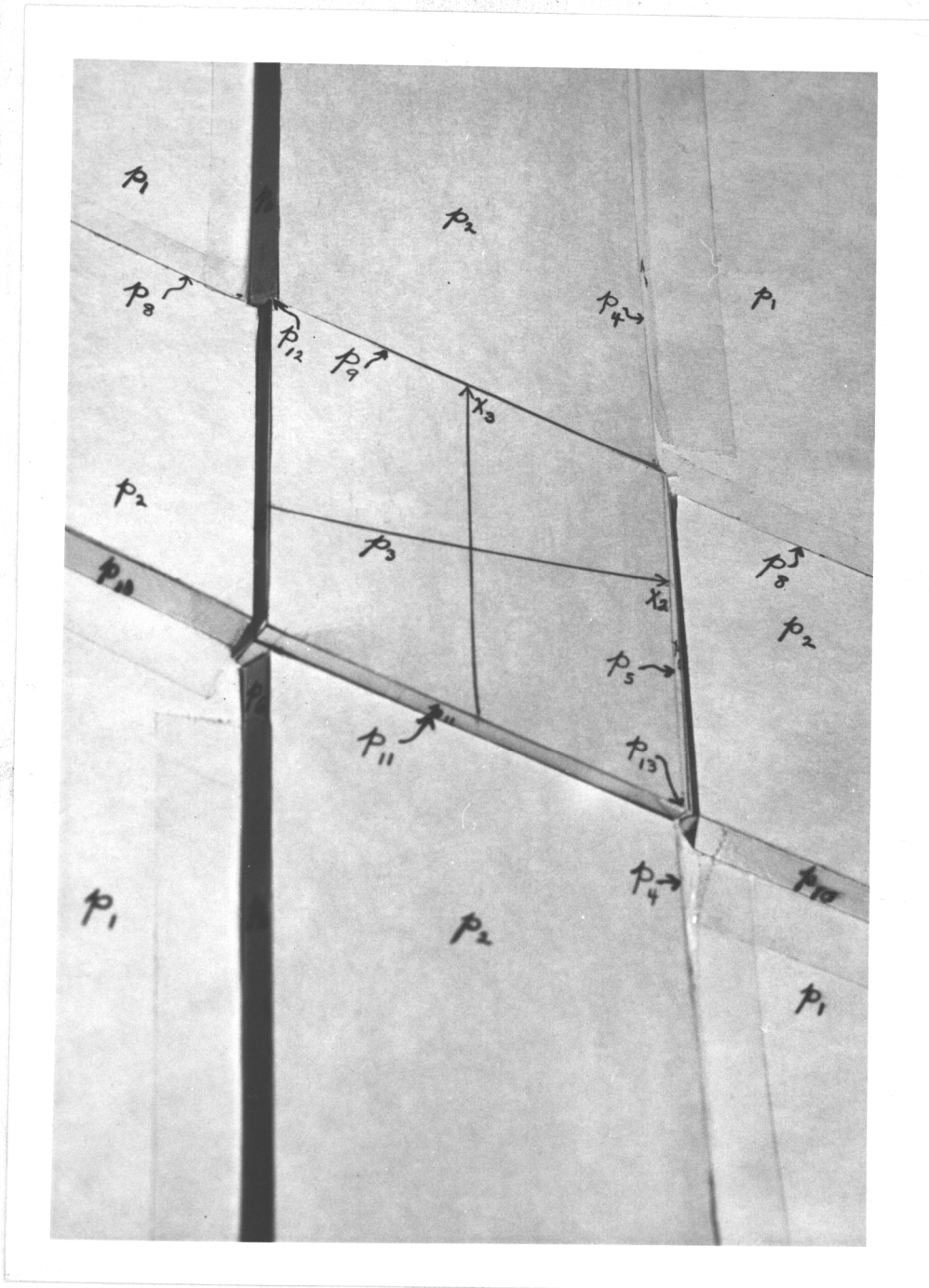
Figure 5

Boundary of Solid Region (2,1).

significant range for a sample of five. Each of the twenty
faces of ($\underline{1,2,3,4,5}$) is a six sided section of one of the
hyperplanes in (8). Any pair of these faces, say the pair
determined from $d_{13}=\pm q_5$, is symmetric about the origin, and
the plane $d_{13}=0$. The region ($\underline{1,2,3,4,5}$) is symmetric about
the origin and each of the ten hyperplanes $d_{ij}=0$, where $i < j$
and $i,j=1,2,3,4,5$.

The region ($\underline{1,2,3,4}$) is a solid figure which is the sum
of two overlapping solids, namely

(9)          ($\underline{1,2,3,4}$) = ($\underline{1,2,3,4,5}$) + ($\overline{1,2,3,4},5$),

where ($\overline{1,2,3,4},5$) is bounded by the twelve hyperplanes

(10)                  $d_{ij} = \pm q_4$,

with $i < j$ and $i,j=1,2,3,4$. The solid bounded by the planes
of (10) is symmetric about the origin and each of the six
planes

(11)                  $d_{ij} = 0$ ,

where $i < j$ and $i,j=1,2,3,4$. Thus the solid region ($\underline{1,2,3,4}$)
is symmetric about the origin and each of the six planes in
(11), since both of the solids of the sum are symmetric about
the origin and these same six planes. The regions ($\underline{1,2,3,5}$)
and ($\underline{1,2,4,5}$) are similarly defined. Their shape and size
are exactly the same as for the region ($\underline{1,2,3,4}$); only the
positions of these three regions differ.

The region ($\underline{1,2,3}$) is a solid figure which is the sum
of three overlapping solids, namely

(12) $(\underline{1,2,3}) = (\underline{1,2,3,4}) + (\underline{1,2,3,5}) + (\overline{1,2,3},4,5)$,

or the four overlapping solids given by

(13) $(\underline{1,2,3}) = (\underline{1,2,3,4},5) + (\overline{1,2,3,4},5) + (\overline{1,2,3,5},4)$

$$+ (\overline{1,2,3},4,5),$$

where $(\overline{1,2,3},4,5)$ is bounded by the six hyperplanes

(14) $\qquad d_{12} = \pm\, q_3, \quad d_{13} = \pm\, q_3, \text{ and } d_{23} = \pm\, q_3,$

and $(\overline{1,2,3,5},4)$ is defined by a method similar to that of (10). The solid bounded by the planes of (14) is symmetric about the origin and each of the three planes

(15) $\qquad d_{12} = 0, \quad d_{13} = 0, \text{ and } d_{23} = 0.$

Thus, the region $(\underline{1,2,3})$ is symmetric about the origin and each of the three planes in (15), since each of the three solids in the sum is symmetric about the origin and these same three planes. The regions $(\underline{1,2,4})$ and $(\underline{1,2,5})$ are similarly defined. Their shape and size are exactly the same as for the region $(\underline{1,2,3})$, but their positions are different.

The region $(\underline{1,2})$ is a solid figure which is the sum of four overlapping solids, namely

(16) $(\underline{1,2}) = (\underline{1,2,3}) + (\underline{1,2,4}) + (\underline{1,2,5}) + (\overline{1,2},3,4,5)$ ,

where $(\overline{1,2},3,4,5)$ is bounded by the two hyperplanes

(17) $\qquad\qquad\qquad d_{12} = \pm\, q_2.$

The solid bounded by the planes of (17) is symmetric about the origin and the plane

(18) $\qquad\qquad\qquad d_{12} = 0$ ,

since each of the four solids in the sum is symmetric about the origin and the plane $d_{12}=0$. Each of the ten regions $(\underline{i,j})$, where $i < j$ and $i,j=1,2,3,4,5$, is defined in this way. The region $(\underline{i,j})$ is symmetric about the origin and the plane $d_{ij}=0$.

The complement of the region $(\underline{i,j})$ is the sum of two disjoint regions, $(i,j)$ and $(j,i)$. In the region $(i,j)$, $d_{ij}$ is always negative, and in region $(j,i)$, $d_{ij}$ is always positive. Thus in the region $(2,1)$, $d_{12}$ is always positive.

For the case of n means, the arguments are similar. The region $(\underline{1,2,\ldots,n})$ is enclosed by the $n(n-1)$ hyperplanes

$$(19) \qquad\qquad d_{ij} = \pm\, q_n ,$$

where $i < j$, $i=1,2,\ldots,n-1$, $j=2,3,\ldots,n$, and $q_n$ is the least significant range for a sample of n. The region $(\underline{1,2,\ldots,n})$ is symmetric about each of the $n(n-1)/2$ hyperplanes

$$(20) \qquad\qquad d_{ij} = 0 ,$$

where i and j are defined as in (19).

Let $(1,2,k_1,k_2,\ldots,k_{n-e-2})$ denote the region for the decision that no pair of the n-e, $e=1,\ldots,n-2$, means involved are significantly different, where $k_1,k_2,\ldots,k_{n-e-2}$ is any selection of n-e-2 means from among the n-2 means $m_2,m_3,\ldots,m_n$. Thus each of the n-2 regions $(1,2,k_1,\ldots,k_{n-3})$ is the sum of two overlapping solids, namely

$$(21) \quad \underline{(1,2,k_1,\ldots,k_{n-3})} = \underline{(1,2,3,\ldots,n)}+(\overline{1,2,k_1,\ldots,k_{n-3}},k_{n-2}),$$

where $(\overline{1,2,k_1,\ldots,k_{n-3}},k_{n-2})$ is bounded by $(n-1)(n-2)$ hyperplanes of the form

(22) $$d_{ij} = \pm\, q_{n-1} \, ,$$

with $j \neq k_{n-2}$, and is symmetric about each of the $((n-1)(n-2))/2$ corresponding hyperplanes $d_{ij}=0$. Each of the $((n-2)(n-3))/2$ regions $(1,2,k_1,\ldots,k_{n-4})$ is the sum of three overlapping solids, namely

(23) $(1,2,k_1,\ldots,k_{n-4}) = (1,2,k_1,\ldots,k_{n-4},k_{n-3})$
$+\ (1,2,k_1,\ldots,k_{n-4},k_{n-2})$
$+\ (\overline{1,2,k_1,\ldots,k_{n-4}},k_{n-3},k_{n-2}),$

where $(\overline{1,2,k,\ldots,k_{n-4}},k_{n-3},k_{n-2})$ is bounded by $(n-2)(n-3)$ hyperplanes of the form

(24) $$d_{ij} = \pm\, q_{n-2} \, ,$$

with $j \neq k_{n-3}, k_{n-2}$, and is symmetric about each of the $((n-2)(n-3))/2$ corresponding hyperplanes $d_{ij}=0$. In general, each of the $((n-2)(n-3)\ldots(n-e-1))/e!$ regions $(1,2,k_1,\ldots,k_{n-e-2})$ is the sum of $e+1$ overlapping solids, namely

(25) $(1,2,k_1,\ldots,k_{n-e-2}) = (1,2,k_1,\ldots,k_{n-e-2},k_{n-e-1})$
$+\ (1,2,k_1,\ldots,k_{n-e-2},k_{n-e}) + \cdots$
$+\ (1,2,k_1,\ldots,k_{n-e-2},k_{n-2})$
$+\ (\overline{1,2,k_1,\ldots,k_{n-e-2}},k_{n-e-1},\ldots,k_{n-2}),$

where $(\overline{1,2,k_1,\ldots,k_{n-e-2}},k_{n-e-1},\ldots,k_{n-2})$ is bounded by $(n-e)(n-e-1)$ hyperplanes of the form

(26) $$d_{ij} = \pm\, q_{n-1},$$

with $j \neq k_{n-e-1},\ldots,k_{n-2}$, and is symmetric about each of the $((n-e)(n-e-1))/2$ corresponding hyperplanes $d_{ij}=0$.

## 3. Derivation of Power Functions.

Since decision region (2,1) is defined for the three multiple range tests under consideration, we may write directly, in integral form, the expression for $P_{21}(\xi)$. $P_{21}(\xi)$ is the power function for the test of the hypothesis $\mu_1 \leq \mu_2$ against the alternative hypothesis $\mu_1 > \mu_2$. Assuming that $\sigma_m = 1$ in the remainder of this paper, the power function will be denoted by $P_{21}(\xi)$, $P[D(2,1)| \xi ]$, or $P[D(2,1)| \xi_1, \ldots, \xi_{n-1} ]$.

Since $m_i$ is normally distributed with true mean $\mu_i$ and variance $\sigma_m^2 = 1$, $x_i$ is normally distributed with true mean $\xi_i$ and variance $\sigma_x^2 = 2$, where $i = 1, 2, \ldots, n-1$. Thus, we let

$$(27) \qquad g_i = g_i(x_i; \xi_i, 2) = \frac{1}{\sqrt{2\pi} \cdot \sqrt{2}} \, e^{-\frac{1}{2} \left( \frac{x_i - \xi_i}{\sqrt{2}} \right)^2} .$$

The $x_i$ are independently distributed. Thus, the joint distribution, $g(x;\xi)$, of $x_1$, $x_2$, ..., and $x_{n-1}$ may be written as the product of the n-1 normal distributions $g_i$, namely

$$(28) \qquad g(x;\xi) \equiv g = g_1 \cdot g_2 \cdots g_{n-1} .$$

For the two mean case, $P_{21}(\xi)$ is the same as the power for the usual one sided normal deviate test for the difference between two means. Thus,

$$(29) \qquad P[D(2,1)| \xi_1] = \int_{q_2}^{\infty} g_1 \, dx_1 ,$$

where $q_2$ is the least significant range for two means.

For the three mean case, see Figure 1 and boundary equations (1). The power function for D(2,1) may be written as

the algebraic sum of two double integrals, that is,

$$(30) \quad P[D(2,1)| \; \xi_1, \xi_2 \,] = \int_{q_2}^{\infty} \int_{-\infty}^{\infty} g \; dx_2 dx_1 \; - \; \int_{q_2}^{q_3} \int_{x_{22}}^{x_{21}} g \; dx_2 dx_1$$

where

$$(31) \qquad \begin{aligned} x_{21} &= (2q_3 - x_1)/\sqrt{3} \text{ and} \\ x_{22} &= (-2q_3 + x_1)/\sqrt{3} = -x_{21} \end{aligned}$$

are obtained by solving for $x_2$ in equations three and four of
(1), respectively. The second subscript is added to distin-
guish between the boundary equations. It should be noted that
the region of integration for the double integral subtracted
in the right member of (30) is that part of the multiple range
region (1,2,3) for which $x_1 > q_2$.

For the four mean case, see Figure 5 and boundary equa-
tions (7). The region of integration for the power function
$P[D(2,1)| \; \xi_1, \xi_2, \xi_3 \,]$ is (2,1). Let $R$ denote the region (set)
of all points for which $x_1 > q_2$. Clearly, $R$ contains (2,1) as
well as all those points in (1,2,3,4), (1,2,3), and (1,2,4)
for which $x_1 > q_2$. If we let

$$(32) \qquad \begin{aligned} R_1 &= R \cdot (1,2,3,4) \; , \\ R_2 &= R \cdot \left[ (1,2,3) - (1,2,3,4) \right] = R \cdot (1,2,3) - R_1, \text{ and} \\ R_3 &= R \cdot \left[ (1,2,4) - (1,2,3) \right] = R \cdot (1,2,4) - (R_1 + R_2) \end{aligned}$$

then we may write

$$(33) \qquad R = (2,1) + R_1 + R_2 + R_3 \; ,$$

where (2,1), $R_1$, $R_2$, and $R_3$ are disjoint sets. Further, since
$R_2$ is the union of two disjoint sets, say $R_2'$ and $R_2''$, and $R_3$ is

the union of two disjoint sets, say $R_3'$ and $R_3''$, (33) may be written as

(34) $\qquad R = (2,1) + R_1 + R_2' + R_2'' + R_3' + R_3''$ .

Hence, the power function for the decision that $m_2$ is significantly less than $m_1$ may be written as

$$P_{21}(\xi) = \int\!\!\int\!\!\int_R g\ dx - \int\!\!\int\!\!\int_{R_1} g\ dx - \int\!\!\int\!\!\int_{R_2'} g\ dx$$

$$- \int\!\!\int\!\!\int_{R_2''} g\ dx - \int\!\!\int\!\!\int_{R_3'} g\ dx - \int\!\!\int\!\!\int_{R_3''} g\ dx,$$

where $dx = dx_3 dx_2 dx_1$. In order to evaluate (35), the five integrals on the right hand side over regions $R_1$, $R_2'$, $R_2''$, $R_3'$, and $R_3''$ should be broken into the algebraic sum of 3, 2, 2, 3, and 3 triple integrals, respectively. In each case, only one of these integrals contributes much to $P_{21}(\xi)$. The other eight triple integral terms are pooled in a remainder term, $r$, whose maximum value is approximated in Appendix B. Hence, a more useful form of the power function in the four mean case is given by

$$P[D(2,1)|\ \xi_1,\ \xi_2,\ \xi_3]$$

$$= \int_{q_2}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g\ dx - \int_{q_2}^{q_4}\int_{x_{27}}^{x_{25}}\int_{x_{3,11}}^{x_{39}} g\ dx$$

(36)

$$- \int_{q_2}^{q_3}\int_{x_{26}}^{x_{24}}\int_{-\infty}^{x_{3,11}} g\ dx - \int_{q_2}^{q_3}\int_{x_{26}}^{x_{24}}\int_{x_{39}}^{\infty} g\ dx$$

$$-\int_{q_2}^{q_3}\int_{-\infty}^{x_{27}}\int_{x_{3,10}}^{x_{38}} g\,dx - \int_{q_2}^{q_3}\int_{x_{25}}^{\infty}\int_{x_{3,10}}^{x_{38}} g\,dx + r,$$

where $x_{24}$, $x_{25}$, $x_{26}$, and $x_{27}$ are obtained by solving for $x_2$ in equations $p_4$, $p_5$, $p_6$, and $p_7$ of (7), respectively, and $x_{38}$, $x_{39}$, $x_{3,10}$, and $x_{3,11}$ are obtained by solving for $x_3$ in equations $p_8$, $p_9$, $p_{10}$, and $p_{11}$, respectively. (The second subscript denotes the number of the boundary equation in (7).) The remainder term is given by

$$(37) \qquad r = \iiint_{\text{region } A} g\,dx + \iiint_{\text{region } B} g\,dx ,$$

where region A is bounded by the six planes $p_1$, $p_4$, $p_5$, $p_{10}$, $p_{11}$, and $p_{13}$ and region B is bounded by the six planes $p_1$, $p_6$, $p_7$, $p_8$, $p_9$, and $p_{12}$.

For more than four means, the method used in writing the power function $P_{21}(\xi)$ is similar to that explained in the four mean case. To illustrate further the general procedure, consider the case of five means. Let R denote the set of all points for which $x_1 > q_2$. Clearly, R contains points of decision regions (1,2,3,4,5), (1,2,3,4), (1,2,3,5), (1,2,4,5), (1,2,3), (1,2,4), (1,2,5), and (2,1) which are defined by (8), (9), (12), and (16) or the method illustrated by those equations. Letting

$$R_1 = R \cdot (\underline{1,2,3,4,5}),$$
$$R_2 = R \cdot (\underline{1,2,3,4}) - R_1,$$
$$R_3 = R \cdot (\underline{1,2,3,5}) - (R_1 + R_2),$$

(38)   $R_4 = R \cdot (\underline{1,2,4,5}) - (R_1 + R_2 + R_3)$ ,

   $R_5 = R \cdot (\underline{1,2,3}) - (R_1 + \ldots + R_4)$ ,

   $R_6 = R \cdot (\underline{1,2,4}) - (R_1 + \ldots + R_5)$ , and

   $R_7 = R \cdot (\underline{1,2,5}) - (R_1 + \ldots + R_6)$ ,

we may write

(39)          $R = (2,1) + \sum_{s=1}^{7} R_s$ ,

where the sets on the right hand side of (39) are disjoint.
Further, since each $R_s$, where $s=2,\ldots,7$, is the union of two
disjoint sets, say $R_s'$ and $R_s''$, (39) may be written as

(40)          $(2,1) = R - (R_1 + \sum_{s=2}^{6} (R_s' + R_s''))$.

Hence the power function for five means may be written as

(41) $P[D(2,1)|\zeta_1, \zeta_2, \zeta_3, \zeta_4] = I_4(R) - I_4(R_1) - \sum_{s=2}^{7} (I_4(R_s') + I_4(R_s''))$,

where $I_4(R_k)$ denotes the fourfold integral over region $R_k$;
$R_k$ being a region on the right hand side of (40).  For com-
putational purposes the right hand side of (41) could be
written in a form similar to the right hand side of (36),
where the regions of integration in the remainder term would
be of two types.  One type region would be bounded, and the
other type would not be bounded in one direction.

## VII.  NOTATION AND A THEOREM ON INTEGRATION

### 1.  Notation and Rules for its Use.

To facilitate handling the integrals in the expression of $P_{21}(\xi)$, it is desirable that a more compact notation be introduced.  For this purpose, let

$$(1) \quad \int_{x_{11}}^{x_{12}} \int_{x_{21}}^{x_{22}} \cdots \int_{x_{n-1,1}}^{x_{n-1,2}} f\, dx = x \left\{ \begin{matrix} 12 & 22 & \cdots & n-1,2 \\ 11 & 21 & \cdots & n-1,1 \ R \end{matrix} \right\}$$

where $f = f(x_1, x_2, \ldots, x_{n-1})$ is a density function, R is the correlation matrix of f and the second subscript on $x_{ik}$, $k=1,2$, is used to distinguish between the lower and upper limits.

In much of the work, the density function is $g(x;\xi)$ as defined in (28) of VI, where the $x_i$ are independent and the correlations are zero.  In this case, write

$$(2) \quad \int_{x_{11}}^{x_{12}} \int_{x_{21}}^{x_{22}} \cdots \int_{x_{n-1,1}}^{x_{n-1,2}} g\, dx = x \left\{ \begin{matrix} 12 & 22 & \cdots & n-1,2 \\ 11 & 21 & \cdots & n-1,1 \end{matrix} \right\} .$$

Some rules for using the symbol $\{ \}$ are now observed.
Since the left hand side of (2) may be written as

$$\int_{x_{11}}^{x_{12}} g_1 \left\{ \int_{x_{21}}^{x_{22}} g_2 \cdots \left( \int_{x_{n-1,1}}^{x_{n-1,2}} g_{n-1}\, dx_{n-1} \right) \cdots dx_2 \right\} dx_1$$

$$= \int_{x_{11}}^{x_{12}} g_1 \left\{ \int_{x_{21}}^{x_{22}} g_2 \cdots \left( \int_{x_{n-1,1}}^{\infty} g_{n-1}\, dx_{n-1} - \int_{x_{n-1,2}}^{\infty} g_{n-1}\, dx_{n-1} \right) \cdots dx_2 \right\} dx_1 \quad ,$$

$$(3) \quad x \begin{Bmatrix} 12 & 22 & \ldots & n{-}1,2 \\ 11 & 21 & \ldots & n{-}1,1 \end{Bmatrix} = x \begin{Bmatrix} 12 & 22 & \ldots & \infty \\ 11 & 21 & \ldots & n{-}1,1 \end{Bmatrix} - x \begin{Bmatrix} 12 & 22 & \ldots & \infty \\ 11 & 21 & \ldots & n{-}1,2 \end{Bmatrix} .$$

For n=3,

$$x \begin{Bmatrix} 12 & 22 \\ 11 & 21 \end{Bmatrix} = x \begin{Bmatrix} 12 & \infty \\ 11 & 21 \end{Bmatrix} - x \begin{Bmatrix} 12 & \infty \\ 11 & 22 \end{Bmatrix}$$

$$= \int_{x_{11}}^{x_{12}} g_1 \left( \int_{x_{21}}^{\infty} g_2 \, dx_2 \right) dx_1 - \int_{x_{11}}^{x_{12}} g_1 \left( \int_{x_{22}}^{\infty} g_2 \, dx_2 \right) dx_1$$

$$= \int_{x_{11}}^{\infty} g_1 \left( \int_{x_{21}}^{\infty} g_2 \, dx_2 \right) dx_1 - \int_{x_{12}}^{\infty} g_1 \left( \int_{x_{21}}^{\infty} g_2 \, dx_2 \right) dx_1$$

$$- \int_{x_{11}}^{\infty} g_1 \left( \int_{x_{22}}^{\infty} g_2 \, dx_2 \right) dx_1 + \int_{x_{12}}^{\infty} g_1 \left( \int_{x_{22}}^{\infty} g_2 \, dx_2 \right) dx_1 ,$$

or

$$(4) \quad x \begin{Bmatrix} 12 & 22 \\ 11 & 21 \end{Bmatrix} = x \begin{Bmatrix} \infty & \infty \\ 11 & 21 \end{Bmatrix} = x \begin{Bmatrix} \infty & \infty \\ 12 & 21 \end{Bmatrix} = x \begin{Bmatrix} \infty & \infty \\ 11 & 22 \end{Bmatrix} + x \begin{Bmatrix} \infty & \infty \\ 12 & 22 \end{Bmatrix} .$$

Similarly, for n=4,

$$x \begin{Bmatrix} 12 & 22 & 32 \\ 11 & 21 & 31 \end{Bmatrix} = x \begin{Bmatrix} \infty & \infty & \infty \\ 11 & 21 & 31 \end{Bmatrix} + x \begin{Bmatrix} \infty & \infty & \infty \\ 12 & 22 & 31 \end{Bmatrix} + x \begin{Bmatrix} \infty & \infty & \infty \\ 12 & 21 & 32 \end{Bmatrix}$$

$$+ x \begin{Bmatrix} \infty & \infty & \infty \\ 11 & 22 & 32 \end{Bmatrix} - x \begin{Bmatrix} \infty & \infty & \infty \\ 12 & 21 & 31 \end{Bmatrix} - x \begin{Bmatrix} \infty & \infty & \infty \\ 11 & 22 & 31 \end{Bmatrix}$$

$$- x \begin{Bmatrix} \infty & \infty & \infty \\ 11 & 21 & 32 \end{Bmatrix} - x \begin{Bmatrix} \infty & \infty & \infty \\ 12 & 22 & 32 \end{Bmatrix} .$$

Thus, for n=i+1, where i=1,2,...,n-1, we may write $x \begin{Bmatrix} 12 & 22 \ldots i2 \\ 11 & 21 \ldots i1 \end{Bmatrix}$

as the algebraic sum of $2^i$ terms with the upper limits all $\infty$. This is done by taking all possible combinations of i objects where each object may have one of two subscripts, 1 or 2. The term is prefixed by a plus sign if an even number of the i objects have the second subscript 2, and by a minus sign if an odd number of the i objects have the second subscript 2.

When the upper limits are all $\infty$, let

$$(6) \quad L_x(1k, 2k, \ldots, n-1,k; R) = \left\{ \begin{matrix} \infty & \infty & \ldots & \infty \\ x \begin{bmatrix} 1k & 2k & \ldots & n-1,k & R \end{bmatrix} \end{matrix} \right\},$$

where $k=1$ or 2. In case $x_1$, $x_2$, $\ldots$, $x_{n-1}$ are independent, $R=0$, and we write $L_x(1k, 2k, \ldots, n-1,k)$.

Thus, (5) may be written as

$$x \begin{matrix} 12 & 22 & 32 \\ 11 & 21 & 31 \end{matrix} = L_x(11,21,31) + L_x(12,22,31) + L_x(12,21,32)$$

$$(7) \qquad + L_x(11,22,32) - L_x(12,21,31) - L_x(11,22,31)$$

$$- L_x(11,21,32) - L_x(12,22,32) \ .$$

If the last t lower limits are $-\infty$, where t is any positive integer less than n-1, then

$$(8) \quad L_x(1k, \ldots, n-1-t,k, n-t,k, \ldots, n-1,k)$$

$$= L_x(1k, \ldots, n-1-t,k) \ .$$

We have $L_x(1k, \ldots, n-1-t,k, -\infty, \ldots, -\infty)$

$$= \int_{x_{1k}}^{\infty} \ldots \int_{x_{n-1-t,k}}^{\infty} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} g \, dx$$

$$= \int_{x_{1k}}^{\infty} g_1 \left( \cdots \int_{x_{n-1-t,k}}^{\infty} g_{n-1-t} \left[ \int_{-\infty}^{\infty} g_{n-t} \left( \cdots \int_{-\infty}^{\infty} g_{n-1} dx_{n-1} \cdots \right) \right. \right.$$

$$\left. \left. dx_{n-t} \right] dx_{n-1-t} \cdots \right) dx_1$$

$$= \int_{x_{1k}}^{\infty} g_1 \left( \cdots \int_{x_{n-1-t,k}}^{\infty} g_{n-1-t} \left[ \int_{-\infty}^{\infty} g_{n-t} dx_{n-t} \cdots \int_{-\infty}^{\infty} g_{n-1} dx_{n-1} \right] \right.$$

$$\left. dx_{n-1-t} \cdots \right) dx_1$$

$$= \int_{x_{1k}}^{\infty} g_1 \left( \cdots \int_{x_{n-1-t,k}}^{\infty} g_{n-1-t} \left[ 1 \cdots 1 \right] dx_{n-1-t} \cdots \right) dx_1$$

$$= \int_{x_{1k}}^{\infty} \cdots \int_{x_{n-1-t,k}}^{\infty} g_1 \cdots g_{n-1-t} \, dx_{n-1-t} \cdots dx_1$$

$$= L_x(1k, \ldots, n-1-t,k).$$

If any lower limit is $\infty$, then

(9) $\qquad L_x(1k, \ldots, n-1,k) = 0.$

Formulas (29), (30), and (36) of VI may be expressed in the L notation. Using the notation of formulas (29), (30), and (36) and the properties of this section, we write

(10) $\quad P[D(2,1)| \xi_1] = L_x(q_2)$ ,

$$P[D(2,1)| \xi_1, \xi_2] = {}_x \left\{ \begin{matrix} \infty & \infty \\ q_2 & -\infty \end{matrix} \right\} - {}_x \left\{ \begin{matrix} q_3 & 21 \\ q_2 & 22 \end{matrix} \right\}$$

$$= L_x(q_2, -\infty) - [ L_x(q_2, 22) + L_x(q_3, 21)$$

$$- L_x(q_2, 21) - L_x(q_3, 22)], \text{ or}$$

(11) $P[D(2,1)| \xi_1, \xi_2] = L_x(q_2) + L_x(q_2, 21) + L_x(q_3, 22)$

$$- L_x(q_2, 22) - L_x(q_3, 21),$$

and

$$P[D(2,1)| \xi_1, \xi_2, \xi_3] = x\begin{Bmatrix} \infty & \infty & \infty \\ q_2 & -\infty & -\infty \end{Bmatrix} - x\begin{Bmatrix} q_4 & 25 & 39 \\ q_2 & 27 & 3,11 \end{Bmatrix}$$

$$- x\begin{Bmatrix} q_3 & 27 & 38 \\ q_2 & -\infty & 3,10 \end{Bmatrix} - x\begin{Bmatrix} q_3 & \infty & 38 \\ q_2 & 25 & 3,10 \end{Bmatrix}$$

$$- x\begin{Bmatrix} q_3 & 24 & 3,11 \\ q_2 & 26 & -\infty \end{Bmatrix} - x\begin{Bmatrix} q_3 & 24 & \infty \\ q_2 & 26 & 39 \end{Bmatrix} + r$$

reduces to

$P[D(2,1)| \xi_1, \xi_2, \xi_3] = L_x(q_2) - L_x(q_2, 27, 3,11)$

$\quad + L_x(q_2, 27, 39) + L_x(q_2, 25, 3,11) - L_x(q_2, 25, 39)$

$\quad + L_x(q_4, 27, 3,11) - L_x(q_4, 27, 39) - L_x(q_4, 25, 3,11)$

$\quad + L_x(q_4, 25, 39) - L_x(q_2, -\infty, 3,10) + L_x(q_3, -\infty, 3,10)$

$\quad + L_x(q_2, 27, 3,10) - L_x(q_3, 27, 3,10) + L_x(q_2, -\infty, 38)$

(12) $\quad - L_x(q_3, -\infty, 38) - L_x(q_2, 27, 38) + L_x(q_3, 27, 38)$

$\quad - L_x(q_2, 25, 3,10) + L_x(q_3, 25, 3,10) + L_x(q_2, 25, 38)$

$\quad - L_x(q_3, 25, 38) - L_x(q_2, 26) - L_x(q_3, 24) + L_x(q_2, 24)$

$\quad + L_x(q_3, 26) + L_x(q_2, 26, 3,11) + L_x(q_3, 24, 3,11)$

$\quad - L_x(q_2, 24, 3,11) - L_x(q_3, 26, 3,11) - L_x(q_2, 26, 39)$

$\quad + L_x(q_3, 26, 39) + L_x(q_2, 24, 39) - L_x(q_3, 24, 39) + r .$

## 2. A Transformation Theorem.

In this paper $x_{1k}$ is constant, $x_{2k}$ a function of $x_1$, $x_{3k}$ a function of $x_1$ and $x_2$, and, in general, $x_{ik}$ a function of $x_1, x_2, \ldots, x_{i-1}$. For reasons which will be made clear later, it is necessary to transform L, which is an expression with variable

lower limits and correlation coefficients zero, to a function
with constant lower limits and at least one non-zero correla-
tion coefficient. With this in mind the important theorem is
given below.

Theorem. Let

(13) $\qquad f_j(x) = f_j(x_1,\ldots,x_{n-1}) = \sum_{k=1}^{j} a_{jk}x_k,$

where $j=1,2,\ldots,n-1$ and $a_{jj}\neq 0$. Let

(14) $\quad L_x = L_x(1v, \ldots, n-1,v) = \int_{x_{1v}}^{\infty} \ldots \int_{x_{n-1,v}}^{\infty} g\,dx,$

where $x_{jv}$ is the function of $x_1,\ldots,x_{j-1}$ obtained by solving

(15) $\qquad f_j(x) = b_j$

for $x_j$ and v is a positive integer denoting the equation
used (v suggests a variable lower limit). Let

(16) $\qquad y_j = (f_j(x) - f_j(\zeta))/(\sqrt{2}D_j)\ ,$

where

(17) $\qquad D_j = \left(\sum_{k=1}^{j} a_{jk}^2\right)^{\frac{1}{2}}\ .$

The integral in (14) may be written as

(18) $\quad L_y = L_y(1c, \ldots, n-1,c;\ R) = \int_{y_{1c}}^{\infty} \ldots \int_{y_{n-1,c}}^{\infty} g(y;R)dy\ ,$

where

(19) $\qquad y_{jc} = (b_j - f_j(\zeta))/(\sqrt{2}D_j)$

and $dy = \prod_{j=1}^{n-1} dy_j$. $g(y;R) = g(y_1,\ldots,y_{n-1};R)$ is a multivari-
ate normal density function with correlation coefficients

given by

(20) $\qquad \rho_{ij}(y) = (\sum_{k=1}^{i} a_{ik}a_{jk})/(D_i D_j)$ ,

where $i < j$.

Proof. Clearly, $g(y;R)$ is a multivariate normal density function since $g$ is a multivariate normal density function and (16) is a non-singular linear transformation. The correlation coefficients given by (20) are obtained by the usual well known method.

The limits of integration as given by (19) are obtained in two steps. First, let

(21) $\qquad t_j = (x_j - \xi_j)/\sqrt{2}$ ,

such that the right member of (14) is transformed to

(22) $\qquad \int_{t_{1v}}^{\infty} \cdots \int_{t_{n-1,v}}^{\infty} g(t)\, dt,$

where $dt = \prod_{k=1}^{n-1} dt_k$,

(23) $\qquad g(t) = \prod_{k=1}^{n-1} g_k(t_k;0,1)$, and

$t_{jv}$ is obtained by solving the j<u>th</u> transformed equation of (15) for $t_j$ in terms of $t_1, \ldots, t_{j-1}$. That is

$\sum_{k=1}^{j} a_{jk}(\sqrt{2}t_k + \xi_k) = b_j$ implies

(24) $\qquad \sum_{k=1}^{j} a_{jk}t_k = (b_j - \sum_{k=1}^{j} a_{jk}\xi_k)/\sqrt{2}$ ,

from which we obtain

$$(25) \qquad t_{jv} = \left[ -\sum_{k=1}^{j-1} a_{jk}t_k + (b_j - \sum_{k=1}^{j} a_{jk}\xi_k)/\sqrt{2} \right] \Big/ a_{jj} .$$

The second step is to let

$$(26) \qquad y_j = \sum_{k=1}^{j} a_{jk}t_k/D_j .$$

Substituting (24) in (26) gives

$$(27) \qquad y_{jc} = (b_j - \sum_{k=1}^{j} a_{jk}\xi_k)/(\sqrt{2}D_j)$$

which is the lower limit on the integral for $y_j$. The upper limits all remain infinite. Thus equation (18) holds.

This theorem is now used to transform (10), (11), and (12) such that the lower limits are constant when $\xi$ is fixed. Thus, (10) may be written as

$$(28) \qquad P[D(2,1)| \xi_1] = L_t[(q - \xi_1)/\sqrt{2}] .$$

Consider $L_x(q_2,21)$ in (11). The bounding equations are $x_1 = q_2$ and $x_1 + \sqrt{3}x_2 = 2q_3$. Hence, $D_1 = 1$, and $D_2 = 2$. Therefore, $\rho = (1 \cdot 1)/(1 \cdot 2) = 1/2$. The limits are $y_{1c} = (q_2 - \xi_1)/(\sqrt{2} \cdot 1)$ and $y_{2c} = (2q_3 - \xi_1 - \sqrt{3}\xi_2)/\sqrt{2} \cdot 2$. Thus $L_x(q_2,21) = L_y(1c,2c;\frac{1}{2})$. The other terms in (11) may be transformed in a similar manner to give

$$(29) \quad P[D(2,1)| \xi_1, \xi_2] = L_t(11) + L_y(11,21;\tfrac{1}{2}) + L_y(11,22;-\tfrac{1}{2})$$
$$- L_y(12,22;-\tfrac{1}{2}) - L_y(12,21;\tfrac{1}{2}),$$

where

$$(30) \quad \begin{aligned} 11 &= (q_2 - \xi_1)/\sqrt{2} , \\ 12 &= (q_3 - \xi_1)/\sqrt{2} , \\ 21 &= (2q_3 - \xi_1 - \sqrt{3}\xi_2)/(2\sqrt{2}) , \text{ and} \\ 22 &= (-2q_3 + \xi_1 - \sqrt{3}\xi_2)/(2\sqrt{2}) . \end{aligned}$$

We use the theorem to transform one of the terms in (12) to illustrate the procedure. Consider $L_x(q_2, 27, 3, 11)$. The bounding equations are

$$
\begin{aligned}
1:\quad & x_1 = q_2 \\
2:\quad & -x_1 + \sqrt{3}x_2 = -2q_4, \text{ and} \\
3:\quad & 3x_1 + \sqrt{3}x_2 + 2\sqrt{6}x_3 = 6q_4.
\end{aligned}
$$

(31)

Hence, $D_1 = 1$, $D_2 = 2$, and $D_3 = 6$, so that $\rho_{12} = -\tfrac{1}{2}$, $\rho_{13} = \tfrac{1}{2}$, and $\rho_{23} = 0$. Thus

(32) $L_x(q_2, 27, 3, 11) = L_y(1c, 2c, 3c; R)$ ,

where

$$
\begin{aligned}
1c &= (q_2 - \xi_1)/\sqrt{2}\,, \\
2c &= (-2q_4 + \xi_1 - \sqrt{3}\,\xi_2)/2\sqrt{2}\,, \\
3c &= (6q_4 - 3\xi_1 - \sqrt{3}\,\xi_2 - 2\sqrt{6}\,\xi_3)/6\sqrt{2}\,, \text{ and} \\
R &= (\rho_{12},\ \rho_{13},\ \rho_{23}) = (-\tfrac{1}{2},\ \tfrac{1}{2},\ 0).
\end{aligned}
$$

(33)

This theorem is useful because it puts the multiple integral (14) in a form (18) which may be evaluated by a series expansion as given by Kendall (1941). He proves that

$$
(34)\quad \int_{h_1}^{\infty}\int_{h_2}^{\infty} g(y;R)\,dy = \left( \sum_{r=0}^{\infty} \rho^r g^{(r-1)}(h_1) g^{(r-1)}(h_2) \right) \Big/ r!
$$

and

$$
(35)\quad \int_{h_1}^{\infty}\int_{h_2}^{\infty}\int_{h_3}^{\infty} g(y;R)\,dy = \sum_{r=j+k+e=0}^{\infty} \frac{\rho_{12}^j \rho_{13}^k \rho_{23}^e}{(j!k!e!)} \cdot g^{(j+k-1)}(h_1) \cdot
$$
$$
g^{(j+e-1)}(h_2) \cdot g^{(k+e-1)}(h_3) \ ,
$$

where $h_1$, $h_2$, $h_3$ are any real constants, the symbol $\sum_{r=j+k+e=0}^{\infty}$

indicates a triple sum on j, k, and e over all non-negative integers, $g^{(s)}(t)$, s=1,2,..., denotes the s<u>th</u> derivative of $g(t;0,1)$, and

$$g^{(-1)}(t) = \int_t^\infty g(x;0,1)dx .$$

In order to write the general expression for the series expansion, let

(36) $$n_j = \sum_{\substack{k=1 \\ k \ne j}}^{m} a_{jk}$$

where j=1, ..., m, $a_{jk}=a_{kj}$, and $a_{jk}$ is a non-negative integer. Let $g^{(s)}(t)$ and $g^{(-1)}(t)$ be defined as above. Let $g(y;R)$ be the multivariate normal density function of the transformation theorem. Let $h_j$ be a real number. Then the general series expression may be written as

(37) $$\int_{h_1}^{\infty} \cdots \int_{h_m}^{\infty} g(y;R)dy = \sum_{r=0}^{\infty} \frac{\rho_{12}^{a_{12}} \rho_{13}^{a_{13}} \cdots \rho_{m-1,m}^{a_{m-1,m}}}{a_{12}! a_{13}! \cdots a_{m-1,m}!} g^{(n_1-1)}(h_1).$$
$$g^{(n_2-1)}(h_2) \cdots g^{(n_m-1)}(h_m),$$

where $r = a_{12} + a_{13} + \cdots + a_{m-1,m}.$

## VIII.   APPROXIMATION METHODS FOR MULTIPLE RANGE TESTS

Before considering the approximation methods, the extent to which exact methods may be used is discussed.   Obviously, for two means, the values of the power function (28) of VII may be read directly from the usual cumulative normal tables.   For three means, tables for the bivariate normal distribution function may be used to obtain (29) of VII.   However, some calculations involving two way interpolation are usually necessary, but they are straightforward and short.   In the case for four means, we must use the series expension, (35) in VII, on the transformed terms of (12) in VII.   This requires a lot of work, particularly since correlation coefficients of $\frac{1}{2}$ make the series converge slowly. For five or more means, (37) of VII could be used to obtain power, but the amount of time and labor required would be prohibitive.

Since the evaluation of power for four means involves long and tedious calculations, we consider finding an upper and lower bound to $P[D(2,1)| \xi_1, \xi_2, \xi_3]$ denoted by $P_U$ and $P_L$, respectively.   Before presenting the arguments and the expressions for the particular $P_L$ and $P_U$ used in the four mean case, it seems desirable that we look at the method as it is used on the three mean case.

The trouble in evaluating power comes from the fact that at least one bounding line or plane of the critical region

is not parallel to some axis or coordinate axes plane.  Thus,
in the three mean case, the bounding line segments AC and
FH in Figure 1 make the calculations difficult.  This dif-
ficulty is eliminated if we let segments between lines $x_1=q_2$
and $x_1=q_3$ and parallel to
the $x_1$ axis pass through
the ends of AC and FH.  Thus
we have two regions which
approximate the region (2,1).
The larger region bounded by
MD, DC, CF, FE, and EN is
the region for $P_U[(2,1)|\xi_1, \xi_2]$
and is denoted by $(2,1)_U$.
The smaller region bounded
by MA, AB, BG, GH, and HN is
the region for $P_L[(2,1)|\xi_1, \xi_2]$
and is denoted by $(2,1)_L$.
For any parameter point, $P_U$
and $P_L$ are easily calculated
as the sum of products of
values taken from a normal
distribution function table.
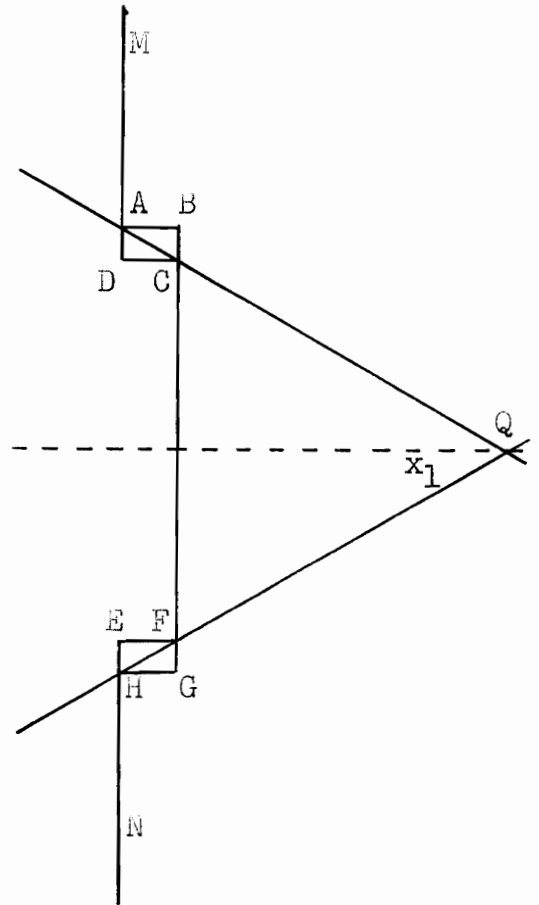
Figure 1

Regions $(2,1)_U$ and $(2,1)_L$

In order to see the relations among the probabilities
$P_L$, $P_{21}$, and $P_U$, let P [region ABC], P [region ACD],
P [region EFH], and P [region FGH] denote the probability of

a sample point falling in region ABC, ACD, EFH, and FGH, respectively. Clearly, $P_{21}$ must always have a value between $P_L$ and $P_U$ since we may write

(1)
$$P_L = P_{21} - PO \quad \text{and}$$
$$P_U = P_{21} + PI \quad ,$$

where

(2)
$$PO = P[\text{region ABC}] + P[\text{region FGH}] \quad \text{and}$$
$$PI = P[\text{region ADC}] + P[\text{region EFH}] \quad .$$

We may determine the relative size of PO and PI by dividing the sample-space into four parts with lines through AC and FH intersecting at Q. It follows that PO is greater than, equal to, or less than PI according as the parameter point falls in the region to the right of Q, at Q, or in the region to the left of Q. If the parameter point falls in the region above or below Q, PO is usually greater than PI. $P_L$ is closer to $P_{21}$ than $P_U$ whenever PO < PI, and the reverse is true whenever PI < PO.

Regardless of the location of the parameter point, $P_{21}$ does not deviate much from the average of $P_U$ and $P_L$. The error in using $(P_L + P_U)/2$ as an estimate of $P_{21}$ is less than $(P_U - P_L)/2$. The maximum error, .00059 as shown in Appendix B, is obtained when the parameter point falls in the center of one of the regions ABCD or EFGH.

Bounds for power in the four mean case are now given. Let $P_{U4}$ and $P_{L4}$ denote the upper and lower bound of

$P[D(2,1)|\xi_1, \xi_2, \xi_3]$, respectively.  Let $(2,1)_U$ be the criti-
cal region for $P_{U4}$ and $(2,1)_L$ be the critical region for
$P_{L4}$.  These regions are defined so that $(2,1)_L$ is contained
in $(2,1)$ and $(2,1)$ is contained $(2,1)_U$.  Furthermore, $(2,1)_U$
and $(2,1)_L$ are defined so that $P_U - P_L$ is small.

The regions $(2,1)_U$ and $(2,1)_L$ are defined as follows:
pass planes parallel to the $x_1$ axis through the edges of the
boundary of critical region $(2,1)$, that is, through the
lines of intersection formed when planes $p_4$, $p_5$, ..., $p_{11}$ of
(7) in VI cut the planes $x_1 = q_2$, $x_1 = q_3$, and $x_1 = q_4$.
These new bounding planes have equations which are obtained
by replacing $x_1$ with $q_2$, $q_3$, or $q_4$ in equations (7) in VI.
The region $(2,1)_L$ is bounded by planes $p_1$, $p_2$, ..., $p_{11}$ of
(7) in VI with $x_1$ replaced by $q_2$ in equation $p_4$, $p_5$, ..., $p_{11}$.
The region $(2,1)_U$ is bounded by planes $p_1$, $p_2$, ..., $p_{11}$ of
(7) in VI with $x_1$ replaced by $q_3$ in $p_4$, $p_6$, $p_8$, and $p_{10}$ and
$x_1$ replaced by $q_4$ in $p_5$, $p_7$, $p_9$, and $p_{11}$.  Thus the bounding
planes of $(2,1)_L$ are

$$
\begin{array}{lll}
p_1: x_1 = q_2 & p_2: x_1 = q_3, & p_3: x_1 = q_4, \\
p_4: \sqrt{3}x_2 = 2q_3 - q_2, & & p_5: \sqrt{3}x_2 = 2q_4 - q_2, \\
p_6: \sqrt{3}x_2 = -2q_3 + q_2, & & p_7: \sqrt{3}x_2 = -2q_4 + q_2,
\end{array}
$$

$$
\begin{aligned}
(3) \quad & p_8: \sqrt{3}x_2 + 2\sqrt{6}x_3 = 6q_3 - 3q_2, \\
& p_9: \sqrt{3}x_2 + 2\sqrt{6}x_3 = 6q_4 - 3q_2, \\
& p_{10}: \sqrt{3}x_2 + 2\sqrt{6}x_3 = -6q_3 + 3q_2, \text{ and} \\
& p_{11}: \sqrt{3}x_2 + 2\sqrt{6}x_3 = -6q_4 + 3q_2.
\end{aligned}
$$

The bounding planes of $(2,1)_U$ are

$$p_1 : x_1 = q_2, \qquad p_2 : x_1 = q_3, \qquad p_3 : x_1 = q_4,$$

$$p_4 : \sqrt{3}x_2 = q_3, \qquad\qquad\qquad p_5 : \sqrt{3}x_2 = q_4,$$

$$p_6 : \sqrt{3}x_2 = -q_3, \qquad\qquad\qquad p_7 : \sqrt{3}x_2 = -q_4,$$

$$(4) \quad p_8 : \sqrt{3}x_2 + 2\sqrt{6}x_3 = 3q_3,$$

$$p_9 : \sqrt{3}x_2 + 2\sqrt{6}x_3 = 3q_4,$$

$$p_{10} : \sqrt{3}x_2 + 2\sqrt{6}x_3 = -3q_3, \text{ and}$$

$$p_{11} : \sqrt{3}x_2 + 2\sqrt{6}x_3 = -3q_4.$$

It is apparent that region $(2,1)_L$ is equal to region $(2,1)$ minus twelve prism shaped regions, and region $(2,1)_U$ is equal to region $(2,1)$ plus twelve other prism shaped regions. The prisms are of two types, those which border the central region and those which extend from this central region infinitely far in one direction. Each prism subtracted from $(2,1)$ in obtaining $(2,1)_L$ has its largest face in common with the largest face of a prism added to $(2,1)$ in obtaining $(2,1)_U$. This common face is part of one of the planes $p_4$, $p_5$, ..., $p_{11}$ of (7) in VI. Together two such prisms form a cylinder shaped solid region with a right cross section which is rectangular. Cross sections of the rectangles illustrating the two type cylinders are shown in Figures 3 and 4 for the particular cylinders bounded by $p_1$, $p_3$, $p_4$, and $p_5$ of (3) and $p_1$, $p_2$, $p_4$, and $p_5$ of (4), respectively. Figure 3 illustrates a cross section of the cylinder bounding one side of the central region, and Figure 4 illustrates
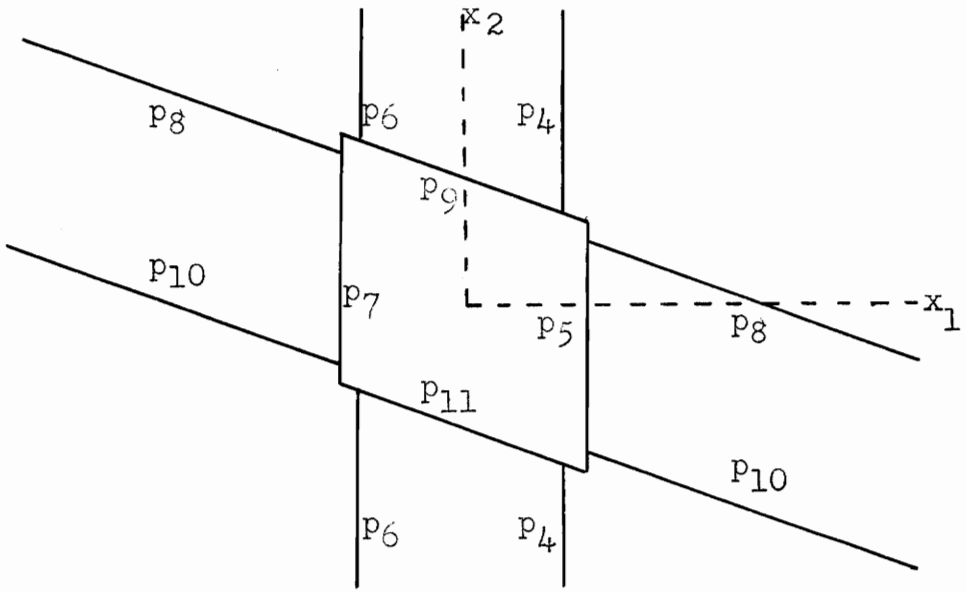
Figure 2

Intersection of Bounding Planes with $x_1 = q_2$.

(This Figure May Be Used For Both $P_U$ and $P_L$).



Figure 3

Cross Section of Cylinder

Bounding the Central Region.
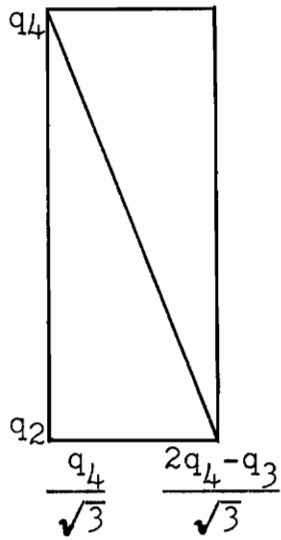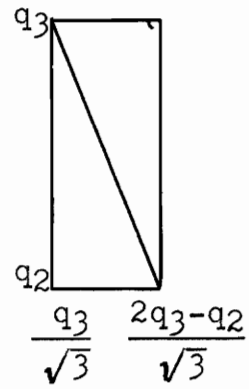


Figure 4

Cross Section of Cylinder

Bounding an Infinite Region.

a cross section of the cylinder bounding one side of an infinite region.

The expression for the lower bound of $P[D(2,1)|\xi_1,\xi_2,\xi_3]$ is given by

$$P_{L4} = x\begin{Bmatrix} \infty \\ q_2 \end{Bmatrix} - x\begin{Bmatrix} q_4 & 25 & 39 \\ q_2 & 27 & 3,11 \end{Bmatrix} - x\begin{Bmatrix} q_3 & 24 & 3,11 \\ q_2 & 26 & -\infty \end{Bmatrix}$$

(5)

$$- x\begin{Bmatrix} q_3 & 27 & 38 \\ q_2 & -\infty & 3,10 \end{Bmatrix} - x\begin{Bmatrix} q_3 & 24 & \infty \\ q_2 & 26 & 39 \end{Bmatrix} - x\begin{Bmatrix} q_3 & \infty & 38 \\ q_2 & 25 & 3,10 \end{Bmatrix},$$

where $2k$, $k = 4,5,6,7$, is obtained by solving for $x_2$ in equation $p_k$ and $3j$, $j = 8,9,10,11$, is obtained by solving for $x_3$ in equation $p_j$ of (3). Using the L notation as well as the fact that $g(x; ) = g_1 \cdot g_2 \cdot g_3$, (5) is written as

$$\begin{aligned}
P_{L4} &= L_x(q_2) - [L_x(q_2) - L_x(q_4)] \cdot [L_x(27, 3,11) \\
&\quad + L_x(25, 39) - L_x(25, 3,11) - L_x(27, 39)] \\
&\quad - [L_x(q_2) - L_x(q_3)] \cdot [L_x(26) - L_x(24) - L_x(26, 3,11) \\
&\quad + L_x(24, 3,11) + L_x(26, 39) - L_x(24, 39) \\
&\quad + L_x(25, 3,10) - L_x(25, 38) \\
&\quad + x\begin{Bmatrix} 27 & \infty \\ -\infty & 3,10 \end{Bmatrix} - x\begin{Bmatrix} 27 & \infty \\ -\infty & 38 \end{Bmatrix}].
\end{aligned}$$

Sometimes it is convenient to let $L_x(a,b) = L(x_a,x_b)$ in the work which follows.

Before applying the transformation theorem, the last two terms of the right hand side of (6) are changed to the L notation. To do this, $x_2$ is replaced by $-x_2'$ in $p_7$, $p_8$, and $p_{10}$ of (3) to obtain $p_7'$, $p_8'$, and $p_{10}'$, respectively, and the

resulting equations are written so that the coefficients of $x_2'$ in $p_7'$, $x_3$ in $p_8'$, and $x_3$ in $p_{10}'$ are positive. The trans-formed equations are

$$
\begin{aligned}
p_7' &: \sqrt{3}x_2' = 2q_4 - q_2 \ , \\
(7) \qquad p_8' &: -\sqrt{3}x_2' + 2\sqrt{6}x_3 = 6q_3 - 3q_2 \ , \text{ and} \\
p_{10}' &: -\sqrt{3}x_2' + 2\sqrt{6}x_3 = -6q_3 + 3q_2 \ .
\end{aligned}
$$

Hence, the last two terms in (6) may be expressed as

$$
\begin{aligned}
x \left\{ \begin{matrix} 27 & \infty \\ -\infty & 3,10 \end{matrix} \right\} &= x' \left\{ \begin{matrix} \infty & \infty \\ 27 & 3,10 \end{matrix} \right\} = L_{x'}(27, \ 3,10) \ , \text{ and} \\
(8) \\
x \left\{ \begin{matrix} 27 & \infty \\ -\infty & 38 \end{matrix} \right\} &= x' \left\{ \begin{matrix} \infty & \infty \\ 27 & 38 \end{matrix} \right\} = L_{x'}(27, \ 38) \ ,
\end{aligned}
$$

where

$$
\begin{aligned}
27 &= (2q_4 - q_2)/\sqrt{3} \ , \\
(9) \qquad 38 &= (6q_3 - 3q_2 + \sqrt{3}x_2')/(2\sqrt{6}) \ , \text{ and} \\
3,10 &= (3q_2 - 6q_3 + \sqrt{3}x_2')/(2\sqrt{6}) \ .
\end{aligned}
$$

Now, applying the transformation theorem, we have

$$
\begin{aligned}
P_{L4} = \ &L_t(a_1) - [L_t(a_1) - L_t(a_3)] \cdot [ \ L_y(27, \ 3,11; \ 1/3) \\
&+ L_y(25, \ 39; \ 1/3) - L_y(25, \ 3,11; \ 1/3) \\
&- L_y(27, \ 39; \ 1/3)] - [L_t(a_1) - L_t(a_2)] \cdot [ \ L_t(26) \\
(10) \qquad &- L_t(24) + L_y(24, \ 3,11; \ 1/3) + L_y(25, \ 3,10; \ 1/3) \\
&+ L_y(26, \ 39; \ 1/3) - L_y(24, \ 39; \ 1/3) - L_y(25, \ 38; \ 1/3) \\
&- L_y(26, \ 3,11; \ 1/3) + L_{y'}(27, \ 3,10; \ -1/3) \\
&- L_{y'}(27, \ 38; \ -1/3)] \ ,
\end{aligned}
$$

where $a_k = (q_{k+1} - \xi_1)/\sqrt{2}$ , $k = 1,2,3$, and $24,25,26,27,38$, $39$, $3,10$, and $3,11$ are the $y_{jc}$'s of equation (19) in VII

obtained by using (3) and (7). From the Bivariate Normal Distribution Tables (1953), we find that

$$L_y(h, -k; r) = L_t(h) - L_y(h, k; -r) ,$$

(11) $\quad L_y(-h, k; r) = L_t(k) - L_y(h, k; -r) ,$ and

$$L_y(-h, -k; r) = 1 + L_y(h, k; r) - L_t(h) - L_t(k) ,$$

where h and k are any real constants and r is the correlation coefficient. Since $y'_{27} = -y_{27}$, $y'_{38} = y_{38}$, and $y'_{3,10} = y_{3,10}$, we may write in place of the last two terms of (10)

$$L'_y(27, 3,10; -1/3) - L_{y'}(27, 38; -1/3)$$

$$= L_y(-27, 3,10; -1/3) - L_y(-27, 38; -1/3)$$

$$= L_t(3,10) - L_t(38) - L_y(27, 3,10; 1/3) + L_y(27, 38, 1/3).$$

The upper limit, $P_{U4}$, is obtained in a similar manner using (4). The only difference is in the constant on the right hand side of equations $p_4$, $p_5$, ..., $p_{11}$. Similarly, the transformation used in obtaining (7) gives

$$p'_7: \sqrt{3}x'_2 = q_4 ,$$

(12) $\quad p'_8: -\sqrt{3}x'_2 + 2\sqrt{6}x_3 = 3q_3 ,$ and

$$p'_{10}: -\sqrt{3}x'_2 + 2\sqrt{6}x_3 = -3q_3 .$$

Further, $P_{U4}$ has exactly the same form as (10) where the h's are determined using the equations of (4) and (12).

For three means, an argument was presented using $(P_L + P_U)/2$ as an estimate of $P_{21}$. Likewise, a good argument can be made for $(P_{L4} + P_{U4})/2$ as an approximation for $P[D(2,1)|\xi_1, \xi_2, \xi_3]$. Again this value is close to the actual value. However, it is usually expected to be nearer the value of $P_{U4}$ than the value of $P_{L4}$. See Appendix B for further

discussion on the approximation of $P_{21}(\xi)$.

## IX.  MULTIPLE F TESTS

The general multiple F test is a test of the significance of all linear comparisons of the form

$$(1) \qquad c = \sum_{i=1}^{n} k_i m_i \ ,$$

where $k_1$, $k_2$, ..., $k_n$ is any set of real constants such that $\sum_{i=1}^{n} k_i = 0$. That is, a general multiple F test is one in which any comparison of the form of (1) is significantly different from zero (i) provided the variance of every subset which contains all the means of (1) with non-zero coefficients is significant according to an F test and (ii) provided (1) differs significantly from zero according to a t-test. This definition is used when we are interested in testing linear comparisons within a set of means as well as differences between single means. However, the power functions defined and discussed in this paper are for tests of differences between pairs of means. Thus, the definition in the Introduction is the one used here since we are interested only in tests of differences between single means.

If we were interested in testing linear comparisons as well as differences between single means, additional power functions would be needed to measure the power of the test with respect to these additional comparisons. Then these additional power functions would need to be included in some sense in the average power obtained for the general multiple

F test. With all these power functions included it seems safe
to assume that a multiple F test would be more powerful than
the companion multiple range test, that is, the multiple
range test with the same system of significance levels. How-
ever, our interest (at this time) is only in tests of dif-
ferences between single means. In this case, the power rela-
tions are more obscure. Thus, for the multiple F tests,
power is determined only for tests of differences between
means.

To illustrate the method used to obtain power for the
multiple F tests involving three means, it is sufficient to
determine the power for Duncan's 5% level multiple compari-
sons test when $n_2 = \infty$ and $\sigma_m = 1$. First, we define the de-
cision region (2,1) for the three and four mean cases.

According to Duncan (1951), the significant ranges,
$R'_{p,n_2,\alpha} = R'_{p,\infty, .05}$, of his multiple comparisons test for
two, three, and four means are 2.772, 3.051, and 3.297,
respectively, where

$$(2) \qquad R'_{p,n_2,\alpha} = \begin{cases} \sqrt{2(p-1)F_{p-1,n_2,\alpha_p}} & \text{when } n_2 > 2 \\[2mm] \sqrt{2F_{1,n_2,\alpha}} = \sqrt{2} \cdot t_{n_2,\alpha} & \text{when } n_2 = 1,2, \end{cases}$$

$$(3) \qquad \alpha_p = 1 - (1 - \alpha)^{p-1}, \text{ and } p=2,3,\ldots,n.$$

Thus, for $n=3$ the multiple F test region (2,1) is that part
of the plane in Figure 1 to the right of the heavy line,
that is, that set of points which satisfy $x_1 > 2.772$ and

$x_1^2 + x_2^2 > (3.051)^2$ (all points for which $d_{12} > 2.772$ and at the same time are a distance of 3.051 from the point determined by $d_{12} = 0$, $d_{13} = 0$, and $d_{23} = 0$). The companion multiple range test region (2,1) is that part of the plane in Figure 1 to the right of the broken dotted line and the line $x_1 = 2.772$ which extends above and below the dotted line.

For n=4, the multiple F test region (2,1) is that set of points in the three dimensional rectangular coordinate system which satisfies the following four conditions:

(i) having points a distance of more than 3.297 units from the origin, i.e. points for which $x_1^2 + x_2^2 + x_3^2 > (3.297)^2$,

(ii) having points a distance of more than 3.051 units from the line determined by $d_{13} = 0$ and $d_{23} = 0$, i.e. points for which $x_1^2 + x_2^2 > (3.051)^2$,



3.05

$x_1 = 2.92$

$x_1$

$x_1 = 2.77$
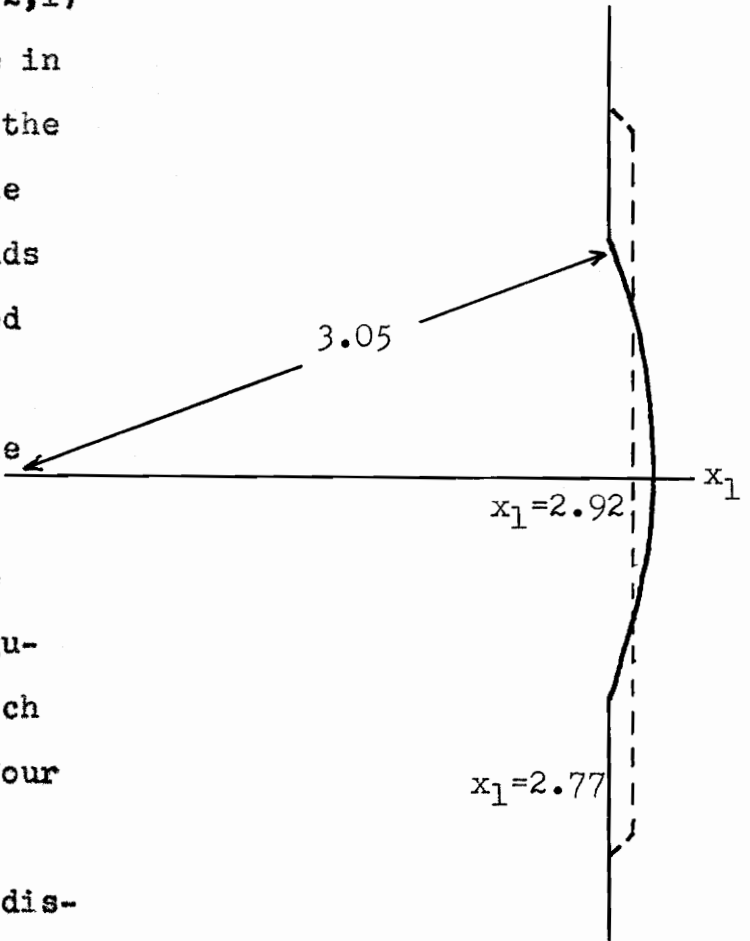
Figure 1

Boundaries of (2,1)

for Duncan's Tests

(iii) having points a distance of more than 3.051

units from the line determined by $d_{14} = 0$

and $d_{24} = 0$, and

(iv) having points more than 2.772 units in a

positive direction from the plane $d_{12} = x_1 = 0$.

The boundary of region (2,1) is shown in Figure 2 with the region (2,1) being above and the origin being below this boundary. $S_1$ is part of the surface of the sphere $x_1^2 + x_2^2 + x_3^2 = (3.297)^2$. $S_2$ is part of the surface of the cylinder $x_1^2 + x_2^2 = (3.051)^2$. $S_3$ is part of the surface of the cylinder defined by (iii) above. $S_4$ is part of the plane $x_1 = 2.772$.

Let

$$(4) \qquad P_c = P[D(2,1)|\xi_1, \xi_2]$$

denote the power function for Duncan's multiple comparisons test. Numerical integration is used to determine $P_c$. Let $R_0$ denote the region for which $x_1 > 2.772$; $R_1$ denote the region above the $x_1$ axis, to the right of $x_1 = 2.772$ and to the left of $x_1^2 + x_2^2 = (3.051)^2$; and $R_2$ denote the region symmetrical to $R_1$ about the $x_1$ axis. Let $I_0$, $I_1$, and $I_2$ denote the integral over region $R_0$, $R_1$, and $R_2$, respectively. Thus

$$(5) \qquad P_c = I_0 - (I_1 + I_2) .$$

$P_c$ may be expressed in a more useful form by transforming to standard coordinates using $t_i = (x_i - \xi_i)/\sqrt{2}$, where $i = 1,2$. The normal density function of $t_i$ is written as

$g_i(t_i) = g_i(t_i; 0,1)$. In this case,

$$I_o = \int_{a_1}^{\infty} g_1(t_1)dt_1 \ ,$$

(6)    $$I_1 = \int_{a_1}^{a_2} y(t_1) \ dt_1 \ , \text{ and}$$

$$I_2 = \int_{a_1}^{a_2} y'(t_1) \ dt_1 \ ,$$

where

$$y(t_1) = g_1(t_1) \int_{-c_2}^{t_2(t_1)-c_2} g_2(t_2)dt_2 \ ,$$

(7)    $$y'(t_1) = g_1(t_1) \int_{-t_2(t_1)-c_2}^{-c_2} g_2(t_2)dt_2 \ ,$$

$$a_1 = 1.9597-c_1, \ a_2 = 2.1577-c_2, \ c_i = \xi_i/\sqrt{2} \ , \text{ and}$$
$$t_2(t_1) = \sqrt{4.655760 - (t_1 + c_1)^2} \ .$$

$I_1$ and $I_2$ of (6) are expressed in a form such that the Newton-Cotes quadrature formulas are immediately applicable.

From Table 4 of Appendix A, we see that Duncan's multiple range test is more powerful everywhere than his multiple comparisons test for the same system of significance levels. For several means, this is not expected to be the case. There should be regions in the parameter-space for which the power

for the multiple comparisons test is higher.

Numerical integration as a method for obtaining power for four or more means does not seem feasible. We could approximate the power by integrating over a region whose boundaries are made up of a set of hyperplanes which closely fit the boundary of the multiple F region. In this way we could get an upper and lower bound, but these bounds would probably be such that the power of the corresponding multiple range test would fall between them. In this case, a comparison between the two tests would not be possible. The best solution to the problem of finding power for multiple F tests of the type of Duncan's test would probably be to use Monte Carlo methods.

## X. SUMMARY

In the study of multiple comparisons tests the following topics were discussed:

(i) extension to the general case of certain properties and results which previously had been given for three and four means only,

(ii) power vectors and average power,

(iii) expressions of power for the multiple range tests and for the multiple F test involving only three means, and

(iv) methods for evaluating power.

These four topics are amplified in order in the four paragraphs below.

A set of recursion formulas was obtained for enumerating the decision patterns for n means. $x_i$ was given by the equation

$$x_i = \sum_{j=1}^{i} d_{j,i+1} \Big/ \left( \frac{i(i+1)}{2} \right)^{\frac{1}{2}},$$

where $i=1,2,\ldots,n-1$. In IV, formulas were derived which express $d_{j,i+1}$ as a function of the $x_i$. This made possible the writing of the bounding equations for any decision region involving n means. The regions $(\underline{1,2})$, $(\underline{1,2,3})$, $(\underline{1,2,4})$, ..., $(\underline{1,2,\ldots,n})$ for the multiple range tests were described in the sample-space of differences among n means.

Two power functions were introduced to describe the power properties of a test of each difference $\delta_{ij}$. This led to the

definition and discussion of a power vector of $n(n-1)$ components in a test of the differences among n means. A weighted average probability of these components was given in order that __the power__ at a parameter point be defined. This was done as an effort to determine which of two tests is more powerful at a given parameter point $\xi$ .

The power functions for the multiple range tests were written after the region (2,1), which depended on a description of regions (1,2), (1,2,3), (1,2,4), ..., (1,2,...,n), was determined. The power expression for the multiple F tests was restricted to the three mean case because of the region, (2,1), of integration.

The multiple range power expressions may always be written as a set of series expansions, but the evaluation of such a series would be prohibitive for $n \geq 5$. Thus, approximations were introduced to give an upper and lower bound for power.

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## XIII.  VITA

R. Lowell Wine was born August 17, 1918, in Indian Springs, Tennessee.  He graduated from Holston High School, Holston, Tennessee, received a Bachelor of Arts degree in Mathematics and Chemistry from Bridgewater College, Bridgewater, Virginia in June 1941, and received a Master of Arts degree in Mathematics from the University of Virginia, Charlottesville, Virginia in June 1945.  He taught college mathematics a total of eleven years at the University of Virginia; Bridgewater College; Amherst College, Amherst, Massachusetts; Oklahoma University, Norman, Oklahoma; and Washington and Lee University, Lexington, Virginia.  After studying mathematical statistics for three summers and one academic year, 1952-1953, at the University of Michigan, Ann Arbor, Michigan, he entered the Virginia Polytechnic Institute in September 1953 to continue his studies in statistics.

He has written several technical reports and one published article.  He is a member of Pi Mu Epsilon, Sigma Xi, and the Institute of Mathematical Statistics.

He and his wife, Ruth, have three children, Ronald, nine, David, seven, and Suellen, five.

R. Lowell Wine

### XIV. APPENDIX A

1.  <u>Computation of Table 1.</u>

The multiple range values were calculated from (11) of
VII with the use of the Bivariate Normal Tables (1953) as
well as the usual Univariate Normal Tables (1924). Two way
linear interpolation was used, producing an error not exceed-
ing one-half unit in the fourth decimal place. The multiple
F values were obtained by evaluating $y(t_1)$ and $y'(t_1)$ of (7)
in IX at eleven points and applying the five point Newton-
Cotes quadrature formula

$$(1) \quad \int_{t_0}^{t_{10}} y(t)dt \doteq \frac{5h}{288}\Big[ 19y_0 + 75y_1 + 50y_2 + 50y_3 + 75y_4 + 38y_5$$
$$+ 75y_6 + 50y_7 + 50y_8 + 75y_9 + 19y_{10}\Big]$$

to (6) of IX, where $h=.02$ and $y_k=y(t_k)$, $k=0,1,...,10$. The
error in the multiple F values is less than $(550y^{(6)}h^7)/12096$,
where $y^{(6)}$ is the sixth derivative of $y(t_1)$ or $y'(t_1)$. Thus,
the error is not greater than that for the multiple range test
values.

It is not necessary that the power be tabulated for $-\xi_2$,
since

$$(2) \quad P[D(2,1)|\ \xi_1,\ \xi_2] = P[D(2,1)|\ \xi_1,\ -\xi_2].$$

For $\xi_1 \le -3$, the power is zero to four decimal places, and
for $\xi_1 \ge 8$, the power is nearly one.

## 2. Computation of Table 2.

The power values in Table 1 were actually calculated for
$D(2,1)$ at each of the parameter points in the grid. Each
parameter point has a fixed position relative to the boundary
of $(2,1)$, that is, the coordinates of each point are located
relative to two perpendicular axes which have a fixed posi-
tion. Consider their position in relation to the line seg-
ment CF in Figure 1 of VIII. The line perpendicular to CF
and passing through the midpoint of CF is the $x_1$ axis, or
$\xi_1$ axis, and the axis parallel to CF and a fixed distance to
the left of $(2,1)$ is the $x_2$ axis, or $\xi_2$ axis. Corresponding
to each of the other regions $(3,1)$, $(3,2)$, $(1,2)$, $(1,3)$, and
$(2,3)$, such a set of axes, $\xi_1'$ and $\xi_2'$, may be defined. Thus,
it is clear that we may think of each point as having six
sets of coordinates. The five sets of coordinates, $\xi_1'$ and
$\xi_2'$, corresponding to a given set ( $\xi_1$, $\xi_2$), are given by
the formulas

$$\xi_1' = \xi_2 \sin(k60^{\circ}) + \xi_1 \cos(k60^{\circ}) \text{ and}$$
$$\xi_2' = \xi_2 \cos(k60^{\circ}) - \xi_1 \sin(k60^{\circ}) \; ,$$

where $k=1,2,3,4$, and 5 according as the coordinates of the
point relative to $(3,1)$, $(3,2)$, $(1,2)$, $(1,3)$, and $(2,3)$ are
required. These are the coordinates given in Table 2 where
the fact that $P[D(2,1)| \xi_1', \xi_2'] = P[D(2,1)| \xi_1', -\xi_2']$ is
taken into account, that is, either $\xi_2'$ or $-\xi_2'$ may be given.

It should be noted that Table 2 does not include the co-

ordinates for points where $\xi_2 = 5$ or $\xi_2 = 6$.

## 3. Computation of Tables 3 and 4.

Power at any parameter point may be computed with the aid of Table 1 and interpolation formulas. The values in Table 3 were obtained for the parameter points in Table 2 by such a procedure.

If a coordinate $\xi_1$ ( $\xi_2$) assumes an integral value, denote this by $[\xi_1]$ ( $[\xi_2]$), otherwise, the brackets are dropped. For fixed values of $[\xi_1]$, changes in $\xi_2$ make very little difference in the value of $P_{21}$. Thus, linear interpolation was used to find $P\big[D(2,1) \mid [\xi_1], \xi_2\big]$ given the probabilities $P\big[D(2,1) \mid [\xi_1], [\xi_{21}]\big]$ and $P\big[D(2,1) \mid [\xi_1], [\xi_{22}]\big]$, where $[\xi_{21}]$ and $[\xi_{22}]$ denote the two integers closest to $\xi_2$. However, for fixed values of $[\xi_2]$,

$$(4) \qquad\qquad P\big[D(2,1) \mid \xi_1, [\xi_2]\big]$$

changes rapidly with $\xi_1$, especially for values of $\xi_1$ between 0 and 5. In this case, the four-point and five-point Lagrangian Interpolation Coefficients (1944) were used. For values of $\xi_1 \leq 0$, $P_{21}$ approaches zero asymptotically as $\xi_1$ becomes smaller. Thus, linear interpolation was used on the transformation $\log (P_{21} + \text{constant})$ to obtain (4). Similarly, for $\xi_1 \geq 7$, linear interpolation was used on the transformation $\log (1 - P_{21} + \text{constant})$ to obtain (4), since $P_{21}$ approaches one asymptotically. For example, in order to find

$P\left[D(2,1)\,\middle|\,\xi_1 = .4, \xi_2\right]$, say, we would determine

(5) $\qquad P\left[D(2,1)\,\middle|\,[\xi_1] = 0, \xi_2\right]$

and

(6) $\qquad P\left[D(2,1)\,\middle|\,[\xi_1] = 1, \xi_2\right]$

by Lagrangian interpolation coefficients or linear interpolation on the log transformation of $P_{21}$. We would then interpolate linearly on (5) and (6) to obtain $P\left[D(2,1)\,\middle|\,\xi_1 = .4, \xi_2\right]$.

The true differences $\delta_{12}$, $\delta_{13}$, and $\delta_{23}$ in Table IV are computed from $\delta_{12} = \xi_1$, $\delta_{13} = (\xi_1 + \sqrt{3}\,\xi_2)/2$, and $\delta_{23} = (-\xi_1 + \sqrt{3}\,\xi_2)/2$. $C_w$ and $E_w$ are found by (39) and (41) of V.

4. <u>Computation of Table 5.</u>

These limits were calculated from (10) and (11) of VIII with the use of the Bivariate Normal Tables and the usual Univariate Normal Tables. Each value is accurate to four decimal places, and their average deviates at most .0001 units from the correct value $P_{21}$.

## Table 1

### Power for Duncan's Multiple Range and
### Multiple F Tests for Three Means
### (Multiple F underlined)

| $\xi_2$ \ $\xi_1$ | -2 | -1 | 0 | 1 | 2 |
|---|---|---|---|---|---|
| 0 | .00028 | .00303 | .02070 | .0914 | .2657 |
|   | .00028 | .00300 | .02057 | .0906 | .2635 |
| 1 | .00030 | .00324 | .02186 | .0950 | .2728 |
|   | .00030 | .00317 | .02143 | .0934 | .2692 |
| 2 | .00035 | .00361 | .02383 | .1013 | .2851 |
|   | .00033 | .00348 | .02312 | .0990 | .2803 |
| 3 | .00037 | .00379 | .02481 | .1045 | .2914 |
|   | .00036 | .00371 | .02437 | .1031 | .2885 |
| 4 | .00037 | .00383 | .02495 | .1051 | .2926 |
|   | .00037 | .00380 | .02487 | .1047 | .2918 |
| 5 | .00037 | .00383 | .02502 | .1052 | .2927 |
|   | .00037 | .00382 | .02500 | .1051 | .2926 |
| 6 | .00037 | .00383 | .02502 | .1052 | .2927 |
|   | .00037 | .00383 | .02502 | .1052 | .2927 |

## Table 1 (cont.)
### Power for Duncan's Multiple Range and
### Multiple F Tests for Three Means
### (Multiple F underlined)

| $\xi_2 \backslash \xi_1$ | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|
| 0 | .5321 | .7844 | .9324 | .9861 | .9982 |
|   | .5288 | .7815 | .9308 | .9856 | .9981 |
| 1 | .5434 | .7905 | .9350 | .9868 | .9983 |
|   | .5357 | .7866 | .9331 | .9862 | .9982 |
| 2 | .5552 | .8010 | .9396 | .9880 | .9984 |
|   | .5492 | .7965 | .9375 | .9875 | .9984 |
| 3 | .5626 | .8064 | .9420 | .9887 | .9986 |
|   | .5591 | .8038 | .9408 | .9883 | .9985 |
| 4 | .5641 | .8074 | .9424 | .9888 | .9986 |
|   | .5631 | .8067 | .9421 | .9887 | .9986 |
| 5 | .5642 | .8075 | .9425 | .9888 | .9986 |
|   | .5640 | .8074 | .9424 | .9888 | .9986 |
| 6 | .5642 | .8075 | .9425 | .9888 | .9986 |
|   | .5642 | .8075 | .9425 | .9888 | .9986 |

## Table 2

## Coordinates of Each Point with Reference to Six Regions

## (Coordinate axes change with region)

| Regions | Points | | | | | |
|---|---|---|---|---|---|---|
| | $\xi_1$ | $\xi_2$ | $\xi_1$ | $\xi_2$ | $\xi_1$ | $\xi_2$ |
| (2,1) | -1 | 0 | 0 | 0 | 1 | 0 |
| (3,1) | -.5000 | .8660 | 0 | 0 | .5000 | .8660 |
| (3,2) | .5000 | .8660 | 0 | 0 | -.5000 | .8660 |
| (1,2) | 1 | 0 | 0 | 0 | -1 | 0 |
| (1,3) | .5000 | .8660 | 0 | 0 | -.5000 | .8660 |
| (2,3) | -.5000 | .8660 | 0 | 0 | .5000 | .8660 |
| (2,1) | -1 | 1 | 0 | 1 | 1 | 1 |
| (3,1) | .3660 | 1.3660 | .8660 | .5000 | 1.3660 | .3660 |
| (3,2) | 1.3660 | .3660 | .8660 | .5000 | .3660 | 1.3660 |
| (1,2) | 1 | 1 | 0 | 1 | -1 | 1 |
| (1,3) | -.3660 | 1.3660 | -.8660 | .5000 | -1.3660 | .3660 |
| (2,3) | -1.3660 | .3660 | -.8660 | .5000 | -.3660 | 1.3660 |
| (2,1) | -1 | 2 | 0 | 2 | 1 | 2 |
| (3,1) | 1.2320 | 1.8660 | 1.7320 | 1 | 2.2320 | .1340 |
| (3,2) | 2.2320 | .1340 | 1.7320 | 1 | 1.2320 | 1.8660 |
| (1,2) | 1 | 2 | 0 | 2 | -1 | 2 |
| (1,3) | -1.2320 | 1.8660 | -1.7320 | 1 | -2.2320 | .1340 |
| (2,3) | -2.2320 | .1340 | -1.7320 | 1 | -1.2320 | 1.8660 |
| (2,1) | -1 | 3 | 0 | 3 | 1 | 3 |
| (3,1) | 2.0981 | 2.3660 | 2.5981 | 1.5000 | 3.0981 | .6340 |
| (3,2) | 3.0981 | .6340 | 2.5981 | 1.5000 | 2.0981 | 2.3660 |
| (1,2) | 1 | 3 | 0 | 3 | -1 | 3 |
| (1,3) | -2.0981 | 2.3660 | -2.5981 | 1.5000 | -3.0981 | .6340 |
| (2,3) | -3.0981 | .6340 | -2.5981 | 1.5000 | -2.0981 | 2.3660 |
| (2,1) | -1 | 4 | 0 | 4 | 1 | 4 |
| (3,1) | 2.9641 | 2.8660 | 3.4641 | 2 | 3.9641 | 1.1340 |
| (3,2) | 3.9641 | 1.1340 | 3.4641 | 2 | 2.9641 | 2.8660 |
| (1,2) | 1 | 4 | 0 | 4 | -1 | 4 |
| (1,3) | -2.9641 | 2.8660 | -3.4641 | 2 | -3.9641 | 1.1340 |
| (2,3) | -3.9641 | 1.1340 | -3.4641 | 2 | -2.9641 | 2.8660 |

## Table 2 (cont.)

### Coordinates of Each Point with Reference to Six Regions

### (Coordinate axes change with region)

| Regions | Points $\xi_1$ | $\xi_2$ | $\xi_1$ | $\xi_2$ | $\xi_1$ | $\xi_2$ |
|---|---|---|---|---|---|---|
| (2,1) | 2 | 0 | 3 | 0 | 4 | 0 |
| (3,1) | 1 | 1.7320 | 1.5000 | 2.5981 | 2 | 3.4641 |
| (3,2) | -1 | 1.7320 | -1.5000 | 2.5981 | -2 | 3.4641 |
| (1,2) | -2 | 0 | -3 | 0 | -4 | 0 |
| (1,3) | -1 | 1.7320 | -1.5000 | 2.5981 | -2 | 3.4641 |
| (2,3) | 1 | 1.7320 | 1.5000 | 2.5981 | 2 | 3.4641 |
| (2,1) | 2 | 1 | 3 | 1 | 4 | 1 |
| (3,1) | 1.8660 | 1.2320 | 2.3660 | 2.0981 | 2.8660 | 2.9641 |
| (3,2) | - .1340 | 2.2320 | - .6340 | 3.0981 | -1.1340 | 3.9641 |
| (1,2) | -2 | 1 | -3 | 1 | -4 | 1 |
| (1,3) | -1.8660 | 1.2320 | -2.3665 | 2.0981 | -2.8660 | 2.9641 |
| (2,3) | .1340 | 2.2320 | .6340 | 3.0981 | 1.1340 | 3.9641 |
| (2,1) | 2 | 2 | 3 | 2 | 4 | 2 |
| (3,1) | 2.7320 | .7320 | 2.2320 | 1.5981 | 3.7320 | 2.4641 |
| (3,2) | .7320 | 2.7320 | .2320 | 3.5981 | - .2680 | 4.4641 |
| (1,2) | -2 | 2 | -3 | 2 | -4 | 2 |
| (1,3) | -2.7320 | .7320 | -3.2320 | 1.5981 | -3.7320 | 2.4641 |
| (2,3) | - .7320 | 2.7320 | - .2320 | 3.5981 | .2680 | 4.4641 |
| (2,1) | 2 | 3 | 3 | 3 | 4 | 3 |
| (3,1) | 3.5981 | .2320 | 4.0981 | 1.0981 | 4.5981 | 1.9641 |
| (3,2) | 1.5981 | 3.2320 | 1.0981 | 4.0981 | .5981 | 4.9641 |
| (1,2) | -2 | 3 | -3 | 3 | -4 | 3 |
| (1,3) | -3.5981 | .2320 | -4.0981 | 1.0981 | -4.5981 | 1.9641 |
| (2,3) | -1.5981 | 3.2320 | -1.0981 | 4.0981 | - .5981 | 4.9641 |
| (2,1) | 2 | 4 | 3 | 4 | 4 | 4 |
| (3,1) | 4.4641 | .2680 | 4.9641 | .5981 | 5.4641 | 1.4641 |
| (3,2) | 2.4641 | 3.7320 | 1.9641 | 4.5981 | 1.4641 | 5.4641 |
| (1,2) | -2 | 4 | -3 | 4 | -4 | 4 |
| (1,3) | -4.4641 | .2680 | -4.9641 | .5981 | -5.4641 | 1.4641 |
| (2,3) | -2.4641 | 3.7320 | -1.9641 | 4.5981 | -1.4641 | 5.4641 |

## Table 2 (cont.)

### Coordinates of Each Point with Reference to Six Regions

### (Coordinate axes change with region)

| Regions | Points | | | | | |
|---|---|---|---|---|---|---|
| | $\xi_1$ | $\xi_2$ | $\xi_1$ | $\xi_2$ | $\xi_1$ | $\xi_2$ |
| (2,1) | 5 | 0 | 6 | 0 | 7 | 0 |
| (3,1) | 2.5000 | 4.3301 | 3 | 5.1962 | 3.5000 | 6.0622 |
| (3,2) | -2.5000 | 4.3301 | -3 | 5.1962 | -3.5000 | 6.0622 |
| (1,2) | -5 | 0 | -6 | 0 | -7 | 0 |
| (1,3) | -2.5000 | 4.3301 | -3 | 5.1962 | -3.5000 | 6.0622 |
| (2,3) | 2.5000 | 4.3301 | 3 | 5.1962 | 3.5000 | 6.0622 |
| (2,1) | 5 | 1 | 6 | 1 | 7 | 1 |
| (3,1) | 3.3660 | 3.8301 | 3.8660 | 4.6962 | 4.3660 | 5.5622 |
| (3,2) | -1.6340 | 4.8301 | -2.1340 | 5.6962 | -2.6340 | 6.5622 |
| (1,2) | -5 | 1 | -6 | 1 | -7 | 1 |
| (1,3) | -3.3660 | 3.8301 | -3.8660 | 4.6962 | -4.3660 | 5.5622 |
| (2,3) | 1.6340 | 4.8301 | 2.1340 | 5.6962 | 2.6340 | 6.5622 |
| (2,1) | 5 | 2 | 6 | 2 | 7 | 2 |
| (3,1) | 4.2320 | 3.3301 | 4.7320 | 4.1962 | 5.2320 | 5.0622 |
| (3,2) | - .7680 | 5.3301 | -1.2678 | 6.1962 | -1.7680 | 7.0622 |
| (1,2) | -5 | 2 | -6 | 2 | -7 | 2 |
| (1,3) | -4.2320 | 3.3301 | -4.7320 | 4.1962 | -5.2320 | 5.0622 |
| (2,3) | .7680 | 5.3301 | 1.2670 | 6.1962 | 1.7680 | 7.0622 |
| (2,1) | 5 | 3 | 6 | 3 | 7 | 3 |
| (3,1) | 5.0981 | 2.8301 | 5.5981 | 3.6962 | 6.0981 | 4.5622 |
| (3,2) | .0981 | 5.8301 | - .4019 | 6.6962 | - .9019 | 7.5622 |
| (1,2) | -5 | 3 | -6 | 3 | -7 | 3 |
| (1,3) | -5.0981 | 2.8301 | -5.5981 | 3.6962 | -6.0981 | 4.5622 |
| (2,3) | - .0981 | 5.8301 | .4019 | 6.6962 | .9019 | 7.5622 |
| (2,1) | 5 | 4 | 6 | 4 | 7 | 4 |
| (3,1) | 5.9641 | 2.3301 | 6.4641 | 3.1962 | 6.9641 | 4.0622 |
| (3,2) | .9641 | 6.3301 | .4641 | 7.1962 | - .0359 | 8.0622 |
| (1,2) | -5 | 4 | -6 | 4 | -7 | 4 |
| (1,3) | -5.9641 | 2.3301 | -6.4641 | 3.1962 | -6.9641 | 4.0622 |
| (2,3) | - .9641 | 6.3301 | - .4641 | 7.1962 | .0359 | 8.0622 |

## Table 3

## Power Components with Reference to Six Regions

## for Duncan's Tests for Three Means

## (Multiple F underlined)

| Parameter Points | | Regions | | | | | |
|---|---|---|---|---|---|---|---|
| $\xi_1$ | $\xi_2$ | (2,1) | (3,1) | (3,2) | (1,2) | (1,3) | (2,3) |
| 0 | 0 | .0207 .0206 | .0207 .0206 | .0207 .0206 | .0207 .0206 | .0207 .0206 | .0207 .0206 |
| 0 | 1 | .0219 .0214 | .0780 .0769 | .0780 .0769 | .0219 .0214 | .0041 .0040 | .0041 .0040 |
| 0 | 2 | .0238 .0231 | .2136 .2110 | .2136 .2110 | .0238 .0231 | .0007 .0006 | .0007 .0006 |
| 0 | 3 | .0248 .0244 | .4378 .4315 | .4378 .4315 | .0248 .0244 | .0000* .0000* | .0000* .0000* |
| 0 | 4 | .0250 .0249 | .6780 .6723 | .6780 .6723 | .0250 .0249 | .0000* .0000* | .0000* .0000* |
| 0 | 5 | .0250 .0250 | .8620 .8591 | .8620 .8591 | .0250 .0250 | .0000* .0000* | .0000* .0000* |
| 1 | 0 | .0914 .0906 | .0485 .0476 | .0085 .0083 | .0030 .0030 | .0085 .0083 | .0485 .0476 |
| 1 | 1 | .0950 .0934 | .1442 .1426 | .0411 .0402 | .0032 .0032 | .0015 .0014 | .0114 .0111 |
| 1 | 2 | .1013 .0990 | .3228 .3200 | .1331 .1303 | .0036 .0035 | .0000* .0000* | .0022 .0022 |
| 1 | 3 | .1045 .1031 | .5661 .5601 | .3113 .3070 | .0038 .0037 | .0000* .0000* | .0000* .0000* |
| 1 | 4 | .1051 .1047 | .7846 .7804 | .5518 .5479 | .0038 .0038 | .0000* .0000* | .0000* .0000* |
| 1 | 5 | .1052 .1051 | .8286 .8264 | .7741 .7721 | .0038 .0038 | .0000* .0000* | .0000* .0000* |

*Values obtained by extrapolation.

## Table 3 (cont.)

## Power Components with Reference to Six Regions

## for Duncan's Tests for Three Means

## (Multiple F underlined)

| Parameter Points | | Regions | | | | | |
|---|---|---|---|---|---|---|---|
| $\xi_1$ | $\xi_2$ | (2,1) | (3,1) | (3,2) | (1,2) | (1,3) | (2,3) |
| 2 | 0 | .2657 | .0996 | .0035 | .0003 | .0035 | .0996 |
|   |   | .2635 | .0975 | .0034 | .0003 | .0034 | .0975 |
| 2 | 1 | .2728 | .2430 | .0188 | .0003 | .0005 | .0302 |
|   |   | .2692 | .2416 | .0183 | .0003 | .0005 | .0294 |
| 2 | 2 | .2851 | .4660 | .0727 | .0004 | .0000* | .0063 |
|   |   | .2803 | .4599 | .0714 | .0003 | .0000* | .0061 |
| 2 | 3 | .2914 | .6946 | .2031 | .0004 | .0000* | .0011 |
|   |   | .2885 | .6909 | .2011 | .0004 | .0000* | .0011 |
| 2 | 4 | .2926 | .8692 | .4125 | .0004 | .0000* | .0000* |
|   |   | .2918 | .8664 | .4109 | .0004 | .0000* | .0000* |
| 2 | 5 | .2927 | .9607 | .6521 | .0004 | .0000* | .0000* |
|   |   | .2926 | .9594 | .6513 | .0004 | .0000* | .0000* |
| 3 | 0 | .5321 | .1824 | .0013 | .0000* | .0013 | .1824 |
|   |   | .5288 | .1797 | .0013 | .0000* | .0013 | .1797 |
| 3 | 1 | .5434 | .3786 | .0077 | .0000* | .0000* | .0640 |
|   |   | .5357 | .3733 | .0075 | .0000* | .0000* | .0631 |
| 3 | 2 | .5552 | .6132 | .0366 | .0000* | .0000* | .0163 |
|   |   | .5492 | .6067 | .0362 | .0000* | .0000* | .0161 |
| 3 | 3 | .5626 | .8107 | .1186 | .0000* | .0000* | .0032 |
|   |   | .5591 | .8070 | .1182 | .0000* | .0000* | .0031 |
| 3 | 4 | .5641 | .9293 | .2841 | .0000* | .0000* | .0004 |
|   |   | .5631 | .9286 | .2837 | .0000* | .0000* | .0004 |
| 3 | 5 | .5642 | .9817 | .4637 | .0000* | .0000* | .0000* |
|   |   | .5640 | .9811 | .4636 | .0000* | .0000* | .0000* |

*Values obtained by extrapolation.

## Table 3 (cont.)

### Power Components with Reference to Six Regions

### for Duncan's Tests for Three Means

### (Multiple F underlined)

| Parameter Points | | Regions | | | | | |
|---|---|---|---|---|---|---|---|
| $\xi_1$ | $\xi_2$ | (2,1) | (3,1) | (3,2) | (1,2) | (1,3) | (2,3) |
| 4 | 0 | .7844 / .7815 | .2920 / .2900 | .0004 / .0004 | .0000* / .0000* | .0004 / .0004 | .2920 / .2900 |
| 4 | 1 | .7905 / .7866 | .5255 / .5219 | .0029 / .0029 | .0000* / .0000* | .0000* / .0000* | .1238 / .1233 |
| 4 | 2 | .8010 / .7965 | .7469 / .7427 | .0153 / .0152 | .0000* / .0000* | .0000* / .0000* | .0388 / .0387 |
| 4 | 3 | .8064 / .8038 | .8963 / .8932 | .0612 / .0611 | .0000* / .0000* | .0000* / .0000* | .0083 / .0083 |
| 4 | 4 | .8074 / .8067 | .9696 / .9685 | .1786 / .1785 | .0000* / .0000* | .0000* / .0000* | .0015 / .0015 |
| 4 | 5 | .8075 / .8074 | .9926 / .9922 | .3767 / .3767 | .0000* / .0000* | .0000* / .0000* | .0000* / .0000* |
| 5 | 0 | .9324 / .9308 | .4228 / .4221 | .0000* / .0000* | .0000* / .0000* | .0000* / .0000* | .4228 / .4221 |
| 5 | 1 | .9350 / .9331 | .6611 / .6597 | .0010 / .0010 | .0000* / .0000* | .0000* / .0000* | .2112 / .2111 |
| 5 | 2 | .9396 / .9375 | .8487 / .8470 | .0060 / .0060 | .0000* / .0000* | .0000* / .0000* | .0165 / .0165 |
| 5 | 3 | .9420 / .9408 | .9500 / .9488 | .0295 / .0295 | .0000* / .0000* | .0000* / .0000* | .0209 / .0209 |
| 5 | 4 | .9424 / .9421 | .9875 / .9870 | .1005 / .1005 | .0000* / .0000* | .0000* / .0000* | .0041 / .0041 |
| 5 | 5 | .9425 / .9424 | .9978 / .9974 | .2261 / .2261 | .0000* / .0000* | .0000* / .0000* | .0006 / .0006 |

*Values obtained by extrapolation.

## Table 3 (cont.)

### Power Components with Reference to Six Regions

### for Duncan's Tests for Three Means

### (Multiple F underlined)

| Parameter Points | | Regions | | | | | |
|---|---|---|---|---|---|---|---|
| $\xi_1$ | $\xi_2$ | (2,1) | (3,1) | (3,2) | (1,2) | (1,3) | (2,3) |
| 6 | 0 | .9861 .9856 | .5642 .5641 | .0000* .0000* | .0000* .0000* | .0000* .0000* | .5642 .5641 |
| 6 | 1 | .9868 .9862 | .7804 .7801 | .0000* .0000* | .0000* .0000* | .0000* .0000* | .3258 .3257 |
| 6 | 2 | .9880 .9875 | .9163 .9160 | .0022 .0022 | .0000* .0000* | .0000* .0000* | .1446 .1446 |
| 6 | 3 | .9887 .9883 | .9781 .9780 | .0119 .0119 | .0000* .0000* | .0000* .0000* | .0476 .0476 |
| 6 | 4 | .9888 .9887 | .9951 .9950 | .0521 .0521 | .0000* .0000* | .0000* .0000* | .0106 .0106 |
| 6 | 5 | .9888 .9888 | .9998* .9998* | .1549 .1549 | .0000* .0000* | .0000* .0000* | .0020 .0020 |
| 7 | 0 | .9982 .9981 | .6951 .6951 | .0000* .0000* | .0000* .0000* | .0000* .0000* | .6951 .6951 |
| 7 | 1 | .9983 .9982 | .8706 .8705 | .0000* .0000* | .0000* .0000* | .0000* .0000* | .4629 .4629 |
| 7 | 2 | .9987 .9984 | .9602 .9602 | .0008 .0008 | .0000* .0000* | .0000* .0000* | .2394 .2394 |
| 7 | 3 | .9986 .9985 | .9905 .9905 | .0043 .0043 | .0000* .0000* | .0000* .0000* | .0928 .0928 |
| 7 | 4 | .9986 .9986 | .9985 .9984 | .0234 .0234 | .0000* .0000* | .0000* .0000* | .0266 .0266 |
| 7 | 5 | .9986 .9986 | .9999* .9999* | .0750 .0750 | .0000* .0000* | .0000* .0000* | .0053 .0053 |

*Values obtained by extrapolation.

Table 4

Average Correct Decision and Error Probabilities with

Differences Corresponding to Each Parameter Point

(Multiple F underlined)

| Parameter Points | | Sets of Differences | | | Probabilities | |
|---|---|---|---|---|---|---|
| $\xi_1$ | $\xi_2$ | $\delta_{12}$ | $\delta_{13}$ | $\delta_{23}$ | $C_w$ | $E_w$ |
| 0 | 0 | .00000 | .00000 | .00000 | .0207 .0206 | .0207 .0206 |
| 0 | 1 | .00000 | .86603 | .86603 | .0780 .0769 | .0041 .0040 |
| 0 | 2 | .00000 | 1.73205 | 1.73205 | .2136 .2110 | .0007 .0006 |
| 0 | 3 | .00000 | 2.59808 | 2.59808 | .4378 .4315 | .0000 .0000 |
| 0 | 4 | .00000 | 3.46410 | 3.46410 | .6780 .6723 | .0000 .0000 |
| 0 | 5 | .00000 | 4.33013 | 4.33013 | .8620 .8591 | .0000 .0000 |
| 1 | 0 | 1.00000 | .50000 | - .50000 | .0699 .0691 | .0057 .0057 |
| 1 | 1 | 1.00000 | 1.36603 | .36603 | .1124 .1109 | .0034 .0034 |
| 1 | 2 | 1.00000 | 2.23205 | 1.23205 | .2208 .2181 | .0014 .0014 |
| 1 | 3 | 1.00000 | 3.09808 | 2.09808 | .4053 .4006 | .0001 .0001 |
| 1 | 4 | 1.00000 | 3.96410 | 2.96410 | .6118 .6083 | .0000 .0000 |
| 1 | 5 | 1.00000 | 4.83013 | 3.83013 | .7321 .7302 | .0000 .0000 |

## Table 4 (cont.)

### Average Correct Decision and Error Probabilities with

### Differences Corresponding to Each Parameter Point

### (Multiple F underlined)

| Parameter Points | | Sets of Differences | | | Probabilities | |
|---|---|---|---|---|---|---|
| $\xi_1$ | $\xi_2$ | $\delta_{12}$ | $\delta_{13}$ | $\delta_{23}$ | $C_w$ | $E_w$ |
| 2 | 0 | 2.00000 | 1.00000 | -1.00000 | .1827 .1805 | .0019 .0018 |
| 2 | 1 | 2.00000 | 1.86603 | - .13398 | .2508 .2483 | .0010 .0010 |
| 2 | 2 | 2.00000 | 2.73205 | .73205 | .3471 .3421 | .0010 .0009 |
| 2 | 3 | 2.00000 | 3.59808 | 1.59808 | .4734 .4703 | .0003 .0003 |
| 2 | 4 | 2.00000 | 4.46410 | 2.46410 | .6140 .6120 | .0001 .0001 |
| 2 | 5 | 2.00000 | 5.33013 | 3.33013 | .7389 .7381 | .0001 .0001 |
| 3 | 0 | 3.00000 | 1.50000 | -1.50000 | .3573 .3542 | .0007 .0006 |
| 3 | 1 | 3.00000 | 2.36603 | - .63398 | .4277 .4217 | .0008 .0008 |
| 3 | 2 | 3.00000 | 3.23205 | .23205 | .5656 .5595 | .0006 .0006 |
| 3 | 3 | 3.00000 | 4.09808 | 1.09808 | .6272 .6240 | .0004 .0004 |
| 3 | 4 | 3.00000 | 4.96410 | 1.96410 | .6913 .6906 | .0001 .0001 |
| 3 | 5 | 3.00000 | 5.83013 | 2.83013 | .7485 .7482 | .0000 .0000 |

## Table 4 (cont.)

### Average Correct Decision and Error Probabilities with

### Differences Corresponding to Each Parameter Point

### (Multiple F underlined)

| Parameter Points | | Sets of Differences | | | Probabilities | |
|---|---|---|---|---|---|---|
| $\xi_1$ | $\xi_2$ | $\delta_{12}$ | $\delta_{13}$ | $\delta_{23}$ | $C_w$ | $E_w$ |
| 4 | 0 | 4.00000 | 2.00000 | -2.00000 | .5382 .5357 | .0002 .0002 |
| 4 | 1 | 4.00000 | 2.86603 | -1.13398 | .6010 .5977 | .0004 .0004 |
| 4 | 2 | 4.00000 | 3.73205 | - .26795 | .7502 .7460 | .0005 .0005 |
| 4 | 3 | 4.00000 | 4.59808 | .59808 | .8029 .8002 | .0005 .0005 |
| 4 | 4 | 4.00000 | 5.46410 | 1.46410 | .8042 .8034 | .0002 .0002 |
| 4 | 5 | 4.00000 | 6.33013 | 2.33013 | .8207 .8205 | .0000 .0000 |
| 5 | 0 | 5.00000 | 2.50000 | -2.50000 | .6776 .6765 | .0000 .0000 |
| 5 | 1 | 5.00000 | 3.36603 | -1.63398 | .7245 .7231 | .0002 .0002 |
| 5 | 2 | 5.00000 | 4.23205 | - .76795 | .8303 .8285 | .0005 .0005 |
| 5 | 3 | 5.00000 | 5.09808 | .09808 | .9372 .9360 | .0001 .0001 |
| 5 | 4 | 5.00000 | 5.96410 | .96410 | .8969 .8965 | .0003 .0003 |
| 5 | 5 | 5.00000 | 6.83013 | 1.83013 | .8741 .8740 | .0001 .0001 |

## Table 4 (cont.)

### Average Correct Decision and Error Probabilities with

### Differences Corresponding to Each Parameter Point

### (Multiple F underlined)

| Parameter Points | | Sets of Differences | | | Probabilities | |
|---|---|---|---|---|---|---|
| $\xi_1$ | $\xi_2$ | $\delta_{12}$ | $\delta_{13}$ | $\delta_{23}$ | $C_W$ | $E_W$ |
| 6 | 0 | 6.00000 | 3.00000 | -3.00000 | .7751 / .7748 | .0000 / .0000 |
| 6 | 1 | 6.00000 | 3.86003 | -2.13398 | .8024 / .8020 | .0000 / .0000 |
| 6 | 2 | 6.00000 | 4.73205 | -1.26795 | .8706 / .8702 | .0002 / .0002 |
| 6 | 3 | 6.00000 | 5.59808 | - .40192 | .9522 / .9520 | .0004 / .0004 |
| 6 | 4 | 6.00000 | 6.46410 | .46410 | .9583 / .9582 | .0004 / .0004 |
| 6 | 5 | 6.00000 | 7.33013 | 1.33013 | .9186 / .9186 | .0002 / .0002 |
| 7 | 0 | 7.00000 | 3.50000 | -3.50000 | .8466 / .8466 | .0000 / .0000 |
| 7 | 1 | 7.00000 | 4.36603 | -2.63398 | .8577 / .8577 | .0000 / .0000 |
| 7 | 2 | 7.00000 | 5.23205 | -1.76795 | .8885 / .8883 | .0001 / .0001 |
| 7 | 3 | 7.00000 | 6.09808 | - .90192 | .9367 / .9367 | .0003 / .0003 |
| 7 | 4 | 7.00000 | 6.96410 | - .03590 | .9960 / .9960 | .0001 / .0001 |
| 7 | 5 | 7.00000 | 7.83013 | .83013 | .9503 / .9503 | .0003 / .0003 |

## Table 5

Lower and Upper Bounds of Power for

Duncan's Multiple Range Test for Four Means

| $\xi_2$ | $\xi_3$ | $\xi_1 = 0$ | | $\xi_1 = 1$ | | $\xi_1 = 2$ | |
|---|---|---|---|---|---|---|---|
| | | $P_L$ | $P_U$ | $P_L$ | $P_U$ | $P_L$ | $P_U$ |
| -4 | 5 | .0238 | .0240 | .1014 | .1018 | .2853 | .2860 |
| -3 | 4 | .0225 | .0227 | .0971 | .0977 | .2768 | .2781 |
| -3 | 5 | .0234 | .0236 | .1000 | .1005 | .2826 | .2835 |
| -2 | 3 | .0199 | .0199 | .0885 | .0886 | .2599 | .2601 |
| -1 | 0 | .0182 | .0185 | .0829 | .0837 | .2482 | .2499 |
| -1 | 1 | .0182 | .0185 | .0829 | .0837 | .2483 | .2499 |
| -1 | 2 | .0189 | .0192 | .0852 | .0861 | .2530 | .2548 |
| 0 | 0 | .0173 | .0180 | .0799 | .0821 | .2422 | .2467 |
| 0 | 1 | .0181 | .0183 | .0824 | .0831 | .2474 | .2488 |
| 0 | 2 | .0189 | .0192 | .0852 | .0860 | .2530 | .2548 |
| 1 | 0 | .0182 | .0185 | .0829 | .0837 | .2482 | .2499 |
| 1 | 1 | .0189 | .0192 | .0851 | .0860 | .2528 | .2546 |
| 2 | 1 | .0204 | .0206 | .0900 | .0909 | .2627 | .2644 |
| 3 | 2 | .0225 | .0232 | .0970 | .0994 | .2765 | .2813 |
| 4 | 2 | .0240 | .0243 | .1020 | .1028 | .2865 | .2881 |
| 4 | 3 | .0245 | .0246 | .1036 | .1038 | .2896 | .2900 |

## Table 5 (cont.)

### Lower and Upper Bounds of Power for
### Duncan's Multiple Range Test for Four Means

| $\xi_2$ | $\xi_3$ | $\xi_1 = 3$ | | $\xi_1 = 4$ | | $\xi_1 = 5$ | |
|---|---|---|---|---|---|---|---|
| | | PL | PU | PL | PU | PL | PU |
| -4 | 5 | .5553 | - .5562 | .8011 | - .8017 | .9397 | - .9399 |
| -3 | 4 | .5452 | - .5468 | .7938 | - .7949 | .9364 | - .9370 |
| -3 | 5 | .5521 | - .5532 | .7988 | - .7996 | .9386 | - .9390 |
| -2 | 3 | .5250 | - .5253 | .7790 | - .7793 | .9299 | - .9300 |
| -1 | 0 | .5104 | - .5125 | .7680 | - .7696 | .9248 | - .9256 |
| -1 | 1 | .5106 | - .5126 | .7682 | - .7697 | .9249 | - .9256 |
| -1 | 2 | .5164 | - .5186 | .7725 | - .7742 | .9269 | - .9277 |
| 0 | 0 | .5031 | - .5085 | .7626 | - .7665 | .9224 | - .9242 |
| 0 | 1 | .5093 | - .5111 | .7672 | - .7685 | .9245 | - .9251 |
| 0 | 2 | .5164 | - .5185 | .7725 | - .7741 | .9269 | - .9277 |
| 1 | 0 | .5104 | - .5125 | .7680 | - .7696 | .9248 | - .9256 |
| 1 | 1 | .5159 | - .5181 | .7721 | - .7737 | .9267 | - .9274 |
| 2 | 1 | .5280 | - .5301 | .7810 | - .7826 | .9307 | - .9314 |
| 3 | 2 | .5448 | - .5505 | .7934 | - .7975 | .9363 | - .9381 |
| 4 | 2 | .5568 | - .5587 | .8022 | - .8035 | .9401 | - .9407 |
| 4 | 3 | .5604 | - .5609 | .8048 | - .8051 | .9413 | - .9414 |

## XV.   Appendix B

Certain small error probabilities are of interest in work of this kind.  Some of these errors are given earlier in this paper.  The method and calculations used in arriving at these error probabilities is now given.

First the maximum error in using $(P_L + P_U)/2$ as an approximation of $P_{21}$ in the three mean case is determined. The error at any parameter point is always considerably less than $(P_U - P_L)/2$.  This value is a maximum when the parameter point falls at the center of gravity of ABCD, say, in Figure 1 of VIII.  The coordinates of A, D, M, F, and H are, clearly

$$
\begin{array}{lll}
\text{A:} & x_1 = q_2 & , \quad x_2 = (2q_3 - q_2)/\sqrt{3}, \\
\text{D:} & x_1 = q_3 & , \quad x_2 = q_3/\sqrt{3}, \\
(1) \quad \text{M:} & x_1 = (q_2 + q_3)/2 & , \quad x_2 = (3q_3 - q_2)/(2\sqrt{3}), \\
\text{F:} & x_1 = q_3 & , \quad x_2 = -q_3/\sqrt{3}, \text{ and} \\
\text{H:} & x_1 = q_2 & , \quad x_2 = (q_2 - 2q_3)/\sqrt{3}.
\end{array}
$$

Let

$$(2) \qquad \xi_{01} = (q_2 + q_3)/2 \text{ and } \xi_{02} = (3q_3 - q_2)/(2\sqrt{3}).$$

Letting $t_k = (x_k - \xi_k)/\sqrt{2}$, k=1,2, the $t_1$ and $t_2$ coordinates of A, D, M, F, and H when the parameter point is given by (2) are

$$
\begin{array}{lll}
\text{A:} & t_1 = (q_2-q_3)/(2\sqrt{2}) \ , & t_2 = (q_3-q_2)/(2\sqrt{6}) \ , \\
\text{D:} & t_1 = (q_3-q_2)/(2\sqrt{2}) \ , & t_2 = (q_2-q_3)/(2\sqrt{6}) \ , \\
(3) \quad \text{M:} & t_1 = 0 & , \quad t_2 = 0 \ ,
\end{array}
$$

$$F: \quad t_1 = (q_3-q_2)/(2\sqrt{2}) \;, \quad t_2 = (q_2-4q_3)/(2\sqrt{6}) \;, \text{ and}$$

$$H: \quad t_1 = (q_2-q_3)/(2\sqrt{2}) \;, \quad t_2 = (3q_2-6q_3)/(2\sqrt{6}) \;.$$

Let the maximum error for parameter point $(\xi_1,\xi_2)$ be given by

$$(4) \quad ME(\xi_1, \xi_2) = (P_U - P_L)/2 \quad \text{or}$$

$$ME(\xi_1, \xi_2) = (P[\text{region ABDC}] + P[\text{region EFGH}])/2.$$

In standard coordinates,

$$(5) \quad 2ME(\xi_{01},\xi_{02}) = \int_{-h_2}^{h_2} g_1(t_1)dt_1 \int_{-h_4}^{h_4} g_2(t_2)dt_2 + \int_{h_5}^{h_6} g_2(t_2)dt_2$$

where

$$(6) \quad \begin{aligned} h_2 &= (q_3 - q_2)/(2\sqrt{2}) \;, \\ h_4 &= (q_3 - q_2)/(2\sqrt{6}) \;, \\ h_5 &= (3q_2 - 6q_3)/(2\sqrt{6}) \;, \text{ and} \\ h_6 &= (q_2 - 4q_3)/(2\sqrt{6} \;. \end{aligned}$$

For Duncan's new multiple range test, $q_2=2.7715$ and $q_3=2.9185$. From (2), $\xi_{01}=2.8450$, and $\xi_{02}=1.7275$. Using (5) and (6), we find

$$ME(2.8450, 1.7275) = \left. t_1 \begin{Bmatrix} .0520 \\ -.0520 \end{Bmatrix} \left[ t_2 \begin{Bmatrix} .0300 \\ -.0300 \end{Bmatrix} + t_2 \begin{Bmatrix} 1.8772 \\ -1.8772 \end{Bmatrix} \right] \right/ 2$$

$$= (0.041471)(0.023933 + 0.004348)/2$$

$$= 0.000587 \;.$$

When $ME(\xi_1,\xi_2)$ is calculated at selected parameter points for Duncan's new multiple range test, the results are as shown in Table 1.

### Table 1

Maximum Error in Using $(P_L+P_U)/2$ to Approximate $P_{21}$

| $\xi_2$ | $\xi_1$ | | | | | |
|---|---|---|---|---|---|---|
| | 0 | 1 | 2 | 3 | 4 | 5 |
| 0 | .000062 | .000201 | .000395 | .000468 | .000338 | .000148 |
| 1 | .000068 | .000218 | .000429 | .000509 | .000367 | .000161 |
| 2 | .000066 | .000214 | .000421 | .000500 | .000360 | .000158 |
| 3 | .000044 | .000142 | .000279 | .000331 | .000239 | .000104 |
| 4 | .000018 | .000058 | .000115 | .000136 | .000098 | .000043 |
| 5 | .000005 | .000015 | .000029 | .000034 | .000024 | .000011 |

The next small probability obtained is an upper bound for the remainder, r, in (37) of VI. Only region A is described since region B is symmetric with region A about the $x_1$ axis.

The boundary planes of region A are $p_1$, $p_4$, $p_5$, $p_{10}$, $p_{11}$, and $p_{13}$ of (7) in VI. Region A is a solid figure, oblique truncated quadrangular prism, with six faces and eight vertices. Let the vertices in $p_{13}$, be $C_1$, $C_2$, $C_3$, and $C_4$, and the vertices in $p_1$ be $D_1$, $D_2$, $D_3$, $D_4$. Denote the coordinates by $(x_1, x_2, x_3)$ so that the vertices are

$$C_1(2q_3-q_4,\ q_4/\sqrt{3},\ -2q_4/\sqrt{6})\ ,$$
$$C_2(q_3,\ q_3/\sqrt{3},\ (-3q_4+q_3)/\sqrt{6}\ ,$$
(7)
$$C_3(q_4,\ q_4/\sqrt{3},\ -2q_4/\sqrt{6})\ ,$$
$$C_4(q_3,\ (2q_4-q_3)/\sqrt{3},\ (-q_4-q_3)/\sqrt{6}\ ,$$
$$D_1(q_2,\ (2q_3-q_2)/\sqrt{3},\ (-4q_3+2q_2)/\sqrt{6})\ ,$$

$$D_2(q_2, \ (2q_3-q_2)/\sqrt{3}, \ (-3q_4-q_3+2q_2)/\sqrt{6}) \ ,$$
$$D_3(q_2, \ (2q_4-q_2)/\sqrt{3}, \ (-4q_4+2q_2)/\sqrt{6}) \ , \text{ and}$$
$$D_4(q_2, \ (2q_4-q_2)/\sqrt{3}, \ (-q_4-3q_3+2q_2)/\sqrt{6}.$$

Clearly the ranges for $x_1$, $x_2$, and $x_3$ in region A are

(8)
$$q_2 \leq x_1 \leq q_4 \ ,$$
$$q_3/\sqrt{3} \leq x_2 \leq (2q_4-q_2)/\sqrt{3} \ , \text{ and}$$
$$(-4q_4+2q_2)/\sqrt{6} \leq x_3 \leq (-q_4-q_3)/\sqrt{6} \ .$$

By symmetry, the ranges in region B are found to be

(9)
$$q_2 \leq x_1 \leq q_4 \ ,$$
$$(-2q_4+q_2)/\sqrt{3} \leq x_2 \leq -q_3/\sqrt{3} \ , \text{ and}$$
$$(q_4+q_3)/\sqrt{3} \leq x_3 \leq (4q_4-2q_2)/\sqrt{3} \ .$$

Pass planes through the end points of the ranges and parallel to the coordinate planes. The figure formed, A', encloses the oblique truncated quadrangular prism, A, with vertices $C_1$, $C_2$, $C_3$, $C_4$, $D_1$, $D_2$, $D_3$, and $D_4$ lying in the boundary of A'. Region B is enclosed by a similar solid figure, B'. Therefore, the probability over both of these regions is greater than r, the probability over regions A and B.

Let the parameter point be at the center of A', say. Thus,

(10)
$$\xi_1 = (q_4+q_2)/2 \ ,$$
$$\xi_2 = (2q_4+q_3-q_2)/(2\sqrt{3}) \ , \text{ and}$$
$$\xi_3 = (-5q_4-q_3+2q_2)/(2\sqrt{6}) \ .$$

In standardized coordinates, the ranges in (8) and (9) become

$$0 \leq t_1 \leq |q_4-q_2|/(2\sqrt{2}) = a \ ,$$

(11)
$$0 \leq t_2 \leq |2q_4-q_3-q_2|/(2\sqrt{6}) = b \ ,$$

$$0 \leq t_3 \leq |3q_4-q_3-2q_2|/(4\sqrt{3}) = c \ ,$$

and

$$0 \leq t_1 \leq |q_4-q_2|/(2\sqrt{2}),$$

(12) $\quad d = (-6q_4-q_3+3q_2)/(2\sqrt{6}) \leq t_2 \leq (-2q_4-3q_3+q_2)/(2\sqrt{6}) = e,$

$\quad f = (7q_4+q_3-2q_2)/(4\sqrt{3}) \leq t_3 \leq (13q_4+q_3-6q_2)/(4\sqrt{3}) = g,$

respectively, where a, b, c, d, e, f, and g are defined as indicated.

For Duncan's new multiple range test, a=.0867, b=.0701, c=.0849, d=-2.5929, e=-2.4528, f=3.5107, and g=3.6806.  Hence,

$$r < \left._{t_1}\left\{^{.0867}_{-.0867}\right]\left[_{t_2}\left\{^{.0701}_{-.0701}\right]._{t_3}\left\{^{.0849}_{-.0849}\right\}+_{t_2}\left\{^{-2.4528}_{-2.5929}\right\}._{t_3}\left\{^{3.6806}_{3.5107}\right\}\right.\right]$$

$$= (.069090) \quad (.055886)(.067659) + (.002329)(.000107)$$

$$= .000261.$$

The remainder is much smaller than .000261, as is indicated by the argument which follows.  The solid regions A' and B' are cylinders.  Regions A and B are truncated cylinders.  The volume of cylinder A is less than for a cylinder, A*, with the same base and same altitude.  It is sufficient to compare the volume of A' with A*, denoted by V(A') and V(A*), respectively.  Now,

$V(A') = (q_4-q_2)(2q_4-q_3-q_2)(3q_4-q_3-2q_2) /(3\sqrt{2})$ , and

$V(A*) = (q_4-q_2)(2q_4-2q_3)(3q_4-3q_3) /(3\sqrt{2})$ .

Thus, $\dfrac{V(A')}{V(A*)}$ = $[(2q_4-q_3-q_2)(3q_4-q_3-2q_2)]/[6(q_4-q_3)^2]$

$\qquad\qquad$ = $[(.3434)(.5886)]/[6(.00964324)]$

$\qquad\qquad$ = 3.4934.

Taking into account the fact that $V(A) + V(B) < V(A*) + V(B*)$, the size of the ratio $V(A')/V(A*)$, and the fact that the normal function is nearly flat over a region as small as A', we may safely conclude that $r < .00001$. As the parameter point moves away from either A' or B', the remainder rapidly decreases.