EXAMPLES AND THEOREMS FOR GENERALIZED PARACOMPACT TOPOLOGICAL SPACES

by

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(ABSTRACT)

In this thesis we answer a number of unsolved problems in generalized paracompact topological spaces. Examples satisfying the T_4 separation axiom are constructed showing the relationship between the properties $B(D, \omega_0)$–refinability, $B(D, \lambda)$–refinability, and weak $\overline{\theta}$–refinability. The properties $B(D, \lambda)$–refinability and weak $\overline{\theta}$–refinability are shown to be strictly weaker than $B(D, \omega_0)$–refinability.

Sum theorems, mapping theorems, and σ–product theorems are obtained for $B(D, \omega_0)$–refinability, weak $\overline{\theta}$–refinability, and several other properties. The σ–product theorem for $B(D, \omega_0)$–refinability, weak $\overline{\theta}$–refinability, and other properties are shown to follow from a new special $B(D, \omega_0)$ sum theorem.

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CHAPTER I

INTRODUCTION AND DEFINITIONS

§1. Introduction.

Since the introduction of the class of metric spaces by M. Frechet in 1906, various properties of metric spaces which do not imply metrizability have been studied. Normality is of primary importance among these properties due to Tietze's Extension Theorem and Urysohn's Lemma. Paracompactness is a property which lies strictly between metrizability and normality in the class of T_1 spaces. R. H. Bing [4] in his 1951 paper entitled "Metrization of Topological Spaces" introduced the properties collectionwise normality (CWN) and metacompactness. CWN and metacompactness are strictly weaker than paracompactness. Diagram 1.1.1 below shows the relationships between these properties.

Diagram 1.1.1.

$$\begin{tikzcd}
 & & \text{CWN} \arrow[r] & \text{normal} \\
\text{metrizable} \arrow[r] & \text{paracompact} \arrow[ur] \arrow[dr] & & \\
 & & \text{metacompact} &
\end{tikzcd}$$

In 1965, A. V. Arkhangel'skii [2] implicitly introduced the notion of mesocompactness and proved that a normal k–space X is paracompact iff X is mesocompact. J. R. Boone [5] later gave this concept the name "mesocompactness" and showed that mesocompactness lies strictly between paracompactness and metacompactness.

R. Arens and J. Dugundji [1] independently introduced the notion of metacompactness in 1950 and proved that countable compactness and compactness are equivalent in the class of metacompact spaces. J. M. Worrell and H. Wicke [32] in 1965 introduced the notion of θ–refinability as a generalization of metacompactness. In 1972, the property weak θ–refinability was introduced by H. R. Bennett and D. J.

Lutzer [3] as a generalization of θ–refinability. J. C. Smith [24] demonstrated that countable compactness and compactness are equivalent in the class of weak θ–refinable spaces, thus generalizing the result above of Arens and Dugundji. Smith [24] also introduced the notion of weak $\overline{\theta}$–refinability, a property which lies strictly between weak θ–refinability and θ–refinability. He then demonstrated that the class of metacompact spaces is exactly the class of almost expandable, weak $\overline{\theta}$–refinable spaces and that CWN, weak $\overline{\theta}$–refinable spaces are paracompact.

In 1980, J. C. Smith [26] defined the property of $B(P,\alpha)$–refinability where α represents a fixed ordinal and P represents any one of several properties which collections of sets may satisfy, such as discreteness (D) or local finiteness (LF). R. H. Price [19] in 1987 obtained a weak $\overline{\theta}$–type characterization of $B(D,\omega_0)$–refinability and used it to demonstrate that $B(D,\omega_0)$–refinability is strictly weaker than θ–refinability. Price also showed that, in the class of CWN spaces, paracompactness is equivalent to $B(LF,\lambda)$–refinability where λ represents any countable ordinal, thus generalizing the result above of Smith.

The question of whether $B(D,\omega_0)$–refinability is equivalent to $B(D,\lambda)$–refinability for some countable ordinal $\lambda > \omega_0$ was asked by J. C. Smith in 1980 [26]. Smith conjectured that this question would be answered in the negative and asked for examples. Chapter II of this thesis gives a method of constructing a normal space that is $B(D,\lambda)$–refinable but not $B(D,\alpha)$–refinable for any ordinal $\alpha < \lambda$. We also give an example of a T_4 space that is weak $\overline{\theta}$–refinable but not $B(D,\omega_0)$–refinable, thus demonstrating that weak $\overline{\theta}$–refinability is strictly weaker than $B(D,\omega_0)$–refinability. Other important and interesting examples are also included in Chapter II.

Diagram 1.1.2 below illustrates the relationships between the properties mentioned above. It is known that implications not indicated in Diagram 1.1.2 are not true in general. Those marked with an "?" remain open problems. Those implications marked with an asterisk are new results in this thesis.

Diagram 1.1.2.

$$
\begin{tikzcd}
\text{paracompact} \arrow[rr] \arrow[drrr] & & \text{mesocompact} \arrow[rr] \arrow[dr] & & \text{metacompact} \arrow[rr] \arrow[dl] & & \theta\text{–refinable} \arrow[dlll] \\
& & & B(D,\omega_0)\text{–refinable} \arrow[dlll, "*"'] \arrow[drrr, "*"] & & & \\
\text{weak } \overline{\theta}\text{–refinable} \arrow[d, shift right=1] & & & & & & B(D,\lambda)\text{–refinable} \arrow[d, shift right=1] \\
B(D,\omega_0^2)\text{–refinable} \arrow[u, shift right=1, "?"'] \arrow[d, shift right=1] & & & & & & B(LF,\lambda)\text{–refinable} \arrow[u, shift right=1, "?"'] \arrow[d] \\
B(LF,\omega_0^2)\text{–refinable} \arrow[u, shift right=1, "?"'] \arrow[rrrrrr] & & & & & & \text{weak } \theta\text{–refinable}
\end{tikzcd}
$$

Chapter III of this thesis contains a discussion of the progress toward obtaining a perfect mapping theorem for the property $B(D,\omega_0)$–refinability. H. J. K. Junnila [16] has demonstrated that closed images of θ–refinable spaces are θ–refinable. D. Burke [7] has shown that perfect images of weak θ–refinable spaces are weak θ–refinable. Thus the natural question is whether such mapping theorems hold for the properties $B(D,\omega_0)$–refinability and weak $\overline{\theta}$–refinability. We demonstrate that the Locally Finite Sum Theorem holds for $B(D,\omega_0)$–refinability on countably metacompact spaces, a necessary condition for the perfect mapping theorem to hold. Also, we demonstrate that if a space is hereditarily $B(CP,\lambda)$–refinable then it is hereditarily weak θ–refinable. It then follows that closed images of hereditarily weak $\overline{\theta}$–refinable spaces are hereditarily weak θ–refinable, a variation of Burke's result above. Other results are also included.

In Chapter IV we obtain several new results concerning σ–products. Recently, there has been much interest in answering the following question. Let P represent some topological property and let X be a σ–product space. If every finite subproduct of X has property P, does X necessarily have property P? If the answer is in the affirmative, P is said to satisfy the "σ–product Theorem". Many properties have been shown to satisfy the σ–product Theorem such as paracompactness [17], metacompactness [29], and θ–refinability [30]. Also, if the σ–product X is normal, the σ–product Theorem holds for the property CWN [10]. On the other hand, the property orthocompactness does not satisfy the σ–product Theorem [30]. We demonstrate that the σ–product Theorem holds for the properties weak $\overline{\theta}$–refinability and $B(D,\omega_0)$–refinability. In fact, we obtain these results as special cases of a new sum theorem which we call a "Special $B(D,\omega_0)$ Sum Theorem". Furthermore, we show that if the σ–product X is normal, the σ–product Theorem holds for the properties mesocompactness, discrete compact finite expandability, para–Lindelöfness, and closed hereditary irreducibility.

§2. Definitions.

In this section we explain the terminology and notation used in this thesis and give definitions and related lemmas which are basic to the subject matter. For the meaning of concepts used without definition in this work, we refer the reader to the texts [13] and [31].

Throughout the following, the word "space" always refers to a T_1 topological space. That is, a space in which each singleton is a closed set. If H is a subset of a space X, we denote the closure of H by "$cl(H)$" or "$\overline{H}$" and the interior of H by "$int(H)$".

The abbreviations "$TFAE$" and "iff" are used to represent the phrases "the following statements are equivalent" and "if and only if" respectively.

We denote the cardinality of a set A by "$|A|$". The symbol "N" represents the set of positive integers. The cardinality of $N = \aleph_0$. The first infinite ordinal is denoted by ω_0 and the first uncountable ordinal by ω_1. We will represent ordinal numbers with lower case Greek letters, and elements of N with lower case English letters. The letter λ will always denote a countably infinite ordinal.

Set Operations 1.2.1. Let $\mathcal{U}$ and $\mathcal{V}$ be collections of subsets of a space X and let $A \subseteq X$. Then

(a) $\cup\mathcal{U} = \cup\{U : U \in \mathcal{U}\}$

(b) $\cap\mathcal{U} = \cap\{U : U \in \mathcal{U}\}$

(c) $st(A,\mathcal{U}) = \cup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$

(d) $st(x,\mathcal{U}) = st(\{x\},\mathcal{U})$ for each $x \in X$, and

(e) $\mathcal{U} \cap \mathcal{V} = \{U \cap V : U \in \mathcal{U} \text{ and } V \in \mathcal{V}\}$.

Definition 1.2.2. Let $\mathcal{H}$ and $\mathcal{U}$ be collections of sets. The collection $\mathcal{H}$ **partially refines** $\mathcal{U}$ provided every member of $\mathcal{H}$ is contained in some member of $\mathcal{U}$. If $\cup\mathcal{H} = \cup\mathcal{U}$ is also the case, we call $\mathcal{H}$ a **refinement** of $\mathcal{U}$.

Properties satisfied by collections of sets 1.2.3. Let $\mathcal{H} = \{H_\alpha : \alpha \in A\}$ be a collection of subsets of a space X. For each $x \in X$, define

$$ord(x,\mathcal{H}) = |\{H \in \mathcal{H} : x \in H\}|.$$

(a) $\mathcal{H}$ is **point finite (PF)** provided $ord(x,\mathcal{H})$ is finite for every $x \in X$.

(b) $\mathcal{H}$ is **point countable (PC)** provided $ord(x,\mathcal{H}) \leq \aleph_0$ for every $x \in X$.

(c) $\mathcal{H}$ is **locally finite (LF)** provided every $x \in X$ has a neighborhood V such that $|\{H \in \mathcal{H} : V \cap H \neq \emptyset\}| < \aleph_0$.

(d) $\mathcal{H}$ is **bded-LF** provided there exists $n \in N$ such that every $x \in X$ has a neighborhood V such that $|\{H \in \mathcal{H} : V \cap H \neq \emptyset\}| \leq n$. In this case we say that $\mathcal{H}$ is **n–bded LF**.

(e) $\mathcal{H}$ is **discrete (D)** provided $\mathcal{H}$ is 1–bded LF.

(f) $\mathcal{H}$ is **locally countable (LC)** provided every $x \in X$ has a neighborhood V such that $|\{H \in \mathcal{U} : V \cap H \neq \emptyset\}| \leq \aleph_0$.

(g) $\mathcal{H}$ is **closure–preserving (CP)** provided for every $A' \subseteq A$, $cl(\cup\{H_\alpha : \alpha \in A'\}) = \cup\{cl(H_\alpha) : \alpha \in A'\}$.

(h) $\mathcal{H}$ is a **special CP family** provided $\mathcal{H}$ is CP and there exists a point finite open collection $\mathcal{U}$ such that for $H \in \mathcal{H}$, $X - \overline{H} = \cup\{U \in \mathcal{U} : U \cap H = \emptyset\}$.

(i) $\mathcal{H}$ is **compact finite (CF)** provided for every compact subset K of X, $|\{H \in \mathcal{H} : H \cap K \neq \emptyset\}| < \aleph_0$.

(j) $\mathcal{H}$ is **interior preserving** provided for each $A' \subseteq A$, $int(\cap\{H_\alpha : \alpha \in A'\}) = \cap\{int(H_\alpha) : \alpha \in A'\}$.

Diagram 1.2.4. The diagram below illustrates general relationships between properties which collections of sets may satisfy.

$$\begin{tikzcd}
 & & & CF \arrow[r] & PF \arrow[d] \\
D \arrow[r] & bded-LF \arrow[r] & LF \arrow[ur] \arrow[r] \arrow[dr] & LC \arrow[r] & PC \\
 & & & CP &
\end{tikzcd}$$

Remarks 1.2.5.

(a) Let $\mathcal{F}$ be a collection of closed subsets of a space X. Then $\mathcal{F}$ is LF iff $\mathcal{F}$ is CP and PF.

(b) Let P represent one of the following properties: D, $bded-LF$, LF or CP. If $\mathcal{H}$ is a P collection of subsets of a space X, then $\{cl(H) : H \in \mathcal{H}\}$ is also a P collection.

Definition 1.2.6. Let $\mathcal{H} = \{H_\alpha : \alpha \in A\}$ and $\mathcal{U}$ be collections of subsets of a space X, and let P represent any property which collections of sets may satisfy.

(a) $\mathcal{H}$ is a **P–(partial) refinement** of $\mathcal{U}$ provided $\mathcal{H}$ is a P collection which (partially) refines $\mathcal{U}$.

(b) $\mathcal{H}$ is a **P–closed (open) refinement** of $\mathcal{U}$ provided $\mathcal{H}$ is a P–refinement of $\mathcal{U}$

and every member of $\mathcal{H}$ is a closed (open) set.

(c) If $\mathcal{H}$ refines $\mathcal{U}$, we refer to $\mathcal{H}$ as a **one–to–one refinement** of $\mathcal{U}$ provided we can index $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ such that $H_\alpha \subseteq U_\alpha$ for every $\alpha \in A$.

(d) A space X is P–**refinable** provided every open cover of X has a P–refinement.

Definition 1.2.7.

(a) Let A be any set. We call $\mathcal{B} = \{B_\gamma : \gamma \in \Gamma\}$ a **partition**, or **decomposition**, of A provided $\cup\mathcal{B} = A$ and $\mathcal{B}$ is pairwise disjoint.

(b) Let $\mathcal{H} = \{H_\alpha : \alpha \in A\}$ be a collection of subsets of a space X. A collection $\mathcal{K}$ of subsets of X is an **amalgamation** of $\mathcal{H}$ provided there exists a partition $\{B_\gamma : \gamma \in \Gamma\}$ of A such that $\mathcal{K} = \{\cup\{H_\alpha : \alpha \in B_\gamma\} : \gamma \in \Gamma\}$.

Amalgamation Lemma 1.2.8. *[19] Let P represent one of the following properties: D, bded–LF, LF, LC, PF, PC, CF, or CP. If $\mathcal{H}$ is a P collection of subsets of a space X and $\mathcal{K}$ is an amalgamation of $\mathcal{H}$, then $\mathcal{K}$ is also a P collection.*

Corollary 1.2.9. *[19] Let P represent one of the following properties: D, bded–LF, LF, LC, PF, PC, CF, or CP. If a cover $\mathcal{U}$ of a space X has a P–refinement, then $\mathcal{U}$ has a one–to–one P–refinement.*

Corollary 1.2.10. *[19] Let P represent one of the following properties: D, bded-LF, LF, LC, PF, PC, CF, or CP. If $\mathcal{U}$ is a cover of a space X, and $\mathcal{U}$ has a P–open refinement, then $\mathcal{U}$ has a one–to–one P–open refinement. If P represents D, bded–LF, LF, or CP, then if a cover $\mathcal{U}$ of a space X has a P–closed refinement, then $\mathcal{U}$ has a one–to–one P–closed refinement.*

Remark 1.2.11. *Throughout this thesis we will often use results 1.2.9 and 1.2.10 above and assume that refinements are one–to–one whenever we can do so without loss of generality.*

Special types of refinements 1.2.12. Let $\mathcal{V}$, $\mathcal{G}$ and $\mathcal{U}$ be collections of subsets of a space X, and let P represent any property which collections of sets may satisfy.

(a) $\mathcal{V}$ is a **star refinement** of $\mathcal{U}$ provided $\{st(V,\mathcal{V}) : V \in \mathcal{V}\}$ refines $\mathcal{U}$.

(b) $\mathcal{V}$ is a **pt–star refinement** of $\mathcal{U}$ provided $\{st(x,\mathcal{V}) : x \in X\}$ refines $\mathcal{U}$.

(c) $\mathcal{V}$ is a **minimal refinement** of $\mathcal{U}$ provided $\mathcal{V}$ refines $\mathcal{U}$ such that no subcollection of members of $\mathcal{V}$ refines $\mathcal{U}$.

(d) $\mathcal{G}$ is a **σ–P–refinement** of $\mathcal{U}$ provided we can write $\mathcal{G} = \cup\{\mathcal{G}_n : n \in N\}$ such that $\mathcal{G}_n$ is a partial refinement of $\mathcal{U}$ satisfying P for each $n \in N$ and $\mathcal{G}$ defines $\mathcal{U}$.

Special types of continuous maps 1.2.13. Let $f : X \to Y$ be a map from a space X to a space Y.

(a) If $f(H)$ is an open (closed) subset of Y for every open (closed) subset H of X, we refer to f as an **open (closed) map.**

(b) The map f is a **perfect (quasi–perfect)** provided f is continuous, closed and $f^{-1}(y) = \{x \in X : f(x) = y\}$ is compact (countably compact) for every $y \in Y$.

(c) We refer to f as a **finite–to–one map** provided $f^{-1}(y)$ is finite for every $y \in Y$.

(d) The map f is **bounded** provided there exists $n \in N$ such that $|f^{-1}(y)| \leq n$ for every $y \in Y$. In this case we sometimes refer to f as a **n–bded map.**

Remark 1.2.14. *It should be clear that every bded-map is finite–to–one, and every closed, continuous, finite–to–one map is perfect.*

Map notation 1.2.15. Let $f : X \to Y$ be a map from a space X to a space Y, and let $\mathcal{U}$ and $\mathcal{V}$ be collections of subsets of X and Y, respectively. Define

(i) $f(\mathcal{U}) = \{f(U) : U \in \mathcal{U}\}$, and

(ii) $f^{-1}(\mathcal{V}) = \{f^{-1}(V) : V \in \mathcal{V}\}$.

Definition 1.2.16. Let $\{X_\alpha : \alpha \in A\}$ be a collection of spaces (not necessarily pairwise disjoint). For each $\alpha \in A$, let $X^*_\alpha = \{(x,\alpha) : x \in X_\alpha\}$, and define a topology on X^*_α in the obvious way to make X^*_α homeomorphic to X_α. By construction, $\{X^*_\alpha :$

$\alpha \in A\}$ is pairwise disjoint. Let $Y = \cup\{X_\alpha^* : \alpha \in A\}$, and define a subset U of Y to be open iff $U \cap X_\alpha^*$ is open in X_α^* for every $\alpha \in A$. We refer to the space Y as the **disjoint sum** of $\{X_\alpha : \alpha \in A\}$ denoted by $\oplus\{X_\alpha : \alpha \in A\}$.

Definition 1.2.17. Let $\{F_\alpha : \alpha \in A\}$ be a cover of a space X. The **canonical map** $f : \oplus\{F_\alpha : \alpha \in A\} \to X$ is defined by $f(x, \alpha) = x$ for each $x \in X$ and $\alpha \in A$.

Remark 1.2.18. *The canonical map defined above is both onto and continuous.*

Sum Theorems 1.2.19. *Let Q represent some topological property such as paracompactness, and let P represent one of the following properties: D, (bded) LF, or CP.*

(a) *Property Q satisfies the* **P Sum Theorem** *provided for every space X, if $\{F_\alpha : \alpha \in A\}$ is a P–closed cover of X such that F_α satisfies property Q for every $\alpha \in A$, then X satisfies property Q.*

(b) *Property Q satisfies the* **Countable Sum Theorem** *provided for every space X, if $\{F_n : n \in N\}$ is a countable closed cover of X such that F_n satisfies property Q for every $n \in N$, then X satisfies property Q.*

Lemma 1.2.20. *Let Q represent a topological property which is preserved under both disjoint sums and closed, continuous (bded) finite–to–one maps. Then Q satisfies the (bded–) LF Sum Theorem.*

Proof: Suppose $X = \cup\{F_\alpha : \alpha \in A\}$ where $\{F_\alpha : \alpha \in A\}$ is a (bded) LF family of closed subsets of X such that each F_α has property Q. Then by hypothesis, $\oplus\{F_\alpha : \alpha \in A\}$ has property Q. Let $f : \oplus\{F_\alpha : \alpha \in A\} \to X$ be the canonical map. It is easy to see that f is a continuous and (bded) finite–to–one map onto X. Furthermore, f is a closed map. To see this, suppose that K is a closed subset of $\oplus(F_\alpha : \alpha \in A\}$ and $x \notin f(K)$. Since $\{F_\alpha : \alpha \in A\}$ is a locally finite family of closed subsets of X, let U be an open set containing x which meets only those finitely many F_α's, say $F_{\alpha_1}, F_{\alpha_2}, \ldots, F_{\alpha_j}$ such that $x \in F_\alpha$. For each $i \in \{1, 2, \ldots, j\}$, let U_i^* be an open (in $F_{\alpha_i} \times \{\alpha_i\}$) subset of $F_{\alpha_i} \times \{\alpha_i\}$ containing (x, α_i) which misses K. Then let U_i be an open (in X) subset of X such that $U_i \cap F_{\alpha_j} = f(U_i^*)$. Then $U \cap (\cap\{U_i : i \in \{1, 2, \ldots, j\}\})$ is an open set containing x that misses $f(K)$. Thus $f(K)$ is a closed set in X. It now follows from the hypothesis that X has property Q.

Definition 1.2.21. Let $\mathcal{U}^* = \{\mathcal{U}_n : n \in N\}$ be a countable family of open covers of a space X. $\mathcal{U}^*$ is a **normal family** of covers provided $\mathcal{U}_{n+1}$ star refines $\mathcal{U}_n$ for every $n \in N$. An open cover $\mathcal{U}$ of a space X is a **normal cover** of X provided $\mathcal{U}$ is a member of some normal family of open covers of X.

Definition 1.2.22. A space X is **fully normal** provided every open cover of X is a normal cover.

Remark 1.2.23. *A. H. Stone [28] proved that a T_1 space X is paracompact iff X is fully normal.*

Definition 1.2.24. A space X is **paracompact (subparacompact, metacompact, meta–Lindelöf, mesocompact, para–Lindelöf, screenable, irreducible, resp. orthocompact)** provided every open cover of X has a LF–open (σ–discrete–closed, PF–open, PC–open, compact finite open, locally countable open, σ–disjoint open, minimal open, resp. interior preserving open) refinement.

Definition 1.2.25. A space X is a **k–space** provided a subset F of X is a closed set iff $F \cap K$ is compact for every compact subset K of X.

Lemma 1.2.26. *Every locally compact, T_2–space is a k–space.*

Proof: Assume that X is a locally compact, T_2–space. If F is a closed subset of X and K is a compact subset of X, then K is closed since X is T_2. Hence $F \cap K$ is compact since compactness is closed hereditary.

Next, suppose that $A \subseteq X$, and $A \cap K$ is compact for every compact subset K of X. We show that A is closed. Let $x \in cl(A)$. There exists a compact neighborhood K_x of x, and since $x \in cl(A)$, $K_x \cap A$ is closed and nonempty. If $x \notin K_x \cap A$, it follows that $x \notin cl(A)$, a contradiction; hence it must be the case that $x \in A$. Therefore, A is a closed set.

Definition 1.2.27. Let $\mathcal{H} = \{H_\alpha : \alpha \in A\}$ and $\mathcal{U}$ be collections of subsets of a space X. We call $\mathcal{U}$ an expansion of $\mathcal{H}$ provided we can index $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ such that $H_\alpha \subseteq U_\alpha$ for every $\alpha \in A$.

Definition 1.2.28. A topological space X is **subnormal** provided that any two disjoint closed subsets of X have a disjoint G_δ expansion.

Definition 1.2.29. Let X be a space.

(a) X is **monotonically normal** provided for each $x \in X$ and neighborhood U of x, there exists a neighborhood $H(x, U)$ such that if any $H(x, U) \cap H(y, V) \neq \emptyset$ then $x \in V$ or $y \in U$. Note that every monotonically normal space is CWN (see Definition 1.2.30 below.)

(b) X is **totally normal** provided that X is normal and every open subset G of X is the union of a locally finite (in G) family $\{V_\alpha : \alpha \in A\}$ of open sets (in X) such that each V_α is an F_σ subset of X.

Definition 1.2.30. Let X be a space.

(a) X is **collectionwise normal (CWN)** provided that every discrete closed family has a pairwise disjoint open expansion.

(b) X is **collectionwise T_2** provided that every discrete family of singleton subsets of X has a pairwise disjoint open expansion.

(c) X is **collectionwise subnormal (CWSN)** provided that every discrete closed family $\mathcal{F} = \{F_\alpha : \alpha \in A\}$ has a G_δ–expansion $\mathcal{G} = \{G_\alpha : \alpha \in A\}$ where $G_\alpha = \cap\{G_{(\alpha,n)} : n \in \omega_0\}$ and for each $x \in X$, there is an $n_x \in \omega_0$ such that x

belongs to at most one member of $\{G_{(\alpha,n_x)} : \alpha \in A\}$.

(d) X is **collectionwise δ–normal (CWδN)** provided that every discrete closed family has a pairwise disjoint G_δ–expansion.

Remark 1.2.31. *Clearly* $CWN \to CWSN \to CW\delta N$.

Definition 1.2.32. Let $\mathcal{G}$ be a collection of open subsets of a space X. We refer to $\mathcal{G}$ as a **θ–collection (almost θ–collection)** provided we can write $\mathcal{G} = \cup\{\mathcal{G}_n : n \in N\}$ such that for every $x \in X$, there exists $n_x \in N$ such that $\mathcal{G}_{n_x}$ is LF (PF) at x and $x \in \cup\mathcal{G}_{n_x}$.

Definition 1.2.33. Let $\mathcal{U}$ be an open cover of a space X, and let $\mathcal{G} = \cup\{\mathcal{G}_n : n \in N\}$ be an open refinement of $\mathcal{U}$.

(a) $\mathcal{G}$ is a **weak θ–refinement** of $\mathcal{U}$ provided $\mathcal{G}$ is an almost θ–collection.

(b) $\mathcal{G}$ is a **finally λ weak θ–refinement** of $\mathcal{U}$ provided we can write $\mathcal{G} = \cup\{\mathcal{G}_\alpha : \alpha < \lambda\}$ such that for each $x \in X$ there exists α_x such that $0 < ord(x, \mathcal{G}_{\alpha_x}) < \aleph_0$ and $x \notin \cup\{\cup\mathcal{G}_\alpha : \alpha_x < \alpha < \lambda\}$.

(c) $\mathcal{G}$ is a **weak $\overline{\theta}$–refinement** of $\mathcal{U}$ provided $\mathcal{G}$ is an almost θ–collection and $\{\cup\mathcal{G}_n : n \in N\}$ is PF.

(d) $\mathcal{G}$ is a **θ–refinement** of $\mathcal{U}$ provided $\mathcal{G}$ is an almost θ–collection and $\mathcal{G}_n$ covers X for every $n \in N$.

Expandability definitions 1.2.34.

(a) A space X is **expandable (almost expandable)** provided every LF collection of closed subsets of X has a LF (PF)–open expansion.

(b) A space X is **discretely expandable (almost discretely expandable)** provided every discrete collection of closed subsets of X has a LF (PF)–open expansion.

(c) A space X is **bded–expandable (almost bded–expandable)** provided every bded–LF collection of closed subsets of X has a LF (PF)–open expansion.

(d) A space X is **compact finite expandable (discretely compact finite expandable)** provided every locally finite (discrete) collection of closed subsets of X has a CF open expansion.

(e) Let $\mathcal{F}$ be a collection of subsets of a space X. We call $\mathcal{G} = \cup\{\mathcal{G}_n : n \in N\}$ an **(almost) θ–expansion** of $\mathcal{F}$ provided

(i) $\mathcal{G}_n$ is an open expansion of $\mathcal{F}$ for every $n \in N$, and

(ii) $\mathcal{G}$ is an (almost) θ–collection.

(f) A space X is **(almost) θ–expandable** provided every LF collection of closed subsets of X has an (almost) θ–expansion.

(g) A space X is **(almost) discretely θ–expandable** provided every discrete collection of closed subsets of X has an (almost) θ–expansion.

Remark 1.2.35. *As written earlier, throughout this thesis we use "λ" to represent a countably infinite ordinal.*

Definition 1.2.36. Let P represent one of the following properties: D, bded–LF, LF, or CP. A space X is **$B(P,\alpha)$–refinable** provided every open cover $\mathcal{U}$ of X has a refinement $\mathcal{B} = \cup\{\mathcal{B}_\gamma : \gamma < \alpha\}$ which satisfies

(i) $\{\cup\mathcal{B}_\gamma : \gamma < \alpha\}$ partitions X,

(ii) for every $\gamma < \alpha$, $\mathcal{B}_\gamma$ is a relatively P collection of closed subsets of the subspace $X - \cup\{\cup\mathcal{B}_\mu : \mu < \gamma\}$.

(iii) for every $\gamma < \alpha$, $\cup\{\cup\mathcal{B}_\mu : \mu < \gamma\}$ is a closed set.

The collection $\mathcal{B}$ is often called a **$B(P,\alpha)$–refinement** of $\mathcal{U}$.

Remark 1.2.37. *Clearly every countable T_1 space is $B(D,\omega_0)$–refinable. Also, it should be clear by the amalgamation lemma (1.2.10) that if $\mathcal{U}$ has a $B(P,\alpha)$–refinement, then $\mathcal{U}$ has a $B(P,\alpha)$–refinement as above such that $\mathcal{B}_\gamma$ is a one–to–one* *partial refinement of $\mathcal{U}$ for every $\gamma < \alpha$.*

Definition 1.2.38. A space X is said to have the **β–property** if, for every infinite cardinal α, and each monotone increasing open cover $\{G_\gamma : \gamma < \alpha\}$ of X, there is a monotone increasing open cover $\{H_\gamma : \gamma < \alpha\}$ of X such that $\overline{H}_\gamma \subseteq G_\gamma$ for each $\gamma < \alpha$.

Definition 1.2.39. A space X is said to have the **weak β–property** (also called the **$\mathcal{D}$–property** in [23]) if, for every infinite cardinal α, and each monotone increasing open cover $\mathcal{G} = \{G_\gamma : \gamma < \alpha\}$ of X, $\mathcal{G}$ has a one–to–one open refinement $\mathcal{H} = \{H_\gamma : \gamma < \alpha\}$ such that $\overline{H}_\gamma \subseteq G_\gamma$ for every $\gamma < \alpha$.

Definition 1.2.40. Let $\mathcal{G} = \{G_\alpha : \alpha \in A\}$ be an open cover of a space X. If $\mathcal{G}$ has a one–to–one open refinement $\mathcal{H} = \{H_\alpha : \alpha \in A\}$ such that $\overline{H}_\alpha \subseteq G_\alpha$, we say that $\mathcal{G}$ is **shrinkable** or that $\mathcal{H}$ is a **shrinking** of $\mathcal{G}$. If every open cover of X is shrinkable, we say that the space X has the **shrinking property**.

Remark 1.2.41. *A space X has the weak β–property iff every monotone open cover of X is shrinkable.*

Definition 1.2.42. A space (X, T), where T is the topology on X, is **semistratifiable** if there is a function $S : N \times T \to \{\text{closed subsets of } X\}$ such that:

(a) if $U \in T$, then $U = \bigcup_{n=1}^{\infty} S(n, U)$

(b) if $U, V \in T$ and $U \subseteq V$, then $S(n, U) \subseteq S(n, V)$ for each $n \in N$.

The function S is called a **semistratification** of X.

Definition 1.2.43. A space X is **stratifiable** if there is a **semistratification** S of X such that for each open set U in X, $U = \bigcup_{n=1}^{\infty} int(S(n, U))$. Such a function S is called a **stratification** of X.

Definition 1.2.44. Let A be a set and α an ordinal such that $|\alpha| \leq |A|$. Then $[A]^{\leq\alpha} = \{B \subseteq A : |B| \leq |\alpha|\}$.

Definition 1.2.45. For each α in an index set A, let X_α be a topological space. Let $X^* = \prod\{X_\alpha : \alpha \in A\}$, and let s^* be a fixed point of X^*. For $x \in X^*$ let $Q(x) = \{\alpha \in A : x_\alpha \neq (s^*)_\alpha\}$. Let $X_n = \{x \in X^* : |Q(x)| \leq n\}$ and $X = \cup\{X_n : n \in \omega_0\}$. We say that s^* is the **base point of the σ–product** X. We sometimes write $X = \sigma\{X_\alpha : \alpha \in A\}$. For $a \in [A]^{<\omega_0}$ the subproduct $X_a = \prod\{X_\alpha : \alpha \in a\}$ is called a **finite subproduct of** X. Define $Y_a = \prod\{X_\alpha : \alpha \in a\} \times \{(s^*)_\alpha : \alpha \in A - a\}$ and the **projection map** p_a from X onto Y_a such that for $x = (x_\alpha)_{\alpha \in A}$,

$$(p_a(x))_\alpha = \begin{cases} x_\alpha & if\ \alpha \in a \\ (s^*)_\alpha & if\ \alpha \notin a \end{cases}.$$

Note that Y_a is homeomorphic to X_a.

Remark 1.2.46. *The projection map p_a in 1.2.45 above is open, onto, and continuous. Note that for every $n \in N$, $X_n = \cup\{Y_a : a \in [A]^n\}$ and X_n is closed. Furthermore, $\{Y_a - X_{n-1} : a \in [A]^n\}$ is discrete in the subspace $X - X_{n-1}$.*

Lemma 1.2.47. *[29] For each $n \in \omega_0$, $\{p_a^{-1}(Y_a - X_{n-1}) : a \in [A]^n\}$ is a point finite collection of open sets in X.*

Proof: Now $p_a^{-1}(Y_a - X_{n-1})$ is obviously open in X since $Y_a - X_{n-1}$ is open in Y_a. We show that for each $x \in X$ and $a \in [A]^n$, if $a \not\subseteq Q(x)$; then $x \notin p_a^{-1}(Y_a - X_{n-1})$. Since there are only finitely many sets $a \in [A]^n$ such that $a \subseteq Q(x)$, the lemma will follow. However, $a \not\subseteq Q(x)$ implies that $|Q(p_a(x))| \leq n - 1$ from the definition of the projection map p_a. Therefore $p_a(x) \in X_{n-1}$ and hence $x \notin p_a^{-1}(Y_a - X_{n-1})$.

Lemma 1.2.48. *[29] Let U be an open subset of a σ–product X such that $X_{n-1} \subseteq U$. Then $\{p_a^{-1}(Y_a - U) : a \in [A]^n\}$ is a locally finite collection of subsets of X.*

Proof: Let $x \in X$ be fixed. For every $b \in [Q(x)]^{\leq n-1}$, it follows that $p_b(x) \in X_{n-1} \subseteq U$. Hence there exists a set $B(b) \in [A]^{<\omega_0}$ such that

(i) $b \subseteq Q(x) \subseteq B(b)$,

(ii) for each $\alpha \in A$ there is an open (in X_α) set $V_\alpha(B(b)) \subseteq X_\alpha$

such that $(p_b(x))_\alpha \in V_\alpha(B(b))$, and

(iii) $p_b(x) \in V(B(b)) = \prod\{V_\alpha(B(b)) : \alpha \in B(b)\} \times \prod\{X_\alpha : \alpha \in A - B(b)\} \subseteq U$.

Let $H = \cup\{B(b) : b \in [Q(x)]^{\leq n-1}\}$. Then $Q(x) \subseteq H$ and $|H| < \aleph_0$.

(1) Define $V_\alpha = \begin{cases} \cap\{V_\alpha(B(b)) : \alpha \in b \text{ and } b \in [Q(x)]^{\leq n-1}\} & \text{for } \alpha \in Q(x) \\ \cap\{V_\alpha(B(b)) : b \in [Q(x)]^{\leq n-1}\} & \text{for } \alpha \in H - Q(x) \\ X_\alpha & \text{for } \alpha \notin H \end{cases}$

(2) Let $V = \prod\{V_\alpha : \alpha \in A\} \cap X$.

Clearly V is an open subset of X. We claim that $x \in V$. If $\alpha \in Q(x)$ then by (ii), for every $b \in [Q(x)]^{\leq n-1}$ such that $\alpha \in b$, it follows that $x_\alpha = (p_b(x))_\alpha \in V_\alpha(B(b))$. Also, if $\alpha \in H - Q(x)$ then by (ii), $x_\alpha = (s^*)_\alpha = (p_b(x))_\alpha \in V_\alpha(B(b))$ for every $b \in [Q(x)]^{\leq n-1}$. Therefore $x \in V_\alpha$ for every $\alpha \in A$ by (1) above.

We now show that for any set $a \in [A]^n$ such that $a \not\subseteq Q(x)$, it follows that $V \cap p_a^{-1}(Y_a - U) = \emptyset$. Since there exist only finitely many sets $a \in [A]^n$ satisfying $a \subseteq Q(x)$, the proof will be complete. Suppose $a \in [A]^n$ such that $a \not\subseteq Q(x)$ and there exists $y \in V \cap p_a^{-1}(Y_a - U)$. Then $p_a(y) \notin U$. However, we now show that it must be the case that $p_a(y) \in U$, hence a contradiction.

Let $b = a \cap Q(x) \in [Q(x)]^{\leq n-1}$.

<u>Case 1</u>: $\alpha \in B(b) \cap a$. Then $(p_a(y))_\alpha = y_\alpha$. Since $y \in V$, by (1) and (2) we have that $(p_a(y))_\alpha = y_\alpha \in V_\alpha(B(b))$.

<u>Case 2</u>: $\alpha \in B(b) - a$. Then since $b = a \cap Q(x)$, we have $(p_a(y))_\alpha = (s^*)_\alpha = (p_b(x))_\alpha \in V_\alpha(B(b))$ by (ii).

It now follows from (iii) that $p_a(y) \in V(B(b)) \subseteq U$.

Definition 1.2.49. Let X be a σ–product space and let Q represent some topological property. Property Q satisfies the σ–**product Theorem** provided that, if every finite subproduct of X satisfies property Q, then X satisfies property Q.

Remark 1.2.50. *In Chapter IV we discuss which topological properties satisfy the σ–product Theorem.*

Definition 1.2.51. Let X be a space and $\mathcal{D}$ a decomposition of X. Then the equivalence relation R of the decompositon $\mathcal{D}$ is the subset of $X \times X$ consisting of all pairs (x, y) such that x and y belong to the same member of $\mathcal{D}$. Also, X/R is defined to be the family of equivalence classes. The **projection map** π (**quotient map**) from X to X/R takes each $x \in X$ to that equivalence class of which x is a member. The quotient space is the family X/R with the quotient topology in which a set $A \subseteq X/R$ is open iff $\pi^{-1}(A)$ is open in X.

Definition 1.2.52. A decomposition $\mathcal{D}$ of a topological space X is said to be **upper semi–continuous** provided that for each $D \in \mathcal{D}$ and each open set U containing D there is an open set V such that $D \subseteq V \subseteq U$ and V is the union of members of $\mathcal{D}$.

The following lemma is found in [31].

Lemma 1.2.53. *A decomposition $\mathcal{D}$ of a topological space X is upper semi–continuous iff the projection map from X onto the quotient space of X given by the decomposition is a closed, continuous map.*

Definition 1.2.54. Let X be a space, $\mathcal{U}$ an open cover of X, and $\mathcal{D}$ a decomposition of X. Then $\mathcal{U}$ is said to be **$\mathcal{D}$–saturated** provided that for each $U \in \mathcal{U}$, $U = \cup\{D \in \mathcal{D} : D \cap U \neq \emptyset\}$.

CHAPTER II

EXAMPLES OF SPACES WITH THE PROPERTIES $B(D,\lambda)$–REFINABILITY AND WEAK $\overline{\theta}$–REFINABILITY

In 1975, J. C. Smith [24] introduced the property weak $\overline{\theta}$–refinability and then proved in [25] that a space X is CWN iff every weak $\overline{\theta}$–cover of X is a normal cover. Recall that a space X is paracompact iff every open cover of X is a normal cover. Smith [26] also proved that the class of metacompact spaces is exactly the class of almost expandable, weak $\overline{\theta}$–refinable spaces and that CWN, weak $\overline{\theta}$–refinable spaces are paracompact.

J. Chaber [9] in an unpublished paper in the mid–1970's introduced the notion of a "$B(LF,\omega_0)$–refinement" of an open cover which generalized the idea of a σ–LF–closed refinement. Chaber named the concept "property b_1." In 1980, J. C. Smith [26] generalized this notion further by defining the concept of a $B(P,\alpha)$–refinement as stated above in Definition 1.2.36. R. H. Price [19, 20, 21, 22] used the property $B(P,\alpha)$–refinability to obtain some new theorems involving the properties weak $\overline{\theta}$–refinability and θ–refinability. For example, Price showed that a space X is CWN iff every open cover of X which has a $B(D,\lambda)$–refinement is a normal cover. Furthermore, Price showed that CWN, $B(LF,\lambda)$–refinable spaces are paracompact, thus generalizing results of Smith listed above. Price also obtained a weak $\overline{\theta}$–type characterization of $B(D,\omega_0)$–refinability and used it to demonstrate that $B(D,\omega_0)$–refinability is strictly weaker than θ–refinability.

In [26], J. C. Smith conjectured that weak $\overline{\theta}$–refinability is strictly weaker than $B(D,\omega_0)$–refinability. Smith also asked for an example showing the relationship between $B(D,\lambda)$–refinability and $B(D,\omega_0)$–refinability where λ represents a countable ordinal. In this chapter we prove Smith's conjecture and give examples demonstrating the relationship between these properties.

In §1 we give a simple example (our first attempt) of a T_1 space that is $B(D,\omega_0^2)$–refinable but not $B(D,\omega_0)$–refinable. Even though this example is not normal, it is important in the sense that it motivated our search for an example demonstrating that $B(D,\omega_0^2)$–refinability is strictly weaker than $B(D,\omega_0)$–refinability in the class of T_4 spaces. In order to construct this T_4 example, we generalized R. H. Bing's example G in [4]. We call such an example a "generalized Bing space".

In §2 we give an example, which we name $\hat{F}$, of a generalized Bing space which is $B(D,\omega_0)$–refinable but not $B(D,n)$–refinable for any $n \in N$. In addition we obtain for each $n \in N$, a normal space that is $B(D,n+1)$–refinable but fails to be $B(D,n)$–refinable.

In §3 we extend the technique used in §2 to build a space, which we call F^*, which is $B(D,\omega_0+1)$–refinable but not $B(D,\omega_0)$–refinable; and in §4 we modify this example to construct a $B(D,\omega_0+1)$–refinable, weak $\overline{\theta}$–refinable space that is not $B(D,\omega_0)$–refinable, thus proving Smith's conjecture above to be true. Also in §4 we show how, for any countable ordinal λ to construct a generalized Bing space that is $B(D,\lambda)$–refinable but not $B(D,\gamma)$–refinable for any ordinal $\gamma < \lambda$. All of the examples given in sections 2 through 4 of this chapter are T_4.

§1. A T_1 space that is $B(D,\omega_0^2)$–refinable but not $B(D,\omega_0)$–refinable.

Theorem 2.1.1. *There exists a T_1 uncountable space X which is $B(D,\omega_0)$–refinable but not $B(D,n)$–refinable for any $n \in N$.*

Proof: Let Y be the space of all ordinals less than ω_0^2 with the topology given by the following basis. A basic open set B is a subset of Y of the form $B = \{\gamma \leq b\} - A$ where $b \in Y$ and A is a finite subset of Y. Since Y is countable and T_1, it is $B(D,\omega_0)$–refinable as noted in Remark 1.2.37 above. Note that Y is not T_2.

Now, suppose Y is $B(D,n)$–refinable for some $n \in N$. Let $\mathcal{B} = \cup\{\mathcal{B}_i : i \leq n\}$ be a $B(D,n)$–refinement of an open cover of Y by basic open sets. Let j_0 be the first integer such that $\left|\left(\bigcup_{i\leq j_0}(\cup\mathcal{B}_i)\right) \cap \{\gamma \in Y : \omega_0 \cdot k \leq \gamma < \omega_0 \cdot (k+1)\}\right| = \aleph_0$ for some $k \in \omega_0$. Let k_0 be the first element of ω_0 such that $\left| \left(\bigcup_{i\leq j_0}(\cup\mathcal{B}_i)\right) \cap \{\gamma \in Y : \omega_0 \cdot k_0 \leq \gamma < \omega_0 \cdot (k_0+1)\} \right| = \aleph_0$. For each $B \in \mathcal{B}_{j_0}$, B is a subset of some basic open set in Y. It follows that there exists some $\gamma \in Y - \left(\left(\bigcup_{i\leq j_0-1}(\cup\mathcal{B}_i)\right) \cup [0, \omega_0 \cdot (k_0+1)) \cup B\right)$ such that γ is an upper bound of $B \cup [0, \omega_0 \cdot (k_0+1))$. Thus, since B is closed in $Y - \left(\bigcup_{i\leq j_0-1}(\cup\mathcal{B}_i)\right)$ and every open set containing γ contains all but finitely many members of B, it must be the case that B is finite. But then infinitely many members of $\mathcal{B}_{j_0}$ must meet $[\omega_0 \cdot k_0, \omega_0 \cdot (k_0+1))$. Hence $\mathcal{B}_{j_0}$ cannot be discrete in $Y - \left(\bigcup_{i<j}(\cup\mathcal{B}_i)\right)$ since every open set containing γ must hit infinitely many members of $\mathcal{B}_{j_0}$. This contradicts the supposition that $\mathcal{B}$ is a $B(D,n)$–refinement. Therefore Y is not $B(D,n)$–refinable.

Next, let $X = \oplus\{Y_\alpha : \alpha \in \omega_1\}$ where $Y_\alpha = Y$ for every $\alpha \in \omega_1$. Note that for every $\alpha \in \omega_1$, $Y_\alpha \times \{\alpha\}$ is closed in X and is homeomorphic to Y. Therefore X is not $B(D,n)$–refinable for any $n \in N$ since $B(D,n)$–refinability is closed hereditary. Clearly X is T_1. Therefore X is a T_1, uncountable, $B(D,\omega_0)$–refinable space that is not $B(D,n)$–refinable for any $n \in N$.

Theorem 2.1.2. *For any ordinal $\gamma > 1$, there exists an uncountable T_1 space S that is $B(D, \omega_0 \cdot \gamma)$–refinable but not $B(D,\omega_0)$–refinable.*

Proof: Let X be the space constructed in Theorem 2.1.1. Let $S = X \times [0,\gamma)$ with the topology given by basic open sets of the form $(A \times \{\sigma\}) \cup B$ where $\sigma \in \gamma$, A is a basic open set in X, and $B \subseteq S - \bigcup_{\tau\leq\sigma}(X \times \{\tau\})$ is cofinite in $X \times \{\delta\}$ for every $\sigma < \delta < \gamma$. (See Diagram 2.1.3 below). Now $X \times \{\sigma\}$ is closed in $S - \left(\bigcup_{\tau<\sigma}(X \times \{\tau\})\right)$ for every $\sigma \in \gamma$, and since X is $B(D,\omega_0)$–refinable, S is $B(D, \omega_0 \cdot \gamma)$–refinable.

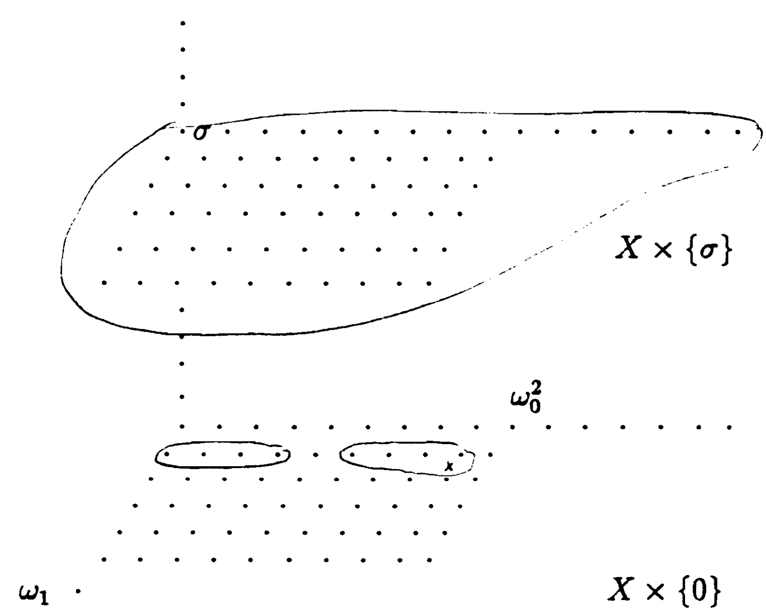
Claim. *S is not $B(D,\omega_0)$-refinable.*

Proof of Claim: Suppose $\mathcal{U}$ is a cover of S by basic open sets and $\mathcal{B} = \cup\{\mathcal{B}_n : n \in \omega_0\}$ is a $B(D,\omega_0)$-refinement of $\mathcal{U}$. Choose the first $n \in \omega_0$ with

$$\left|\left(\bigcup_{i\leq n}(\cup\mathcal{B}_i)\right) \cap (X \times \{1\})\right| = \aleph_0.$$

Since $X \times \{0\}$ is not $B(D,n)$-refinable, it must be the case that $\bigcup_{i\leq n}(\cup\mathcal{B}_i)$ does not cover $X \times \{0\}$. Thus $\bigcup_{i\leq n}(\cup\mathcal{B}_i)$ is not closed in S, contradicting the definition of a $B(D,\omega_0)$-refinement. Therefore S is not $B(D,\omega_0)$-refinable.

Diagram 2.1.3. *A basic open set in S.*



Corollary 2.1.4. *There exists an uncountable, T_1 space that is $B(D,\omega_0^2)$-refinable but not $B(D,\omega_0)$-refinable.*

Proof: Let $\gamma = \omega_0$ in Theorem 2.1.2.

Corollary 2.1.5. *For any ordinal γ and $n \in N$ there exists an uncountable T_1 space K that is $B(D,\omega_0\cdot\gamma+1)$-refinable but not $B(D,\omega_0)$-refinable. In particular, K can be $B(D,\omega_0+1)$-refinable but not $B(D,\omega_0)$-refinable.*

Proof: Let S be the space constructed in Theorem 2.1.2 that is $B(D, \omega_0 \cdot \gamma)$–refinable (but not $B(D, \omega_0)$–refinable if $\gamma > 1$.) Define $K = S \cup T$ where $T = [0, \omega_1)$ with the following topology. A basic open set B about a point $s \in S$ is any set such that $B \cap S$ is a basic open set about s in S and $|\alpha \in T : \alpha \notin B| < \aleph_0$. For every $s \in T$, $\{s\}$ is an open subset of K. Since S is a closed subset of K, which is not $B(D, \omega_0)$–refinable, then K is not $B(D, \omega_0)$–refinable.

Claim. *K is $B(D, \omega_0 \cdot \gamma + 1)$–refinable.*

Proof: Let $\mathcal{U} = \{U_\delta : \delta \in \Delta\}$ be an open cover of K and let $\mathcal{B}^* = \cup\{\mathcal{B}^*_\alpha : \alpha \in \omega_0 \cdot \gamma\}$ be a $B(D, \omega_0 \cdot \gamma)$–refinement of $\mathcal{U} \mid S$. Then since S is a closed subset of K and T is relatively discrete in $K - S$, it is the case that K is $B(D, \omega_0 \cdot \gamma + 1)$–refinable.

§2. Construction of the generalized Bing space $\hat{F}$.

In [4], R. H. Bing gave an example of a normal topological space that is not collectionwise T_2. We call such an example a *Bing space* and give the construction of such a space below.

Let $Q = P(\omega_1)$ =the set of all subsets of ω_1. Let $G = \{f \in \prod_{q \in Q}\{0,1\} \mid f \in \prod^{-1}_{\{\alpha\}}(1)$ for some $\alpha \in \omega_1\}$, where $\{0,1\}$ is the two-point discrete space. For $\alpha \in \omega_1$, define f_α by

$$f_\alpha(q) = \left\{ \begin{array}{ll} 1 & \text{if } \alpha \in q \\ 0 & \text{if } \alpha \notin q. \end{array} \right\}$$

Define $F = \{f_\alpha | \alpha \in \omega_1\}$ so that $F \subseteq G$.

Topologize G by adding to the induced Tychonoff product topology all singleton sets $\{g\}$ where $g \in G \setminus F$.

Clearly G is T_1. To show that G is T_4, suppose H_1 and H_2 are two disjoint closed subsets of G. Let $A_k = H_k \cap F$ for $k = 1, 2$. If $A_1 = \emptyset$, then H_1 and $G \setminus H_1$ are two disjoint open sets separating H_1 and H_2. Hence we assume that $A_k \neq \emptyset$ for $k = 1, 2$. In what follows we use the identification of A_k with $\{\alpha \in \omega_1 | f_\alpha \in A_k\}$.

Let

$$D_1 = \{f \in G | f(A_1) = 1 \text{ and } f(A_2) = 0\},$$

$$D_2 = \{f \in G | f(A_1) = 0 \text{ and } f(A_2) = 1\}.$$

Then $A_k \subseteq D_k$ and $D_1 \cap D_2 = \emptyset$. Thus $(D_1 \setminus H_2) \cup (H_1 \setminus A_1)$ and $(D_2 \setminus H_1) \cup (H_2 \setminus A_2)$ are disjoint open sets in G containing H_1 and H_2 respectively.

To prove that G is not collectionwise T_2 we show that no uncountable collection of singleton subsets of F can be separated by pairwise disjoint open sets in G. Suppose $T = \{f_\alpha \in F | \alpha \in A\}$ is uncountable and that for each $\alpha, \gamma \in A$, D_α is a basic open set about f_α such that $D_\alpha \cap D_\gamma = \emptyset$ if $\alpha \neq \gamma$. For each α, let r_α be the finite subset of Q such that $D_\alpha = \bigcap_{q \in r_\alpha} \prod_q^{-1}(f_\alpha(q))$.

Since A is uncountable there exists an integer m and an uncountable subset A_1' of A such that r_α has exactly m elements for every $\alpha \in A_1'$. For any two members $\alpha, \gamma \in A_1'$, we have $r_\alpha \cap r_\gamma \neq \emptyset$; otherwise $D_\alpha \cap D_\gamma \neq \emptyset$ in contradiction to our choice of D_α and D_γ. It then follows that there exists an uncountable subset A_1 of $A_1' \subseteq A$, $q_1 \in Q$, and t_1 with value either 1 or 0 with $f_\alpha(q_1) = t_1$ for every $\alpha \in A_1$. Similarly there exists an uncountable subset A_2 of A_1, $q_2 \in Q$, and t_2 with value either 1 or 0 with $f_\alpha(q_2) = t_2$ for every $\alpha \in A_2$. Continuing inductively in this way we get q_k, t_k, A_k for $k = 1, 2, \ldots, m$. Let $r = \{q_1, q_2, \ldots, q_m\}$ and D be the set of all $f_\alpha \in T$ with $f_\alpha(q_k) = t_k$ for $k = 1, 2, \ldots, m$. Then $r_\alpha = r$ and $D_\alpha = D$ for every $\alpha \in A_m$. Hence the D_α's could not be pairwise disjoint, so G is not collectionwise T_2.

We are now ready to construct our generalized Bing space $\hat{F}$. Let $F_1 = F, G_1 = G$, $F_2^* = G_1 \setminus F_1$, and $Q_1 = Q$. Next, define $Q_2 = P(F_2^*)$. Let $G_2 = \{f \in \prod_{q \in Q_2}\{0,1\} | f \in \prod_{\{p\}}^{-1}(1)$ for some $p = F_2^*\}$. For $p \in F_2^*$, define $f_p(q) = \left\{ \begin{matrix} 1 & \text{if } p \in q \\ 0 & \text{if } p \notin q. \end{matrix} \right\}$. Define $F_2 = \{f_p | p \in F_2^*\}$, $F_3^* = G_2 \setminus F_2$. Topologize G_2 by adding to the induced Tychonoff topology all singleton sets $\{g\}$ where $g \in G_2 \setminus F_2$. Continue inductively in this way to define F_n, G_n for every $n \in N$. Note that $|F_1| = \omega_1, |F_2| = 2^{\omega_1}, |F_3| = 2^{2^{\omega_1}}$, etc.

Define $\hat{F} = \cup\{F_n | n \in N\}$. Using the natural identification of F_n with F_n^* for $n > 1$, define a basic open set U in $\hat{F}$ about a point g in F_n as follows:

(1) $U \cap G_n$ is a basic open set about g in G_n;

(2) for each $g^* \in U \cap F_{n+1}$, U contains a basic open set about g^* in G_{n+1};

(3) assume that for $k < m$, U has been defined so that for each $g^* \in U \cap F_{n+k}$, U contains a basic open set about g^* in G_{n+k}. Then for each $g^* \in F_{n+m}$, U contains a basic open set about g^* in G_{n+m}.

We will refer to a basic open set in $\hat{F}$ about a point g as a *funnel* about g. Note also that F_n is a discrete closed collection in $\hat{F} \setminus \left(\bigcup_{i<n} F_i\right)$.

For each $n \in N$, define $\hat{F}_n = \cup\{F_i | i \leq n+1\}$. Topologize $\hat{F}_n$ with the induced subspace topology from $\hat{F}$.

Properties of $\hat{F}$.

I. $\hat{F}$ is T_4. Suppose that H_1 and H_2 are disjoint closed subsets of $\hat{F}$. Let n_k be the first integer such that $H_k \cap F_{n_k} \neq \emptyset$ for $k = 1, 2$. Without loss of generality we may assume that $n_1 \leq n_2$. Since $H_1 \cap G_{n_1}$ and $H_2 \cap G_{n_1}$ are disjoint closed subsets of G_{n_1} they can be separated by disjoint subsets $V_{n_1}^1$ and $V_{n_1}^2$ of G_{n_1} which are open in G_{n_1} and contain $H_1 \cap G_{n_1}$ and $H_2 \cap G_{n_1}$, respectively. Then $(V_{n_1}^1 \cup H_1) \cap G_{n_1+1}$ and $(V_{n_1}^2 \cup H_2) \cap G_{n_1+1}$ are disjoint closed subsets of G_{n_1+1} and can be separated by disjoint subsets $V_{n_1+1}^1$ and $V_{n_1+1}^2$ of G_{n_1+1} which are open in G_{n_1+1} and contain $(V_{n_1}^1 \cup H_1) \cap G_{n_1+1}$ and $(V_{n_1}^2 \cup H_2) \cap G_{n_1+1}$, respectively. Continue by induction to obtain for each $j \in N$ disjoint subsets $V_{n_1+j}^1$ and $V_{n_1+j}^2$ of G_{n_1+j} which are open in G_{n_1+j} and contain $(V_{n_1+j-1}^1 \cup H_1) \cap G_{n_1+j}$ and $(V_{n_1+j-1}^2 \cup H_2) \cap G_{n_1+j}$, respectively. Then $\cup\{V_{n_1+j}^1 | j \in \omega_0\}$ and $\cup\{V_{n_1+j}^2 | j \in \omega_0\}$ are disjoint open sets in $\hat{F}$ which contain H_1 and H_2, respectively. So $\hat{F}$ is normal. It is easy to see that $\hat{F}$ is T_1.

II. $\hat{F}$ is $B(D, \omega_0)$-refinable. Observe that if we let $\mathcal{B}_n = \{\{f\} | f \in F_n\}$ for each

$n \in N$, then $\mathcal{B} = \cup\{\mathcal{B}_n | n \in N\}$ is a $B(D, \omega_0)$ refinement of every open cover of $\hat{F}$. Thus $\hat{F}$ is $B(D, \omega_0)$-refinable.

Definition 2.2.1. For each $f_\alpha \in F_1$, define the subbasic funnel U_{f_α} about f_α as follows:

(1) $U_{f_\alpha} \cap G_1 = \prod^{-1}_{\{\alpha\}}(1)$;

(2) $U_{f_\alpha} \cap G_2 = \cup\{\prod^{-1}_{\{g\}}(1) | g \in U_{f_\alpha} \cap F_2\}$;

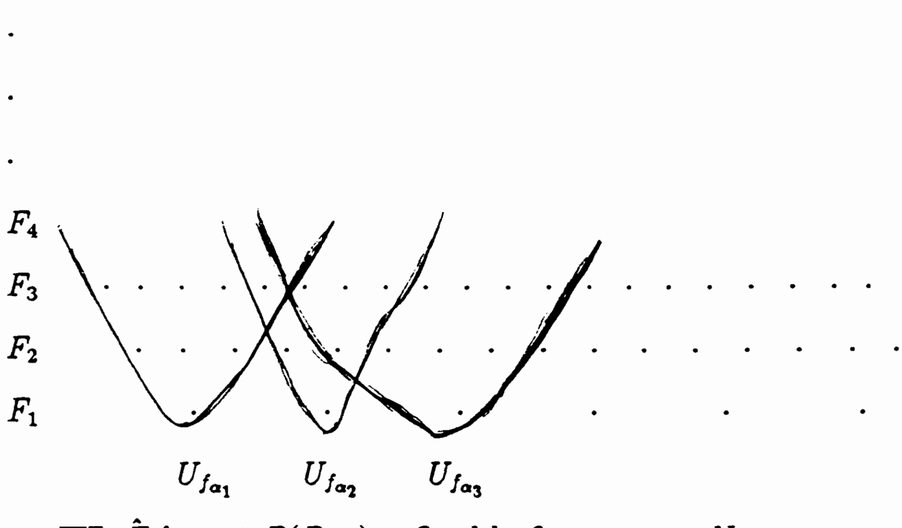
(3) assume that for $k < n$ that U_{f_α} has been defined such that

$$U_{f_\alpha} \cap G_k = \cup\{\prod^{-1}_{\{g\}}(1) | g \in U_{f_\alpha} \cap F_k\}.$$

Then let $U_{f_\alpha} \cap G_n = \cup\{\prod^{-1}_{\{g\}}(1) | g \in U_{f_\alpha} \cap F_n\}$. Then $\mathcal{U} = \{U_{f_\alpha} | f_\alpha \in F_1\}$ is called the *standard subbasic open cover of* $\hat{F}$. We will refer to each U_{f_α} as the *standard open funnel about* $f_\alpha \in F_1$. For an element f in F_n, the standard subbasic open funnel about f is defined in the analogous way.

The fact that $\mathcal{U}$ covers $\hat{F}$ follows from the definition of $G_n = \{f \in \prod_{q \in Q_n}\{0,1\} | f \in \prod^{-1}_{\{p\}}(1)$ for some $p \in F^*_n\}$. Note that if $\alpha, \gamma \in \omega_1$ with $\alpha \neq \gamma$, then $f_\alpha \notin U_{f_\gamma}$. Diagram 2.2.2 below is an illustration of the standard cover.

Diagram 2.2.2.



III. $\hat{F}$ is not $B(D, n)$-refinable for any $n \in N$.

The following lemmas are needed to prove this result. The first lemma is a generalization of the fact that G_1 is $B(D,2)$-refinable but not $B(D,1)$-refinable. To see this, suppose that $\mathcal{B}$ is a discrete closed refinement of the standard subbasic open cover of G_1. Then each $f_\alpha \in F_1$ is contained in a member B_{f_α} of $\mathcal{B}$ which contains no other member of F_1. For each such $f_\alpha \in F_1$, we pick a basic open subset V_{f_α} of the standard subbasic open funnel U_{f_α} such that $f_\alpha \in V_{f_\alpha}$ and $V_{f_\alpha} \cap B = \emptyset$ if $B \neq B_{f_\alpha}$. Since G_1 is not collectionwise T_2 it must be the case that for some $\alpha, \gamma \in \omega_1$, there exists an $x \in V_{f_\alpha} \cap V_{f_\alpha} \cap G_1 \neq \emptyset$. Since $\mathcal{B}$ covers G_1, we have $x \in B_{f_\alpha} \cap B_{f_\alpha}$, contradicting the discreteness of $\mathcal{B}$. It follows that G_1 is not $B(D,1)$-refinable. The strategy of Lemma 2.2.3 is similar to the proof we have just given.

Lemma 2.2.3. *Let $\mathcal{U}$ be the standard subbasic open cover of $\hat{F}$. Fix $n \in N$. Suppose that X is a closed subset of $\hat{F}$ and $\mathcal{B} = \{B_{f_\alpha} | f_\alpha \in F_1\}$ is a relatively discrete closed collection in $\hat{F} \setminus X$ which partially refines $\mathcal{U}$ such that the following condition holds:*

(A) There exists an $\alpha_1 \in \omega_1$ such that for every $\gamma > \alpha_1$

(1) there exists some $g(n,\gamma) \in (U_{f_\gamma} \cap F_n) \setminus (X \cup (\cup\{U_{f_\tau} | \tau < \gamma\}))$

and

(2) for each $g(n,\gamma)$ in (1) we can choose a funnel $V(n,\gamma) \subseteq U_{f_\gamma}$ about $g(n,\gamma)$ such that $V(n,\gamma) \cap X = \emptyset$ and $V(n,\gamma)$ hits at most one member of $\mathcal{B}$; i.e. $V(n,\gamma) \cap B \neq \emptyset$ iff $g(n,\gamma) \in B$ for each $B \in \mathcal{B}$. (*)

Then we have the following:

(B) There exists an $\alpha_2 \in \omega_1$ such that for every $\rho > \alpha_2$ we have

$$[V(n,\rho) \setminus ((\cup\mathcal{B}) \cup (\cup\{U_{f_\tau} | \tau < \rho\}))] \cap F_{n+1} \neq \emptyset.$$

Remark. *Note that in (A), if (1) holds then (2) follows.*

Proof: Assume (A) and suppose (B) is false; that is, no such α_2 exists. Choose

$\gamma_0 > \alpha_1$ and a funnel $V(n, \gamma_0)$ such that,

$$[V(n,\gamma_0) \setminus ((\cup\mathcal{B}) \cup (\{U_{f_\tau} | \tau < \gamma_0\}))] \cap F_{n+1} = \emptyset.$$

By our supposition we can choose $\gamma_1 > \gamma_0$ such that $\gamma_1 > \tau$ if $V(n,\gamma_0) \cap B_{f_\tau} \neq \emptyset$ and

$$[V(n,\gamma_1) \setminus (\cup\mathcal{B}) \cup (\cup\{U_{f_\tau} | \tau < \gamma_1\})] \cap F_{n+1} = \emptyset.$$

Assume that for $\rho < \Gamma$ that γ_ρ has been chosen such that the following conditions hold:

(i) $\gamma_\rho > \gamma_\delta$ if $\delta < \rho$;

(ii) $\gamma_\rho > \tau$ if $V(n,\gamma_\delta) \cap B_{f_\tau} \neq \emptyset$ for any $\delta < \rho$;

(iii) $[V(n,\gamma_\rho) \setminus ((\cup\mathcal{B}) \cup (\cup\{U_{f_\tau} | \tau < \gamma_\rho\}))] \cap F_{n+1} = \emptyset$.

By our supposition there exists $\gamma_\Gamma > \sup(\{\gamma_\rho | \rho < \Gamma\} \cup \{\tau \in \omega_1 | B_{f_\tau} \cap V(n,\gamma_\rho) \neq \emptyset \text{ for some } \rho < \Gamma\})$ such that (i), (ii), and (iii) hold. Thus we can continue the induction on ω_1.

Since the singletons $\{g(n,\gamma_\delta) | \delta \in \omega_1\}$ cannot be separated by pairwise disjoint open sets in G_n, and since $V(n,\gamma_\delta) \cap G_n$ is open in G_n for every $\delta \in \omega_1$, there exists $\delta_1, \delta_2 \in \omega_1$, $\delta_1 < \delta_2$, such that $V(n,\gamma_{\delta_1}) \cap V(n,\gamma_{\delta_2}) \cap F_{n+1} \neq \emptyset$. Now $V(n,\gamma_{\delta_1}) \cap V(n,\gamma_{\delta_2}) = \bigcap_{j=1}^{k} \prod_{q_k}^{-1}(t_k)$ for some $q_1, q_2, \ldots, q_k \in Q_n$ and each t_j has the value 1 or 0. Since $g(n,\gamma_{\delta_1}) \in V(n,\gamma_{\delta_1})$ and $g(n,\gamma_{\delta_2}) \in V(n,\gamma_{\delta_2})$, by (A) if any $q_j = \{h\}$ where $h \in \cup\{U_{f_\tau} | \tau < \gamma_{\delta_1}\} \cap F_n$, it follows that $t_j = 0$. Hence $[V(n,\gamma_{\delta_1}) \cap V(n,\gamma_{\delta_2}) \setminus (\cup\{U_{f_\tau} | \tau < \gamma_{\delta_1}\})] \cap F_{n+1} \neq \emptyset$. Choose $x \in [V(n,\gamma_{\delta_1}) \cap V(n,\gamma_{\delta_2}) \setminus (\cup\{U_{f_\tau} | \tau < \gamma_{\delta_1}\})] \cap F_{n+1}$. By (iii) above we must have that $x \in \cup\mathcal{B}$. Choose $B_x \in B$ with $x \in B_x$. Then

$$x \in V(n,\gamma_{\delta_1}) \cap V(n,\gamma_{\delta_2}) \cap B_x. \tag{**}$$

By (*), $g(n,\gamma_{\delta_1}) \in B_x$. Thus it follows from (ii) and the assumption (A) that $g(n,\gamma_{\delta_2}) \notin B_x$. However by (*) we have that $V(n,\gamma_{\delta_2}) \cap B_x = \emptyset$, contradicting (**). Therefore the lemma is proved.

Lemma 2.2.4. *Let $\mathcal{U}$ be the standard subbasic open cover of $\hat{F}$ and let $\mathcal{B} = \cup\{\mathcal{B}_n | n \in N\}$ be a $B(D,\omega_0)$-refinement of $\mathcal{U}$. Then there exists an $\alpha \in \omega_1$ such that for every $\gamma > \alpha$, there exists a sequence of ordered pairs $\{(x_n^\gamma, V_n^\gamma) | n \in N\}$ satisfying the following conditions:*

(1) $x_n^\gamma \in F_n$ and V_n^γ is a funnel about x_n for every $n \in N$;

(2) $V_n^\gamma \subseteq V_{n-1}^\gamma$ for $n > 1$; and

(3) $V_n^\gamma \cap (\bigcup_{i=1}^{n-1}(\cup\mathcal{B}_i)) = \emptyset$, for every $n > 1$.

Proof: For $n = 1$, let $X = \emptyset$. With $\mathcal{B}_1 = \mathcal{B}$ in Lemma 2.2.3, and $\alpha_1 = 0$, let $g(1,\gamma) = f_\gamma$ for every $\gamma \in \omega_1$. Note that

$$g(1,\gamma) \in (U_{f_\alpha} \cap F_1) \setminus (X \cup (\cup\{U_{f_\tau} | \tau < \gamma\})).$$

For each $g(1,\gamma)$, choose a funnel $V(1,\gamma)$ about $g(1,\gamma)$ such that $V(1,\gamma) \subseteq U_{f_\gamma}$ and $V(1,\gamma) \cap B \neq \emptyset$ iff $g(1,\gamma) \in B$. Thus condition (A) of Lemma 2.2.3 is satisfied for $n = 1$. By the conclusion of Lemma 2.2.3 there exists $\gamma_2 \in \omega_1$ such that for every $\gamma > \gamma_2$ there exists

$$g(2,\gamma) \in [V(1,\gamma) \setminus ((\cup\mathcal{B}_1) \cup (\cup\{U_\tau | \tau < \gamma\}))] \cap F_2 \neq \emptyset.$$

For $n = 2$, let $X = \cup\mathcal{B}_1$. Also, let $\mathcal{B}_2 = \mathcal{B}$ in Lemma 2.2.3. For every $\gamma > \gamma_2$, choose a funnel $V(2,\gamma)$ about $g(2,\gamma)$ such that $V(2,\gamma) \cap X = \emptyset$, $V(2,\gamma) \subseteq V(1,\gamma)$, and $V(2,\gamma) \cap B \neq \emptyset$ iff $g(2,\gamma) \in B$ for every $B \in \mathcal{B}_2$. Then by Lemma 2.2.3 again there exists γ_3 such that for every $\gamma > \gamma_3$ we can pick $g(3,\gamma) \in \Big[V(2,\gamma) \setminus ((\bigcup_{i=1}^{2}(\cup\mathcal{B}_i)) \cup (\cup\{U_\tau | \tau < \gamma\})) \Big] \cap F_3$. Continue in this way so that for every $n > 1$ we have $V(n,\gamma) \subseteq V(n-1,\gamma) \subseteq \cdots \subseteq V(1,\gamma) \subseteq U_{f_\gamma}$ for $\gamma > \gamma_n$. Now, let $\alpha^* > \sup\{\gamma_n | n \in N\}$. Then for every $\gamma > \alpha^*$, $n \in N$, let $V_n^\gamma = V(n,\gamma)$ and $x_n^\gamma = g(n,\gamma)$. It is easy to see that $\{(x_n^\gamma, V_n^\gamma) | n \in N\}$ is the desired nested sequence.

Theorem 2.2.5. *$\hat{F}$ is not $B(D,n)$-refinable for any $n \in N$ while $\hat{F}$ is $B(D,\omega_0)$-refinable.*

Proof: Let $\mathcal{U}$ be the standard cover of $\hat{F}$. As in the proof of Lemma 2.2.4 we see that for any $n \in N$, $F_{n+1} \not\subseteq \bigcup_{i=1}^{n}(\cup\mathcal{B}_i)$ if $\mathcal{B}_i$ is a discrete closed collection in $\hat{F} \setminus \cup\{\cup\mathcal{B}_j | j < i\}$. Hence $\hat{F}$ cannot be $B(D,n)$-refinable for any $n \in N$. We have already seen above in Part II that $\hat{F}$ is $B(D,\omega_0)$-refinable.

IV. $\hat{F}_n$ is $B(D,n+1)$–refinable but not $B(D,j)$–refinable for any $j < n+1$.

This fact follows from the proof of Theorem 2.2.5.

§3. Construction of the generalized Bing space F^*.

We now extend $\hat{F}$ to construct a space F^* which is $B(D,\omega_0+1)$-refinable but is not $B(D,\omega_0)$-refinable and hence not $B(D,\alpha)$-refinable for any $\alpha < \omega_0 + 1$.

<u>**Construction:**</u>

Define $F_{\omega_0} = \{f : N \to \hat{F} | f(n) \in F_n \text{ for every } n \in N\}$. Let $F^* = \hat{F} \cup F_{\omega_0}$ and topologize as follows:

(1) If V is a basic open set (funnel) in $\hat{F}$, then

$$V^* = V \cup \{f \in F_{\omega_0} | V \text{ contains a tail of } f\}$$

is a basic open set in F^* and

(2) $\{f\}$ is open in F^* for every $f \in F_{\omega_0}$.

Note that $\hat{F}$ is a closed subspace of F^*. Thus if H_1 and H_2 are disjoint closed subsets of F^*, there exist disjoint open subsets A_1 and A_2 in $\hat{F}$ which separate $\hat{F} \cap H_1$ and $\hat{F} \cap H_2$, respectively. Since the tail of any member of F_{ω_0} cannot be contained in both A_1 and A_2, by (1) there exist disjoint sets U_1 and U_2 which are open in F^* separating $\hat{F} \cap H_1$ and $\hat{F} \cap H_2$. Therefore there exist disjoint open subsets of F^* which separate H_1 and H_2. It is similarly shown that F^* is T_1, so that F^* is T_4.

Note that $\mathcal{B} = \cup\{\mathcal{B}_\lambda | \lambda < \omega_0 + 1\}$ where $\mathcal{B}_n = \{\{f\} | f \in F_{n+1}\}$ for $n \in \omega_0$ and $\mathcal{B}_{\omega_0} = \{\{f\} | f \in F_{\omega_0}\}$ is a $B(D, \omega_0 + 1)$ refinement of every open cover of F^*.

Theorem 2.3.1. *F^* is $B(D, \omega_0 + 1)$-refinable. However, for every $\lambda < \omega_0 + 1$, F^* is not $B(D, \lambda)$-refinable.*

Proof: We have shown above that F^* is $B(D, \omega_0 + 1)$-refinable. Furthermore, since $B(D, n)$-refinability is hereditary for closed subspaces, and since $\hat{F}$ is not $B(D, n)$-refinable for any $n \in N$ by Theorem 2.2.5, F^* cannot be $B(D, n)$-refinable. It remains to show that F^* is not $B(D, \omega_0)$-refinable.

Let $\mathcal{U}^* = \hat{\mathcal{U}} \cup \{\{f\} | f \in F_{\omega_0}\}$ where $\hat{\mathcal{U}}$ is the natural extension of $\mathcal{U}_1 = \{U_{f_\alpha} : f_\alpha \in F_1\}$, the standard subbasic open cover of $\hat{F}$. Clearly $\mathcal{U}^*$ is an open cover of F^*. Suppose $\mathcal{B} = \cup\{\mathcal{B}_n | n \in N\}$ is a $B(D, \omega_0)$-refinement of $\mathcal{U}^*$. Then $\mathcal{B}$ restricted to $\hat{F}$ refines $\mathcal{U}^* | \hat{F} = \mathcal{U}$. It follows from Lemma 2.2.4 and the definition of the topology on F^* that there exists an $\alpha^* \in \omega_1$ such that for $\gamma > \alpha^*$, the point $f^\gamma \in F_{\omega_0}$ (defined by the sequence $f^\gamma(n) = x_n^\gamma$) cannot be covered by $\bigcup_{n=1}^{\infty}(\cup \mathcal{B}_n)$. But this contradicts the assumption that $\mathcal{B} = \cup\{\mathcal{B}_n | n \in N\}$ is a $B(D, \omega_0)$-refinement of $\mathcal{U}^*$. Hence F^* is not $B(D, \omega_0)$-refinable, and the theorem is proved.

Question 2.3.2. Is the space F^* above weak $\overline{\theta}$-refinable? We conjecture that it is not. If this is the case then F^* is a normal, $B(D, \omega_0 + 1)$-refinable space that is not weak $\overline{\theta}$-refinable.

§4. More generalized Bing spaces.

We now modify the spaces $\hat{F}$ and F^* to obtain T_4 spaces $\hat{K}$ and K^* with the following properties:

(1) $\hat{K}$ is mesocompact;

(2) $\hat{K}$ is $B(D, \omega_0)$-refinable but not $B(D, n)$-refinable for any n;

(3) K^* is weak $\overline{\theta}$-refinable;

(4) K^* is $B(D,\omega_0+1)$–refinable but is not $B(D,\lambda)$-refinable for any $\lambda < \omega_0 + 1$.

We begin our inductive construction of $\hat{K}$. Let $K_1 = F_1$ and let $K_2^* = \{f \in F_2^* | f(q) = 1$ for at least one but only finitely many $q \in Q_1\}$. Put $Q_2^* = P(K_2^*)$. For each α in K_2^*, define k_α by

$$k_\alpha(q) = \begin{cases} 1 & \text{if } \alpha \in q \text{ for } q \in Q_2^* \\ 0 & \text{if } \alpha \notin q \end{cases}$$

Now let $K_2 = \{k_\alpha | \alpha \in K_2^*\}$, and define

$K_3^* = \{f \in \prod_{q \in Q_2^*}\{0,1\} | f(q) = 1$ for at least one but at most finitely many $q \in Q_2^*\}$.

Continue in this way to define K_n for all n. Define and topologize $\hat{K}$ and K^* similarly to $\hat{F}$ and F^* in §2 and §3 above. For $n \in N$, let $\tilde{G}_n = K_n \cup K_{n+1}$ with the topology inherited from $\hat{K}$. In [18] it is shown that $\tilde{G}_1$ is metacompact. We extend this result to show that $\hat{K}$ is metacompact.

Theorem 2.4.1. *$\tilde{G}_n$ is metacompact for each $n \in N$.*

Proof: Let $\mathcal{U}$ be any open cover of $\tilde{G}_n$. For each $f \in K_n$, choose a member U_f of $\mathcal{U}$ that contains f. For each $f \in K_n$, let V_f represent the standard funnel about f. Then $\{U_f \cap V_f | f \in K_1\} \cup \{\{g\} | g \in \tilde{G}_n \setminus K_n\}$ is a point finite refinement of $\mathcal{U}$.

Theorem 2.4.2. *$\hat{K}$ is a metacompact.*

Proof: Let $\mathcal{V} = \{V_\alpha | \alpha \in A\}$ be an open cover of $\hat{K}$. Then, since $\tilde{G}_1$ is metacompact, $\mathcal{V}$ has a 1-1, open in $\hat{K}$, partial refinement $\mathcal{T} = \{T_\alpha | \alpha \in A\}$ which is point-finite on $\tilde{G}_1$ and covers $\tilde{G}_1$. For each $x \in \tilde{G}_n \cap T_\alpha$, let $C(x,n,\alpha)$ be the intersection of the standard funnel about x with T_α. Define the open set S_α^1 inductively as follows:

1) $S_\alpha^1 \cap \tilde{G}_1 = T_\alpha \cap \tilde{G}_1$;

2) for $n > 1$, $S_\alpha^1 \cap \tilde{G}_n = (S_\alpha^1 \cap K_n) \cup \{y \in \tilde{G}_n | y \in C(x,n,\alpha)$ for some $x \in S_\alpha^1 \cap K_n\}$.

Note that $\mathcal{S}^1 = \{S_\alpha^1 | \alpha \in A\}$ is an open in $\hat{K}$, point-finite, partial refinement of $\mathcal{V}$ which covers $\tilde{G}_1$. To see this, suppose $x \in K_3$. Then $x \in \prod_q^{-1}(1)$ for only finitely many q in $P(K_2)$. Hence x is a member of only finitely many standard funnels about elements in K_2. Since $K_2 \subseteq \tilde{G}_1$, $\mathcal{S}^1$ is point-finite on K_2. Thus by 2) it must be the case that $\mathcal{S}^1$ is point-finite on K_3. Continuing in this way it follows that $\mathcal{S}^1$ is point-finite on K_n for every n.

Next, since $K_1 \cup K_2 \cup \cdots \cup K_{n-1}$ is a closed subset of $\hat{K}$, and since $\tilde{G}_n$ is metacompact for each n, we can construct a 1-1 point-finite open partial refinement $\mathcal{S}^n$ of $\mathcal{V}$ that covers K_n and misses $K_1 \cup K_2 \cup \cdots \cup K_{n-1}$. It then follows that $\mathcal{S} = \cup\{\mathcal{S}^n : n \in N\}$ is a point-finite open refinement of $\mathcal{V}$. Hence $\hat{K}$ is metacompact.

To show that $\hat{K}$ is mesocompact it suffices to show that every compact subset of $\hat{K}$ is finite. In [5], J. R. Boone shows that every compact subset of $\tilde{G}_1$ is finite. We now extend this result to obtain the following lemma.

Lemma 2.4.3. *For each $n \in N$, if C is a compact subset of $\hat{K}$, then $C \cap (\cup\{\tilde{G}_i | i \leq n\})$ is finite.*

Proof: The proof is by induction on N. For $n = 1$ observe that $\tilde{G}_1 \cap C$ is closed in $\hat{K}$ and therefore compact. Suppose that we can choose distinct elements $f_1, f_2, f_3, \ldots, f_n, \ldots$ in $C \cap \tilde{G}_1$. Since K_1 is discrete, we may assume that each f_i belongs to $K_2 \cap C$. Since $|\{q \in Q_1 | f_i(q) \neq 0 \text{ for some } i \in N\}| \leq \aleph_0$, for each $f \in K_1$ we can choose a basic open funnel about f, say V_f, that misses $\{f_1, f_2, f_3, \ldots,\}$, since $|\{q \in Q_1 | f(q) \neq 0\}| > \aleph_0$. Thus $\{V_f | f \in K_1 \cap C\} \cup \{\{f_i\} | i \in N\}$ is an open cover (open in $\tilde{G}_1$) of $C \cap \tilde{G}_1$ with no finite subcover, contradicting the compactness of $C \cap \tilde{G}_1$. Hence $C \cap \tilde{G}_1$ must be finite.

Assume that for all $k < n$, $C \cap [\cup\{\tilde{G}_i : i \leq k\}]$ is finite.

By inductive hypothesis, $C \cap (\cup\{\tilde{G}_i | i \leq n-1\})$ is compact and therefore finite. Suppose we can choose distinct elements $f_1, f_2, f_3, \ldots$ in $C \cap K_{n+1}$. Since $|\{q \in Q_n | f_i(q) \neq 0 \text{ for some } i \in N\}| \leq \aleph_0$, for each $f \in C \cap (\cup\{\tilde{G}_i | i \leq n-1\})$ we can choose a funnel V_f about f which misses $\{f_1, f_2, f_3, \dots\} \subseteq C$. Thus $\{V_f | f \in C \cap (\cup\{\tilde{G}_i | i \leq n-1\})\} \cup \{\{f_i\} | i \in N\}$ is an open cover of $C \cap (\cup\{G_i | i \leq n\})$ with no finite subcover, in contradiction to the compactness of $C \cap (\cup\{\tilde{G}_i | i \leq n\})$, so the lemma follows.

Lemma 2.4.4. *If C is a compact subset of $\hat{K}$, then C is finite.*

Proof: Suppose C is an infinite compact subset of $\hat{K}$. Then by Lemma 2.4.3 there must exist a sequence $f : N \to C$ such that $f(n) \notin \cup\{K_j | f(i) \in K_j \text{ for some } i < n\}$. Since $\{f(n) | n \in N\}$ is a closed subset of C, it is compact. It is easy to show that this sequence is also discrete and thus cannot be compact. Hence the lemma is proved.

From Theorem 2.4.2 and Lemma 2.4.4 we now have the following.

Theorem 2.4.5. *$\hat{K}$ is mesocompact.*

Theorem 2.4.6.

(i) $\hat{K}$ is $B(D, \omega_0)$-refinable but not $B(D, n)$-refinable for any n.

(ii) K^* is $B(D, \omega_0 + 1)$-refinable, but not $B(D, \omega_0)$-refinable.

(iii) K^* is weak $\overline{\theta}$-refinable and T_4.

Proof: The proof if (i) and (ii) follow in the analogous fashion as the properties of $\hat{F}$ in §2 above. Since $\hat{K}$ is metacompact, (iii) follows.

We now show, for any countable ordinal λ, how to construct a space K^*_λ, which is $B(D, \lambda)$-refinable but not $B(D, \lambda)$-refinable for any $\gamma < \lambda$. We extend the space K^* above to accomplish this result. The proof that our extended space has the desired properties is again analogous to the previous results for K^* and is left to the reader.

For $n \in N$, let K_n be defined as above and $K_{\omega_0} = \{f : N \to \hat{K} | f(n) \in K_n \text{ for every } n \in N\}$. Let λ be a countable ordinal greater than or equal to $\omega_0 + 1$ and let $\alpha < \lambda$. If α is a successor ordinal, say $\alpha = \beta + 1$, construct (the corresponding Bing space) K_α from K_β in the same manner as K_{n+1} was constructed from K_n above.

If α is a limit ordinal, let $S_\alpha = \{\gamma_1, \gamma_2, \dots\}$ be a cofinal sequence in α. Let K_α be the set of all sequences of the form $f : N \to \cup\{K_{\gamma_n} : n \in N\}$ such that $f(n) \in K_{\gamma_n}$.

Assume that for $\gamma < \alpha$ that K_γ^* has been defined in the natural way and is a T_4, $B(D,\lambda)$-refinable space that is not $B(D,\tau)$-refinable for any $\tau < \gamma$,

Case 1: α is not the successor of a limit ordinal.

Subcase 1: α is a limit ordinal.

Let $K_\alpha^* = \cup\{K_\beta^* : \beta < \alpha\}$ with the natural identification of the levels and topology.

Subcase 2: α is a successor ordinal.

Let $K_\alpha^* = (\cup\{K_\beta^* : \beta < \alpha\}) \cup K_\alpha$ with the natural topology.

Case 2: α is the successor of a limit ordinal, say $\alpha = \beta + 1$.

Let $K_\alpha^* = K_\beta^* \cup K_\beta$, topologized as follows.

If $f \in K_\beta$, then $\{f\}$ is an open set. If U is a basic open funnel in K_β^*, then $U \cup \{f \in K_\beta$: a tail of f is in $U\}$ is a basic open set in K_β^*.

The proof of the next theorem follows easily from the proof of Theorem 2.4.2 and the above construction.

Theorem 2.4.7. *$K_{\omega_0^2}^*$ is $B(D,\omega_0^2)$-refinable and weak $\overline{\theta}$-refinable, but not $B(D,\omega_0)$-refinable.*

CHAPTER III

SUM AND MAPPING THEOREMS

In 1978, H. J. K. Junnila [16] proved that closed maps preserve θ–refinability. In 1984, D. K. Burke [7] proved that perfect maps preserve weak θ–refinability, and in [8] he showed that closed maps preserve subparacompactness. It is natural to ask whether $B(D,\omega_0)$–refinability and weak $\bar{\theta}$–refinability are preserved under perfect or closed maps. Unfortunately however, the techniques of Junnila and Burke do not seem to be adaptable for use in obtaining a similar result for these properties. In fact, the following is still an open question.

Question 3.1.1. Are the properties $B(D,\omega_0)$–refinability and weak $\bar{\theta}$–refinability preserved under closed, finite-to-one maps?

In [19], R. H. Price proved the following theorem.

Theorem 3.1.2. *Let $f : X \to Y$ be a continuous, closed, and bded-map from a space X onto a space Y. If X is $B(D,\omega_0)$–refinable, then Y is $B(D,\omega_0)$–refinable.*

In [26], J. C. Smith proved the following sum theorem.

Theorem 3.1.3. *The property $B(D,\omega_0)$–refinability satisfies the Countable Sum Theorem.*

Remark 3.1.4. *It is easy to see that the property $B(D,\omega_0)$–refinability is preserved under disjoint sums. Thus, in light of Lemma 1.2.20, we see that the LF Sum Theorem must hold for the property $B(D,\omega_0)$–refinability if closed, finite-to-one maps preserve $B(D,\omega_0)$–refinability. Our next two theorems pertain to this result.*

Theorem 3.1.5. *The property $B(D,\omega_0)$–refinability satisfies the bded LF Sum Theorem.*

Proof: Since $B(D,\omega_0)$–refinability is preserved under disjoint sums, the theorem follows from Theorem 3.1.2 and Lemma 1.2.20.

In [15] F. Ishikawa proved the following lemma.

Lemma 3.1.6. *A space X is countably metacompact iff every countable monotone increasing open cover of X has a monotone increasing closed refinement.*

Theorem 3.1.7. *The property $B(D,\omega_0)$-refinability satisfies the LF Sum Theorem for countably metacompact spaces.*

Proof: Let $\mathcal{F} = \{F_\alpha : \alpha \in A\}$ be a closed, locally finite cover of X where each F_α is $B(D,\omega_0)$-refinable. Now $\mathcal{G} = \{G_n : n \in N\}$ where $G_n = \{x \in X : ord(x,\mathcal{F}) \leq n\}$ is a countable monotone increasing open cover of X. Since X is countably metacompact, by Lemma 3.1.6 above, $\mathcal{G}$ has a monotone increasing closed refinement, say $\mathcal{K} = \{K_n : n \in N\}$ where $K_n \subseteq G_n$ for each $n \in N$. Note that $\mathcal{F}_{|K_n}$ is an n–bded LF closed cover of K_n for each $n \in N$. Hence by Theorem 3.1.5, K_n is $B(D,\omega_0)$–refinable for each $n \in N$. Therefore, X is $B(D,\omega_0)$–refinable by Theorem 3.1.3.

Theorem 3.1.8. *Let K^*_λ be the generalized Bing space constructed in Chapter II which is $B(D,\lambda)$–refinable. Then every closed, continuous image of K^*_λ is $B(D,\lambda)$–refinable. In particular, every closed, continuous image of the space $\hat{K}$ constructed in Section 2.4 is $B(D,\omega_0)$–refinable. Likewise, every closed continuous image of the space $\hat{F}$ constructed in Section 2.2 is $B(D,\omega_0)$-refinable.*

Proof: Suppose Y is an image of K^*_λ under a closed, continuous map f. Let $\mathcal{V}$ be an open cover of Y and $\mathcal{U} = f^{-1}(\mathcal{V})$. Then $\mathcal{U}$ is an open cover of K^*_λ and $\mathcal{B}^* = \cup\{\mathcal{B}^*_\gamma : \gamma < \lambda\}$, where each $\mathcal{B}^*_\gamma = \{\{x\} : x \in K_\gamma\}$, is a $B(D,\lambda)$-refinement of $\mathcal{U}$. For each $\gamma \in \lambda$ let $\mathcal{B}_\gamma = \left\{\{y\} : y \in f(K_\gamma) - \left(\bigcup_{\mu<\gamma} f(K_\mu)\right)\right\}$. Note that for each $\{y\} \in \mathcal{B}_\gamma$ we have that $[f^{-1}(\mathcal{B}_\gamma - \{\{y\}\})] \cap \mathcal{B}^*_\gamma$ is a closed subset of $K^*_\lambda - \cup\{\mathcal{B}^*_\mu : \mu < \gamma\}$, since each $\mathcal{B}^*_\gamma$ is a relatively discrete collection of closed subsets of $K^*_\lambda - \left(\bigcup_{\mu<\gamma} K_\mu\right)$. Since f is a closed map, it follows that each $\mathcal{B}_\gamma$ is a relatively discrete closed collection in $Y - \cup\{\cup\mathcal{B}_\mu : \mu < \gamma\}$. It is now easy to see that $\mathcal{B} = \cup\{\mathcal{B}_\gamma : \gamma < \lambda\}$ is a $B(D,\lambda)$-refinement of $\mathcal{V}$.

Definition 3.1.9. Let each of X and Y be a topological space and $f : X \to Y$ a map. Let $\mathcal{H} = \{H_\alpha : \alpha \in A\}$ be a collection of subsets of X. Then we define $\hat{H}_\alpha = f^{-1}(f(H_\alpha))$ for each subset H_α of X and define $\hat{\mathcal{H}} = \{\hat{H}_\alpha : \alpha \in A\}$.

Remark 3.1.10. *In [7], D. K. Burke proved that perfect maps preserve weak θ–refinability. Theorem 3.1.11 below is the result of our attempt to obtain a corresponding closed mapping theorem.*

Theorem 3.1.11. *The closed continuous image of a hereditarily finally λ weak θ–refinable space is hereditarily weak θ–refinable.*

Proof: Let f be a closed, continuous map from a hereditarily finally λ weak θ–refinable space X onto a space Y. Suppose $\mathcal{V} = \{V_\gamma : \gamma \in \Gamma\}$ is an open cover of Y. Then $\mathcal{U} = \{U_\gamma : \gamma \in \Gamma\}$ where $U_\gamma = f^{-1}(V_\alpha)$ is an open cover of X. Now $\mathcal{U}$ has a finally λ weak θ refinement, say $\mathcal{G} = \cup\{\mathcal{G}^*_{(\mu)} : \mu \in \lambda\}$ where $\mathcal{G}^*_{(\mu)} = \{G_{(\mu,\gamma)} : \gamma \in \Gamma\}$. Note that we may assume that the levels are amalgamated so that $G(\mu,\gamma) \subseteq U_\gamma$ for every $\gamma \in \Gamma$. Recall from Definition 1.2.33 (b) that for each $x \in X$ there exists $\mu_x \in \lambda$ such that $0 < ord(x, \mathcal{G}^*_{(\mu_x)}) < \aleph_0$ and $x \notin \bigcup_{\mu_x < \mu}(\cup\mathcal{G}^*_\mu)$. For $\mu \in \lambda$ define $K_{(\mu)} = \{x \in X : \mu_x \leq \mu\}$. Note that each set $K_{(\mu)}$ is closed in X. For $\gamma \in \Gamma$ define $H_{(\mu,\gamma)} = \left(X - \bigcup_{\gamma \leq \beta} G_{(\mu,\beta)}\right) \cap K_{(\mu)}$, which is a closed set in X. Define $U_{(\mu,\gamma)} = U_\gamma - \hat{H}_{(\mu,\gamma)}$ for each $\gamma \in \Gamma$ and $\mathcal{U}_{(\mu)} = \{U_{(\mu,\gamma)} : \gamma \in \Gamma\}$. Now, let $\mathcal{G}_{(\mu)}$ be a final λ weak θ–refinement of $\mathcal{U}_{(\mu)}$, say $\mathcal{G}_{(\mu)} = \cup\{\mathcal{G}^*_{(\mu,\tau)} : \tau \in \lambda\}$ where $\mathcal{G}^*_{(\mu,\tau)} = \{G_{(\mu,\tau,\gamma)} : \gamma \in \Gamma\}$. For $\tau \in \lambda$ let $K_{(\mu,\tau)} = \left(\bigcup_{\delta<\mu} K_{(\delta)}\right) \cup \{x \in K_{(\mu)} : 0 \leq ord(x, \mathcal{G}^*_{(\mu,\tau)}) < \aleph_0$ and $0 = ord(x, \mathcal{G}^*_{(\mu,\beta)})$ for all $\beta > \tau\}$. For $\gamma \in \Gamma$, define $H_{(\mu,\tau,\gamma)} = (X - \bigcup_{\beta \geq \gamma} G_{(\mu,\tau,\beta)}) \cap K_{(\mu,\tau)}$, a closed set in X. Let $U_{(\mu,\tau,\gamma)} = U_\gamma - \hat{H}_{(\mu,\tau,\gamma)}$ for $\gamma \in \Gamma$, and $\mathcal{U}_{(\mu,\tau)} = \{U_{(\mu,\tau,\gamma)} : \gamma \in \Gamma\}$. Continue this process by induction for all finite tuples in $\bigcup_{n \in N} \lambda^n$.

Claim. *For each $y \in Y$ there exists an $n \in N$ and an n–tuple $(\alpha_1, \alpha_2, \ldots, \alpha_n)$, where $\alpha_i \in \lambda$, and $\gamma \in \Gamma$ such that $f^{-1}(y) \cap K_{(\alpha_1,\alpha_2,\ldots,\alpha_n)} \subseteq G_{(\alpha_1,\alpha_2,\ldots,\alpha_n,\beta)}$ for only finitely many elements β of Γ, one of which is γ.*

Proof of Claim: Let $y \in Y$ and α_1 be the first element of λ for which there exists $x \in f^{-1}(y) \cap K_{(\alpha_1)}$. For each $x \in f^{-1}(y) \cap K_{(\alpha_1)}$, let $m_{x,1} = \max\{\gamma \in \Gamma : x \in G_{(\alpha,\gamma)}\}$. Let $m_1 = \inf\{m_{x,1} : x \in f^{-1}(y) \cap K_{(\alpha_1)}\}$. Note that $f^{-1}(y) \cap U_{(\alpha,\gamma)} = \emptyset$ for $\gamma > m_1$. Next, let α_2 be the first element of λ for which there is some $x \in f^{-1}(y) \cap K_{(\alpha_1,\alpha_2)}$. For each $x \in f^{-1}(y) \cap K_{(\alpha_1,\alpha_2)}$ let $m_{x,2} = \max\{\gamma \in \Gamma : x \in G_{(\alpha_1,\alpha_2,\gamma)}\}$. Let $m_2 = \inf\{m_{x,2} : x \in f^{-1}(y) \cap K_{(\alpha_1,\alpha_2)}\}$. Note that $m_2 \leq m_1$ and $f^{-1}(y) \cap U_{(\alpha_1,\alpha_2,\gamma)} = \emptyset$ if $\gamma > m_2$. By induction define $m_3, m_4, m_5, \ldots,$ etc. Now there exists $n \in N$ such that $m_{n-1} = \inf\{m_j : j \in N\}$. Then $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is the n–tuple we need with $\gamma = m_n$. The claim is thus proved.

Next, let

$$\mathcal{S}_{(\alpha_1,\alpha_2,\ldots,\alpha_n)} = \left\{ \, (Y - f((X - G_{(\alpha_1,\alpha_2,\ldots,\alpha_n,\gamma)}) \cap K_{(\alpha_1,\alpha_2,\ldots,\alpha_n)})) \cap V_\gamma : \gamma \in \Gamma \, \right\}.$$

Then for each $y \in Y$ it follows from the claim that there exists $n \in N$, $\gamma \in \Gamma$, and an $(n+1)$-tuple $(\alpha_1, \alpha_2, \ldots, \alpha_n, \gamma)$ such that $f^{-1}(y) \cap K_{(\alpha_1,\alpha_2,\ldots,\alpha_n)} \subseteq G_{(\alpha_1,\alpha_2,\ldots,\alpha_n,\gamma)}$. Then by the definition of $\mathcal{S}_{(\alpha_1,\alpha_2,\ldots,\alpha_n)}$, it follows that $\mathcal{S}_{(\alpha_1,\alpha_2,\ldots,\alpha_n)}$ is an open partial refinement of $\mathcal{V}$ that has positive finite order at y. Since $\left|\bigcup_{n \in N} \lambda^n\right| = \aleph_0$, it follows that Y is hereditarily weak θ–refinable.

In [26], J. C. Smith proved the following theorem.

Theorem 3.1.12. *Every weak $\bar{\theta}$–refinable space is $B(D, \omega_0^2)$–refinable.*

Theorem 3.1.13. *Every $B(D, \lambda)$–refinable space is finally λ weak θ–refinable.*

Proof: Let $\mathcal{U} = \{U_\gamma : \gamma \in \Gamma\}$ be an open cover of a space X and let $\mathcal{B} = \cup\{\mathcal{B}_\mu : \mu \in \lambda\}$ be a $B(D, \lambda)$-refinement of $\mathcal{U}$. For each $\gamma \in \Gamma$ and $\mu \in \lambda$ choose $V_{(\mu,\gamma)}$ to be an open set satisfying;

(i) $B_{(\mu,\gamma)} \subseteq V_{(\mu,\gamma)} \subseteq U_\gamma$ and

(ii) $V_{(\mu,\gamma)} \cap \left[(\cup\{\mathcal{B}_\tau : \tau < \mu\}) \cup (\cup\{B_{(\mu,\psi)} : \psi \neq \gamma\})\right] = \emptyset$.

It is easy to see that $\mathcal{V} = \cup\{\mathcal{V}_\mu : \mu \in \lambda\}$ where $\mathcal{V}_\mu = \{V_{(\mu,\gamma)} : \gamma \in \Gamma\}$ is a finally λ weak θ–refinement of $\mathcal{U}$. In fact, for each $x \in X$, $ord(x, \mathcal{V}_{\mu_x}) = 1$.

Theorem 3.1.14. *The closed continuous image of a hereditarily weak $\bar{\theta}$–refinable space X is hereditarily weak θ-refinable.*

Proof: By Theorem 3.1.12, X is hereditarily $B(D, \omega_0^2)$-refinable. Thus by Theorem 3.1.13, X is hereditarily finally ω_0^2 weak θ-refinable. Therefore, every closed continuous image of X is hereditarily weak θ-refinable by Theorem 3.1.11.

In 1969, R. E. Hodel [14] obtained the following result for totally normal spaces.

Theorem 3.1.15. *Let Q denote a class of topological spaces that satisfies the following conditions.*

(a) If X is a topological space and $\mathcal{F} = \{F_\alpha : \alpha \in A\}$ is a locally finite closed cover of X such that each $F_\alpha \in Q$, then $X \in Q$.

(b) If X is a topological space such that every open subset of X belongs to Q, then every subset of X belongs to Q.

(c) If X is a topological space which belongs to Q, then every closed subset of X belongs to Q.

Then if X is a totally normal space in Q, every subset of X is in Q.

Lemma 3.1.16. *The class of countably metacompact, $B(D, \omega_0)$-refinable spaces satisfies conditions (a), (b), and (c) of Theorem 3.1.15 above.*

Proof: By Theorem 3.1.7, condition (a) is satisfied. It is easily observed that if every open subset of a space is countably metacompact and $B(D, \omega_0)$-refinable then every subset of the space is countably metacompact and $B(D, \omega_0)$-refinable, so (b) is satisfied. Also, countable metacompactness and $B(D, \omega_0)$-refinability are closed hereditary, and hence (c) is satisfied.

Remark 3.1.17. *By Theorem 3.1.15 and Lemma 3.1.16, we now have the following theorem which establishes sufficient conditions for a space to be hereditarily $B(D, \omega_0)$-refinable.*

Theorem 3.1.18. *Let X be a totally normal, countably metacompact $B(D, \omega_0)$-refinable space. Then X is hereditarily $B(D, \omega_0)$-refinable.*

Corollary 3.1.19. *The closed, continuous image of a countably metacompact, totally normal, $B(D, \omega_0)$-refinable space X is hereditarily weak θ-refinable.*

Proof: By Theorem 3.1.18, X is hereditarily $B(D, \omega_0)$-refinable. Then by Theorem 3.1.13, X is hereditarily finally ω_0 weak θ-refinable. Therefore, by Theorem 3.1.11, every closed continuous image of X is hereditarily weak θ-refinable.

Remark 3.1.20. *Theorem 3.1.15 can be used to obtain, for the class of totally normal spaces, a corollary to Theorem 3.1.21 below.*

R. H. Price [19] has shown the following result.

Theorem 3.1.21. *The perfect image of a hereditarily countably metacompact, $B(D, \omega_0)$-refinable space is $B(D, \omega_0^2)$-refinable.*

We now show the following corollary to Theorem 3.1.21 above.

Corollary 3.1.22. *The perfect image of a countably metacompact, totally normal, $B(D, \omega_0)$-refinable space X is hereditarily $B(D, \omega_0^2)$-refinable.*

Proof: Since countable metacompactness satisfies the conditions of Theorem 3.1.15, X is hereditarily countably metacompact. Thus the corollary follows from Theorem 3.1.15 and 3.1.21.

Remark 3.1.23. *In examining the proof of Theorem 3.1.11, we see that the main step in the construction of weak θ-refinement of the cover of the image space was* *building partial refinements of the cover of the domain space with "small enough" sets such that for each member of the image, its inverse image met only finitely many members of some partial refinement. With this observation in mind, we can obtain the following generalization of Theorem 3.1.11.*

Theorem 3.1.24. *Every hereditarily $B(CP,\lambda)$-refinable space X is hereditarily weak θ-refinable.*

Proof: Let $\mathcal{U} = \{U_\gamma : \gamma \in \Gamma\}$ be an open cover of X. Let $\mathcal{B} = \cup\{\mathcal{B}^*_{(\mu)} : \mu \in \lambda\}$, where $\mathcal{B}^*_{(\mu)} = \{B_{(\mu,\gamma)} : \gamma \in \Gamma\}$ be a $B(CP,\lambda)$-refinement of $\mathcal{U}$. For $\mu \in \lambda$ let $K_{(\mu)} = \{x \in X : x \in \bigcup_{\tau<\mu}(\cup\mathcal{B}^*_{(\mu)})\}$. Then $K_{(\mu)}$ is a closed set in X. Let $F_{(\mu,\gamma)} = K_{(\mu)} \cup \left(\bigcup_{\delta<\gamma} B_{(\mu,\delta)}\right)$ and let $U_{(\mu,\gamma)} = U_\gamma - F_{(\mu,\gamma)}$. Define $\mathcal{U}_{(\mu)} = \{U_{(\mu,\gamma)} : \gamma \in \Gamma\}$. Then by hypothesis there exists a $B(CP,\lambda)$-refinement of $\mathcal{U}_{(\mu)}$, say $\mathcal{B}_{(\mu)} = \cup\{\mathcal{B}^*_{(\mu,\psi)} : \psi \in \lambda\}$ where $\mathcal{B}^*_{(\mu,\psi)} = \{B_{(\mu,\psi,\gamma)} : \gamma \in \Gamma\}$. Then for $\psi \in \lambda$, let $K_{(\mu,\psi)} = K_{(\mu)} \cup \{x \in X : x \in \bigcup_{\tau<\psi}(\cup\mathcal{B}^*_{(\mu,\tau)})\}$. Clearly $K_{(\mu,\psi)}$ is a closed subset of X. Let $F_{(\mu,\psi,\gamma)} = K_{(\mu,\psi)} \cup \left(\bigcup_{\delta<\gamma} B_{(\mu,\psi,\delta)}\right)$ and let $U_{(\mu,\psi,\gamma)} = U_{(\mu,\gamma)} - F_{(\mu,\psi,\gamma)}$. Define $\mathcal{U}_{(\mu,\psi)} = \{U_{(\mu,\psi,\gamma)} : \gamma \in \Gamma\}$. Then $\mathcal{U}_{(\mu,\psi)}$ is an open partial refinement of $\mathcal{U}$ with a $B(CP,\lambda)$-refinement, say $\mathcal{B}_{(\mu,\psi)} = \cup\{\mathcal{B}^*_{(\mu,\psi,\epsilon)} : \epsilon \in \lambda\}$ where $\mathcal{B}^*_\epsilon = \{B_{(\mu,\psi,\epsilon,\gamma)} : \gamma \in \Gamma\}$. Continue this procedure by induction on $\bigcup_{n\in N}\lambda^n$. As in the proof of Theorem 3.1.11, for each $x \in X$ there exists an $n \in N$, an n-tuple $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ in λ^n, and a $\gamma \in \Gamma$ such that

(1) $x \in B_{(\alpha_1,\alpha_2,\ldots,\alpha_n,\beta)}$ iff $\beta = \gamma$.

For each $n \in N$ and $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \lambda^n$,

(2) define $\mathcal{S}_{(\alpha_1,\alpha_2,\ldots,\alpha_n)} = \{U_{(\alpha_1,\alpha_2,\ldots,\alpha_n,\gamma)} - \left(\left(\bigcup_{\delta\neq\gamma} B_{(\alpha_1,\ldots,\alpha_n,\delta)}\right) \cup K_{(\alpha_1,\alpha_2,\ldots,\alpha_n)} \cup \left(\bigcup_{\mu<\alpha_n} \cup\mathcal{B}^*_{(\alpha_1,\alpha_2,\ldots,\mu)} \right)\right) : \gamma \in \Gamma\}$.

Now, let $\mathcal{S} = \cup\{\mathcal{S}_{(\alpha_1,\ldots,\alpha_n)} : (\alpha_1, \ldots, \alpha_n) \in \bigcup_{n\in N}\lambda^n\}$. It follows from (1) and (2) that $\mathcal{S}$ is a weak θ-refinement of $\mathcal{U}$.

Since it is clear that closed maps preserve CP collections, we have the next theorem.

Theorem 3.1.25. *The closed continuous image of a $B(CP,\lambda)$-refinable space is $B(CP,\lambda)$-refinable.*

Corollary 3.1.26. *The closed continuous image a hereditarily $B(CP,\lambda)$-refinable space is hereditarily weak θ-refinable.*

Proof: By Theorem 3.1.25 it is the case that the image is hereditarily $B(CP,\lambda)$-refinable and hence hereditarily weak θ-refinable by Theorem 3.1.24.

Theorem 3.1.27. *Every finally λ weak θ-refinable space X is $B(D,\omega_0\cdot\lambda)$-refinable.*

Proof: Let $\mathcal{U}=\{U_\gamma:\gamma\in\Gamma\}$ be an open cover of X. Let $\mathcal{G}=\cup\{\mathcal{G}_\mu:\mu\in\lambda\}$ be a finally λ weak θ-refinement of $\mathcal{U}$. Recall from Definition 1.2.33 (b) that for each $x\in X$ there exists $\mu_x\in\lambda$ such that $0<ord(x,\mathcal{G}_{\mu_x})<\aleph_0$ and $x\notin\bigcup_{\delta>\mu_x}(\cup\mathcal{G}_\delta)$. Then for $\mu\in\lambda$, let $K_\mu=\{x\in X:\mu_x=\mu\}$. For $\mu\in\lambda$ and $n\in N$, define $\mathcal{B}_{(\mu,n)}=\{(G_{(\mu,\gamma_1)}\cap G_{(\mu,\gamma_2)}\cap\cdots\cap G_{(\mu,\gamma_n)}\cap K_\mu)-\{x\in\cup\mathcal{G}_\mu:ord(x,\mathcal{G}_\mu)>n\}:\gamma_1,\gamma_2,\ldots,\gamma_n$ are distinct elements of $\Gamma\}$. Then $\mathcal{B}_{(\mu,n)}$ is a relatively discrete closed collection in $X-((\bigcup_{\delta<\mu}(\bigcup_{i\in N}(\cup\mathcal{B}_{(\delta,i)})))\cup(\bigcup_{i<n}(\cup\mathcal{B}_{(\mu,i)})))$. Also, $\bigcup_{\delta<\mu}(\bigcup_{i\in N}(\cup\mathcal{B}_{(\delta,i)}))=\bigcup_{\delta<\mu}K_\delta$ is a closed set in X. It follows that $\mathcal{B}=\cup\{\mathcal{B}_{(\mu,n)}:\mu\in\lambda,\ n\in N\}$ is a $B(D,\omega_0\cdot\lambda)$-refinement of $\mathcal{U}$.

Remark 3.1.28. *The rest of this chapter gives necessary and sufficient conditions, in terms of upper semi-continuous decompositions, for $B(D,\omega_0)$-refinability to be preserved by closed, continuous maps.*

Lemma 3.1.29. *Let X represent a topological space that satisfies the following condition.*

(*) For every upper semi-continuous decomposition $\mathcal{D}$ of X and $\mathcal{D}$-saturated open cover $\mathcal{U}=\{U_\gamma:\gamma\in\Gamma\}$ of X, there exists a partial refinement $\mathcal{F}=\cup\{\mathcal{F}_n:n\in$

$\omega_0\}$ of $\mathcal{U}$ where for each $n \in \omega_0$ we have $\mathcal{F}_n = \{F_{(n,\gamma)} : \gamma \in \Gamma\}$ with $F_{(n,\gamma)} \subseteq U_\gamma$ for each $\gamma \in \Gamma$ and such that

(1) $\mathcal{F}_n$ is a relatively discrete collection of closed sets in $X - \bigcup_{i<n}(\cup\hat{\mathcal{F}}_i)$; and

(2) for each $D \in \mathcal{D}$ there exists $n_D \in \omega_0$ with $D \cap \bigcup_{i<n_0}(\cup\mathcal{F}_i) = \emptyset$ and $|\{\gamma \in \Gamma : D \cap F_{(n_D,\gamma)} \neq \emptyset\}| = 1$.

Then every closed continuous image of X is $B(D,\omega_0)$-refinable.

Proof: Suppose $f : X \to Y$ is a closed, continuous, onto map, and $\mathcal{V} = \{V_\gamma : \gamma \in \Gamma\}$ is an open cover of Y. Define $\mathcal{U} = \{U_\gamma = f^{-1}(V_\gamma) : \gamma \in \Gamma\}$. Define $\mathcal{D} = \{f^{-1}(y) : y \in Y\}$. Then $\mathcal{U}$ is a $\mathcal{D}$ -saturated open cover of X. Let $\mathcal{F}$ be a partial refinement of $\mathcal{U}$ satisfying (*) above. Define $\mathcal{B} = \cup\{\mathcal{B}_n : n \in \omega_0\}$ where $\mathcal{B}_n = \{B_{(n,\gamma)} = f(F_{(n,\gamma)}) - f(\bigcup_{i<n}(\cup\mathcal{F}_i)) : \gamma \in \Gamma\}$ for each $n \in \omega_0$.

Claim. *$\mathcal{B}$ is a $B(D,\omega_0)$-refinement of $\mathcal{V}$.*

Proof: Let $y \in Y$ and let $D_y = f^{-1}(y)$. Since $\mathcal{F}_{n_{D_y}}$ is a discrete closed collection in $X - (\bigcup_{i<n_{D_y}} \hat{\mathcal{F}}_i)$ it is the case that $\mathcal{B}_{n_{D_y}}$ is a relatively closed discrete collection in $Y - \bigcup_{i<n_{D_y}}(\cup\mathcal{B}_i)$ and $ord(y, \mathcal{B}_{n_{D_y}}) = 1$.

Lemma 3.1.30. *Let X represent a topological space such that every closed continuous image of X is $B(D,\omega_0)$-refinable. Then for every upper semi-continuous decomposition $\mathcal{D}$ of X and every $\mathcal{U} = \{U_\gamma : \gamma \in \Gamma\}$ a $\mathcal{D}$-saturated open cover of X, there exists a $B(D,\omega_0)$-refinement $\mathcal{B} = \cup\{\mathcal{B}_n : n \in \omega_0\}$ of $\mathcal{U}$ such that for each $D \in \mathcal{D}$ there exists $n_D \in \omega_0$ with $D \subseteq B_{(n_D,\gamma)}$ for some $\gamma \in \Gamma$.*

Proof: Let π be the projection map from X to the quotient space given by the decomposition $\mathcal{D}$. Let $\mathcal{V} = \{V_\gamma = \pi(U_\gamma) : \gamma \in \Gamma\}$. By Lemma 1.2.53, $\mathcal{V}$ is an open cover of the quotient space. Then by hypothesis, $\mathcal{V}$ has a $B(D,\omega_0)$-refinement, say $\mathcal{B}^* = \cup\{\mathcal{B}^*_n : n \in \omega_0\}$. Let $\mathcal{B} = \cup\{\mathcal{B}_n : n \in \omega_0\}$ where $\mathcal{B}_n = \{B_{(n,\gamma)} : \gamma \in \Gamma\}$ and $B_{(n,\gamma)} = \pi^{-1}(B^*_{(n,\gamma)})$ for each $\gamma \in \Gamma$. It is easy to see that $\mathcal{B}$ is a $B(D,\omega_0)$-refinement of $\mathcal{U}$ with the desired property.

We now have the following theorem.

Theorem 3.1.31. *Every closed continuous image of a space X is $B(D, \omega_0)$-refinable iff X satisfies the condition (*) in Lemma 3.1.29.*

Question 3.1.32. Can the hereditary condition in Theorem 3.1.14 be removed?

CHAPTER IV

σ–PRODUCT THEOREMS

Recently [10, 11, 12, 17, 27, 29, 30], there has been much interest in answering the following question. Let P represent some topological property. If every finite subproduct of a σ-product X has property P, does X necessarily have property P? Diagram 4.0.1 below lists many results regarding this question. Results numbered 20–26 are new results in this thesis. In addition, results numbered 7 and 10 follow directly as corollaries from a special $B(D, \omega_0)$ sum theorem proved in §1.

Diagram 4.0.1. σ–product theorems.

	Property	σ-product	Reference
1.	paracompact	yes (regular)	[17]
2.	shrinking	yes (normal)	[10]
3.	CWN	yes (normal)	[10]
4.	weak β-property	yes (normal)	[11]
5.	screenable	yes (normal)	[12]
6.	countably paracompact	yes (normal)	[11]
7.	metacompact	yes	[29]
8.	Lindelöf	yes	[17]
9.	expandability	yes (normal)	[29]
10.	almost expandable (discretely)	yes	[29]
11.	CWSN	yes (subnormal)	[29]
12.	subparacompact	yes (subnormal)	[29]

	Property	σ-product	Reference
13.	CWδN	yes (subnormal)	[29]
14.	stratifiable*	yes (monotone normal)	[29]
15.	θ-refinable	yes	[30]
16.	orthocompact	no	[30]
17.	β-property	yes (normal)	[30]
18.	CP cover by compact sets	yes	[27]
19.	special CP cover by compact sets	yes	[27]
20.	weak $\bar{\theta}$-refinable	yes	
21.	$B(D, \omega_0)$-refinable	yes	
22.	mesocompact	yes (normal)	
23.	discretely CF expandable	yes (normal)	
24.	para-Lindelöf	yes (normal)	
25.	meta-Lindelöf	yes	
26.	closed hereditary irreducibility	yes	

* plus some additional conditions.

Remark 4.0.2. *Let $\{X_\alpha : \alpha \in A\}$ be an infinite collection of T_2 spaces containing more than one point. Then every σ-product $X = \sigma\{X_\alpha : \alpha \in A\}$ is not compact since X is not closed in $\prod\{X_\alpha : \alpha \in A\}$.*

In §1 of this chapter, we establish the σ-product theorem for the properties weak

$\bar{\theta}$-refinability and $B(D,\omega_0)$-refinability. We obtain these results as special cases of a more general sum theorem established for a class of spaces with a "$B(D,\omega_0)$-breakdown" with some particular properties. We will refer to this kind of theorem as a "Special $B(D,\omega_0)$ Sum Theorem". We then show that the σ-product theorem for properties such as metacompactness and almost (discrete) expandability easily follows from this Special $B(D,\omega_0)$ Sum Theorem.

In §2 we obtain σ-product theorems for mesocompactness, discrete CF expandability, para-Lindelöfness, and closed hereditary irreducibility. We demonstrate how, in the case of mesocompactness, the Special $B(D,\omega_0)$ Sum Theorem can be used to provide a more general setting for a σ-product theorem.

§1. σ-product theorem for weak $\bar{\theta}$-refinability and $B(D,\omega_0)$-refinability.

We begin this section with a proof of a result which we call the Special $B(D,\omega_0)$ Sum Theorem for weak $\bar{\theta}$-refinability. The σ-product theorem for the properties weak $\bar{\theta}$-refinability and $B(D,\omega_0)$-refinability will follow as special cases of this result.

Definition 4.1.1. Let X be a space. Then $\mathcal{B} = \cup\{\mathcal{B}_n : n \in \omega_0\}$ is said to be a **$\mathbf{B(D,\omega_0)}$-breakdown of** X provided that $\mathcal{B}_n = \{B(n,\alpha) : \alpha \in A_n\}$ is a relatively discrete collection of closed subsets of $X - \bigcup_{i<n}(\cup\mathcal{B}_i)$ and $\cup\mathcal{B} = X$.

Special $\mathbf{B(D,\omega_0)}$ Sum Theorem 4.1.2. *Suppose X is a space which has a $B(D,\omega_0)$-breakdown $\mathcal{B} = \cup\{\mathcal{B}_n : n \in \omega_0\}$, where $\mathcal{B}_n = \{B(n,\alpha) : \alpha \in A_n\}$, satisfying the following properties:*

(1) For each $n \in \omega_0$, $\mathcal{B}_n$ has a point finite open expansion $\mathcal{G}_n = \{G(n,\alpha) : \alpha \in A_n\}$ in X.

Note that we may assume $\cup\mathcal{G}_n \cap [\bigcup_{i<n}(\cup\mathcal{B}_i)] = \emptyset$.

(2) For each $n \in \omega_0$ and $\alpha \in A_n$, if $\mathcal{S}_{(n,\alpha)} = \{S(n,\alpha,\delta) : \delta \in \Delta\}$ is a point finite open collection in $B(n,\alpha)$ then $\mathcal{S}_{(n,\alpha)}$ has a point finite open expansion in X. Note that

again we may assume that the expansion of $\mathcal{S}_{(n,\alpha)}$ does not meet $\left(\bigcup_{i<n}(\cup\mathcal{B}_i)\right) \cup \left(\bigcup_{\psi\neq\alpha} B_{(n,\psi)}\right)$.

(3) For every $n \in \omega_0$ and $\alpha \in A_n$ the set $B(n,\alpha)$ is weak $\bar{\theta}$-refinable.

Then X is weak $\bar{\theta}$-refinable.

Proof. Let $\mathcal{U} = \{U_\gamma : \gamma \in \Gamma\}$ be an open cover of X. For each $n \in \omega_0$ and $\alpha \in A_n$, let

(i) $\mathcal{S}_{(n,\alpha)} = \cup\{\mathcal{S}_{(n,\alpha,i)} : i \in \omega_0\}$ be a weak $\bar{\theta}$-refinement of $\mathcal{U}|B(n,\alpha)$, with $\mathcal{S}_{(n,\alpha,i)} = \{S(n,\alpha,i,\gamma) : \gamma \in \Gamma\}$. Assume by (2) that the collections $\mathcal{S}_{(n,\alpha,i)}$ have been expanded to open collections in X that partially refine $\mathcal{U}$ such that $\{\cup\mathcal{S}_{(n,\alpha,i)} : i \in \omega_0\}$ is point finite in X.

(ii) Define $H_{(n,\alpha,i,\gamma)} = S(n,\alpha,i,\gamma) \cap G(n,\alpha)$ for $\gamma \in \Gamma$, let $\mathcal{H}_{(n,\alpha,i)} = \{H(n,\alpha,i,\gamma) : \gamma \in \Gamma\}$, and for $i \in \omega_0$, let $\mathcal{H}_{(n,i)} = \cup\{\mathcal{H}_{(n,\alpha,i)} : \alpha \in A_n\}$.

(*) By (i) and (ii), $\{\cup\mathcal{H}_{(n,i)} : i \in \omega_0\}$ is a point finite open collection in X. Furthermore if $j < n$ and $y \in \cup\mathcal{B}_j$, then $0 = ord(y, \mathcal{H}_{(n,i)})$ for every $i \in \omega_0$ by (i).

Now, for each $n \in \omega_0$ and $x \in \cup\mathcal{B}_n$, we have $0 < ord(x, \mathcal{H}_{(n,i)})$ for some $i \in \omega_0$ by (i) and (ii). Next consider the map $f : \omega_0 \times \omega_0 \to \omega_0$ where $f(0,0) = 0$, $f(0,1) = 1$, $f(1,0) = 2$, $f(0,2) = 3$, $f(1,1) = 4$, $f(2,0) = 5$, etc.

Claim. *$\mathcal{H} = \cup\{\mathcal{H}_{f^{-1}(k)} : k \in \omega_0\}$ is a weak $\bar{\theta}$-refinement of $\mathcal{U}$.*

Proof. Let $x \in X$. Then there exists $n \in \omega_0$ such that $x \in \cup\mathcal{B}_n$.

By (*), $\{\cup\mathcal{H}_{f^{-1}}(k) : k \in \omega_0\}$ is a PF collection of open subsets of X.

Next, there exists an $\alpha \in A_n$ such that $x \in B(n,\alpha)$. Then by (i) and (ii) there exists an $i \in \omega_0$ such that $0 < ord(x, \mathcal{H}_{(n,i)}) < \aleph_0$. In fact, $ord(x, \mathcal{H}_{(n,i)}) = ord(x, \mathcal{H}_{(n,\alpha,i)})$.

Remark 4.1.3. *Hypothesis (2) in Theorem 4.1.2 can actually be weakened to the following: For each $n \in \omega_0$ and $\alpha \in A_n$, if $\mathcal{S}_{(n,\alpha)} = \{S(n,\alpha,i) : i \in \omega_0\}$ is a point* *finite open collection in $B(n,\alpha)$ then $\mathcal{S}_{(n,\alpha)}$ has a point finite open expansion in X. However, hypothesis (2) will be needed later to obtain the Special $B(D,\omega_0)$ Sum Theorem for properties such as metacompactness.*

Lemma 4.1.4. *[19] A space X is $B(D,\omega_0)$-refinable iff it is 1-bded weak $\bar{\theta}$-refinable.*

From the proof of Theorem 4.1.2 and Lemma 4.1.4, we now have a result similar to Theorem 4.1.2 for $B(D,\omega_0)$-refinability.

Theorem 4.1.5. *Let X be a space with a $B(D,\omega_0)$ breakdown satisfying conditions (1) and (2) in Theorem 4.1.2. If $B(n,\alpha)$ is $B(D,\omega_0)$-refinable for each $n \in \omega_0$ and $\alpha \in A_n$, then X is $B(D,\omega_0)$-refinable.*

Theorem 4.1.6. *Let $X = \sigma\{X_\alpha : \alpha \in A\}$ be a σ-product space. Then X satisfies conditions (1) and (2) in the hypothesis of Theorem 4.1.2 with $\mathcal{B}_n = \{Y_a - X_{n-1} : a \in [A]^n\}$ for each $n \in \omega_0$.*

Proof. Recall from Definition 1.2.45 that X_n is closed in X for every $n \in \omega_0$, and $\{Y_a - X_{n-1} : a \in [A]^n\}$ is a discrete closed collection in $X - X_{n-1}$. Also, by Lemma 1.2.47, $\{p_a^{-1}(Y_a - X_{n-1}) : a \in [A]^n\}$ is a point finite open expansion of $\mathcal{B}_n$ in X, so condition (1) is satisfied. Likewise, if $\mathcal{S}_{(n,a)} = \{S(n,a,\gamma) : \gamma \in \Gamma\}$ is a point finite open collection in $B(n,\alpha) = Y_a - X_{n-1}$ for some $a \in [A]^n$ then $\mathcal{S}^*_{(n,a)} = \{p_a^{-1}(S(n,a,\gamma)) : \gamma \in \Gamma\}$ is a point finite open expansion of $\mathcal{S}_{(n,a)}$ in X and (2) is satisfied.

We now have the following result.

Theorem 4.1.7. *The σ-product theorem holds for the property weak $\bar{\theta}$-refinability (resp., $B(D,\omega_0)$-refinability.)*

Theorem 4.1.8. *The Special $B(D,\omega_0)$ Sum Theorem holds for the properties metacompactness (meta-Lindelöfness) and almost (discrete) expandability.*

Proof.

(i) To prove the theorem for metacompactness (meta-Lindelöfness), suppose $\mathcal{U} = \{U_\gamma : \gamma \in \Gamma\}$ is an open cover of X. Then for each $n \in \omega_0$ and $\alpha \in A_n$, let $\mathcal{U}_{(n,\alpha)} = \{U(n,\alpha,\gamma) : \gamma \in \Gamma\}$ be a refinement of $\mathcal{U}|B(n,\alpha)$ that is open and point finite (point countable) on $B(n,\alpha)$. Then by hypothesis (2) of Theorem 4.1.2, there exists an open collection $\mathcal{V}_{(n,\alpha)} = \{V(n,\alpha,\gamma) : \gamma \in \Gamma\}$ that is a point finite (point countable) open expansion in X of $\mathcal{U}_{(n,\alpha)}$ that misses $\left(\bigcup_{i<n}(\cup\mathcal{B}_i)\right) \cup \left(\bigcup_{\psi\neq\alpha} B(n,\psi)\right)$ and partially refines $\mathcal{U}$. It follows that $\{V(n,\alpha,\gamma)\cap G(n,\alpha)\cap U_\gamma : \gamma \in \Gamma,\ n \in \omega_0,\ \alpha \in A_n\}$ is an open in X point finite (point countable) refinement of $\mathcal{U}$, where for each $n \in \omega_0$, $\mathcal{G}_n = \{G(n,\alpha) : \alpha \in A_n\}$ is the point finite (point countable) open expansion of $\mathcal{B}_n$ given in hypothesis (1) of Theorem 4.1.2.

(ii) To prove the theorem for almost (discrete) expandability, let $\mathcal{F} = \{F_\gamma : \gamma \in \Gamma\}$ be a locally finite (discrete) family of closed sets in X. We note that for each $n \in \omega_0$, $\mathcal{F}|B(n,\alpha)$ is a locally finite (discrete) family of closed sets in $B(n,\alpha)$. Hence $\mathcal{F}|B(n,\alpha)$ has a point finite expansion, $\mathcal{V}_{(n,\alpha)} = \{V(n,\alpha,\gamma) : \gamma \in \Gamma\}$ that is open in $B(n,\alpha)$. Now by hypothesis (2) of Theorem 4.1.2, $\mathcal{V}_{(n,\alpha)}$ has a point finite open expansion, $\mathcal{V}^*_{(n,\alpha)} = \{V^*(n,\alpha,\gamma) : \gamma \in \Gamma\}$ in X that misses $\left(\bigcup_{i<n}(\cup\mathcal{B}_i)\right) \cup \left(\bigcup_{\psi\neq\alpha} B_{(n,\psi)}\right)$. For each $n \in \omega_0$, let $\mathcal{G}_n = \{G(n,\alpha) : \alpha \in A_n\}$ be the point finite open in X expansion of $\mathcal{B}_n$ as in condition (1) of the hypothesis of Theorem 4.1.2. It is now easy to see that $\mathcal{V} = \{(\cup\{V^*(n,\alpha,\gamma) \cap G(n,\alpha) : n \in \omega_0,\ \alpha \in A_n\}) : \gamma \in \Gamma\}$ is a point finite open (in X) expansion of $\mathcal{F}$.

Remark 4.1.9. *Now two σ-product theorems proved in [29] and the σ-product theorem for meta-Lindelöfness follow directly from Lemma 4.1.6 and Theorem 4.1.8 above.*

Theorem 4.1.10. *The σ-product theorem holds for the properties metacompactness, meta-Lindelöfness, and almost (discrete) expandability.*

§2. σ–Product Theorem for mesocompactness, discrete compact finite expandability, para–Lindelöfness, and closed hereditary irreducibility.

Lemma 4.2.1. *Let $n \in w_0$. Suppose V and U are open subsets of $X = \sigma\{X_\alpha : \alpha \in A$ such that $X_n \subseteq V \subseteq \overline{V} \subseteq U$ and that each finite subproduct of X is CF expandable. If $\mathcal{F} = \{F_\gamma : \gamma \in \Gamma\}$is a LF collection of closed sets in X, then $\mathcal{F}|\,(X_{n+1} - U)$ has a CF open expansion in X.*

Proof. For each $a \in [A]^{n+1}$, Y_a is CF expandable. Hence $\mathcal{F}|(Y_\alpha - U)$ has a CF expansion in Y_a, say $\mathcal{G}^*_{(n+1,a)} = \{G^*(n+1, a, \gamma) : \gamma \in \Gamma\}$. Let $\mathcal{G}_{(n+1,a)} = \{G(n+1, a, \gamma) = p_a^{-1}(G^*(n+1, a, \gamma)) \cap p_a^{-1}(Y_a - \overline{V}) : \gamma \in \Gamma\}$. Now define $\mathcal{G}_{(n+1)} = \{G(n+1, \gamma) = (\cup G(n+1, a, \gamma) : a \in [A]^{n+1}) : \gamma \in \Gamma\}$. Clearly $\mathcal{G}_{(n+1)}$ is an open family in X. We now show that $\mathcal{G}_{(n+1)}$ is an open expansion of $\mathcal{F}|\,(X_{n+1} - U)$.

Let $x \in X_{n+1} \cap (F_\gamma - U)$ for some $\gamma \in \Gamma$. Let $a = Q(x)$. Then $a \in [A]^{n+1}$. Note that $p_a(x) = x \notin \overline{V}$. Hence $x \in p_a^{-1}(Y_a - \overline{V})$. Therefore $x \in (p_a^{-1}(G^*(n+1, a, \gamma)) \cap p_a^{-1}(Y_a - \overline{V})) = G(n+1, a, \gamma) \subseteq G(n+1, \gamma) \in \mathcal{G}_{(n+1)}$. It follows that $\mathcal{G}_{(n+1)}$ is an expansion of $\mathcal{F}|\,(X_{n+1} - U)$.

Finally, we claim that $\mathcal{G}_{(n+1)}$ is CF. Let K be a compact subset of X. Then $(K \cap p_a^{-1}(Y_a - \overline{V})) \neq \emptyset$ for only finitely many $a \in [A]^{n+1}$ by Lemma 1.2.48. Since $p_a(K)$ is compact for each $a \in [A]^{n+1}$, it follows that K meets only finitely many members of $\mathcal{G}_{(n+1)}$. Hence $\mathcal{G}_{(n+1)}$ is a CF open expansion of $\mathcal{F}$ in X.

Theorem 4.2.2. *Suppose $X = \sigma\{X_\alpha : \alpha \in A$ is normal and that every finite subproduct of X is discretely CF expandable. Then X is discretely CF expandable.*

Proof. Let $\mathcal{F} = \{F_\gamma : \gamma \in \Gamma\}$ be a discrete family of closed subsets of X. To prove the theorem we construct by induction a family $\mathcal{H}^* = \cup\{\mathcal{H}_i : i \in w_0\}$ of open subsets of X such that each $\mathcal{H}_i = \{H(i, \gamma) : \gamma \in \Gamma\}$ is CF, $\{\cup\mathcal{H}_i : i \in w_0\}$ is CF, and each $F_\gamma \subseteq \cup\{H(i, \gamma) : i \in \omega_0\}$. Then $\mathcal{H} = \{\cup\{H(i, \gamma) : i \in w_0\} : \gamma \in \Gamma\}$ will be a CF open expansion of $\mathcal{F}$. We may assume that $s^* \in F_\gamma$ for some $\gamma \in \Gamma$.

Step 1. Choose an open set $U_0 \subseteq X$ such that $s^* \in U_0$ and $U_0 \cap F_\gamma = \emptyset$ if $s^* \notin F_\gamma$ for all $\gamma \in \Gamma$. Let $H(0,\gamma) = U_0 \cup (X - \cup\mathcal{F})$ if $s^* \in F_\gamma$. Let $H(0,\gamma) = \emptyset$ otherwise. Then clearly $\mathcal{H}_{(0)} = \{H(0,\gamma) : \gamma \in \Gamma\}$ is an open CF expansion of $\mathcal{F} \mid X_0$.

Step 2. Fix $n \in w_0$. Assume that we have constructed an open CF expansion $\mathcal{H}_{(n)} = \{H(n,\gamma) : \gamma \in \Gamma\}$ of $\mathcal{F} \mid X_n$ such that

(i) $H(n,\gamma) \cap (\bigcup_{\beta\neq\gamma} F_\beta) = \emptyset$ for all $\gamma \in \Gamma$.

(ii) $\cup\mathcal{H}_{(n)} \supseteq \cup\mathcal{H}_{(n-1)}$.

(iii) There is an open set $V_{n-1} \subseteq X$ such that $X_{n-1} \subseteq V_{n-1} \subseteq \overline{V}_{n-1} \subseteq \cup\mathcal{H}_{(n-1)}$ with $V_k \subseteq V_{n-1}$ if $k \leq n-1$.

Step 3. By the normality of X, let V_n be an open subset of X such that $X_n \subseteq V_n \subseteq \overline{V}_n \subseteq \cup\mathcal{H}_{(n)}$ and $V_k \subseteq V_n$ for $k \leq n$.

By Lemma 4.2.1 above and the discreteness of $\mathcal{F}$, there exists a CF open expansion $\mathcal{G}_{(n+1)} = \{G(n+1,\gamma) : \gamma \in \Gamma\}$ of $\mathcal{F} \mid (X_{n+1} - \cup\mathcal{H}_{(n)})$ which does not meet $\overline{V}_n$ and such that $(G(n+1,\gamma) \cap (\bigcup_{\beta\neq\gamma} F_\beta)) = \emptyset$ for all $\gamma \in \Gamma$. Let $\mathcal{H}_{(n+1)} = \{H(n+1,\gamma) = H(n,\gamma) \cup G(n+1,\gamma) : \gamma \in \Gamma\}$. Then $\mathcal{H}_{(n+1)}$ is an open CF expansion of $\mathcal{F} \mid X_{n+1}$ such that $(H(n+1,\gamma) \cap (\bigcup_{\beta\neq\gamma} F_\beta)) = \emptyset$ for every $\gamma \in \Gamma$.

Let $\mathcal{H} = \{H_\gamma = \cup\{H(n,\gamma) : n \in w_0\} : \gamma \in \Gamma\}$.

Clearly $\mathcal{H}$ is a collection of open subsets of X. To see that $\mathcal{H}$ is an expansion of $\mathcal{F}$, let $x \in F_\gamma$ for some $\gamma \in \Gamma$. If $x = s^*$, then $x \in H_\gamma$ by Step 1. If $x \neq s^*$, choose the first $n \in w_0$ such that $x \in X_n$. From Step 2 we must have $x \in H(n,\gamma) \subseteq H_\gamma$. It follows that $\mathcal{H}$ is an open expansion of $\mathcal{F}$.

Finally, we show that $\mathcal{H}$ is CF; for if $K \subseteq X$ is compact, then there exists $n \in w_0$ such that $K \subseteq V_n$. By the construction of $\mathcal{G}_{(i)}$ above, we have $\cup\mathcal{G}_{(i)} \cap V_n = \emptyset$, for all $i > n$. Hence $\cup\mathcal{G}_{(i)} \cap K = \emptyset$ for $i > n$. By the construction of $\mathcal{H}_{(i)}$, K hits only finitely many members of $\{\cup\mathcal{H}_{(i)} : i \in w_0\}$; and since each $\mathcal{H}_{(i)}$ is CF, K hits only finitely many members of $\mathcal{H}$.

In 1973, J. R. Boone [6] proved the following characterization of mesocompactness. We give a proof below for completeness.

Theorem 4.2.3. *A normal space X is mesocompact iff X is metacompact and discretely CF expandable.*

Proof. If X is mesocompact, then X is clearly metacompact. Let $\mathcal{F} = \{F_\gamma : \gamma \in \Gamma\}$ be a discrete family of closed subsets of X. For $\gamma \in \Gamma$, define $U_\gamma = X - \bigcup_{\beta \neq \gamma} F_\beta$ and $\mathcal{U} = \{U_\gamma : \gamma \in \Gamma\} \cup \{U^* = X - \bigcup_{\gamma \in \Gamma} F_\gamma\}$.

Since X is mesocompact, $\mathcal{U}$ has a CF open refinement $\mathcal{V} = \{V_\gamma : \gamma \in \Gamma\} \cup \{U^*\}$ where $V_\gamma \subseteq U_\gamma$ for all $\gamma \in \Gamma$. Clearly $\{V_\gamma : \gamma \in \Gamma\}$ is the desired CF open expansion of $\mathcal{F}$.

Now, suppose X is metacompact and discretely CF expandable. Let $\mathcal{U} = \{U_\gamma : \gamma \in \Gamma\}$ be a PF open cover of X. Define $\mathcal{K}_1 = \{\{x \in U_\gamma : ord(x, \mathcal{U}) = 1\} : \gamma \in \Gamma\}$. It is easy to see that $\mathcal{K}_1$ is a discrete collection of closed subsets of X and has a CF open expansion, $\mathcal{T}_1 = \{T(1,\gamma) : \gamma \in \Gamma\}$, such that $T(1,\gamma) \subseteq U_\gamma$ for every $\gamma \in \Gamma$. By the normality of X, let G_1 be an open set such that $\cup\mathcal{K}_1 \subseteq G_1 \subseteq \overline{G}_1 \subseteq \cup\mathcal{T}_1$.

Assume that for $k < n$ we have constructed a compact finite open expansion $\mathcal{T}_k = \{T(k,\gamma) : \gamma \in \Gamma\}$ of $\mathcal{K}_k = \{U_{\gamma_1} \cap U_{\gamma_2} \cap U_{\gamma_3} \cap \cdots \cap U_{\gamma_k} \cap \{x \in X : ord(x, \mathcal{U}) = k\} - \cup\mathcal{T}_{k-1} : \gamma_1, \ldots, \gamma_k$ are distinct members of $\Gamma\}$ and open sets G_k such that

(i) $\mathcal{T}_k$ misses $\overline{G}_{k-1}$ and partially refines $\mathcal{U}$

(ii) $\overline{G}_{k-2} \subseteq G_{k-1}$ and

(iii) G_{k-1} is an open set such that $\cup\mathcal{K}_{k-1} \subseteq G_{k-1} \subseteq \overline{G}_{k-1} \subseteq \cup\mathcal{T}_{k-1}$.

Again by the normality of X we can choose an open set G_{n-1} such that $\cup\mathcal{K}_{n-1} \subseteq G_{n-1} \subseteq \overline{G}_{n-1} \subseteq \cup\mathcal{T}_{n-1}$ and $\overline{G}_{n-2} \subseteq G_{n-1}$. Furthermore, since $\mathcal{K}_n = \{U_{\gamma_1} \cap \cdots \cap U_{\gamma_n} \cap \{x \in X : ord(x, \mathcal{U}) = n\} - \cup\mathcal{T}_{n-1} : \gamma_1, \ldots, \gamma_n$ are distinct members of $\Gamma\}$ is discrete in X, there exists a CF open expansion $\mathcal{T}_n = \{T(n,\gamma) : \gamma \in \Gamma\}$ of $\mathcal{K}_n$ that misses $\overline{G}_{n-1}$ and partially refines $\mathcal{U}$. Now define $\mathcal{T} = \{T_\gamma = \cup\{T(n,\gamma) : n \in N\} : \gamma \in \Gamma\}$. It is easy to show that $\mathcal{T}$ is a CF open refinement of $\mathcal{U}$, and hence X is mesocompact.

We are now ready to obtain some new σ–product theorems.

Theorem 4.2.4. *Let $X = \sigma\{X_\alpha : \alpha \in A\}$ be normal such that each finite subproduct of X is mesocompact. Then X is mesocompact.*

Proof. By Theorem 4.1.10 above, X is metacompact; and by Theorem 4.2.2 X is discretely CF expandable. Hence X is mesocompact by Theorem 4.2.3.

Question 4.2.5. Can the normality condition in Theorem 4.2.4 above be weakened?

Remark 4.2.6.. *In [12] K. Chiba proved a σ–product theorem for para–Lindelöfness which required that the space be normal and countably paracompact. The next theorem generalizes this result by eliminating the requirement of countable paracompactness.*

Theorem 4.2.7. *Let $X = \sigma\{X_\alpha : \alpha \in A$ be normal and such that each finite subproduct of X is para–Lindelöf. Then X is para–Lindelöf.*

Proof. Let $\mathcal{U} = \{U_\gamma : \gamma \in \Gamma\}$ be an open cover of X. Choose an open subset V_0 of X and a member γ_0 of Γ such that $s^* \in V_0 \subseteq \overline{V}_0 \subseteq U_{\gamma_0}$.
Define $U(0,\gamma) = \begin{cases} \emptyset & if\ \gamma \neq \gamma_0 \\ U_{\gamma_0} & \text{otherwise} \end{cases}$ and let $\mathcal{U}_0 = \{U(0,\gamma) : \gamma \in \Gamma\}$.

Assume that for each $k \leq n$, we have constructed a LC open partial refinement $\mathcal{U}_k = \{U(n,\gamma) : \gamma \in \Gamma\}$ of $\mathcal{U}$ that covers X_k. Furthermore assume that we have chosen open sets V_k such that $X_k \subseteq V_k \subseteq \overline{V}_k \subseteq \cup\mathcal{U}_k$ and $\overline{V}_{k-1} \subseteq V_k$ for all $k \leq n$.

Since Y_a is para–Lindelöf for each $a \in [A]^{n+1}$, we have a LC open (in Y_a) refinement $\mathcal{U}^*_{(n+1,a)}$ of $\mathcal{U}|\left(Y_a - \overline{V}_n\right)$. Let $\mathcal{U}^*_{(n+1,a)} = \{U^*(n+1,a,\gamma) : \gamma \in \Gamma\}$. Now define,

$$U(n+1,a,\gamma) = [p_a^{-1}(U^*(n+1,a,\gamma) - \overline{V}_n)] \cap (U_\gamma - \overline{V}_n)$$

and $U^*(n+1,\gamma) = \cup\{U_{(n+1,a,\gamma)} : a \in [A]^{n+1}\}$. By Lemma 1.2.48 above, $\mathcal{U}^*_{n+1} = \{U^*(n+1,\gamma) : \gamma \in \Gamma\}$ is a LC family of open subsets of X which covers $X_{n+1} - (\cup\mathcal{U}_n)$ and partially refines $\mathcal{U}$. It now follows that $\mathcal{U}_{n+1} = \{U(n+1,\gamma) = U^*(n+1,\gamma) \cup U(n,\gamma) : \gamma \in \Gamma\}$ is an open, LC (in X) partial refinement of $\mathcal{U}$ that covers X_{n+1}.

Continue the procedure above by induction, and let $\mathcal{H} = \{H_\gamma = \cup\{U(n,\gamma) : n \in w_0\} : \gamma \in \Gamma\}$. Clearly $\mathcal{H}$ is an open refinement of $\mathcal{U}$. Furthermore, for $x \in X$ choose the first $n \in w_0$ such that $x \in X_n$. Note that for $m > n$, $x \in \overline{V}_n \subseteq \overline{V}_{m-1}$ and hence $x \notin \cup\mathcal{U}^*_m$. It now follows that $\mathcal{H}$ is LC; and hence X is para–Lindelöf.

Remark 4.2.8. *Alternate forms of the Special $B(D,w_0)$ Sum Theorem can be obtained to provide a more general setting for many σ–product theorems listed in this thesis. We give an example below in the case of mesocompactness.*

Second Special $\mathbf{B(D,\omega_0)}$ Sum Theorem. 4.2.9. *Let X be a normal space with a $B(D,w_0)$-breakdown $\mathcal{B} = \cup\{\mathcal{B}_n : n \in w_0\}$, where $\mathcal{B}_n = \{B(n,\gamma) : \gamma \in A_n\}$, which has the following properties:*

(1) For each $n \in w_0$, if U is an open set such that $\bigcup_{i<n}(\cup\mathcal{B}_i) \subseteq U$ then $\{B(n,\gamma) - U : \gamma \in A_n\}$ has a CF open expansion in X.

(2) For each $n \in w_0$ and $\gamma \in A_n$, if $\mathcal{G}_{(n,\gamma)} = \{G(n,\gamma,\delta) : \delta \in \Delta\}$ is a CF collection of open subsets of $B(n,\gamma)$, then $\mathcal{G}_{(n,\gamma)}$ has a CF open expansion in X.

(3) For every $n \in w_0$ and $\gamma \in A_n$ the set $B(n,\gamma)$ is mesocompact.

Then X is mesocompact.

Proof. Let $\mathcal{U} = \{U_\delta : \delta \in \Delta\}$ be an open cover of X. Let $\mathcal{H}_0 = \{H(0,\gamma) : \gamma \in A_0\}$ be a CF open (in X) expansion of $\mathcal{B}_0$. Now for each $\gamma \in A_0$ let $\mathcal{U}_{(0,\gamma)}$ be a CF open (in $B(0,\gamma)$) refinement of $\mathcal{U} \mid B(0,\gamma)$. By (2) above there exists a CF open (in X) expansion $\mathcal{U}^*_{(0,\gamma)}$ of $\mathcal{U}_{(0,\gamma)}$. Define $\mathcal{G}_{(0,\gamma)} = \{U^* \cap H(0,\gamma) : U^* \in \mathcal{U}_{(0,\gamma)}\}$. Then $\mathcal{G}_0 = \cup\{\mathcal{G}_{(0,\gamma)} : \gamma \in A_0\}$ is a CF open (in X) partial refinement of $\mathcal{U}$ that covers $\cup\mathcal{B}_0$.

By the normality of X, there exists an open set V_0 such that $\cup\mathcal{B}_0 \subseteq V_0 \subseteq \overline{V}_0 \subseteq \cup\mathcal{G}_0$.

Now by (1) and the same procedure above, we construct a CF open (in X) collection, $\mathcal{G}_1 = \cup\{\mathcal{G}_{(1,\gamma)} : \gamma \in A_1\}$ which misses $\overline{V}_0$, covers $(\cup\mathcal{B}_1) - \cup\mathcal{G}_0$, and partially refines $\mathcal{U}$. Continue this procedure by induction to obtain for each $n \in w_0$ an open (in X) collection $\mathcal{G}_n$, which is a CF partial refinement of $\mathcal{U}$ that covers $(\cup\mathcal{B}_n) - \bigcup_{i<n}(\cup\mathcal{G}_i)$, and an open set V_n, such that $\overline{V}_i \subseteq V_n$ if $i < n$ and $\bigcup_{i\leq n}(\cup\mathcal{B}_i) \subseteq V_n \subseteq \overline{V}_n \subseteq \bigcup_{i\leq n}(\cup\mathcal{G}_i)$. Now for any compact subset $K \subseteq X$, there exists an $n \in w_0$ such that $K \subseteq V_n$ and $K \cap (\cup\mathcal{G}_i) = \emptyset$ if $i > n$. It is easy to see that K meets only finitely many members of $\bigcup_{i\leq n}(\cup\mathcal{G}_i)$. It follows that $\mathcal{G} = \cup\{\mathcal{G}_i : i \in w_0\}$ is a CF open refinement of $\mathcal{U}$, and hence X is mesocompact.

Corollary 4.2.10. *Mesocompactness satisfies the σ–product theorem for normal σ–products.*

$\mathbf{B(D,\omega_0)}$ Sum Theorem for closed hereditary irreducibility 4.2.11. *Let X be a space with a $B(D,\omega_0)$ breakdown $\mathcal{B} = \cup\{\mathcal{B}_n : n \in \omega_0\}$ where each $\mathcal{B}_n = \{B(n,\gamma) : \gamma \in A_n\}$ is such that every closed subset of $B(n,\gamma)$ is irreducible. Then X is irreducible.*

Proof: Let $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ be an open cover of X. For $n \in \omega_0$, let $\mathcal{G}_n = \{G(n,\gamma) : \gamma \in A_n\}$ be an open expansion of $\mathcal{B}_n$ that is discrete in $\cup\mathcal{B}_n$ and is such that $\cup\mathcal{G}_n \cap \left(\bigcup_{i<n}(\cup\mathcal{B}_i)\right) = \emptyset$.

For each $\gamma \in A_1$, let $\mathcal{H}_{(1,\gamma)} = \{H(1,\gamma,\alpha) : \alpha \in A\}$ be a minimal collection of open sets (in X) which refines $\mathcal{U}|B(1,\gamma)$ and is such that each $H(1,\gamma,\alpha) \subseteq G(1,\gamma)$. Let $\mathcal{H}_1 = \cup\{\mathcal{H}_{(1,\gamma)} : \gamma \in A_1\}$. Assume, for $k < n$, that for each $\gamma \in A_k$ there exists a minimal collection $\mathcal{H}_{(k,\gamma)} = \{H(k,\gamma,\alpha) : \alpha \in A\}$ of open sets which refines $\mathcal{U}|\left[B(k,\gamma) - \bigcup_{i<k}(\cup\mathcal{H}_i)\right]$ and such that

(i) $H(k,\gamma,\alpha) \subseteq G(k,\gamma)$, and

(ii) $\mathcal{H}_k = \cup\{\mathcal{H}_{(k,\gamma)} : \gamma \in A_k\}$.

Since $B_{(n,\gamma)} - \left(\bigcup_{i<n}(\cup\mathcal{H}_i)\right)$ is closed and hence irreducible, for each $\gamma \in A_n$, there exists a minimal collection $\mathcal{H}_{(n,\gamma)} = \{H(n,\gamma,\alpha) : \alpha \in A\}$ of open sets which refines $\mathcal{U}|B(n,\gamma) - \left(\bigcup_{i<n}(\cup\mathcal{H}_i)\right)$ where each $H(n,\gamma,\alpha) \subseteq G(n,\gamma)$.

Define $\mathcal{H}_n = \cup\{\mathcal{H}_{(n,\gamma)} : \gamma \in A_n\}$. Then it is easy to show that $\mathcal{H} = \cup\{\mathcal{H}_n : n \in \omega_0\}$ is a minimal open refinement of $\mathcal{U}$, and hence X is irreducible.

Remark 4.2.12. *By an easy transfinite induction argument, the proof of Theorem 4.2.11 can be extended to show that, for any ordinal α, the $B(D,\alpha)$ Sum Theorem holds for closed hereditary irreducibility.*

We now have the following corollary.

Corollary 4.2.13. *The σ-product theorem holds for closed hereditary irreducibility.*

Question 4.2.14. Many interesting questions regarding σ-products remain open. For example, does the σ-product theorem hold for the properties $B(D,\lambda)$-refinability and $B(LF,\lambda)$-refinability?

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